Foundations of Physics (2022) 52:18
https://doi.org/10.1007/s10701-021-00535-9

Isolated Objects and Their Evolution: A Derivation of the Propagator's Path Integral for Spinless Elementary Particles

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Received: 21 May 2021 / Accepted: 22 December 2021

Abstract
We formalize the notion of isolated objects (*units*), and we build a consistent theory to describe their evolution and interaction. We further introduce a notion of indistinguishability of distinct spacetime paths of a unit, for which the evolution of the state variables of the unit is the same, and a generalization of the equivalence principle based on indistinguishability. Under a time reversal condition on the whole set of indistinguishable paths of a unit, we show that the quantization of motion of spinless elementary particles in a general potential field can be derived in this framework, in the limiting case of weak fields and low velocities. Extrapolating this approach to include weak relativistic effects, we explore possible experimental consequences. We conclude by suggesting a primitive ontology for the theory of isolated objects.

Keywords Foundations of quantum mechanics · Equivalence principle · Path integral formulation

1 Introduction

In this paper we build a theory where physical events are defined with respect to the identification of distinct isolated objects (*units*): elementary particles, or more complex physical objects, whenever they do not significantly affect their surrounding environment.

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Published online: 05 January 2022

Since in isolation a unit is the only object validating the evolution of its own state variables, we start from the premise that such evolution is a physically valid local representation of reality, until the unit matches its locally available information in interaction with another unit. As we shall see, this premise has far-reaching implications in terms of how an isolated object is localized in an external frame of reference and of how it responds to external potential fields. To emphasize this point, we use the unusual term *experience* to refer to the evolution of the state variables of an isolated object.

We will demonstrate that some basic additional assumptions on the notion of experience of isolated objects, and on the intermittent matching of the experiences of distinct isolated objects, are sufficient to derive the quantum mechanical propagator for spinless elementary particles in the limiting case of weak fields and low velocities.

Note that the theory of isolated objects does not rely on any preferred role of conscious observers (or even less, of conscious objects) in determining the outcome of physical processes. Indeed, the notion of experience of a unit and the notion of matching of experiences that we propose in Sect. 2 simply assume that to properly understand physical evolution and interaction of units we must first consider the restricted information available to the objects directly involved. The theory of isolated objects can then be seen as a contribution to the understanding of the role of information and computation in the foundations of quantum systems [24, 39].

Additionally, the theory we develop is relational, since it relies on matching distinct experiences, and it allows for multiple and compatible possibilities of motion of a unit to be meaningfully considered real physical quantities, as we will show in Sect. 2.2. These general characteristics are shared respectively with relational quantum mechanics [32] and with the transactional interpretation of quantum mechanics [9, 25]. We also note that the emphasis on the perspective of distinct objects has been already suggested in a different form in [35], in the broader context of the real ensemble interpretation of quantum mechanics [34]. In this approach, every element of a physical system is associated with a "view" encoding the knowledge of the system according to that specific element.

Broadly speaking, the paper is divided into three parts with different emphases: Sects. 2, 3, 4, and 5.

The longer Sect. 2 is concerned with the general theory of isolated units, that establishes the objects of interest and their properties. We introduce the notions of isolated unit and of experience of a unit and we list their basic properties, including the assumption of an internal time state, which we then relate to de Broglie's hypothesis on the existence of an internal clock frequency, first postulated in [10, Chapter 1] and expanded in its scope in recent years [5, 11, 12, 26, 28]. We further introduce a notion of indistinguishability of distinct spacetime paths of a unit, for which the evolution of its state variables is the same, and a generalization of the equivalence principle based on indistinguishability.

In Sects. 3 and 4, we develop the mathematical and physical apparatus implicit in the axioms and principles from Sect. 2. More particularly, the main result of Sect. 3 is the construction of a relativistically invariant path integral propagator for *simple units*, i.e. units whose state variables do not change in time. This construction is dependent on a time reversal condition on the whole set of indistinguishable paths of a unit and on identifying the frames of reference that can be meaningfully defined for isolated units. In Sect. 3.3 we show that the propagator for simple units approximates the quantum mechanical propagator of a spinless elementary particle in a potential field, in the limiting case of weak potentials and low velocities. In Sect. 4 we extrapolate the results of Sect. 3.3 to include weak relativistic effects on the propagator of a charged simple unit and we suggest the outline of an experimental setting that could detect small deviations from the predictions of standard quantum mechanics, under restrictive conditions on the extremal trajectory of the charged unit to minimize field radiation.

In Sect. 5.1 we suggest a possible primitive ontology (PO) for isolated objects, following the general definition of primitive ontologies outlined in [3], that emphasizes quantities in a theory that can be put in direct correspondence with the world we experience. Finally, in Sect. 5.2 we offer some concluding remarks on future developments of the approach developed in this paper.

The appropriate PO for isolated objects will be based on a localization of the matching of experiences for a wide class of units that have internal structure. The identification of the PO will also stress some implications of our theory for actual structured physical objects, and it will underline that the objects of interest in our theory are essentially objects as we experience them, as spacetime localizations of mass. The possibility of quantum-like behavior for isolated objects is a consequence of their being isolated and of the restricted information carried by their state variables, as we will show in Sects. 3 and 4. The phenomenological observations we will make, when developing the PO of isolated objects, will also provide a counterbalance to the more abstract approach we take at the beginning of the paper.

Indeed, abstracting from the specific state variables of concrete objects allows to highlight the structure that any object in isolation must have, and to deduce the principles that must apply uniformly to all isolated objects, irrespectively of their structure or size. At the same time, it is important to keep in mind from the very beginning that simple units, having a fixed and unchanging set of state variables, are in correspondence with spinless elementary particles (seen as dimensionless point particles with fixed rest mass), *whenever they are isolated*, while more complex isolated objects (*structured units*) with internal structure can be naturally put in correspondence with isolated elementary particles with variable spin direction, atoms, molecules or larger solid compounds.

2 The Evolution of Isolated Objects

2.1 Isolated Objects

We now formally define isolated objects, and we introduce the key properties necessary to understand their evolution. The purpose of this section is to explore the possibility of a consistent theory for isolated physical objects, assuming that their evolution (the “experience” of Definition 3) is subordinate to the restricted information available to the objects themselves.

Definition 1 An *isolated object* (or *unit*) is a physical object that is affected by potential fields generated by other objects and that does not significantly affect its surrounding environment.

This definition of units gives a basic criterion to establish when an object counts as distinct, by stressing the asymmetry in its interactions with the surrounding environment. The terms "interaction" and "potential field" are assumed to be given primitive notions and in the following they will be clarified case by case in specific physical contexts. Note that, in principle, there is no restriction on the size or structure of an isolated object, as units are defined relationally: *any* physical object can be, however briefly, a unit if it does not significantly affect its surrounding environment (and following Definition 2 we give some examples of units of particular interest). The expression "does not significantly affect" can be made precise by saying that the state of the surrounding environment in the presence of the unit cannot be distinguished from the state of the surrounding environment when the unit is absent.

If we consider macroscopic changes of the surrounding environment, this definition can be seen as a variation on the view that changes in the "position of things" [7, Chapter 19.2] are the relevant variables of a meaningful physical theory. However, according to Definition 1, a unit is not isolated also when there are changes of the microscopic states of another object (say, its spin direction), and not only when there are macroscopic changes in the state of the surrounding environment.

Assumption 1 Every unit has a set of *internal state variables* that define its physical properties and an associated *internal time state* with respect to which changes of the internal state variables are defined.

Internal state variables include the list of physical properties that define the characteristics of an object needed to study its evolution. According to the interaction that is considered in the theory, different sets of internal state variables may be considered. For example, if we consider gravitational potential fields, we will take rest mass as one of the internal state variables of a unit, and if in addition electromagnetic fields are considered, the charge of the unit will also be an internal state variable, and its spin in more general settings. Because of the equivalence of gravitational and inertial mass, and because of the fundamental role of the latter in describing motion of an object in any potential field, we will always assume that a unit has rest mass as one of its internal states.

Changes in the values of the internal state variables (when such changes are permissible) are established with respect to an internal time state τ, and we note that in principle each interaction that we consider in the theory can separately affect the internal state variables of the unit. In this section and in Sect. 3.1 it is helpful to think of τ as the proper time along the trajectory of the unit. We will see in Sect. 3.2 that τ needs to satisfy some specific mathematical constraints to be compatible with the setting of isolated units, and we will give its analytical form, distinct from proper time, in Definition 9 of Sect. 3.2.

Definition 2 A unit whose internal state variables cannot change with respect to internal time is a *simple unit*.

The only physical embodiments of simple units are isolated spinless elementary particles, with internal state variables defined by mass and charge. Any unit that is not simple is a *structured unit*. Structured units include the following: isolated composite units, such as atoms, made of simple units; complex molecules and macroscopic, composite objects; isolated elementary particles that can have spin states with varying directions; entangled elementary particles. This paper will be concerned mostly with simple units, structured units will be addressed separately. Still, whenever a result will not be specifically restricted to simple units, we may assume it to be valid for structured units as well. As already noted, the theory we are building affirms the primacy of the notion of isolated objects (whenever they arise) in understanding physical objects *irrespective of the size or complexity of the units*. Even though our main results in this paper concern simple units, and therefore spinless elementary particles, the theory of isolated objects is not specifically a theory of elementary particles.

In the following definition we address the relation of the evolution of the internal state variables of a unit and its surrounding environment.

Definition 3 A *labelled experience* of a unit is the evolution of its internal state variables over some period of internal time, for a given surrounding environment and a given spacetime localization (a path) within the surrounding environment.

When looking at the evolution of the internal state variables of a unit, there is in general no way from their changes to allow us to reconstruct the surrounding environment in which they are evolving. For example, a particle with spin may change its spin direction, which possibly implies the presence of a variable magnetic field, but the simple fact that the spin direction has changed does not allow us to reconstruct the surrounding field. However, a surrounding environment is implied nevertheless; note that two evolutions can be identical, and still need to be considered as distinct according to the environment in which they occur. This is the reason we refer to the evolutions of the internal state variables as "labelled" according to their localization in their environment.

We use the unconventional term "experience" in relation to the evolution of the internal state variables to stress that the restricted information available to an isolated object through its internal state variables determines, as long as the object is isolated, the ways it can be localized in its surrounding environment and the way it is affected by external potential fields. This simple observation, brought to its extreme consequences, will have a crucial impact in the theory, and it will lead in Sect. 2.2 to the notion of indistinguishable paths of a unit and to a generalization of the equivalence principle for isolated units.

For ease of reading, we will often drop the qualification “labelled” for experiences, and we will refer to them as “experiences” tout court.

Note that a specific embodiment for the internal time state of Assumption 1 is provided by the hypothesis of De Broglie of an internal frequency being attached to any finite mass body [10, Chapter 1]. In particular, de Broglie conjectured that for any body of finite rest mass $M_0 > 0$ there is an internal phenomenon (a “clock”) of frequency $\nu_0 = M_0 c^2 / h$ attached to the body itself, where h is Planck’s constant and c is the speed of light. Alternatively, we can write $\omega_0 = M_0 c^2 / \hbar$, if for simplicity we switch to units of measure such that angular frequency is measured in radians per second and we indicate with $\hbar = \frac{h}{2\pi}$ the reduced Planck constant.

In Sect. 3.2 we will derive a relativistically invariant propagator of simple units that uses in an essential way the internal clock frequency and in Remark 14 we will argue that the analytic form of the propagator requires that the internal clock frequency for the internal time state must be a multiple α of the frequency conjectured by de Broglie. In Sect. 3.3 we will then show that our framework reduces to standard quantum mechanics (under the conditions in which the latter applies) if we choose $\alpha = \frac{1}{2}$. Accordingly, already in this section we redefine ω_0 as follows:

Definition 4 The *internal clock frequency* ω_0 of an isolated unit of rest mass $M_0 > 0$ is

$$\omega_0 = \frac{\alpha M_0 c^2}{\hbar}, \tag{1}$$

where α is an appropriate scaling constant.

Remark 1 A direct probing of the reality of the internal clock frequency for a particle such as an electron would require the measurement of effects of the order of its Compton wavelength, something that is not yet feasible. However, the physical reality of the internal clock frequency has been indirectly tested by electron channeling in silicon crystals [8, 21]. These preliminary experimental results seem consistent with a value of the internal frequency that is double the one predicted by de Broglie. Note however that the specific experimental deviation from de Broglie’s internal frequency has no direct role in what follows and does not contradict our choice of $\alpha = \frac{1}{2}$ in Sect. 3.3, as the internal clock frequency will always appear in our work in relation to internal time, and not time as measured in an external frame of reference. Additionally, regardless of the possibility of a direct experimental confirmation of the existence of the internal frequency, in this paper we show that there are concrete physical consequences of assuming its existence, such as: the possibility (explored in Sect. 3.3) of relating an extension of the equivalence principle to the path integral for spinless particles; as well as the experimental possibility (in Sect. 4) of small deviations from the predictions of standard quantum mechanics for the trajectory of slowly moving electrons. These deviations would be small, but they would still be at a much larger scale than the Compton wavelength of an electron.

The periodic representation of τ compatible with the internal clock frequency ω_0 can be taken to be $\mathcal{P}(\tau) = e^{i\omega_0 \tau}$. Note that, in principle, any periodic function of period ω_0 would be suitable to define $\mathcal{P}(\tau)$, in the setting we have described so far. The significance of the complex exponential in the setting of isolated units will become clear when evaluating differences of internal time states for distinct trajectories of the unit, as we will do in the next section. Note also that the dependence of $\mathcal{P}(\tau)$ on the rest mass of the unit elevates mass to a special status among internal states, and this is another reason we assumed in the discussion of Assumption 1 that all units have rest mass as an internal state.

Remark 2 The notion of internal clock frequency has been used in several works exploring the interfacing of quantum mechanics with the theory of relativity and in the presence of gravitational or more general potential fields [5, 11, 12, 26, 28]. The most comprehensive approach to quantum mechanics based on de Broglie's hypothesis can be found in the body of work exemplified by [11, 12], where a Lorentz invariant description of elementary particles is given in terms of cyclic Minkowski spacetime coordinates associated to the internal clock frequency, and in the context of the deterministic dynamics of one-dimensional classical closed strings vibrating in a four-dimensional spacetime. In particular, in [12] the classical evolution of all de Broglie's internal clock dynamics (that satisfy periodic boundary conditions in a cyclic time dimension) is proven to be equivalent to the Feynman path integral propagator in ordinary spacetime.

As it is the case for all recent works that assume de Broglie's hypothesis, the crucial impact of the internal clock frequency in our theory is that it endows the notion of internal time state with a periodic representation. Nevertheless, the main objective of this work is to explore some of the concrete and most direct physical consequences of the theory of isolated units. Accordingly, the notion of isolated unit will continue to be the dominant thread in the way the internal clock frequency will be used, and, in particular, in establishing Lorentz invariance for the propagator of simple units in Sect. 3.2.

2.2 Indistinguishability and Isolated Equivalence

We established in Sect. 2.1 the basic notions of isolated unit and of experience of a unit. We now address how the type of internal state variables of a unit affect the possibility of internally distinguishing distinct labelled experiences.

Definition 5 *Indistinguishability.* Any two labelled experiences of a unit are internally indistinguishable if they cannot be distinguished through changes in the unit's internal state variables over internal time. The corresponding spacetime localizations of the two experiences are defined as the internally *indistinguishable paths* of the unit.

With Definition 5 we move to a crucial point: when the internal state variables are not changed (or are equally changed) by any pair of experiences, there is no way for the unit to establish in isolation which of the two has happened, they are not distinguishable.

Since the unit is an isolated object and it is the only object validating the reality of a specific evolution of its internal state variables, indistinguishable experiences that cannot be physically discriminated before interaction are all valid, real experiences for the unit itself, and must eventually have a physical impact in interaction. For a simple unit, whose internal state variables are unchanging, all externally labelled and distinct experiences will be internally indistinguishable and therefore physically valid in isolation.

Specifically, in the experience of a simple unit, *its motion at any instant can be taken to be in any direction and at any velocity*. Indeed, a simple unit has no ability to distinguish these distinct paths, since its internal state variables are unchanging over internal time; therefore, all these possibilities of motion are fully consistent with the evolution of its internal state variables, even in the presence of potential fields that exert a force on the unit, as long as the unit itself does not need to match its experience with other units. Because motion at each instant could be at any velocity and in any direction, the labelled indistinguishable paths of a simple unit are all continuous, not differentiable paths. Even moving at a speed higher than light is logically allowed in this setting, as long as the unit is not forced to confirm this possibility with the external environment.[1]

Remark 3 Note that the “reality of possibility” is a basic tenet of the transactional interpretation of quantum mechanics [9, 25], where it is derived from the formalism of quantum mechanics itself. In the setting of isolated units, the reality of the entire range of indistinguishable possibilities of motion is derived from the notion of isolation and from the type of internal state variables of the unit, before establishing a formal connection with quantum mechanics. In Sect. 3.1 we will argue that, during interaction, the impact of the whole range of indistinguishable paths of a unit is always mediated by the assessment of the significance of each path with respect to all the others. In this respect, indistinguishable paths are not to be considered classical paths, because we cannot fully separate them from each other in determining their relational impact on external events such as localization at a specific spacetime point.

[1] When dealing with structured units, care must be given to the identification of the appropriate set of indistinguishable paths that change the internal state variables in the same way. This set will be generally smaller than the corresponding set for simple units.

We also note that the notion of indistinguishability, though set here as a property of the labelled experiences of general isolated units, has a direct antecedent in the sum-over-histories formalism of quantum theories with its distinction of variables in observables and unobserved labels (see for example [22]). In this formalism, probability amplitudes in path integrals are summed over the range of unobserved labels of a given object, if a corresponding "experiment does not determine the end of a history precisely, as most will not" [22]. Effectively, unobserved labels can be considered in our terminology as states defining indistinguishable labelled experiences.

This paper does not specifically address the mechanism of interaction, still, the theory of isolated objects would not be complete without the inclusion of a general prescription on how the labelled experiences of units, described by the evolution of their internal state variables, are affected in interaction. To this purpose, we define a notion of matching of experiences.

Assumption 2 *Matching of experiences.* When units interact, they match their labelled experiences to be reciprocally compatible.

This assumption basically asserts that the range of indistinguishable paths of a unit is reset every time it interacts with another unit. Consider, for example, the range of all indistinguishable paths of a simple isolated unit S; we have seen in the discussion of Definition 5 that all types of motions are allowed for such a unit in isolation. Assume now that a macroscopic structured unit U_M (say, a measuring device) does not have multiple indistinguishable paths. Then any interaction of S with U_M will reset the range of labelled experiences of S, localizing them to the position of their interaction with respect to U_M, and only indistinguishable paths of S compatible with this localization will be preserved. Without this assumption, it would not be meaningful to speak of the propagation of a unit from a specific point A to another point B, as we will do in Sect. 3.3.

The definition of matching of experiences has also important implications on the isolated status of units. If a unit U_1, because of some changes in its internal state variables, experiences being no longer separated from unit U_2, then unit U_2 will no longer be isolated, even if in its experience there is no change in its internal state variables. We will refer to this observation in Remarks 4 and 18 when evaluating under which conditions moving charged particles can be considered isolated units.

By assuring that every time units interact their labelled experiences are reciprocally consistent, we preserve a form of objectivity, while at the same time acknowledging that such objectivity is always mediated by the act of matching the experiences of the units to each other. Note that the moment two distinct units interact, i.e. they exert a reciprocal influence on each other, they cease to be distinct in the sense of Definition 1 for the whole duration of their interaction. Note also that no limitation is put on the conservation of the number of units before and after the interaction.

The theory of isolated units allows a reformulation and an extension of the equivalence principle that emphasizes its relation to indistinguishability. We introduce therefore the following principle:

Principle 1 *Isolated equivalence:* If a unit cannot internally distinguish the labelled experience of being in a stationary frame in a potential field from the labelled experience of being at rest in a non-inertial frame with uniform acceleration, then the two experiences are physically equivalent.

The theory of general relativity was motivated by the realization that locally there is no way to distinguish a frame of reference in uniformly accelerated motion from one that is stationary within a corresponding gravitational field. However, this is true only in the limit of infinitesimal systems, that do not experience the discriminating tidal effects in a gravitational field [29]. Under these limiting conditions, a single isolated charged particle can be effectively considered a simple unit (notwithstanding possible spin states). For such a unit, a frame in uniformly accelerated motion will also be indistinguishable, for example, from a static frame in a Coulomb field. The unit will not be able to discriminate the nature of the field by detecting changes to its internal state variables.

Isolated equivalence assumes that the standard equivalence principle can be extended to all cases, such as this, where the unit cannot discriminate the nature of the field acting on it, *if only in the isolated experience of a simple unit*.[2]

Indeed, in the case of simple units, to be "physically equivalent" refers essentially to dilation effects on internal time, since all other internal state variables are unchanging; internal time dilation is the same for the two indistinguishable experiences with respect to time in an external frame of reference. Note that a surrounding environment is always implied when dealing with labelled experiences, in that we still need an external frame of reference to be able to quantify time dilation.[3]

The fact that the experience of the unit must be compatible with what can be quantified in an external frame of reference has an important role in ensuring transitivity of the notion of isolated equivalence. Indeed, let E_s be the experience of the unit being in a stationary frame in a general potential field. To avoid the possibility for two experiences equivalent to E_s not to be equivalent to each other, time dilation must be the same for all non-inertial frames that are deemed equivalent to the stationary frame in the potential field, despite the fact that all non-inertial frames of reference are in principle indistinguishable for a simple unit. Since, additionally, the simple unit cannot distinguish gravitational fields from other potential fields, the

[2] We focus on simple units in the justification of the principle of isolated equivalence, because of their significance in subsequent sections. However, the principle is formulated to apply to general isolated units.

[3] An external frame of reference can be defined with respect to the position and state of motion of the unit itself at the time of matching of experiences, as we explain in Sect. 3.2.

non-inertial frames equivalent to the stationary frames in a general potential field must be the same as those of the corresponding gravitational fields. We can conclude that the only experiences of the simple unit in non-inertial frames that can meaningfully be considered equivalent to E_s are those with the same acceleration as the one induced on the unit by the potential field.

Remark 4 Isolated equivalence does not contradict the fact that static reference frames in a general potential field are not in general equivalent to uniformly accelerated frames. Such equivalence is only valid for an isolated unit. The motion of continuously interacting units in an electromagnetic field will depend as expected on relativistic effects on the mass.[4] In particular, radiation may affect the unit by making it no longer isolated, as we argue in Remark 18 of Sect. 4. Lorentz invariance of electromagnetic laws is preserved in this case, consistently with the standard equivalence principle and its experimental validations [38].

Unlike the standard equivalence principle of general relativity, isolated equivalence is a conditional principle that depends on the inability of the unit to distinguish experiences. The dependence of isolated equivalence on the type of internal state variables of the unit, and especially on the unit being isolated, sets our contribution apart from other generalizations of the equivalence principle to general fields such as the one developed in [18]. We will show in Sect. 3.3 that this dependence is also the key for the application of isolated equivalence to quantum mechanics.

3 Propagation of Simple Units

3.1 Relative Significance and Time Reversal Condition

We now explore the question of *propagation* of a simple unit S in a reference frame, that is, of how a unit S that is localized at a point $A = (x_a, t_a)$ is then found at point $B = (x_b, t_b)$. The localization of the simple unit at these two points is established by assuming interaction by matching of experiences with other units U_A at A and U_B at B. Even though this question applies to any isolated unit, we assume here that we deal only with simple isolated units such as spinless elementary particles, with mass and charge as internal state variables.

Remark 5 As already noted in Sect. 2.1, we continue in Sect. 3.1 to assume that τ is analogous to proper time and we use the internal clock frequency ω_0 for the periodic representation $\mathcal{P}(\tau)$. In Sect. 3.2 we give a specific representation for the internal

[4] Note that the Abraham–Lorentz–Dirac force (due to the interaction of the charged unit with its own electromagnetic field and to the corresponding radiating field) does not break the equivalence in the isolated perspective since it cannot affect the internal state variables of a simple unit. Such self-force can be interpreted in the experience of the unit as an effect of a varying potential field.

time state, distinct from proper time, and we define a relativistic form for the internal clock frequency.

To take all indistinguishable experiences and their corresponding paths as real, means that, in interacting with another unit, they all should have a physical impact. By considering the relation of indistinguishability and isolated equivalence of the experiences of a unit, we clarify the modalities of this impact in Definitions 6 and 7 that together establish the notion of relative significance of the experience of a unit. They quantify to which extent a given experience of an isolated unit can be trusted to be physically significant in the process of matching of experiences.

Definition 6 Let γ_1 and γ_2 be two indistinguishable paths of a (simple) unit S starting at A and ending at B. Assume S takes internal time τ_1 to reach B from A along γ_1, and τ_2 along γ_2. The *measure of equivalence* $\mathcal{M}(\gamma_1, \gamma_2)$ of the labelled indistinguishable experiences associated to the two paths γ_1 and γ_2 is the periodic representation of the difference of their internal time durations,

$$\mathcal{M}(\gamma_1, \gamma_2) = \mathcal{P}(\tau_1 - \tau_2) = e^{i\omega_0(\tau_1 - \tau_2)}. \tag{2}$$

Since for indistinguishable experiences internal state variables evolve in the same way, internal time differences become the only way to partially assess the impact of the surrounding environment on distinct indistinguishable experiences of the unit. With respect to the experience of the unit at B, any time interval, including the time difference $\tau_1 - \tau_2$, can only be assessed through the periodization of internal time, as $\mathcal{P}(\tau_1 - \tau_2) = e^{i\omega_0(\tau_1 - \tau_2)}$. This representation of time differences assures that they are compared, across indistinguishable paths, with periodicity ω_0 corresponding to the unit's internal clock frequency.

The measure of equivalence assesses the differential impact of the surrounding environment on the indistinguishable experiences of the unit. This impact may be due to the geometric characteristics of each path (say, its length), and to time dilation effects due to potential fields via isolated equivalence.

Remark 6 As anticipated in Sect. 1.1 when introducing the periodic representation of internal time $\mathcal{P}(\tau)$, the relevance of the complex exponential function for isolated units is in great part due to the possibility of expressing the periodic representation of the time difference $\tau_1 - \tau_2$ in terms of the periodic representation of τ_1 and τ_2 respectively, via the relation $e^{i\omega_0(\tau_1 - \tau_2)} = e^{i\omega_0\tau_1} e^{-i\omega_0\tau_2}$. This property is the key in Eq. (6) to an analytical form for the relative significance of the unit going from A to B.

We are now ready to quantify the significance of each experience in interactions.

Definition 7 The *relative significance* of an experience E with respect to a set $\mathcal{S}$ of experiences indistinguishable from E is the sum of all pairwise measures of equivalence of E with the elements of $\mathcal{S}$.

The extent to which a specific experience of a unit can be trusted to be physically meaningful in an interaction is validated by all other indistinguishable experiences that are equally involved in the interaction. Therefore, the totality of all internal time differences of an experience with its set of indistinguishable experiences becomes a proxy for the significance of the experience itself. This concept will play a crucial role in understanding the propagation of units.

Note that a notion of "distinctiveness" of the views of elements of a system plays an important role in [35], similar to the one played by the relative significance and the measure of equivalence for isolated units.

Consider now several indistinguishable paths $\{\gamma_1, \dots, \gamma_n\}$ starting at A and ending at B with internal time durations $\{\tau_1, \dots, \tau_n\}$ respectively. In Definition 7 the relative significance of an indistinguishable experience E of the unit S is defined as the sum of all pairwise measures of equivalence of experience E with respect to the other indistinguishable experiences. Each path γ_i, $i = 1, \dots, n$ identifies one experience of S, therefore the (normalized) relative significance of γ_i with respect to the remaining indistinguishable paths is

$$W(\gamma_i) = \frac{1}{n-1} \sum_{j \neq i} \mathcal{M}(\gamma_i, \gamma_j) = \frac{1}{n-1} \sum_{j \neq i} e^{i\omega_0(\tau_i - \tau_j)}. \tag{3}$$

We can now express the normalized relative significance $W(A, B)$ of the unit going from A to B as the sum of the relative significance of all paths starting at A and ending at B, scaled by n:

$$W(A, B) = \frac{1}{n} \sum_{i} W(\gamma_i). \tag{4}$$

We can calculate the relative significance of a path $\tilde{\gamma}_{AB}$ in the limit of infinitely many paths in analogy to Eq. (3), by assuming a suitable measure $D\gamma_{AB}$ on the space of all paths from A to B. We define $\tau_{\gamma_{AB}}$ to be the total internal time along a generic path γ_{AB} and we write

$$W(\tilde{\gamma}_{AB}) = \int e^{i\omega_0(\tau_{\tilde{\gamma}_{AB}} - \tau_{\gamma_{AB}})} D\gamma_{AB}. \tag{5}$$

Then, the relative significance $W(A, B)$ of the unit going from A to B becomes the integral over $\tilde{\gamma}_{AB}$ of $W(\tilde{\gamma}_{AB})$,

$$\begin{aligned} W(A, B) &= \int \left(\int e^{i\omega_0(\tau_{\tilde{\gamma}_{AB}} - \tau_{\gamma_{AB}})} D\gamma_{AB} \right) D\tilde{\gamma}_{AB} \\ &= \int e^{i\omega_0 \tau_{\tilde{\gamma}_{AB}}} D\tilde{\gamma}_{AB} \cdot \int e^{-i\omega_0 \tau_{\gamma_{AB}}} D\gamma_{AB}, \end{aligned} \tag{6}$$

where we assume that the integrals can be separated.

We set $\int_{\gamma_{AB}} d\tau$ to be an integral parametrization of the total internal time $\tau_{\gamma_{AB}}$ along the path γ_{AB} starting at A and ending at B, and we write

$$W(A,B) = \int e^{i\omega_0 \int_{\tilde{\gamma}_{AB}} d\tau} D\tilde{\gamma}_{AB} \cdot \int e^{-i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB}. \tag{7}$$

Now, in general the two integrals in Eq. (7) will be different. We note that

$$\int e^{-i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} = \int e^{i\omega_0 \int_{\gamma_{BA}} d\tau} D\gamma_{BA}, \tag{8}$$

which can be interpreted to mean that $W(A, B)$ depends both on paths from A to B and on those from B to A.

The time-symmetric structure of $W(A, B)$ is reminiscent of the dependence of a quantum system on the boundary conditions in the future and in the past as posited by the two-state-vector formalism of quantum mechanics [1, 2], or by the transactional interpretation of quantum mechanics [25].

We expect that the expression in Eq. (8) will be significant when exploring causality constraints for structured units. However, in this paper we make the following simplifying assumption that leads to a mathematical form for $W(A, B)$ closer to the standard quantum mechanical propagator.

Time Reversal Condition Given the space Γ_{AB} of indistinguishable paths from A to B, there is a map $\Sigma : \Gamma_{AB} \to \Gamma_{AB}$ that satisfies the following conditions:

1. *Time reversal along each path:* for every path γ_{AB}

$$\int_{\Sigma(\gamma_{AB})} d\tau = -\int_{\gamma_{AB}} d\tau. \tag{9}$$

2. *Density:* the image of Σ is dense in Γ_{AB}.
3. *Invariance:* the measure $D\gamma_{AB}$ in

$$\int e^{-i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} \tag{10}$$

   is invariant (up to sign) with respect to Σ.

Under the Time Reversal Condition, we have that

$$\begin{aligned} W(A,B) &= \int e^{i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} \cdot \int e^{-i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} \\ &= \int e^{i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} \cdot \int e^{i\omega_0 \int_{\Sigma(\gamma_{AB})} d\tau} D\Sigma(\gamma_{AB}) \\ &= \pm \Big(\int e^{i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} \Big)^2. \end{aligned} \tag{11}$$

If we want to compare the relative significance of different sets of indistinguishable paths, as for example the set of paths from A to B with those from A to B', we can provide a partial ordering of relative significance by taking the module of W in Eq. (11):

$$|W(A,B)| = \left| \int e^{i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} \right|^2. \tag{12}$$

This partial ordering of the relative significance makes sense only from an external perspective. Indistinguishable paths cannot be separated into subgroups unless distinct matchings of experiences are considered.

If we scale uniformly $|W(A, B)|$ to make it into a probability distribution, we can equate it with the likelihood $P(A, B)$ of a simple unit that interacted at A with another unit to then interact again with another unit at B. Up to a rescaling hidden in the measure of the integral, we can finally write

$$P(A,B) = \left| \int e^{i\omega_0 \int_{\gamma_{AB}} d\tau} D\gamma_{AB} \right|^2. \tag{13}$$

As much as the formal connection of $|W(A, B)|$ with the computation of probabilities via path integrals in quantum mechanics is obvious, conceptually the process that led to Eq. (13) is very different.

We established how localization of a simple unit at two spacetime points A and B affects the relative significance of all compatible indistinguishable paths. We defined the relative significance of going from A to B and finally we imposed an ordering on the relative significance of distinct choices of the pair of points (A, B). The interplay of indistinguishability and equivalence naturally led to a representation of the likelihood $P(A, B)$ of propagating from A to B as the square of the module of a complex number.

Remark 7 The description of simple units by their internal time and its phase representation is reminiscent of the description of light propagation suggested in [14], with its stress on stopwatches associated with each possible path of a photon. However, the dependence of probability amplitudes from elapsed time in the photon detection is not a fundamental characteristic of the physics of photons. It is rather a consequence of the excitation and decaying processes of atoms involved in emitting and detecting the photons themselves [16, Sect. 4], [17]. We also note that the framework developed for simple units does not apply to photons in its current form, but rather to elementary particles of nonzero rest mass.

3.2 A Relativistically Invariant Propagator for Simple Units

As noted in Sect. 2.2, a simple isolated unit S experiences motion on the set of all its indistinguishable paths. Accordingly, it is not possible to reduce the integral in Eq. (13) only to paths that are time-like: the velocity v measured along the path must be allowed in principle to be higher than the speed of light c. While in general allowing paths with speed greater than the speed of light leads to acausal effects [30], causality is definable only when units interact, and instead here we allow acausal paths only for simple isolated units. The physical mechanism that enforces $v < c$ for a simple unit must be relational, and dependent on the matching of experiences.

Remark 8 On this last point, we note that a relational interpretation of quantum mechanics (RQM) was first introduced in [32]. RQM assumes a radical form of relationality according to which all physical quantities continue to have different values in the perspective of different reference systems. In our setting, the experiences of the units directly involved in the matching are privileged and determine which outcomes are mutually compatible. Relationality ultimately is dependent on the type of internal state variables of the specific units that are matching their experiences.

Regardless, to be able to apply properly the principle of isolated equivalence, it is still necessary to consider possible relativistic effects on the scaled module of the measure of relative significance $P(A, B)$, that we equated in Eq. (13) to the likelihood of a unit propagating from A to B.

The consideration of relativistic effects will require us to choose a specific form for the internal time differential $d\tau$ in Eq. (13), and in turn of internal time intervals $\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} d\tau$, such that: it respects the properties of simple isolated units; it satisfies the Time Reversal Condition; and it ensures relativistic invariance for $P(A, B)$ for the frames of reference that are physically meaningful for the unit S under consideration.

Proper time does not satisfy the Time Reversal Condition. A possible choice for $d\tau$ in the exponent of Eq. (13) could be to set it equal to the square root of the Lorentzian line element, corresponding to relativistic invariant proper time, i.e. $d\tau = (ds^2/c^2)^{1/2}$. This is the choice we implicitly made throughout Sects. 2 and 3.1 and it is standard in Lagrangian approaches to relativity [23, Chapter 3.19], [31].

However, $d\tau = (ds^2/c^2)^{1/2}$ is not truly compatible with the theory of isolated units, and a different expression for $d\tau$ and $\tau_{\gamma_{AB}}$ will be given in Definition 9.

The first issue with the choice $d\tau = (ds^2/c^2)^{1/2}$ is that it is explicitly incompatible with the Time Reversal Condition. If $d\tau = (ds^2/c^2)^{1/2}$, then for all paths $\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} d\tau = a + ib$, with $a, b > 0$ (if in addition a path has some portions with $v > c$ then $b \neq 0$). Under these conditions, we cannot find a map of the space of paths onto itself to account for the negative sign in the integral on the right-hand side of Eq. (7).

Second, this choice of $d\tau$ is not fully compatible with the setting of isolated units. To see this, recall that, by definition, indistinguishable paths cannot be discriminated within the experience of the unit. Suppose now we have a path γ with internal time interval $\tau_{\gamma_{AB}} = a + ib$, then the periodized internal time would be $e^{i\omega_0(a+ib)} = e^{-\omega_0 b} e^{i\omega_0 a}$. The exponentially decaying factor $e^{-\omega_0 b}$, would break the uniformity of internal time states across indistinguishable paths: simply by having $e^{-\omega_0 b} < 1$ we would be able to ascertain whether that path had in its past portions with $v > c$.

Note that this second issue with $d\tau = (ds^2/c^2)^{1/2}$ stands also when the relative significance $W(A, B)$ is defined according to its form in Eq. (7), that does not depend on the Time Reversal Condition.

We conclude that, despite its relativistic invariance, proper time $d\tau = (ds^2/c^2)^{1/2}$ cannot be used to define internal time: it is not compatible with the Time Reversal Condition on the analytic form of the relative significance; and it does not guarantee for simple units the indistinguishability of paths, because of the discriminant introduced by complex valued internal time values dependent on the path.

Remark 9 Note that relativistic path integrals based on $d\tau = (ds^2/c^2)^{1/2}$ recover causality for distances that are large enough [30], exactly because of the exponential decay e^{-b} associated with paths where $v > c$. Nevertheless, the breakdown of causality at short distances is unavoidable even for this choice of $d\tau$. A similar conclusion was already reached in [22], where it is shown that simple quantum systems with no preferred time parameter lead to path integral propagators that must include acausal paths.

Frames of reference at points of matching of experiences. To define a suitable $d\tau$ that preserves relativistic invariance for $P(A, B)$, we need first to establish which frames of reference are physically meaningful for interacting units.

Recall that the simple unit S under consideration is localized at A and B by interaction respectively with units U_A and U_B. We argue that it is possible to define an external frame of reference that is relevant to the propagation of S, and whose relative velocity with respect to S is *uniquely* defined, only from the perspective of the experience of units such as U_A and U_B.[5] Indeed, S is localized at A and all its indistinguishable paths will agree on the position and velocity of S with respect to U_A. Denote by u the relative speed of U_A and S at A, we can then uniquely define, with respect to S, the frame of reference $\mathcal{F}_u$ at rest with U_A at A (the same argument can be made for frames of references defined by U_B at B).

We call this privileged set of frames of references defined at matching points the *matching frames of reference* of the unit S, and we only refer to such frames in the following arguments.[6] We can now define:

Definition 8 The *relativistic internal clock frequency* of a unit S as observed from a matching frame of reference $\mathcal{F}_u$ in motion at speed u with respect to unit S as localized at A is

$$\omega_u = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} \omega_0. \tag{14}$$

All experiences of the unit are defined with respect to the periodic representation of internal time $\mathcal{P}(\tau) = e^{i\omega_0 \tau}$, dependent on the rest mass M_0 of a unit. It follows that relativistic effects on ω_0 must be defined directly in terms of relativistic effects on the rest mass M_0.

Remark 10 Note the discrepancy of our definition with respect to the definition of the relativistic internal clock frequency given in [10, Chapter 1.1]. De Broglie,

[5] Or from the perspective of localized units that are in an unbroken chain of matchings of experiences with U_A and U_B.

[6] We simplify our analysis in this section by assuming that the interaction at B is with a unit U_B that is also at rest in $\mathcal{F}_u$.

building on the inverse relation of period and frequency and on time dilation, postulates that

$$\omega_u = \left(1 - \frac{u^2}{c^2}\right)^{1/2} \omega_0,$$

while the relativistic form in Eq. (14) is reserved in de Broglie's work to the *matter wave* associated with the body and propagating in space. This suggestion of de Broglie does not allow the construction of a relativistic propagator for simple units (see also Remark 11).

We now define a particular choice of internal time differential $d\tau$ compatible with the setting of isolated units, with the corresponding likelihood $P(A, B)$ and internal time path integral.

We denote by s spacetime point coordinates, and we parametrize s on all indistinguishable paths with respect to time t' as measured in a matching frame of reference $\mathcal{F}_u$. We further denote by $\frac{ds}{dt'} * \frac{ds}{dt'}$ the Lorentzian inner product of $\frac{ds}{dt'}$ with itself.

Definition 9 Given a matching frame of reference $\mathcal{F}_u$ in motion at speed u with respect to the simple unit S as localized at A, the internal time differential is defined as

$$d\tau = \frac{1}{c^2}\frac{ds}{dt'} * \frac{ds}{dt'} dt'. \tag{15}$$

The corresponding internal time interval $\tau_{\gamma_{AB}}$ along the path γ_{AB} is

$$\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt'} * \frac{ds}{dt'} dt'. \tag{16}$$

Remark 11 Recall that with respect to time t' as measured in an external matching frame of reference we can write

$$(ds^2/c^2)^{1/2} = \frac{(ds^2/c^2)^{1/2}}{dt'} dt' = \left(\frac{1}{c^2}\frac{ds}{dt'} * \frac{ds}{dt'}\right)^{1/2} dt'.$$

The internal time differential in Definition 9 is simply a frame dependent representation of the differential of proper time where the square root in $\left(\frac{1}{c^2}\frac{ds}{dt'} * \frac{ds}{dt'}\right)^{1/2}$ has been removed. We will show in Proposition 1 that, even though the internal time differential is frame dependent, the corresponding periodized internal time interval $\mathcal{P}(\tau_{\gamma_{AB}})$ is relativistically invariant for matching frames of reference. In turn, $\mathcal{P}(\tau_{\gamma_{AB}})$ is the only physical quantity that is significant for computing the measurable probability distribution $P(A, B)$ we give in Eq. (17). Note that the original suggestion of de Broglie on the Lorentz transformation of the internal clock frequency (reported in Remark 10) would not allow relativistic invariance for $\mathcal{P}(\tau_{\gamma_{AB}})$.

Definition 9 allows us to further define the corresponding expressions for the propagator of a simple unit S.

Definition 10 The likelihood $P(A, B)$ of S propagating from A to B is defined in $\mathcal{F}_u$ as[7]

$$P(A, B) = \left| \int_{\gamma_{AB}} e^{i\omega_u \int_{\gamma_{AB}} \frac{1}{c^2} \frac{ds}{dt'} * \frac{ds}{dt'} dt'} D\gamma_{AB} \right|^2. \tag{17}$$

The *internal time path integral* (or *the propagator*) associated to $P(A, B)$ is

$$I(A, B) = \int_{\gamma_{AB}} e^{i\omega_u \int_{\gamma_{AB}} \frac{1}{c^2} \frac{ds}{dt'} * \frac{ds}{dt'} dt'} D\gamma_{AB}. \tag{18}$$

A relativistic periodic representation of internal time. One of the main conclusions of this section is the following proposition:

Proposition 1 *Let the periodic representation of the internal time interval $\tau_{\gamma_{AB}}$ of a single unit along a path γ_{AB} be*

$$\mathcal{P}(\tau_{\gamma_{AB}}) = e^{i\omega_u \int_{\gamma_{AB}} \frac{1}{c^2} \frac{ds}{dt'} * \frac{ds}{dt'} dt'}.$$

Then $\mathcal{P}(\tau_{\gamma_{AB}})$ and the integrals $P(A, B)$ and $I(A, B)$, as in Definition 10, are all relativistically invariant with respect to matching frames of reference defined at the points of interaction.

Proof It is enough to show relativistic invariance of the argument of the exponent in $\mathcal{P}(\tau_{\gamma_{AB}})$, namely of the quantity

$$\mathcal{T}_u(\gamma_{AB}) = \omega_u \int_{\gamma_{AB}} d\tau = \omega_u \int_{\gamma_{AB}} \frac{1}{c^2} \frac{ds}{dt'} * \frac{ds}{dt'} dt'. \tag{19}$$

We denoted by t' a parametrization of time in $\mathcal{F}_u$, and similarly we denote by t a parametrization of time in $\mathcal{F}_0$, the frame of reference at rest with respect to S at A. From the perspective of $\mathcal{F}_0$, $t' = \epsilon t$ with $\epsilon = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$, and $\omega_u = \epsilon \omega_0$, therefore

[7] This definition follows closely the one of the scaled relative significance in Eq. (13).

$$
\begin{aligned}
\mathcal{T}_u(\gamma_{AB}) &= \omega_u \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt'} * \frac{ds}{dt'}dt' \\
&= \epsilon\omega_0 \int_{\gamma_{AB}} \frac{1}{c^2}\left(\frac{ds}{dt}\frac{dt}{dt'} * \frac{ds}{dt}\frac{dt}{dt'}\right)\frac{dt'}{dt}dt \\
&= \epsilon\omega_0 \int_{\gamma_{AB}} \frac{1}{c^2}\left(\frac{ds}{dt}\epsilon^{-1} * \frac{ds}{dt}\epsilon^{-1}\right)\epsilon dt \\
&= \epsilon\omega_0 \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}\epsilon^{-1}dt \\
&= \omega_0 \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt = \mathcal{T}_0(\gamma_{AB}).
\end{aligned}
\tag{20}
$$

For any speed u, $\mathcal{T}_u(\gamma_{AB})$ is identical with $\mathcal{T}_0(\gamma_{AB})$, i.e. the same quantity as defined in the matching frame of reference $\mathcal{F}_0$ at rest with respect to unit S at A, and therefore $\mathcal{T}_u(\gamma_{AB})$ and $\mathcal{P}(\tau_{\gamma_{AB}})$ are relativistically invariant. Since $P(A, B)$ and $I(A, B)$ are also defined in terms of $\mathcal{T}_u(\gamma_{AB})$, they are relativistically invariant as well. □

We will now show that the choice of $\tau_{\gamma_{AB}}$ in Definition 9 is compatible with the experience of simple units, and also with the Time Reversal Condition. We work for simplicity in the frame of reference $\mathcal{F}_0$.

First, note that it may happen that the internal time interval $\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt < 0$, but the periodization $\mathcal{P}(\tau_{\gamma_{AB}}) = e^{i\omega_0\tau_{\gamma_{AB}}}$ makes the distinction of positive and negative values of $\tau_{\gamma_{AB}}$ meaningless, so that no inconsistencies in the experience of the unit arise. Indeed, in the definition of indistinguishability, we considered changes in the internal state variables with respect to internal time, irrespective of the direction of flow of the internal time state.

We now address the compatibility of $\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$ with the Time Reversal Condition.

Proposition 2 *For every path γ_{AB} of total internal time τ, there is a path γ'_{AB} of total internal time $-\tau$ as close as we want to γ_{AB} itself, such that the map $\Sigma(\gamma_{AB}) = \gamma'_{AB}$ satisfies the Time Reversal Condition.*

A full proof of this proposition is beyond the scope of this paper, as the technical nature of such proof contrasts with the informal way we have handled path integrals so far. However, we sketch a justification here.

Sketch of Proof. Let us work in the frame of reference $\mathcal{F}_0$. To show that $\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$ satisfies the Time Reversal Condition, consider a piecewise linear path $\gamma_{(AB,N)}$ made of N small segments $[A_i, A_{i+1}]$ with $A_1 = A$ and $A_N = B$ and such that all segments' endpoints are on γ_{AB}.

Let $\tau_{[A_i,A_{i+1}]} = \delta\tau_i > 0$. For small $\delta\tau_i$ we can approximate $\delta\tau_i = \frac{1}{c^2}\frac{\delta s_i}{\delta t_i} * \frac{\delta s_i}{\delta t_i}\delta t_i = \frac{1}{c^2}\frac{\delta s_i^2}{\delta t_i^2}\delta t_i$. Assuming without loss of generality that $\delta t_i > 0$, we conclude that $\delta s_i^2 > 0$, i.e. $[A_i, A_{i+1}]$ is a time-like interval.

We can now build two connected segments $[A_i, H_i]$, $[H_i, A_{i+1}]$ such that $\tau_{[A_i,H_i]} + \tau_{[H_i,A_{i+1}]} = -\delta\tau_i$. For simplicity, and again without loss of generality, we show the existence of H_i on the $1 + 1$ spacetime subspace identified by the segment $[A_i, A_{i+1}]$ and by the t-axis.

We first select a time value $\delta\tilde{t}$ such that $0 < \delta\tilde{t} < \delta t_i$, and we build the line of simultaneity $\mathcal{L}$ such that $t = \delta\tilde{t}$. We choose $\delta\tilde{t}$ small enough that the intersection point $\bar{H}$ of $\mathcal{L}$ and the light cone of A_i makes the segment $[\bar{H}, A_{i+1}]$ time-like. Under these conditions, $\tau_{[A_i,\bar{H}]} = 0$ and $\tau_{[\bar{H},A_{i+1}]} > 0$, so that $\tau_{[A_i,\bar{H}]} + \tau_{[\bar{H},A_{i+1}]} > 0$.

If, on the other hand, we select $\tilde{H}$ on $\mathcal{L}$ that is outside the light cone of A_i and sufficiently far from A_i and A_{i+1}, the segments $[A_i, \tilde{H}]$ and $[\tilde{H}, A_{i+1}]$ will be space-like, with large negative spacetime intervals that we can denote respectively as δs_1^2 and δs_2^2.

Since $0 < \delta\tilde{t} < \delta t_i$, the time differences $\delta t_1 = \delta\tilde{t} - 0$ (the time interval in $\mathcal{F}_0$ along $[A_i, \tilde{H}]$) and $\delta t_2 = \delta t - \delta\tilde{t}$ (the corresponding time interval along $[\tilde{H}, A_{i+1}]$) will be positive. It follows that $\tau_{[A_i,\tilde{H}]} = \frac{\delta s_1^2}{\delta t_1^2}\delta t_1 < 0$ and $\tau_{[\tilde{H},A_{i+1}]} = \frac{\delta s_2^2}{\delta t_2^2}\delta t_2 < 0$. More particularly, $\tilde{H}$ can be chosen so that $\tau_{[A_i,\tilde{H}]} + \tau_{[\tilde{H},A_{i+1}]} << -\delta\tau_i$. We can then find an intermediate point H_i between $\bar{H}$ and $\tilde{H}$ on $\mathcal{L}$ such that $\tau_{[A_i,H_i]} + \tau_{[H_i,A_{i+1}]} = -\delta\tau_i$, as needed.

A similar argument can be used to show that if $\tau_{[A_i,A_{i+1}]} = -\delta\tau_i < 0$, we can always build two connected segments $[A_i, H_i]$, $[H_i, A_{i+1}]$ such that their total internal time is $\delta\tau_i$.

If the segment $[A_i, A_{i+1}]$ is null, then $\tau_{[A_i,A_{i+1}]} = 0$. In this case, we simply replace $[A_i, A_{i+1}]$ with the pair of segments $[A_i, H_i]$, $[H_i, A_{i+1}]$, with H_i any point in $[A_i, A_{i+1}]$, and we have two null segments with $\tau_{[A_i,H_i]} + \tau_{[H_i,A_{i+1}]} = 0$.

By working on each segment $[A_i, A_{i+1}]$, we build in this way a new piecewise path $\gamma'_{(AB,2N-1)}$ of $2N - 1$ segments denoted by the sequence of points $\{A_1, H_1, \dots, H_{N-1}, A_N\}$ and such that if $\tau_{\gamma_{(AB,N)}} = \tau_N$, then $\tau_{\gamma'_{(AB,2N-1)}} = -\tau_N$.

Denote by $\Gamma_{(AB,N)}$ the space of piecewise linear paths made of N segments, and define the map $\Sigma_N : \Gamma_{AB,N} \to \Gamma_{(AB,2N-1)}$ such that $\gamma'_{(AB,2N-1)}$ is the image of $\gamma_{(AB,N)}$.

In the limit of $N \to \infty$, γ_{AB} is the limit of $\gamma_{(AB,N)}$. Let Σ be the limit of Σ_N, and note that $\Gamma_{(AB,N)}$, $\Gamma_{(AB,2N-1)}$ both converge to the space of (continuous and not differentiable) indistinguishable paths Γ_{AB}.

Then $\Sigma_N(\gamma_{(AB,N)})$ will also converge pointwise to the path γ_{AB} (even though its stochastic structure will be different in terms of distribution of time-like, space-like and null local approximations). Since this is true for every path, and distances among paths are respected in the limit, we expect the measure $D\Sigma(\gamma_{AB})$ to be the same as $D\gamma_{AB}$. □

We conclude that the choice of total internal time along a path given in Definition 9 is compatible with the experience of simple units, and also with the Time Reversal Condition.

Remark 12 Clearly, if $\tau = \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$ satisfies the Time Reversal Condition, $-\tau$ will also satisfy it. We will use this simple observation again in Sect. 3.3 while deriving the standard path integral formulation of quantum mechanics for spinless elementary particles from the general framework of internal time path integrals.

Remark 13 In the definition $d\tau = \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$, we could replace $L = \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}$ with any polynomial in L, and $d\tau$ would still be compatible, in principle, with the general setting of isolated units. However, polynomials in L of degree > 1 would not lead to a relativistic invariant $\mathcal{P}(\tau_{\gamma_{AB}})$, as it can be easily seen by replacing L in Eq. (20) with a polynomial in L of higher degree.

Remark 14 There is a crucial consequence of taking $d\tau = \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$ as the correct expression for the internal time differential of isolated units. Since all measurable quantities are derived experimentally in the context of a time differential $d\tau = (ds^2/c^2)^{1/2}$, it follows that the value of the internal frequency ω_0 to be used in the internal time path integral cannot be measured directly from experimental evaluations. Internal frequency at rest must be defined with respect to internal time differential $d\tau = \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$. For this reason, in Definition 4 we allowed ω_0 to be a multiple α of the de Broglie frequency. We will show in Sect. 3.3 that enforcing consistency with standard quantum mechanics in the limit of weak fields and low velocities implies $\alpha = \frac{1}{2}$.

Remark 15 For a realistic isolated charged spinless elementary particle, the propagator $I(A, B)$ in Definition 10 would not fully describe its evolution in a general non-gravitational potential. As we already pointed out in Remark 4, since accelerating charged particles radiate, they are likely to intermittently stop being isolated at some points between A and B. We will further explain this point in Remark 18 in Sect. 4, and we will point out that these radiating effects actually play a crucial role, as they make sure that the dynamics of motion of a simple unit subject to a non-gravitational potential retains dependence from relativistic mass effects, and, under most experimental conditions, does not display significant induced metrical effects when subject to strong non-gravitational potentials.

The connection between the propagator of simple isolated units and quantum mechanical propagator of spinless elementary particles is fully fleshed out in Sect. 3.3, where we shall assume that only low energy classical paths are associated to the propagation of a unit, so that higher order relativistic effects can be neglected and the internal clock frequency at rest ω_0 can be used as an approximation of ω_u in all calculations.

3.3 The Propagator's Path Integral for Spinless Elementary Particles

Let us assume now that a simple isolated unit S is subject to a scalar potential $\phi(x)$. The principle of isolated equivalence implies that, irrespective of the nature of the potential, the effects of $\phi(x)$ on the internal time state along indistinguishable paths of S are the same of those of a corresponding scalar gravitational potential Φ, what we call the *induced potential* as experienced by the isolated unit.

However, when we represent a general potential as an induced potential, we need to scale it appropriately by the mass of the unit subject to the potential. For example, in the case of a charged simple unit S of charge e and mass m_e subject to a static electric potential ϕ, the corresponding potential energy is $V(x) = e\phi$, we define then the induced potential experienced by the unit in isolation as $\boldsymbol{\Phi} = \frac{e\phi}{m_e}$, so that $V(x) = \boldsymbol{\Phi} m_e$ as it would be the case for a gravitational potential.

Given that for simple isolated units there is this simple correspondence between a general, non-gravitational scalar potential ϕ and its induced potential $\boldsymbol{\Phi}$, in the following we can work directly and without loss of generality with a gravitational potential $\boldsymbol{\Phi}$.

We first restrict ourselves in this section to the case of a static, radially symmetric gravitational potential $\boldsymbol{\Phi}$, and we further assume that $\boldsymbol{\Phi}$ is weak, that is, such that $|\boldsymbol{\Phi}| << c^2$. Under this last assumption, and following closely [23, Sect. 17.9], we can approximate the line element $\frac{ds^2}{c^2}$ as

$$\frac{ds^2}{c^2} = \left(1 + \frac{2\boldsymbol{\Phi}}{c^2}\right)dt^2 - \left(1 - \frac{2\boldsymbol{\Phi}}{c^2}\right)\frac{1}{c^2}d\sigma^2, \tag{21}$$

where $d\sigma^2 = dx^2 + dy^2 + dz^2$. This expression for the line element provides the necessary information on the local metric to be able to compute the Lorentzian inner product in $d\tau = \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$. In particular, the Lorentzian inner product of two vectors $P_1 = (t_1, x_1, y_1, z_1)$ and $P_2 = (t_2, x_2, y_2, z_2)$ will be defined locally by $P_1 * P_2 = g_{11}t_1t_2 + g_{22}x_1x_2 + g_{33}y_1y_2 + g_{44}z_1z_2$ where $(g_{11}, g_{22}, g_{33}, g_{44}) = \left(1 + \frac{2\boldsymbol{\Phi}}{c^2}, -(1 - \frac{2\boldsymbol{\Phi}}{c^2}), -(1 - \frac{2\boldsymbol{\Phi}}{c^2}), -(1 - \frac{2\boldsymbol{\Phi}}{c^2})\right)$.

Note that the principle of isolated equivalence ensures that, for a simple isolated unit, a unique local *induced metric* can always be defined for any scalar potential, simply by establishing the corresponding induced potential at that point.

The total internal time along an indistinguishable path from a point A to a point B is

$$\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt = \int_{\gamma_{AB}} \left[\left(1 + \frac{2\boldsymbol{\Phi}}{c^2}\right) - \left(1 - \frac{2\boldsymbol{\Phi}}{c^2}\right)\frac{v^2}{c^2}\right]dt, \tag{22}$$

where we substituted $v^2 = (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2$, and we assume that t is measured in an external frame of reference $\mathcal{F}_A$ at rest with respect to the potential field.

If $v << c$, then $\tau_{\gamma_{AB}}$ can be approximated as

$$\tau_{\gamma_{AB}} = \int_{\gamma_{AB}} \left(1 + \frac{2\boldsymbol{\Phi}}{c^2} - \frac{v^2}{c^2}\right)dt, \tag{23}$$

in which case the internal time path integral is approximately

$$I(A, B) = \int e^{i\omega_u \int_{\gamma_{AB}} \left(1+\frac{2\boldsymbol{\Phi}}{c^2}-\frac{v^2}{c^2}\right)dt} D\gamma_{AB}. \tag{24}$$

Since we assume a non-relativistic setting, the speed u of the simple unit at A is such that $u << c$, and we can approximate $\omega_u \approx \omega_0$. Define

$$K = e^{i\omega_0 \int_{\gamma_{AB}} 1 dt},$$

which is independent of the specific choice of the path γ_{AB}, and substitute $\omega_0 = \frac{\alpha M_0 c^2}{\hbar}$ from Eq. (1), then

$$I(A,B) = K \int e^{i\frac{\alpha}{\hbar}\int_{\gamma_{AB}} 2\Phi M_0 - M_0 v^2 dt} D\gamma_{AB}. \tag{25}$$

We set $V = \Phi M_0$ and divide and multiply by 2 to write

$$I(A,B) = K \int e^{i\frac{\alpha}{\hbar/2}\int_{\gamma_{AB}} V(x) - \frac{1}{2}M_0 v^2 dt} D\gamma_{AB}. \tag{26}$$

Recalling the time symmetry argument in Remark 12, the sign of the integrand in

$$\int_{\gamma_{AB}} V(x) - \frac{1}{2}M_0 v^2 dt$$

does not affect the value of $P(A,B) = |I(A,B)|^2$, and we can conclude that

$$P(A,B) = |I(A,B)|^2 = \left| \int e^{i\frac{\alpha}{\hbar/2}\int_{\gamma_{AB}} -V(x) + \frac{1}{2}M_0 v^2 dt} D\gamma_{AB} \right|^2, \tag{27}$$

where, since we deal with a probability distribution, we have absorbed the constant K into the measure $D\gamma_{AB}$.

We recall now that the probability distribution $P_{qm}(A,B)$ of a quantum spinless point particle with Lagrangian $-V(x) + \frac{1}{2}v^2 M_0$ and propagating from A to B can be described as follows [15]:

$$P_{qm}(A,B) = \left| \int e^{i\frac{1}{\hbar}\int_{\gamma_{AB}} -V(x) + \frac{1}{2}M_0 v^2 dt} D\gamma_{AB} \right|^2. \tag{28}$$

There is an obvious difference between $P(A, B)$ and $P_{qm}(A,B)$: in $P(A, B)$, the constant $\hbar$ is replaced by $\hbar/2$. The only way to make $P(A, B)$ consistent with quantum mechanics in the limit of weak potentials and low velocities is to set $\alpha = \frac{1}{2}$ in $\omega_0 = \frac{\alpha M_0 c^2}{\hbar}$ in Eq. (1). That is, if we compute physical quantities by using the standard definition of time and therefore we use the probability density $P_{qm}(A,B)$, the observed internal frequency of spinless elementary particles will be close to twice the internal frequency to be used in $P(A, B)$.

In the limit of weak potentials and velocity $v << c$, we conclude that the square module of the internal time path integral for a simple isolated unit can be reduced to the square module of the path integral describing the propagator of quantum spinless point particles.

Remark 16 In moving from the internal time path integral to its approximation, we assume that we can set $v << c$ in the argument of the path integral while keeping the measure $D\gamma_{AB}$ unchanged. This is essentially equivalent to a regularization of the integral argument, and to the assumption that the integral is well approximated under such regularization. We make the same approximation in Sect. 4 when we calculate possible effects of a weak Coulomb potential on the local metric defined along each indistinguishable path.

The arguments derived in this section for scalar potentials can be adapted in a straightforward way to more general vector potentials, if we continue to restrict ourselves to simple units with no spin.[8] More particularly, for a spinless elementary particles of rest mass m_e in a non-relativistic regimen subject to a general electromagnetic field, we note that $V(x)$ along a path $x(t)$ takes the form $V(t) = -e\phi + \frac{e}{c}\boldsymbol{A} \cdot \dot{x}$, where ϕ is the scalar potential, $\boldsymbol{A}$ is the vector field associated to the external charges distribution and motion, e is the charge of the simple unit, and we denote by $\dot{x}$ the derivative of x and by "$\cdot$" the Euclidean inner product.

Because the internal state variables of the simple unit cannot change in time, paths subject to vector and scalar potentials are not distinguishable for simple units. At each instant t, the simple unit behaves as if it is subject to a scalar potential $\boldsymbol{\Phi}$ such that $V(t) = -e\phi + \frac{e}{c}\boldsymbol{A} \cdot \dot{x} = \boldsymbol{\Phi} m_e$. The computation of $I(A, B)$ in Eq. (25) is modified (in a non-relativistic scenario) by setting the induced potential as $\boldsymbol{\Phi} = (-e\phi + \frac{e}{c}\boldsymbol{A} \cdot \dot{x})/m_e$ and allowing a dependence of $\boldsymbol{\Phi}$ on t in the path integral.

The possibility of this straightforward generalization to vector potentials also implies that the range of indistinguishable paths of a simple unit in a scalar potential is the same as the range of indistinguishable paths in a general potential. On the other hand, the spacetime metric induced by the vector potential at each point is not uniquely defined: the metric of the line element is dependent not just on the specific test unit, but also on the indistinguishable path of the unit.

Remark 17 In [28] the propagator of a quantum mechanical particle was also derived from de Broglie's internal clock hypothesis in the context of weak gravitational potentials, while in [11, 12] the Feynman path integral propagator in the presence of general gauge fields is derived from spacetime geometrodynamics. Another geometric interpretation of quantum mechanics valid for general potentials was proposed in [36], starting from Bohm's pilot wave interpretation of quantum mechanics.

In our work the effects on the local metric induced by a general potential are deduced from first principles; the structure of isolated units and the principle of isolated equivalence allow to extend the analysis of weak gravitational potentials to scalar and vector potentials for simple units. Moreover, our theory is not an interpretation of quantum mechanics, as it reduces to the latter only under the condition of weak potentials and low velocities, when the propagator of a simple unit in Eq. (18) approximates the quantum mechanical propagator of a spinless elementary particle.[9]

[8] We note however that our results extend, at least formally, to the more complex case of units with spin. The discreteness of spin states preserves the possibility of having a suitable space of continuous indistinguishable paths on which to build an internal time path integral.

[9] As noted in Remark 18, for a charged unit in a general electromagnetic potential the predictions of the standard theory are preserved under most experimental conditions because of the intermittent loss of isolated status of the unit due to its radiating field.

We shall see in Sect. 4 that the condition of isolated evolution of a unit can be harnessed to devise simple experimental tests of our approach.

4 Weak Relativistic Effects and Experimental Considerations

We now explore whether it is possible to experimentally detect induced effects due to a non-gravitational potential on the local spacetime metric defined along the indistinguishable paths of a simple unit.

To connect our results on simple units to possible experimental predictions for non-gravitational potentials, the correspondence (noted at the end of Sect. 1) of simple units with isolated spinless elementary particles is not enough. We need to move a step closer to real objects that are affected by non-gravitational potentials, and we assume that an isolated electron can be a suitable physical embodiment of an isolated charged spinless elementary particle. The implication of this assumption is that we can only consider experimental scenarios where the spin direction of the electron can be neglected, such as, for example, in determining the motion of an electron in an electrostatic Coulomb potential.

Additionally, the whole experimental setting we suggest here is meaningful only under restrictive conditions on the potential and the extremal trajectories of the electron, seen as a simple unit. These conditions are meant to ensure that the radiating field of the electron is negligible and does not impact its isolated status, and that the results of Sect. 3.3 can be extended to include weak relativistic effects of the order of $\frac{u^2}{c^2}$, with u the velocity of the electron at the beginning of propagation.

We avoid any scenario that would require an accurate evaluation of $P(A, B) = |I(A, B)|^2$, because $d\tau = \frac{1}{c^2}\frac{ds}{dt} * \frac{ds}{dt}dt$ indirectly defines a local Lorentzian geometry, but in general the internal time path integral is not defined on a single Lorentzian manifold. A potential that depends on the velocity of the unit (such as, for example, a vector potential) will induce on each indistinguishable path its own local metric. A proper perturbation theory for such types of path integrals is not available.

In light of these computational limitations, in what follows we work only with static potentials and on measurements of the shape of extremal paths associated to the internal time path integral that are not essentially dependent on a complete perturbative analysis of the integral itself. As we said at the beginning of this section, this limitation to static potentials is also in line with the restrictions on the experimental setting that we must enforce to be able to approximate the behavior of an electron as the one of a simple unit.

Weak Relativistic Effects. Our derivation of quantum mechanics for spinless elementary particles in Sect. 3.3 assumes a non-relativistic regimen where the relativistic expression ω_u for the internal clock frequency was replaced by the approximation ω_0. However, the most general expression for the internal time path integral in Eq. (18) used ω_u and was relativistically invariant for the frames of reference defined at the matching points. This raises the possibility that our formalism could be extrapolated to consider weak relativistic effects, while eschewing the difficulties of a fully relativistic setting.

Accordingly, in this section we continue to assume that $v << c$, but we seek very small effects of the order of $\frac{v^2}{c^2}$, and in particular of the order of $\frac{u^2}{c^2}$, where u is the velocity of the simple unit as established at the point A in an external frame of reference at rest with respect to the potential field and with the whole experimental apparatus. We further assume that we are in the presence of static, radially symmetric Coulomb potential ϕ, and that the induced potential Φ due to the Coulomb potential is weak, i.e. $|\Phi| << c^2$.

For an electron of charge e and rest mass m_e, the relativistic internal clock frequency is $\omega_u = \epsilon \frac{\alpha m_e c^2}{\hbar}$, with $\alpha = \frac{1}{2}$ (as we established in Sect. 3.3), and $\epsilon = \left(1 - \frac{u^2}{c^2}\right)^{-1/2}$. Similarly, if we wish to consider weak relativistic effects, the relativistic mass ϵm_e will appear in the computation of the induced potential due to the static electric potential, as $\Phi = \phi \frac{e}{\epsilon m_e}$.

Following the expression for the total internal time along a path given in Eq. (22), the corresponding approximate internal time path integral for the induced weak potential is

$$I(A,B) = \int e^{i\omega_u \int_{\gamma_{AB}} \left[\left(1+\frac{2\Phi}{c^2}\right)-\left(1-\frac{2\Phi}{c^2}\right)\frac{v^2}{c^2}\right]dt} D\gamma_{AB}. \tag{29}$$

We can assume that there is only one extremal path from A to B for $I(A, B)$. Multiple extremal paths among points A and B may arise for strong potentials. However, the energy of the particle along extremal paths that do not correspond to the classical trajectory between A and B would generally be high. In these cases, relativistic effects would be significant, while we are considering only the case of paths with initial velocity $u << c$ and subject to weak potentials.

The relativistic mass affects the extremal trajectories of the path integral in Eq. (29) through the equality $\Phi = \phi \frac{e}{\epsilon m_e}$. The relativistic internal clock frequency ω_u is also dependent on the relativistic mass, but the specific value of ω_u affects only the value of the path integral, and not its extremal trajectories.

Remark 18 The integral in Eq. (29) has extremal trajectories that differ in general from the classical trajectory of a charged particle in a static electric potential, even in the limiting case of weak potentials: the impact of the relativistic mass is fixed at the point of matching of experiences in A and does not change as long as the unit is isolated. This is a consequence of the principle of isolated equivalence that we need to reconcile with the predictions of the classical theory of electromagnetism, where the equations of motion depend on the relativistic mass.

To this end, we note that the electromagnetic radiation of the electron is unavoidable in the context of any significant acceleration. This radiation would affect the state of the surrounding environment and it would make the unit, at least intermittently, not isolated. The intermittent matching of experiences would achieve two crucial results: break the isolated equivalence that gives rise to induced metric effects; reset every time the value of u in the relativistic mass $\epsilon m_e = \left(1 - \frac{u^2}{c^2}\right)^{-1/2} m_e$ appearing in Eq. (29) in ω_u and Φ. The intermittent breaking of isolated equivalence (and of the induced metric effects) and the resetting of the relativistic mass ensure that a correspondence with the standard classical trajectory of a charge accelerated in an electromagnetic field is preserved under most experimental scenarios, as already pointed out in Remarks 4 and 15 . Indeed, radiating effects are relevant when considering accelerated motion of a simple unit subject to any realistic non-gravitational potential.

This caveat implies that there is no possible equivalent for high speed simple units of a Klein-Gordon theory for spinless elementary particles (seen as test particles in a general potential): any fully relativistic extension of the theory we developed in this paper (as opposed to a relativistically invariant formulation, as in Sect. 3) must explicitly include the possibility of radiating fields for any object of the theory that can be affected by non-gravitational potentials. However, a full account of radiating interaction with other units can be properly addressed only by generalizing the current work to include structured units.

Experimental Considerations. The experimental setting we assume in this section is essentially a single slit diffraction experiment (see for example [15, Chapter 3.2]), where we pay particular attention to the distance of screen and detector and where we add a radially symmetric weak Coulomb potential between screen and detector.

Let x_a be the position of the mouth of an electron gun pointing right at a screen that is orthogonal to the line l that goes through the gun mouth. Set a slit on the screen at the intersection of the screen and the line l. Position a detector plane parallel to the screen and to its right. Suppose that electrons are emitted with initial velocity that is very narrowly distributed around some value u, so that without loss of generality we can consider only paths with initial velocity exactly u. Assume also that a radially symmetric Coulomb potential is centered at a distance D from the line l and to the right of the screen. Let $A = (x_a, t_a)$ with x_a at the mouth of the electron gun, and $B = (x_b, t_b)$ with x_b a point on the detector and $t_b > t_a$.

The velocity of the electron is assumed to be large enough that the potential field only deflects the electron, and it does not significantly change the norm of the velocity. These assumptions are crucial also to ensure that the transversal acceleration is small, so that the relative loss of kinetic energy due to electromagnetic radiation can be neglected as well. We stress that a very low radiating field is necessary to ensure the electron will be an isolated unit until it reaches the detector.

We first analyze a particular path in this experimental setting, the one corresponding to the classical trajectory with initial velocity u. We then argue that the maximum of the distribution of the electron's hits on the detector is located at the intersection of this path with the detector itself.

Note that since the velocity along the extremal trajectory is assumed to be nearly constant, the relativistic mass is constant as well. Under this condition, the caveats of Remark 18 do not apply, and the computation of the deflection of the extremal path associated to the path integral in Eq. (29), with starting point x_a at the mouth of the electron gun and velocity equal to u, follows the calculations for the deflection of finite mass particles in a weak gravitational potential.

In a radial, static gravitational potential that is flat at infinity, the total angle deflection of the trajectory of a particle of initial velocity u can be approximated as [27, Sect. 25.5]

$$\delta\alpha_G \approx 2\frac{GM}{D}\frac{1}{u^2}\left(1+\frac{u^2}{c^2}\right), \tag{30}$$

where D is the distance of the closest point of the trajectory to the potential field source (the impact parameter), M is the total mass of the field source, and we assume that u is large enough to allow the particle to escape the field.

The scalar, induced potential for a given static Coulomb potential is, in radial coordinates,

$$\Phi(R) = k_e\frac{Qe}{R}\frac{1}{\epsilon m_e} = k_e\frac{Qe}{R}\frac{1}{m_e}\left(1-\frac{u^2}{c^2}\right)^{1/2}, \tag{31}$$

with k_e the Coulomb constant, Q the charge of the Coulomb field, e the charge of the electron, and m_e its mass.

Let Γ be the unique geodesic (of the geometry induced by the potential Φ in Eq. (31)) from x_a to the detector plane with initial velocity u. Substituting $\Phi(D)$ for $\frac{GM}{D}$ in Eq. (30) we have this corresponding formula for the deflection angle of Γ:

$$\delta\alpha \approx 2k_e\frac{Qe}{D}\frac{1}{m_e}\left(1-\frac{u^2}{c^2}\right)^{1/2}\frac{1}{u^2}\left(1+\frac{u^2}{c^2}\right). \tag{32}$$

Let now B_Γ be the intersection of Γ and the detector plane, and denote by $P_A(x_b, \Phi)$ the restriction of the probability distribution $P(A, B)$ on the detector plane in the presence of the potential Φ.

Since Γ is the only permissible classical path in our experimental setting, it is the only extremal path that will contribute to the corresponding WKB semiclassical limit. In this limit, it will be the path reaching the detector with the largest constructive interference of nearby paths. Accordingly, the maximum of $P_A(x_b, \Phi)$ is located at B_Γ and the deflection angle $\delta\alpha$ is measurable from $P_A(x_b, \Phi)$.

In a classical setting, without induced effects on the local metric, the deflection angle due to the Coulomb field would be [27, p. 671]

$$\delta\alpha_C \approx 2k_e\frac{Qe}{D}\frac{1}{m_e}\left(1-\frac{u^2}{c^2}\right)^{1/2}\frac{1}{u^2}. \tag{33}$$

This calculation considers first-order effects in $\frac{u^2}{c^2}$ and strictly speaking we cannot apply standard non-relativistic quantum mechanics here. However, since $u << c$ and since in the WKB limit we expect to recover the classical trajectory also for relativistic quantum mechanics, we can assume that the deflection angle $\delta\alpha_C$ will correspond to the peak of the distribution of hits on the detector plane as predicted by a standard quantum mechanical analysis of this setting.

By approximating $\left(1-\frac{u^2}{c^2}\right)^{1/2} = 1-\frac{1}{2}\frac{u^2}{c^2}$ and retaining only terms of the order of $\frac{u^2}{c^2}$ we can write $\delta\alpha - \delta\alpha_C \approx 2k_e\frac{Qe}{Dm_ec^2}$. This difference will be small in absolute terms, because the condition of weak induced potential $|\Phi| << c^2$ implies that $2k_e\frac{Qe}{Dm_ec^2} << 1$. Still, $\delta\alpha - \delta\alpha_C$ gives an estimate of the interval of accuracy for the measurement of $\delta\alpha$ that would be required to validate experimentally the weak relativistic effects conjectured in this section.

We conclude that the maximum of the probability distribution of hits on the detector plane should be deflected by the Coulomb field by an angle $\delta\alpha$ and not by $\delta\alpha_C$, as it would be predicted by standard quantum mechanics. Once more, we stress this result would hold only assuming the radiating field of the electron can be neglected and does not affect its isolated status, and extrapolating the results of Sect. 3.3 to hold when effects of the order $\frac{u^2}{c^2}$ are very small, but not negligible.

5 Further Developments

5.1 A Primitive Ontology for Isolated Objects

In Sects. 3 and 4 we have analyzed in depth only the propagation of a single simple unit. The abstract approach of Sect. 2, defining simple units independently of their specific physical embodiment as isolated spinless elementary particles, emphasized the general role of the notion of internal state variables in determining the labelled indistinguishable paths associated to any isolated object (unit), including composite, structured units. We first explored the general structure of the matching of experiences in light of the equivalence principle, by defining the relative significance of labelled indistinguishable paths, and we constructed a relativistically invariant propagator for simple units. Only then we showed how these ideas could be used to derive in Sect. 3.3 the path integral formulation of Schrödinger equation for a spinless elementary particle.

Note that all our results on the propagator of a simple unit were derived under the assumption of a surrounding environment for the unit, to assess whether the unit was indeed isolated. However isolated objects, whether simple or structured, should be identifiable independently of an arbitrary partition of a physical system into a unit and its surrounding environment. This is especially significant if we want to establish a satisfactory primitive ontology (PO) for the theory of isolated objects, to better understand how multiple indistinguishable paths of isolated objects can be related to our direct experience of a classical world.

To be more precise, we recall that the PO of a theory is the set of elements of the theory "that make direct contact with the world of our experience", as stated in [3, Sect. 4.1] (a key reference on POs on which we rely heavily in this section). Note that this definition of a PO implies that its entities must be at least expressible in classical terms, expanding on the idea that the "positions of things" are ultimately the relevant variables of any meaningful physical theory [7, Chapter 19].

We shall see that the identification of a suitable PO for isolated objects will lead to valuable insight into how changes of acceleration affect the range of indistinguishable paths of structured units, and later explore the relation of the PO of isolated objects with two POs of the Gherardi-Rimini-Weber collapse theory [6, 19, 20, 37], that share some interesting points of contact with the PO of isolated objects we are about to define.

Our starting point to build a PO for isolated units is an argument we made in Sect. 2 when motivating Assumption 2 on the matching of experiences of distinct units. We stressed that talking of propagation of a unit S from a point A to a point B

implies the possibility of localizing the unit by matching of experiences, and this can only be done if there are additional structured units U_A and U_B with single indistinguishable paths, i.e. fully localized units. Then interaction of S with U_A at A and U_B at B would effectively localize S as well because of their matching of experiences.

We call the spacetime event corresponding to a matching of experiences with a fully localized unit a *local matching of experiences (local ME)*, and we claim that the PO of the theory of isolated objects is the set of all local MEs. Indeed, local MEs are local events shared and validated by all units that partake of the matching and they are described in terms of classical notions of space and time.

Interestingly, the localized isolated objects that allow the local MEs to exist do not directly belong to the PO: in isolation they are not accessible to other units and do not make up a shared reality; during the matching of experiences, a unit is only assumed to change its experience (defined by the evolution of its internal state variables) to be compatible with the experience of another interacting unit, it does not directly have access to the experience of the other unit. Matching of experiences is only a process of verification of compatibility of experiences: what is not compatible simply does not appear in the matching.

These are the reasons we assume that only the set of all local MEs can be part of the PO of isolated objects, and not the isolated objects themselves. However, since the set of all local MEs depends on the interaction of isolated objects, it follows that, given an external frame of reference, all objects that are isolated at each time t need to be identified for the PO to be completely defined, even if the specific set of isolated objects can and likely will change in time.

We do not directly address in this section relativistic invariance of the set of isolated objects, as the notion of isolation is inherently non-local. We will simply prove here that for each given frame of reference all isolated objects can be identified at time t, and assume that the same set of local MEs is generated regardless of the frame of reference used to identify the isolated objects.

Note however that Proposition 1 in Sect. 3.2 establishes relativistic invariance of the general propagator of simple units. This result already ensures that local MEs are relativistically invariant when restricted to simple units, and it is likely that the PO specified by all local MEs is relativistically invariant as well, conditionally to the following assumptions: we equate a local ME of a structured unit to a collection of local MEs for its constituent elementary particles; the results in Proposition 1 for the propagator of simple units can be extended to units with variable spin direction.

We can take the existence of local MEs as an additional *a priori* element of the theory, and in this case the PO of isolated objects will be fully independent from phenomenological assumptions. However, the set of local MEs does indirectly introduce a phenomenological aspect to the PO, in the sense that we do not describe the microscopic mechanism leading some units to have single indistinguishable paths and be localized. Localization is nevertheless a concrete and plausible assumption for structured, macroscopic objects, describing a property that we can experimentally validate and see in our own experience.

Moreover, we will shortly see that the existence of a localized structured unit is dependent on the geometry of the bonds among the constituent parts of the unit itself, and its impact on the mass of the unit. In other words, the phenomenological understanding of bonds and molecular geometry in molecules and crystal structures will be used to deduce general properties of isolated objects.

Note that the only plausible PO of the orthodox interpretation of quantum mechanics (OQM) also has a phenomenological component: visible changes of the macroscopic states of measuring devices, even though not fully understood with respect to the underlying microscopic description of the system, can be considered the PO of OQM [3, Sect. 4].

We stress that not all partitions of a system into separate objects allow, even in principle, the existence of the local MEs that we need to define the PO of isolated objects. For example, imagine a partition that is only made of unconstrained elementary particles (i.e. a partition that has only simple units and no structured units): they will all, always, have multiple indistinguishable paths in isolation, and their matching of experiences can potentially be anywhere, without the need of being localized in any specific point. Indeed, the difficulty of localizing the matching of experiences for simple units is just the way the measurement problem and the need for a quantum state reduction present themselves in the context of isolated objects.

Since localized structured units are essential to the definition of local MEs, it is not possible to have a PO of simple units on their own, and, more particularly, a PO only for spinless elementary particles that are not interacting and bonding with each other in significant ways to generate structured units. Because of the importance of localized structured units in establishing local ME and a proper PO, we will suggest now which physical objects can be assumed to be essentially localized when isolated.

First of all, we stress that when we spoke of objects in the definition of isolated objects in Sect. 2, we meant objects as we see them in our experience: as spacetime localizations of mass. We further assume now that each such localization of mass is the largest possible aggregate of elementary particles (each with a fixed rest mass) related to each other by some constraints (corresponding in practice to atomic and intramolecular forces[10]). This idea is formalized in the following definition:

Definition 11 An isolated set U of elementary particles is a structured isolated object (structured unit) at time t if:

1. The energy of U is in a stable (locally minimal) equilibrium at time t.
2. The mass M of U is not equal to the sum of the mass of any partition of U into two subsets of elementary particles.
3. U is the maximal, i.e. there is no larger set of particles that includes U that satisfy properties 1 and 2.

In other words, with Definition 11 we are essentially assuming that a structured unit is a distinct atom, molecule or larger solid object with energy in a stable equilibrium, and such that the mass of the unit depends also on the energy of its bonds, a

[10] We neglect here for simplicity possible weaker intermolecular forces that may also exist in liquid and gas states.

dependence that usually manifests itself as mass defect. Note moreover that realistic elementary particles with variable spin direction can be considered as well, even though we do not specifically focus on them in this section.

Definition 11 allows for all structured units at time t to be uniquely identified (simply by assuming that each elementary particle is included into its unique maximal set of particles connected by bonds), and it confirms the special role of mass, since we argued in the justification of Assumption 1 in Sect. 2 that mass must always be one of the internal state variables of a unit.

Of course, Definition 11 is not a dynamical description of structured units, since the mechanism that leads a set of particles to be in a stable energy equilibrium is not explained. Our point is that, given the existence of such stable sets of particles, a specific and unique partition of all elementary particles into simple or structured objects is possible at each time t, and their status as isolated or not can then be ascertained.

We further assume now that there are at least some structured objects that are sensitive to acceleration in the sense clarified by the following definition.

Definition 12 A structured object S is *acceleration-sensitive* if it is not perfectly rigid and if its molecular geometry is such that any change of acceleration the object is subject to can be orthogonally decomposed to have at least one component parallel to one of its bonds.

Note that with Definition 12 we move one step closer to real objects, by using the notion of molecular geometry, i.e. the spatial arrangement of atoms within a molecule that is in a stable energy equilibrium.

Any molecule or solid compound with a three dimensional crystal structure satisfies Definition 12, and since an acceleration-sensitive object is not perfectly rigid, a sudden change of acceleration along one of its possible indistinguishable paths would change the distances of at least some of the particles in the object and therefore the energy stored in its bonds.[11] This means that the total mass of the object, dependent in part on the energy of the bonds, would also be changed for any change of acceleration that is large enough to deform the object and affect at least the energy of one of its bonds. Moreover, the larger the object, the greater the chance that at least one of its bonds will be affected by a change of acceleration.

Note that we are making here a non-degeneracy assumption that changes of acceleration along a possible indistinguishable path are not uniform within an object with extended volume, either because of the presence of tidal effects, or because of forces being applied unevenly to the particles within the object. We focus on changes of acceleration instead of acceleration itself because in a uniformly accelerated object its particles will eventually be in equilibrium and at rest one with respect

[11] To be more precise, for an object with extended volume we should speak of indistinguishable world tubes in spacetime generated by the motion of the object, rather than indistinguishable spacetime paths (i.e. world lines).

to the other, so that its mass will not be subject to changes due to variations in the energy in its bonds.

Assume now that an acceleration-sensitive object is at rest with respect to an inertial frame of reference from point A to point B. In such a state, it is not subject to changes of acceleration and its rest mass is constant in isolation along the corresponding spacetime path γ_1 from A to B. Any other spacetime path γ_2 starting at A and ending at B but distinct from γ_1 will have a change of acceleration at some point. If this change of acceleration is enough to locally deform the object, it will lead the object to have a different mass because of the arguments above on the correspondence between bonds' energy and mass. Since mass is one of the internal state variables, γ_2 will be distinguishable from γ_1.

The range of indistinguishable paths of the object reduces in this case to the path associated to its inertial state, and possibly to a narrow range of paths that are so close to γ_1, and with such slight changes of acceleration, to be unable to deform the object. Under these conditions the object is effectively localized, because any change of acceleration that could affect the position of the center of mass of the object, would also deform it. Note that this simple argument on the mass of an object is relevant for determining localization only because we assume that the internal state variables of the object *as a whole* do matter in establishing its indistinguishable paths.

We can conclude that any matching of experiences with an acceleration-sensitive object in inertial state will localize other objects, whether microscopic or not: the local ME of a localized unit with an elementary particle will be single points in spacetime, while the local ME with a structured unit will be a larger set of points corresponding to all the elementary particles in the unit. Since in practice most large solid objects have a three-dimensional molecular geometry that makes them sensitive to changes of acceleration, and none is perfectly rigid, we expect local MEs to be exceedingly common events.

Remark 19 Note that if the molecular geometry of a configuration of particles is essentially two dimensional, the energy of its bonds would not immediately be affected by a slight change of acceleration in the direction perpendicular to the plane established by the configuration. Similarly, simpler, one-dimensional molecular geometries are possibly deformed by changes of acceleration without immediate impact on the energy of the bonds.

Remark 20 The PO of isolated objects specified by local MEs shares some significant similarities with a PO of the Gherardi-Rimini-Weber theory (GRW), a well-known modification of quantum mechanics that assumes a stochastic, discontinuous collapse of the wave function [20]. In [6] it was suggested that the PO of GRW is comprised of the spacetime events corresponding to the spontaneous collapses of the wave function, and this suggestion was further developed in [37] where these events were labelled "flashes" and relativistic invariance of the resulting GRW flash ontology (GRWf) was proven (but see [13] for a critical examination of the claim of relativistic invariance in [37]). Indeed, local MEs and flashes are both events related to the localization of objects, however flashes within GRWf are completely determined at the microscopic level, while the physical plausibility of the PO specified by local MEs is essentially dependent on a realistic world that includes multiple acceleration-sensitive structured objects. Moreover, in the PO specified by local MEs, no separate mechanism for collapse is assumed, but simply a matching of experiences in interaction with localized objects, and the probability of a local ME arising is derived, at least for matching with simple units, from the concept of relative significance described in Sect. 3.1.

Remark 21 Our emphasis on mass changes within the isolated object, albeit mediated by changes in the bonds' energy, established an interesting link with yet another PO of GRW that sets the density of mass in space for each time *t* as the PO of the theory, usually referred to as GRWm [3, Sect. 3.1, 19]. In particular, our discussion on possible changes of the center of mass of the object and distinguishability of spatially distinct mass distributions is paralleled by similar concerns in GRWm. Note however that the key property of objects that we needed to localize them is a lack of rigidity in their structure, while quasi-rigid solids are emphasized in the arguments establishing the definite state of macroscopic objects in [19, Sect. 3.3 (ii)]. Additionally, we dealt with geometric configurations of discrete point particles in our analysis, as opposed to continuous mass distributions in GRWm. At a more fundamental level, mass has a different ontological status in the PO of isolated objects and GRWm. Mass distributions appear directly as entities in GRWm, while mass appears as an internal state variable of isolated objects in our framework: changes to mass distributions have only an indirect an effect on the local MEs, being the key to define acceleration-sensitive objects, and therefore to establish that local MEs are physically plausible events.

5.2 Concluding Remarks

This work started from the assumption that the evolution of a physical object (of any size) is subordinate to the restricted information available to the object itself through the evolution of its state variables (the experience of the unit), whenever no other object validates this information. From this broad assumption, we developed a theory of isolated units, i.e. objects that do not significantly affect their surrounding environment. Such units were then assumed to intermittently match their experiences during interaction.

The main physical implications of the theory of isolated units are the following: the equivalence principle can be conditionally extended to non-gravitational potentials whenever an isolated unit cannot distinguish the nature of the potentials via its state variables; similarly, all spacetime paths of an isolated unit that cannot be distinguished must be considered physically real in isolation, and must eventually have an impact when the unit interacts with other units.

From these assumptions, we derived a relativistically invariant propagator for simple units (isolated units whose state variables do not change), under an additional assumption on the existence of an internal periodic representation of time for nonzero mass units (adapted from de Broglie's internal clock frequency hypothesis). We then demonstrated that the propagator for simple units reduces to the standard quantum mechanical propagator of spinless elementary particles in the limiting case of weak potentials and low velocities. By extrapolating these results to include weak relativistic effects, and under restrictive conditions meant to minimize field radiation and ensure that a moving electron can be considered an isolated unit, we suggested an experimental setting that can detect small deviations from the predictions of standard quantum mechanics for the propagator of an electron slowly moving in a weak Coulomb potential.

In fact, as pointed out in Remark 18, radiating charged particles only intermittently retain isolated status, and the setting of simple units is insufficient to account for a fully relativistic theory of moving charged units, so that it will need to be extended to include structured units. This extension will be necessary also for a proper treatment of units with spin.

Note however that the isolation of a unit is a non-local, relational property, and in principle it is amenable to experimental manipulation, by controlling its surrounding environment. We expect therefore that an experimental and phenomenological exploration of the notions of isolated unit and of the matching of experiences will yield further interesting results, even before a complete theory for structured units is developed.

A phenomenological approach was already evident when we proposed a primitive ontology (PO) for isolated objects in Sect. 5.1. The PO we suggested depended on the matching of experiences with localized structured objects, each having a single indistinguishable path. In turn, the physical plausibility of a localized structured object was deduced simply on the basis of general phenomenological considerations on its possible molecular geometry, and the way such geometry would respond to variations in acceleration along a path and affect the mass of the object.

In particular, we expect the range of permissible indistinguishable paths of large molecules to be restricted by the geometry of their molecular configurations (along the lines of the observations in Remark 19), and by the way these configurations respond to variable acceleration effects induced by general potentials. A better understanding of this relation may allow to establish qualitative constraints on the observed quantum superposition that persists for large, composite objects [4], and contribute a different viewpoint on the transition from quantum to classical behavior.

Acknowledgements We thank all our colleagues who read and commented on previous versions of this paper. Above all, we would like to express our deep gratitude to A. Gulian for his steadfast encouragement, insightful suggestions, and careful review of our work. This paper is dedicated to Nello Senatore, remembering his goodness and love.

Funding The authors received no specific funding for this work.

Availability of Data and Material Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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