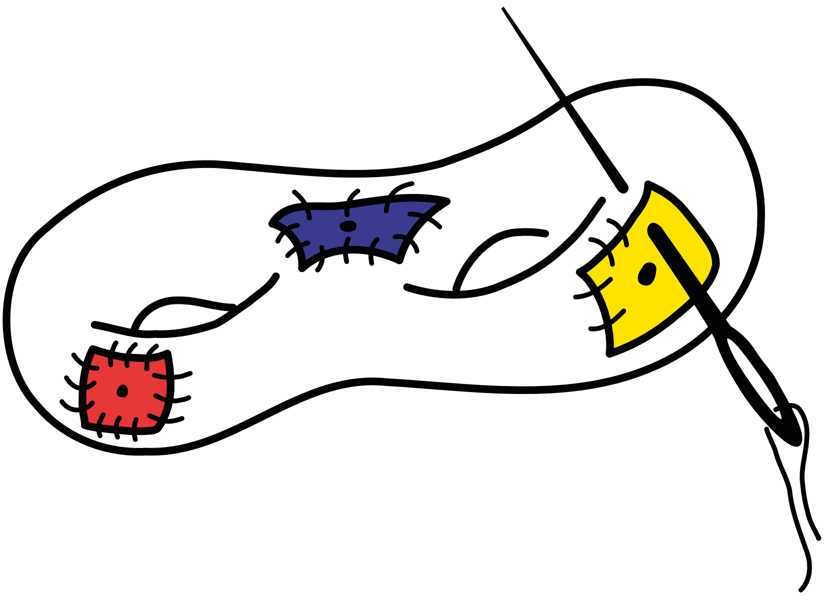
Cohomological field theories and global spectral curves

Popolitov, A.

Cohomological field theories and global spectral curves



Aleksandr Popolitov

Cohomological field theories and global spectral curves

ACADEMISCH PROEFSCHRIFT
ter verkrijging van de graad van doctor
aan de Universiteit van Amsterdam
op gezag van de Rector Magnificus
prof. dr. ir. K. I. J. Maex
ten overstaan van een door het College voor Promoties ingestelde
commissie, in het openbaar te verdedigen in de Agnietenkapel
op dinsdag 13 juni 2017, te 10:00 uur

door

Aleksandr Popolitov

geboren te Khimki, Rusland

Promotiecommissie:

Promotor:	prof. dr. S. V. Shadrin	Universiteit van Amsterdam
Copromotor:	dr. H. B. Posthuma	Universiteit van Amsterdam
Overige leden:	dr. G. Carlet	Universiteit van Amsterdam
	prof. dr. N. Y. Reshetikhin	Universiteit van Amsterdam
	prof. dr. J. V. Stokman	Universiteit van Amsterdam
	dr. G. Borot	Max-Planck Institute, Bonn
	prof. dr. L. O. Chekhov	Steklov Institute, Moscow
	prof. dr. V. Rubtsov	Angers University, France

Faculteit: Faculteit der Natuurwetenschappen, Wiskunde en Informatica

This research was funded by the Netherlands Organization for Scientific Research

Acknowledgements

First of all, I would like to thank Sergey, my advisor. Your unique aikido way of doing mathematics, where not a single problem is attacked directly – but rather approached in a careful meandering fashion, proved extremely fruitful time and time again. Whenever I had a wild idea, maybe not yet exactly focused and rigorous, you were always there to listen, to encourage me to go with it, to see where it leads. Likewise, whenever I was stuck or puzzled, you were always there to discuss, to brainstorm, to encourage open-mindedness. Perhaps, most importantly, you helped me see, feel and appreciate, that mathematics is really a collective effort, where teamwork is central to success. For all this, and more, I thank you with all my heart.

I would also like to thank Hessel, my co-supervisor, for his geniune interest in my research and for his numerous valuable pieces of advice.

I am forever grateful to my coauthors on the papers, that are on the subject of my thesis. Petya, Danilo, Motohico, Paul, Nicolas and Dima, it was a pleasure collaborating with you, learning from you, sharing together the exciting moment of discovery.

I also want to thank my coauthors on the papers which are not subject of my thesis, yet, which were done during my stay in Amsterdam, or are nearing completion. Clay, Ben, Alexei, Andrei, Reinier and Shamil, I enjoyed collaborating with you a lot, you made me think about science from unusual angles and made my life during these four years more colorful.

Apo, Arie, Arie, Dirk, Lesha, Kirsten, Wessel, Zijian, thank you for being such amazing roommates, for the result-oriented, yet informal and warm, culture that we have in our room.

I also would like to thank all the bachelor and master students, whom I had luxury to teach during my time in Amsterdam. Your questions helped me see novelty even in well-established branches of mathematics.

I thank Korteweg-de Vries Institute for Mathematics of the University of Amsterdam and all its staff for unique atmosphere of creativity, that made working on my thesis such a lot of fun.

I thank the Netherlands Organisation for Scientific Research, whose support made my work in Amsterdam possible.

I am eternally grateful to my parents, Victor and Olga. Your patience, warmth and love is what made all of this possible. My dear sister, Sveta, thank you for your empathy and for constantly inspiring me. My nephews, Sasha and Andrei, your simple and naive questions kept reminding me to be a little bit playful about everything I do and I thank you for this.

Finally, Sonya, thank you for being there, for the emotional connection that we have and for your infinite patience and understanding.

MIX
Paper from responsible sources
FSC® C128610

ISBN: 978-94-028-0648-9

Contents

1
Introduction

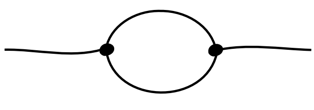
In my thesis I consider a system of relations between two objects that lie on the intersection between algebraic geometry and mathematical physics: *cohomological field theory* and *global spectral curve*. These relations are a part of a larger web of relations between several important concepts: *Frobenius manifold*, *Landau-Ginzburg superpotential*, *Givental group* and *quantum curve*. The motivation to consider these concepts can be informally traced back to various attempts to develop a rigorous approach to the quantum field theory that forms the framework of elementary particle physics. These attempts are very different in nature and their quantum field theory origin doesn't provide any natural mathematical context for their interaction. Nevertheless such a context exists and this thesis partly reconstructs it.

1.1 Basic concepts

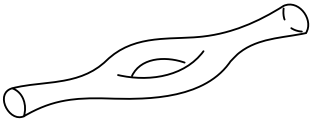
In this section I informally introduce some of the key concepts considered in my thesis, with forward references to the precise definitions in other chapters. This part is specially targeted for a layman.

Let us start with cohomological field theory (CohFT for brevity). Cohomological field theories are mathematically rigorous versions of the topological string theory, whose motivation comes from theoretical physics, see [75]. An informal way to introduce CohFT comes from quantum field theory.

One of the mathematical problems of quantum field theory is that certain integrals corresponding to decorated graphs, the so-called Feynman diagrams, diverge whenever the graph has loops. This happens already for the simplest example of the trivalent graph with one loop in a scalar QFT:



There are many ways to deal with this problem. In the approach suggested by the string theory we replace a one-dimensional graph with a two-dimensional surface given by its tubular neighborhood in three-dimensional space:

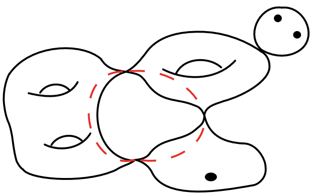


(1.1.1)

A very naive way to explain the construction of topological string theory is the following: to associate to this surface a well-defined integral we define a space over which we are integrating and a measure of integration, which in the QFT case was the product of propagators, and we have to specify the theory of integration that we are using. There is a way to do this that is natural both from the physics point of view and the point of view of algebraic geometry.

The integration space is the space of all possible structures of a one-dimensional complex manifold on the given two-dimensional surface. In other words, we consider the moduli space of algebraic curves of prescribed topology. The basic topological invariants are the genus of the surface denoted by g and the number of boundaries denoted by n. One can choose how to consider the boundaries. They can be either actual geodesic boundaries, or punctures, or just marked points on a compact curve. The latter way suits our purposes the best. Thus, the non-compact integration space is the *moduli space* $\mathcal{M}_{g,n}$ of all possible complex structures on a given two-dimensional surface of a fixed genus g with n marked points.

The space $\mathcal{M}_{g,n}$ has a canonical compactification $\overline{\mathcal{M}}_{g,n}$. The curves we add at the compactification divisor of $\mathcal{M}_{g,n}$ are no longer smooth. Their possible singularities are *nodes* that locally look like $xy = 0$ and are disjoint from the marked points. Furthermore, all the curves have finitely many automorphisms. Such curves are called *stable*.



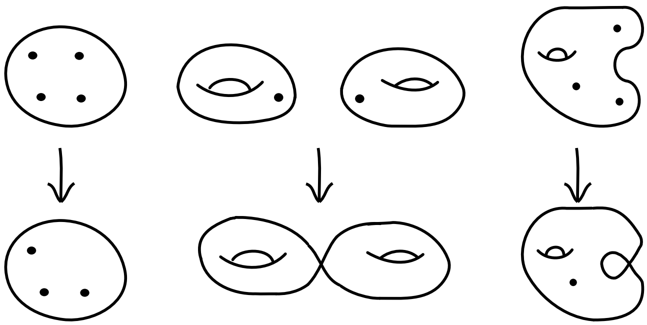
An example of a stable nodal curve.
Its arithmetic genus is $3 + 1$.
Extra 1 comes from the degenerate handle denoted with red dashed circle.
It has 3 marked points and 4 nodes.

(1.1.2)

The moduli space $\overline{\mathcal{M}}_{g,n}$ of stable nodal curves of genus g with n marked points is the integration space of a CohFT. It is a smooth orbifold. For an introduction into the theory of the moduli spaces of curves, see [94].

By the integration over $\overline{\mathcal{M}}_{g,n}$ we understand the intersection of the cohomology classes considered with rational coefficients. We can integrate either over the fundamental class of $\overline{\mathcal{M}}_{g,n}$, or over a system of the so-called tautological classes that we introduce in the next paragraph. These tautological classes can be considered as analogs of observables of the QFT.

Let us introduce the system of the subalgebras of the cohomology algebras of the spaces $\overline{\mathcal{M}}_{g,n}$ called the *tautological ring*. We consider three natural operations on stable nodal curves: one can forget a marked point, one can glue two curves together at two marked points and one can glue two marked points of the same curve together to form a degenerate handle:



(1.1.3)

These operations result in maps π, σ and ρ between corresponding moduli spaces (see Section 2.2.3). The tautological ring is the minimal algebra of classes that is closed under pushforwards and pullbacks with respect to maps π, σ, ρ. In particular, the tautological ring contains the so-called psi-classes $\psi_1, \dots, \psi_n$, see Section 2.2.3, also called gravitational descendants. An introduction to the tautological ring can be found in [98]. The elements of the tautological ring are called tautological classes.

A cohomological field theory, introduced by Kontsevich and Manin in [67], is a collection of polylinear maps $\alpha_{g,n} : V^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$, where V is a vector space equipped with an inner product, called the space of primary fields. In other words, for every n-tuple of vectors from V we get a cohomology class on $\overline{\mathcal{M}}_{g,n}$. Moreover, these classes should satisfy certain natural factorization properties with respect to maps π, ρ and σ (see Section 2.2.3 for the precise definition of the CohFT and these properties).

One can consider the *correlators* of a CohFT: this is the set of all possible integrals of the classes $\alpha_{g,n}$ intersected with all possible monomials in psi-classes. Thus, when each marked point i of the surface is decorated with a non-negative integer d_i and a vector e_i from V, we integrate the class

$$\alpha_{g,n}(e_1, \dots, e_n)\psi_1^{d_1} \dots \psi_n^{d_n} \tag{1.1.4}$$

over the fundamental cycle of the moduli space $\overline{\mathcal{M}}_{g,n}$.

The free energy function is a formal generating function for the correlators of a CohFT. Clearly, the free energy function contains á priori less information than the classes of the underlying CohFT. However, under certain mild assumptions, explained below, the cohomology classes $\alpha_{g,n}$ can be fully reconstructed from the correlators.

First of all, genus 0 correlators are always completely determined by the integrals of the classes of a CohFT with no ψ-classes inserted. These are the so-called primary invariants of a CohFT. These primary invariants can be considered as the coefficients of the prepotential of a formal Frobenius manifold (see Section 2.2.1 for the definition).

Frobenius manifolds were introduced by Dubrovin in his systematic study of the structure of 2d topological field theories [27]. The main part of the structure is the function F of n formal variables $t_1, \dots, t_n$ that satisfies the celebrated WDVV equation (summation over repeated indices is assumed)

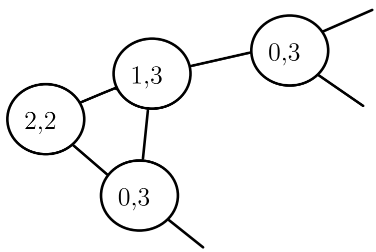
$$\frac{\partial^3 F(\vec{t})}{\partial t^\alpha \partial t^\beta \partial t^\epsilon}\eta^{\epsilon\zeta}\frac{\partial^3 F(\vec{t})}{\partial t^\zeta \partial t^\gamma \partial t^\delta} = \frac{\partial^3 F(\vec{t})}{\partial t^\alpha \partial t^\gamma \partial t^\epsilon}\eta^{\epsilon\zeta}\frac{\partial^3 F(\vec{t})}{\partial t^\zeta \partial t^\beta \partial t^\delta} \tag{1.1.5}$$

This function is called the prepotential.

Assume that the prepotential F is a smooth function defined on an open disk. By *Frobenius manifold* we mean this disk together with the differential geometric structure induced by the prepotential. Triple derivatives of the prepotential define an algebra structure in each tangent space of the disk and this structure smoothly depends on a point. When this algebra has no nilpotents in an open subset of the disk, the Frobenius manifold is called *semisimple*. A set of points of the manifold where multiplication is degenerate is called the *discriminant*. If, furthermore, the prepotential satisfies a certain homogeniety property, the Frobenius manifold is called *conformal*. This concept plays, in particular, the central role in the theory of integrable hierarchies [30].

The celebrated reconstruction theorem by Teleman [91] states that a CohFT is determined by its set of correlators in genus 0 if the following conditions are satisfied. Its underlying Frobenius manifold should be semisimple and conformal, and the CohFT itself should also be *homogeneous* in higher genera. The reconstruction is performed with the help of the *Givental group action*. The formula for this action is very explicit. It was first discovered by Faber, Shadrin and Zvonkine [53] as the group action on the formal free energy functions of CohFTs, and later understood as the action on the systems of cohomology classes $\alpha_{g,n}$ forming CohFTs independently by several groups of people (Teleman [91], Katzarkov-Kontsevich-Pantev (unpublished), and Kazarian (unpublished)).

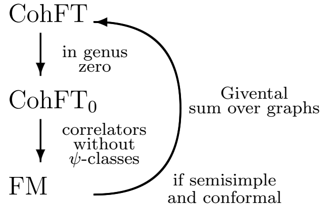
In the reconstruction every class $\alpha_{g,n}^{\mathrm{new}}$ of the resulting CohFT is a sum over so-called *stable graphs*. Stable graphs count topologically distinct stable curves. For example, the degeneration type of the curve on Figure 1.1.2 is given by the following stable graph (the vertices are represented by circles, so that the labels fit in)



At the vertices of the stable graph we put classes $\alpha_{g,n}^0$ that form a cohomological field theory of cohomological degree zero and are completely determined by the choice of a semisimple point of the underlying Frobenius manifold. At the edges we put some explicit contribution that is also determined by the choice a semisimple point of the Frobenius manifold (see Section 2.2.3).

The Givental group action is a generalization of this construction. It acts transitively on the subspace of semisimple CohFTs with a fixed semisimple Frobenius algebra structure at the origin [56].

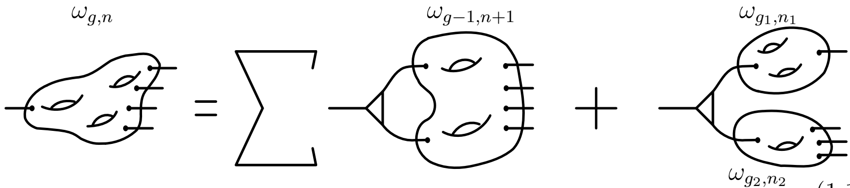
We have the following diagram of relations between CohFTs, their restrictions to genus 0, the underlying Frobenius manifold structures, and Givental group action:



In this thesis we mostly concentrate only on semisimple, homogeneous CohFTs. So, for us, a cohomological field theory and the underlying Frobenius manifold are the often equivalent concepts.

The global spectral curve, the second main character of my thesis, is an algebraic curve Σ, with some additional data. Namely, we have two meromorphic functions x and y on the curve, as well as a symmetric meromorphic bi-differential B called Bergman kernel, with prescribed pole structure. There are some further technical conditions on these objects, see Section 2.2.5. Out of this initial data there is a unique way to construct a system of *correlation differentials* $\omega_{g,n}$, where $2g-2+n>0$, which are symmetric meromorphic n-differentials on the cartesian powers of the curve. Every $\omega_{g,n}$ is recursively expressed through other correlation differentials with lower values of $2g-2+n$. The differentials $\omega_{0,1} := ydx$ and $\omega_{0,2} := B$ are the base of the recursion.

The essence of the recursive reconstruction is captured by the following drawing (see Equation (2.2.30) for precise algebraic expression):



(1.1.6)

This figure represents the so-called *topological recursion.* This recursion procedure was developed in works of Chekhov, Eynard and Orantin [17, 50, 52] hence we sometimes refer to it as CEO-recursion.

The CEO-recursion has first appeared in the context of matrix models as a method of computation of the coefficients of the resolvents, also called correlators. The correlators of a matrix model are certain integrals over a space of finite-dimensional matrices of some particular type, for example, over the space of $N \times N$ Hermitean matrices. Direct computation of the correlators of a matrix model is a difficult task. So in practice one substitutes this problem by a different one: to find a solution to the set of constraints called Ward identities that the correlators must satisfy [76]. In some cases one can show that the CEO-recursion provides a solution to these constraints. Moreover, under some assumptions this solution is unique. As for the space of *all* solutions to the Ward identities that have genus-expansion form, it is conjecturally expressed with help of the check-operators of Alexandrov-Mironov-Morozov (see [74] and references therein). Furthermore, for some matrix models one can compute all correlators under extra special assumptions and they do not even have a genus-expansion form [22]. While the CEO-recursion emerged as one of the approaches in the theory of matrix models and proved to be very useful there, the whole theory of the CEO-recursion has outgrown its original context and became a subject of its own with applications in multiple diverse contexts.

The global spectral curve together with the CEO-recursion procedure on it gives a universal way to encode solutions to various enumerative geometric and combinatorial problems. For specific choices of the spectral curve data (Σ, x, y, B) the differentials $\omega_{g,n}$ are the generating functions for particular problems, such as for instance the number of the ramified coverings of $\mathbb{P}^1$ of certain ramification type. The way to obtain the system of correlation differentials $\omega_{g,n}$ from the initial data (Σ, x, y, B) is, however, always the same and does not dependent on the enumerative problem at hand.

The notion of *global* spectral curve can be compared with the notion of *local* spectral curve [37]. A local spectral curve is not a Riemann surface, instead, it is a union of open disks. Accordingly, the functions x and y, and the bi-differential B are defined only locally, their arguments taking values inside these disks, via their Taylor or Laurent expansions. One can informally think of these disks as coordinate charts near the critical points of the function x on *some* Riemann surface Σ. Since the recursion step of the topological recursion depends only on the local information near each critical point (see Equation (2.2.30)), it is defined for a local spectral curve as well. So, starting from a local spectral curve we can also obtain the system of correlation differentials $\omega_{g,n}$. In general, it is not known, whether for a given local spectral curve there exists such a global spectral curve (Σ, x, y, B) whose expansions of x, y and B near critical points of x coincide with the data of the local spectral curve, or whether such a global spectral curve is unique if it exists.

Semisimple, homogeneous cohomological field theories and global, or local, spectral curves are similar: for both concepts we can reconstruct an infinite system of objects, parametrized by genus g and number of points n, from some simple initial data. In the former case this data is the structure of a Frobenius manifold and in the latter case it is given by the functions x, y and B. One of the main motivations for this thesis is an identification result of Dunin-Barkowski et al. [37]. It establishes a certain equivalence between semisimple CohFTs and local spectral curves. More precisely, for every semisimple CohFT one can explicitly construct a local spectral curve, such that the correlation differentials $\omega_{g,n}$, obtained from the initial data of this curve, are generating functions for the correlators of the CohFT. The proof of this result is done by direct comparison of Givental's sum over graphs and the sum over graphs that arises when one applies topological recursion step repeatedly, decomposing a given $\omega_{g,n}$ into elementary components.

The data of the local spectral curve (coefficients of Taylor expansion of the function y and the Bergman kernel B in the local coordinates of the disks) is arguably no simpler than the explicit form of all the ingredients of the Givental formula. So, if one wants to encode a given CohFT in a compact way, one needs to find a matching global spectral curve. The identification result does not give any recipe for this. What it allows to do is to check, whether a given candidate global spectral curve is indeed a global spectral curve for a given CohFT. But one must come up with this candidate global spectral curve first, mostly by educated guessing. It would be much better if there was an algorithm that directly produced a global spectral curve for a given CohFT.

Such direct association of the global spectral curve to a CohFT is desirable for one more reason. From the point of view of Frobenius manifolds, the identification works point-wise: for each semisimple point of the Frobenius manifold there is a local spectral curve. But it does not specify what to associate to non-semisimple points. Suppose, however, that we have a family of global spectral curves over the semisimple points of the Frobenius manifold and, moreover, dependence on the point of the Frobenius manifold is sufficiently good. Then we can analyze what happens as we approach a non-semisimple point. Sometimes, as we take the limit, two or more simple critical points of function x merge to form a higher-order critical point. For such spectral curves with higher-order critical points the CEO-recursion procedure needs to be substituted by the much more involved Bouchard-Eynard recursion procedure [11], so proving directly that this degenerate spectral curve corresponds to this non-semisimple CohFT is difficult. However, there is a general statement that taking limits in suitable families of global spectral curves commutes with evaluating the correlation differentials. Therefore, in this way we get the correct global spectral curve for some non-semisimple points of Frobenius manifolds. In this thesis we use this logic to give an easy alternative proof for the correspondence between the r-spin Witten class and the A_r singularity.

So, the main focus of this thesis is the following question: how to go from a cohomological field theory to a global spectral curve and vice versa and when it is possible? The answer that we give is very general but not exhaustive. We identify sufficient conditions that guarantee that some particular ways of direct and reverse transition work. This results in plenty of interesting applications.

1.2 Outline

In this section I present the main results of my thesis.

1.2.1 Dubrovin's superpotential as a global spectral curve

In Chapter 2 we start from a cohomological field theory and (under some assumptions) construct the corresponding global spectral curve. We do this using Dubrovin's construction of a superpotential (see Section 2.2.4). Dubrovin's superpotential is a particular case of a more general Landau-Ginzburg superpotential (see Section 2.2.2).

In the most interesting examples the Landau-Ginzburg (LG) superpotential is a family of functions from a ball around origin in $\mathbb{C}^n$ to $\mathbb{C}$, such that one function in the family has isolated degenerate singularity at the origin and for the other functions in the family this singularity is resolved. An LG superpotential can be used to define the structure of Frobenius manifold on the space of deformation parameters of the singularity. The structure constants and the metric of the Frobenius manifold are given by certain integrals, in the simplest examples by the residues at the critical points of the superpotential.

Suppose we are given a Frobenius manifold. A natural question would be: can we interpret this manifold as the space of deformations of *some* 1-dimensional singularity?

Dubrovin [28] answers this question by explicitly constructing a family of curves $\mathcal{D}_\tau = \mathcal{D}(\tau)$, parametrized by the semisimple points τ of the Frobenius manifold, equipped with two meromorphic functions, λ_τ and p_τ and showing that it satisfies all the properties of an LG superpotential. His construction depends on some choices, which we fix in this chapter in some particular way.

The main results of this chapter, Theorems 2.5.1 and 2.6.1, are devoted to proving that topological recursion applied to

$$\Sigma = \mathcal{D}_\tau, \quad x = \lambda_\tau, \quad y = p_\tau,$$

and some choice of B gives, under the correspondence from [37], exactly the CohFT we started with.

One of the main tools in the proofs of Theorems 2.5.1 and 2.6.1 is the identification result of Dunin-Barkowski et al. [37]. The local spectral curve that is obtained from a CohFT by their procedure is not arbitrary: there is a system of consistency equations on x, y and B. Our main task is, therefore, to show that λ_τ, p_τ and an appropriately chosen B satisfy these constraints. Here we give a brief outline of the proof.

The identification of x and λ up to some topological properties is the starting point since the CohFT is based on a vector space formally spanned by the zeros of dx, respectively the zeros of $d\lambda$. On the side of topological recursion there is one requirement that we need, namely we have to assume that there is exactly one critical point on $\mathcal{D}_\tau$ over each critical value of $x = \lambda_\tau$. This gives a restriction on the possible choices of analytic continuation in Dubrovin's superpotential.

The relation of y with structure constants in the Frobenius manifold required in [37] leads to an identification of $y = p_\tau$. This theorem (Theorem 2.3.1) is heavily based on the computations done by Dubrovin in [28]. Next we need to find a good choice of B that will make either theorem work. In genus zero we find that the unique possible Bergman kernel B satisfies the conditions required by [37] which we present in a form that can be checked (or used as a condition) for the superpotentials. This is Theorem 2.4.1 and its corollaries. It allows us to conclude that topological recursion applied to the superpotential produces *some* CohFT and it remains to prove that this CohFT is the one associated to the Frobenius manifold defined by the superpotential. We show that in fact it is sufficient to know that we get *homogeneous* CohFT from the Bergman kernel – then the correct CohFT is reproduced automatically. This leads to a general theorem that Dubrovin's construction indeed gives the right global spectral curve for a given CohFT, provided it results in genus 0 spectral curve (Theorem 2.5.1). This theorem is key to several important examples that we discuss in this chapter as well (we mention these examples in the list of applications in Section 2.1.2).

In higher genera, the Bergman kernel is not canonical and we need to choose the correct one. In order to have a suitable shape of the Laplace transform of the Bergman kernel (required for correspondence with Givental graphs), we have to use the Bergman kernel normalized on a basis of $\mathcal{A}$-cycles for some Torelli marking, using results of Eynard [46]. We show, using the Rauch variational formulae, that the homogeneity property is also satisfied in this case, and this allows us to make a general statement about the correspondence of a global spectral curve to a cohomological field theory in any genus (Theorem 2.6.1). This is a conditional statement requiring Theorem 2.4.1

that needs to be checked in particular examples. As an application, we work out an elliptic example in detail (Theorem 2.9.2).

Finally, we develop a theory for the case when the extra assumptions on the choice of analytic continuation of Dubrovin's superpotential are dropped. In this case we have to generalize the setup of topological recursion in order to take into account the action of the reflection group associated with Frobenius manifold. The correspondence that we obtain in this case (Theorem 2.10.6) is parallel to the ideas of Milanov [73].

1.2.2 Primary Invariants of Hurwitz Frobenius Manifolds

In Chapter 3 we start with a global spectral curve. We identify sufficient set of conditions under which this curve corresponds to a CohFT and, moreover, this CohFT has a separate geometric meaning.

We observe that the map x from the compact curve Σ to $\mathbb{P}^1$ defines a point in a *Hurwitz space* $H_{g,\mu}$ – the space of equivalence classes of meromorphic maps from genus g surface to $\mathbb{P}^1$, with prescribed ramification profile μ over infinity. Dubrovin [27] showed that the cover $\tilde{H}_{g,\mu}$ of this space, where we additionally fix the basis of $\mathcal{A}$- and $\mathcal{B}$-cycles on the curve, naturally carries the structure of a Frobenius manifold.

The main theorem of this chapter, Theorem 3.1.6, states that it is this, natural, CohFT that agrees with the global spectral curve we started with, provided some conditions hold true. Namely, the Bergman kernel B should be a unique Bergman kernel normalized on the set of chosen $\mathcal{A}$-cycles and the differential of the function y should be the *primary differential* on the curve, meaning it should agree with B and x in some specific way.

Furthermore, we show that correlators of this CohFT can be extracted from the multidifferentials $\omega_{g,n}$. Theorem 3.1.4 gives explicit formula for the *primary invariants*, which are the CohFT correlators without insertions of ψ-classes, in terms of multiple integrations of $\omega_{g,n}$ over the contours from some canonical set of contours, associated to $\tilde{H}_{g,\mu}$. Moreover, in Proposition 3.4.12 we state that correlators with insertions of ψ-classes can also be extracted from correlation multidifferentials with help of multiple integration over some contours, though we do not give an explicit formula for these contours.

The results of this chapter also hold in a bit more general setting. The Bergman kernel B need not be the canonically normalized one. It is sufficient that it belongs to Shramchenko's $g \times g$-parametric family of deformed Bergman kernels [88].

1.2.3 Chiodo formulas for the r-th roots and topological recursion

In Chapter 4 we establish the correspondence between cohomological field theory and global spectral curve for a particular case: the system of cohomology classes defined by Chiodo in [19]. This system of classes depends on two parameters r and s, which are non-negative integers.

This particular CohFT is important in applications. It is the main building block in the proof of the Pixton conjecture by Janda et al. [62]. Moreover, at particular values of parameters r and s Chiodo's CohFT is conjectured to be related to the so-called Hurwitz numbers with completed cycles. Completed cycles are one of the central objects in the theory of shifted symmetric functions (see, e.g. [81]). Therefore, it is interesting to look at the Chiodo CohFT from different angles, in particular, from the point of view of the corresponding global spectral curve.

The main theorem of this chapter, Theorem 4.4.6, states that Chiodo's CohFT is related to the following global spectral curve $\Sigma = \mathbb{P}^1$

$$\begin{cases} x = \log z - z^r, \\ y = z^s, \\ B = \frac{dzdz'}{(z-z')^2} \end{cases} \tag{1.2.7}$$

In the case $s = 1$ our theorem reduces to the theorem of Shadrin-Spitz-Zvonkine [87] about the equivalence of the r-spin Bouchard-Marino and the r-spin ELSV conjectures.

In the case $s = r$ the topological recursion on the spectral curve is known to produce generating functions for the so-called *orbifold Hurwitz numbers* [34]. Using our theorem, we obtain an independent proof of this statement. Namely, we show that at these values of parameters Chiodo's class is equal to the class used in the Johnson-Pandharipande-Tseng formula (the ELSV-type formula for the orbifold Hurwitz numbers). Since Chiodo's CohFT is related to the desired global spectral curve, this implies the topological recursion for the orbifold Hurwitz numbers.

1.2.4 Quantum spectral curve for the Gromov-Witten theory of $\mathbb{P}^1$

In Chapter 5 we prove that a *quantum spectral curve* exists for a particular cohomological field theory – the Gromov-Witten theory of the complex projective line.

A quantum spectral curve for a given cohomological field theory is a differential, or difference, operator $P(x, \hbar\frac{d}{dx})$, together with the "wave function" $\Psi(x)$ that is annihilated by this operator and is assembled from the correlators of the CohFT in some specific way (see Section 5.1).

When the quantum spectral curve exists, it gives us a guess about the global spectral curve. Namely, by substituting $\hbar\frac{d}{dx}$ by y in the expression for the differential operator, we get an implicit equation $P(x, y) = 0$ of the global spectral curve. Then we only need to find the correct Bergman kernel B.

It is not known, whether a quantum spectral curve exists for every CohFT and whether it is unique when it does. However, Bouchard and Eynard [12] showed, that if the corresponding global spectral curve is known and is plane algebraic and, moreover, its Newton polygon has no interior points, then at least one quantum spectral curve can be explicitly constructed.

The particular example of Gromov-Witten theory of $\mathbb{P}^1$ is interesting, because the global spectral curve is not algebraic: $y = \log z$, where z is the global coordinate on the curve. Hence, it is not covered by the general Bouchard-Eynard argument.

The main theorem of this chapter, Theorem 5.1.1, states the precise form of the quantum curve in this case. The correct differential operator (quantization) of the global spectral curve is simply

$$e^{\hbar\frac{d}{dx}} + e^{-\hbar\frac{d}{dx}} - x,$$

that is, out of two equations $x(z)$ and $y(z)$ we obtain the equation $f(x, y) = 0$ in the most naive way and then quantize it by simply substituting $y \to \hbar \frac{d}{dx}$. The correct wave function $\Psi(x)$, which is annihilated by this differential operator, is assembled from individual Gromov-Witten correlators in a specific way. Namely, there are logarithmic corrections to the unstable terms (see formulas (5.1.4) and (5.1.5)). Also, there are explicit insertions of the identity vector field (see formulas (5.1.3) and (5.1.5)).

1.2.5 Combinatorics of loop equations for branched covers of sphere

In Chapter 6 we discuss in detail one of the combinatorial problems, that appeared in as an application in Chapter 2 – the enumeration of hypermaps. We construct a global spectral curve for this problem, as well as a quantum spectral curve.

The main theorem of the chapter, Theorem 6.3.8, states that certain global genus zero spectral curve encodes the number of hypermaps (equivalence classes of covers of $\mathbb{P}^1$ of specific ramification type, see Section 6.2).

Theorem 6.5.1, another important theorem of the chapter, states, how the quantum curve for this combinatorial problem looks. The quantization is extremely simple: differential operator is obtained from equation of the spectral curve by simple substitution $y \to \hbar \frac{d}{dx}$. The expression for the wave function $\Psi(x)$ involves logarithmic corrections to the unstable terms, similarly to the case of Gromov-Witten theory of the complex projective line.

In principle, since in this case the global spectral curve is plane algebraic and its Newton polygon has no interior points, Theorem 6.5.1 follows from Theorem 6.3.8 and the argument of Bouchard-Eynard [12]. However, results presented in Chapter 6 appeared before the paper of Bouchard-Eynard, so Theorem 6.5.1 can be considered as an important step towards their general theory.

Proofs of both Theorem 6.3.8 and Theorem 6.5.1 make heavy use of the loop equations (6.4.1),(6.4.3). In the work of Chekhov, Eynard and Orantin [17] these loop equations were derived by the physics argument with help of a certain formal 2-matrix model. This argument is hard to make precise: there are convergence issues as well as issues of analytic continuation. So, in Chapter 6 we prove loop equations (6.4.1) and (6.4.3) directly from the cut-and-join combinatorics of the numbers of the hypermaps (see Section 6.4.2), without the need to resort to matrix model arguments. Then, in the proof of Theorem 6.3.8 we are able to re-use the rest of the argument of Chekhov, Eynard and Orantin, which is completely rigorous.

References

The content of Chapters 2-6 is based on the following papers [1], in order. Only small changes have been made to increase readability. Hence, the chapters are self-contained and in principle can be read independently. Therefore, they sometimes introduce

[1] Formal remark on co-authorship, required by the *Promotiereglement 2014* of the University of Amsterdam: the authors of these papers have equally contributed to the obtained results.

the same mathematical objects, though each one does it a bit differently, as different aspects of these objects are needed.

- P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, S. Shadrin
  Dubrovin's superpotential as a global spectral curve
  accepted for publication in **J. Inst. Math. Jussieu**; arXiv:1509.06954

- P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, S. Shadrin
  Primary invariants of Hurwitz Frobenius manifolds
  submitted; arXiv:1605.07644

- D. Lewanski, A. Popolitov, S. Shadrin, D. Zvonkine
  Chiodo formulas for the r-th roots and topological recursion
  accepted for publication in **Lett. Math. Phys.**; arXiv:1504.07439

- P. Dunin-Barkowski, M. Mulase, P. Norbury, A. Popolitov, S. Shadrin
  Quantum spectral curve for the Gromov-Witten theory of $\mathbb{C}\mathrm{P}^1$
  accepted for publication in **J. Reine Angew. Math.**; arXiv:1312.5336

- P. Dunin-Barkowski, N. Orantin, A. Popolitov, S. Shadrin
  Combinatorics of loop equations for branched covers of sphere
  accepted for publication in **Int. Math. Res. Not.**; arXiv:1412.1698

2

Dubrovin's superpotential as a global spectral curve

Abstract

We apply the spectral curve topological recursion to Dubrovin's universal Landau-Ginzburg superpotential associated to a semi-simple point of any conformal Frobenius manifold. We show that under some conditions the expansion of the correlation differentials reproduces the cohomological field theory associated with the same point of the initial Frobenius manifold.

2.1 Introduction

2.1.1 The goal

A semi-simple (conformal) Frobenius manifold is an important algebro-geometric structure, introduced by Dubrovin, that appears naturally in a circle of questions related to classical mirror symmetry. Closely related to a semi-simple conformal Frobenius manifold is a cohomological field theory, that is, a system of cohomology classes on the moduli space of stable curves introduced by Kontsevich and Manin in order to capture the main universal properties of Gromov-Witten theory. Via Givental-Teleman theory, these two concepts (semi-simple conformal Frobenius manifolds and semi-simple homogeneous cohomological field theories) are essentially equivalent.

The theory of Landau-Ginzburg superpotentials associates to a Riemann surface (or a family of Riemann surfaces) equipped with a meromorphic function and a meromorphic differential 1-form (or a meromorphic function whose differential is this 1-form) structure that is essentially equivalent to the concept of a semi-simple Frobenius manifold, after work of Dubrovin [27]. It is part of a more general theory of Landau-Ginzburg models that exists in any dimension, not necessarily on a curve.

The theory of spectral curve topological recursion, initially developed for computation of the correlation differentials of matrix models, uses a very similar input: a Riemann surface (or a family of Riemann surfaces) equipped with a meromorphic function, a meromorphic differential 1-form (or a meromorphic function, whose differential is this 1-form), and a symmetric bi-differential. It produces a system of symmetric differentials on the cartesian powers of the underlying Riemann surface.

Under some extra conditions these symmetric differentials can be expressed in terms of the correlators of a cohomological field theory.

To summarize, we have the following system of relations:

$$\begin{array}{ccc} \text{semi-simple conformal} & \leftrightarrow & \text{Landau-Ginzburg} \\ \text{Frobenius manifolds (FM)} & & \text{superpotentials (LG)} \\ \Updownarrow & & \\ \text{semi-simple homogeneous} & & \text{spectral curve} \\ \text{cohomological field theories (CohFT)} & \leftrightarrow & \text{topological recursion (TR)} \end{array} \tag{2.1.1}$$

We give precise definitions of all geometric structures involved in this diagram and explain the precise statements about their relations in Section 2.2. In all cases the rigorous formulation of these correspondences requires extra conditions and is not a one-to-one correspondence or an equivalence of categories. It is more like a dictionary that allows one to translate from one language to another under various extra assumptions.

The theory of Landau-Ginzburg superpotentials and spectral curve topological recursion use almost the same input data, namely a Riemann surface equipped with a meromorphic function and a meromorphic differential 1-form. This input data is used in a completely different way in these two theories, nevertheless the natural question is whether one can add a vertical arrow so that the diagram commutes. More explicitly, if a Landau-Ginzburg superpotential and spectral curve topological recursion produce the same Frobenius manifold/CohFT structure on the left hand side of this diagram, do we expect that the input data for the LG model and TR to be the same?

This chapter is devoted to an affirmative answer to this question. As in the case of all other correspondences in this diagram, it is not an equivalence of categories or one-to-one correspondence, but rather a system of general statements that allows one to connect the input data of LG and TR in a large class of examples.

2.1.2 Contributions to the theory of topological recursion

In order to establish a correspondence with the Landau-Ginzburg theory and to work out several basic examples, we obtain a number of results that are of independent interest for the theory of topological recursion, and here we collect them all.

Global spectral curve for the CohFT-TR correspondence

One way to present our main result is the following. The correspondence between CohFT and TR obtained in [37] uses a local version of topological recursion, that is when the spectral curve is just a union of disks. An important open question is whether we can glue all these open disks into a global spectral curve. This would allow one to use a variety of analytical methods developed in the theory of topological recursion that are applicable only in the case of a global curve [50, 52]. The main result of this chapter is an affirmative answer to this question, that is, for a large class of CohFTs we can indeed claim the existence of a global spectral curve. In this form this question was also considered by Milanov for singularity theory [73].

Bouchard-Eynard recursion locally

Topological recursion requires the spectral curve to have simple critical points. There is an extension of the theory of topological recursion for the curves with higher order critical points, due to Bouchard and Eynard [11]. A fundamental question is to identify the correlation functions of their generalized recursion in the elementary case of one point of order $r+1$. Bouchard and Eynard have announced [10] a theorem that in this case the correlators are expanded in terms of the string tau-function of the r-Gelfand-Dickey hierarchy (or, equivalently, in terms of the intersection theory of the Witten top Chern class on the moduli space of r-spin structures, [95, 53]).

An application of the main theorem of this chapter, i.e. where topological recursion applied to Dubrovin's construction of a superpotential produces the same CohFT is the case of the A_n singularity. Careful analysis of this example in its limit at the zero point implies immediately the theorem of Bouchard and Eynard.

Enumeration of hypermaps

Each time a particular combinatorial problem is solved in terms of topological recursion, there occurs a natural question whether this leads to an interesting CohFT inside this combinatorial problem, and, as a consequence, to an interesting ELSV-type formula for it. This logic is explained in detail in [34, Introduction]. In particular, the topological recursion was proved in [36] for the enumeration of *hypermaps*, see also [26].

In the case of hypermaps the correspondence between LG and TR gives us immediately a full description of the Frobenius manifold structure behind this combinatorial problem; it is a particular simple example of a so-called Hurwitz Frobenius manifold. In the simplest case one can say that the Frobenius manifold with the prepotential $t_1^2t_2/2 + t_2^2 \log t_2$ resolves, via its associated CohFT and the ELSV-type formula, the combinatorial problem known, in different versions, as generalized Catalan numbers, discrete volumes of moduli spaces, or discrete surfaces [4, 32, 52, 79]. This explains, in a conceptual way, some observations already made in [8, 54].

Bergman kernel and Torelli marking

Another important application of this chapter is to prove a form of independence of the output of topological recursion from the choice of the bidifferential B for a global spectral curve. Topological recursion depends on B and there are many ways to normalize B depending on a choice of Torelli marking on the Riemann surface. We show that for a global spectral curve satisfying a compatibility condition, topological recursion gives rise to a so-called homogeneous CohFT with flat identity *independent* of the choice of normalisation of B.

2.1.3 Guide to the chapter

In Section 2.2 we give a full description of all concepts mentioned in Diagram (2.1.1) and explain the known relations between them.

In Section 2.3 we prove that Dubrovin's superpotential always gives the right y-function for the topological recursion. Then in Section 2.4 we revisit in geometric terms the necessary compatibility conditions between y and B on the spectral curve from [37]. This allows us to prove the two main theorems of this chapter. Namely, in Section 2.5 we prove the LG-TR correspondence in the genus 0 case, and in Section 2.6 we generalize this result to higher genera.

Then we discuss several important series of examples, where Dubrovin's superpotential can be computed explicitly. In Section 2.7 we discuss A_n singularities, with an application to the Bouchard-Eynard generalisation of topological recursion. In Section 2.8 we present in detail a computation for a special class of Hurwitz Frobenius manifolds, corresponding to the case of meromorphic functions on the Riemann sphere with two poles, one of which is of order 1. In this case the corresponding topological recursion resolves enumeration of hypermaps. In Section 2.9 we describe a higher genera case, namely, we consider the case of elliptic curve, where the superpotential is given by the Weierstrass function.

Section 2.10 is devoted to a general theory where we use a universal construction of analytic continuation instead of the rather particular constructions of Sections 2.7, 2.8, and 2.9. This essentially reproduces, in our context, the main ideas of the work of Milanov [73] initially applied by him to the case of simple singularities.

In Section 2.11 we explicitly construct global spectral curves for two rank 2 CohFTs. We need to vary the construction slightly due to degeneracy of the Gauss-Manin system. These examples satisfy the conditions of Theorem 2.6.1 and hence topological recursion produces the CohFT associated to the Frobenius manifold.

2.2 Recollection of basic facts

The purpose of this Section is to recall all necessary definitions and facts on Frobenius manifold, moduli spaces of curves, cohomological field theories, Dubrovin's universal construction of Landau-Ginburg superpotentials, and topological recursion.

2.2.1 Frobenius manifolds

In this Section we recall, following [27, 28], the definition of Frobenius manifold and recollect some basic facts about its structures.

Consider a function $F(t^1,\dots,t^n)$ defined on a ball $B\subset\mathbb{C}^n$ and a constant inner product $\eta^{\alpha\beta}$ such that the triple derivatives of F with one shifted index,

$$C^{\gamma}_{\alpha\beta} := \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\gamma}, \tag{2.2.2}$$

are the structure constants of a commutative associative Frobenius algebra with the scalar product given by $\eta_{\alpha\beta}$. We can think about this structure as defined on the tangent bundle of $B\subset\mathbb{C}^n$ (and we denote the corresponding multiplication of vector field by $\cdot$), and we require that ∂_{t^1} is the unit of the algebra in each fiber.

Consider a vector field $E:=\sum_{\alpha=1}^n((1-q_\alpha)t^\alpha+r_\alpha)\partial_{t^\alpha}$, here q_α and r_α are some constants, $\alpha=1,\dots,n$. We require that $q_1=0$ and $r_\alpha\neq 0$ only in the case $1-q_\alpha=0$. We require that there exists a constant d such that $E.F-(3-d)F$ is a polynomial of order at most 2 in $t^1,\dots,t^n$.

The triple (F, η, E) that satisfies all conditions above gives us the structure of a (conformal) Frobenius manifold of rank n and conformal dimension d. The function F is called the prepotential; the vector field E is called the Euler vector field. Of course, there are coordinate-free descriptions of this structure as well, we refer to [27, 28] for details.

Two important structures associated to Frobenius manifolds are the second metric η' on TB and the extended flat connection $\tilde{\nabla}$ on $B \times \mathbb{C}$. The second metric η' on TB is defined in the following way. The first metric η can be considered as an isomorphism between $\eta\colon TB \to T^*B$. For any two vector fields ∂' and ∂'' we define $\eta'(\partial', \partial'')$ to be $E \vdash \eta(\partial' \cdot \partial'')$. The extended connection $\tilde{\nabla}$ is defined as

$$\tilde{\nabla}_{\partial'}\partial'' := \nabla^{\eta}_{\partial'}\partial'' + z\partial' \cdot \partial''; \tag{2.2.3}$$

$$\tilde{\nabla}_{\partial'}\partial_z := 0; \tag{2.2.4}$$

$$\tilde{\nabla}_{\partial_z}\partial_z := 0; \tag{2.2.5}$$

$$\tilde{\nabla}_{\partial_z}\partial' := \partial_z(\partial') + E \cdot \partial' - \frac{1}{z}\mu\partial', \tag{2.2.6}$$

where ∇^{η} is the Levi-Civita connection of η, and the endomorphism $\mu : TB \to TB$ is defined by

$$\mu(v) := (1 - d/2)v - \nabla^{\eta}_{v}E. \tag{2.2.7}$$

In the flat basis, $\mu = \mathrm{diag}(\mu_1, \dots, \mu_n)$ for constants $\mu_\alpha = q_\alpha - d/2$.

In this chapter we only consider semi-simple Frobenius manifolds, that is, we require that the algebra structure on an open subset $B^{ss} \subset B$ is semi-simple. In a neighborhood of a semi-simple point we have a system of canonical coordinates $u_1, \dots, u_n$, defined up to permutations, such that the vector fields ∂_{u_i}, $i = 1, \dots, n$, are the idempotents of the algebra product, and the Euler vector field has the form $E = \sum_{i=1}^{n} u_i\partial_{u_i}$.

The geometric structure that is equivalent to the notion of conformal Frobenius manifolds can be described in canonical coordinates [27]. The canonical coordinate vector fields ∂_{u_i} are orthogonal but not orthonormal. We can normalize them to produce a so-called normalized canonical frame in each tangent space, that is, if $\Delta_i^{-1} = \eta(\partial_{u_i}, \partial_{u_i})$, then the orthonormal basis is given by $\Delta_i^{1/2}\partial_{u_i}$, $i = 1, \dots, n$. By Ψ we denote the transition matrix from the flat basis to the normalized canonical one. Hence the columns of Ψ are given by the coordinates of the flat vectors ∂_{t_α} in the basis $\Delta_i^{1/2}\partial_{u_i}$, with first column $\Psi_{i1} = \Delta_i^{-1/2}$ representing the unit vector. We have the relation

$$E \cdot \Psi = \Psi\mu$$

where $E\cdot$ is differentiation with respect to E.

Define the matrix V to be the endomorphism μ with respect to the normalized canonical basis, hence $V = \Psi \cdot \mathrm{diag}(\mu_1, ..., \mu_n) \cdot \Psi^{-1}$ and $V + V^T = 0$. Covariant constancy of μ implies that V satisfies

$$dV = [V, d\Psi \cdot \Psi^{-1}].$$

Define $V_i = \partial_{u_i}\Psi \cdot \Psi^{-1}$ so $\sum_i u_iV_i = V$.

Remark 2.2.1. Note that Givental [56] (and [37]) uses a different convention for matrices than what is used here. Givental's convention uses a right action of matrices on vectors which is the transpose of the convention we use here.

2.2.2 Superpotential

A convenient way to describe a Frobenius structure is in terms of a so-called Landau-Ginzburg superpotential. We recall the definition from [27, 28]. A *superpotential* is a function $\lambda(p, u_1, \ldots, u_n)$ of a variable $p \in \mathcal{D}$ in some domain $\mathcal{D}$ that depends on points $(u_1, \ldots, u_n) \in B_0 \subset B^{ss}$ in a ball in the semisimple part of the Frobenius manifold, and satisfies the following properties:

1. The critical values of λ as a function on $\mathcal{D}$ are $u_1, \ldots, u_n$.

2. The critical points are non-degenerate.

3. If there are several critical points in the inverse image $\lambda^{-1}(u_i)$, then the Hessians of λ at these points must coincide.

4. For any choice $p_1, \ldots, p_n \in \mathcal{D}$ of the critical preimages of $u_1, \ldots, u_n$ (that is, $\lambda(p_i, u_1, \ldots, u_n) = u_i$) and for any choice of the vector fields ∂', ∂'', and ∂''' on B_0 we have:

$$\eta(\partial', \partial'') = -\sum_{i=1}^{n} \operatorname*{Res}_{p \to p_i} \frac{\partial'(\lambda dp)\partial''(\lambda dp)}{d_p\lambda}; \tag{2.2.8}$$

$$\eta'(\partial', \partial'') = -\sum_{i=1}^{n} \operatorname*{Res}_{p \to p_i} \frac{\partial'(\log \lambda dp)\partial''(\log \lambda dp)}{d_p \log \lambda}; \tag{2.2.9}$$

$$\eta(\partial' \cdot \partial'', \partial''') = -\sum_{i=1}^{n} \operatorname*{Res}_{p \to p_i} \frac{\partial'(\lambda dp)\partial''(\lambda dp)\partial'''(\lambda dp)}{dp\, d_p\lambda} \tag{2.2.10}$$

where $\partial'(\lambda dp)$ gives the action of the vector field by derivation in the parameters u_i. In particular, the map $\partial' \mapsto \partial'(\lambda dp)$ from vector fields on B to meromorphic differentials on $\mathcal{D}$ quotiented out by $d_p\lambda$ is injective.

5. There exist some cycles $Z_1, \ldots, Z_n$ in $\mathcal{D}$ such that the integrals

$$\frac{1}{\sqrt{z}} \int_{Z_\alpha} e^{z\lambda} dp, \qquad \alpha = 1, \ldots, n \tag{2.2.11}$$

converge and give a non-degenerate system of flat coordinates for $\tilde{\nabla}$.

In these terms, the identity vector field ∂_0 of the Frobenius manifold is represented by dp, i.e. $\partial_0(\lambda dp) = dp$. Indeed, since $\eta(\partial_0 \cdot \partial, \partial') = \eta(\partial, \partial')$ for all vector fields ∂, ∂', then non-degeneracy of η implies that $\partial_0 \cdot \partial = \partial$ for all ∂. The Euler vector field is represented in these terms by λdp, i.e. $E(\lambda dp) = \lambda dp$.

2.2.3 Cohomological field theories

In this Section we recall all basic definitions that are necessary to introduce the concept of a cohomological field theory. It is an algebraic structure on a given vector space that captures the main properties of Gromov-Witten theories, and there is a natural group action on these structures, due to Givental. The main sources for this Section are [67, 91, 56, 85, 82].

A stable curve of genus g with k marked points is a possibly reducible curve with nodal singularities, of arithmetic genus g and k non-singular marked points, such that the group of its automorphisms is finite. By $\overline{\mathcal{M}}_{g,k}$ we denote the moduli space of stable curves of genus g with k ordered marked points. There are natural line bundles $L_i \to \overline{\mathcal{M}}_{g,k}$, $i = 1, \dots, k$, whose fiber of the point $[(C_g, x_1, \dots, x_k)] \in \overline{\mathcal{M}}_{g,k}$ represented by the curve C_g with the marked points $x_1, \dots, x_k \in C_g$ is given by $T^*_{x_i} C_g$. The first Chern class of L_i is denoted by $\psi_i \in H^2(\overline{\mathcal{M}}_{g,k}, \mathbb{C})$.

There are a number of natural maps between the moduli spaces. By $\pi \colon \overline{\mathcal{M}}_{g,k+1} \to \overline{\mathcal{M}}_{g,k}$ we denote the map that forgets the last marked point and stabilizes the curve. By $\sigma \colon \overline{\mathcal{M}}_{g_1,k_1+1} \times \overline{\mathcal{M}}_{g_2,k_2+1} \to \overline{\mathcal{M}}_{g,k}$ we denote the map that sews the last marked points on the source curves into a node on the target curve, $g = g_1 + g_2$, $k = k_1 + k_2$. By $\rho \colon \overline{\mathcal{M}}_{g-1,k+2} \to \overline{\mathcal{M}}_{g,k}$ we denote the map that sews the two last marked points on the source curve into a node on the target curve.

Consider a vector space $V = \mathbb{C}\langle e_1, \dots, e_n \rangle$ with a scalar product η. A cohomological field theory with the target (V, η) is a system of cohomology classes $\alpha_{g,k} \colon V^{\otimes k} \to H^*(\overline{\mathcal{M}}_{g,k}, \mathbb{C})$ satisfying the following conditions:

1. The form $\alpha_{g,k}$, $g \geq 0$, $k \geq 0$, $2g - 2 + k > 0$, is invariant under the action of S_k that simultaneously reshuffle $V^{\otimes k}$ and relabel the marked points on the curves in $\overline{\mathcal{M}}_{g,k}$.

2. We have:

$$\pi^* \alpha_{g,k} = e_1 \vdash \alpha_{g,k+1}; \tag{2.2.12}$$

$$\sigma^* \alpha_{g,k+1} = \eta^{\alpha\beta} e_\alpha \otimes e_\beta \vdash \alpha_{g_1,k_1+1} \alpha_{g_2,k_2+1}; \tag{2.2.13}$$

$$\rho^* \alpha_{g,k} = \eta^{\alpha\beta} e_\alpha \otimes e_\beta \vdash \alpha_{g-1,k+2}. \tag{2.2.14}$$

   Here by $\vdash$ we denote the substitution of the vector e_1 at the $(k+1)$-st argument in the first equation, and the substitution of the bivector corresponding to the scalar product at the marked points that are sewed into the nodes under the maps σ and ρ.

Note that if all classes $\{\alpha_{g,k}\}$ are of degree 0, then the structure that we get is called a topological field theory (TFT), and it is equivalent to a Frobenius algebra structure on (V, η).

Correlators, or ancestor invariants, of the CohFT are defined by:

$$\int_{\overline{\mathcal{M}}_{g,k}} \alpha_{g,k}(e_{\nu_1}, \dots, e_{\nu_k}) \cdot \prod_{j=1}^{k} \psi_j^{m_j} \tag{2.2.15}$$

for $m_i \in \mathbb{N}$, $\{e_\nu, {}_{\nu=1,\dots,N}\} \subset H$.

There is a group action on CohFTs with a fixed target space (V, η). The group is the group of matrices $R(z) \in \mathrm{End}(V) \otimes \mathbb{C}[[z]]$ such that $R = \mathbf{I} + O(z)$ and $R(z)R^*(-z) = \mathbf{I}$. The action is defined as follows. The classes $\{\alpha'_{g,k}\} = R.\{\alpha_{g,k}\}$ are defined as the sums over so-called stable graphs.

A stable graph is a graph with a set of vertices V, a set of edges E, and a set of unbounded edges (leaves) $L \sqcup D$. The vertices are labeled by non-negative integers, that is, we have a map $V \to \mathbb{Z}_{\geq 0}$, $v \mapsto g(v)$. The stability condition means that for each vertex v of valency $k(v)$ we require $2g(v) - 2 + k(v) > 0$. We say that the stable graph Γ has genus g and k leaves if $b_1(\Gamma) + \sum_{v \in V} g(v) = g$ and $|L| = k$. So, we allow an arbitrary number of unbounded leaves in D (these leaves are called dilaton leaves), that is, the set of stable graphs of genus g with k leaves is infinite. The leaves in L are labeled from 1 to k.

A stable graph Γ gives us a map f_Γ from the Cartesian product of the spaces $\overline{\mathcal{M}}_{g(v),k(v)}$, $v \in V$, to $\overline{\mathcal{M}}_{g,k}$. Namely, we associate to each vertex v a curve of genus $g(v)$, and to all attached half-edges we associate the marked points on the curve. Then we first apply the maps π on each space $\overline{\mathcal{M}}_{g(v),k(v)}$, $v \in V$, in order to forget all marked points corresponding to the dilaton leaves, and then we apply a sequence of maps σ and ρ, indexed by the edges E of the graph, such that each edge determines the sewing of the corresponding curves.

We associate to a stable graph Γ a map from $V^{\otimes k}$ to $\otimes_{v \in V} H^*(\overline{\mathcal{M}}_{g(v),k(v)}, \mathbb{C})$. That is, a map from $e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_k}$ to the following class. We decorate by $R^{-1}(\psi) e_{\alpha_i}$ the leaf labeled by i. We decorate each dilaton leaf by $-\psi(\mathbf{I} - R^{-1}(\psi)) e_1$. We decorate each edge by

$$\left(\frac{\mathbf{I} \otimes \mathbf{I} - R^{-1}(\psi') \otimes R^{-1}(\psi'')}{\psi' + \psi''} \right) \eta^{\alpha\beta} e_\alpha \otimes e_\beta, \tag{2.2.16}$$

where by ψ' and ψ'' we denote the ψ-classes associated with the marked points that correspond to the ends of the edge. Each vertex v is decorated by $\alpha_{g(v),k(v)}$ considered as an element of $(V^*)^{\otimes k(v)} \otimes H^*(\overline{\mathcal{M}}_{g(v),k(v)}, \mathbb{C})$. We contract the tensor product of the vectors corresponding to edges and leaves with the tensor product of covectors corresponding to the vertices according to the graph. This gives us a class α_Γ in $\otimes_{v \in V} H^*(\overline{\mathcal{M}}_{g(v),k(v)}, \mathbb{C})$.

By definition, the class $\alpha'_{g,k}$ is given by $\sum_\Gamma (f_\Gamma)_* \alpha_\Gamma$, where the sum is taken over all stable graphs of genus g with k leaves. Though there is an infinite number of graphs like that, one can check that only a finite number of them can contribute to this sum for dimensional reasons. It is indeed a group action on CohFTs, see e. g. [82].

There is a canonical way to associate a CohFT to a semi-simple point of a Frobenius manifold. Namely, we associate to a point $b \in B^{ss}$ of a Frobenius manifold the topological field theory $\{\alpha_{g,k}\}$ with values in $(T_b B, \eta|_b)$. The equation for the flat sections of the connection $\tilde{\nabla}$ has essential singularity at $z = \infty$. The asymptotic fundamental solution near $z = \infty$ can be represented in a neighborhood of b as $\Psi^{-1} R(z^{-1}) e^{zU}$, where all involved matrices are functions on B^{ss}, and the matrix R satisfies all properties required in the definition of the group action. We can construct a CohFT applying the group element $R(z)|_b$ to the topological field theory on $(T_b B, \eta|_b)$.

2.2.4 Dubrovin's superpotential

In this Section we recall a construction of a particular Landau-Ginzburg superpotential due to Dubrovin [28].

Given a manifold M equipped with a flat metric, a locally defined function t is a *flat coordinate* at $p \in M$, if

(i) $dt(p) \neq 0$ and

(ii) dt is covariantly constant with respect to to the Levi-Civita connection.

Condition (i) guarantees that t is a local coordinate, i.e. we can find a coordinate system $(t^1, ..., t^n)$ with $t^1 = t$ and an open neighbourhood $B \subset M$ of p such that $(t^1, ..., t^n) : B \to B_0 \subset \mathbb{R}^n$ is a homeomorphism onto an open set B_0 of $\mathbb{R}^n$. Condition (ii), which uses the induced connection on the cotangent bundle, guarantees that $(t^1, ..., t^n)$ can be chosen so that the metric is represented by a constant matrix with respect to $(t^1, ..., t^n)$.

We now consider a flat coordinate $\rho(\lambda, u)$ with respect to the pencil of metrics $\eta' - \lambda\eta$. We study covariant constancy of $d\rho$ via its gradient vector field $\phi(\lambda, u) = \nabla\rho(\lambda, u)$ defined by

$$(\eta' - \lambda\eta)(\phi, \cdot) = d\rho.$$

The Levi-Civita connection of η' with respect to flat coordinates (for η) is given in [28, Equation (5.5)]). This leads to the following system of equations for vector fields ϕ expressed in canonical coordinates on a Frobenius manifold (the extended Gauss-Manin system [28, Equations (5.31) and (5.32)]):

$$d\phi = -(U - \lambda)^{-1} d(U - \lambda) \left(\frac{1}{2} + V \right) \phi + d\Psi \cdot \Psi^{-1} \phi. \tag{2.2.17}$$

Here $d = d_\lambda + d_u$ is the total de Rham differential; $U = \mathrm{diag}(u_1, ..., u_n)$ and V and Ψ are naturally associated to a Frobenius manifold as defined in Section 2.2.1. Abusing notation, we use λ for the matrix of multiplication by λ. So (2.2.17) encodes the system of PDEs giving covariant constancy of $\phi(\lambda, u) = \nabla\rho(\lambda, u)$ in directions $\partial/\partial\lambda, \partial/\partial u_i$.

One can retrieve ρ from its gradient vector field via

$$\rho(\lambda, u) = \frac{\sqrt{2}}{1 - d} \phi^T (U - \lambda) \Psi \mathbb{1}. \tag{2.2.18}$$

This is proved in [29, Section 2].

This equation has poles at $\lambda = u_1, \ldots, u_n$ on the λ-plane, so we choose parallel cuts $L_1, \ldots, L_n$ from the points u_i to infinity (we assume that $u_j \notin L_i$ for $i \neq j$). On $\mathbb{C} \setminus \cup_{i=1}^n L_i$ we choose branches of functions $\sqrt{u_i - \lambda}$, $i = 1, \ldots, n$. We denote by $\mathcal{R}_i$ the monodromy of the space of solutions of Equation (2.2.17) corresponding to following a small loop around u_1.

Dubrovin proves that there exists a unique system of solutions $\phi^{(1)}, \ldots, \phi^{(n)}$ to equation (2.2.17) satisfying the following properties:

$$\begin{aligned}
&\mathcal{R}_j\phi^{(j)} = -\phi^{(j)}, && j=1,\dots,n; && (2.2.19)\\
&\phi^{(j)}_j = \frac{1}{\sqrt{u_j-\lambda}} + O(\sqrt{u_j-\lambda}) \text{ for } \lambda\to u_j, && j=1,\dots,n; && (2.2.20)\\
&\phi^{(j)}_a = \sqrt{u_j-\lambda}\cdot O(1) \text{ for } \lambda\to u_j, && a\neq j;\ a,j=1,\dots,n; && (2.2.21)\\
&\mathcal{R}_j\phi^{(i)} = \phi^{(i)} - 2G^{ij}\phi^{(j)}, && i,j=1,\dots,n; && (2.2.22)
\end{aligned}$$

where $G^{ij} := (\phi^{(i)})^T(U-\lambda)\phi^{(j)}$ is a bilinear form that doesn't depend on λ and $u_1,\dots,u_n$.

Assume that G^{ij} is non-degenerate and denote by G_{ij} the inverse matrix. Note that non-degeneracy of G^{ij} is a property of the Frobenius manifold M which holds generically. In fact the proof of Theorem 2.3.1 does not require the non-degeneracy of G^{ij}—see Remark 2.3.2. Consider a special solution of Equation (2.2.17) given by $\phi := \sum_{i,j=1}^n G_{ij}\phi^{(j)}$. The main property of this solution is that ϕ has the local behavior

$$\begin{aligned}
&\phi_j = \frac{1}{\sqrt{u_j-\lambda}} + O(1) \text{ for } \lambda\to u_j, && j=1,\dots,n; && (2.2.23)\\
&\phi_a = \sqrt{u_j-\lambda}\cdot O(1) \text{ for } \lambda\to u_j, && a\neq j; a,j=1,\dots,n. && (2.2.24)
\end{aligned}$$

We consider the function $p=p(\lambda,u)$ given by the formula

$$p(\lambda,u) := \frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\mathbb{1}. \tag{2.2.25}$$

This function is analytic in $\mathbb{C}\setminus\cup_{i=1}^n L_i$, with a regular singularity at infinity, and its local behavior for $\lambda\to u_i$ is given by

$$p(\lambda,u) = p(u_i,u) + \Psi_{i,\mathbb{1}}\sqrt{2(u_i-\lambda)} + O(u_i-\lambda), \qquad i=1,\dots,n. \tag{2.2.26}$$

The 1-form $d_\lambda p$ has at most a finite number of zeros. We denote them by $r_1,\dots,r_N$ and we assume that they do not belong to the cuts L_i, $i=1,\dots,n$. Let D be the image of $\mathbb{C}\setminus\cup_{i=1}^n L_i$ under the map $p(\lambda,u)$. This domain has a boundary given by the unfolding of the cuts L_i, $i=1,\dots,n$. The inverse function $\lambda=\lambda(p,u)$ is a multivalued function on D. Consider the points $p(r_c,u)$, $c=1,\dots,N$. We glue a finite number of copies of D along the cuts from the points $p(r_c,u)$ to infinity, $c=1,\dots,N$. In this way we obtain a domain $\hat{D}$, where the function λ is single-valued.

We analytically continue the function λ on $\hat{D}$ beyond the boundary. This procedure is not unique; for instance, we can glue several copies of $\hat{D}$ along the boundaries that are the images of the same cuts on the λ-plane. In any case, we can perform this construction uniformly over a small ball in the space of parameters $u_1,\dots,u_n$. This way we obtain a (not necessarily compact) Riemann surface $\mathcal{D}$, with a function $\lambda=\lambda(\tilde{p},u)\colon \mathcal{D}\to\mathbb{C}$ (by $\tilde{p}$ we denote some local coordinate on $\mathcal{D}$).

Dubrovin proves in [28] that the family of functions $\lambda(\tilde{p},u)$ defined this way is a superpotential of the Frobenius manifold which was the input of this construction.

2.2.5 Spectral curve topological recursion

In this Section, we recall the basic set-up of the topological recursion procedure, which originated in the computation of the correlation functions of matrix models [50, 40].

Consider a Riemann surface Σ with meromorphic functions $x, y\colon \Sigma \to \mathbb{C}$ such that x has a finite number of critical points, $c_1, \dots, c_n$, and y is holomorphic near these points with a non-vanishing derivative. Let B be a symmetric bi-differential on $\Sigma \times \Sigma$, with a double pole on the diagonal, the double residue equal to 1, and no further singularities.

We define a sequence of symmetric n-forms $\omega_{g,k}(z_1, \dots, z_k)$ on $\Sigma^{\times k}$, known as *correlation differentials* for the spectral curve, by the following recursion:

$$\omega_{0,1}(z) := y(z)dx(z); \tag{2.2.27}$$

$$\omega_{0,2}(z_1, z_2) := B(z_1, z_2); \tag{2.2.28}$$

$$\omega_{g,k+1}(z_0, z_1, \dots, z_k) := \sum_{i=1}^{n} \operatorname*{Res}_{z\to c_i} \frac{\int_z^{\sigma_i(z)} \omega_{0,2}(\bullet, z_0)}{2(\omega_{0,1}(\sigma_i(z)) - \omega_{0,1}(z))} \tilde{\omega}_{g,2|k}(z, \sigma_i(z)|z_1, \dots, z_k), \tag{2.2.29}$$

where σ_i is the deck transformation for the function x near the point c_i, $i = 1, \dots, n$, and $\tilde{\omega}_{g,2|k}$ is defined by the following formula:

$$\tilde{\omega}_{g,2|k}(z', z''|z_1, \dots, z_k) := \omega_{g-1,n+2}(z', z'', z_1, \dots, z_k) + \sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{1,\dots,k\} \\ 2g_1-1+|I_1|\geq 0 \\ 2g_2-1+|I_2|\geq 0}} \omega_{g_1,|I_1|+1}(z', z_{I_1}) \omega_{g_2,|I_2|+1}(z'', z_{I_2}). \tag{2.2.30}$$

Here we denote by z_I the sequence $z_{i_1}, \dots, z_{i_{|I|}}$ for $I = \{i_1, \dots, i_{|I|}\}$.

Remark 2.2.2. In the global recursion we also allow y to be the (multivalued) primitive of a differential ω on Σ. The ambiguity in y consists of periods and residues of ω and hence the ambiguity is locally constant. Since y appears in the recursion only via $y(\sigma_i(z)) - y(z)$ (and there are no poles of ω at the zeros of dx) the locally constant ambiguity disappears and the recursion is well-defined.

Remark 2.2.3. A local version of the recursion was defined in [46] as follows. Consider some small neighborhoods $U_i \subset \Sigma$ of the points c_i. If we look at just the restrictions of $\omega_{g,k}$ to the products of these disks, $U_{i_1} \times \dots \times U_{i_k}$, we can still proceed by topological recursion, using as an input the restrictions of $\omega_{0,1}$ to U_i, $i = 1, \dots, n$, and $\omega_{0,2}$ to $U_i \times U_j$, $i, j = 1, \dots, n$. Indeed, Equation (2.2.29) uses only local data for the recursion.

Remark 2.2.4. There is a variation of the usual (global) topological recursion that will also be important in this chapter, especially in Section 2.10. Namely, we can assume that there is more than one critical point in the fiber of the function x over a critical value u_i. Then we require that the local behavior of the function x near these points is the same (that is, the Hessians are the same), and in this case it is still possible to define a version of topological recursion, see Section 2.10. Note that this more general critical behavior of the function x is exactly the one that is allowed for the function λ in the definition of the Landau-Ginzburg superpotential of a Frobenius manifold in Section 2.2.1.

2.2.6 Spectral curve topological recursion via CohFTs

In this Section we recall a relation of the (local version of) spectral curve topological recursion to the Givental formulae for cohomological field theories obtained in [37]. A more convenient exposition is given in [70], so we follow the presentation given there.

We choose the local coordinates w_i in the domains U_i such that $x|_{U_i} = -w_i^2/2+x(c_i)$, $i=1,\dots,n$. The identification with the data of a CohFT then goes as follows:

$$\Delta_i^{-\frac{1}{2}} = \frac{dy}{dw_i}(0); \tag{2.2.31}$$

$$R^{-1}(\zeta^{-1})_i^j = -\frac{1}{\sqrt{2\pi\zeta}}\int_{-\infty}^{\infty} \left.\frac{B(w_i,w_j)}{dw_i}\right|_{w_i=0} \cdot e^{(x(w_j)-x(c_j))\zeta}; \tag{2.2.32}$$

$$\sum_{k=1}^{n}(R^{-1}(\zeta^{-1}))_k^i \Delta_k^{-\frac{1}{2}} = \frac{\sqrt{\zeta}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} dy(w_i)\cdot e^{(x(w_i)-x(c_i))\zeta}. \tag{2.2.33}$$

Note that Equation (2.2.31) is in fact a consequence of Equation (2.2.33).

There is an extra condition on the bi-differential B that can be formulated as a requirement on decomposition of its Laplace transform as

$$\begin{aligned}&\frac{\sqrt{\zeta_1\zeta_2}}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} B(w_i,w_j)e^{(x(w_i)-x(c_i))\zeta_1+(x(w_j)-x(c_j))\zeta_2} \\ &= \frac{\sum_{k=1}^{n} R^{-1}(\zeta_1^{-1})_k^i R^{-1}(\zeta_2^{-1})_k^j}{\zeta_1^{-1}+\zeta_2^{-1}}.\end{aligned} \tag{2.2.34}$$

This assumption is always satisfied if the curve is compact and the differential dx is meromorphic. This uses a general finite decomposition for $B(p,q)$ proven by Eynard in Appendix B of [46] together with (2.2.32).

This data (the constants $\Delta_i^{-\frac{1}{2}}$ and the matrix $R^{-1}(\zeta^{-1})_i^j$) determine for us a semi-simple CohFT $\{\alpha_{g,k}\}$ with an n-dimensional space of primary fields $V := \langle e_1,\dots,e_n\rangle$. The differentials $\omega_{g,k}$ can be written in terms of the auxiliary functions

$$\xi^i(z) := \int^z \left.\frac{B(w_i,\bullet)}{dw_i}\right|_{w_i=0} \tag{2.2.35}$$

as

$$\omega_{g,k} = \sum_{\substack{i_1,\dots,i_k \\ d_1,\dots,d_k}} \int_{\overline{\mathcal{M}}_{g,k}} \alpha_{g,k}(e_{i_1},\dots,e_{i_k})\prod_{j=1}^{k}\psi_j^{d_j} d\left(\left(\frac{d}{dx}\right)^{d_j}\xi^{i_j}\right). \tag{2.2.36}$$

(These kind of formulas are typically of ELSV-type, see [34] for explanation.) In terms of the underlying Frobenius manifold structure, the basis $e_1,\dots,e_n$ corresponds to the normalized canonical basis.

2.3 Superpotential and function y

The goal of this Section is to prove that Dubrovin's superpotential provides us with a Riemann surface with two functions, $x := \lambda$ and $y := p$, such that the local expansion of y near the critical points of x reproduces the unit vector at the point $(u_1, \dots, u_n)$ of the underlying Frobenius manifold as well as the value of the matrix R^{-1} on the unit vector. These two local properties of y are precisely equivalent to the equations (2.2.31) and (2.2.33).

Consider Dubrovin's construction of a superpotential on the Riemann surface $\mathcal{D}$ described in Section 2.2.4. It is associated to a Frobenius manifold with given constants $\Delta_i^{-\frac{1}{2}}$ and the matrix $R^{-1}(\zeta^{-1})_i^j$ at the point with canonical coordinates $u_1, \dots, u_n$. Consider the points $c_i = p(u_i, u) \in \mathcal{D}$. These points are the critical points of the function $x := \lambda$.

Theorem 2.3.1. *Given a semi-simple Frobenius manifold M, and Dubrovin's construction of a superpotential $\mathcal{D}$ for M, define spectral curve data by $\Sigma = \mathcal{D}$, $x := \lambda$, $y := p$ (with B yet to be defined). Then equations* (2.2.31) *and* (2.2.33) *are satisfied for the constants $\Delta_i^{-\frac{1}{2}}$ and the matrix $R^{-1}(\zeta^{-1})_i^j$ associated to M.*

Proof. Let us prove the first statement, namely, Equation (2.2.31) (though it is a corollary of Equation (2.2.33), it is convenient to check it directly). Indeed, Equation (2.2.26) states that near the points c_i the function p looks like

$$p = c_i + \Psi_{i,1\!\!1}(u)\sqrt{2(u_j - \lambda)} + O(u_j - \lambda).$$

Therefore, the derivative of p with respect to the local coordinate $w_i = \sqrt{2(u_i - \lambda)}$ at the point c_i is equal to $\Psi_{i,1\!\!1}(u) = \Delta_i^{-\frac{1}{2}}$.

Now we prove Equation (2.2.33). We can assume that the contour of integration on the right hand side in Equation (2.2.33) is the image of L_i under the map p. Then,

$$\frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{p(L_i)} dp \cdot e^{(\lambda - u_i)\zeta} = \frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{p(L_i)} \frac{dp}{d\lambda} \cdot e^{(\lambda - u_i)\zeta} d\lambda. \tag{2.3.37}$$

Here we treat dp and $d\lambda$ as 1-forms defined on the surface $\mathcal{D}$.

Observe that from equation (2.2.17) we have

$$\frac{d\phi^T}{d\lambda} = \phi^T \left(\frac{1}{2} - V\right)(U - \lambda)^{-1}. \tag{2.3.38}$$

Therefore, using definition (2.2.25), we get

$$\begin{aligned} \frac{dp}{d\lambda} &= \frac{d}{d\lambda}\frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi 1\!\!1 = \frac{\sqrt{2}}{1-d}\phi^T\left(\frac{1}{2} - V\right)\Psi 1\!\!1 - \frac{\sqrt{2}}{1-d}\phi^T\Psi 1\!\!1 \qquad (2.3.39)\\ &= \frac{\sqrt{2}}{1-d}\phi^T\Psi\Psi^{-1}\left(-\frac{1}{2} - V\right)\Psi 1\!\!1 = \frac{\sqrt{2}}{1-d}\phi^T\Psi\left(-\frac{1}{2} - \mu\right)1\!\!1 \\ &= \frac{\sqrt{2}}{1-d}\phi^T\Psi\left(-\frac{1}{2} + \frac{d}{2}\right)1\!\!1 = -\frac{1}{\sqrt{2}}\phi^T\Psi 1\!\!1. \end{aligned}$$

(In this computation we used the fact that $\mu 1\!\!1 = (-d/2)1\!\!1$.)

Equations (2.2.23) and (2.2.24) imply that on the contour $p(L_i)$ the vector ϕ is equal to $\phi^{(i)} + E_i$, where E_i is some holomorphic function of $(u_i - \lambda)$. Recall also that $(\Psi \mathbb{1})_k = \Delta_k^{-\frac{1}{2}}$. Therefore,

$$\frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{p(L_i)} \frac{dp}{d\lambda} \cdot e^{(\lambda - u_i)\zeta} d\lambda = \sum_{k=1}^{n} \Delta_k^{-\frac{1}{2}} \cdot \frac{-\sqrt{\zeta}}{2\sqrt{\pi}} \int_{p(L_i)} \phi_k^{(i)} \cdot e^{(\lambda - u_i)\zeta} d\lambda. \tag{2.3.40}$$

Dubrovin shows in [28, Proof of Lemma 5.4] that the second factor in this expression is $(R^{-1}(\zeta^{-1}))^i_k$. Thus the right hand side of Equation (2.3.40) coincides with the left hand side of Equation (2.2.33). This completes the proof of the Theorem. □

Remark 2.3.2. Note that we have not used the specific formula for ϕ in the proof. We used only Equation (2.2.17) and the fact that the local expansion of ϕ for $\lambda \to u_i$ coincides with the local expansion of $\phi^{(i)}$ up to some holomorphic non-branching term. Thus, if we have a solution for (2.2.17) satisfying this property, we can use it directly in the formula for the superpotential (2.2.25), bypassing the requirement for G^{ij} to be non-degenerate. This will be important below in certain applications.

Remark 2.3.3. *Flat identity.* Topological recursion satisfies the string equation.

$$\sum_{i=1}^{n} \operatorname*{Res}_{p=c_i} y(p)\omega_{g,k+1}(p, p_1, ..., p_k) = -\sum_{j=1}^{k} d_{p_j} \partial z_j \left(\frac{\omega_{g,k}(p_1, ..., p_k)}{dx(p_j)} \right) \tag{2.3.41}$$

where the sum is over the zeros $dx(c_i) = 0$ and d_{p_j} is exterior derivative in the variables p_j. The operator $\omega \mapsto \sum_i \operatorname{Res}_{p=c_i} y(p)\omega(p)$ acts on differentials ω. It is non-zero (and evaluates to 1) on the auxiliary differential $\sum_j a_j d\xi^j$ corresponding to the flat identity and annihilates all others. In particular

$$\sum_i \operatorname*{Res}_{p=c_i} d \left(\left(\frac{d}{dx} \right)^{d_j} \xi^{i_j} \right) = 0, \quad d_j > 0.$$

This corresponds to insertion/removal of the identity vector in ancestor invariants.

2.4 Compatibility between B and y

In this section we discuss a necessary condition on a spectral curve to be able to apply the inverse construction of [37], i.e. so that a CohFT can be reconstructed from this spectral curve.

More precisely, for a given data of a spectral curve (Σ, x, y, B) (maybe, local) Equations (2.2.32) and (2.2.33), (2.2.31) imply some relation for x, y, and B, and we want to state this relation in a direct geometric way rather than in terms of the Laplace transform.

The compatibility condition below is equivalent to differentiation of the potential of a CohFT by t_1 producing the string equation. In the language of [50], $\delta(ydx) = \int (dy/dx)(p')\, B(p, p') = d(dy/dx)$ gives rise to variations of $\omega_{g,k}$ corresponding to the string equation (2.3.41).

Recall that x defines a local involution σ_i near each zero c_i of dx, $i = 1, \ldots, n$.

Theorem 2.4.1. *If a CohFT can be reconstructed from a spectral curve* (Σ, x, y, B) *via the inverse construction of [37] described in Section 2.2.6, then the 1-form on* Σ

$$\eta(z) = d\left(\frac{dy}{dx}(z)\right) + \sum_{i=1}^{n} \operatorname*{Res}_{z'=c_i} \frac{dy}{dx}(z')B(z,z'). \tag{2.4.42}$$

is invariant under each local involution σ_i, $i = 1, \dots, n$.

Proof. The construction of [37] requires equations (2.2.31), (2.2.32), and (2.2.33) to hold. We will prove that the 1-form (2.4.42) is invariant under each local involution σ_i, $i = 1, \dots, n$ if and only if equations (2.2.31), (2.2.32), and (2.2.33) are compatible (as equations for the unknown variables R^{-1} and $\Delta_i^{-\frac{1}{2}}$, $i = 1, \dots, n$).

Recall that $x = x(c_i) - w_i^2/2$ in a neighborhood of c_i. Note that

$$\begin{aligned} &\operatorname*{Res}_{w_i=c_i} \frac{dy}{dx}(w_i)B(z,w_i) = \operatorname*{Res}_{w_i=c_i} \frac{dy}{dw_i}(w_i) \cdot \frac{dw_i}{dx} \cdot B(z,w_i) \\ &= -\operatorname*{Res}_{w_i=c_i} \frac{dy}{dw_i}(w_i) \cdot \frac{dw_i}{w_i} \cdot \frac{B(z,w_i)}{dw_i} = \frac{dy}{dw_i}(0) \cdot \left.\frac{B(z,w_i)}{dw_i}\right|_{w_i=0}. \end{aligned} \tag{2.4.43}$$

An equivalent way to say that η is σ_i-invariant is to say that the following Laplace transform of η is equal to zero:

$$\int_{-\infty}^{\infty} \eta(w_i) e^{(x(w_i)-x(c_i))\zeta} = 0. \tag{2.4.44}$$

On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} \eta(w_i) e^{(x(w_i)-x(c_i))\zeta} = &-\zeta \int_{-\infty}^{\infty} \frac{dy}{dx}(w_i) e^{(x(w_i)-x(c_i))\zeta} dx \\ &- \sum_{j=1}^{n} \frac{dy}{dw_j}(0) \int_{-\infty}^{\infty} \left.\frac{B(w_i,w_j)}{dw_j}\right|_{w_j=0} e^{(x(w_i)-x(c_i))\zeta}. \end{aligned} \tag{2.4.45}$$

Thus, Equation (2.4.44) is satisfied if and only if

$$\begin{aligned} &\frac{\sqrt{\zeta}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy(w_i) e^{(x(w_i)-x(c_i))\zeta} \\ &= \sum_{j=1}^{n} \frac{dy}{dw_j}(0) \cdot \frac{-1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \left.\frac{B(w_i,w_j)}{dw_j}\right|_{w_j=0} e^{(x(w_i)-x(c_i))\zeta}, \end{aligned} \tag{2.4.46}$$

which is precisely the compatibility condition for Equations (2.2.31), (2.2.32), and (2.2.33). ☐

We can state (2.4.42) in simpler terms when the spectral curve is connected.

Corollary 2.4.2. *For a connected spectral curve, equations* (2.2.31), (2.2.32), *and* (2.2.33) *are compatible if and only if the 1-form defined in* (2.4.42) *is a pull-back of a 1-form downstairs, i.e.* $\eta(z) = x^*\omega$.

Proof. If $\eta(z) = x^*\omega$ for ω a differential downstairs then it is invariant under local involutions hence Theorem 2.4.1 applies. On a comnnected spectral curve Σ the converse is also true. This follows from the more general fact that any $\eta(z)$ which is invariant under local involutions defined around simple ramification points of $x : \Sigma \to \mathbb{C}$ is the pull-back of a differential downstairs. Take any regular point of x $p \in \Sigma$ and a path γ from p to a zero b of dx. Then $x(\gamma)$ is covered by a path $\tilde{\gamma} \subset \Sigma$ that contains p and p' where $x(p) = x(p')$. The local involution defined by x in a neighbourhood of b can be analytically continued along $\tilde{\gamma}$. Since $\eta(z)$ is invariant under the local involution at b, it is invariant under the continued involution above a neighbourhood of $x(\gamma)$. So $\eta(z)$ agrees (via identification of cotangent bundles using x) around p and p'. Connectedness of Σ guarantees that the monodromy of the cover defined by x is transitive and generated by local involutions. Hence we can find paths γ_i that can be used to show that $\eta(z)$ agrees around p and any point in the fibre over $x(p)$. Hence $\eta(z) = x^*\omega$ locally and this pieces together to give the global result. The result isn't true on disconnected curves, in particular local curves, because monodromy is not transitive. □

Let us show how this compatibility test can be used.

Proposition 2.4.3. *The differential $\eta \equiv 0$, hence Equation (2.4.44) is satisfied, when Σ is a global curve equipped with a canonical bidifferential B normalized so that $\int_{p'\in\alpha_i} B(p,p') = 0$ for a choice of A-cycles α_i, and one of the following holds:*

1. *Σ is rational with global coordinate z chosen so that $x(z = \infty) = \infty$;*
2. *dy is a meromorphic differential such that $\frac{dy}{dx}$ has poles only at the zeros of dx, for example dy is a holomorphic differential.*

Note that in case (2) above, we take y to be the (multiply-defined) primitive of a differential which is sufficient for the purposes of topological recursion—see Remark (2.2.2).

Proof. Recall the property that for any function f on Σ, $\text{Res}_{p'=p} f(p')B(p,p') = df(p)$ (independent of the choice of A-cycles along which B is normalized). For example, in the rational case $B = \frac{dzdz'}{(z-z')^2}$ and this property is the Cauchy integral formula. Since $\frac{dy}{dx}$ has poles only at the zeros of dx

$$\sum_{i=1}^{n} \operatorname*{Res}_{p'=c_i} \frac{dy}{dx}(p')B(p,p') = -\operatorname*{Res}_{p'=p} \frac{dy}{dx}(p')B(p,p') = -d\left(\frac{dy}{dx}(p)\right)$$

hence $\eta \equiv 0$. □

Example 2.4.4. *Consider $x = z + 1/z$, $y = p(z)$ a polynomial. Then $\frac{dy}{dx} = \frac{z^2 p'(z)}{z^2-1}$ has poles at $z = \pm 1$ and possibly $z = \infty$. Hence $\eta(z) = dq(z)$ where $q(z)$ is a polynomial given by the principal part of dy/dx at $z = \infty$. A non-trivial polynomial has poles only at $z = \infty$ so if $\eta \neq 0$ it cannot be the pull-back of a differential form downstairs since it would necessarily require poles at $x^{-1}(\infty) = \{0, \infty\}$. Hence this fails the compatibility test, unless $\eta(z) \equiv 0$ i.e. $\deg p(z) \leq 1$. If $\deg p(z) = 1$ then Equation (2.4.44) is satisfied.*

Example 2.4.5. *Consider $x = z + 1/z$, $y = \ln z$. Then $\frac{dy}{dx} = \frac{z}{z^2-1}$ has poles only at $z = \pm 1$ so Equation* (2.4.44) *is satisfied.*

Example 2.4.6. *Since the compatibility test is a linear condition in y, $x = z + 1/z$, $y = \ln z + cz$ also satisfies the compatibility test and leads to a CohFT with a flat unit. This was also observed in [54].*

2.5 Superpotential as a global spectral curve in genus 0 case

In this Section we discuss a special case of Dubrovin's superpotential defined in Section 2.2.4 and show that it indeed gives a proper spectral curve for the correponding cohomological field theory.

More precisely, we start with a homogeneous cohomological field theory. Its genus zero part without descendants defines a Frobenius manifold that we assume to be semi-simple. Consider Dubrovin's construction in Section 2.2.4. Assume that this construction goes through in such a way that

1. The form $d_\lambda p$ has no zeros in $\mathbb{C} \setminus \cup_{i=1}^n L_i$;
2. $\lambda(p = \infty) = \infty$;
3. The resulting curve $\mathcal{D}$ is a compact curve of genus 0 and p is a global coordinate on it;
4. There is exactly one critical point in each singular fiber of function λ.

Theorem 2.5.1. *Under the conditions* (1)-(4) *above, the correlators of the CohFT are related by Equation* (2.2.36) *to the correlation differentials obtained through spectral curve topological recursion on a curve $\mathcal{D}$ with $x = \lambda$, $y = p$ and $B(p_1, p_2) = dp_1 dp_2/(p_1 - p_2)^2$.*

In other words, in this case the ancestor potential of CohFT is reproduced by global topological recursion related to Dubrovin's superpotential. Note that this identification happens over an open ball in the underlying Frobenius manifold.

Proof. First of all, note that since p is a global coordinate and $\lambda(p = \infty) = \infty$, this spectral curve satisfies the compatibility condition of Theorem 2.4.1, which means that one can reconstruct a CohFT such that Equation (2.2.36) is satisfied. We only need to prove that this CohFT is the same as the original one.

Theorem 2.3.1 implies that we have the right function y, so, in particular, the functions $\Delta_i^{-\frac{1}{2}}(u)$ are correctly reproduced on an open ball in the space of parameters $u_1, \ldots, u_n$. Note that these functions determine completely the structure of Frobenius multiplication, so we can conclude that the CohFT reconstructed from the spectral curve data coincides with the original one in genus zero.

Higher genera correlators of a semi-simple CohFT are determined uniquely by genus 0 data in homogeneous cases [91]. Therefore, it is sufficient to prove that the CohFT reconstructed from the spectral curve data is homogeneous. We do this by proving the Euler equation for the corresponding R-matrix. Namely, a CohFT with an R-matrix $R(\xi)$ is homogeneous if and only if the R-matrix satisfies the Euler equation [56]:

$$\left(\xi\frac{d}{d\xi}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)R(\xi,u)=0 \tag{2.5.47}$$

(or, equivalently, we can consider the same equation for $R^{-1}(\xi,u)=R(-\xi,u)^T$). Using Equation (2.2.34), the Euler equation for the R-matrix can be rewritten as

$$\left(1+\xi_1\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)\check{B}=0 \tag{2.5.48}$$

for $\check{B}=\check{B}^{ij}(\xi_1,\xi_2)$ given by

$$\frac{e^{-\frac{u_i}{\xi_1}-\frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1\xi_2}}\int_{p(L_i)}\int_{p(L_j)}B\cdot e^{\frac{\lambda_1}{\xi_1}+\frac{\lambda_2}{\xi_2}}. \tag{2.5.49}$$

Recall that we consider the case when $d_\lambda p$ does not have zeros in $\mathbb{C}\setminus\cup_{i=1}^n L_i$, and the Riemann surface $\mathcal{D}$ that we get through Dubrovin's construction has genus 0. The Bergman kernel $B(p_1,p_2)$ has the form $dp_1dp_2/(p_1-p_2)^2$.

Proposition 2.5.2. *Under these conditions Equation* (2.5.48) *is satisfied.*

We prove this proposition below. It implies that the R-matrix associated to the Bergman kernel in this case satsifies the Euler equation, and, therefore, the corresponding CohFT is homogeneous. This proposition completes the proof of Theorem 2.5.1. □

For the proof of Proposition 2.5.2 we need the following technical lemma:

Lemma 2.5.3. *We have:*

$$\left(\lambda\frac{d}{d\lambda}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)p(\lambda,u)=\frac{1-d}{2}p(\lambda,u). \tag{2.5.50}$$

Proof. Recall Equation (2.3.39):

$$d_\lambda p(\lambda,u)=-\frac{1}{\sqrt{2}}\phi^T d\lambda\Psi\mathbb{1}. \tag{2.5.51}$$

In the same way we prove that

$$d_u p(\lambda,u)=\frac{1}{\sqrt{2}}\phi^T dU\Psi\mathbb{1} \tag{2.5.52}$$

(this is [28, equation (5.66)]; note that there is a misprint in this equation in [28]). Combining these equations, we get

$$\left(\lambda\frac{d}{d\lambda}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)p(\lambda,u)=\frac{1}{\sqrt{2}}\phi^T(U-\lambda)\Psi\mathbb{1}=\frac{1-d}{2}p(\lambda,u). \tag{2.5.53}$$

□

Proof of Proposition 2.5.2. We have:

$$\begin{aligned}&\left(1+\xi_1\frac{\partial}{\partial\xi_1}+\xi_2\frac{\partial}{\partial\xi_2}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)\check{B} \\ &=\frac{e^{-\frac{u_i}{\xi_1}-\frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1\xi_2}}\iint\frac{d\lambda_1 d\lambda_2}{(p(\lambda_1)-p(\lambda_2))^2}\frac{dp}{d\lambda}(\lambda_1)\frac{dp}{d\lambda}(\lambda_2)\,e^{\frac{\lambda_1}{\xi_1}+\frac{\lambda_2}{\xi_2}}X,\end{aligned}\tag{2.5.54}$$

where

$$\begin{aligned}X=&-\frac{\lambda_1}{\xi_1}-\frac{\lambda_2}{\xi_2}-2\cdot\frac{\left(\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)(p(\lambda_1)-p(\lambda_2))}{p(\lambda_1)-p(\lambda_2)}\\ &+\frac{\left(\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)\frac{dp}{d\lambda}(\lambda_1)}{\frac{dp}{d\lambda}(\lambda_1)}+\frac{\left(\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)\frac{dp}{d\lambda}(\lambda_2)}{\frac{dp}{d\lambda}(\lambda_2)}.\end{aligned}$$

Applying the integration by parts to the terms $-\lambda_1/\xi_1$ and $-\lambda_2/\xi_2$, we can rewrite the right hand side of Equation (2.5.54) as

$$\frac{e^{-\frac{u_i}{\xi_1}-\frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1\xi_2}}\iint\frac{d\lambda_1 d\lambda_2}{(p(\lambda_1)-p(\lambda_2))^2}\frac{dp}{d\lambda}(\lambda_1)\frac{dp}{d\lambda}(\lambda_2)\,e^{\frac{\lambda_1}{\xi_1}+\frac{\lambda_2}{\xi_2}}Y,\tag{2.5.55}$$

where

$$\begin{aligned}Y=2&+\frac{\left(\lambda_1\frac{d}{d\lambda_1}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)\frac{dp}{d\lambda}(\lambda_1)}{\frac{dp}{d\lambda}(\lambda_1)}+\frac{\left(\lambda_2\frac{d}{d\lambda_2}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)\frac{dp}{d\lambda}(\lambda_2)}{\frac{dp}{d\lambda}(\lambda_2)}-\\ &-2\cdot\frac{\left(\lambda_1\frac{d}{d\lambda_1}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)p(\lambda_1)-\left(\lambda_2\frac{d}{d\lambda_2}+\sum_{i=1}^{n}u_i\frac{\partial}{\partial u_i}\right)p(\lambda_2)}{p(\lambda_1)-p(\lambda_2)}.\end{aligned}$$

Using Equation (2.5.50), we rewrite Y as

$$\begin{aligned}Y=2&+\frac{\left(-1+\frac{1-d}{2}\right)\frac{dp}{d\lambda}(\lambda_1)}{\frac{dp}{d\lambda}(\lambda_1)}+\frac{\left(-1+\frac{1-d}{2}\right)\frac{dp}{d\lambda}(\lambda_2)}{\frac{dp}{d\lambda}(\lambda_2)}-\\ &-2\cdot\frac{\frac{1-d}{2}p(\lambda_1)-\frac{1-d}{2}p(\lambda_2)}{p(\lambda_1)-p(\lambda_2)}\\ =2&+\left(-1+\frac{1-d}{2}\right)+\left(-1+\frac{1-d}{2}\right)-2\cdot\frac{1-d}{2}=0,\end{aligned}$$

which proves the proposition. □

2.6 Superpotential as a global spectral curve for arbitrary genus

In this Section we extend the result of the previous section to the case of a compact global curve of arbitrary genus.

Theorem 2.6.1. *Given a conformal Frobenius manifold, construct a superpotential* $p(\lambda; u)$ *which defines the Riemann surface* $\mathcal{D}$ *according to Dubrovin's construction of Section 2.2.4. Assume the following:*

- $\mathcal{D}$ *is a compact curve of genus* g*;*
- *there is exactly one critical point in each singular fiber of* $\lambda : \mathcal{D} \to \mathbb{C}$*.*

Fix a symplectic basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$ *of* $H_1(\mathcal{D}, \mathbb{Z})$ *and define* $B(p_1, p_2)$ *as the only Bergman kernel on* $\mathcal{D}$ *normalized by*

$$\forall i = 1, \ldots, g\,, \oint_{p_1 \in \mathcal{A}_i} B(p_1, p_2) = 0. \tag{2.6.56}$$

Further assume that:

- *the pair* $(p, B(p_1, p_2))$ *passes the compatibility test of Section 2.4 in any of its possible forms (given by Theorem 2.4.1, Corollary 2.4.2, or Proposition 2.4.3).*

Then the correlators of the CohFT associated to the Frobenius manifold are related by Equation (2.2.36) *to the correlator differentials obtained through spectral curve topological recursion on the Riemann surface* $\mathcal{D}$ *with* $x = \lambda$*,* $y = p$ *and* $B(p_1, p_2)$*.*

Remark 2.6.2. This result extends Theorem 2.5.1 to an arbitrary compact curve. The new feature is that one needs to normalize the Bergman kernel on an arbitrary basis of cycles. In particular, for each basis, we recover a total ancestor potential for the same CohFT.

Proof. The proof is very similar to the proof of the genus 0 case presented in the preceding section. However, it is important to remark that this proof only relies on Rauch's variational formula, i.e. it is valid for any compact curve presented as a ramified cover of the Riemann sphere with simple branch points. It does not require any knowledge about an auxiliary meromorphic form such as the super-potential.

Let us first show that the $(0, 3)$ correlators are independent of choice of normalisation cycles for B. $\omega_{0,3}$ depends on these choices, but when decomposed into linear combinations of auxiliary differentials $d\xi^j = B/ds_j$ (for s_j defined by $x = (1/2)s_j^2 + a_j$) the coefficients are independent of A-cycles. By reconstruction, as in the proof of Theorem 2.5.1, this means that all correlators are the same. The formula

$$\begin{aligned}
\omega_{0,3}(z_1, z_2, z_3) &= \sum_{i=1}^{n} \operatorname*{Res}_{p=c_i} B(p, z_1)B(p, z_2)B(p, z_3)/dx(p)dy(p) \\
&= \sum_{i=1}^{n} B(a_i, z_1)B(a_i, z_2)B(a_i, z_3)/x''(a_i)y'(a_i) \\
&= \sum_{i=1}^{n} \langle \ldots . \rangle d\xi^i(z_1)d\xi^i(z_2)d\xi^i(z_3)
\end{aligned}$$

shows the independence of the coefficients $\langle ... \rangle$ on the choice of B.

For the rest of the proof, the only part differing from the genus 0 case is the proof of the homogeneity of the CohFT, i.e. the fact that the R-matrix satisfies the Euler equation.

The first step consists in proving that there exist a R-matrix. This is due to a lemma of Eynard [46]:

Lemma 2.6.3. *If $d\lambda$ is a meromorphic form on $\mathcal{D}$ and B the Bergman kernel normalized on a basis of $\mathcal{A}$-cycles as above, then the Laplace transform of the Bergman kernel satisfies Equation* (2.2.34) .

The Euler equation for the R-matrix is then equivalent to the following equation for the Laplace transform of B:

$$\left(1 + \xi_1 \frac{\partial}{\partial \xi_1} + \xi_2 \frac{\partial}{\partial \xi_2} + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i}\right) \check{B} = 0 \tag{2.6.57}$$

for $\check{B} = \check{B}^{ij}(\xi_1, \xi_2)$ given by

$$\frac{e^{-\frac{u_i}{\xi_1} - \frac{u_j}{\xi_2}}}{2\pi\sqrt{\xi_1 \xi_2}} \int_{p(L_i)} \int_{p(L_j)} B \cdot e^{\frac{\lambda_1}{\xi_1} + \frac{\lambda_2}{\xi_2}}. \tag{2.6.58}$$

By inverting the Laplace transform and integration by part, this is equivalent to

$$d_1 \left(\frac{\lambda_1 \, B(p_1, p_2)}{d\lambda_1}\right) + d_2 \left(\frac{\lambda_2 \, B(p_1, p_2)}{d\lambda_2}\right) + \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} B(p_1, p_2). \tag{2.6.59}$$

In order to prove this equation, we remind Rauch's variational formula which expresses the variations of the Bergman kernel under deformation of the spectral curve. In particular

$$\frac{\partial B(p_1, p_2)}{\partial u_i} = \operatorname*{Res}_{r \to a_i} \frac{B(p_1, r) \, B(p_2, r)}{d\lambda(r)} \tag{2.6.60}$$

which implies that

$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} B(p_1, p_2) = \sum_{i=1}^{n} \operatorname*{Res}_{r \to a_i} \frac{\lambda(r) B(p_1, r) \, B(p_2, r)}{d\lambda(r)}. \tag{2.6.61}$$

Moving the integration contours around the other poles of the integrands and reminding that the $\mathcal{A}$-periods of $B(p, r)$ are vanishing, this reads

$$\sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} B(p_1, p_2) = - \operatorname*{Res}_{r \to p_1, p_2} \frac{\lambda(r) B(p_1, r) \, B(p_2, r)}{d\lambda(r)} = - \left(\frac{d}{d\lambda_1} + \frac{d}{d\lambda_2}\right) B(p_1, p_2) \tag{2.6.62}$$

proving Equation 2.6.59.

□

2.7 Global curves for A_n singularities

In this Section we apply the results of Sections 2.3, 2.4, and 2.5 in order to construct the spectral curve for the ancestor potential of A_n-singularities, $n = 1, 2, \dots$. The structure of this Frobenius manifold is described in terms of Saito's theory on the space of polynomials

$$f(p,\tau) = p^{n+1} + \tau_1 p^{n-1} + \cdots \tau_n. \tag{2.7.63}$$

We refer to [27, 58] for the detailed description of the structure of this Frobenius manifold. In particular, it is enough to say that $\lambda = f(p,\tau)$ is a superpotential of this Frobenius manifold.

The corresponding CohFT is well-studied. It was a subject of Witten's conjecture [95] proved in [53]. We refer to [82] for an exposition of this CohFT that includes an overview of its constructions; the CohFT whose correlators give the ancestor potential at the point τ of this Frobenius manifold is called there the shifted Witten class of A_n singularity.

Theorem 2.7.1. *The correlation differentials of the global spectral curve data $\Sigma := \mathbb{CP}^1$, $y := p$ (the global coordinate), $x := f(p,\tau)$, $B := dp_1 dp_2/(p_1 - p_2)^2$ are expressed via Equation* (2.2.36) *in terms of the shifted Witten class of A_n singularity.*

Proof. As we have already mentioned, the function $\lambda = f(p,\tau)$ is known to be a superpotential of the corresponding Frobenius manifold. We have to show that this superpotential can be obtained by Dubrovin's construction in Section 2.2.4. Then it is easy to see that all conditions of Theorem 2.5.1 are satisfied, which implies this theorem.

We construct solutions of Equation (2.2.17) in terms of the integrals over the vanishing cycles. Namely, consider the tangent bundle over the space of polynomials parametrized by $\tau \in \mathcal{T}$. It is identified with the space $\mathbb{C}[p]/(d_p f(p,\tau)/dp)$ by the map $v \mapsto (d_\tau f(p,\tau))(v)$ and equipped with a flat metric given by

$$(v_1, v_2) := \operatorname*{Res}_{p=\infty} \frac{d_\tau f(v_1) d_\tau f(v_2)}{d_p f(p,\tau)/dp} dp. \tag{2.7.64}$$

For a cycle $\beta \in H_0(f^{-1}(\lambda), \mathbb{C})$ we denote by $I_\beta(\lambda,\tau)$ the section of the tangent bundle specified by the following formula:

$$(I_\beta(\lambda,\tau), v) := \int_\beta d_\tau f(v) \cdot \frac{dp}{d_p f}. \tag{2.7.65}$$

In normalized canonical coordinates $I_\beta(\lambda,\tau)$ is represented by the vector $\phi^\beta(\lambda,\tau)$ with components given by

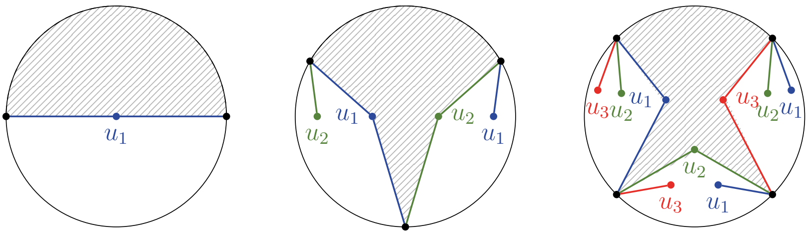
$$\phi_i^\beta(\lambda,\tau) := \left(I_\beta(\lambda,\tau), \frac{\Delta_i^{\frac{1}{2}}}{\sqrt{2}} \frac{\partial}{\partial u_i} \right) \tag{2.7.66}$$

is a solution of Equation (2.2.17) (see [58]). Let us discuss the singularities of this solution, depending on β.

Consider the λ-plane as the image of the map $\lambda = f(p,\tau)$. Let $u_1, \dots, u_n$ be the critical values of $f(p,\tau)$. We can always choose a system of cuts L_i, $i = 1, \dots, n$, from u_i to infinity such that the preimage $f^{-1}(\mathbb{C} \setminus \cup_{i=1}^n L_i)$ is a union of $n+1$ disks, $D_0, D_1, \dots, D_n$, glued along the boundary cuts in the following way:

- D_0 is glued to D_i along the boundary that is a double cover of L_i; in particular, their common boundary contains the critical preimage of u_i;
- All other lifts of the cut L_i are just cuts inside D_j, $j \neq 0, i$; the endpoints of these cuts are non-critical preimages of u_i.

For $n = 1, 2, 3$ we give the corresponding pictures for a real orientable blow-up at infinity (that is, the boundary circle on the picture corresponds to the infinity point of the source sphere). The domain D_0 is shadowed there.



(a) A_1 (b) A_2 (c) A_3

Figure 2.1: A_n-singularities, $1 \leq n \leq 3$

Consider the vanishing cycles $\beta_i \in H_0(f^{-1}(\lambda), \mathbb{C})$ given by $\beta_i := p_0 - p_i$, where $\lambda = f(p_0, \tau) = f(p_i, \tau)$, and $p_0 \in D_0$, $p_i \in D_i$. Then the system of solutions of Equation (2.2.17) given by $\phi^{(i)}(\lambda, \tau) := \phi^{\beta_i}(\lambda, \tau)$ satisfies the properties given by Equations (2.2.19)–(2.2.22). In particular, $G^{ij} = 1/2$ for $i \neq j$ and $G^{ii} = 1$. The inverse matrix is given by $G_{ii} = 2n/(n+1)$ and $G_{ij} = -2/(n+1)$ for $i \neq j$. Therefore, Dubrovin's solution $\phi = \sum_{i,j=1}^n G_{ij}\phi^{(j)}$ is equal to ϕ^{β_0} for

$$\beta_0 = \sum_{i=1}^n \left(\frac{2n}{n+1} - (n-1)\frac{2}{n+1} \right) \beta_i = 2p_0 - \frac{2}{n+1} \sum_{i=0}^n p_i. \tag{2.7.67}$$

Recall that for the Frobenius structure A_n, $d = (n-1)/(n+1)$. Also we recall that for the Euler vector field $E = \sum_{i=1}^n u_i \frac{\partial}{\partial u_i}$ and the unit vector field $e = \sum_{i=1}^n \frac{\partial}{\partial u_i}$ so that we have:

$$Ef(p,\tau) = f(p,\tau) - \frac{p}{n+1} \frac{d_p f(p,\tau)}{dp}, \tag{2.7.68}$$

$$ef(p,\tau) = 1. \tag{2.7.69}$$

The formula $\phi^T(U - \lambda)\Psi\mathbb{1}$ can be written as $\big(I_{\beta_0}(\lambda,\tau), (E - \lambda e)/\sqrt{2}\big)$. Therefore,

$$\frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\mathbb{1} = \frac{n+1}{2}\int_{\beta_0} (E - \lambda e) f(p,\tau) \cdot \frac{dp}{d_p f(p,\tau)} \tag{2.7.70}$$
$$= \frac{n+1}{2}\int_{\beta_0} \left(f(p,\tau) - \frac{p}{n+1}\frac{d_p f(p,\tau)}{dp} - \lambda \right) \cdot \frac{dp}{d_p f(p,\tau)}.$$

Since the cycle β_0 lies in $f^{-1}(\lambda)$, then $(f(p,\tau)-\lambda)|_{\beta_0}=0$. Therefore, the last integral can be rewritten as

$$\frac{n+1}{2}\int_{\beta_0}\frac{p}{n+1}=p_0(\lambda,\tau)-\frac{1}{n+1}\sum_{i=0}^{n}p_i(\lambda,\tau). \qquad (2.7.71)$$

Since $\sum_{i=0}^{n}p_i(\lambda,\tau)=0$ (recall the form of the polynomial $f(p,\lambda)$), we conclude that the function $\frac{\sqrt{2}}{1-d}\phi^T(U-\lambda)\Psi\mathbb{1}$ is equal to the branch D_0 of $p=f^{-1}(\lambda,\tau)$.

So, $p(\mathbb{C}\setminus\cup_{i=1}^{n}L_i)=D_0$. It is obvious that $d_\lambda p$ has no zeros in $\mathbb{C}\setminus\cup_{i=1}^{n}L_i$, so $\hat{D}=D$, and one of the possible analytic continuation of the function $\lambda=f(p,\tau)|_{D_0}$ is its extension to the polynomial $f(p,\tau)$ defined on $\mathbb{CP}^1$. All condition of Theorem 2.5.1 are satisfied, so we apply it here to complete the proof. □

Remark 2.7.2 (Relation to Milanov's spectral curve). The global spectral curve that we constructed differs from the one constructed by Milanov in [73]. Milanov gets a spectral curve with the same local behavior as $x=f(y,\tau)$ near the critical points, but, in our terms, he chooses a different analytic continuation of $\lambda|_D$. He constructs an analytic continuation using the action of the Weyl group (we revisit his construction in our terms in Section 2.10), and obtains a curve where all preimages of the critical points in the x-plane are critical. In our terms, this can be achieved by gluing $n!$ copies of the curve $x=f(y,\tau)$ along the cuts connecting the non-critical preimages of the points u_i, $i=1,\dots,n$ such that each point belongs to exactly one cut. This makes all preimages of $u_1,\dots,u_n$ critical and will produce a curve of genus $1+\frac{n!}{2}(\frac{n^2}{2}-\frac{n}{2}-2)$ where the function x has degree $(n+1)!$, and it has $n!$ poles of degree $(n+1)$ each (cf. computation in [73] and further explanation in Section 2.10).

2.7.1 Bouchard-Eynard recursion

In this Section we discuss an application of Theorem 2.7.1. There is a more general formulation of topological recursion that works for functions x with higher order singular points [11]. Locally, a higher order singularity is given by $x=y^{n+1}$, $B=dy_1dy_2/(y_1-y_2)^2$. Bouchard and Eynard announced a theorem [10] that identifies the coefficients of the local expansion in y at $y=0$ of the correlation differentials of this spectral curve with the coefficients of the string solution of the $(r+1)$-Gelfand-Dickey hierarchy, also known as the total descendant potential of the A_r singularity. The proof of Bouchard and Eynard goes through analysis of matrix models. Here we give a new proof of their theorem, namely, we derive it directly from Theorem 2.7.1.

Theorem 2.7.3. *[10] The Bouchard-Eynard recursion applied to $x=p^{n+1}$, $y=p$, $B=dp_1dp_2/(p_1-p_2)^2$ produces differentials $\omega_{g,k}$, whose expansions near infinity are given by*

$$\omega_{g,k}(p_1,\dots,p_k)=\sum_{\substack{0\le a_1,\dots,a_k\le n-1\\ d_1,\dots,d_k}}\langle\tau_{d_1a_1}\cdots\tau_{d_ka_k}\rangle_{g,k}\times \qquad (2.7.72)$$
$$\prod_{j=1}^{k}\left(\frac{(a_j+1)(a_j+1+(n+1))\cdots(a_j+1+d_j(n+1))}{(-1)^{d_j}(n+1)^{d_j+1}}\frac{dp_j}{p_j^{(n+1)d_j+a_j+2}}\right),$$

where $\langle\tau_{d_1a_1}\cdots\tau_{d_ka_k}\rangle_{g,k}$ *are the coefficients of the string solution of the* $(n+1)$*-Gelfand-Dickii hierarchy [95, 58, 53].*

Note that we don't recall and don't use the definition of the Bouchard-Eynard recursion. The only property that we are using here is that it is compatible with the usual recursion on the curves with simple singularities and the limits [11]. In this case, we know that in the neighborhood of infinity the correlation differentials of the Bouchard-Eynard recursion are the limits for $\epsilon\to 0$ of the correlation differentials of the usual recursion applied to $x=y^{n+1}+\epsilon y$.

Let us now prove theorem 2.7.3.

Proof. The flat coordinates $t_0=t_{1\!\!1},t_1,\dots,t_{n-1}$ are given on the space of polynomials $f(p,\tau)$ defined in Equation (2.7.63) by the following formula:

$$f(p,\tau)^{\frac{1}{n+1}}=p+\frac{1}{n+1}\left(\frac{t_{n-1}}{p}+\frac{t_{n-2}}{p^2}+\dots+\frac{t_0}{p^n}\right)+O\left(\frac{1}{p^{n+1}}\right). \tag{2.7.73}$$

Recall that the canonical coordinates are the critical values $u_1,\dots,u_n$ of $f(p,\tau)$ and it is obvious that $\partial u_i/\partial t_{1\!\!1}=1$. We denote by $c_1,\dots,c_n$ the positions of the critical points of function $f(p,\tau)$; so $u_i=f(c_i,\tau)$, $i=1,\dots,n$.

We perform all computations only on a special curve in the space of polynomials, namely, $f(p,\tau)=p^{r+1}+\epsilon p$, and we are interested in all results only up $O(\epsilon)$ for $\epsilon\to 0$. In particular, we note that $t_a=O(\epsilon)$, $a=0,\dots,n-1$.

The full Jacobian of the change from the canonical to flat coordinates is then given by the following computation:

$$\frac{\partial u_i}{\partial t_a}=\frac{\partial f(c_i,\tau)}{\partial t_a}=f(c_i,\tau)^{\frac{n}{n+1}}\left(\frac{1}{c_i^{n-a}}+O(\epsilon)\right)=c_i^a\frac{\partial u_i}{\partial t_0}+O(\epsilon)=c_i^a+O(\epsilon). \tag{2.7.74}$$

The correlation differentials, written in terms of a CohFT considered in normalized canonical frame in Equation (2.2.36), can be rewritten in terms of the correlators of the ancestor potential of Givental [58] $\mathcal{A}_t(\{t_{d,\alpha}\})$ considered at the point t in flat coordinates $t_{d,\alpha}$, $d=0,1,2,\dots,$, $\alpha=0,\dots,n-1$ in the following way:

$$\begin{aligned}\omega_{g,k}&=\sum_{\substack{i_1,\dots,i_k\\ d_1,\dots,d_k}}\int_{\overline{\mathcal{M}}_{g,k}}\alpha_{g,k}\left(\Delta_{i_1}^{\frac12}\frac{\partial}{\partial u_{i_1}},\dots,\Delta_{i_k}^{\frac12}\frac{\partial}{\partial u_{i_k}}\right)\prod_{j=1}^k\psi_j^{d_j}d\left(\left(\frac{d}{dx}\right)^{d_j}\xi_{i_j}\right). \quad (2.7.75)\\ &=\sum_{\substack{\alpha_1,\dots,\alpha_k\\ d_1,\dots,d_k}}\langle\tau_{d_1\alpha_1}\cdots\tau_{d_k\alpha_k}\rangle_{g,k}(t)\prod_{j=1}^k d\left(\left(\frac{d}{dx}\right)^{d_j}\frac{-1}{p^{\alpha_j+1}}\right)+O(\epsilon).\end{aligned}$$

(by $\langle\tau_{d_1\alpha_1}\cdots\tau_{d_k\alpha_k}\rangle_{g,k}(t)$ we denote the coefficients of the expansion of $\log\mathcal{A}_t$). Indeed, let us expand the vector $\sum_{i=1}^n\xi_i(p)\Delta_i^{\frac12}\frac{\partial}{\partial u_i}$ near $p=\infty$. Recall that we denote by $c_1,\dots,c_n$ the positions of the critical points of function $f(p,\tau)$. We have:

$$\begin{aligned}\sum_{i=1}^n\xi_i(p)\Delta_i^{\frac12}\frac{\partial}{\partial u_i}&=\sum_{i=1}^n\left(\frac{1}{z-p}\frac{dz}{d\sqrt{f(p,\tau)-u_i}}\right)\Bigg|_{z=c_i}\Delta_i^{\frac12}\frac{\partial}{\partial u_i} \quad (2.7.76)\\ &=-\sum_{k=0}^\infty\frac{1}{p^{k+1}}\sum_{i=1}^n c_i^k\frac{\partial}{\partial u_i}=-\sum_{k=0}^{n-1}\frac{1}{p^{k+1}}\frac{\partial}{\partial t_k}+O(\epsilon).\end{aligned}$$

(we use Equation (2.7.74) and the fact that $c_i^n = O(\epsilon)$ for the last equality).

Recall [82] that the correlators of the ancestor potential $\mathcal{A}_t$ are represented in terms of the correlators of the descendant potential (which is exactly the string solution of the $(n+1)$-Gelfand-Dickey hierarchy) as

$$\langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k} \rangle_{g,k}(t) = \langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k} \rangle_{g,k} + O(\epsilon). \tag{2.7.77}$$

Thus we see that

$$\omega_{g,k} = \sum_{\substack{\alpha_1,\dots,\alpha_k \\ d_1,\dots,d_k}} \langle \tau_{d_1\alpha_1} \cdots \tau_{d_k\alpha_k} \rangle_{g,k} \prod_{j=1}^{k} d\left(\left(\frac{d}{dx}\right)^{d_j} \frac{-1}{p^{\alpha_j+1}}\right) + O(\epsilon). \tag{2.7.78}$$

In the limit $\epsilon \to 0$ we get exactly Equation (2.7.72). □

2.8 Frobenius manifolds for hypermaps

In this Section we construct a global spectral curve for the Frobenius manifold given by the superpotential $\lambda = f(p,a)$, where $a = (a_0, \dots, a_{n+1})$, $n \geq 1$, and

$$f(p,a) = p^n + a_2 p^{n-2} + a_3 p^{n-3} + \cdots + a_n + \frac{a_{n+1}}{p - a_1}. \tag{2.8.79}$$

This superpotential defines a semi-simple Frobenius manifold (this Frobenius manifold is studied in [27, Section 5]). Furthermore the spectral curve

$$(\Sigma, x, y, B) = \left(\mathbb{CP}^1, f(p,a), p, \frac{dp_1 dp_2}{(p_1 - p_2)^2}\right)$$

satisfies equations (2.2.31)-(2.2.34) hence it stores the correlators of a CohFT via Equation (2.2.36). The following theorem answers the question of whether these two CohFTs coincide.

Theorem 2.8.1. *The CohFT associated to the Frobenius manifold given by the superpotential $\lambda = f(p,a)$ coincides with the one reconstructed from the spectral curve $(\mathbb{CP}^1, f(p,a), p, dp_1dp_2/(p_1-p_2)^2)$.*

Remark 2.8.2. The correlation differentials for this spectral curve considered for the particular values of the parameters a enumerate hypermaps on the curves. This is proved in [36], see also [26], where some special case of that was conjectured.

So, Theorem 2.8.1 is to be used in the converse way: We start with a combinatorial problem that is known to be solved by global topological recursion. It appears that the correlators of this global topological recursion are expressed in terms of a CohFT. This CohFT appears to be homogeneous, so it is associated to a Frobenius manifold, and this Theorem describes precisely the underlying Frobenius manifold.

Proof. The proof is completely parallel to the proof of Theorem 2.7.1. Note that as in the case of A_n-singularity, we claim that the spectral curve is the superpotential itself. We use Theorem 2.5.1, so it is enough to show that we can reproduce the superpotential $\lambda = f(p,a)$ via Dubrovin's construction from Section 2.2.4.

The canonical coordinates $u_1, \dots, u_{n+1}$ of this Frobenius manifold are the critical values of $f(p,a)$; the Euler vector field is given by

$$E = \sum_{i=1}^{n} u_i \frac{\partial}{\partial u_i} = \sum_{i=1}^{n+1} \frac{i}{n} a_i \frac{\partial}{\partial a_i}; \tag{2.8.80}$$

the unit vector field is equal to

$$e = \sum_{i=1}^{n} \frac{\partial}{\partial u_i} = \frac{\partial}{\partial a_n}; \tag{2.8.81}$$

and the constant d is equal to $(n-2)/n$. Note that

$$Ef(p,a) = f(p,a) - \frac{p}{n}\frac{df(p,a)}{dp}. \tag{2.8.82}$$

As in the case of A_n singularity, the solutions to the Equation (2.2.17) are given by the integrals over the cycles $\beta \in H_0(f^{-1}(\lambda), \mathbb{C})$, where the components of the solutions are given by

$$\phi_i^\beta := \int_\beta \frac{\Delta_i^{\frac{1}{2}}}{\sqrt{2}} \frac{\partial f(p,a)}{\partial u_i} \cdot \left(\frac{df(p,a)}{dp} \right)^{-1}. \tag{2.8.83}$$

Consider the λ-plane as the image of the map $\lambda = f(p,\tau)$. Recall that $u_1, \dots, u_{n+1}$ are the critical values of $f(p,\tau)$. We can alway choose a system of cuts L_i, $i = 1, \dots, n+1$, from u_i to infinity such that the preimage $f^{-1}(\mathbb{C} \setminus \cup_{i=1}^n L_i)$ is a union of $n+1$ disks, $D_0, D_1, \dots, D_n$, glued along the boundary cuts in the following way:

- D_0 is glued to D_i, $i = 2, \dots, n$, along the boundary that is a double cover of L_i; in particular, their common boundary contains the critical preimage of u_i;

- D_0 is glued to D_1 along two components of the boundary that are double covers of L_1 and L_{n+1} and these boundary components have common point $p = a_1$. In particular, these boundary components contain the critical preimages of u_1 and u_{n+1};

- All other lifts of the cut L_i are just cuts inside D_j, $j \neq 0, i$ for $i = 2, \dots, n$ and $j \neq 0, 1$ for $i = 1, n+1$; the endpoints of these cuts are non-critical preimages of u_i.

For $n = 1, 2, 3$ we give the corresponding pictures for a real orientable blow-up at infinity (that is, the external boundary circle on the picture corresponds to the infinity point of the source sphere, and the internal circle corresponds to $p = a_1$). The domain D_0 is shadowed.

Consider the vanishing cycles $\beta_i \in H_0(f^{-1}(\lambda), \mathbb{C})$ given by $\beta_i := p_i - p_0$, where $\lambda = f(p_0, \tau) = f(p_i, \tau)$, and $p_0 \in D_0$, $p_i \in D_i$. Then the system of solutions of Equation (2.2.17) given by $\phi^{(i)}(\lambda, \tau) := \phi^{\beta_i}(\lambda, \tau)$, $i = 1, \dots, n$, $\phi^{(n+1)} = \phi^{(1)}$, satisfies the properties given by Equations (2.2.19)–(2.2.22). In particular, $G^{ii} = 1$

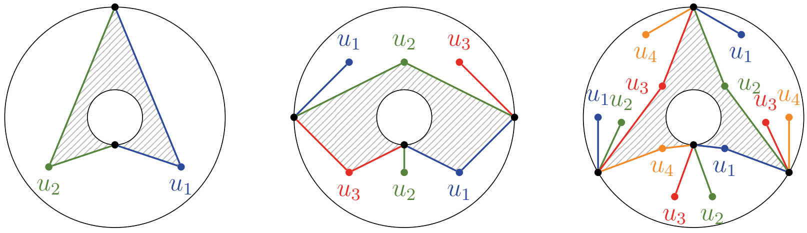


Figure 2.2: Hypermaps $z^n + \dfrac{1}{z}$, $1 \le n \le 3$

for $i = 1, \dots, n+1$, $G^{1,n+1} = G^{n+1,1} = 1$, and for all other $i \neq j$ $G^{ij} = 1/2$. So, this matrix is degenerate.

However, Remark 2.3.2 specifies the properties of ϕ that are sufficient for Theorems 2.3.1 and 2.5.1. Note that

$$\phi := \frac{2}{n+1} \sum_{i=1}^{n} \phi^{(i)} = \phi^{\beta_0} \tag{2.8.84}$$

for $\beta_0 := \frac{2}{n+1} \sum_{i=0}^{n} p_i - 2p_0$ satisfies all condition of Remark 2.3.2. With this choice of ϕ and, therefore, β_0, Dubrovin's superpotential can be presented as

$$\frac{\sqrt{2}}{1-d} \phi^T (U - \lambda) \Psi \mathbb{1} = \frac{n}{2} \int_{\beta_0} (E - \lambda e) f(p,a) \cdot \left(\frac{df(p,a)}{dp} \right)^{-1}. \tag{2.8.85}$$

Using Equation (2.8.82), we have:

$$\frac{\sqrt{2}}{1-d} \phi^T (U - \lambda) \Psi \mathbb{1} = \frac{n}{2} \int_{\beta_0} \left(f(p,a) - \frac{p}{n} \frac{df(p,a)}{dp} - \lambda \right) \cdot \left(\frac{df(p,a)}{dp} \right)^{-1} \tag{2.8.86}$$
$$= \int_{\beta_0} -\frac{p}{2} = p_0 - \frac{1}{n+1} \sum_{i=0}^{n} p_i = \begin{cases} p_0 - \frac{\lambda + a_1}{2} & n = 1; \\ p_0 - \frac{a_1}{n+1} & n > 1. \end{cases}$$

(in the last equality we used that we know the sum of all roots of the equation $f(p,a) = \lambda$).

Let us now discuss the cases that we get. For $n > 1$ Dubrovin's function $p_{\mathrm{Dub}} = p_{\mathrm{Dub}}(\lambda, a)$ is the branch D_0 of the inverse function of $\lambda = f(p,a)$ shifted by a constant. Obviously, $d_\lambda p_{\mathrm{Dub}}$ has no zeros in $\mathbb{C} \setminus \cup_{i=1}^{n+1} L_i$, and we can choose as the analytic extension of $\lambda|_{D_0}$ the function $\lambda = f(p,a)$ defined on $\mathbb{CP}^1$. Then Theorem 2.5.1 is applied. We get, therefore, not precisely the statement that we want to prove, but we have instead $y = p_{\mathrm{Dub}} = p - a_1/(n+1)$ (the Bergman kernel is still the same). However, it doesn't change anything in topological recursion if we shift y by a constant.

The case $n = 1$ is even more interesting. One can easily check by direct computation that Dubrovin's function $p_{\mathrm{Dub}} = p_{\mathrm{Dub}}(\lambda, a)$ is equal to $\sqrt{(\lambda - u_1)(\lambda - u_2)}/2$. Further construction of the curve gives the following equation:

$$p_{\mathrm{Dub}}^2 - \frac{1}{4}(\lambda - a_1)^2 + a_2 = 0$$

It is a rational curve, and it has a global coordinate $p = p_{\mathrm{Dub}} + (\lambda + a_1)/2$, which is our original coordinate p, that is, $\lambda = p + a_2/(p - a_1)$. Theorem 2.5.1 can not be applied directly, but in this case we can just check by hand that we get the statement that we want to prove. See Appendix 2.11.2.

Note that Theorem 2.3.1 suggests that the right choice of function y is $y = p_{\mathrm{Dub}} = p - (\lambda + a_1)/2$ rather than $y = p$. However, it doesn't change anything in topological recursion if we shift y by a function of $x = \lambda$, so there is no contradiction. □

Since in this example we rather start from a combinatorial problem of enumeration of hypermaps and use Theorem 2.8.1 in order to clarify the structure of the ELSV-type formula (2.2.36) for this combinatorial problem, it is interesting to have a description of the underlying Frobenius manifolds (given by superpotentials) in terms of their prepotentials. We know an algorithm which can produce this prepotential for any given n (this algorithm follows from Dubrovin's construction found in [27]), but we do not know a general formula which would describe these prepotentials for all $n \geq 1$. Here we list the formulas for cases $n = 1, 2, 3$:

$$\begin{aligned}
n = 1: \quad & \frac{a_1^2 a_2}{2} + \frac{a_2^2}{2} \log a_2; \\
n = 2: \quad & \frac{a_1^3}{6} + a_1 a_2 a_3 + \frac{a_3^2}{2} \log a_3 + \frac{a_1^3 a_3}{6} - \frac{3a_3^2}{4}; \\
n = 3: \quad & \frac{a_1^2 a_4}{2} + a_1 a_2 a_3 - \frac{3a_2^2}{4} + \frac{a_2^2}{2} \log (a_2) + \frac{a_2 a_3^4}{4} + \frac{3a_2 a_3^2 a_4}{2} + \frac{3a_2 a_4^2}{2} - \frac{3a_4^4}{8}
\end{aligned}$$

Note that in the case $n = 1$ the corresponding combinatorial problem has also interpretation in terms of the discrete volumes of the moduli space of curves [79] and discrete surfaces/generalized Catalan numbers [4, 32, 52]. The relation of these combinatorial problems to a CohFT is also discussed in [8, 54], though it is not mentioned there that the underlying Frobenius manifold is given by the prepotential $a_1^2 a_2/2 + a_2^2/2 \cdot \log a_2$.

2.9 Elliptic example

In this section we give an example of a superpotential that satisfies the conditions of Theorem 2.6.1.

Consider the spectral curve defined by the Weierstrass $\wp$-function

$$\lambda = \wp(z), \quad p = z, \quad B(z, z') = (\wp(z - z') + b)\, dz dz' \tag{2.9.87}$$

where $b \in \mathbb{C}$ and p is only defined locally—it is the primitive of a holomorphic differential on the curve—which is sufficient for topological recursion. The compatibility condition (2.4.42) is satisfied by Proposition 2.4.3. It is equivalent to the elliptic identity:

$$\frac{\wp''(z)}{\wp'(z)^2} = \sum_{i=1}^{3} \frac{\wp(z - \omega_i)}{\wp''(\omega_i)} \tag{2.9.88}$$

where the sum is over the zeros ω_i of $\wp'(z)$. Hence the spectral curve defines a CohFT.

Introduce three parameters ω, ω' and c into the spectral curve to define the following superpotential taken from [27]:

$$\lambda = \wp(z;\omega,\omega') + c, \quad p = \frac{z}{\omega} \tag{2.9.89}$$

where

$$\wp(z;\omega,\omega') = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \frac{1}{(z-2m\omega-2n\omega')^2} - \frac{1}{(2m\omega+2n\omega')^2}. \tag{2.9.90}$$

The Frobenius manifold structure on $M = \{(\omega,\omega',c)\}$ is given by the formulae (2.2.8), (2.2.9), (2.2.10) where the vector fields ∂ on M are given by, for example ∂_ω, $\partial_{\omega'}$, ∂_c. Note that in (2.9.88), $\omega_1 = \omega$, $\omega_2 = \omega'$, $\omega_3 = \omega + \omega'$.

Remark 2.9.1. Note that we know that the superpotential (2.9.89) defines a Frobenius manifold due to the existence of flat coordinates, proven in [27], and also given below. The CohFT produced by topological recursion applied to (2.9.87) is homogeneous if we choose b in (2.9.87) so that $\int_A B(z,z') = 0$, i.e. $b = \eta/\omega$ where the A and B periods are $2\omega = \oint_A dz$, $2\omega' = \oint_B dz$ and

$$\eta = -\frac{1}{2}\oint_A \wp(z)dz, \quad \eta' = -\frac{1}{2}\oint_B \wp(z)dz.$$

The periods satisfy Legendre's relation

$$\eta\omega' - \eta'\omega = \frac{i\pi}{2}.$$

The homogeneous CohFT corresponds to a conformal Frobenius manifold which gives rise to a superpotential via Dubrovin's construction (actually, since $d = 1$ it is a variant of the construction). What needs to be proven is that the two superpotentials agree.

Theorem 2.9.2. *The superpotential* (2.9.89) *can be obtained via (a variant of) Dubrovin's construction described in Section 2.2.4 applied to the Frobenius manifold M. The conditions of Theorem 2.6.1 are satisfied for this superpotential. Hence the two cohomological field theories—obtained from the superpotential* (2.9.89) *and topological recursion applied to the spectral curve* (2.9.87) *with $b = \eta/\omega$—agree.*

Proof. To apply Dubrovin's construction to M we construct a solution of the Gauss-Manin system as in the proof of Theorem 2.7.1.

The flat metric (2.2.8) for the superpotential (2.9.89) is given by

$$(\partial,\partial') := \sum_{i=1}^{3} \operatorname*{Res}_{z=\omega_i} \frac{\partial\lambda\cdot\partial'\lambda}{\wp'}dp. \tag{2.9.91}$$

We use this to construct a vector field $I_\beta(\lambda;u)$ on M for any cycle $\beta \in H_0(\lambda^{-1}(\mathrm{pt}),\mathbb{C})$ specified by:

$$(I_\beta(\lambda;u),\partial) := \int_\beta \frac{\partial(\lambda)}{d_p\lambda/dp}. \tag{2.9.92}$$

The elliptic curve (2.9.89) is built by gluing two copies of the disk $D = \mathbb{C} \setminus \cup_{i=1}^3 L_i$ in the λ plane along L_i. Choose β to be the cycle given by $p_0 - p_1$, for p_0 and p_1 the pre-images of λ in each of the two disks. In normalized canonical coordinates $I_\beta(\lambda; u)$ is represented by a solution $\phi^\beta(\lambda; u) = \sum_i \phi_i^\beta(\lambda; u)\partial_{v_i}$ of the Gauss-Manin system (2.2.17) which has components given by

$$\begin{aligned}\phi_i^\beta(\lambda; u) &= \left(I_\beta(\lambda; u), \frac{1}{\sqrt{2}}\Delta_i^{\frac{1}{2}}\partial_{u_i}\right)\\ &= \int_\beta \frac{\Delta_i^{\frac{1}{2}}\partial_{u_i}\lambda}{\sqrt{2}\cdot d_p\lambda/dp}\\ &= \frac{\sqrt{2}\cdot\Delta_i^{\frac{1}{2}}\partial_{u_i}\lambda}{d_p\lambda/dp}\end{aligned}$$

where the integral over β simply doubles the integrand since the integrand is skew symmetric. Since $d = 1$, we cannot use the inversion formula (2.2.18) so we directly check that $\frac{1}{\sqrt{2}}\phi^\beta = \nabla_u p$ as follows.

$$\nabla_u p = \eta^{-1} d_u p = \frac{1}{\omega}\sum_{i=1}^3 \Delta_i \cdot \frac{\partial_{u_i}\lambda}{\wp'(z)}\partial_{u_i} = \frac{1}{\omega}\sum_{i=1}^3 \frac{\Delta_i^{\frac{1}{2}}\partial_{u_i}\lambda}{\wp'(z)}\partial_{v_i} = \frac{1}{\sqrt{2}}\phi^\beta.$$

Using

$$\partial_{u_i}\lambda = \frac{1}{2\wp''(\omega_i)}\frac{\wp'(z)^2}{\wp(z) - \wp(\omega_i)} + \frac{z\wp(\omega_i) + \zeta(z)}{\wp''(\omega_i)}\wp'(z)$$

which can be proven from the known variations $\sum_i u_i^k \partial_{u_i}$, $k = 0, 1, 2$, we see that the solution $\phi^\beta(\lambda; u)$ satisfies

$$\begin{aligned}\frac{1}{\sqrt{2}}\phi^T \Psi \mathbb{1} &= \frac{1}{\omega}\sum_{i=1}^3 \frac{\partial_{u_i}\lambda}{\wp'(z)}\\ &= \frac{1}{\omega}\sum_{i=1}^3 \frac{1}{2\wp''(\omega_i)}\frac{\wp'(z)}{\wp(z) - \wp(\omega_i)} + \frac{z\wp(\omega_i) + \zeta(z)}{\wp''(\omega_i)}\\ &= \frac{1}{\omega\wp'(z)} = \frac{dp}{d\lambda}\end{aligned}$$

which is (2.3.39) and hence via Remark 2.3.2 we see that the properties of ϕ are sufficient for Theorems 2.3.1 and 2.5.1. Hence the theorem follows. □

Theorem 2.9.2 states that we can study the CohFT obtained from the superpotential (2.9.89) via topological recursion applied to the spectral curve (2.9.87). We will need the three-point function of this CohFT in calculations below. We calculate it in two ways to demonstrate the proof, although we know from the theorem that they coincide.

2.9.1 Three-point function.

Superpotential. Introduce the canonical coordinates

$$u_i = \wp(\omega_i) + c, \quad i = 1, 2, 3$$

where, as usual, $\omega_1 = \omega$, $\omega_2 = \omega'$, $\omega_3 = \omega + \omega'$. The three-point calculations take place in the ring $\mathbb{C}[E]/\wp' = \mathbb{C}[\wp]/\wp'$ and we have

$$\frac{\partial \lambda}{\partial u_1} \equiv \frac{(\lambda - u_2)(\lambda - u_3)}{(u_1 - u_2)(u_1 - u_3)}, \quad \frac{\partial \lambda}{\partial u_1}\frac{\partial \lambda}{\partial u_j} \equiv \delta_{1j}\frac{\partial \lambda}{\partial u_1}, \; j = 2, 3.$$

and cyclic permutations of the above. This is quite general and also can be proven via elliptic identities. Hence the three-point function for the superpotential is

$$\left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i} \right\rangle = \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i} \right\rangle = \left\langle \frac{\partial}{\partial u_i} \right\rangle = \sum_{j=1}^{3} \operatorname*{Res}_{z=\omega_j} \frac{\frac{\partial \lambda}{\partial u_i}}{\wp'(z)}\frac{dz}{\omega^2} = \frac{1}{\omega^2 \wp''(\omega_i)}.$$

Thus

$$\left\langle \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_i}, \frac{\partial}{\partial v_i} \right\rangle = \omega\sqrt{\wp''(\omega_i)} \tag{2.9.93}$$

where $\frac{\partial}{\partial v_i} = \Delta_i^{\frac{1}{2}} \frac{\partial}{\partial u_i}$ for $\Delta_i^{-1} = \left\langle \frac{\partial}{\partial u_i}, \frac{\partial}{\partial u_i} \right\rangle = \omega^2 \cdot \wp''(\omega_i)$ give the normalized canonical coordinates.

Topological recursion. The three-point function obtained via topological recursion is

$$\begin{aligned} \omega_{0,3}(z_1, z_2, z_3) &= \sum_{j=1}^{3} \operatorname*{Res}_{z=\omega_j} \frac{\omega dz}{\wp'(z)} \prod_{i=1}^{3} (\wp(z_i - z) + b) dz_i \\ &= \sum_{j=1}^{3} \frac{\omega}{\wp''(\omega_j)} \prod_{i=1}^{3} (\wp(z_i - \omega_j) + b) dz_i \\ &= \sum_{j=1}^{3} \omega \sqrt{\wp''(\omega_j)} V_0^j(z_1) V_0^j(z_2) V_0^j(z_3) \end{aligned}$$

for

$$V_0^j(z) = \left. \frac{(\wp(z - \omega_j) + b) dz dz_j}{ds_j} \right|_{s_j=0} = \frac{\wp(z - \omega_j) + b}{\sqrt{\wp''(\omega_i)}} dz$$

where $\lambda = \wp(z) + c = \frac{1}{2}s_j^2 + \wp(\omega_j) + c$ defines the local coordinate s_j. The coefficients of $V_0^j(z_i)$ define the three-point function of the cohomological field theory which agree with (2.9.93).

2.9.2 Flat coordinates

The cohomological field theory is defined on the three-dimensional vector space $\mathbb{C}[\wp]/\wp'$ equipped with its natural ring structure and gives rise to a Frobenius manifold structure on the family M of such rings parametrized by $\{\omega, \omega', c\}$. It will be convenient to express the metric on M with respect to a natural basis of vector fields on M corresponding to the basis $\{1, \wp, \wp^2\}$ of $\mathbb{C}[\wp]/\wp'$ since the metric requires knowledge of the variation of $\wp$ under the action of vector fields on the Frobenius manifold. We will see that $\{\omega, \omega', c\}$ are not flat coordinates and find in Lemma 2.9.4 flat coordinates $\{t_1, t_2, t_3\}$ on M, i.e. so that the metric on M is constant with respect to them.

Recall that correlation functions of the cohomological field theory arising from topological recursion applied to a spectral curve appear as coefficents of auxiliary differentials on the spectral curve. Proposition 2.9.6 gives the auxiliary differentials that correspond to the flat basis for the metric.

In the following lemma we calculate the vector fields on M that correspond to the basis elements $1, \wp, \wp^2$ of $\mathbb{C}[\wp]/\wp'$. This uses $g_2 = g_2(\omega, \omega')$ defined by $\wp'(z)^2 = 4\wp(z) - g_2\wp - g_3$.

Lemma 2.9.3. *Under the map $TM \to \mathbb{C}[\wp]/\wp'$ defined by $\partial \mapsto \partial\lambda (mod\ \wp')$ for $\lambda = \wp(z; \omega, \omega') + c$*

$$\partial_c \mapsto 1, \quad -\tfrac{1}{2}(\omega\partial_\omega + \omega'\partial_{\omega'}) \mapsto \wp, \quad -\tfrac{1}{2}(\eta\partial_\omega + \eta'\partial_{\omega'}) + \tfrac{1}{6}g_2\partial_c \mapsto \wp^2. \tag{2.9.94}$$

Proof. The variation $\partial_c\lambda = 1$ is obvious. The identity

$$\omega\partial_\omega\wp(z) + \omega'\partial_{\omega'}\wp(z) + z\wp'(z) = -2\wp(z) \tag{2.9.95}$$

follows immediately from the expansion (2.9.90) of $\wp$ and yields $-\frac{1}{2}(\omega\partial_\omega + \omega'\partial_{\omega'}) \mapsto \wp$. The final identification uses the identity proven in [55]

$$\eta\partial_\omega\wp(z) + \eta'\partial_{\omega'}\wp(z) + \zeta(z)\wp'(z) = -2\wp(z)^2 + \tfrac{1}{3}g_2 \tag{2.9.96}$$

where $\zeta(z)$ is the Weierstrass ζ-function

$$\zeta(z; \omega, \omega') = \frac{1}{z} + \sum_{(m,n)\neq(0,0)} \frac{1}{z - 2m\omega - 2n\omega'} + \frac{1}{2m\omega + 2n\omega'} + \frac{z}{(2m\omega + 2n\omega')^2}$$

which is not an elliptic function (C.63 in [27]). Note that $\eta = \zeta(\omega)$, $\eta' = \zeta(\omega')$. □

The metric

$$\langle\wp^j, \wp^k\rangle = \sum_{i=1}^{3} \operatorname*{Res}_{z=\omega_i} \frac{\wp^{j+k}}{\wp'(z)} \frac{dz}{\omega^2} = -\operatorname*{Res}_{z=0} \frac{\wp^{j+k}}{\wp'(z)} \frac{dz}{\omega^2}$$

is given by

	1	$\wp$	$\wp^2$
1	0	0	$1/2\omega^2$
$\wp$	0	$1/2\omega^2$	0
$\wp^2$	$1/2\omega^2$	0	$g_2/8\omega^2$

Lemma 2.9.4 (Dubrovin [27]). *Flat coordinates for the metric are given by*

$$t_1 = c - \frac{\eta}{\omega}, \quad t_2 = \frac{1}{\omega}, \quad t_3 = \frac{\omega'}{\omega}.$$

Proof. This is (5.95) in [27]. We simply use change of coordinates given by (2.9.94) and the metric calculated above. We have $\partial_c = \partial_{t_1}$. From the identity

$$(\omega\partial_\omega + \omega'\partial_{\omega'})\frac{\eta}{\omega} = -2\frac{\eta}{\omega}$$

which uses the fact that $\omega\partial_\omega+\omega'\partial_{\omega'}$ is the degree operator, $\frac{\eta}{\omega}$ is homogeneous of degree -2 we have $\omega\partial_\omega+\omega'\partial_{\omega'}=\frac{2\eta}{\omega}\partial_{t_1}-\frac{1}{\omega}\partial_{t_2}$. The identity

$$(\eta\partial_\omega+\eta'\partial_{\omega'})\frac{\eta}{\omega}=-\frac{1}{12}g_2-\frac{\eta^2}{\omega^2}$$

appearing as (C.69) in [27] gives $\eta\partial_\omega+\eta'\partial_{\omega'}=\left(\frac{1}{12}g_2+\frac{\eta^2}{\omega^2}\right)\partial_{t_1}-\frac{\eta}{\omega^2}\partial_{t_2}-\frac{i\pi}{2\omega^2}\partial_{t_3}$. Hence we have

$$\partial_{t_1}\mapsto 1,\quad -\frac{\eta}{\omega}\partial_{t_1}+\frac{1}{2\omega}\partial_{t_2}\mapsto\wp,\quad \left(\tfrac{1}{8}g_2-\frac{\eta^2}{2\omega^2}\right)\partial_{t_1}+\frac{\eta}{2\omega^2}\partial_{t_2}+\frac{i\pi}{4\omega^2}\partial_{t_3}\mapsto\wp^2. \quad (2.9.97)$$

Hence the metric is given by:

	∂_{t_1}	∂_{t_2}	∂_{t_3}
∂_{t_1}	0	0	$2/i\pi$
∂_{t_2}	0	2	0
∂_{t_3}	$2/i\pi$	0	0

which is constant so that $\{t_1,t_2,t_3\}$ are flat coordinates. □

Remark 2.9.5. As mentioned in Section 2.1.2 we can choose a different $(0,2)$ term $B(z,z')$ on the spectral curve (2.9.87) which still satisfies the compatibility condition (2.4.42) by varying $b\in\mathbb{C}$. For each b it gives rise to a CohFT with the same genus 0 three-point function since ancestor invariants are coefficients of B-dependent differentials. When b is chosen so that $B(z,z')$ is normalised along a choice of cycle, e.g. $b=\eta'/\omega'$ so $\int_B B(z,z')=0$, then the CohFT is homogeneous and hence the same CohFT as for $b=\eta/\omega$. Other choices of b gives rise to non-homogeneous CohFTs.

Proposition 2.9.6. *The flat coordinates correspond to the following auxiliary differentials:*

$$dt_1\longleftrightarrow T_0^1=(\omega\wp+b)dz-2\omega d\left(\frac{\wp^2}{\wp'}\right)+2\eta d\left(\frac{\wp}{\wp'}\right)+\left(\frac{\omega g_2}{4}+\frac{\eta^2}{\omega}\right)d\left(\frac{1}{\wp'}\right)$$

$$dt_2\longleftrightarrow T_0^2=-d\left(\frac{\wp}{\wp'}\right)-\frac{\eta}{\omega}d\left(\frac{1}{\wp'}\right)$$

$$dt_3\longleftrightarrow T_0^3=-\frac{i\pi}{2\omega}d\left(\frac{1}{\wp'}\right).$$

Proof. The auxiliary differentials on the spectral curve corresponding to the normalized canonical basis are straighforward. They are given by $V_k^i dz$ where

$$V_0^i=\frac{\wp(z-\omega_i)}{\sqrt{\wp''(\omega_i)}}$$

and for $k>0$, V_k^i is the principal part of the kth derivative of V_0^i with respect to $\wp(z)$. We also have the canonical basis $U_0^i=\omega\wp(z-\omega_i)$. The auxiliary differentials $T_k^i dz$ corresponding to flat coordinates are linear combinations of $V_k^i dz$

$$V_k^i=\Psi_\mu^i\cdot T_k^\mu$$

where we recall that Ψ^i_μ is the transition matrix from flat coordinates labeled by μ to normalized canonical coordinates labeled by i. We can calculate Ψ via

$$\begin{pmatrix} 1 & 1 & 1 \\ \wp(\omega_1) & \wp(\omega_2) & \wp(\omega_3) \\ \wp(\omega_1)^2 & \wp(\omega_2)^2 & \wp(\omega_3)^2 \end{pmatrix} \begin{pmatrix} \partial_{u_1} \\ \partial_{u_2} \\ \partial_{u_3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{\eta}{\omega} & \frac{1}{2\omega} & 0 \\ \frac{1}{8}g_2 - \frac{\eta^2}{2\omega^2} & \frac{\eta}{2\omega^2} & \frac{i\pi}{4\omega^2} \end{pmatrix} \begin{pmatrix} \partial_{t_1} \\ \partial_{t_2} \\ \partial_{t_3} \end{pmatrix}$$

which we write as $M\frac{\partial}{\partial u} = T\frac{\partial}{\partial t}$ hence $M^{-1}T = \Delta^{1/2}\Psi^T$. The auxiliary differentials corresponding to $1, \wp, \wp^2$ in the Landau-Ginzburg model are:

$$\begin{aligned} &[U^1, U^2, U^3] \cdot M^{-1} = \\ &2\omega dz \left[\frac{\wp(z-\omega_1)+b}{\wp''(\omega_1)}, \frac{\wp(z-\omega_2)+b}{\wp''(\omega_2)}, \frac{\wp(z-\omega_3)+b}{\wp''(\omega_3)}\right] \begin{pmatrix} \wp(\omega_1)^2 - \frac{1}{4}g_2 & \wp(\omega_1) & 1 \\ \wp(\omega_2)^2 - \frac{1}{4}g_2 & \wp(\omega_2) & 1 \\ \wp(\omega_3)^2 - \frac{1}{4}g_2 & \wp(\omega_3) & 1 \end{pmatrix} \\ &= \left[-2\omega d\left(\frac{\wp^2}{\wp'}\right) + (\omega\wp + b)dz + \frac{\omega g_2}{2} d\left(\frac{1}{\wp'}\right), -2\omega d\left(\frac{\wp}{\wp'}\right), -2\omega d\left(\frac{1}{\wp'}\right)\right] \end{aligned}$$

which is proven using the elliptic identities

$$\frac{\wp(z)^k}{\wp'(z)^2} = \sum_{i=1}^3 \frac{\wp(\omega_i)^k \wp(z-\omega_i)}{\wp''(\omega_i)^2}, \quad k = 0, 1, 2$$

and slight generalisations for $k > 2$. Hence

$$\begin{aligned} [T^1, T^2, T^3] &= [U^1, U^2, U^3] \cdot M^{-1} \cdot T \\ &= \Big[-2\omega d\left(\frac{\wp^2}{\wp'}\right) + (\omega\wp + b)dz + \left(\frac{\omega g_2}{4} + \frac{\eta^2}{\omega}\right) d\left(\frac{1}{\wp'}\right) + 2\eta d\left(\frac{\wp}{\wp'}\right), \\ &\qquad - d\left(\frac{\wp + \eta/\omega}{\wp'}\right), -\frac{i\pi}{2\omega} d\left(\frac{1}{\wp'}\right) \Big] \end{aligned}$$

□

The following lemma allows us to apply equation (2.2.36) to obtain ancestor invariants for the CohFT.

Lemma 2.9.7. *The following kernels* K_0^i

$$K_0^1 = y(z), \quad K_0^2 = -2\omega\zeta(z) + 2\eta, \quad K_0^3 = \frac{4}{i\pi}\left(\omega z\wp(z)^2 - \left(\frac{\eta^2}{2\omega} + \frac{\omega}{8}g_2\right) z + \eta\zeta(z)\right)$$

are dual (as linear functionals) to T_0^i *for* $i = 1, 2, 3$, *i.e.*

$$\sum_{j=1}^3 \operatorname*{Res}_{z=\omega_j} K_0^j(z) T_k^i(z) = \delta_{ij}\delta_{k0}.$$

Proof. Each kernel is analytic at $z = \omega_i$, $i = 1, 2, 3$ and hence annihilates differentials analytic at $z = \omega_i$. Consider the action of each kernel on $d(\wp^k/\wp')$ for $k = 0, 1, 2$.

$$\sum_{j=1}^3 \operatorname*{Res}_{z=\omega_j} K_0^i(z) d\left(\frac{\wp^k}{\wp'}\right) = -\sum_{j=1}^3 \operatorname*{Res}_{z=\omega_j} dK_0^i(z)\frac{\wp^k}{\wp'} = \operatorname*{Res}_{z=0} dK_0^i(z)\frac{\wp^k}{\wp'}$$

so $K_0^1 = y(z) = z/\omega$ annihilates $d(\wp^k/\wp')$ for $k = 0, 1$ and sends $d(\wp^2/\wp')$ to $-1/2\omega$. Similarly $K_0^2 = \zeta(z)$ annihilates $d(\wp^k/\wp')$ for $k = 0, 2$ and sends $d(\wp/\wp')$ to $1/2$. Apply the kernels to T_0^i given in Proposition 2.9.6 as linear combinations of $d(\wp^k/\wp')$ (and terms analytic at $z = \omega_i$) to achieve the result.

The kernels K_0^1 and K_0^2 annihilate exact differentials that vanish to order 2 at $z = 0$, in particular T_k^i for $k > 0$ by integration by parts. One can also check that K_0^3 annihilates T_k^i for $k > 0$. □

Remark. One can also produce kernels K_j^i dual to each T_j^i.

The 3-point function in flat coordinates leads to the prepotential given in [27] (C.87):

$$F_0 = \frac{1}{i\pi}t_1^2 t_3 + t_1 t_2^2 - \frac{i\pi}{2}t_2^4\left(\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right), \quad q = e^{2\pi i t_3}.$$

Proposition 2.9.8.

$$\exp F_1 = t_2^{1/8}\eta(q)^{1/4}, \quad \eta(q) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

Proof. Topological recursion—defined in Section 2.2.5—applied to the spectral curve (2.9.87) uses the kernel

$$\begin{aligned} K(z_1, z) &= \frac{\omega}{2}\frac{\int_{\sigma_i(z)}^{z}(\wp(z_1 - w) + b)dw dz_1}{(z - \sigma_i(z))\wp'(z)dz} \\ &= \frac{\omega}{4}\frac{(\zeta(z_1 + z) - \zeta(z_1 - z) + 2\eta_i + 2b(z - \omega_i))dz_1}{(z - \omega_i)\wp'(z)dz} \end{aligned}$$

where $\sigma_i(z) = 2\omega_i - z$. Hence

$$\begin{aligned} \omega_{1,1}(z_1) &= \sum_{j=1}^{3}\operatorname*{Res}_{z=\omega_j} K(z_1, z)\wp(2z) = \sum_{j=1}^{3}\operatorname*{Res}_{z=\omega_j}\frac{\omega}{4}\frac{(\zeta(z_1 + z) - \zeta(z_1 - z))}{(z - \omega_i)\wp'(z)}\wp(2z)dz dz_1 \\ &= \frac{\omega}{8}\left(2\sum_{j=1}^{3}\frac{\wp(\omega_i)\wp(z_0 - \omega_i)}{\wp''(\omega_i)} - \sum_{j=1}^{3}\frac{\wp(z_0 - \omega_i)^2}{\wp''(\omega_i)}\right)dz_1 \\ &= \frac{\omega}{8}\left(\frac{2\wp\wp''}{(\wp')^2} - \frac{(\wp'')^3}{(\wp')^4} + 10g_2\frac{\wp}{(\wp')^2} + 15g_3\frac{1}{(\wp')^2} + 11\right)dz_1 \end{aligned}$$

where $\eta_i \in \mathbb{C}$ and b are annihilated by the residues. Integrate the kernels K_j^i against $\omega_{1,1}$ to get

$$\omega_{1,1} = 0 \cdot T_0^1 + \frac{\omega}{8}T_0^2 + \frac{i\eta\omega}{4\pi}T_0^3 + \frac{1}{8}T_1^1 + \frac{\eta}{4}T_1^2 + \frac{g_2\omega^2 - 12\eta^2}{48 i\pi}T_1^3.$$

The primary part uses only T_0^k and yields

$$F_1 = \frac{1}{8}\log t_2 + f(t_3), \quad f'(t_3) = \frac{i\pi}{2}\left(\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right)$$

which is obtained from $\omega_{1,1}$ since $\frac{\partial F_1}{\partial t_2} = \frac{1}{8t_2} = \frac{\omega}{8}$ agrees with the coefficient of T_0^2 and $\frac{\partial F_1}{\partial t_3} = \frac{i\pi}{2}\left(\frac{1}{24} - \sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right) = \frac{i\eta\omega}{4\pi}$ agrees with the coefficient of T_0^2.

□

2.10 General theory

In the preceding sections, we have investigated the construction of a global spectral curve producing the ancestor potential of a Frobenius manifold by topological recursion in some examples or assuming some additional properties of the curve defined by Dubrovin's superpotential. In the present section, we begin with the data of a semi-simple Frobenius manifold, and produce a global curve in a general setup not coming from the superpotential but rather from a family of curves built out of the reflection group generated by the monodromies of the solutions of our Fuchsian system. In particular, it explains how our setup is related to the spectral curve built by Milanov in [73].

2.10.1 Spectral curves from reflection group

Here we define a family of spectral curves associated to the reflection group defined by the monodromies of the Fuchsian system given by Equation (2.2.17). The spectral curve defined by Dubrovin's superpotential is a particular point in this family.

Definition 2.10.1. For any $\gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{C}^n$, let us define a function $\phi^{[\gamma]} : \mathbb{C}\backslash\{L_i\} \to \mathbb{C}$ by

$$\phi^{[\gamma]}(\lambda; u) := \sum_{i=1}^{\mu} \gamma_i \phi^{(i)}(\lambda; u) \tag{2.10.98}$$

where $\phi^{(i)}$ are solutions to Equation 2.2.17 defined as in section 2.2.4.

We define the corresponding function $p^{[\gamma]}$ analytic on $\mathbb{C}\backslash\{L_i\}$ by

$$p^{[\gamma]}(\lambda, u) := \frac{\sqrt{2}}{1-d} \left(\phi^{[\gamma]}\right)^T (U - \lambda)\Psi \mathbb{1}. \tag{2.10.99}$$

Finally, let us define the pairing

$$\forall (\gamma, \gamma') \in \mathbb{C}^{2n}\,,\ (\gamma|\gamma') := -2 \sum_{i,j} \gamma_i G^{ij} \gamma'_j. \tag{2.10.100}$$

The main property of these functions is that $\phi^{[\gamma]}$ has the local behavior

$$\phi_j^{[\gamma]} = \frac{\sum_{i=1}^n \gamma_i G^{ij}}{\sqrt{u_j - \lambda}} + O(1) \text{ for } \lambda \to u_j, \qquad j = 1, \ldots, n; \tag{2.10.101}$$

$$\phi_a^{[\gamma]} = \sum_{i=1}^n \gamma_i G^{ij} \sqrt{u_j - \lambda} \cdot O(1) \text{ for } \lambda \to u_j, \qquad a \neq j; a, j = 1, \ldots, n \tag{2.10.102}$$

and $p^{[\gamma]}$ has a local behavior for $\lambda \to u_i$ given by

$$p^{[\gamma]}(\lambda, u) = p^{[\gamma]}(u_i, u) + \sum_{j=1}^n \gamma_j G^{ji} \Psi_{i,\mathbb{1}} \sqrt{2(u_i - \lambda)} + O(u_i - \lambda), \quad i = 1, \ldots, n. \tag{2.10.103}$$

Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{C}^n$. We have $\phi^{[e_i]}(\lambda; u) = \phi^{(i)}(\lambda; u)$.

Remark 2.10.2. Dubrovin's standard superpotential defined in Section 2.2.4 is obtained by considering the particular case $\gamma_j = \sum_{i=1}^n G_{ij}$.

From now on, we assume that the reflections $\mathcal{R}_i$ generate a finite group W. Infinite families of Frobenius manifolds with finite group W are given in [27].

For any $\gamma \in \mathbb{C}^n$, one can define a Riemann surface $\mathcal{D}^{[\gamma]}$ as a cover $\lambda^{[\gamma]} : \mathcal{D}^{[\gamma]} \to \mathbb{C}$ where $\lambda^{[\gamma]}(\tilde{p}^{[\gamma]}, u)$ is the inverse function to $\tilde{p}^{[\gamma]}$ defined out of $p^{[\gamma]}$ by resolving the zeroes of $dp^{[\gamma]}$ as in Section 2.2.4. It is important to remark that the construction of $D^{[\gamma]}$ as a branched cover of $\mathbb{C}\backslash\{L_i\}$ does not depend on γ but rather on a choice of gluing for the different sheets—see Remark 2.10.3 for a discussion of these choices. In this section, we consider the most naive gluing and the resulting spectral curve.

We consider the reflection $\mathcal{R}_i$ as a linear map on the space $\mathbb{C}^n$ changing the coordinates of the vectors by the following rule:

$$\gamma_j \to \begin{cases} \gamma_j & \text{if} \quad j \neq i \\ \gamma_i + (\gamma|e_i) & \text{if} \quad j = i \end{cases} . \tag{2.10.104}$$

We denote $w\gamma$ the image of a vector γ under the action of an element $w \in W$.

We build the spectral curve $\mathcal{D}^{[\gamma]}$ as follows. A point $z \in \mathcal{D}^{[\gamma]}$ is defined by a pair $(\lambda, p) \in \hat{D} \times \mathbb{C}$ such that $p^{[\gamma]}(\lambda, u) = p$. By definition of $p^{[\gamma]}(\lambda, u)$, this defines a cover of $\hat{D}$ with ramification points in the fibres above the critical values $u_1, \ldots, u_n$. We now glue in the most naive way, meaning that each point in the fibre above any of the u_i is a simple ramification point. Let us now describe this sheeted cover.

Our spectral curve is obtained by analytic continuation of $p^{[\gamma]}$ from $\hat{D}$ through the (pre-images of the) cuts L_i. Each copy of $\hat{D}$ is then viewed as a sheet of a branched cover of the λ plane. We can analytically continue $p^{[\gamma]}$ through L_i seen as a cut on a Riemann surface giving rise to a new function of λ

$$p^{[\mathcal{R}_i\gamma]}(\lambda) := \mathcal{R}_i p^{[\gamma]}(\lambda; u) := \frac{\sqrt{2}}{1-d} \left(\mathcal{R}_i \phi^{[\gamma]}\right)^T (U - \lambda)\Psi \mathbb{1} \tag{2.10.105}$$

where

$$\mathcal{R}_i \phi^{[\gamma]}(\lambda; u) = \sum_{j=1}^{\mu} \gamma_j \mathcal{R}_i \phi^{(j)}(\lambda; u) = \phi^{[\gamma]}(\lambda; u) + (\gamma|e_i)\phi^{[e_i]}(\lambda; u). \tag{2.10.106}$$

In other terms, we glue along the images of the cut L_i the sheets given by $p^{[\mathcal{R}_i\delta]}(\lambda)$ and $p^{[\delta]}(\lambda)$ for all δ in the W-orbit of the initial vector γ.

The above procedure defines a $|W|$ sheeted cover $\mathcal{D}$ of the λ plane such that the fiber above a point λ is $\{p^{[w\gamma]}(\lambda, u)\}_{w\in W}$. The different sheets of this cover can thus be labelled by elements $w \in W$ and we denote by $\lambda^{[w]}$ the unique point in the fiber above a generic point λ belonging to the sheet labelled by w. We define by p the unique function on $\mathcal{D}$ such that

$$\forall w \in W\,,\ p\left(\lambda^{[w]}\right) = p^{[w\gamma]}(\lambda, u) \tag{2.10.107}$$

for a generic λ.

This cover is branched over all the points in the fibres above the points u_i, $i = 1, \ldots, n$, and a ramification point above u_i joins the sheets labelled by w and $\mathcal{R}_i w$ for some $w \in W$.

This branched cover is our spectral curve. It has $|W|/2$ simple ramification points over u_i, $i = 1, \dots, n$. We denote by $u_i^{[w]}$, $w \in W$, the point in the fiber above u_i such that $p\left(u_i^{[w]}\right) = p^{[w\gamma]}(u_i)$. This notation is ambiguous, so we denote by W_i the minimal set such that

$$\lambda^{-1}(u_i) = \{u_i^w \mid w \in W_i\}. \tag{2.10.108}$$

By definition, one has the important relation

$$\forall w \in W_i\,,\; p\left(\lambda^{[w]}\right) - p\left(\lambda^{[\mathcal{R}_i w]}\right) = \frac{(w\gamma|e_i)}{(\gamma|e_i)}\left[p\left(\lambda^{[Id]}\right) - p\left(\lambda^{[\mathcal{R}_i]}\right)\right]. \tag{2.10.109}$$

Thanks to our assumption of finiteness of W, $\mathcal{D}$ can be compactified by introducing ramification points of higher order above ∞.

The order of these ramification points above ∞ deserves some investigation. Since the reflection group W is finite, then the ramification index of such a point is equal to the Coxeter number $h(W)$, i.e. the order of a Coxeter transformation. In such a case, there exists a longest positive root $\sum_i m_i\alpha_i$ (reminding that the set $\{\alpha_i\}$ is a set of simple roots) and the Coxeter number is equal to $1 + \sum_i m_i$.

Let us recall as well that a Coxeter transformation is a product of all simple reflections. The different order for this product leading to different transformations, all with the same order. The different Coxeter transformations correspond to the different points in the fiber above ∞. In the case of an infinite group, this order is infinite and the different ramification points in the fiber above ∞ correspond to different conjugacy classes of Coxeter transformations.

We now have a Riemann surface Σ which is a branched cover of the λ plane. In our case, when the group is finite, its genus is given by the Riemann-Hurwitz formula:

$$2 - 2g(\Sigma) = 2|W| - \frac{|W|}{2}n - (h(W) - 1)\frac{|W|}{h(W)}. \tag{2.10.110}$$

Remark 2.10.3. We have built **a** curve using this procedure. There exist two ways of changing the cover built in this way. First by specifying some particular value for the vector γ. Second, by choosing a different gluing procedure for building the cover: for each point in the fibre above a critical value u_i, one can decide whether it is a ramification point or not. We followed here the most naive procedure where all the points are ramification points, recovering the spectral curve built by Milanov in the case of simple singularities [73]. This procedure is the most general but gives the highest possible genus of the curve.

In the preceding sections, we had chosen a particular value of γ prescribed by Dubrovin's construction as well as the simplest possible curve by considering covers where only one point in the fibre above each critical value is a ramification point. This leads to the lowest genus spectral curve possible but requires one to study the gluing procedure carefully case by case.

2.10.2 Global topological recursion and correlation functions of a CohFT

Global topological recursion

We remark that we are not in the cases discussed in the preceding sections since the spectral curve has $|W|/2$ ramification points in the fibre above one critical value. This implies that the topological recursion has to be modified a little in order to take the right form.

Definition 2.10.4. We define the correlation functions defined by the global topological recursion applied to $\mathcal{D}$ as the differential forms defined by induction through

$$\omega_{g,k}(z_1,\ldots,z_k) = \sum_{i=1}^{n}\sum_{w\in W_i}\operatorname*{Res}_{z\to u_i^{[w]}} \frac{\int_z^{\sigma_{i,w}(z)} B(z_1,\cdot)}{2\left(\omega_{0,1}(z)-\omega_{0,1}(\sigma_i(z))\right)}\Bigg[\omega_{g-1,k+1}(z,\sigma_{i,w}(z),z_2,\ldots,z_k) + \sum_{A\sqcup B=\{2,\ldots,k\}}\sum_{h=0}^{g}\omega_{h,|A|+1}(z,\vec{z}_A)\omega_{g-h,|B|+1}(\sigma_{i,w}(z),\vec{z}_B)\Bigg],$$

where

$$\omega_{0,1}(z) := p(z)d\lambda(z), \tag{2.10.111}$$

$$\omega_{0,2}(z_1,z_2) = \sum_{w\in W}(\gamma|w\gamma)B(z_1,z_2) \tag{2.10.112}$$

and $\sigma_{i,w}$ is the local involution exchanging the two sheets meeting at u_i^w. In the right hand side, all the contributions involving a factor of $\omega_{0,1}$ are set to 0.

Note that, in this recursion, the recursion kernel does not involve $\omega_{0,2}$ itself but rather B. This might seem to break the usual symmetry between the different arguments of $\omega_{g,k}$ but, as we shall see in the next section, it is not the case.

From global to local

In [37], the correspondence between topological recursion and CohFT was discussed only at the local level. In order to match the correlation functions defined by the global topological recursion with those of the CohFT, let us translate the global recursion into a local one written in terms of integrals in the λ-plane around the critical values u_i.

Lemma 2.10.5. *The global topological recursion on the spectral curve $\mathcal{D}$ with $x=\lambda$, $y=p$ and $B(p_1,p_2)$ is equivalent to the local recursion with local spectral curve*

$$\forall i=1,\ldots n\,,\ \omega_{0,1}^{[i]}(\lambda) = \Delta_{i,\lambda}p(\lambda^{[Id]})d\lambda \tag{2.10.113}$$

and

$$\forall i,j=1,\ldots n\,,\ \omega_{0,2}^{[i,j]}(\lambda_1,\lambda_2) = \Delta_{i,\lambda_1}\Delta_{j,\lambda_2}\omega_{0,2}(\lambda_1^{[Id]},\lambda_2^{[Id]}) \tag{2.10.114}$$

where

$$\Delta_{i,\lambda} f(\lambda^{[w]}) = \frac{f(\lambda^{[w]}) - f(\lambda^{[\mathcal{R}_i w]})}{2} \tag{2.10.115}$$

for a meromorphic form f on $\mathcal{D}$.

In other words, the discontinuities

$$\omega_{g,k}^{[i_1,\dots,i_k]}(\lambda_1,\dots,\lambda_k) := \prod_{j=1}^{k} \Delta_{i_j,\lambda_j}\omega_{g,k}(\lambda_1^{[Id]},\dots,\lambda_k^{[Id]}) \tag{2.10.116}$$

of the correlation functions $\omega_{g,k}$ produced by the global recursion satisfy the corresponding local recursion.

Proof. It is first important to note that

$$\Delta_{i,\lambda_1}\omega_{g,k+1}(\lambda^{[w]},\lambda_1^{[w_1]},\dots,\lambda_k) = \frac{(w\gamma|e_i)}{(\gamma|e_i)}\Delta_{i,\lambda_1}\omega_{g,k+1}(\lambda^{[Id]},\lambda_1^{[w_1]},\dots,\lambda_k). \tag{2.10.117}$$

This is proved by induction and follows from the definition of $\omega_{0,2}$ in terms of the Bergman kernel. This property allows us to rewrite the topological recursion in a local version where one sums only over one of the ramification points in the fiber above each of the critical values u_i.

Writing $z = \lambda^{[w]}$, one can rewrite the term $\operatorname{Res}_{z\to u_i^{[w]}}$ as a residue when $\lambda \to u_i$ in the following way:

$$\operatorname*{Res}_{\lambda^{[w]}\to u_i^{[w]}} = 2\operatorname*{Res}_{\lambda\to u_i}. \tag{2.10.118}$$

This gives

$$\begin{aligned}
&\omega_{g,k}(z_1,\dots,z_k) = \\
&\sum_{i=1}^{n}\sum_{w\in W_i}\operatorname*{Res}_{\lambda\to u_i}\frac{\int_{\lambda^{[w]}}^{\lambda^{[\mathcal{R}_i w]}} B(z_1,\cdot)}{2\Delta_{i,\lambda}\omega_{0,1}(\lambda^{[w]})}\Delta_{i,\lambda}\Delta_{i,\lambda'}\Bigg[\omega_{g-1,k+1}(\lambda^{[w]},\lambda'^{[w]},z_2,\dots,z_k) \\
&\sum_{A\sqcup B=\{2,\dots,k\}}\sum_{h=0}^{g}\omega_{h,|A|+1}(\lambda^{[w]},\vec{z}_A)\omega_{g-h,|B|+1}(\lambda'^{[w]},\vec{z}_B)\Bigg]\Bigg|_{\lambda'=\lambda}.
\end{aligned}$$

Plugging property (2.10.117) into this equation, the global recursion reads

$$\begin{aligned}
&\omega_{g,k}(z_1,\dots,z_k) = \\
&\sum_{i=1}^{n}\operatorname*{Res}_{\lambda\to u_i}\frac{\sum_{w\in W_i}\frac{(w\gamma|e_i)}{(\gamma|e_i)}\int_{\lambda^{[w]}}^{\lambda^{[\mathcal{R}_i w]}} B(z_1,\cdot)}{2\Delta_{i,\lambda}\omega_{0,1}(\lambda^{[Id]})}\Delta_{i,\lambda}\Delta_{i,\lambda'}\Bigg[\omega_{g-1,k+1}(\lambda^{[Id]},\lambda'^{[Id]},z_2,\dots,z_k) \\
&\sum_{A\sqcup B=\{2,\dots,k\}}\sum_{h=0}^{g}\omega_{h,|A|+1}(\lambda^{[Id]},\vec{z}_A)\omega_{g-h,|B|+1}(\lambda'^{[Id]},\vec{z}_B)\Bigg]\Bigg|_{\lambda'=\lambda}.
\end{aligned}$$

Finally, using the fact that $2\sum_{w\in W_i} = \sum_{w\in W}$ in the expression above, one gets

$$\omega_{g,k}(z_1,\dots,z_k) = \frac{1}{4}\sum_{i=1}^{n}\operatorname*{Res}_{\lambda\to u_i}\frac{\Delta_{i,\lambda}\int^{\lambda^{[Id]}}\omega_{0,2}(z_1,.)}{\Delta_{i,\lambda}\omega_{0,1}(\lambda^{[Id]})}\Delta_{i,\lambda}\Delta_{i,\lambda'}\left[\omega_{g-1,k+1}(\lambda^{[Id]},\lambda'^{[Id]},z_2,\dots,z_k)\right.$$
$$\left.\sum_{A\sqcup B=\{2,\dots,k\}}\sum_{h=0}^{g}\omega_{h,|A|+1}(\lambda^{[Id]},\vec{z}_A)\omega_{g-h,|B|+1}(\lambda'^{[Id]},\vec{z}_B)\right]\Bigg|_{\lambda'=\lambda}.$$

Acting with the operators $\prod_{j=1}^{k}\Delta_{i_j,\lambda_j}$ on both sides proves the lemma. □

Identification of the local initial data with a CohFT

Now that we have derived a local topological recursion equivalent to the global one, one only needs to identify its initial data with the data of a CohFT following the dictionary of [37]. For this purpose, we will follow exactly the same steps as in the preceding sections. Let us first state precisely the identification that we want to find since it is slightly different from the usual setup where one has only one ramification point in each fiber and a specific value for γ.

First of all, let us remind that, according to [46], the Laplace transform of the local two point function reads

$$\frac{1}{2\pi\sqrt{\zeta_1\zeta_2}}\iint\limits_{\substack{\lambda_1-u_i\in\mathbb{R}\\ \lambda_2-u_j\in\mathbb{R}}}\omega_{0,2}^{[ij]}(\lambda_1,\lambda_2)e^{\frac{\lambda_1-u_i}{\zeta_1}+\frac{\lambda_2-u_j}{\zeta_2}} = \frac{\sum_{k=1}^{n} f(\zeta_1)_k^i\, f(\zeta_2)_k^j}{\zeta_1+\zeta_2}, \tag{2.10.119}$$

where

$$f(\zeta)_k^i := -\frac{1}{\sqrt{2\pi\zeta}}\int\limits_{\lambda_1-u_i\in\mathbb{R}}\frac{\omega_{0,2}^{[ij]}(\lambda_1,\lambda_2)}{d\sqrt{-2\lambda_2+2u_j}}\Bigg|_{\lambda_2=u_k} e^{\frac{\lambda_1-u_i}{\zeta}}. \tag{2.10.120}$$

In these terms, the identification consists in showing that

$$f(\zeta)_k^i = \sum_{j=1}^{n}(\gamma|e_j)G_{ji}R(\zeta)_k^i \tag{2.10.121}$$

and

$$\sum_{j=1}^{n}(\gamma|e_j)G_{ji}\sum_{k=1}^{n}R(\zeta)_k^i\Delta_k^{-\frac{1}{2}} = \frac{1}{\sqrt{2\pi\zeta}}\int\limits_{\lambda-u_i\in\mathbb{R}}\omega_{0,1}^{[i]}(\lambda)\cdot e^{\frac{\lambda-u_i}{\zeta}} \tag{2.10.122}$$

where $R(\zeta)$ is the R-matrix defining the CohFT we started from for deriving our Fuchsian system.

Note that the proof of Equation (2.10.122) is a simple verbatim of the proof of Section 2.5 by replacing $\phi^{(i)}$ by $\sum_{j=1}^{n}(\gamma|e_j)G_{ji}\phi^{(i)}$. A corollary of this identification is the following theorem:

Theorem 2.10.6. *The correlation functions $\omega_{g,k}$ produced by the global recursion generate the correlation functions of the original CohFT through*

$$\omega_{g,k}(\lambda(z_1)^{[Id]},\ldots,\lambda(z_k)^{[Id]}) = \sum_{\substack{i_1,\ldots,i_k\\ j_1,\ldots,j_k\\ d_1,\ldots,d_k}} \prod_{l=1}^{k} [(\gamma|e_j)G_{ji}] \int_{\overline{\mathcal{M}}_{g,k}} \alpha_{g,k}(e_{i_1},\ldots,e_{i_k}) \prod_{l=1}^{k} \psi_l^{d_l} d\left(\left(\frac{d}{dx}\right)^{d_l} \xi_{i_l}(z_l)\right).$$

Compatibility condition and homogeneity

Let us now remark the compatibility between Equation (2.10.121) and Equation (2.10.122) can be written as the usual compatibility condition for the Bergman kernel by considering all the ramification points, i.e.

$$\eta(z) = \sum_{i=1}^{n} \sum_{w\in W_i} \operatorname*{Res}_{z'=u_i^{[w]}} \frac{dp}{d\lambda}(z')B(z,z') + \operatorname*{Res}_{z'=z} \frac{dp}{d\lambda}(z')B(z,z') \tag{2.10.123}$$

is invariant any local involution $\sqrt{\lambda(z)-u_i} \to -\sqrt{\lambda(z)-u_i}$.

Finally, it is an easy exercise to prove the homogeneity at the level of $\omega_{0,2}$ by using Rauch's variational formula as in Section 2.6.

2.11 Frobenius manifolds of rank 2

In this Section we explicitly construct global spectral curves for two rank 2 CohFTs. We begin with the prepotential $F(t_1,t_2)$ which one uses to produce the structure of a Frobenius manifold. We follow Dubrovin's construction to produce a superpotential. In both cases we need to vary the construction slightly due to degeneracy of the Gauss-Manin system. The two examples satisfy the conditions of Theorem 2.6.1 and hence topological recursion produces the CohFT associated to the Frobenius manifold. Note that although the two examples are of genus zero, they do not satisfy the conditions of Theorem 2.5.1.

2.11.1 Gromov-Witten invariants of $\mathbb{CP}^1$

$$F = \frac{t_1^2 t_2}{2} + e^{t_2}, \quad E = t_1\partial_{t_1} + 2\partial_{t_2}, \quad E\cdot F = 2F\ (+t_1^2)$$

$$\eta^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{-t_2/4} & e^{t_2/4} \\ -ie^{-t_2/4} & ie^{t_2/4} \end{pmatrix}$$

$$\mu = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}, \quad V = \Psi\mu\Psi^{-1} = \frac{i}{2}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$u_1 = t_1 + 2e^{t_2/2}, \quad u_2 = t_1 - 2e^{t_2/2}$$

$$V_1 = \partial_{u_1}\Psi\cdot\Psi^{-1} = \frac{1}{u_1-u_2}\cdot V = -V_2$$

The vector fields ϕ given in canonical coordinates satisfy (2.2.17) which is equivalent to the Fuchsian system:

$$(U-\lambda)\partial_\lambda\phi = (\tfrac{1}{2}+V)\phi \tag{2.11.124}$$

and

$$\partial_{u_i}\phi = \left(-\frac{B_i}{\lambda-u_i}+V_i\right)\phi, \quad B_i = -E_i(\tfrac{1}{2}+V), \quad V_i = \partial_{u_i}\Psi\cdot\Psi^{-1}. \tag{2.11.125}$$

This has general solution

$$\phi = \frac{c_1}{(u_1-u_2)^{1/2}}\begin{pmatrix}\sqrt{\dfrac{u_2-\lambda}{u_1-\lambda}}\\ -i\sqrt{\dfrac{u_1-\lambda}{u_2-\lambda}}\end{pmatrix} + \frac{c_2}{(u_1-u_2)^{1/2}}\begin{pmatrix} i\\ 1\end{pmatrix}.$$

We choose the solution $c_1 = 1$, $c_2 = 0$. Since $d = 1$ (2.2.25) does not apply. Nevertheless, ϕ is the gradient of p so we can calculate

$$dp(\lambda,u) = \frac{1}{u_1-u_2}\left(\sqrt{\frac{u_2-\lambda}{u_1-\lambda}}du_1 + \sqrt{\frac{u_1-\lambda}{u_2-\lambda}}du_2\right).$$

In this example, we will also go through the equivalent treatment in terms of flat coordinates for the pencil of metrics. The vector fields ϕ are gradient vector fields of the flat coordinates

$$\phi_i = \Psi_{i\alpha}\eta^{\alpha\beta}\partial_\beta x(t_1-\lambda, t_2, ..., t_n)$$

for the pencil of metrics $g-\lambda\eta$ where

$$g^{\alpha\beta} = \begin{pmatrix} 2e^{t_2} & t_1\\ t_1 & 2\end{pmatrix}.$$

The flat coordinates for the pencil are of the form $x(t_1-\lambda, t_2, ..., t_n)$ so it is enough to consider the case $\lambda = 0$, i.e. find flat coordinates for the intersection form. These are given by solutions of the Gauss-Manin system of linear differential equations ((5.9) in [28]):

$$g^{\alpha\gamma}\partial_\beta\xi_\gamma + \sum_\gamma(\tfrac{1}{2}-\mu_\gamma)c^{\alpha\gamma}\xi_\gamma = 0, \quad \xi_\beta = \partial_\beta x.$$

$$\begin{array}{ccccccc} 2e^{t_2}\partial_1^2 x & + & t_1\partial_1\partial_2 x & + & 0 & = & 0\\ 2e^{t_2}\partial_1\partial_2 x & + & t_1\partial_2^2 x & + & e^{t_2}\partial_1 x & = & 0\\ t_1\partial_1^2 x & + & 2\partial_1\partial_2 x & + & \partial_1 x & = & 0\\ t_1\partial_1\partial_2 x & + & 2\partial_2^2 x & + & 0 & = & 0\end{array}$$

$$\Rightarrow x = c_1\cdot\arccos\left(\tfrac{1}{2}t_1e^{-t_2/2}\right) + c_2\cdot t_2$$

Choose

$$p = i\arccos\left(\tfrac{1}{2}(t_1-\lambda)e^{-t_2/2}\right)$$

$$\lambda = t_1 - 2e^{t_2/2}\cos(-ip) = t_1 - e^{t_2/2}(e^p+e^{-p})$$

Note that the critical points of λ are indeed $t_1 \pm 2e^{t_2/2} = u_{1/2}$. It was proven in [37] that the curve $p = \ln z$, $\lambda = a + b(z+1/z)$ does indeed produce the CohFT for Gromov-Witten invariants of $\mathbb{CP}^1$.

2.11.2 Discrete surfaces

The 2-dimensional Hurwitz-Frobenius manifold $H_{0,(1,1)}$ of double branched covers of the sphere, with two branch points and unramified at infinity was defined by Dubrovin [27]. Its potential is

$$F = \frac{t_1^2 t_2}{2} + \frac{1}{2} t_2^2 \log t_2, \quad E = t_1 \partial_{t_1} + 2t_2 \partial_{t_2}, \quad E \cdot F = 4F \; (+t_2^2)$$

$$\eta^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} t_2^{1/4} & t_2^{-1/4} \\ -it_2^{1/4} & it_2^{-1/4} \end{pmatrix}$$

$$\mu = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad V = \Psi \mu \Psi^{-1} = \frac{i}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$u_1 = t_1 + 2t_2^{1/2}, \quad u_2 = t_1 - 2t_2^{1/2}$$

$$V_1 = \partial_{u_1} \Psi \cdot \Psi^{-1} = \frac{-1}{u_1 - u_2} \cdot V = -V_2$$

The general solution of (2.11.124) and (2.11.125) is

$$\phi = \frac{c_1}{(u_1 - u_2)^{1/2}} \begin{pmatrix} \sqrt{\dfrac{u_2 - \lambda}{u_1 - \lambda}} \\ i\sqrt{\dfrac{u_1 - \lambda}{u_2 - \lambda}} \end{pmatrix} + \frac{c_2}{(u_1 - u_2)^{1/2}} \begin{pmatrix} i \\ 1 \end{pmatrix}.$$

The solutions of Dubrovin described in (2.2.19)-(2.2.22) yield $\phi^{(1)} = \phi^{(2)}$ hence G^{ij} is degenerate. We use one of the solutions $\phi = \phi^{(1)}$ in (2.2.25) to get

$$p(\lambda, u) = \frac{t_2^{1/4}}{2} \frac{\sqrt{(u_1 - \lambda)(u_2 - \lambda)}}{(u_1 - u_2)^{1/2}} \begin{pmatrix} 1 & i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \frac{1}{2} \sqrt{(u_1 - \lambda)(u_2 - \lambda)}.$$

This corresponds to the spectral curve $\lambda = t_1 + z + t_2/z$, $p = z - t_2/z$ which arises from the well-studied Hermitian matrix model with Gaussian potential hence discrete maps [52] and was shown to correspond to the given CohFT in [8].

3

Primary Invariants of Hurwitz Frobenius Manifolds

Abstract

Hurwitz spaces parameterizing covers of the Riemann sphere can be equipped with a Frobenius structure. In this chapter, we recall the construction of such Hurwitz Frobenius manifolds as well as the correspondence between semisimple Frobenius manifolds and the topological recursion formalism. We then apply this correspondence to Hurwitz Frobenius manifolds by explaining that the corresponding primary invariants can be obtained as periods of multidifferentials globally defined on a compact Riemann surface by topological recursion. Finally, we use this construction to reply to the following question in a large class of cases: given a compact Riemann surface, what does the topological recursion compute?

3.1 Introduction

Consider a diagonal flat metric on a complex manifold M with local coordinates $u = (u_1, ..., u_n)$

$$ds^2 = \sum_{i=1}^{N} \eta_i(u) du_i^2 \quad \text{is flat,} \tag{3.1.1}$$

generated by a potential $H : M \to \mathbb{C}$

$$\eta_i(u) = \partial_{u_i} H, \quad i = 1, ..., N. \tag{3.1.2}$$

Metrics satisfying (3.1.1) and (3.1.2) are known as Darboux-Egoroff metrics [27]. Condition (3.1.1) is equivalent to a nonlinear PDE in $\eta_i(u)$ which gives vanishing of the Riemann curvature tensor $R_{ijkl} = 0$. The PDE becomes integrable when condition (3.1.2) is added. By a metric we mean a smooth family of complex non-degenerate symmetric bilinear forms on the tangent space $T_m M$, so in particular it is not Riemannian.

Analogous to K. Saito's construction [84] of flat coordinates on unfolding spaces of singularities, Dubrovin [27] and Krichever [68] produced beautiful families of Darboux-Egoroff metrics on moduli spaces of pairs (Σ, x) consisting of an algebraic curve Σ equipped with a meromorphic function $x : \Sigma \to \mathbb{C}$. Such a pair (Σ, x) is a point in a Hurwitz space $H_{g,\mu}$ which parametrises covers $x : \Sigma \to \mathbb{P}^1$ of genus g with points above infinity marked and with fixed ramification profile $(\mu_1, \ldots, \mu_d)$. Further, choose a symplectic basis of cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i=1,\ldots,g}$ on Σ to define a point in a cover $\widetilde{H}_{g,\mu}$ of a Hurwitz space. Namely, $\widetilde{H}_{g,\mu}$ consists of the data of a point in a Hurwitz space together with the data of a Torelli marking. One goes from one sheet of the cover $\widetilde{H}_{g,\mu}$ to another one through the action of modular transformations $Sp(2g, \mathbb{Z})$.

Definition 3.1.1. Given $(\Sigma, x, \{\mathcal{A}_i, \mathcal{B}_i\}_{i=1,\ldots,g})$ define a set of generalised contours $\mathcal{D}$ on Σ as follows. Choose representatives for $\{\mathcal{A}_i, \mathcal{B}_i\}_{i=1,\ldots,g}$ in $H_1(\Sigma \setminus x^{-1}(\infty))$ and choose a set of relative homology classes $\gamma_i \in H_1(\Sigma, x^{-1}(\infty))$, $i = 2, ..., d$ such that $\gamma_i \subset \Sigma \setminus \{\mathcal{A}_1, ...\mathcal{A}_g, \mathcal{B}_1, ..., \mathcal{B}_g\}$ runs from ∞_i to ∞_1 where the poles of x are given by $x^{-1}(\infty) = \{\infty_1, ..., \infty_d\}$ with respective orders $\{\mu_1, ..., \mu_d\}$. Let $\mathcal{C}_{\infty_i}$, $i = 1, ..., d$ be small circles around each pole ∞_i of x. Then define

$$\mathcal{D} = \{x\mathcal{A}_1, ..., x\mathcal{A}_g, \mathcal{B}_1, ..., \mathcal{B}_g\} \bigcup_{i=2,\ldots,d} \{\gamma_i, x\mathcal{C}_{\infty_i}\} \bigcup_{\substack{k = 1, ..., \mu_i - 1,\\ j = 1, ..., d}} \{x^{k/\mu_j}\mathcal{C}_{\infty_j}\}. \tag{3.1.3}$$

If x has only simple poles, so each $\mu_i = 1$, then the contours are built out of classes in $H_1(\Sigma \setminus x^{-1}(\infty))$ and $H_1(\Sigma, x^{-1}(\infty))$. A contour $\mathcal{C}$ acts on a differential ω by $\omega \mapsto \int_{\mathcal{C}} \omega$, and by $x\mathcal{C}$ we mean $\omega \mapsto \int_{x\mathcal{C}} \omega := \int_{\mathcal{C}} x\omega$. We often enumerate the elements of $\mathcal{D}$ by $\mathcal{C}_\alpha \in \mathcal{D}$ for $\alpha = 1, ..., N = |\mathcal{D}|$. Note that $N = \dim H_{g,\mu}$—see (3.4.39).

Definition 3.1.2. On any compact Riemann surface $(\Sigma, \{\mathcal{A}_i\}_{i=1,\ldots,g})$ with a given set of $\mathcal{A}$-cycles, define a *Bergman kernel* $B(p, p')$ to be a symmetric bidifferential, i.e. a tensor product of differentials on $\Sigma \times \Sigma$, uniquely defined by the properties that it has a double pole on the diagonal of zero residue, double residue equal to 1, no further singularities and normalised by $\int_{p\in\mathcal{A}_i} B(p, p') = 0$, $i = 1, ..., g$. It satisfies the Cauchy property for any meromorphic function f on Σ

$$df(p) = \operatorname*{Res}_{p'=p} f(p')B(p, p'). \tag{3.1.4}$$

On Σ, choose a Bergman kernel $B(p, p')$ normalised to have zero periods over the $\mathcal{A}$-cycles in the Torelli marking $\{\mathcal{A}_i, \mathcal{B}_i\}$. For any $\mathcal{C}_\alpha \in \mathcal{D}$ define a *primary differential* by

$$\phi_\alpha(p) = \oint_{p'\in\mathcal{C}^*_\alpha} B(p, p') \tag{3.1.5}$$

where $\mathcal{C}^*_\alpha$ is a cycle dual to $\mathcal{C}_\alpha$ defined in section 4. Each primary differential is locally holomorphic on $\Sigma \setminus x^{-1}(\infty)$.

Denote by $\mathcal{P}_i \in \Sigma$ the finite critical points of x, i.e. $dx(\mathcal{P}_i) = 0$. For a generic point in $\widetilde{H}_{g,\mu}$ the critical points of x are simple and the critical values $u_i = x(\mathcal{P}_i)$, $i = 1, ..., N$ of x are local coordinates in the open dense domain of $\widetilde{H}^s_{g,\mu} \subset \widetilde{H}_{g,\mu}$ defined by $u_i \neq u_j$ for $i \neq j$ and $\{\mathcal{A}_1, ...\mathcal{A}_g, \mathcal{B}_1, ..., \mathcal{B}_g\}$ avoid $x^{-1}(\infty)$.

Define a metric on $\widetilde{H}^s_{g,\mu}$ by

$$\eta = \sum_{i=1}^{N} du_i^2 \cdot \operatorname*{Res}_{\mathcal{P}_i} \frac{\phi \cdot \phi}{dx} \tag{3.1.6}$$

for any choice of primary differential $\phi = \phi_\alpha$ on Σ obtained from $\mathcal{C}_\alpha \in \mathcal{D}$ via (3.1.5).

Theorem 3.1.3 (Dubrovin [27]). *(i) The metric η defined in* (3.1.6) *is flat with local flat coordinates given by*

$$t_\beta = \int_{\mathcal{C}_\beta} \phi, \quad \mathcal{C}_\beta \in \mathcal{D}, \quad \beta = 1, ..., N \tag{3.1.7}$$

i.e. the metric is constant with respect to the coordinates t_β.

(ii) The flat metric η forms part of a Frobenius manifold structure on $\widetilde{H}_{g,\mu}$ with multiplication on the tangent space of $\widetilde{H}^s_{g,\mu}$ defined using the local basis of vector fields ∂_{u_i} by

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij} \partial_{u_i}. \tag{3.1.8}$$

The theorem was proven by Dubrovin in [27] using a definition of primary differential via deformations of $y_\alpha dx$—see Lemma 3.4.5. As stated here we use an equivalent definition of primary differential (3.1.5) proven by Shramchenko [89].

Recall that a Frobenius manifold M comes equipped with a a flat metric η together with a commutative, associative product $\cdot$ on its tangent space satisfying the compatibility condition $\eta(u \cdot v, w) = \eta(u, v \cdot w)$ for all $u, v, w \in T_pM$. Associated to each semi-simple point p of a Frobenius manifold M is a cohomological field theory [53, 57, 71, 91] defined on $(H, \eta) = (T_pM, \eta|_{T_pM})$, which is a sequence of S_n-equivariant maps

$$I_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

that satisfy gluing conditions on boundary divisors in $\overline{\mathcal{M}}_{g,n}$ given explicitly in Section 3.2.2. For any collection of vectors $v_1, ..., v_n \in T_pM$, the integral $\int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}(v_1 \otimes ... \otimes v_n) \in \mathbb{C}$ (which is a function of $p \in M$) is known as a primary invariant of M.

Recently, [37] explained that one can compute the primary (and ancestor) invariants of a semisimple Frobenius manifold efficiently using the topological recursion procedure of [50]. In this chapter we apply this result to Hurwitz Frobenius manifolds.

The main observation is that just as the flat coordinates can be obtained as periods of a primary differential along cycles taken from $\mathcal{D}$ via (3.1.7), the primary invariants of the Hurwitz Frobenius manifolds can also be obtained as periods along cycles taken from $\mathcal{D}$. Since we need multiple insertions of vectors into the primary invariants, we need to take periods of symmetric *multidifferentials* on Σ which are tensor products of differentials on $\Sigma^n = \Sigma \times ... \times \Sigma$.

Theorem 3.1.4. *Given a point $p = (\Sigma, x, \{\mathcal{A}_i, \mathcal{B}_i\}) \in \widetilde{H}^s_{g,\mu}$ and a choice of primary differential dy_α that determines a Frobenius manifold structure on $\widetilde{H}^s_{g,\mu}$, there exist multidifferentials $\omega_{g,n}$ defined on Σ whose periods along contours in $\mathcal{D}$ give the primary invariants of the Frobenius manifold at p. More precisely, for flat coordinates $\{t_1, ..., t_N\}$, put $e_\alpha = \partial_{t_\alpha}$ and define the dual vector with respect to the metric* (3.1.6) *by $e^\alpha = \sum_\beta \eta^{\alpha\beta} e_\beta$. Then*

$$\int_{\mathcal{C}_{\alpha_1}} \cdots \int_{\mathcal{C}_{\alpha_n}} \omega_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}\Big(e^{\alpha_1} \otimes ... \otimes e^{\alpha_n}\Big)$$

where $\mathcal{C}_{\alpha_i}$ and $e_{\alpha_i} = \partial_{t_{\alpha_i}}$ are related by (3.1.7)

Theorem 3.1.4 is a consequence of the more general Theorem 3.1.6 that proves that the $\omega_{g,n}$ store all ancestor invariants of the Frobenius manifold, using a larger class of cycles than those in $\mathcal{D}$.

Remark 3.1.5. The statement and conclusion of Theorem 3.1.4 can be made for any point in $\widetilde{H}_{g,\mu}$ not just the semisimple points $\widetilde{H}^s_{g,\mu} \subset \widetilde{H}_{g,\mu}$. It would be interesting to prove the theorem with these weaker hypotheses. There are candidate multidifferentials, such as those defined in [11] where the zeros of dx are not required to be simple, or in the case of Dubrovin's superpotential, studied from the perspective of topological recursion in [35], which applies to *any* semi-simple Frobenius manifold, and where there may be multiple zeros of dx above a critical value.

The multidifferentials $\omega_{g,n}$ in Theorem 3.1.4 are obtained from the topological recursion procedure associated to the spectral curve (Σ, x, y_α, B) where $B = B(p, p')$ is the Bergman kernel defined in Definition 3.1.2 using the Torelli marking and y_α is a function defined on $\Sigma \setminus \{\mathcal{A}_i, \mathcal{B}_i\}$ such that locally $\phi_\alpha = dy_\alpha$. In general [50], the $\omega_{g,n}$ are a family of symmetric multidifferentials on the spectral curve that encode solutions of a wide array of problems from mathematical physics, geometry and combinatorics. By a spectral curve[1] we mean the data of (Σ, x, y, B) given by a Riemann surface Σ equipped with a meromorphic function x and a locally defined meromorphic function $y\colon \Sigma \to \mathbb{C}$ such that the zeros of dx given by $\{\mathcal{P}_1, ..., \mathcal{P}_N\}$ are simple and dy is analytic and non-vanishing on $\{\mathcal{P}_1, ..., \mathcal{P}_N\}$, and equipped with a symmetric bidifferential B on $\Sigma \times \Sigma$, with a double pole on the diagonal of zero residue, double residue equal to 1, and no further singularities. The spectral curve may be a collection of N open disks, known as a *local spectral curve*, because $\omega_{g,n}$ are defined using only local information about x, y and B around zeros of dx—see Section 3.3. On a compact spectral curve we relax the condition on y being globally defined, and instead require that dy is a locally defined meromorphic differential (a connection) ambiguous up to $dy + df(x)$ for any rational function f. This gives rise to a locally defined function y on Σ which is sufficient to apply topological recursion.

In [37, 72], it was proven that, starting from a semi-simple CohFT, or equivalently a semi-simple Frobenius manifold M, it is possible to compute its correlation functions by the topological recursion procedure applied to a specific local spectral curve:

$$\{\text{semisimple CohFT}\} \longrightarrow \{\text{topological recursion applied to a local spectral curve}\} \tag{3.1.9}$$

Under this correspondence whose details are reviewed in Section 3.3.1, the number of zeros of dx on the local spectral curve (Σ, x, y, B) is equal to the dimension N of the Frobenius manifold M. It was then proven in [35] that, under some additional assumptions on the Frobenius manifold M, it is possible to arrange that the image of (3.1.9) is a *compact* spectral curve producing the same correlation functions. This compact Riemann surface is given by Dubrovin's superpotential [27, 28] which is a family of compact Riemann surfaces parametrised by the semi-simple points of M and constructed out of flat coordinates of a pencil of metrics on M.

[1]The term spectral curve is inherited from the matrix model origin of this formalism. In the general formalism, this term is expected to make sense due to the probable existence of an associated integrable system.

We can now try to reverse the direction of the arrow in (3.1.9). Given a *compact* spectral curve, when does it lie in the image of (3.1.9), and can we reconstruct the corresponding CohFT (or, equivalently, the Frobenius manifold)? The following theorem answers this question. It begins with the observation that a compact spectral curve gives a point (Σ, x) in a Hurwitz space $H_{g,\mu}$.

Theorem 3.1.6. *Given a generic point $(\Sigma, x) \in H_{g,\mu}$, equip it with a bidifferential B normalised over a given set of $\mathcal{A}$-cycles, and choose $\mathcal{C} \in \mathcal{D}$ to define a locally defined function y on Σ by $dy(p) := \oint_{\mathcal{C}} B(p,p')$. The topological recursion procedure applied to the spectral curve (Σ, x, y, B) computes the ancestor invariants of the CohFT associated to Dubrovin's Frobenius manifold structure on the cover $\widetilde{H}^s_{g,\mu}$ via:*

$$\omega_{g,n}(p_1, ..., p_n) = \sum_{\substack{i_1,...,i_n \\ d_1,...,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}(e_{i_1}, \dots, e_{i_n}) \prod_{j=1}^{n} \psi_j^{d_j} \cdot \bigotimes_{j=1}^{n} V^{i_j}_{d_j}(p_j) \tag{3.1.10}$$

where $V^i_k(p)$, $i = 1, ..., N$, $k = 0, 1, ...$, are canonical differentials on Σ defined by (3.3.36) *in Section 3.3.1.*

The proof of Theorem 3.1.6—contained in Section 3.4.3—is a simple combination of results from [28, 37, 56, 90] which we recall below. The main tool in the proof is the map (3.1.9) from [37] which shows how topological recursion relates to Givental's construction [56] of the total ancestor potential associated to each semi-simple point of a Frobenius manifold. To apply the reverse construction of (3.1.9) one needs a specific relationship between the Bergman kernel B on the spectral curve and the R-matrix of the Frobenius manifold which is proven in [90].

Remark 3.1.7. Theorem 3.1.6 also answers the following question. Given a compact spectral curve, what does the topological recursion procedure compute? For a large class of spectral curves—where B and y are determined almost canonically by Σ and x—the answer is that it produces generating functions for ancestor invariants of a Hurwitz space to which the branched cover underlying the spectral curve belongs. In particular, it completes the picture drawn by Zhou in [96] for relating Frobenius manifolds and spectral curves.

Remark 3.1.8. Theorem 3.1.4 concerns only the primary invariants in (3.1.10), corresponding to $d_j = 0$, $j = 1, ..., n$. One can also construct generalised contours $\mathcal{C}_{\alpha,k} = p_k(x)\mathcal{C}_\alpha$, for $\mathcal{C}_\alpha \in \mathcal{D}$ and $p_k(x) = x^k + ...$ a monic polynomial of degree k in x, so that the ancestor invariants, corresponding to $d_j \geq 0$, appear as periods thus generalising Theorem 3.1.4:

$$\int_{\mathcal{C}_{\alpha_1,k_1}} ... \int_{\mathcal{C}_{\alpha_n,k_n}} \omega_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}\Big(e^{\alpha_1} \otimes ... \otimes e^{\alpha_n}\Big) \cdot \prod_{j=1}^{n} \psi_j^{k_j}. \tag{3.1.11}$$

Theorems 3.1.4 and 3.1.6 enable one to generate primary invariants and all ancestor invariants of $\widetilde{H}^s_{g,\mu}$ of all genera. Previously only genus 0 and genus 1 primary invariants were known. Theorems 3.1.4 and 3.1.6 also have applications to the topological recursion procedure. Using the generalised contours in $\mathcal{D}$ one gets a direct map from $\omega_{g,n}$ to primary invariants via integration over the cycles.

The chapter is organised as follows. In Sections 3.2 and 3.3, we remind the reader of the general theory of Frobenius manifolds and topological recursion, as well as the correspondence between the two, following [37]. In Section 3.4, we describe the construction of Dubrovin of Frobenius manifold structures on covers of Hurwitz spaces and prove Theorems 3.1.4 and 3.1.6. In Section 3.5, we discuss an extension of the results to non-semi-simple points.

3.2 Frobenius manifolds

In this section we give a short introduction to Frobenius manifolds. An important construction for this chapter is Givental's R-matrix defined in Section 3.2.3.

3.2.1 Frobenius manifold

Definition 3.2.1. A Frobenius algebra $(H, \eta, \cdot)$ is a finite-dimensional vector space H equipped with a metric $\eta = \langle\ ,\ \rangle$ and a commutative, associative product $\cdot$ satisfying $\langle u \cdot v, w\rangle = \langle u, v \cdot w\rangle$.

Example 3.2.2.

$$H \cong \mathbb{C} \oplus \mathbb{C} \oplus ... \oplus \mathbb{C}, \quad \langle e_i, e_j\rangle = \delta_{ij}\eta_i, \quad e_i \cdot e_j = \delta_{ij} e_i$$

for any $\eta_i \in \mathbb{C} \setminus \{0\}$, $i = 1, ..., N$ and where $\{e_i\}$ is the standard basis. Conversely any semisimple *Frobenius algebra is determined uniquely by N non-zero complex numbers $\{\eta_i\}$ and is isomorphic to this example.*

A Frobenius manifold is defined by the data of a Frobenius algebra on the tangent space at each point of the manifold and such that the metric is flat. In terms of flat coordinates $\{t^\alpha\}$ a Frobenius manifold can be defined locally as follows. Consider a function $F(t^1, \dots, t^N)$ defined on a ball $B \subset \mathbb{C}^N$ and a constant inner product $\eta^{\alpha\beta}$ such that the triple derivatives of F with one raised index,

$$C_{\alpha\beta}^{\gamma} := \frac{\partial^3 F}{\partial t^\alpha \partial t^\beta \partial t^\lambda}\eta^{\lambda\gamma}, \tag{3.2.12}$$

are the structure constants of a commutative associative Frobenius algebra with the scalar produce given by $\eta_{\alpha\beta}$. We can think about this structure as defined on the tangent bundle of $B \subset \mathbb{C}^N$ (and we denote the corresponding multiplication of vector field by $\cdot$), and we require that ∂_{t^1} is the unit of the algebra in each fibre.

We further consider structures (almost) homogeneous under a vector field $E := \sum_{\alpha=1}^{N}((1-q_\alpha)t^\alpha + r_\alpha)\partial_{t^\alpha}$, where q_α and r_α are constants for $\alpha = 1, \dots, N$, satisfying $q_1 = 0$ and $r_\alpha \neq 0$ only in the case $1 - q_\alpha = 0$. We require that there exists a constant d such that $E.F - (3-d)F$ is a polynomial of order at most 2 in $t^1, \dots, t^N$.

The triple (F, η, E) that satisfies all conditions above gives us the structure of a (conformal) Frobenius manifold of rank N and conformal dimension d with flat unit. The function F is called the prepotential; the vector field E is called the Euler vector field. The coordinate-free description of this structure requires a flat metric with associated Levi-Civita connection, unit and Euler vector fields satisfying compatibility conditions—see [27] for details.

In this chapter we only consider semi-simple Frobenius manifolds, that is, we require that the algebra structure at each point on an open subset $B^{ss} \subset B$ is semi-simple hence isomorphic to Example 3.2.2. In a neighborhood of a semi-simple point we have a system of canonical coordinates $u_1, \ldots, u_N$, defined up to permutations, such that the vector fields ∂_{u_i}, $i = 1, \ldots, N$, are the idempotents of the algebra product, and the Euler vector field has the form $E = \sum_{i=1}^N u_i \partial_{u_i}$. This gives rise to two important systems of coordinates: flat coordinates, leading to a fixed metric and varying product, and canonical coordinates, leading to a fixed product and varying metric. With respect to the canonical coordinates, the flat metric on M is diagonal with diagonal terms generated by a potential $H : M \to \mathbb{C}$ via (3.1.2) which satisfies (3.1.1).

Define the rotation coefficients

$$\beta_{ij} = \frac{\partial_{u_j} \eta_i}{\sqrt{\eta_i \eta_j}}. \tag{3.2.13}$$

Then (3.1.1) and (3.1.2) imply that $\beta_{ij}(u)$ satisfy the Darboux-Egoroff system

$$\beta_{ij} = \beta_{ji}, \tag{3.2.14}$$

$$\partial_{u_k} \beta_{ij} = \beta_{ik} \beta_{jk}. \tag{3.2.15}$$

Flatness of the identity and conformality imply

$$\sum_k \partial_{u_k} \beta_{ij} = 0, \tag{3.2.16}$$

$$\sum_k u_k \partial_{u_k} \beta_{ij} = -\beta_{ij}. \tag{3.2.17}$$

Assemble the rotation coefficients into a symmetric $N \times N$ matrix $\Gamma = \Gamma(u)$—whose diagonal is not a part of the structure—by $\Gamma_{ij} = \beta_{ij}$. Then equations (3.2.15-3.2.17) are equivalent to the Darboux-Egoroff equation

$$d[\Gamma, U] = [[\Gamma, U], [\Gamma, dU]] \tag{3.2.18}$$

where $U = \mathrm{diag}(u_1, \ldots, u_N)$.

The rotation coefficients give less information than the metric, i.e. there are different solutions of (3.1.1) and (3.1.2) that give rise to the same rotation coefficients. The system

$$\partial_{u_j} \psi_i = \beta_{ij} \psi_j, \quad i \neq j$$

$$\sum_{j=1}^N \partial_{u_j} \psi_i = 0, \quad i = 1, ..., N$$

has an N-dimensional space of solutions $\psi = (\psi_1(u), ..., \psi_N(u))$ which enables one to retrieve a metric for each solution from the rotation coefficients. Put N independent solutions of this system into the columns of a matrix Ψ, so the system becomes

$$d\Psi = [\Gamma, dU]\Psi. \tag{3.2.19}$$

The canonical coordinate vector fields ∂_{u_i} are orthogonal but not orthonormal. We can normalise them to produce a so-called normalised canonical frame in each tangent space, that is, if $\eta_i = \eta(\partial_{u_i}, \partial_{u_i})$, then the orthonormal basis is given by $\partial_{v_i} := \eta_i^{-1/2}\partial_{u_i}$, $i = 1, \dots, N$. The matrix Ψ in (3.2.19) is the transition matrix from the flat basis to the normalised canonical basis.

3.2.2 Cohomological field theory

A *cohomological field theory* is a pair (H, η) composed of a finite-dimensional complex vector space H equipped with a metric η and a sequence of S_n-equivariant maps.

$$I_{g,n} : H^{\otimes n} \to H^*(\overline{\mathcal{M}}_{g,n})$$

that satisfy compatibility conditions from inclusion of strata:

$$\psi : \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n}, \quad \phi_I : \overline{\mathcal{M}}_{g_1,|I|+1} \times \overline{\mathcal{M}}_{g_2,|J|+1} \to \overline{\mathcal{M}}_{g,n}, \quad I \sqcup J = \{1, ..., n\}$$

given by

$$\phi_I^* I_{g,n}(v_1 \otimes ... \otimes v_n) = I_{g_1,|I|+1} \otimes I_{g_2,|J|+1}\big(\bigotimes_{i\in I} v_i \otimes \Delta \otimes \bigotimes_{j\in J} v_j\big) \tag{3.2.20}$$

$$\psi^* I_{g,n}(v_1 \otimes ... \otimes v_n) = I_{g-1,n+2}(v_1 \otimes ... \otimes v_n \otimes \Delta) \tag{3.2.21}$$

where $\Delta \in H \otimes H$ is dual to the metric. In local coordinates it is given by $\Delta = \eta^{\alpha\beta} e_\alpha \otimes e_\beta$.

The metric $\eta = \langle \cdot, \cdot \rangle$ and the 3-point function $I_{0,3}$ induce a product $\cdot$ on H via

$$\langle u \cdot v, w \rangle = I_{0,3}(u, v, w) \in H^*(\overline{\mathcal{M}}_{0,3}) \cong \mathbb{C}.$$

Correlators, or ancestor invariants, of the CohFT make use of the Chern classes $\psi_j = c_1(L_j)$ of the tautological line bundles $L_j, j = 1, ..., n$ over $\overline{\mathcal{M}}_{g,n}$. The correlators are defined by:

$$\langle \tau_{k_1}(e_{\nu_1}) ... \tau_{k_n}(e_{\nu_n}) \rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}(e_{\nu_1}, ..., e_{\nu_n}) \cdot \prod_{j=1}^n \psi_j^{k_j} \tag{3.2.22}$$

for $k_i \in \mathbb{N}$, $\{e_\nu,\ {}_{\nu=1,\dots,N}\} \subset H$. When $k_i = 0$, $i = 1, ..., n$ the ancestor invariants are also known as *primary invariants* of the CohFT.

Givental [56] introduced a group action on genus 0 potentials of a CohFT, and used it to propose a formula for higher genera. Faber, Shadrin, Zvonkine [53] proved that the higher genera formula satisfies all properties that might be imposed to correlators of CohFT, hence the Givental group acts on partition functions of CohFTs in all genera. The interpretation of the action on correlators as an action on cohomology classes was constructed by several people independently, namely, by Teleman [91], Katzarkov-Kontsevich-Pantev (unpublished), and Kazarian (unpublished)—see [85]. There is a good account of this action on cohomology classes by Pandharipande-Pixton-Zvonkine [82]. Hence we can associate a CohFT to a semi-simple point of a Frobenius manifold. Conversely a CohFT gives rise to a Frobenius manifold structure on (a neighborhood inside) H using the constant metric η as the flat metric and a varying family of products using $I_{0,n}$ in place of $I_{0,3}$. See [71] for details.

If the Frobenius manifold has flat identity—meaning that the identity vector field for the product on the tangent bundle is parallel with respect to the Levi-Civita connection of the flat metric η—then this is realised on the CohFT level by an extra relation involving the forgetful map

$$\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

given by

$$I_{g,n+1}(v_1 \otimes \cdots \otimes v_n \otimes \mathbb{1}) = \pi^* I_{g,n}(v_1 \otimes \cdots \otimes v_n), \quad I_{0,3}(v_1 \otimes v_2 \otimes \mathbb{1}) = \eta(v_1 \otimes v_2) \quad (3.2.23)$$

where $\mathbb{1}$ is the unit vector for the product.

3.2.3 Classification of semi-simple cohomological field theories

The Givental-Teleman theorem [56, 91] states that a semi-simple CohFT is equivalent to the pair (H, η) together with a so-called R-matrix. An R-matrix

$$R(z) = \sum_{k=0}^{\infty} R_k z^k$$

is a formal series whose coefficients are $N \times N$ matrices where $N = \dim H$ is the rank of the Frobenius manifold. Givental used $R[z]$ to produce a differential operator, a so-called quantisation of $R[z]$, which acts on a known tau-function to produce a generating series for the correlators of the CohFT.

The coefficients R_k are defined using Ψ, the transition matrix from flat coordinates to normalised canonical coordinates determined by (3.2.19), via $R_0 = I$ and the inductive equation

$$d\left(R(z)\Psi\right) = \frac{[R(z), dU]}{z}\Psi \quad (3.2.24)$$

which uniquely determines $R(z)$ up to left multiplication by a diagonal matrix $D(z)$ independent of u with $D(0) = I$. We recall that this equation is a consequence of the fact that $R(z)$ is the regular part of the expansion of the solution of a linear system associated by Dubrovin to any semisimple Frobenius manifold around its essential singularity (see for example lecture 3 in [27] for more details).

Using $d\Psi = [\Gamma, dU]\,\Psi$ one can write

$$d\left(R(z)\Psi\right) = d\left[R(z)\right]\Psi + R(z)d\Psi = d\left[R(z)\right]\Psi + R(z)\left[\Gamma, dU\right]\Psi.$$

Together with equation (3.2.24) and the invertibility of Ψ, this gives

$$dR(z) = \frac{[R(z), dU]}{z} - R(z)\left[\Gamma, dU\right]. \quad (3.2.25)$$

This re-expresses the equation for $R(z)$ in terms of the rotation coefficients, which uses less information than the full metric, encoded in Ψ. Since $dU(\mathbb{1}) = I$, an immediate consequence of (3.2.25) is

$$\mathbb{1} \cdot R(z) = 0. \tag{3.2.26}$$

If the theory is homogenous, then invariance under the action of the Euler field

$$(z\partial_z + E) \cdot R(z) = 0 \tag{3.2.27}$$

fixes the diagonal ambiguity in $R(z)$.

The first non-trivial term R_1 of $R(z)$ is given by the rotation coefficients

$$R_1 = \Gamma. \tag{3.2.28}$$

This follows from comparing the constant (in z) term in (3.2.24) which is

$$d\Psi = [R_1, dU]\Psi$$

to equation (3.2.19) given by $d\Psi = [\Gamma, dU]\,\Psi$. Since dU is diagonal with distinct diagonal terms, we see that (3.2.28) holds for off-diagonal terms, and the ambiguity in the diagonal term for both is unimportant—it can be fixed in R_1 by (3.2.24) together with $E \cdot R_1 = -R_1$.

3.3 Topological recursion and CohFT

In this section, we give a brief overview of topological recursion defined in [50]. Consider a Riemann surface Σ equipped with meromorphic functions $x, y\colon \Sigma \to \mathbb{C}$ such that the zeros of dx, given by $\{\mathcal{P}_1, ..., \mathcal{P}_N\}$ are simple and dy is analytic and non-vanishing on $\{\mathcal{P}_1, ..., \mathcal{P}_N\}$. Let B be a Bergman kernel on $\Sigma \times \Sigma$ as in definition 3.1.2.

Define a sequence of symmetric multidifferentials $\omega_{g,n}(p_1, \ldots, p_n)$ on $\Sigma^{\times n}$ by the following recursion:

$$\omega_{0,1}(p) := y(p)dx(p); \tag{3.3.29}$$

$$\omega_{0,2}(p_1, p_2) := B(p_1, p_2); \tag{3.3.30}$$

$$\begin{aligned}&\omega_{g,m+1}(p_0, p_1, \ldots, p_n) := \\ &\sum_{i=1}^{N} \operatorname*{Res}_{p=\mathcal{P}_i} \frac{\int_p^{\sigma_i(p)} \omega_{0,2}(\bullet, p_0)}{2(\omega_{0,1}(\sigma_i(p)) - \omega_{0,1}(p))} \tilde{\omega}_{g,2|n}(p, \sigma_i(p)|p_1, \ldots, p_n),\end{aligned} \tag{3.3.31}$$

where σ_i is the local involution defined by x near the point $\mathcal{P}_i$, $i = 1, \ldots, N$, and $\tilde{\omega}_{g,2|n}$ is defined by the following formula:

$$\begin{aligned}\tilde{\omega}_{g,2|n}(p', p''|p_1, \ldots, p_n) :=&\omega_{g-1,n+2}(p', p'', p_1, \ldots, p_n) + \\ &\sum_{\substack{g_1+g_2=g \\ I_1 \sqcup I_2 = \{1,\ldots,n\} \\ 2g_1-1+|I_1|\geq 0 \\ 2g_2-1+|I_2|\geq 0}} \omega_{g_1,|I_1|+1}(p', p_{I_1})\omega_{g_2,|I_2|+1}(p'', p_{I_2}).\end{aligned} \tag{3.3.32}$$

Here we denote by p_I the sequence $p_{i_1}, \ldots, p_{i_{|I|}}$ for $I = \{i_1, \ldots, i_{|I|}\}$.

Remark 3.3.1. The recursion was defined on so-called *local* spectral curves in [42] as follows. Consider small neighborhoods $U_i \subset \Sigma$ of the points $\mathcal{P}_i$. If we look at just the restrictions of $\omega_{g,n}$ to the products of these disks, $U_{i_1} \times \cdots \times U_{i_n}$, we can still proceed by topological recursion, using as an input the restrictions of $\omega_{0,1}$ to U_i, $i = 1, \ldots, N$, and $\omega_{0,2}$ to $U_i \times U_j$, $i, j = 1, \ldots, N$. Indeed, Equation (3.3.31) uses only local expansion data around the points $\mathcal{P}_i$. Hence, the word local refers to the unique knowledge of these local data.

Remark 3.3.2. In the topological recursion on a compact spectral curve we also allow y to be the (multivalued) primitive of a differential ω on Σ. The ambiguity in y consists of periods and residues of ω and hence the ambiguity is locally constant. Since y appears in the recursion formula (3.3.32) only via $y(\sigma_i(p)) - y(p)$ (and there are no poles of ω at the zeros of dx) the locally constant ambiguity disappears and the recursion is well-defined. We go even further and allow ω to be a locally defined meromorphic differential (a connection) ambiguous up to $dy + df(x)$ for any rational function f. In this case the ambiguity $y \mapsto y + f(x)$ is no longer constant, but again $y(\sigma_i(p)) - y(p)$ is unchanged.

3.3.1 Topological recursion from CohFTs

We recall the relation (3.1.9) of topological recursion on a local spectral curve to the Givental formulae for cohomological field theories obtained in [37].

Definition 3.3.3. For a Riemann surface equipped with a meromorphic function (Σ, x) we define evaluation of any meromorphic differential ω at a simple zero $\mathcal{P}$ of dx by

$$\omega(\mathcal{P}) := \operatorname*{Res}_{p=\mathcal{P}} \frac{\omega(p)}{\sqrt{2(x(p) - x(\mathcal{P}))}}$$

where we choose a branch of $\sqrt{x(p) - x(\mathcal{P})}$ once and for all at each $\mathcal{P}$ to remove the ± 1 ambiguity.

Theorem 3.3.4. *[37] Given a semi-simple CohFT presented via the R-matrix $R(z) = \sum_{k=0}^{\infty} R_k z^k$ and constants $\eta_1, \ldots, \eta_N$ define a local spectral curve (Σ, x, y, B), presented as (the Laplace transform of) local series for $dy(p)$ and $B(p, p')$ around each zero $p = \mathcal{P}_i$, $p' = \mathcal{P}_j$ of dx (which is locally canonical) as follows:*

$$\left[R^{-1}(z)\right]^i_j = -\frac{\sqrt{z}}{\sqrt{2\pi}} \int_{\Gamma_j} B(\mathcal{P}_i, p) \cdot e^{\frac{(u_j - x(p))}{z}} \tag{3.3.33}$$

$$\sum_{k=1}^{N} \left[R^{-1}(z)\right]^k_i \cdot \eta_k^{1/2} = \frac{1}{\sqrt{2\pi z}} \int_{\Gamma_i} dy(p) \cdot e^{\frac{(u_i - x(p))}{z}} \tag{3.3.34}$$

$$\frac{1}{2\pi\sqrt{z_1 z_2}} \int_{\Gamma_i} \int_{\Gamma_j} B(p_1, p_2) e^{\frac{(u_i - x(p_1))}{z_1} + \frac{(u_j - x(p_2))}{z_2}} = -\frac{\sum_{k=1}^{N} \left[R^{-1}(z_1)\right]^k_i \left[R^{-1}(z_2)\right]^k_j}{z_1 + z_2} \tag{3.3.35}$$

where Γ_i is a path containing $u_i = x(\mathcal{P}_i)$. Then the multidifferentials $\omega_{g,n}(p_1, \ldots, p_n)$ obtained via topological recursion applied to the local spectral curve (Σ, x, y, B) are *polynomials in differentials $V^i_k(p_j)$ defined by*

$$V^i_0(p) = B(\mathcal{P}_i, p), \quad V^i_{k+1}(p) = d\left(\frac{V^i_k(p)}{dx(p)}\right), \ k = 0, 1, 2, ... \tag{3.3.36}$$

with coefficients given by ancestor invariants of the CohFT:

$$\omega_{g,n}(p_1, ..., p_n) = \sum_{\substack{i_1,...,i_n \\ d_1,...,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}(e_{i_1}, \ldots, e_{i_n}) \prod_{j=1}^n \psi_j^{d_j} \cdot \bigotimes_{j=1}^n V^{i_j}_{d_j}(p_j).$$

Remark 3.3.5. The spectral curve thus obtained is local, i.e. a collection of open sets U_i each containing a unique zero $\mathcal{P}_i$ of dx. Thus Γ_i is defined only locally, which is fine since we are interested only in the asymptotic expansion for R around $z = 0$. Let us also remind the reader that this result is valid for any semisimple Frobenius manifold. We shall see in the next section that, in the case of Hurwitz Frobenius manifolds, one can make these Laplace transform globally well-defined by choosing carefully the integration cycles to consider.

Remark 3.3.6. This data (the constants η_i and the matrix $R(z)^j_i$) determine for us a semi-simple CohFT $\{I_{g,n}\}$ with an N-dimensional space of primary fields $V := \langle e_1, \ldots, e_N\rangle$ corresponding to a chosen point $(u_1, ..., u_N)$ on a Frobenius manifold—see Section 3.2.3. In terms of the underlying Frobenius manifold structure, the basis $e_1, \ldots, e_N$ corresponds to the normalised canonical basis

Remark 3.3.7. Note that the limit of (3.3.34) at $z = 0$ yields:

$$\eta_i^{1/2} = dy(\mathcal{P}_i) \tag{3.3.37}$$

which tells us that dy encodes the metric.

Remark 3.3.8. Compatibility of (3.3.33) and (3.3.35) is a condition on the bidifferential B, not satisfied in general, nevertheless always satisfied if the spectral curve is compact and the differential dx is meromorphic. Compatibility for compact spectral curves uses a general finite decomposition for $B(p_1, p_2)$ proven by Eynard in Appendix B of [46] together with (3.3.33). This is recalled in section 3.5.1.

Theorem 3.3.4 produces a map

$$\{\text{semisimple CohFT}\} \longrightarrow \{\text{topological recursion applied to a local spectral curve}\}$$

with image consisting of spectral curves with B and y necessarily satisfying compatibility conditions—compatibility of (3.3.33), (3.3.34) and (3.3.35). A general spectral curve will not satisfy such compatibility conditions, i.e. in general one can choose B and y independently. For example, the rational spectral curve $(\mathbb{P}^1, x, y, B)$ for $x = z + 1/z$, $B = dzdz'/(z - z')^2$, $dy = z^m dz$, $m \in \{-1, 0, 1, 2, ...\}$, lies in the image of the map only for $m = -1$ or 0.

Compatibility of (3.3.33) and (3.3.35) is discussed in Remark 3.3.8 and compatibility of (3.3.33) and (3.3.34) is characterised by the following theorem.

Theorem 3.3.9 ([35]). *Equations* (3.3.33) *and* (3.3.34) *are compatible (as equations for the unknown variables* R^{-1} *and* η_i*),* $i = 1, \ldots, N$*) if and only if the 1-form*

$$\omega(p) = d\left(\frac{dy}{dx}(p)\right) + \sum_{i=1}^{N} \operatorname*{Res}_{p'=\mathcal{P}_i} \frac{dy}{dx}(p')B(p,p'). \tag{3.3.38}$$

is invariant under each local involution σ_i*,* $i = 1, \ldots, N$*.*

The characterisation in Remark 3.3.8 and Theorem 3.3.9 allows a *converse* construction of semisimple CohFTs from compact spectral curves. The following is a sufficient condition for compatibility of (3.3.33) and (3.3.34).

Definition 3.3.10. We say that a compact spectral curve (Σ, x, y, B) is *dominant* if x and dy are meromorphic and the poles of dx dominate the poles of dy.

Corollary 3.3.11. *A dominant compact spectral curve* (Σ, x, y, B) *lies in the image of* (3.1.9) *and hence gives rise to a semisimple CohFT.*

Proof. Any Bergman kernel satisfies the Cauchy property (3.1.4). If the poles of dx dominate the poles of dy then dy/dx has poles only at the zeros $\mathcal{P}_i$ of dx. Then $\omega(p) \equiv 0$ since

$$\sum_{i=1}^{N} \operatorname*{Res}_{p'=\mathcal{P}_i} \frac{dy}{dx}(p')B(p,p') = -\operatorname*{Res}_{p'=p} \frac{dy}{dx}(p')B(p,p') = -d\left(\frac{dy}{dx}(p)\right)$$

and hence it is invariant under each local involution σ_i. Since the Riemann surface Σ is compact it automatically satisfies (3.3.35) hence the claim is proven. ☐

Remark 3.3.12. In fact Corollary 3.3.11 allows a weaker hypothesis which we will need. We can instead allow dy to be a locally defined meromorphic differential, essentially a connection, which is ambiguous up to $dy \mapsto dy + \lambda dx$. The conclusion of Corollary 3.3.11 still holds since $d\left(\frac{dy}{dx}\right)$ is globally defined.

3.4 Hurwitz Frobenius manifolds

In this section we first remind the reader of Dubrovin's construction of a Frobenius manifold on a cover of Hurwitz space and then prove a number of deformation lemmas, which will be useful in the following sections.

3.4.1 Dubrovin's construction

As defined in the introduction, denote by $\widetilde{H}_{g,\mu}$ the moduli space of tuples $(\Sigma, x, \{\mathcal{A}_i, \mathcal{B}_i\}_{i=1,\ldots,g})$ consisting of covers $x : \Sigma \to \mathbb{P}^1$ of genus g with fixed ramification profile above infinity $\mu = (\mu_1, \ldots, \mu_n)$ together with a choice of a symplectic basis of cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i=1,\ldots,g}$ and marked branches of x at each point above ∞.

Given such a generic cover x, we denote its simple branch points

$$\forall i = 1, \ldots, N\,,\ u_i = x(\mathcal{P}_i) \qquad \text{where} \qquad dx|_{\mathcal{P}_i} = 0.$$

By the Riemann-Hurwitz formula:

$$N = 2g - 2 + n + |\mu|$$

and since an element of the Hurwitz space is defined up to a finite information by its critical values, this gives the dimension

$$\dim_{\mathbb{C}} \widetilde{H}_{g,\mu} = 2g - 2 + n + |\mu|.$$

In the introduction we claimed that

$$\#\mathcal{D} = N = \#\{p \mid dx(p) = 0\} \tag{3.4.39}$$

i.e. the number of generalised contours, defined by $\mathcal{D}$ in (3.1.3), coincides with $\dim_{\mathbb{C}} \widetilde{H}_{g,\mu}$. This follows from the fact that dx is a meromorphic differential so its divisior $(dx) = Z - P$ has degree $2g - 2$, where Z and P are the zeros and poles of dx. Hence

$$\dim_{\mathbb{C}} \widetilde{H}_{g,\mu} = |Z| = 2g - 2 + |P| = \dim H_1(\Sigma, x^{-1}(\infty)) - 1 + \sum_{i=1}^{d} \mu_i = \#\mathcal{D}.$$

The last equality is clear since the elements of $\mathcal{D}$ consist of firstly $\{x\mathcal{A}_1, ..., x\mathcal{A}_g, \mathcal{B}_1, ..., \mathcal{B}_g, \gamma_i, i = 2, ..., d\}$ which has cardinality equal to $\dim H_1(\Sigma, x^{-1}(\infty)) = 2g-1+d$, together with $-1+\sum_i \mu_i = |P|-d-1$ extra elements $x^{k/\mu_i}\mathcal{C}_{\infty_i}, k = 1, ..., \mu_i, i = 1, ..., d$ remove $x\mathcal{C}_{\infty_1}$.

We use the critical values u_i as local coordinates in an open dense domain of $\widetilde{H}^s_{g,\mu} \subset \widetilde{H}_{g,\mu}$ where $u_i \neq u_j$ for $i \neq j$. The vector fields ∂_{u_i} give a basis of $T\widetilde{H}_{g,\mu}$ and define a multiplication $\cdot$ given by:

$$\partial_{u_i} \cdot \partial_{u_j} = \delta_{ij}\partial_{u_i}. \tag{3.4.40}$$

We denote the unity and the Euler vector fields:

$$e = \sum_{i=1}^{N} \partial_{u_i}, \quad E = \sum_{i=1}^{N} u_i \partial_{u_i}. \tag{3.4.41}$$

Let us now define one-forms on $\widetilde{H}_{g,\mu}$. For any quadratic differential Q on Σ, define the one-form

$$\Omega_Q = \sum_{i=1}^{N} du_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{Q(p)}{dx}.$$

Dubrovin defines a set of differentials ϕ on Σ, defined in (3.1.5) and described in more detail below, which have poles dominated by the poles of dx. They are known as *primary differentials* and used to produce a quadratic differential $Q = \phi^2$.

Theorem 3.4.1 (Dubrovin [27]). *For any primary differential ϕ, $\widetilde{H}^s_{g,\mu} \bigcap \{u \mid \phi(\mathcal{P}_i) \neq 0\}$ is equipped with a structure of a Frobenius manifold with multiplication* (3.4.40)*, unity and Euler vector fields* (3.4.41) *and metric*

$$\eta := \sum_{i=1}^{N} du_i^2 \cdot \operatorname*{Res}_{p=\mathcal{P}_i} \frac{\phi^2(p)}{dx(p)} = \sum_{i=1}^{N} du_i^2 \cdot \phi(\mathcal{P}_i)^2 \tag{3.4.42}$$

where we used the notation of definition 3.3.3 for the evaluation of a one-form at a point. In addition, the corresponding flat coordinates $(t_\alpha)_{\alpha=1,\dots,N}$ *can be explicitly written in terms of periods of* ϕ *via*

$$t_\alpha = \int_{\mathcal{C}_\alpha} \phi$$

for any $\mathcal{C}_\alpha \in \mathcal{D}$.

This means that the data of such a Frobenius manifold structure on $\widetilde{H}_{g,\mu}$ is given by the choice of a primary differential ϕ. The definition of a primary differential uses the Torelli marking of Σ as follows. Fix a point in $\widetilde{H}_{g,\mu}$, i.e. a pair (Σ, x) (a point in a Hurwitz space) together with a basis $(\mathcal{A}_i, \mathcal{B}_i)_{i=1,\dots,g}$ (a sheet of $\widetilde{H}_{g,\mu}$ seen as a cover). Recall from the introduction that there is a unique Bergman Kernel $B(p,p')$ which is a bidifferential of the second kind normalised to have zero periods over the $\mathcal{A}$-cycles in the Torelli marking $\{\mathcal{A}_i, \mathcal{B}_i\}$. For any generalised contour $\mathcal{C}_\alpha \in \mathcal{D}$ we define a primary differential by

$$\phi_\alpha(p) = dy_\alpha(p) = \oint_{\mathcal{C}^*_\alpha} B(p,p')$$

which is locally holomorphic on $\Sigma \setminus x^{-1}(\infty)$. Here, the dual $\mathcal{C}^*_\alpha = \eta_{\alpha\beta}\mathcal{C}_\beta$ with respect to the metric η.

Following Dubrovin, let us classify these cycles in 5 types:

- Type (1): for $i = 1, \dots, d$ and $k = 1, \dots, \mu_i - 1$:

$$\int_{p \in \mathcal{C}_{t_{i,k}}} f(p) = \frac{1}{\mu_i - 1} \operatorname*{Res}_{p \to \infty_i} x(p)^{\frac{k}{\mu_i}} f(p);$$

- Type (2) : for $i = 2, \dots, d$:

$$\int_{p \in \mathcal{C}_{v_i}} f(p) = \operatorname*{Res}_{p \to \infty_i} x(p)\, f(p);$$

- Type (3): for $i = 2, \dots, d$:

$$\int_{p \in \mathcal{C}_{w_i}} f(p) = \text{v.p.} \int_{\infty_1}^{\infty_i} f(p);$$

- Type (4): for $i = 1, \dots, g$:

$$\int_{p \in \mathcal{C}_{r_i}} f(p) = -\oint_{\mathcal{A}_i} x(p)\, f(p);$$

- Type (5): for $i = 1, \dots, g$:

$$\int_{p \in \mathcal{C}_{s_i}} f(p) = \frac{1}{2i\pi} \oint_{\mathcal{B}_i} f(p).$$

We see that the two important systems of coordinates—flat coordinates and canonical coordinates—correspond to cycles in $\mathcal{D}$, respectively zeros of dx. These sets have the same cardinality by (3.4.39).

3.4.2 Vector fields, cycles and meromorphic differentials.

Let us now introduce a correspondence between vector fields and meromorphic forms using the Bergman kernel B which allows us to express all the quantities defining the Hurwitz Frobenius manifold in terms of meromorphic forms. For flat coordinates

$$\partial_{t_\alpha} \mapsto \phi_\alpha(p) = \int_{p' \in \mathcal{C}_\alpha^*} B(p, p').$$

By linearity, for any vector field v, we can define a cycle $\mathcal{C}_v$ by

$$\mathcal{C}_v = \sum_\alpha \langle v, \partial_{t_\alpha} \rangle_\eta \mathcal{C}_\alpha \tag{3.4.43}$$

a meromorphic differential

$$\phi_v(p) = \int_{p' \in \mathcal{C}_v} B(p, p')$$

and the metric η by

$$\langle v_1, v_2 \rangle_\phi = \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{\phi_{v_1} \phi_{v_2}}{dx(p)}. \tag{3.4.44}$$

Note that (3.4.43) and (3.4.44) are proven by verifying them on a basis. We choose the flat basis, to prove (3.4.43). Substitute $v = \partial_{t_\alpha}$ into (3.4.43) to get $\mathcal{C}_{\partial_{t_\alpha}} = \sum_\beta \left\langle \partial_{t_\alpha}, \partial_{t_\beta} \right\rangle_\eta \mathcal{C}_\beta = \sum_\beta \eta_{\alpha\beta} \mathcal{C}_\beta = \mathcal{C}_\alpha^*$ as required. We choose the canonical basis to prove (3.4.44) as follows.

Apply (3.4.43) to the canonical vector fields to get

$$\mathcal{C}_{\partial_{u_i}} = \sum_\alpha \langle \partial_{u_i}, \partial_{t_\alpha} \rangle_\eta \mathcal{C}_\alpha = \sum_\alpha \phi(\mathcal{P}_i) \Psi_\alpha^i \mathcal{C}_\alpha$$

and hence

$$\begin{aligned} \phi_{\partial_{u_i}}(p) = \int_{\mathcal{C}_{\partial_{u_i}}} B(p, p') &= \sum_\alpha \phi(\mathcal{P}_i) \Psi_\alpha^i \int_{\mathcal{C}_\alpha} B(p, p') \\ &= \sum_{\alpha,\beta} \phi(\mathcal{P}_i) \Psi_\alpha^i \eta^{\alpha\beta} \int_{\mathcal{C}_\beta^*} B(p, p') = \sum_{\alpha,\beta} \phi(\mathcal{P}_i) \Psi_\alpha^i \eta^{\alpha\beta} \phi_\beta(p) \end{aligned}$$

We will study $\phi_{\partial_{u_i}}$ via evaluation at $\mathcal{P}_j$.

$$\phi_{\partial_{u_i}}(\mathcal{P}_j) = \sum_{\alpha,\beta} \phi(\mathcal{P}_i) \Psi_\alpha^i \eta^{\alpha\beta} \phi_\beta(\mathcal{P}_j) = \sum_{\alpha,\beta} \phi(\mathcal{P}_i) \Psi_\alpha^i \eta^{\alpha\beta} \Psi_\beta^j = \delta_{ij} \phi(\mathcal{P}_i)$$

which uses the relation $\phi_\beta(\mathcal{P}_j) = \Psi_\beta^j$ proven in Proposition 3.4.6. Since $\phi_{\partial_{u_i}}(p)$ vanishes at $\mathcal{P}_j$ for $j \neq i$, (3.4.44) becomes rather simple:

$$\left\langle \partial_{u_i}, \partial_{u_j} \right\rangle_\phi = \sum_k \operatorname*{Res}_{p=\mathcal{P}_k} \frac{\phi_{\partial_{u_i}} \phi_{\partial_{u_j}}}{dx(p)} = \delta_{i,j} \operatorname*{Res}_{p=\mathcal{P}_i} \frac{\phi^2(p)}{dx(p)}$$

in agreement with (3.4.42) and hence proving (3.4.44) for all vector fields.

The product in terms of the canonical basis gives us a formula in terms of the matrix Ψ of change of basis from flat to canonical which takes the form of the Verlinde formula, or Krichever formula depending on the context. (The Verlinde formula is actually for the degree 0 part of the theory.) This can be written for example following [27] equation (5.61)

$$C_{\alpha\beta\gamma} = \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{\phi_\alpha(p)\phi_\beta(p)\phi_\gamma(p)}{dx(p)\phi(p)}. \tag{3.4.45}$$

This depends on the choice of Frobenius structure through ϕ which appears in the denominator and a point in the Frobenius manifold through the dependence on x.

Let us finally identify the identity and the Euler field. The consistency condition for the identity vector field $\mathbb{1} = \partial_{t_{\alpha_0}}$

$$\left\langle \partial_{t_\alpha}, \partial_{t_\beta} \right\rangle_\phi = C_{\alpha\beta\alpha_0}$$

imposes

$$\phi_{\mathbb{1}} = \phi_{\alpha_0} = \phi$$

and the Euler vector field

$$\phi_E = -E \cdot ydx|_{x \text{ fixed}} = E \cdot xdy|_{y \text{ fixed}} = xdy = x\phi$$

uses variations of the structures which are described below.

3.4.3 Rauch variational formula

An important tool used in this chapter is Rauch variational formula expressing the variation of the Bergman kernel with respect to the position of the critical values.

$$\frac{\partial}{\partial u_i} B(p_1, p_2) = \operatorname*{Res}_{p=\mathcal{P}_i} \frac{B(p,p_1)B(p,p_2)}{dx(p)}. \tag{3.4.46}$$

Rauch originally derived the dependence of the Riemann matrix of periods of a Hurwitz cover on the critical values of the covering map in [83]. It later led to the expression of the variation of the Bergman kernel in [65].

In the present context, the meaning of the variation is as follows. Over the Frobenius manifold $M = \widetilde{H}_{g,\mu}$ we have a universal curve $\pi : \tilde{C} \to M$ and a function $x : \tilde{C} \to M \times \overline{\mathbb{C}}$ satisfying:

(i) Each fibre $C = C_u = \pi^{-1}(u)$ is a Riemann surface.

(ii) x is meromorphic on each fibre C.

(iii) The critical values $\{u_1, ..., u_n\}$ of x on each fibre above a semi-simple point are canonical coordinates for M.

For any vector field $\partial \in \Gamma(TM)$, we choose a lift $\tilde{\partial} \in \Gamma(TC)$ so that $\tilde{\partial} x = 0$. We abuse terminology and write $\tilde{\partial} = \partial$. Hence we make sense of variations of a function $f(p_1, p_2)$ on $C \times C$ by identifying $p_i \in C_u$ with $p_i' \in C_{u'}$ when $x(p_1) = x(p_1')$.

Rauch's variational formula for the Bergman kernel leads to variational formulae for other quantities, in particular primary differentials.

$$\partial_{u_i} dy(p) = \partial_{u_i} \int_{\mathcal{C}_\alpha} B(p,p') = \int_{\mathcal{C}_\alpha} \partial_{u_i} B(p,p') = \int_{\mathcal{C}_\alpha} B(p,\mathcal{P}_i)B(p',\mathcal{P}_i) = dy(\mathcal{P}_i)B(p,\mathcal{P}_i). \tag{3.4.47}$$

We apply this to give a short proof of flatness of the metric (3.4.42) and refer to [27] for the full proof of Theorem 3.4.1 which gives a different proof of flatness. The tangent space to $\widetilde{H}_{g,\mu}$ is spanned by primary differentials constructed from contours in $\mathcal{D}$. Hence the following lemma proves flatness of the metric.

Lemma 3.4.2. *When $\mathcal{C},\mathcal{C}' \in \mathcal{D}$ then $\langle \phi_\mathcal{C}, \phi_{\mathcal{C}'}\rangle_\phi$ is constant in $\{u_1,...,u_N\}$.*

Proof. From the Rauch's variational formula (3.4.46), we have
$\partial_{u_i}\phi_\mathcal{C}(p) = \phi_\mathcal{C}(\mathcal{P}_i)B(p,\mathcal{P}_i)$. This uses the fact that the contour $\mathcal{C}$ depends only on a geometric contour independent of the choice of u_i, and possibly a function of x which is constant, i.e. $\partial_{u_i}x = 0$ by assumption. Hence

$$\begin{aligned}\partial_{u_j}\langle \phi_\mathcal{C}, \phi_{\mathcal{C}'}\rangle &= \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{\partial_{u_j}(\phi_\mathcal{C}(p)\phi_{\mathcal{C}'}(p))}{dx(p)} \\ &= \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{B(p,\mathcal{P}_j)(\phi_\mathcal{C}(\mathcal{P}_j)\phi_{\mathcal{C}'}(p) + \phi_\mathcal{C}(p)\phi_{\mathcal{C}'}(\mathcal{P}_j))}{dx(p)} = 0.\end{aligned}$$

Note that the integrand potentially has poles at $\mathcal{P}_j$ and ∞_k but since each $\phi_\mathcal{C}(p)$ is dominated by $dx(p)$ at each $p=\infty_k$ the poles at ∞_k are removable. Hence the last equality uses the fact that the integrand has poles only at $\mathcal{P}_i$, $i=1,...,N$ so that the sum of its residues at $\mathcal{P}_i$ is 0. □

The following theorem proven by Shramchenko identifies the $R(z)$ matrix of the Hurwitz Frobenius manifold with the Laplace transform of the Bergmann kernel. It uses the Rauch's variational formula.

Theorem 3.4.3 (Shramchenko, [90])**.** *Given a point $(\Sigma, x, (\mathcal{A}_i,\mathcal{B}_i)_{i=1,...,g})$ in the cover of a Hurwitz space with $B(p,p')$ normalised on the $\mathcal{A}$-cycles together with a choice of admissible differential ϕ the $R(z)$ matrix of the Hurwitz Frobenius manifold is given by:*

$$\left[R^{-1}(z)\right]^i_j := -\frac{\sqrt{z}}{\sqrt{2\pi}}\int_{\Gamma_j} e^{-\frac{(x(p)-u_j)}{z}} B(p,\mathcal{P}_i). \tag{3.4.48}$$

The resemblance of (3.4.48) and (3.3.33) means we are now in a position to prove Theorem 3.1.6. Let us also remark that Shramchenko's result goes further than a formal series in z. Indeed, [90] defines integration cycles Γ_i such that $R(z)$ is the regular part of the expansion of a solution to Dubrovin's linear system which is well defined in a half plane.

Proof of Theorem 3.1.6. The proof combines Theorem 3.3.4, Theorem 3.3.9 and Theorem 3.4.3.

Define the spectral curve (Σ, x, y, B) by a generic point $(\Sigma, x) \in H_{g,\mu}$ equipped with a bidifferential B normalised over a given set of $\mathcal{A}$-cycles, and a primary differential by

$dy(p) := \oint_{\mathcal{C}} B(p,p')$ for some $\mathcal{C} \in \mathcal{D}$, defined in (3.1.3). If the spectral curve satisfies the conditions (3.3.33)-(3.3.35) of Theorem 3.3.4 for the $R(z)$ matrix of the Hurwitz Frobenius manifold then topological recursion applied to the spectral curve produces the ancestor invariants of the Frobenius manifold via the decomposition of $\omega_{g,n}$ given by (3.1.10) and the theorem is proven.

By Theorem 3.4.3 the $R(z)$ matrix of the Hurwitz Frobenius manifold is given by (3.4.48) hence condition (3.3.33) is satisfied. Next we need to show that the choice of y is the correct one. But since $dy(p) := \oint_{\mathcal{C}} B(p,p')$ the poles of dy are dominated by the poles—the pole behaviour of the integrals over generalised cycles described in Section 3.4.1 is given in [27]—hence the spectral curve is dominant and Corollary 3.3.11 applies, proving that condition (3.3.34) is satisfied. Finally condition (3.3.35) is satisfied by Lemma 3.5.4 since Σ is compact and x is meromorphic.

□

3.4.4 Shramchenko's deformation.

Following methods of Kokotov-Korotkin [66], Shramchenko [88] defined deformations of Dubrovin's Frobenius manifold structures on $\widetilde{H}_{g,\mu}$. See also Buryak-Shadrin [15]. Recall that once we are given $(\Sigma, x, \{\mathcal{A}_i, \mathcal{B}_i\}_{i=1,\ldots,g})$ and $\mathcal{D}$, we define a Bergman kernel and use that to define primary differentials ϕ_α for $\alpha \in \mathcal{D}$. Instead of the Bergman kernel $B(p,p')$ Shramchenko considered arbitrary Bergman kernels $\omega_{0,2}^{[\kappa]}(p,p')$ on Σ which is a symmetric bidifferential on $\Sigma \times \Sigma$, with a double pole on the diagonal of zero residue, double residue equal to 1, and no further singularities. The set of such kernels is parameterised by symmetric matrices κ of size $g \times g$. We denote by $\omega_{0,2}^{[0]} = B$ the Bergman kernel normalised in the basis of cycles chosen, i.e.

$$\forall i = 1, \ldots, g\,, \oint_{\mathcal{A}_i} \omega_{0,2}^{[0]} = 0.$$

The key ingredients in the proofs of Theorems 3.1.4 and 3.1.6 are Rauch's variational principle for $B(p,p')$ which holds more generally for Bergman kernels normalised on geometric cycles and Eynard's formula (3.3.35) which is valid for any $B = \omega_{0,2}^{[\kappa]}$.

Theorem 3.4.4. *The conclusion of Theorems 3.1.4 and 3.1.6 holds for Frobenius manifold structures on $\widetilde{H}_{g,\mu}$ defined by $\omega_{0,2}^{[\kappa]}(p,p')$ when κ is such that there exist a basis of geometric cycles $\left(\mathcal{A}_i^{[\kappa]}, \mathcal{B}_i^{[\kappa]}\right)_i$ satisfying*

$$\forall i = 1, \ldots, g\,, \oint_{p' \in \mathcal{A}_i^{[\kappa]}} \omega_{0,2}^{[\kappa]}(p,p') = 0.$$

3.4.5 Landau-Ginzburg model.

In Section 3.4.2 we described a map from the tangent space at $p \in \widetilde{H}_{g,\mu}$ to the vector space spanned by primary differentials, denoted by V_p^{prim}. It was defined via a map to contours which are linear combinations of contours in $\mathcal{D}$. For $v \in T_p\widetilde{H}_{g,\mu}$ we defined

$$v \mapsto \mathcal{C}_v \mapsto \phi_v(p)$$

A more direct path uses variations. It is known as a Landau-Ginzburg model for (Σ, x, dy) and defined by:

$$\begin{array}{ccc} T_p\widetilde{H}_{g,\mu} & \to & V_p^{\text{prim}} \\ v & \mapsto & v\cdot(-ydx) \end{array}.$$

So the claim is that the variation gives the composition of the two maps above, i.e. $\cdot(-ydx(p)) = \phi.(p)$. We prove this relation in terms of flat coordinates.

Lemma 3.4.5. *For $\mathcal{C}_\alpha \in \mathcal{D}$, the coordinate $t_\alpha = \int_{\mathcal{C}_\alpha} dy$ is associated to the differential $\phi_\alpha(p)$ via*

$$\partial_{t_\alpha}[-y(p)dx(p)] = \phi_\alpha(p) = \int_{\mathcal{C}_\alpha^*} B(p,p').$$

Proof. The main idea of the proof is to consider evaluation of $\partial_{t_\alpha} ydx(p)$ at $p = \mathcal{P}_i$ in order to be able to integrate by parts. From the variation of dy with respect to canonical coordinates given in (3.4.47) we have

$$\begin{aligned} \partial_{t_\alpha} dy(p) &= \sum_i \Psi^i_\alpha \partial_{v_i} dy(p) = \sum_i \frac{\Psi^i_\alpha}{dy(\mathcal{P}_i)}\partial_{u_i} dy(p) \\ &= \sum_i \frac{\Psi^i_\alpha}{dy(\mathcal{P}_i)} dy(\mathcal{P}_i)B(p,\mathcal{P}_i) = \sum_i \Psi^i_\alpha B(p,\mathcal{P}_i). \end{aligned} \tag{3.4.49}$$

Then

$$\begin{aligned} \partial_{t_\alpha}[-ydx](\mathcal{P}_i) = -\operatorname*{Res}_{p=\mathcal{P}_i} \frac{1}{\sqrt{2(x(p)-u_i)}}\partial_{t_\alpha}[ydx] &= \operatorname*{Res}_{p=\mathcal{P}_i} \sqrt{2(x(p)-u_i)}\partial_{t_\alpha} dy \\ &= \operatorname*{Res}_{p=\mathcal{P}_i} \sqrt{2(x(p)-u_i)} \sum_j \Psi^j_\alpha B(p,\mathcal{P}_j) \\ &= \operatorname*{Res}_{p=\mathcal{P}_i} \sqrt{2(x(p)-u_i)} B(p,\mathcal{P}_i)\Psi^i_\alpha \\ &= \Psi^i_\alpha = \phi_\alpha(\mathcal{P}_i) \end{aligned}$$

where the second line uses (3.4.47), the third line uses the fact that $B(p,\mathcal{P}_j)$ has no pole at $\mathcal{P}_i$ for $j \neq i$, the third line uses $\operatorname{Res}_{p=\mathcal{P}_i} \sqrt{2(x(p)-u_i)}B(p,\mathcal{P}_i) = 1$, and the final equality uses Proposition 3.4.5.

Hence

$$\partial_{t_\alpha}[ydx](\mathcal{P}_i) = \phi_\alpha(\mathcal{P}_i), \quad i = 1, ..., N$$

which is nearly enough to guarantee that the differentials $\partial_{t_\alpha} ydx$ and ϕ_α agree. Define the function on Σ by

$$f(p) = \frac{\partial_{t_\alpha} ydx(p) - \phi_\alpha(p)}{dx(p)}.$$

Then $f(p)$ has no poles since the numerator of $f(p)$ vanishes at $p = \mathcal{P}_i$ and $dx(p)$ has simple zeros at $p = \mathcal{P}_i$. Also, from (3.4.49) we see that $\partial_{t_\alpha} ydx$ has no poles at $x = \infty$ hence $\partial_{t_\alpha} ydx - \phi_\alpha$ has poles only at $x = \infty$, dominated by poles of dx, since this is true of ϕ_α. In particular $f(p)$ has no poles at $x = \infty$.

Thus $f(p) = c$ constant and $\partial_{t_\alpha} y dx(p) = \phi_\alpha(p) + c dx(p)$. In [27] Dubrovin proves that the differential $\phi_\alpha(p)$ is either strictly dominated by dx at at least one point $\infty_i \in x^{-1}(\infty)$, in which case $f(\infty_i) = 0$, or $\phi_\alpha(p)$ is a connection with ambiguity given by cdx for any constant c. Hence we may assume $c = 0$ and the lemma is proven. □

We can now identify the transition matrix Ψ between flat and normalised canonical vector fields in an elegant way. Flat coordinates correspond to periods along generalised contours while canonical coordinates correspond to (finite) critical points of x. The Bergman kernel allows a natural marriage of the two.

Proposition 3.4.6 ([88]). *The transition matrix Ψ between flat and normalised canonical vector fields, defined in* (3.2.19) *is given by*

$$\Psi^i_\alpha = \int_{p \in \mathcal{C}^*_\alpha} B(p, \mathcal{P}_i) = \phi_\alpha(\mathcal{P}_i).$$

As usual the indices $i = 1, ..., N$ are associated to the canonical coordinates and $\alpha = 1, ..., N$ are associated to the flat coordinates.

Proof. We have

$$\partial_{u_i} t_\alpha = \partial_{u_i} \int_{\mathcal{C}_\alpha} dy = \int_{\mathcal{C}_\alpha} \partial_{u_i} dy = dy(\mathcal{P}_i) \int_{\mathcal{C}_\alpha} B(p, \mathcal{P}_i)$$

where the last equality uses (3.4.47), hence

$$\partial_{v_i} = \sum_\alpha \int_{\mathcal{C}_\alpha} B(p, \mathcal{P}_i)\, \partial_{t_\alpha}. \tag{3.4.50}$$

Now

$$[\partial_{v_1}, ..., \partial_{v_N}]\Psi = [\partial_{t_1}, ..., \partial_{t_N}]$$

and since $\Psi^T \Psi = \eta$, or $\Psi \eta^{-1} \Psi^T = I$ we have

$$[\partial_{v_1}, ..., \partial_{v_N}] = [\partial_{t_1}, ..., \partial_{t_N}]\eta^{-1}\Psi^T$$

hence

$$\partial_{v_i} = \sum_{\alpha,\beta} \eta^{\alpha\beta} \Psi^i_\beta \cdot \partial_{t_\alpha}$$

and comparing this with (3.4.50) we see that

$$\sum_\beta \eta^{\alpha\beta} \Psi^i_\beta = \int_{\mathcal{C}_\alpha} B(p, \mathcal{P}_i)$$

so

$$\Psi^i_\gamma = \sum_{\alpha,\beta} \eta_{\gamma\alpha} \eta^{\alpha\beta} \Psi^i_\beta = \sum_\alpha \eta_{\gamma\alpha} \int_{\mathcal{C}_\alpha} B(p, \mathcal{P}_i) = \int_{\sum_\alpha \eta_{\gamma\alpha} \mathcal{C}_\alpha} B(p, \mathcal{P}_i) = \int_{\mathcal{C}^*_\gamma} B(p, \mathcal{P}_i)$$

as required. The second equality in the statement of the proposition simply uses the definition $\phi_\alpha(p) := \int_{p \in \mathcal{C}^*_\alpha} B(p, p')$. □

Remark 3.4.7. The column of Ψ corresponding to the vector field gives the square root of the diagonal coefficients $\eta_i^{1/2}$ of the metric η in canonical coordinates. From Proposition 3.4.6, we have $\eta_i = \phi(\mathcal{P}_i)^2$ which agrees with (3.4.42).

The transition matrix Ψ gives rise to the R matrix of the Frobenius manifold built from a choice of point $(\Sigma, x, (\mathcal{A}_i, \mathcal{B}_i)_{i=1,\dots,g})$ in $\widetilde{H}_{g,\mu}$ given in Theorem 3.4.3 together with a choice of admissible differential η.

3.4.6 Flat coordinates

Let us now explain how to recover the expression of the correlators of the CohFT in flat coordinates out of integration along contours in $\mathcal{D}$.

Lemma 3.4.8. *For any generalised contour $\mathcal{C} \in \mathcal{D}$ and any $(g,n) \in \mathbb{N} \times \mathbb{N}^*$, the map*

$$\omega_{g,n} \mapsto \int_{\mathcal{C}} \omega_{g,n}$$

defining the action of integration of the correlation functions is well defined.

Proof. Since $\mathcal{C}$ is only an isotopy class of contours (with coefficients that are functions of x) in $\Sigma \setminus x^{-1}(\infty)$ and $\omega_{g,n}$ has poles in $\Sigma \setminus x^{-1}(\infty)$ we need to prove that the integral is independent of the choice of contour. This is a consequence of the fact that $\omega_{g,n}$ and $x\omega_{g,n}$ have zero residues at $\mathcal{P}_i$. Note that the residues at ∞_j might not be zero, but the contours are not allowed to deform through ∞_j. □

Proposition 3.4.9. *For any $\mathcal{C} \in \mathcal{D}$ define $\phi_{\mathcal{C}}(p) = \int_{\mathcal{C}} B(p,p') = df_{\mathcal{C}}$ where $f_{\mathcal{C}}$ is locally valued. As operators acting on $\omega_{g,n}$ for $2g-2+n>0$,*

$$\sum_i \operatorname*{Res}_{\mathcal{P}_i} f_{\mathcal{C}} \cdot = \int_{\mathcal{C}} \cdot$$

in other words,

$$\sum_i \operatorname*{Res}_{p=\mathcal{P}_i} f_{\mathcal{C}}(p)\omega_{g,n}(p,p_2,\dots,p_n) = \int_{p\in\mathcal{C}} \omega_{g,n}(p,p_2,\dots,p_n).$$

Proof. Recall Riemann's bilinear relation. For meromorphic differentials ϕ and ω such that ϕ is residueless

$$\sum_P \operatorname*{Res}_P f \cdot \omega = \frac{1}{2\pi i} \sum_{j=1}^{g} \left[\oint_{\mathcal{A}_i} \phi \oint_{\mathcal{B}_i} \omega - \oint_{\mathcal{B}_i} \phi \oint_{\mathcal{A}_i} \omega \right] \tag{3.4.51}$$

where $df = \phi$ for a locally defined primitive f and the sum is over all poles P of ϕ and ω.

Primary differentials $\phi_{\mathcal{C}}$ of types 1, 2 and 5, with respect to the classification given in Section 3.4.1, are residueless so apply (3.4.51) to $\phi = \phi_{\mathcal{C}}$ and $\omega = \omega_{g,n}$.

For $\mathcal{C} = \mathcal{B}_j$, $i = j, \dots, g$, $\phi_{\mathcal{C}}(p) = \int_{\mathcal{B}_j} B(p,p') = \theta_j = df_{\mathcal{C}}$ (f defined locally) is a holomorphic differential satisfying $\int_{\mathcal{A}_k} \theta_j = 2\pi i \cdot \delta_{jk}$. Then (3.4.51) becomes:

$$\sum_i \operatorname*{Res}_{\mathcal{P}_i} f_{\mathcal{C}} \cdot \omega_{g,n} = \frac{1}{2\pi i} \sum_{k=1}^{g} \left[\oint_{\mathcal{A}_k} \phi_{\mathcal{C}} \oint_{\mathcal{B}_k} \omega_{g,n} - \oint_{\mathcal{B}_k} \phi_{\mathcal{C}} \oint_{\mathcal{A}_k} \omega_{g,n} \right] = \oint_{\mathcal{B}_j} \omega_{g,n} = \oint_{\mathcal{C}} \omega_{g,n}$$

since $\oint_{\mathcal{A}_k}\omega_{g,n}=0$.

For $\mathcal{C}=x^{k/(n_i+1)}\mathcal{C}_i$, $k=1,...,n_i+1$, $i=1,...,d$, (which this includes both types 1 and 2) then $\phi_{\mathcal{C}}=\operatorname{Res}_{\infty_i}x^{k/(n_i+1)}B=df_{\mathcal{C}}$ is residueless and normalised so that $\int_{\mathcal{A}_i}\phi_{\mathcal{C}}=0$. Then (3.4.51) becomes

$$
\begin{aligned}
\sum_{P=\mathcal{P}_k,\infty_\ell}\operatorname*{Res}_{P} f_{\mathcal{C}}\cdot\omega_{g,n}&=\frac{1}{2\pi i}\sum_{j=1}^{g}\left[\oint_{\mathcal{A}_j}\phi_{\mathcal{C}}\oint_{\mathcal{B}_j}\omega_{g,n}-\oint_{\mathcal{B}_j}\phi_{\mathcal{C}}\oint_{\mathcal{A}_j}\omega_{g,n}\right]=0\\
\Rightarrow\sum_{k}\operatorname*{Res}_{\mathcal{P}_k} f_{\mathcal{C}}\cdot\omega_{g,n}&=-\operatorname*{Res}_{\infty_i} f_{\mathcal{C}}\cdot\omega_{g,n}=\oint_{\mathcal{C}}\omega_{g,n}
\end{aligned}
$$

where the last equality uses the fact that $f_{\mathcal{C}}\sim -x^{k/(n_i+1)}$ near ∞_i.

For $\mathcal{C}=x\mathcal{C}_i$, $i=1,...,d$, $\phi_{\mathcal{C}}=\int_{\Gamma_i}B$ is a differential of the 3rd kind with simple poles at ∞_1 and ∞_i normalised so that $\int_{\mathcal{A}_k}\phi_{\mathcal{C}}=0$. Since $\omega_{g,n}$ is residueless we switch the roles of ϕ and ω in (3.4.51). Choose $F_{g,n}$ such that $dF_{g,n}=\omega_{g,n}$, i.e. a primitive with respect to one variable. Then

$$
\begin{aligned}
\sum_{p=\mathcal{P}_k,\infty_\ell}\operatorname*{Res}_{p} F_{g,n}(p,p_2,...,p_n)\phi_{\mathcal{C}}(p)&=\frac{1}{2\pi i}\sum_{j=1}^{g}\left[\oint_{\mathcal{A}_j}\phi_{\mathcal{C}}\oint_{\mathcal{B}_j}\omega_{g,n}-\oint_{\mathcal{B}_j}\phi_{\mathcal{C}}\oint_{\mathcal{A}_j}\omega_{g,n}\right]=0\\
\Rightarrow\sum_{k}\operatorname*{Res}_{\mathcal{P}_k} f_{\mathcal{C}}(p)\omega_{g,n}(p,p_2,...,p_n)&=-\sum_{k}\operatorname*{Res}_{\mathcal{P}_k} F_{g,n}(p,p_2,...,p_n)\phi_{\mathcal{C}}(p)\\
&=\operatorname*{Res}_{p=\infty_i} F_{g,n}(p,p_2,...,p_n)\phi_{\mathcal{C}}(p)\\
&\quad+\operatorname*{Res}_{p=\infty_1} F_{g,n}(p,p_2,...,p_n)\phi_{\mathcal{C}}(p)\\
&=F_{g,n}(\infty_i,p_2,...,p_n)-F_{g,n}(\infty_1,p_2,...,p_n)\\
&=\int_{\infty_1}^{\infty_i}\omega_{g,n}=\int_{\mathcal{C}}\omega_{g,n}
\end{aligned}
$$

For $\mathcal{C}=x\mathcal{A}_i$, $i=1,...,g$, we cannot apply (3.4.51) directly since $\phi_{\mathcal{C}}$ is not a globally defined differential. Instead we need to apply the proof of (3.4.51) as follows. Cut Σ along $\mathcal{A}$ and $\mathcal{B}$ cycles meeting at a common point P_0 to leave a simply-connected region $R\subset\Sigma$ on which $\phi_{\mathcal{C}}$ and a primitive (with respect to one variable) $F_{g,n}(p)$ of $\omega_{g,n}(p)$ (suppress variables $p_2,...,p_n$) are well-defined. As in the proof of (3.4.51) integrate $\frac{1}{2\pi i}\phi_{\mathcal{C}}F_{g,n}$ along the boundary of R given by the $\mathcal{A}$ and $\mathcal{B}$ cycles to get

$$\begin{aligned}\sum_{p=\mathcal{P}_k}\operatorname*{Res}_{p} F_{g,n}(p)\phi_{\mathcal{C}}(p) &= \sum_{j=1}^{g}\left[\oint_{\mathcal{A}_j}\phi_{\mathcal{C}}\oint_{\mathcal{B}_j}\omega_{g,n} - \oint_{\mathcal{B}_j}\phi_{\mathcal{C}}\oint_{\mathcal{A}_j}\omega_{g,n}\right] \\ &\quad - \int_{P_0+\mathcal{B}_i}^{P_0+\mathcal{B}_i+\mathcal{A}_i} F_{g,n}(p)dx(p) \\ &= -\int_{P_0+\mathcal{B}_i}^{P_0+\mathcal{B}_i+\mathcal{A}_i} F_{g,n}(p)dx(p) \\ &= -\int_{\mathcal{A}_i} x(p)\omega_{g,n}(p) = -\int_{\mathcal{C}}\omega_{g,n}(p) \\ \Rightarrow \sum_{p=\mathcal{P}_k}\operatorname*{Res}_{p} f_{\mathcal{C}}(p)\omega_{g,n}(p) &= \int_{\mathcal{C}}\omega_{g,n}(p).\end{aligned}$$

□

Theorem 3.1.6 proved that topological recursion applied to the spectral curve $(\Sigma, x, (\mathcal{A}_i, \mathcal{B}_i)_{i=1,\dots,g})$ with a choice of admissible differential $\phi = dy$ and $\omega_{0,2} = B$, stores the ancestor invariants of the Hurwitz Frobenius manifold and hence proves Theorem 3.1.4. We now prove the remainder of the statement of Theorem 3.1.4 by showing how to extract the ancestor invariants via integration over generalised contours.

Proposition 3.4.10. *Integration over flat contours $\mathcal{C}_\alpha \in \mathcal{D}$ produces primary invariants:*

$$\int_{\mathcal{C}_{\alpha_1}}\dots\int_{\mathcal{C}_{\alpha_n}}\omega_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}\Big(e^{\alpha_1}\otimes\dots\otimes e^{\alpha_n}\Big)$$

Proof. We will prove the dual statement

$$\int_{\mathcal{C}^*_{\alpha_1}}\dots\int_{\mathcal{C}^*_{\alpha_n}}\omega_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}\Big(e_{\alpha_1}\otimes\dots\otimes e_{\alpha_n}\Big). \tag{3.4.52}$$

For $k > 0$,

$$\sum_i \operatorname*{Res}_{\mathcal{P}_i} y_\alpha\cdot V^i_k = 0, \quad k>0$$

where V^i_k are defined in (3.3.36) and $dy_\alpha = \phi_\alpha$. Hence the operator $\sum_i \operatorname{Res}_{\mathcal{P}_i} y_\alpha\cdot$ only detects coefficients of $V^i_0(p) = B(\mathcal{P}_i, p)$ in $\omega_{g,n}$ which stores the primary invariants by (3.1.10). Now

$$\operatorname*{Res}_{\mathcal{P}_j} y_\alpha(p)\cdot V^i_0(p) = \operatorname*{Res}_{\mathcal{P}_j} y_\alpha(p)\cdot B(\mathcal{P}_i,p) = \delta_{ij}\phi_\alpha(\mathcal{P}_j) = \Psi^i_\alpha$$

since B acts as a Cauchy kernel which sends y_α to evaluation of dy_α. Hence $\sum_i \operatorname*{Res}_{\mathcal{P}_i} y_\alpha\cdot$ acts as insertion of the vector

$$\Psi^i_\alpha\cdot\partial_{v_i} = \partial_{t_\alpha} = e_\alpha$$

into the ancestor invariant. Thus, using Proposition 3.4.9 we see that as an operator on $\omega_{g,n}$

$$\int_{\mathcal{C}^*_\alpha} \cdot = \sum_i \operatorname*{Res}_{\mathcal{P}_i} y_\alpha \cdot$$

acts as insertion of the vector e^α into the ancestor invariant and in particular (3.4.52) holds. Note that since $\mathcal{C}^*_\alpha = \sum_\beta \eta_{\alpha\beta}\mathcal{C}_\beta$ is a constant linear combination of contours in $\mathcal{D}$, then Proposition 3.4.9 applies also to $\mathcal{C}^*_\alpha$. □

Remark 3.4.11. Let us apply Proposition 3.4.10, or more precisely (3.4.52), to the simplest case of $\omega_{0,3}$ to get the following.

$$\begin{aligned}
C_{\alpha\beta\gamma} &= \int_{\overline{\mathcal{M}}_{0,3}} I_{0,3}\Big(e_\alpha \otimes e_\beta \otimes e_\gamma\Big) \\
&= \int_{\mathcal{C}^*_\alpha}\int_{\mathcal{C}^*_\beta}\int_{\mathcal{C}^*_\gamma} \omega_{0,3} \\
&= \int_{\mathcal{C}^*_\alpha}\int_{\mathcal{C}^*_\beta}\int_{\mathcal{C}^*_\gamma} \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{B(p_1,p)B(p_2,p)B(p_3,p)}{dx(p)dy(p)} \\
&= \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \int_{\mathcal{C}^*_\alpha}\int_{\mathcal{C}^*_\beta}\int_{\mathcal{C}^*_\gamma} \frac{B(p_1,p)B(p_2,p)B(p_3,p)}{dx(p)dy(p)} \\
&= \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{\phi_\alpha(p)\phi_\beta(p)\phi_\gamma(p)}{dx(p)dy(p)}
\end{aligned}$$

which agrees with (3.4.45) as expected. Here we have used the formula

$$\omega_{0,3}(p_1,p_2,p_3) = \sum_i \operatorname*{Res}_{p=\mathcal{P}_i} \frac{B(p_1,p)B(p_2,p)B(p_3,p)}{dx(p)dy(p)}$$

proven in [50].

The following proposition generalises Theorem 3.1.4.

Proposition 3.4.12. *There exist generalised contours $\mathcal{C}_{\alpha,k} = p_k(x)\mathcal{C}_\alpha$, for $\mathcal{C}_\alpha \in \mathcal{D}$ and $p_k(x) = x^k + ...$ a monic polynomial of degree k in x, so that the ancestor invariants, corresponding to $d_j \geq 0$, appear as periods.*

$$\int_{\mathcal{C}_{\alpha_1,k_1}} ... \int_{\mathcal{C}_{\alpha_n,k_n}} \omega_{g,n} = \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}\Big(e^{\alpha_1} \otimes ... \otimes e^{\alpha_n}\Big) \cdot \prod_{j=1}^n \psi_j^{k_j}. \tag{3.4.53}$$

Proof. Using integration by parts, we see that the contour $x^k\mathcal{C}_i$ acts on the differential $V^i_k(p)$ by

$$\int_{x^k\mathcal{C}_i} V^i_k(p) = \int_{\mathcal{C}_i} V^i_0(p).$$

Hence there exists a monic polynomial $p_k(x) = x^k + ...$ of degree k in x such that

$$\int_{p_k(x)\mathcal{C}_i} V^j_m(p) = \delta_{ij}\delta_{km}.$$

Define $\mathcal{C}^*_{\alpha,k} = p_k(x)\mathcal{C}^*_\alpha$, for $\mathcal{C}_\alpha \in \mathcal{D}$ then we have

$$\int_{\mathcal{C}^*_{\alpha_1,k_1}} \cdots \int_{\mathcal{C}^*_{\alpha_n,k_n}} \omega_{g,n} = \langle \prod_{j=1}^{n} \tau_{k_j}(e_{\alpha_j}) \rangle. \tag{3.4.54}$$

□

3.5 Topological recursion in compact case

In this section we associate to a spectral curve a so-called $\hat{R}(z)$ matrix, which in the case of spectral curves lying in the image of the map (3.1.9) from CohFTs to spectral curves, coincides with the $R(z)$ matrix of the Frobenius manifold.

Definition 3.5.1. Given a spectral curve (Σ, x, y, B) define a formal series

$$\hat{R}(z) = \sum_{k=0}^{\infty} \hat{R}_k z^k$$

with coefficients $N \times N$ matrices where $N =$ number of zeros of dx by

$$\left[\hat{R}^{-1}(z)\right]_j^i := -\frac{\sqrt{z}}{\sqrt{2\pi}} \int_{\Gamma_j} e^{-\frac{(x(p)-u_j)}{z}} B(p, \mathcal{P}_i). \tag{3.5.55}$$

In fact $\hat{R}(z)$ depends only on (Σ, x, B). This definition begins with the spectral curve and produces $\hat{R}(z)$ which reverses the direction of (3.3.33) where one begins with a Frobenius manifold and its associated $R(z)$ and produces a spectral curve. In general $\hat{R}(z)$ will not arise out of a Frobenius manifold.

Remark 3.5.2. Note that $\left[\hat{R}^{-1}(z)\right]_j^i$ is well-defined for $i = j$ because the integrand has a pole of residue zero at $\mathcal{P}_i$, so Γ_i can be deformed to avoid $\mathcal{P}_i$ in a well-defined manner.

Remark 3.5.3. The paths Γ_i were defined only locally in a neighbourhood of $\mathcal{P}_i$ in Section 3.3.1. That is also sufficient here, because again we are only concerned with the asymptotic expansion of $\hat{R}(z)$ at $z = 0$. Nevertheless, we can choose paths along which $x/z \to \infty$ in both directions, such as a path of steepest descent of $-x/z$ so that the series $\hat{R}(z)$ converges.

Let us denote $\left[\hat{R}(z)\right]_j^i = \sum_k \left[\hat{R}_k\right]_j^i z^{-k}$. In particular, one has

$$\left[\hat{R}_1\right]_j^i = B_{0,0}^{i,j} = B(\mathcal{P}_i, \mathcal{P}_j). \tag{3.5.56}$$

3.5.1 Factorisation property

On a compact spectral curve $\hat{R}(z)$ shares the symplectic property of any $R(z)$ associated to a Frobenius manifold. This is proven below as a consequence of a factorisation formula for the (Laplace transform of the) bidifferential B in terms of $\hat{R}(z)$. The factorisation formula is also required in the proof of Corollary 3.3.11.

Lemma 3.5.4 (Eynard, [46]). *Whenever the spectral curve is a Hurwitz cover of* $\mathbb{P}^1$ *with* dx *a meromorphic form with simple zeroes,* $\hat{R}(z)$*—defined in* (3.5.55)*—satisfies the symplectic condition*

$$\hat{R}(z)\hat{R}^T(-z) = Id. \tag{3.5.57}$$

Furthermore, the Laplace transform of a Bergman kernel

$$\check{B}^{i,j}(z_1,z_2) = \frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1 z_2}} \int_{\Gamma_i}\int_{\Gamma_j} B(p,p')e^{-\frac{x(p)}{z_1}-\frac{x(p')}{z_2}}.$$

satisfies

$$\check{B}^{i,j}(z_1,z_2) = -\frac{\sum_{k=1}^N \left[\hat{R}^{-1}(z_1)\right]_i^k \left[\hat{R}^{-1}(z_2)\right]_j^k}{z_1+z_2}. \tag{3.5.58}$$

This means that the coefficients $B_{k,l}^{i,j}$ of the expansion of the Bergman kernel around the branch points $\mathcal{P}_i$ and $\mathcal{P}_j$ can be defined recursively in terms of the initial data $B_{k,0}^{i,j}$. We give a proof here that differs from the proof in [46].

Proof. We have

$$\begin{aligned}\sum_{i=1}^N \operatorname*{Res}_{q=\mathcal{P}_i} \frac{B(p,q)B(p',q)}{dx(q)} &= -\operatorname*{Res}_{q=p}\frac{B(p,q)B(p',q)}{dx(q)} - \operatorname*{Res}_{q=p'}\frac{B(p,q)B(p',q)}{dx(q)} \\ &= -d_p\left(\frac{B(p,p')}{dx(p)}\right) - d_{p'}\left(\frac{B(p,p')}{dx(p')}\right)\end{aligned} \tag{3.5.59}$$

where the first equality uses the fact that the only poles of the integrand are $\{p,p',\mathcal{P}_i, i = 1,...,N\}$, and the second equality uses the Cauchy formula (3.1.4) satisfied by the Bergman kernel. The Laplace transform of the LHS of (3.5.59) is

$$\begin{aligned}&\frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1 z_2}}\int_{\Gamma_i}\int_{\Gamma_j} e^{-\frac{x(p)}{z_1}-\frac{x(p')}{z_2}} \sum_{k=1}^N \operatorname*{Res}_{q=\mathcal{P}_k}\frac{B(p,q)B(p',q)}{dx(q)} \\ &\qquad = \sum_{k=1}^N \frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1 z_2}}\int_{\Gamma_i} e^{-\frac{x(p)}{z_1}} B(p,\mathcal{P}_k)\int_{\Gamma_j} e^{-\frac{x(p')}{z_2}} B(p',\mathcal{P}_k) \\ &\qquad = \sum_{k=1}^N \frac{\left[\hat{R}^{-1}(z_1)\right]_i^k \left[\hat{R}^{-1}(z_2)\right]_j^k}{z_1 z_2}\end{aligned}$$

and the Laplace transform of the RHS of (3.5.59) is

$$\begin{aligned}&-\frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1 z_2}}\int_{\Gamma_i}\int_{\Gamma_j} e^{-\frac{x(p)}{z_1}-\frac{x(p')}{z_2}}\left\{d_p\left(\frac{B(p,p')}{dx(p)}\right) + d_{p'}\left(\frac{B(p,p')}{dx(p')}\right)\right\} \\ &\qquad = -\left(\frac{1}{z_1}+\frac{1}{z_2}\right)\check{B}^{i,j}(z_1,z_2)\end{aligned}$$

since the Laplace transform satisfies

$$\int_{\Gamma_i} d\left(\frac{\omega(p)}{dx(p)}\right) e^{-\frac{x(p)}{z}} = \frac{1}{z}\int_{\Gamma_i} \omega(p) e^{-\frac{x(p)}{z}}.$$

for any differential $\omega(p)$, by integration by parts. Hence we see that the Laplace transform of (3.5.59) gives (3.5.58) as required. Then (3.5.57) is a consequence of (3.5.58) and the finiteness of $\check{B}^{i,j}(z_1, z_2)$ at $z_2 = -z_1$. □

3.5.2 Defining equation for $\hat{R}_{ij}(z)$

In the preceding section we defined an $\hat{R}$-matrix from which is equivalent to the Bergman kernel. When the Bergman kernel is normalised on a basis of geometric cycles, we can go further and compute all the terms $\left[\hat{R}_k\right]_{ij}$ in terms of some minimal quantities. This uses the Rauch variational formula (3.4.46) which allows us to derive an equation for the $\hat{R}$ matrix.

When a spectral curve lies in the image of the map (3.1.9) from CohFTs to spectral curves, the following theorem is a consequence of the properties (3.2.25), (3.2.26), (3.2.27) of the R matrix of a CohFT. For compact spectral curves, by Theorem 3.4.3 generically $\hat{R} = R$, but $\hat{R}$ is a little more general, and for example exists when critical values u_i coincide and R is problematic, since it is defined over the semi-simple part of the Frobenius manifold. The outcome of the following theorem is that $\hat{R}$ resembles R and it can be used to give an alternative proof of Theorem 3.1.6.

Theorem 3.5.5. *Given a triple* (Σ, x, B) *consisting of a compact Riemann surface* Σ*, a meromorphic function* $x : \Sigma \to \mathbb{C}$ *with zeros of* dx *simple, and a Bergman kernel* B*, then* $\hat{R}(z)$ *satisfies* (3.2.25), (3.2.26), (3.2.27)*, i.e.*

$$d\hat{R}(z) = \frac{\left[\hat{R}(z), dU\right]}{z} - \hat{R}(z)\left[\Gamma, dU\right], \tag{3.5.60}$$

$$\mathbb{1} \cdot \hat{R}(z) = 0, \tag{3.5.61}$$

$$(z\partial_z + E) \cdot \hat{R}(z) = 0. \tag{3.5.62}$$

Proof. Although (3.5.61) is a consequence of (3.5.60) we first prove (3.5.61) and use this to prove (3.5.60).

Proof of (3.5.61): Differentiate Eynard's formula (3.5.58)

$$\check{B}^{i,j}(z_1, z_2) = \frac{e^{\frac{u_i}{z_1} + \frac{u_j}{z_2}}}{2\pi\sqrt{z_1 z_2}} \int_{\Gamma_i}\int_{\Gamma_j} B(p, p') e^{-\frac{x(p)}{z_1} - \frac{x(p')}{z_2}} = \sum_{k=1}^{N} \frac{\left[\hat{R}^{-1}(z_1)\right]_k^i \left[\hat{R}^{-1}(z_2)\right]_k^j}{z_1 + z_2}.$$

to get

$$\sum_{k=1}^{N} \frac{\partial}{\partial u_k} \check{B}^{i,j}(z_1,z_2) = \sum_{k=1}^{N} \frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1 z_2}} \int_{\Gamma_i} B(p,\mathcal{P}_k) e^{-\frac{x(p)}{z_1}} \int_{\Gamma_j} B(p',\mathcal{P}_k) e^{-\frac{x(p')}{z_2}}$$
$$+ \left(\frac{1}{z_1}+\frac{1}{z_2}\right) \check{B}^{i,j}(z_1,z_2)$$
$$= \sum_{k=1}^{N} \frac{\left[\hat{R}^{-1}(z_1)\right]_k^i \left[\hat{R}^{-1}(z_2)\right]_j^k}{z_1 z_2} - \sum_{k=1}^{N} \frac{\left[\hat{R}^{-1}(z_1)\right]_k^i \left[\hat{R}^{-1}(z_2)\right]_k^j}{z_1 z_2} = 0.$$

Since $\left[\hat{R}^{-1}(z_1)\right]_j^i = -z\check{B}^{i,j}(z,0)$, we have

$$\mathbb{1} \cdot \left[\hat{R}^{-1}(z_1)\right]_j^i = \sum_{k=1}^{N} \frac{\partial}{\partial u_k} \left[\hat{R}^{-1}(z_1)\right]_j^i = -z \sum_{k=1}^{N} \frac{\partial}{\partial u_k} \check{B}^{i,j}(z,0) = 0, \quad \forall i,j$$

and since $\mathbb{1} \cdot \hat{R}(z) = 0 \Leftrightarrow \mathbb{1} \cdot \hat{R}^{-1}(z) = 0$ this proves (3.5.61).

Proof of (3.5.60): For $k \neq i$,

$$\frac{\partial \left[\hat{R}^{-1}(z)\right]_j^i}{\partial u_k} = -\frac{\partial}{\partial u_k} \frac{\sqrt{z}}{\sqrt{2\pi}} \int_{\Gamma_j} e^{-\frac{(x(p)-u_j)}{z}} B(p,\mathcal{P}_i) \tag{3.5.63}$$
$$= \delta_{k,j} \frac{1}{z} \left[\hat{R}^{-1}(z)\right]_j^i - \frac{\sqrt{z}}{\sqrt{2\pi}} \int_{\Gamma_j} e^{-\frac{(x(p)-u_j)}{z}} B(p,\mathcal{P}_k) B(\mathcal{P}_k,\mathcal{P}_i)$$
$$= \delta_{k,j} \frac{\left[\hat{R}^{-1}(z)\right]_j^i}{z} + \left[\hat{R}^{-1}(z)\right]_j^k \beta_{ki}.$$

For $k = i$, by (3.5.61),

$$\frac{\partial \left[\hat{R}^{-1}(z)\right]_j^i}{\partial u_i} = -\sum_{m \neq i} \frac{\partial \left[\hat{R}^{-1}(z)\right]_j^i}{\partial u_m} = (\delta_{i,j}-1) \frac{\left[\hat{R}^{-1}(z)\right]_j^i}{z} - \sum_{m \neq i} \left[\hat{R}^{-1}(z)\right]_j^m \beta_{mi}$$

which gives the (i,j) component of the equation

$$d\hat{R}^{-1}(z) = \frac{\left[\hat{R}^{-1}(z), dU\right]}{z} + [\Gamma, dU]\, \hat{R}^{-1}(z).$$

Hence

$$d\hat{R}(z) = -\hat{R}(z) d\hat{R}^{-1}(z) \hat{R}(z) = -\hat{R}(z) \left(\frac{\left[\hat{R}^{-1}(z), dU\right]}{z} + [\Gamma, dU]\, \hat{R}^{-1}(z) \right) \hat{R}(z)$$
$$= \frac{\left[\hat{R}(z), dU\right]}{z} - \hat{R}(z)\, [\Gamma, dU]$$

and (3.5.60) holds.

Proof of (3.5.62): We begin with a variation of the proof of (3.5.58), replacing the identity vector field with the Euler vector field. We have

$$\sum_i u_i \frac{\partial}{\partial u_i} B(p,p') = \sum_{i=1}^N u_i \operatorname*{Res}_{q=\mathcal{P}_i} \frac{B(p,q)B(p',q)}{dx(q)} = \sum_{i=1}^N \operatorname*{Res}_{q=\mathcal{P}_i} \frac{x(q)B(p,q)B(p',q)}{dx(q)} \tag{3.5.64}$$

$$= -\operatorname*{Res}_{q=p} \frac{x(q)B(p,q)B(p',q)}{dx(q)} - \operatorname*{Res}_{q=p'} \frac{x(q)B(p,q)B(p',q)}{dx(q)}$$

$$= -d_p\left(\frac{x(p)B(p,p')}{dx(p)}\right) - d_{p'}\left(\frac{x(p')B(p,p')}{dx(p')}\right).$$

Then

$$\left(1 + z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \frac{u_i}{z_1} + \frac{u_j}{z_2}\right)\check{B}^{i,j}(z_1,z_2)$$

$$= \frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1z_2}}\left(z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2}\right)\int_{\Gamma_i}\int_{\Gamma_j} e^{-\frac{x(p)}{z_1}-\frac{x(p')}{z_2}} B(p,p')$$

$$= -\frac{e^{\frac{u_i}{z_1}+\frac{u_j}{z_2}}}{2\pi\sqrt{z_1z_2}}\int_{\Gamma_i}\int_{\Gamma_j}\sum_k u_k\frac{\partial}{\partial u_k}B(p,p')$$

$$= \left(\frac{u_i}{z_1} + \frac{u_j}{z_2} - \sum_k u_k\frac{\partial}{\partial u_k}\right)\check{B}^{i,j}(z_1,z_2)$$

where the second equality uses (3.5.64) and integration by parts to show that for any differential $\omega(p)$ the Laplace transform satisfies

$$\int_{\Gamma_i} e^{-\frac{x(p)}{z}} d\left(\frac{x(p)\omega(p)}{dx(p)}\right) = z\frac{d}{dz}\int_{\Gamma_i} e^{-\frac{x(p)}{z}}\omega(p).$$

Hence we are left with the following equation which is essentially the Laplace transform of (3.5.64):

$$\left[1 + z_1\frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_2} + \sum_i u_i\frac{\partial}{\partial u_i}\right]\check{B}(z_1,z_2) = 0. \tag{3.5.65}$$

We will now take the $z_2 \to 0$ limit of (3.5.65). From Eynard's formula (3.5.58) we see that $\frac{\partial}{\partial z_2}\check{B}(z_1,z_2)$ is well-defined at $z_2 = 0$, hence $\lim_{z_2\to 0} z_2\frac{\partial}{\partial z_2}\check{B}(z_1,z_2) = 0$. We also have $\check{B}^{j,k}(z_1,0) = -\frac{1}{z_1}\left[\hat{R}^{-1}(z_1)\right]_k^j$. Thus the $z_2 \to 0$ limit of (3.5.65) becomes

$$0 = \left[1 + z_1\frac{\partial}{\partial z_1} + \sum_i u_i\frac{\partial}{\partial u_i}\right]\frac{1}{z_1}\left[\hat{R}^{-1}(z_1)\right]_k^j = \frac{1}{z_1}\left[z_1\frac{\partial}{\partial z_1} + \sum_i u_i\frac{\partial}{\partial u_i}\right]\left[\hat{R}^{-1}(z_1)\right]_k^j$$

which gives (3.5.62). □

Remark 3.5.6. A spectral curve (Σ, x, y, B) with dy a primary differential is dominant—see Definition 3.3.10—hence by Corollary 3.3.11 it corresponds to a CohFT, which we have identified with the Hurwitz Frobenius manifold corresponding to primary differential. More generally we can take dy to be any linear combination of primary differentials, which is no longer a primary differential hence Theorem 3.1.6 does not apply, but the spectral curve is still dominant and hence corresponds to a CohFT.

3.6 Topological recursion for families of spectral curves

Vector fields on the Frobenius manifold $\widetilde{H}_{g,\mu}$ can give rise to recursion relations between ancestor invariants. In this section we show how the vector fields act on the multidifferentials $\omega_{g,n}$ arising out of topological recursion and give rise to the recursion relations between ancestor invariants.

Over the Frobenius manifold $\widetilde{H}_{g,\mu}$ is a *universal curve* which is a family of spectral curves constructed via the underlying Hurwitz map (Σ, x) together with natural cycles on Σ used to define the full spectral curve (Σ, x, y, B). Note that topological recursion applied to a single spectral curve produces a CohFT which extends uniquely to a family of CohFTs, nicely encoded in a Frobenius manifold, and each giving rise to a corresponding spectral curve. Hence in this way the family of spectral curves is reconstructed from any single spectral curve in the family.

Consider the family of multidifferentials $\omega_{g,n}$ obtained by applying topological recursion to the universal curve. We can differentiate the multidifferentials $\omega_{g,n}$ with respect to vector fields on the Frobenius manifold $\widetilde{H}_{g,\mu}$. As usual, for any vector field $v \in \Gamma(T\widetilde{H}_{g,\mu})$ we choose a lift $\tilde{v} \in \Gamma(TC)$ where C is the universal curve over $\widetilde{H}_{g,\mu}$, so that $\tilde{v} \cdot x = 0$. We abuse terminology and write $\tilde{v} = v$.

First order deformations of topological recursion are described in [50]. There it is shown that deformations of $\omega_{0,1}$ propagate via the recursion to determine deformations of $\omega_{g,n}$. Specifically, for v a vector field on $\widetilde{H}_{g,\mu}$, if we can express the variation of ydx as an integral of B over a generalised contour $\mathcal{C}$, then the variation of $\omega_{g,n}$ uses the same contour as follows.

$$v \cdot ydx(p) = \int_{\mathcal{C}} B(p', p) \quad \Rightarrow \quad v \cdot \omega_{g,n}(p_1, ..., p_n) = \int_{\mathcal{C}} \omega_{g,n+1}(p', p_1, ..., p_n). \quad (3.6.66)$$

Deformations with respect to natural vector fields on the Frobenius manifold correspond to relations between correlators in the CohFT. In the remainder of this section we describe the dictionary between deformations by the unit and Euler vector fields and their realisations via topological recursion.

3.6.1 Identity vector field.

When $v = \mathbb{1}$ is the identity vector, we have

$$\mathbb{1} \cdot ydx|_{x \text{ fixed}} = -\mathbb{1} \cdot xdy|_{y \text{ fixed}} = -dy = -\operatorname*{Res}_{p'=p} y(p')B(p, p') = \sum_P \operatorname*{Res}_{p'=P} y(p')B(p, p')$$

where the sum is over the poles P of y. Hence by (3.6.66)

$$\mathbb{1}\cdot\omega_{g,n}=\sum_P \operatorname*{Res}_{p'=P} y(p')\omega_{g,n+1}=-\sum_i \operatorname*{Res}_{p'=\mathcal{P}_i} y(p')\omega_{g,n+1}. \tag{3.6.67}$$

We can calculate the action of $\mathbb{1}$ on $\omega_{g,n}$ in a different way via the lift of $\mathbb{1}$ to the universal curve. Note that there are flat coordinates $t_1,.,,.t_N$ such that $\mathbb{1}=\partial/\partial t_1$ where t_1 appears in x as $x=x_0+t_1$ for x_0 independent of t_1. The lift $\mathbb{1}$ necessarily annihilates x so with respect to a local parameter z on Σ

$$0=\mathbb{1}\cdot x=x'(z)\mathbb{1}\cdot z+1 \quad\Rightarrow\quad \mathbb{1}\cdot z=-1/x'(z)$$

where we used the explicit partial derivative $\partial_{t_1}x=1$. Hence for any differential ξ with no explicit t_1 dependence, locally $\xi=df$ so we have

$$\mathbb{1}\cdot\xi(z)=d\mathbb{1}\cdot f(z)=d(f'(z)\mathbb{1}\cdot z)=-d(f'(z)/x'(z))=-d(df/dx)=-d(\xi/dx).$$

In other words the lift of $\mathbb{1}$ coincides on fibres with the operator

$$\mathbb{1}=-\frac{d}{dx}$$

which acts on functions or differentials. In particular, ξ_k^α have no explicit t_1 dependence so

$$\mathbb{1}\cdot\xi_k^\alpha=-d\left(\frac{\xi_k^\alpha}{dx}\right)=-\xi_{k+1}^\alpha.$$

Furthermore $\omega_{g,n}$ has no explicit t_1 dependence since topological recursion is unchanged under $x\mapsto x+t_1$. Hence

$$\mathbb{1}\cdot\omega_{g,n}=-\sum_{j=1}^n d\left(\frac{\omega_{g,n}(p_1,...,p_n)}{dx(p_j)}\right).$$

The relation

$$\sum_i \operatorname*{Res}_{p'=\mathcal{P}_i} y(p')\omega_{g,n+1}=-\sum_{j=1}^n d\left(\frac{\omega_{g,n}(p_1,...,p_n)}{dx(p_j)}\right)$$

is proven in a different way in [50] as a direct consequence of topological recursion. Here we have shown it to be a consequence of the action of the lift of the identity vector field on the universal curve.

The Hurwitz Frobenius manifold from Section 3.4 have flat identity hence the CohFT satisfies the pull-back relation (3.2.23). A consequence of (3.2.23) on correlators is known as the string equation which expressed in tau notation:

$$\left\langle\prod_{i=1}^n \tau_{k_i}(v_i)\right\rangle_g := \int_{\overline{\mathcal{M}}_{g,n}} I_{g,n}(v_1\otimes\cdots\otimes v_n)\prod_{i=1}^n \psi_i^{k_i}$$

is given by

$$\langle\tau_0(1)\tau_{k_1}(v_1)\cdots\tau_{k_n}(v_n)\rangle_g=\sum_{i=1}^n\langle\tau_{k_1}(v_1)\cdots\tau_{k_i-1}(v_i)\cdots\tau_{k_n}(v_n)\rangle_g.$$

But this is precisely equivalent to the relation (3.6.67) since

$$d\left(\frac{\xi_k^\alpha}{dx(p_j)}\right) = \xi_{k+1}^\alpha$$

where coefficients of ξ_k^α correspond to insertions of the vector field $\partial/\partial t_\alpha$.

3.6.2 Euler vector field.

When $v = E$ is the Euler vector field, we have

$$\begin{aligned} E \cdot ydx|_{x \text{ fixed}} &= -E \cdot xdy|_{y \text{ fixed}} = -xdy \\ &= \operatorname*{Res}_{p'=p} (\Phi - xy)(p')B(p,p') = -\sum_P \operatorname*{Res}_{p'=P} (\Phi - xy)B(p,p') \end{aligned} \tag{3.6.68}$$

where $d\Phi = ydx$ and the sum is over the poles P of $\Phi - xy$. Hence

$$\begin{aligned} E \cdot \omega_{g,n} &= -\sum_P \operatorname*{Res}_{p'=P} (\Phi - xy)\omega_{g,n+1} = \sum_i \operatorname*{Res}_{p'=\mathcal{P}_i} (\Phi - xy)\omega_{g,n+1} \\ &= (2g-2+n)\omega_{g,n}(p_1,...,p_n) - \sum_{j=1}^n d\left(\frac{x(p_j)\omega_{g,n}(p_1,...,p_n)}{dx(p_j)}\right) \end{aligned} \tag{3.6.69}$$

where the last equality uses the dilaton and second string equation satisfied quite generally by the $\omega_{g,n}$, proven in [50]. Analogous to the string equation above, which enables one to remove or insert the identity vector field, this last expression enables one to remove or insert the Euler vector field in correlators. For example, in the Gromov-Witten case, it is given by the divisor equation.

A conformal Frobenius manifold corresponds to a homogeneous CohFT. A CohFT is *homogeneous* of weight d if

$$\begin{aligned} ((g-1)d+n)I_{g,n} &= \deg I_{g,n}(v_1 \otimes ... \otimes v_n) \\ &\quad - \sum_{j=1}^n I_{g,n}(v_1 \otimes ... \otimes [E, v_j] \otimes ... \otimes v_n) \\ &\quad + \pi_* I_{g,n+1}(v_1 \otimes ... \otimes v_s \otimes E) \end{aligned} \tag{3.6.70}$$

where E is the Euler vector field and $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ is the forgetful map. Equation (3.6.70) allows one to remove or insert the Euler vector field in correlators and (3.6.68) is equivalent to this relation.

4

Chiodo formulas for the r-th roots and topological recursion

Abstract

We analyze Chiodo's formulas for the Chern classes related to the r-th roots of the suitably twisted integer powers of the canonical class on the moduli space of curves. The intersection numbers of these classes with ψ-classes are reproduced via the Chekhov-Eynard-Orantin topological recursion.

As an application, we prove that the Johnson-Pandharipande-Tseng formula for the orbifold Hurwitz numbers is equivalent to the topological recursion for the orbifold Hurwitz numbers. In particular, this gives a new proof of the topological recursion for the orbifold Hurwitz numbers.

4.1 Introduction

4.1.1 Topological recursion

The topological recursion in the sense of Chekhov, Eynard, and Orantin (see, e.g., [50]) takes as an input a spectral curve (Σ, x, y, B), i.e., the data of a Riemann surface Σ, two functions x and y on Σ with some compatibility condition, and the choice of a bi-differential B on the surface (which is canonical in the case $\Sigma = \mathbb{C}\mathrm{P}^1$, so we will omit it in this case). The recursion produces a collection of symmetric n-differentials $\mathcal{W}_{g,n}$ (called correlation differentials) defined again on the surface whose expansion can generate solutions to enumerative geometric problems.

In particular, under some conditions the expansion of $\mathcal{W}_{g,n}$ is related to the correlators of semi-simple cohomological field theories [37].

4.1.2 Chiodo's formula

In [78], Mumford derived a formula for the Chern character of the Hodge bundle on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$ in terms of the tautological classes and Bernoulli numbers. In [19], Chiodo generalizes Mumford's formula. The moduli stack $\overline{\mathcal{M}}_{g,n}$ is substituted with $\overline{\mathcal{M}}^{r,s}_{g;a_1,\ldots,a_n}$, the proper moduli stack of rth roots of the line bundle

$$\omega_{\log}^{\otimes s}\left(-\sum_{i=1}^{n} a_i x_i\right)$$

where $\omega_{\log} = \omega(\sum x_i)$, the integers s, $a_1, \ldots, a_n$ satisfy

$$(2g-2+n)s - \sum_i a_i \in r\mathbb{Z},$$

and the x_i's are the marked points on the curves. Let $\pi\colon \mathcal{C} \to \overline{\mathcal{M}}^{r}_{g,n}$ be the universal curve and denote by $\mathcal{S} \to \mathcal{C}$ the universal r-th root. Chiodo's formula computes the Chern character $\mathrm{ch}(R^\bullet \pi_* \mathcal{S})$, again in terms of tautological classes and values of Bernoulli polynomials at rational points with denominator r. The push-forward of the corresponding Chern class to the moduli space of curves will be called the *Chiodo class*.

In one particular case we know a relation between the Chiodo classes and the topological recursion. Namely, the coefficients of some expansion of the differentials $\mathcal{W}_{g,n}$ for the spectral curve data $(\Sigma = \mathbb{C}\mathrm{P}^1, x = \log z - z^r, y = z)$ are expressed in terms of the intersection numbers of the Chiodo classes for $s = 1$, $r = 1, 2, \ldots$. The main result of this chapter is an extension of this correspondence to arbitrary $s \geq 0$.

4.1.3 Chiodo classes and topological recursion

We consider the spectral curve

$$(\Sigma = \mathbb{C}\mathrm{P}^1, x(z) = -z^r + \log z, y(z) = z^s). \tag{4.1.1}$$

We prove that (see Theorem 4.4.6)

the expansion of the corresponding correlation differentials in some auxiliary basis of 1-forms is given by the intersection numbers of the corresponding Chiodo class for these particular $r, s \geq 1$.

The case $s = 0$ is exceptional. In this case, the intersection numbers are the same as in the case $s = r$, so we still have to use the spectral curve $(\Sigma = \mathbb{C}\mathrm{P}^1, x(z) = -z^r + \log z, y(z) = z^r)$.

These spectral curves are known in the literature, in some particular cases, in relation to various versions of Hurwitz numbers.

4.1.4 Hurwitz numbers

Hurwitz numbers play an important role in the interaction of combinatorics, representation theory of symmetric groups, integrable systems, tropical geometry, matrix models, and intersection theory on moduli spaces of curves.

There are several kinds of Hurwitz numbers. Simple Hurwitz numbers enumerate finite degree d coverings of the 2-sphere by a genus g connected surface, with a fixed ramification profile $(\mu_1, \ldots, \mu_n)$ over infinity, $\sum_{i=1}^n \mu_i = d$ while the remaining $2g-2+n+d$ ramifications over fixed points are simple.

These Hurwitz numbers are known to be the coefficients of the expansions of the correlation forms of the spectral curve (4.1.1) for $r = s = 1$. This was conjectured in [13] and proved in several different ways, see, e.g., [49, 33].

Chiodo's formula in this case is reduced to the standard Mumford formula, so the Chiodo class is the Chern class of the dual Hodge bundle on the moduli space of curves. The fact that the same correlation differentials are related, in different expansion, to simple Hurwitz numbers and to the intersection numbers, implies that there is a formula for simple Hurwitz numbers in terms of the intersection numbers. Indeed, it is the celebrated ELSV formula [39]. The equivalence between the topological recursion and the ELSV formula is proved in [42], see also [33, 87].

Another example is r-spin Hurwitz numbers. In this case, the definition is a bit involved; roughly speaking, we still consider the maps of genus g algebraic curves to $\mathbb{C}\mathrm{P}^1$, with a fixed profile over infinity, but the remaining simple ramifications are replaced by more complicated singularities, so-called completed cycles. We refer to [86, 87] for the precise definition.

In this case, the r-spin Hurwitz numbers are conjecturally related by the spectral curve (4.1.1) for that particular r and $s = 1$, see [77, 87]. The same logic as for the simple Hurwitz numbers implies that this conjecture is equivalent to an ELSV-type formula that expresses the r-spin Hurwitz numbers in terms of intersection numbers [87]. The corresponding ELSV-type formula was conjectured in [99] and is still open.

4.1.5 Orbifold Hurwitz numbers

A case of special interest for us is the r-orbifold Hurwitz numbers. They enumerate finite degree d, $r|d$, coverings of the 2-sphere by a genus g connected surface, with a fixed ramification profile $(\mu_1, \ldots, \mu_n)$ over the infinity, $\sum_{i=1}^{n} \mu_i = d$, the fixed ramification profile $(r, r, \ldots, r)$ over zero, while the remaining $2g - 2 + n + d/r$ ramifications over fixed points are simple.

It is proved in [14, 25] that the r-orbifold Hurwitz numbers satisfy the topological recursion for the spectral curve (4.1.1) with this particular r and $s = r$. Johnson-Pandharipande-Tseng [63] exhibited an ELSV-type formula that can be restricted to express r-orbifold Hurwitz numbers in terms of intersection numbers. As an application of the general correspondence between the Chiodo formulas and topological recursion, we prove the equivalence of these two statements (see Theorem 5.1).

Since the Johnson-Pandharipande-Tseng formula (the JPT formula, for brevity) is proved independently, our equivalence result implies a proof of the topological recursion of r-orbifold Hurwitz numbers.

It is a new proof of the topological recursion; the existing proofs [14, 25] do use the JPT formula, but only its combinatorial structure, and not the geometry of the classes. The topological recursion is then derived in [14, 25] from an additional recursion relation for r-orbifold Hurwitz numbers called cut-and-join equation.

4.1.6 Further remarks

A natural question is whether we can use the equivalence between the topological recursion and the JPT formula for r-orbifold Hurwitz numbers in order to give a new proof of the JPT formula, as it is done in [33] for the simple Hurwitz numbers. This approach requires a new proof of the topological recursion that wouldn't use the JPT formula. This is done in [34], so we refer there for further details.

Another natural question is whether there is any natural combinatorial and/or geometric problem of Hurwitz type related to the other spectral curves (4.1.1) for arbitrary r and s. The only indication of a possible relation that we know is that similar spectral curves are used in [77] for the so-called mixed Hurwitz numbers in the context of the quantum spectral curve theory.

4.1.7 Plan of the chapter

In Section 2 we review the semi-simple cohomological field theories, possibly with a non-flat unit, that correspond to Chiodo classes. In Section 3 we recall the general formula of the differentials $\mathcal{W}_{g,n}$ in terms of integrals over moduli spaces of curves as described in [37, 46], while in Section 4 we compute explicitly all the ingredients of that formula and prove our main theorem, Theorem 4.4.6. Finally, in Section 5 we identify the particular Chiodo class with the one used in the JPT formula and prove the equivalence of the JPT formula and the topological recursion for r-orbifold Hurwitz numbers.

4.2 Chiodo classes

In this Section we recall the definition and some simple properties of the Chiodo classes. These classes are defined on the moduli spaces of tensor rth roots of the line bundle $\omega_{\log}^{\otimes s}(-\sum m_i x_i)$, but here we will only need their push-forward to the space of curves $\overline{\mathcal{M}}_{g,n}$. A more detailed discussion of the space of rth roots in the case $s = 0$ is contained in Section 4.5.2. We also refer the reader to [19, 21, 20, 87] for all necessary background and origin of the lemmas in this section.

4.2.1 Definition

Let $r \geq 1$ be an integer and $1 \leq a_1, \dots, a_n \leq r$, $0 \leq s$ be integers satisfying

$$(2g-2+n)s - \sum_{i=1}^{n} a_i \in r\mathbb{Z} \tag{4.2.2}$$

Consider the morphisms

$$\overline{\mathcal{C}} \xrightarrow{\pi} \overline{\mathcal{M}}^{r,s}_{g;a_1,\dots,a_n} \xrightarrow{\epsilon} \overline{\mathcal{M}}_{g,n},$$

where $\overline{\mathcal{M}}^{r,s}_{g;a_1,\dots,a_n}$ is the space of rth roots $\mathcal{S}^{\otimes r} \xrightarrow{\sim} \omega_{\log}^{\otimes s}(-\sum a_i x_i)$, $\overline{\mathcal{C}}$ is its universal curve, and ϵ is the forgetful morphism to the space of curves.

While the boundary strata of $\overline{\mathcal{M}}_{g,n}$ are described by stable graphs, those of $\overline{\mathcal{M}}^{r,s}_{g;a_1,\dots,a_n}$ are described by stable graphs with a remainder mod r assigned to each half-edge in such a way that the sum of residues on each edge vanishes and that

Condition (4.2.2) is satisfied for each vertex. The boundary divisors correspond to one-edged graphs with two opposite remainders mod r assigned the two half-edges.

The Chern characters of the derived push-forward $R^*\pi_*\mathcal{S}$ are given by Chiodo's formula [19]

$$\mathrm{ch}_m(R^*\pi_*\mathcal{S}) = \frac{B_{m+1}(\frac{s}{r})}{(m+1)!}\kappa_m - \sum_{i=1}^{n} \frac{B_{m+1}(\frac{a_i}{r})}{(m+1)!}\psi_i^m \tag{4.2.3}$$
$$+\frac{r}{2}\sum_{a=0}^{r-1} \frac{B_{m+1}(\frac{a}{r})}{(m+1)!}(j_a)*\frac{(\psi')^m + (-1)^{m-1}(\psi'')^m}{\psi'+\psi''},$$

where j_a is the boundary map corresponding to the boundary divisor with remainder a at one of the two half-edges and ψ', ψ'' are the ψ-classes at the two branches of the node.

We are interested in the Chiodo classes

$$\begin{aligned}
&\mathrm{C}_{g,n}(r,s;a_1,\dots,a_n) = \\
&\epsilon_* c(-R^*\pi_*\mathcal{S}) = \\
&\epsilon_* \left[c(R^1\pi_*\mathcal{S})/c(R^0\pi_*\mathcal{S})\right] = \\
&\epsilon_* \exp\left(\sum_{m=1}^{\infty}(-1)^m(m-1)!\mathrm{ch}_m(R^*\pi_*\mathcal{S})\right) \in H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}).
\end{aligned} \tag{4.2.4}$$

An explicit expression of the classes $\mathrm{C}_{g,n}(r,s;a_1,\dots,a_n)$ in terms of stable graphs, obtained by expanding the exponential in the expression above, is given in [62], Corollary 4.

Consider $\mathrm{C}_{g,n}(r,s;a_1,\dots,a_n)$ as a coefficient of a map

$$\mathrm{C}_{g,n}(r,s)\colon V^{\otimes n} \to H^{\mathrm{even}}(\overline{\mathcal{M}}_{g,n}), \tag{4.2.5}$$

where $V = \langle v_1,\dots,v_r\rangle$, and

$$\mathrm{C}_{g,n}(r,s)\colon v_{a_1}\otimes\cdots\otimes v_{a_n} \mapsto \mathrm{C}_{g,n}(r,s;a_1,\dots,a_n). \tag{4.2.6}$$

4.2.2 Cohomological field theories

Lemma 4.2.1. *For $0 \le s \le r$ the classes $\{\mathrm{C}_{g,n}(r,s)\}$ form a semi-simple cohomological field theory.*

A semi-simple cohomological field theory (CohFT) is obtained via the action of an element of the upper-triangular Givental group on a topological field theory. In order to determine a topological field theory $\{\omega_{g,n}\}$, we have to fix its scalar product η and $\omega_{0,3}$. An element of the upper-triangular Givental group is determined by a matrix $R(\zeta) \in \mathrm{End}(V)[[\zeta]]$ that should satisfy the symplectic conditions with respect to η.

In the case of $\{\mathrm{C}_{g,n}(r,s)\}$ we have the following description.

Lemma 4.2.2. *For $0 \leq s \leq r$ the classes $\{\mathrm{C}_{g,n}(r,s)\}$ are given by Givental's action of the R-matrix $R(\zeta)$ on the topological field theory ω with metric η on V, where*

$$V = \langle v_1, \dots, v_r \rangle, \tag{4.2.7}$$

$$R(\zeta) = \exp\left(\sum_{m=1}^{\infty} \frac{\mathrm{diag}_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)} (-\zeta)^m \right), \tag{4.2.8}$$

$$R^{-1}(\zeta) = \exp\left(- \sum_{m=1}^{\infty} \frac{\mathrm{diag}_{a=0}^{r-1} B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)} (-\zeta)^m \right), \tag{4.2.9}$$

$$\eta(v_a, v_b) = \frac{1}{r} \delta_{a+b \mod r}, \tag{4.2.10}$$

$$\omega_{0,3}(v_a \otimes v_b \otimes v_c) = \frac{1}{r} \delta_{a+b+c-s \mod r}, \tag{4.2.11}$$

$$\omega_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = r^{2g-1} \delta_{a_1+\cdots+a_n - s(2g-2+n) \mod r}. \tag{4.2.12}$$

4.2.3 Cohomological field theories with a non-flat unit

Let us discuss now what happens for $s > r$. We need an extension of the notion of cohomological field theory, namely, we have to consider the cohomological field theories with a non-flat unit, CohFT/1 for brevity.

The CohFT/1s are obtained by an extension of the Givental group by translations, which allows one to use the dilaton leaves (in the terminology of [38, 37]) or κ-legs (in the terminology of [82]) with arbitrary coefficients. We refer to the exposition in [82] for further details.

One of the possible descriptions of a CohFT/1 is in terms of stable graphs without any κ-legs. The vertices, leaves, and edges of these graphs are decorated in exactly the same way as in the case of a usual CohFT, but in addition every vertex is also decorated by $\exp(\sum_{m=1}^{\infty} T_m \kappa_m)$ for some constants T_m, $m = 1, 2, \dots$.

In the case of Chiodo classes (4.2.4) for $s > r$, we have the following:

Lemma 4.2.3. *For $s > r$ the classes $\{\mathrm{C}_{g,n}(r,s)\}$ form a CohFT/1. The corresponding element of the extended Givental group coincides with the one described in Lemma 4.2.2, but instead of the dilaton shift, we decorate each vertex by*

$$\exp\left(\sum_{m=1}^{\infty} (-1)^m \frac{B_{m+1}(\frac{s}{r})}{m(m+1)} \kappa_m \right). \tag{4.2.13}$$

4.3 Topological recursion and Givental group

In this Section we revisit the main result of [37, 46]. We present a bit refined version of it, in order to make precise relation that incorporates a torus action on cohomological field theories.

4.3.1 General background

The input of the local topological recursion consists of a local spectral curve $\Sigma = \sqcup_{i=1}^{r} U_i$, which is a disjoint union of open disks with the center points p_i, $i = 1, \dots, r$, holomorphic function $x\colon \Sigma \to \mathbb{C}$ such that the zeros of its differential dx are $p_1, \dots, p_r$, holomorphic function $y\colon \Sigma \to \mathbb{C}$, and a symmetric bidifferential B defined on $\Sigma \times \Sigma$ with a double pole on the diagonal with residue 1.

The output is a set of symmetric differentials $\mathcal{W}_{g,n}$ on Σ^n. This set of differentials is canonically associated to the input data via the topological recursion procedure. Under some conditions (for example, when Σ is an open submanifold of a Riemann surface, where dx is a globally defined meromorphic differential, see [46], and we should assume some relation between y and B, see [37] and below), we can represent this set of differentials in terms of the correlators of a CohFT multiplied by some auxiliary differentials. This representation is not canonical, the choice of it is controlled by the action of the group $(\mathbb{C}^*)^r$.

Our goal is to make this action on all ingredients of the formula (that is, the matrix R of a CohFT, its underlying TFT, and the auxiliary differentials) precise.

4.3.2 The formula

We fix a point $(C_1, \dots, C_r) \in (\mathbb{C}^*)^r$. We also fix some additional constant $C \in \mathbb{C}^*$. All constructions in this Section depend on these choices.

We choose a local coordinate w_i on U_i, $i = 1, \dots, r$, such that $w_i(p_i) = 0$ and

$$x = (C_i w_i)^2 + x_i. \tag{4.3.14}$$

In this case, the underlying TFT is given by

$$\begin{aligned} &\eta(e_i, e_j) = \delta_{ij}, \\ &\alpha^{Top}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \delta_{i_1 \dots i_n} \left(-2C_i^2 C \frac{\mathrm{d}y}{\mathrm{d}w_i}(0) \right)^{-2g+2-n}. \end{aligned} \tag{4.3.15}$$

In particular, the unit vector is equal to $\sum_{i=1}^r \left(-2C_i^2 C \frac{\mathrm{d}y}{\mathrm{d}w_i}(0) \right) e_i$.

The matrix $R(\zeta)$ is given by

$$-\frac{1}{\zeta} R^{-1}(\zeta)_i^j = \frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \left. \frac{B(w_i, w_j)}{\mathrm{d}w_i} \right|_{w_i=0} \cdot e^{-\frac{w_j^2}{2\zeta}}. \tag{4.3.16}$$

We have to check that the function y satisfies the condition

$$\frac{2C_i^2 C}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \mathrm{d}y \cdot e^{-\frac{w_i^2}{2\zeta}} = \sum_{k=1}^{r} (R^{-1})_k^i \left(2C_k^2 C \frac{\mathrm{d}y}{\mathrm{d}w_k}(0) \right) \tag{4.3.17}$$

Finally, the auxiliary functions $\xi_i\colon \Sigma \to \mathbb{C}$ are given by

$$\xi_i(x) := \int^x \left. \frac{B(w_i, w)}{\mathrm{d}w_i} \right|_{w_i=0} \tag{4.3.18}$$

Using Formulas (4.3.15) and (4.3.16) we define a CohFT, whose classes we denote by $\alpha^{Coh}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n})$.

Theorem 4.3.1. *[46, 37] The differentials $\mathcal{W}_{g,n}$ produced by the topological recursion from the input (Σ, x, y, B) are equal to*

$$\mathcal{W}_{g,n} = C^{2g-2+n} \sum_{\substack{i_1,\dots,i_n \\ d_1,\dots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \alpha^{Coh}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{d}\left(\left(-\frac{1}{w_j}\frac{\mathrm{d}}{\mathrm{d}w_j} \right)^{d_j} \xi_{i_j} \right). \tag{4.3.19}$$

In particular, this formula doesn't depend on the choice of $(C_1, \dots, C_r) \in (\mathbb{C}^*)^r$ and $C \in \mathbb{C}^*$, though all its ingredients do.

The proof of this theorem is given by exactly the same argument as in [46, 37], with a different choice of local coordinates near the points p_i, so we omit it here.

Remark 4.3.2. Let us discuss what happens if the condition (4.3.17) is not satisfied. Still, under the same conditions a version of Theorem 4.3.1 holds. Namely, we can represent the correlation differentials as

$$\mathcal{W}_{g,n} = C^{2g-2+n} \sum_{\substack{i_1,\dots,i_n \\ d_1,\dots,d_n}} \int_{\overline{\mathcal{M}}_{g,n}} \alpha^{Coh/1}_{g,n}(e_{i_1} \otimes \cdots \otimes e_{i_n}) \prod_{j=1}^{n} \psi_j^{d_j} \mathrm{d}\left(\left(-\frac{1}{w_j}\frac{\mathrm{d}}{\mathrm{d}w_j} \right)^{d_j} \xi_{i_j} \right), \tag{4.3.20}$$

where the classes $\alpha^{Coh/1}_{g,n}$ are described, in terms of the graphical formalism recalled in Section 4.2.3, via the same TFT and R-matrix as $\alpha^{Coh}_{g,n}$ in Theorem 4.3.1, but instead of the dilaton leaves, we decorate each vertex labeled by i (that is, the one that is decorated by $\alpha^{Top}_{g,n}(e_i \otimes \cdots \otimes e_i)$) with the κ-class

$$\exp\left(\sum_{k=1}^{\infty} T_{i,k}\kappa_k \right), \tag{4.3.21}$$

where the constants $T_{i,k}$ are given by

$$\frac{\mathrm{d}y}{\mathrm{d}w_i}(0) \exp\left(\sum_{k=1}^{\infty} T_{i,k}(-\zeta)^k \right) = \frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \mathrm{d}y \cdot e^{-\frac{w_i^2}{2\zeta}}. \tag{4.3.22}$$

This is a direct corollary of [42, Theorem 3.2], see also [37, Lemma 3.5].

4.4 Computations with the spectral curve

Consider the following initial data on the spectral curve $\Sigma = \mathbb{CP}^1$ with a global coordinate z:

$$\begin{aligned} x(z) &= -z^r + \log z; \\ y(z) &= z^s; \\ B(z, z') &= \frac{\mathrm{d}z\,\mathrm{d}z'}{(z-z')^2}. \end{aligned} \tag{4.4.23}$$

In this section we compute all ingredients of the Formula (4.3.19) for this initial data with a special choice of the torus point. In particular, for $1 \leq s \leq r$ we prove that the correlation differentials are controlled by a CohFT, and the corresponding CohFT coincides with the one given by Chiodo classes (4.2.4) considered in the normalized canonical frame.

4.4.1 Local expansions

As it was computed in [87], we can associate with this curve the following local data.

The critical points are

$$p_i := r^{-1/r}\mathrm{J}^i, \qquad i = 0, \dots, r-1, \tag{4.4.24}$$

and the critical values of the function x at these points are

$$x_i := x(p_i) = -\frac{1}{r} + \frac{2\pi i\sqrt{-1}}{r} - \frac{\log r}{r}, \qquad i = 0, \dots, r-1. \tag{4.4.25}$$

If we choose a local coordinate w_i near the point p_i such that $w_i(p_i) = 0$ and $-w_i^2/2r + x_i = x$, $i = 0, 1, \dots, r-1$, then there are two possible choices for the expansion of the function z in w_i. We fix it to be

$$z(w_i) = r^{-1/r}\mathrm{J}^i + \left(r^{-1-\frac{1}{r}}\mathrm{J}^i\right) w_i + O(w_i^2), \tag{4.4.26}$$

With this choice we also fix the expansion of $y = z^s$, namely,

$$y(w_i) = r^{-s/r}\mathrm{J}^{si} + \left(sr^{-1-\frac{s}{r}}\mathrm{J}^{is}\right) w_i + O(w_i^2). \tag{4.4.27}$$

Lemma 4.4.1. *We have:*

$$\frac{1}{\sqrt{2\pi\zeta}} \int_{-\infty}^{\infty} \mathrm{d}y(w_i) \cdot e^{-\frac{w_i^2}{2\zeta}} \sim \left(sr^{-1-\frac{s}{r}}\mathrm{J}^{is}\right) \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{s}{r}\right)}{m(m+1)}(-\zeta)^m\right). \tag{4.4.28}$$

Proof. This Lemma is analogous to [87, Lemma 4.3]. Indeed, we introduce a new coordinate $t = rz^r$. In this coordinate we have:

$$z = t^{\frac{1}{r}} r^{-\frac{1}{r}} \mathrm{J}^i; \tag{4.4.29}$$

$$-x_i - z^r + \log z = \frac{1}{r} - \frac{t}{r} + \frac{\log t}{r}; \tag{4.4.30}$$

$$\mathrm{d}z = t^{\frac{1-r}{r}} r^{-1-\frac{1}{r}} \mathrm{J}^i \mathrm{d}t. \tag{4.4.31}$$

We can then make a change of variables and use the standard asymptotic expansion of the gamma function, cf. the proof of Lemma 4.3 in [87]:

$$\begin{aligned} \frac{\sqrt{-2r}}{\sqrt{2\pi\zeta}} \int \mathrm{d}y \cdot e^{2r\cdot\frac{(x-x_i)}{2\zeta}} &= \frac{sr^{-\frac{1}{2}-\frac{s}{r}}\mathrm{J}^{si} e^{\frac{1}{\zeta}}}{\sqrt{-\pi\zeta}} \int \mathrm{d}t \cdot t^{\frac{s-r}{r}+\frac{1}{\zeta}} e^{-\frac{t}{\zeta}} \\ &\sim \left(s\sqrt{-2}r^{-\frac{1}{2}-\frac{s}{r}}\mathrm{J}^{si}\right) \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{s}{r}\right)}{m(m+1)}(-\zeta)^m\right). \end{aligned} \tag{4.4.32}$$

□

Lemma 4.4.2. *We have:*

$$\frac{1}{\sqrt{2\pi\zeta}}\int_{-\infty}^{\infty}\left.\frac{B(w_i,w_j)}{\mathrm{d}w_i}\right|_{w_i=0}\cdot e^{-\frac{w_j^2}{2\zeta}} \sim \sum_{c=0}^{r-1}\frac{\mathrm{J}^{cj-ci}}{r}\frac{\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right)}{(-\zeta)}. \tag{4.4.33}$$

Proof. This Lemma is just a refined version of Lemma 4.4 in [87], so the proof is exactly the same as there. □

Note that this Lemma means that we have to consider the Givental group action defined by the matrix $R(\zeta)$, where

$$R^{-1}(\zeta)_i^j := \sum_{c=0}^{r-1}\frac{\mathrm{J}^{cj-ci}}{r}\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right). \tag{4.4.34}$$

We choose the constants $C_1 = \cdots = C_r := 1/\sqrt{-2r}$ and $C := r^{1+s/r}/s$. In particular, with this choice the structure constants of the underlying TFT are given by

$$-2C_i^2 C\frac{\mathrm{d}y}{\mathrm{d}w_i}(0) = \frac{\mathrm{J}^{is}}{r} \tag{4.4.35}$$

Lemma 4.4.3. *For* $1 \le s \le r$ *the condition* (4.3.17) *is satisfied.*

Proof. This is a direct computation. We have:

$$\begin{aligned}\frac{2C_i^2 C}{\sqrt{2\pi\zeta}}\int_{-\infty}^{\infty}\mathrm{d}y\cdot e^{-\frac{w_i^2}{2\zeta}} &= -\frac{\mathrm{J}^{is}}{r}\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{s}{r}\right)}{m(m+1)}(-\zeta)^m\right) \\ &= \sum_{k=1}^{r}\sum_{c=0}^{r-1}\frac{\mathrm{J}^{ci-ck}}{r}\exp\left(-\sum_{m=1}^{\infty}\frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right)\left(-\frac{\mathrm{J}^{ks}}{r}\right) \\ &= \sum_{k=1}^{r}(R^{-1})_k^i\left(2C_k^2 C\frac{\mathrm{d}y}{\mathrm{d}w_k}(0)\right)\end{aligned} \tag{4.4.36}$$

The second equality is true for $0 \le s \le r-1$, and also for $s = r$, since $B_{m+1}(1) = B_{m+1}(0)$ for $m \ge 1$. □

This Lemma implies that we indeed have correlators of a cohomological field theory inside Formula (4.3.19) in this case.

Finally, Definition (4.3.18) implies that

$$\xi_i = \frac{r^{-1-\frac{1}{r}}\mathrm{J}^i}{r^{-\frac{1}{r}}\mathrm{J}^i - z}, \tag{4.4.37}$$

and it is easy to see that

$$-\frac{1}{w}\frac{\mathrm{d}}{\mathrm{d}w} = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}x}. \tag{4.4.38}$$

This completes the description of all the ingredient of the Formula (4.3.19) for the correlation differentials $\mathcal{W}_{g,n}$.

4.4.2 Correlation differentials in flat basis

In the previous section we described all ingredients of the formula for the correlation differentials (4.3.19) for the case of the spectral curve data (4.4.23). In particular, for $1 \leq s \leq r$ we proved that there are the correlators of a CohFT inside this formula, otherwise we have a CohFT/1. Our goal now is to show that the cohomological field theories obtained in the previous Section is the one given by the same formulas as in Lemmas 4.2.2 and 4.2.3. In order to do that we apply a linear change of variables to the basis $e_0, \dots, e_{r-1}$ used in the previous Section.

We use the change of basis from $e_0, \dots, e_{r-1}$ to $v_1, \dots, v_r$ given by the formula

$$e_i = \sum_{a=1}^{r} \mathrm{J}^{-ai} v_a; \qquad v_a = \sum_{i=0}^{r-1} \frac{\mathrm{J}^{ai}}{r} e_i \tag{4.4.39}$$

Lemma 4.4.4. *In the basis $v_1, \dots, v_r$ we have:*

- *The underlying TFT $\alpha_{g,n}^{Top}$ (4.3.15) with the choice of constants given by Equation (4.4.35) is given by*

$$\eta(v_a, v_b) = \frac{1}{r}\delta_{a+b \mod r}; \tag{4.4.40}$$

$$\omega_{0,3}(v_a \otimes v_b \otimes v_c) = \frac{1}{r}\delta_{a+b+c-s \mod r}$$

$$\omega_{g,n}(v_{a_1} \otimes \cdots \otimes v_{a_n}) = r^{2g-1}\delta_{a_1+\cdots+a_n-s(2g-2+n) \mod r}$$

- *The R-matrix is given by*

$$R(\zeta) = \exp\left(\sum_{m=1}^{\infty} \frac{\mathrm{diag}_{a=1}^{r} B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)}(-\zeta)^m\right) \tag{4.4.41}$$

- *The auxiliary functions ξ_a are given by*

$$\xi_a = r^{\frac{r-a}{r}} \sum_{p=0}^{\infty} \frac{(pr+r-a)^p}{p!} e^{(pr+r-a)x}. \tag{4.4.42}$$

Proof. The computation of the underlying TFT is fairly simple:

$$\eta(v_a, v_b) = \sum_{i,j=0}^{r-1} \frac{\mathrm{J}^{ai+bj}}{r^2}\eta(e_i, e_j) = \sum_{i=0}^{r-1} \frac{\mathrm{J}^{(a+b)i}}{r^2} = \frac{1}{r}\delta_{a+b \mod r}, \tag{4.4.43}$$

$$\omega_{0,3}(v_a \otimes v_b \otimes v_c) = \sum_{i=0}^{r-1} \frac{\mathrm{J}^{ai+bi+ci}}{r^3}\omega_{0,3}(e_i \otimes e_i \otimes e_i)$$

$$= \sum_{i=0}^{r-1} \frac{\mathrm{J}^{ai+bi+ci-si}}{r^2} = \frac{1}{2}\delta_{a+b+c-s \mod r},$$

and the other correlators of the underlying TFT are determined uniquely.

The change of basis for the matrix R^{-1} reads:

$$R^{-1}(\zeta)^b_a = \sum_{i,j=0}^{r-1} \frac{\mathrm{J}^{-jb+ia}}{r} \sum_{c=0}^{r-1} \frac{\mathrm{J}^{cj-ci}}{r} \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right) \tag{4.4.44}$$
$$= \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{c}{r}\right)}{m(m+1)}(-\zeta)^m\right) \cdot \delta_{c-b \mod r} \cdot \delta_{c-a \mod r}$$
$$= \exp\left(-\sum_{m=1}^{\infty} \frac{B_{m+1}\left(\frac{a}{r}\right)}{m(m+1)}(-\zeta)^m\right) \cdot \delta_{a-b},$$

which implies Equation (4.4.41).

Finally, Equation (4.4.42) follows from Lemma 4.6 in [87]. □

Remark 4.4.5. Observe that Equations (4.4.40) and (4.4.41) and Lemma 4.4.3 imply that for $s \leq r$ the cohomological field theory that we have in the flat basis coincides with the one given in Lemma 4.2.2. For $s > r$, where Lemma 4.4.3 does not apply, we have obtained the topological field theory and the R-matrix as in Lemma 4.2.3, but we still have to compare the power series that determines the κ-legs.

Lemma 4.4.4 allows us to rewrite formula (4.3.19) for the correlation differentials of the spectral curve data (4.4.23) in the following way.

Theorem 4.4.6. *The correlation differentials of the spectral curve* (4.4.23) *are equal to*

$$\mathcal{W}_{g,n} = \sum_{\mu_1,\ldots,\mu_n=1}^{\infty} \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \, e^{\sum_{j=1}^n \mu_j x_j} \tag{4.4.45}$$
$$\times \int_{\overline{\mathcal{M}}_{g,n}} \frac{\mathrm{C}_{g,n}\left(r, s; r - r\left\langle\frac{\mu_1}{r}\right\rangle, \ldots, r - r\left\langle\frac{\mu_n}{r}\right\rangle\right)}{\prod_{j=1}^n (1 - \frac{\mu_i}{r}\psi_i)}$$
$$\times \prod_{j=1}^{n} \frac{\left(\frac{\mu_j}{r}\right)^{\left\lfloor\frac{\mu_j}{r}\right\rfloor}}{\left\lfloor\frac{\mu_j}{r}\right\rfloor!} \times \frac{r^{2g-2+n+\frac{(2g-2+n)s+\sum_{j=1}^n \mu_j}{r}}}{s^{2g-2+n}},$$

where $\frac{\mu}{r} = \lfloor\frac{\mu}{r}\rfloor + \langle\frac{\mu}{r}\rangle$ *is the decomposition into the integer and the fractional parts.*

Proof. First, consider the case $s \leq r$. Using Equation (4.3.19), together with

Lemma 4.4.4, Remark 4.4.5, Equation (4.4.38) and $C = r^{1+s/r}/s$, we have:

$$\begin{aligned}
&\mathcal{W}_{g,n}(x_1,\dots,x_n) \qquad (4.4.46)\\
&= \sum_{\substack{d_1,\dots,d_n\geq 0\\ 1\leq a_1,\dots,a_n\leq r}} \frac{r^{2g-2+n+\frac{(2g-2+n)s}{r}}}{s^{2g-2+n}} \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{C}_{g,n}(r,s;a_1,\dots,a_n)\\
&\times \prod_{j=1}^{n} \psi_j^{d_j} r^{-d_j} r^{\frac{r-a_j}{r}} \mathrm{d}\left[\left(\frac{\mathrm{d}}{\mathrm{d}x_j}\right)^{d_j} \sum_{p=0}^{\infty} \frac{(pr+r-a_j)^p}{p!} e^{(pr+r-a_j)x_j}\right]\\
&= \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \sum_{\substack{d_1,\dots,d_n\geq 0\\ 1\leq a_1,\dots,a_n\leq r}} \int_{\overline{\mathcal{M}}_{g,n}} \mathrm{C}_{g,n}(r,s;a_1,\dots,a_n) \prod_{j=1}^{n} \psi_j^{d_j}\\
&\times \frac{r^{2g-2+2n-\sum_{j=1}^n d_j + \frac{(2g-2+n)s-\sum_{j=1}^n a_j}{r}}}{s^{2g-2+n}}\\
&\times \prod_{j=1}^{n} \sum_{p=0}^{\infty} \frac{(pr+r-a_j)^{p+d_j}}{p!} e^{(pr+r-a_j)x_j}.
\end{aligned}$$

Equation (4.4.45) is just a way to rewrite the last formula using a summation over the parameter $\mu_i = p_i r + r - a_i$ instead of a double summation over p_i and a_i.

In the case $s > r$, we should compute separately the κ-classes. In this case, Remark 4.3.2 and Equation (4.4.28) imply that the κ-class attached to the vertex of index i (in the basis $e_0,\dots,e_{r-1}$) is equal to $\exp\left(\sum_{m=1}^{\infty}(-1)^m \frac{B_{m+1}(\frac{s}{r})}{m(m+1)}\kappa_m\right)$. Since it doesn't depend on i, it remains the same in the basis $v_1,\dots,v_r$, where it coincides with the one given by Lemma 4.2.3. □

Remark 4.4.7. Note that in the case $s = 1$ we reproduce Theorem 1.7 in [87].

4.5 Johnson-Pandharipande-Tseng formula and topological recursion

In this Section we consider a special case of the correspondence between the Chiodo formulas and the spectral curve topological recursion. We assume that $s = r$. In this case, the correlation differentials of this spectral curve are known to give the so-called r-orbifold Hurwitz numbers in some expansion.

An r-orbifold Hurwitz number $h_{g;\vec{\mu}}$ is just a double Hurwitz number that enumerates ramified coverings of the sphere by a genus g surface, where one special fiber is arbitrary (given by the partition $\vec{\mu}$ of length n) and one has ramification indices $(r, r, \dots, r)$. Therefore, the degree of the covering $\sum_{i=1}^n \mu_i$ is divisible by r and there are $b = 2g - 2 + n + \sum_{i=1}^n \mu_i/r$ simple critical points.

The r-orbifold Hurwitz numbers are also known to satisfy the Johnson-Pandharipande-Tseng (JPT) formula that expresses them in terms of the intersection theory of the moduli space of curves. The main goal of this Section is to show that the JPT formula is equivalent to the topological recursion for r-orbifold

Hurwitz numbers. In particular, this gives a new proof of the topological recursion for r-orbifold Hurwitz numbers.

4.5.1 The JPT formula

The formula of Johnson, Pandharipande and Tseng is presented in [63] for a general abelian group G, its particular finite representation U and a vector of monodromies γ. Here we consider only the case of $G = \mathbb{Z}/r\mathbb{Z}$, the representation U sends $1 \in \mathbb{Z}/r\mathbb{Z}$ to $e^{\frac{2\pi i}{r}}$, and γ is empty. In this case the JPT formula reads

$$\frac{h_{g;\vec{\mu}}}{b!} = r^{1-g+\sum\langle\frac{\mu_i}{r}\rangle} \prod_{i=1}^{n} \frac{\mu_i^{\lfloor\frac{\mu_i}{r}\rfloor}}{\lfloor\frac{\mu_i}{r}\rfloor!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\epsilon_* \sum_{i\geq 0}(-r)^i\lambda_i}{\prod_{j=1}^{n}(1-\mu_j\psi_j)}, \tag{4.5.47}$$

where the class $\epsilon_* \sum_{i\geq 0}(-r)^i\lambda_i$ is described in detail below.

4.5.2 Two descriptions of rth roots

Let $G = \mathbb{Z}/r\mathbb{Z}$ be the abelian group of rth roots of unity. The space $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ is the space of stable maps to the stack $\mathcal{B}G$ with monodromies $a_i \in \{0,\dots,r-1\}$ at the marked points. This space, and the natural cohomology classes on it, can be constructed in several ways, see, for instance, [1, 18]. Johnson, Pandharipande, and Tseng [63] use the description via admissible covers. Chiodo [19] uses the moduli space of rth roots of the line bundle $\mathcal{O}(-\sum a_i x_i)$. Here, we apply Chiodo's formulas to a result of Johnson, Pandharipande, and Tseng, so we recall and briefly explain the equivalence between the two approaches.

The r-stable curves.

An *r-stable curve* is an orbifold stable curve whose only nontrivial orbifold structure appears at the nodes and at the markings. The neighborhood of a marking is isomorphic to Δ/G, where an rth root of unity $\rho \in G$ acts on the disc Δ by $z \mapsto \rho z$. The neighborhood of a node in a family of r-stable curves is isomorphic to $(\Delta\times\Delta)/G$, where $\rho \in G$ acts by $(z,w) \mapsto (\rho z, \rho^{-1}w)$.

The moduli space of r-stable curves has the same coarse space as $\overline{\mathcal{M}}_{g,n}$, but an extra factor of G appears in the stabilizer for every node of the curve.

Line bundles over r-stable curves.

A line bundle L over an r-stable curve has a particular structure at the neighborhoods of markings and nodes. At a marking it can be given by the chart $\Delta\times\mathbb{C}$ with the action of an element $\rho \in G$ given by $(z,s) \mapsto (\rho z, \rho^a s)$. Thus the number $a \in \{0,\dots,r-1\}$ describes the local structure of L at a marking. At a node L can be given a by a chart $(\Delta\times\Delta)\times\mathbb{C}$ with the action of an element $\rho\in G$ given by $(z,w,s) \mapsto (\rho z, \rho^{-1}w, \rho^a s)$. Note, however, that the number a is replaced with $-a \pmod r$ if we exchange z and w. Thus the local structure of L at node is described by assigning to the branches of the node two numbers $a', a'' \in \{0,\dots,r-1\}$ such that $a'+a'' = 0 \mod r$.

Roots of $\mathcal{O}$.

In [19] an element of $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ is an r-stable curve $\mathcal{C}$ with an orbifold line bundle $L \to \mathcal{C}$ endowed with an identification $L^{\otimes r} \xrightarrow{\sim} \mathcal{O}$. The integers $a_i \in \{0, \dots, r-1\}$ prescribe the structure of L at the markings.

From r-th roots to G-bundles.

To make the connection with the description of $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ in [63] we look at the multi-section of L that maps to the section 1 of $\mathcal{O}$ when raised to the power r. This multi-section is a principal G-bundle $\pi : D \to C$ ramified over the markings and the nodes. At a marking with label a the G-bundle has the monodromy given by adding a in $\mathbb{Z}/r\mathbb{Z}$. This can be seen from the G-action $(z, s) \mapsto (\rho z, \rho^a s)$. If we choose $\rho = e^{2\pi i/r}$, a path from z to ρz in the chart corresponds to a loop around the marking in the stable curve and its lifting leads from s to $\rho^a s$ in the fiber of L.

Similarly, at the node the G-bundle has monodromies a' and a'' at the two branches, satisfying $a' + a'' = 0 \mod r$.

Note that, because D is formed by a multi-section of L, the pull-back of L to D has a tautological section. We will denote this section by ϕ_0.

From G-bundles to r-th roots.

In [63] an element of $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ is G-cover $\pi : D \to C$ ramified over the markings and the nodes and satisfying the "kissing condition": the monodromies of the G-action over two branches of a node are opposite modulo r. The integers $a_i \in \{0, \dots, r-1\}$ prescribe the monodromies at the markings. Suppose we are given a principal G-bundle $\pi : D \to C$ like that. Using this data it is easy to construct a line bundle L over the r-stable curve $\mathcal{C}$ corresponding to C. Over any contractible open set $U \subset C$ that does not contain markings and nodes we create a chart $U \times \mathbb{C}$ and identify the r-roots of unity in $\mathbb{C}$ with the sheets of the G-bundle in an arbitrary way that preserves the G-action. At the markings we create the orbi-chart $\Delta \times \mathbb{C}$ endowed with the G-action $(z, s) \mapsto (\rho z, \rho^a s)$ as above and also identify the r-th roots of unity with the sheets of the bundle. The transition maps between the charts are obtained from the matching of the sheets over different charts (every transition map is the multiplication by a locally constant r-th root of unity).

Sections of L and of $K \otimes L^*$.

Let ϕ be a section of L over an open set $U \subset \mathcal{C}$. Then $\pi^*\phi/\phi_0$ is a holomorphic function on $\pi^{-1}(U) \subset D$. Moreover, the G-action on this function has the form $f(\rho z) = \rho^{-1} f(z)$. A global section of L gives rise to a global holomorphic function on D satisfying the above transformation rule. It follows that L has no global sections over $\mathcal{C}$, with the exception of the case where all a_i's vanish, L is the trivial line bundle and $D = C \times G$.

Similarly, let ϕ be a section of $K \otimes L^*$ on an open set $U \subset \mathcal{C}$. Then $\alpha = \pi^*\phi \cdot \phi_0$ is a section of the canonical line bundle K_D over $\pi^{-1}(U)$. Moreover, the G-action on this function has the form $\alpha(\rho z) = \rho\alpha(z)$. In particular, the space of global sections of $K \otimes L^*$ coincides with the space of holomorphic differentials on D satisfying the transformation rule $\alpha(\rho z) = \rho\alpha(z)$.

Two ways of writing R^*p_*L.

Chiodo's formula expresses the Chern character of R^*p_*L, where $p : \overline{\mathcal{C}}_{g;a_1,\dots,a_n}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ is the universal curve. Using this formula one can also easily express the total Chern class of $-R^*p_*L$.

According to our remarks above, if there is at least one positive a_i then $R^0p_*L = 0$. In that case R^1p_*L is a vector bundle, and we have $c(-R^*p_*L) = c(R^1p_*L)$.

If all the a_i's vanish, the space $\overline{\mathcal{M}}_{g;a_1,\dots,a_n}(\mathcal{B}G)$ has a special connected component on which the line bundle L is trivial. Over this component $R^0p_*L = \mathbb{C}$. On the other connected components we have, as before, $R^0p_*L = 0$. Therefore the total Chern class of R^0p_*L is equal to 1 and we have, once again, $c(-R^*p_*L) = c(R^1p_*L)$.

Johnson, Pandharipande, and Tseng use the Chern classes λ_i of the vector bundle of equivariant sections of K_D. Our analysis above shows that this vector bundle is the dual of R^1p_*L. In other words, we have

$$c(-R^*p_*L) = \sum(-1)^i\lambda_i, \tag{4.5.48}$$

which is the equality that we use in our computations.

Remark 4.5.1. In the Johnson-Pandharipande-Tseng formula the monodromies at the markings are given by the remainders modulo r of $-\mu_i$, that is, minus the parts of the ramification profile. Thus if we denote by $a_i = \mu_i \mod r$, we will use Chiodo's formula with remainders $r - a_1, \dots, r - a_n$ at the markings. If an a_i is equal to 0, we can plug either 0 or r in Chiodo's formula. Indeed, we have $B_k(0) = B_k(1)$ for any $k > 1$, thus replacing 0 by r will only affect the Chern character of degree 0, that is not used in the expression for the total Chern class.

In particular, in Equation (4.5.47) we use the push-forward of $\sum(-1)^i\lambda_i$ to $\overline{\mathcal{M}}_{g,n}$, for monodromies equal to minus the remainders of $\mu_1, \dots, \mu_n$. This class coincides with $\mathrm{C}_{g,n}(r, s; r - a_1, \dots, r - a_n)$ defined by Equation (4.2.4).

4.5.3 The equivalence

Now we are armed to prove the following

Theorem 4.5.2. *The expansion of the correlation differentials of the spectral curve* (4.4.23) *for $s = r$ is given by*

$$\mathcal{W}_{g,n} = \sum_{\mu_1,\dots,\mu_n=1}^{\infty} \mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \; e^{\sum_{j=1}^n \mu_j x_j} \frac{h_{g;\vec{\mu}}}{b!}, \tag{4.5.49}$$

if and only if the numbers $h_{g;\vec{\mu}}$ are given by the Johnson-Pandharipande-Tseng formula (4.5.47).

Proof. The proof is indeed very simple. First, Equation (4.5.48) allows us to replace Chiodo class in (4.4.45) with the push-forward of the linear combination of λ-classes. Then we notice the following rescaling of the integral

$$\int_{\overline{\mathcal{M}}_{g,n}} \frac{\pi_* \sum_{i\geq 0}(-r)^i \lambda_i}{\prod_{j=1}^n (1-\mu_i\psi_i)} = r^{3g-3+n} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\pi_* \sum_{i\geq 0}(-1)^i \lambda_i}{\prod_{j=1}^n (1-\frac{\mu_i}{r}\psi_i)}. \tag{4.5.50}$$

The equivalence then follows from comparison of coefficients in front of particular $\mathrm{d}_1 \otimes \cdots \otimes \mathrm{d}_n \, e^{\sum_{j=1}^n \mu_j x_j}$ in (4.5.49) and (4.4.45), which is obvious, modulo the following simple computation of the powers of r. For $s=r$,

$$\prod_{j=1}^n \frac{\left(\frac{\mu_j}{r}\right)^{\left\lfloor \frac{\mu_j}{r}\right\rfloor}}{\left\lfloor \frac{\mu_j}{r}\right\rfloor !} \frac{r^{2g-2+n+\frac{(2g-2+n)s+\sum_{j=1}^n \mu_j}{r}}}{s^{2g-2+n}} = \prod_{j=1}^n \frac{\mu_j^{\left\lfloor \frac{\mu_j}{r}\right\rfloor}}{\left\lfloor \frac{\mu_j}{r}\right\rfloor !} r^{2g-2+n+\sum_{j=1}^n \left\langle \frac{\mu_i}{r}\right\rangle}$$

is the coefficient in Equation (4.4.45). This is equal to

$$r^{3g-3+n} r^{1-g+\sum \left\langle \frac{\mu_i}{r}\right\rangle} \prod_{i=1}^n \frac{\mu_i^{\left\lfloor \frac{\mu_i}{r}\right\rfloor}}{\left\lfloor \frac{\mu_i}{r}\right\rfloor !},$$

which is the coefficient of (4.5.47) after rescaling (4.5.50). □

5

Quantum spectral curve for the Gromov-Witten theory of $\mathbb{P}^1$

Abstract

We construct the quantum curve for the Gromov-Witten theory of the complex projective line.

5.1 Introduction

The purpose of this chapter is to construct the *quantum curve* which is a *Schrödinger-like* equation

$$P(x,\hbar)\Psi(x,\hbar) = 0 \tag{5.1.1}$$

for the Gromov-Witten invariants of the complex projective line $\mathbb{P}^1$. Quantum curves are conceived in the physics literature, including [3, 23, 24, 48, 59, 61]. They quantize the *spectral curves* of the theory, and are conjectured to capture the information of many topological invariants, such as certain Gromov-Witten invariants, quantum knot invariants, and cohomology of instanton moduli spaces for 4-dimensional gauge theory. In this chapter we show that the conjecture is indeed true for the Gromov-Witten theory of $\mathbb{P}^1$. This gives the first rigorous example of a direct connection between Gromov-Witten theory and quantum curves. Our construction requires the fermionic Fock space representation of the Gromov-Witten invariants [81], and a subtle combinatorial analysis based on representation theory of symmetric groups.

5.1.1 Main theorem

Let $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)$ denote the moduli space of stable maps of degree d from an n-pointed genus g curve to $\mathbb{P}^1$. This is an algebraic stack of dimension $2g-2+n+2d$. The dimension can be understood via the Riemann-Hurwitz formula applied to a generic map from an algebraic curve to $\mathbb{P}^1$ which has only simple ramifications. The descendant Gromov-Witten invariants of $\mathbb{P}^1$ are defined by

$$\left\langle \prod_{i=1}^{n} \tau_{b_i}(\alpha_i) \right\rangle_{g.n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \psi_i^{b_i} ev_i^*(\alpha_i), \tag{5.1.2}$$

where $[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)]^{vir}$ is the virtual fundamental class of the moduli space,

$$ev_i : \overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \longrightarrow \mathbb{P}^1$$

is a natural morphism defined by evaluating a stable map at the i-th marked point of the source curve, $\alpha_i \in H^*(\mathbb{P}^1, \mathbb{Q})$ is a cohomology class of the target $\mathbb{P}^1$, and ψ_i is the tautological cotangent class in $H^2(\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d), \mathbb{Q})$. We denote by 1 the generator of $H^0(\mathbb{P}^1, \mathbb{Q})$, and by $\omega \in H^2(\mathbb{P}^1, \mathbb{Q})$ the Poincaré dual to the point class. Classes $\tau_k(\omega)$ are known as *stationary* since the pull-back $ev_i^*(\omega) \subset \mathcal{M}_{g,n}(\mathbb{P}^1, d)$ restricts to stable maps f with $f(p_i) = x_i$ for a given point $x_i \in \mathbb{P}^1$ representing the Poincaré dual of $\omega \in H^2(\mathbb{P}^1)$ and hence $f(p_i)$ is stationary.

We assemble the Gromov-Witten invariants into particular generating functions as follows. For every (g, n) in the stable sector $2g - 2 + n > 0$, we define the *free energy* of type (g, n) by

$$F_{g,n}(x_1, \ldots, x_n) := \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n}. \tag{5.1.3}$$

Here the degree d is determined by the dimension condition of the cohomology classes to be integrated over the virtual fundamental class. We note that (5.1.3) contains the class $\tau_0(1)$ known as the *puncture operator*. For unstable geometries, we introduce two functions

$$S_0(x) := x - x \log x + \sum_{d=1}^{\infty} \left\langle -\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}} \right\rangle_{0,1}^{d}, \tag{5.1.4}$$

$$S_1(x) := -\frac{1}{2} \log x + \frac{1}{2} \sum_{d=0}^{\infty} \left\langle \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x^{b+1}} \right)^2 \right\rangle_{0,2}^{d}. \tag{5.1.5}$$

The appearance of the extra terms, in particular the $\log x$ terms, will be explained in Section 5.3. We shall prove the following.

Theorem 5.1.1 (Main Theorem)**.** *The wave function*

$$\Psi(x, \hbar) := \exp\left(\frac{1}{\hbar} S_0(x) + S_1(x) + \sum_{2g-2+n>0} \frac{\hbar^{2g-2+n}}{n!} F_{g,n}(x, \ldots, x) \right) \tag{5.1.6}$$

satisfies the quantum curve equation of an infinite order

$$\left[\exp\left(\hbar \frac{d}{dx} \right) + \exp\left(-\hbar \frac{d}{dx} \right) - x \right] \Psi(x, \hbar) = 0. \tag{5.1.7}$$

Moreover, the free energies $F_{g,n}(x_1, \ldots, x_n)$ as functions in n-variables, and hence all the Gromov-Witten invariants (5.1.2), *can be recovered from the equation* (5.1.7) *alone, using the mechanism of the* ***topological recursion*** *of [17, 50].*

Remark 5.1.2. It was proven in [37, 80] that the stationary Gromov-Witten theory of $\mathbb{P}^1$ satisfies the topological recursion of [17, 50] with respect to the spectral curve

$$\begin{cases} x = z + \frac{1}{z} \\ z = e^y \end{cases} . \tag{5.1.8}$$

The WKB analysis provides a perturbative quantisation method of a classical mechanical problem. We can recover the classical problem corresponding to (5.1.7) by taking its semi-classical limit, which is the singular perturbation limit

$$\lim_{\hbar \to 0} \left(e^{-\frac{1}{\hbar} S_0(x)} \left[\exp\left(\hbar \frac{d}{dx}\right) + \exp\left(-\hbar \frac{d}{dx}\right) - x \right] e^{\frac{1}{\hbar} S_0(x)} e^{\sum_{m=1}^{\infty} \hbar^{m-1} S_m(x)} \right) \\ = \left(e^{S_0'(x)} + e^{-S_0'(x)} - x \right) e^{S_1(x)} = 0. \tag{5.1.9}$$

In terms of new variables $y(x) = S_0'(x)$ and $z(x) = e^{y(x)}$, the semi-classical limit gives the spectral curve (5.1.8).

Remark 5.1.3. A key part of the proof of Theorem 5.1.1 shows that the quantum curve equation (5.1.7) is equivalent to a recursion equation

$$\frac{x}{\hbar} \left(e^{-\hbar \frac{d}{dx}} - 1 \right) X_d(x, \hbar) + \frac{1}{1 + \frac{x}{\hbar}} e^{\hbar \frac{d}{dx}} X_{d-1}(x, \hbar) = 0 \tag{5.1.10}$$

for a rational function

$$X_d(x, \hbar) = \sum_{\lambda \vdash d} \left(\frac{\dim \lambda}{d!} \right)^2 \prod_{i=1}^{\ell(\lambda)} \frac{x + (i - \lambda_i)\hbar}{x + i\hbar}. \tag{5.1.11}$$

Here λ is a partition of $d \geq 0$ with parts λ_i and $\dim \lambda$ denotes the dimension of the irreducible representation of the symmetric group S_d characterized by λ.

Remark 5.1.4. Put

$$S_m(x) := \sum_{2g-2+n=m-1} \frac{1}{n!} F_{g,n}(x, \ldots, x). \tag{5.1.12}$$

Then the wave function (5.1.6) is of the form

$$\Psi(x, \hbar) = \exp\left(\sum_{m=0}^{\infty} \hbar^{m-1} S_m(x) \right), \tag{5.1.13}$$

which provides the WKB approximation of the solution of the quantum curve equation (5.1.7). The significance of Theorem 5.1.1 is that the exponential generating function (5.1.6) of the descendant Gromov-Witten invariants of $\mathbb{P}^1$ gives the solution to the *exact* WKB analysis for the difference equation (5.1.7).

5.1.2 Organization of the chapter

This chapter is organized as follows. In Section 5.2 we start with a solution $\mathcal{W}_{g,n}$ to the topological recursion equation with respect to the spectral curve Σ of (5.1.8). It is a symmetric differential form of degree n on Σ^n. We then propose a unique mechanism to integrate $\mathcal{W}_{g,n}$ into a rational function. The goal of this section is to show that this primitive function is identical to (5.1.3). Then in Section 5.3, we re-write $\Psi(x,\hbar)$ in a different manner, only involving stationary Gromov-Witten invariants of $\mathbb{P}^1$. This formula allows us to express it in terms of a *semi-infinite wedge product* in Section 5.4. Using this formalism, we reduce the quantum curve equation (5.1.7) to a combinatorial equation (5.1.10) in Section 5.5. Equation (5.1.10) is then proved in Section 5.6 using representation theory of S_d, which in tern establishes (5.1.7).

5.2 The functions $F_{g,n}$ in terms of Gromov-Witten invariants

The significance of the idea of a quantum curve which is a *Schrödinger-like* equation (5.1.1) is that it captures all information of the topological invariants of the theory. The key process from this single equation to the topological invariants is the *integral form* of the mechanism known as the topological recursion of [17, 50]. We refer to [31, 32, 80] for mathematical formulation of the topological recursion. This section is devoted to providing the unique mechanism to integrate the topological recursion, in the context of the Gromov-Witten theory of $\mathbb{P}^1$.

Let us begin with a solution $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ to the topological recursion of [17, 37, 50] associated with the spectral curve $\Sigma = \mathbb{C}^*$ defined by

$$\begin{cases} x(z) = z + \frac{1}{z} \\ y(z) = \log z \end{cases} . \tag{5.2.1}$$

This means that symmetric differential forms $\mathcal{W}_{g,n}(z_1,\dots,z_n)$ of degree n on Σ^n for (g,n) in the stable range $2g-2+n>0$ are inductively defined by the following recursion formula:

$$\begin{aligned} &\mathcal{W}_{g,n}(z_1,\dots,z_n) \\ &\quad = \frac{1}{2\pi i}\oint_{z=\pm 1} \frac{\int_z^{1/z}\mathcal{W}_{0,2}(\,\cdot\,,z_1)}{\mathcal{W}_{0,1}(1/z)-\mathcal{W}_{0,1}(z)} \Bigg[\mathcal{W}_{g-1,n+1}(z,1/z,z_2,\dots,z_n) \\ &\qquad\qquad + \sum_{\substack{g_1+g_1=g \\ I\sqcup J=\{2,\dots,n\}}}^{\text{stable}} \mathcal{W}_{g_1,|I|+1}(z,z_I)\mathcal{W}_{g_2,|J|+1}(1/z,z_J)\Bigg], \end{aligned} \tag{5.2.2}$$

where the residue integral is taken with respect to the variable $z \in \Sigma$ on two small, positively oriented, closed loops around $z=1$ and $z=-1$, and for the index set $I \subset \{2,\dots,n\}$, we denote by $|I|$ its cardinality, and $z_I = (z_i)_{i\in I}$. For (g,n) in the unstable range, we define

$$\mathcal{W}_{0,1}(z) := y(z)dx(z), \tag{5.2.3}$$

$$\mathcal{W}_{0,2}(z_1, z_2) := \frac{dz_1 dz_2}{(z_1 - z_2)^2} - \frac{dx(z_1)dx(z_2)}{(x(z_1) - x(z_2))^2}. \tag{5.2.4}$$

The goal of this section is to derive the integral $F_{g,n}(z_1, \ldots, z_n)$ of $\mathcal{W}_{g,n}(z_1, \ldots, z_n)$ in a consistent and unique way that has the x-variable expansion (5.1.3).

Remark 5.2.1. The second term of the right-hand side of (5.2.4) does not play any role in the topological recursion (5.2.2). It is included here for the consistency of the primitive $F_{0,2}(z_1, z_2)$ to be discussed in Section 5.3.

Definition 5.2.2. For $2g - 2 + n > 0$, we define the *primitive* $F_{g,n}(z_1, \ldots, z_n)$ of the n-form $\mathcal{W}_{g,n}(z_1, \ldots, z_n)$ to be a rational function on Σ^n that satisfies the following conditions:

$$d_1 \cdots d_n F_{g,n}(z_1, \ldots, z_n) = \mathcal{W}_{g,n}(z_1, \ldots, z_n); \tag{5.2.5}$$

$$F_{g,n}(z_1, \ldots, z_{i-1}, 1/z_i, z_{i+1}, \ldots, z_n) = -F_{g,n}(z_1, \ldots, z_n), \quad i = 1, \ldots, n; \tag{5.2.6}$$

$$F_{g,n}(z_1, \ldots, z_n)\big|_{z_1 = \cdots = z_n = 0} = 0. \tag{5.2.7}$$

If it exists, then it is unique.

It is established in [37, 80] that the solution $\mathcal{W}_{g,n}$ of the topological recursion has the following x-variable expansion in terms of the stationary Gromov-Witten invariants of $\mathbb{P}^1$:

$$\mathcal{W}_{g,n}(x_1, \ldots, x_n) = \left\langle \prod_{i=1}^{n} \left(\sum_{b=0}^{\infty} (b+1)!\, \tau_b(\omega) \frac{dx_i}{x_i^{b+2}} \right) \right\rangle_{g,n}. \tag{5.2.8}$$

There is no systematic mechanism to integrate this expression to obtain (5.1.3). Instead, we establish the following theorem in this section.

Theorem 5.2.3. *For every (g, n) in the stable sector $2g - 2 + n > 0$, there exists a primitive $F_{g,n}(z_1, \ldots, z_n)$ of (5.2.8) satisfying the conditions of Definition 5.2.2, such that its x-variable expansion is given by*

$$F_{g,n}(x_1, \ldots, x_n) = \left\langle \prod_{i=1}^{n} \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x_i^{b+1}} \right) \right\rangle_{g,n}. \tag{5.2.9}$$

Remark 5.2.4. We need a different treatment for the unstable primitives $F_{0,1}(z)$ and $F_{0,2}(z_1, z_2)$. They are calculated in Section 5.3.

The rest of this section is devoted to proving this theorem. We start with recalling some results of [37]. The most important one is the formula for $\mathcal{W}_{g,n}(z_1, \ldots, z_n)$ in terms of the auxiliary functions $W_d^i(z)$ (defined below) with the *ancestor* Gromov-Witten invariants as its coefficients. We will then prove the existence of the anti-symmetric primitives of the functions W_d^i, and their x-expansions. This will then lead us to the proof of the above theorem, where we will also utilize the known relations between the ancestor and the descendant Gromov-Witten invariants.

5.2.1 Some results from [37]

The ancestor Gromov-Witten invariants of $\mathbb{P}^1$ we need are

$$\left\langle \prod_{i=1}^{n} \bar{\tau}_{b_i}(\alpha_i) \right\rangle_{g,n}^{d} := \int_{[\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1,d)]^{vir}} \prod_{i=1}^{n} \bar{\psi}_i^{b_i} ev_i^*(\alpha_i), \tag{5.2.10}$$

where $\bar{\psi}_i$ denotes the pull back of the cotangent class on $\overline{\mathcal{M}}_{g,n}$ by the natural forgetful morphism

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d) \longrightarrow \overline{\mathcal{M}}_{g,n}.$$

Since we adopt a quantum field theoretic point of view in calculating Gromov-Witten invariants, we often call them *correlators* in this chapter. The ancestor and descendant correlators do not agree. We will give a formula to determine one from the other in (5.2.15).

Let us define

$$W_0^1(z) := \frac{dz}{(1-z)^2}, \tag{5.2.11}$$

$$W_0^2(z) := \frac{idz}{(1+z)^2}, \tag{5.2.12}$$

$$W_k^i(z) := d\left(\left(-2\frac{d}{dx(z)}\right)^k \int W_0^i(z)\right), \qquad i = 1, 2; \quad k \geq 0. \tag{5.2.13}$$

Then for $g \geq 0$ and $n \geq 1$ with $2g - 2 + n > 0$, from Theorem 4.1 of [37] (as shown in the proof of Theorem 5.2 of [37]), we have

$$\mathcal{W}_{g,n}(z_1, \ldots, z_n) = \sum_{\vec{d},\, \vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1}) \ldots \bar{\tau}_{d_n}(\tilde{e}_{i_n}) \rangle_g \frac{W_{d_1}^{i_1}(z_1)}{2^{d_1}\sqrt{2}} \cdots \frac{W_{d_n}^{i_n}(z_n)}{2^{d_n}\sqrt{2}}. \tag{5.2.14}$$

Here the sum over $\vec{d}$ and $\vec{i}$ are taken over all integer values $0 \leq d_k$ and $i_k = 1, 2$. Note that the coefficients of this expansion are the *ancestor* Gromov-Witten invariants. The cohomology basis for $H^1(\mathbb{P}^1, \mathbb{Q})$ is normalized as follows. First we denote by $e_1 = 1$ and $e_2 = \omega$. Using the normalisation matrix

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix},$$

we define

$$\tilde{e}_i = \left(A^{-1}\right)_i^{\mu} e_{\mu}.$$

In this section we use the Einstein convention and take summation over repeated indices.

With the help of the Givental formula, Proposition 5.1 of [37] relates the ancestor and the descendant correlators for $\mathbb{P}^1$ by

$$\begin{aligned} &\sum_{\vec{d},\, \vec{i}} \langle \bar{\tau}_{d_1}(\tilde{e}_{i_1}) \ldots \bar{\tau}_{d_n}(\tilde{e}_{i_n}) \rangle_g v^{d_1, i_1} \ldots v^{d_n, i_n} \\ =& \sum_{\vec{d},\, \vec{\mu}} \langle \tau_{d_1}(e_{\mu_1}) \ldots \tau_{d_n}(e_{\mu_n}) \rangle_g t^{d_1, \mu_1} \ldots t^{d_n, \mu_n}, \end{aligned} \tag{5.2.15}$$

where $v^{d,i}$ and $t^{d,\mu}$ are formal variables related by the following formula:

$$v^{d,i} = A^i_\mu \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu t^{m,\nu}. \tag{5.2.16}$$

Here $(\mathcal{S}_k)^\mu_\nu$ are the matrix elements that define the Givental loop group action [56] and defined by

$$\begin{aligned} \mathcal{S}(\zeta^{-1}) = \sum_{k=0}^{\infty} \mathcal{S}_k \zeta^{-k} &= \mathrm{Id} + \zeta^{-1} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k}}{(k!)^2} \begin{pmatrix} 1 - 2k\left(\frac{1}{1} + \cdots + \frac{1}{k}\right) & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \sum_{k=1}^{\infty} \frac{\zeta^{-2k-1}}{(k!)^2} \begin{pmatrix} 0 & -2\left(\frac{1}{1} + \cdots + \frac{1}{k}\right) \\ \frac{1}{k+1} & 0 \end{pmatrix}. \end{aligned} \tag{5.2.17}$$

In the proof of Theorem 5.2 of [37] it was shown that the x^{-1}-expansion of $W^i_d(z)$ near $z = 0$ is given by the following formula:

$$W^i_d(z) = 2^d \sqrt{2}\, A^i_\mu \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \; \delta^\nu_2 \, (m+1)! \, \frac{dx}{x^{m+2}}, \tag{5.2.18}$$

where δ^i_j is the Kronecker delta symbol. The above formula, together with formulas (5.2.14)-(5.2.17), implies (5.2.8).

To define a primitive of $\mathcal{W}_{g,n}$ satisfying the conditions of Definition 5.2.2 we first identify suitable primitives of the differential 1-forms $W^i_d(z)$. Given the property

$$W^i_d(1/z) = -W^i_d(z)$$

if there exists a uniquely defined rational function $\theta^i_d(z)$ on Σ such that

$$d\theta^i_d(z) = W^i_d(z), \tag{5.2.19}$$

$$\theta^i_d(1/z) = -\,\theta^i_d(z) \tag{5.2.20}$$

then it is unique. The following proposition supplies existence of $\theta^i_d(z)$.

Proposition 5.2.5. *For given $i = 1, 2$ and $d \geq 0$, there exists a unique rational function $\theta^i_d(z)$ on Σ satisfying* (5.2.19) *and* (5.2.20)*. It is recursively defined by*

$$\theta^1_0 := \frac{1}{1-z} - \frac{1}{2}, \qquad \theta^2_0 := -\frac{i}{1+z} + \frac{i}{2} \tag{5.2.21}$$

$$\theta^i_d(z) = \left(-2\frac{d}{dx(z)}\right)^d \theta^i_0(z). \tag{5.2.22}$$

Moreover, the x^{-1}-expansion of $\theta^i_d(z)$ near $z = 0$ is given by

$$\theta^i_d(z(x)) = 2^d \sqrt{2}\, A^i_\mu \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \; \left(-\delta^\nu_1 \delta^m_0 \, \frac{1}{2} - \delta^\nu_2 \, m! \, \frac{1}{x^{m+1}}\right). \tag{5.2.23}$$

5.2.2 Proof of Proposition 5.2.5

It is easy to verify by direct computation that (5.2.21) are the unique solutions of (5.2.19) and (5.2.20) for $d = 0$.

Equation (5.2.13), together with condition (5.2.19), implies that if $\theta^i_d(z)$ exists, then it has to satisfy (5.2.22). Since x is symmetric under the coordinate change $z \longmapsto 1/z$, we see that the right-hand side of equation (5.2.22) satisfies the skew-invariance property (5.2.20). This means that $\theta^i_d(z)$ *defined* by (5.2.22) is, for given i and d, indeed the unique solution of (5.2.19) and (5.2.20).

It remains to prove the expansion (5.2.23) of $\theta^i_d(z)$ at $x = \infty$. We denote by $\tilde{\theta}^i_d$ the right-hand side of (5.2.23). We wish to prove that the x^{-1}-expansion of $\theta^i_d(z)$ near $z = 0$ is given by $\tilde{\theta}^i_d$. Let us introduce the following notation:

$$\eta^\mu_d := \frac{1}{2^d\sqrt{2}} \left(A^{-1}\right)^\mu_i \theta^i_d. \tag{5.2.24}$$

Then we have

$$\eta_0 = \left(\frac{1}{1-z^2} - \frac{1}{2}, \frac{z}{1-z^2}\right), \tag{5.2.25}$$

$$\eta^\mu_k(z) = \left(-\frac{d}{dx(z)}\right)^k \eta^\mu_0, \tag{5.2.26}$$

and condition (5.2.23) becomes equivalent to the condition that the x^{-1}-expansion of η^μ_d near $z = 0$ is equal to $\tilde{\eta}^\mu_d$, where

$$\tilde{\eta}^\mu_d := \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^\mu_\nu \left(-\delta^\nu_1\delta^m_0 \frac{1}{2} - \delta^\nu_2\, m!\, \frac{1}{x^{m+1}}\right). \tag{5.2.27}$$

Let us prove formula (5.2.27) for $d = 0$. Note that $\mathcal{S}_0 = \mathrm{Id}$, so for the constant term of $\tilde{\eta}_0$ we have

$$\left[\frac{1}{x^0}\right] \tilde{\eta}^\mu_0 = -\delta^\mu_1 \frac{1}{2}. \tag{5.2.28}$$

It is easy to see from (5.2.25) that η^i_0 has the same constant term at $z = 0$.

For $k \geq 1$ we have

$$\begin{aligned}
\left[\frac{1}{x^{2k-1}}\right] \tilde{\eta}^1_0 &= -(2k-2)!\,(\mathcal{S}_{2k-2})^1_2 = 0,\\
\left[\frac{1}{x^{2k-1}}\right] \tilde{\eta}^2_0 &= -(2k-2)!\,(\mathcal{S}_{2k-2})^2_2 = -\frac{(2k-2)!}{((k-1)!)^2},\\
\left[\frac{1}{x^{2k}}\right] \tilde{\eta}^1_0 &= -(2k-1)!\,(\mathcal{S}_{2k-1})^1_2 = -\frac{(2k-1)!}{k!\,(k-1)!},\\
\left[\frac{1}{x^{2k}}\right] \tilde{\eta}^2_0 &= -(2k-1)!\,(\mathcal{S}_{2k-1})^2_2 = 0.
\end{aligned} \tag{5.2.29}$$

For the corresponding coefficients in the x^{-1}-expansion of η_0^μ near $z=0$ we have $(k \geq 1)$:

$$\begin{aligned}
&\operatorname*{Res}_{z=0} x^{2k-2}(z)\,\eta_0^1\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k}\left(1+z^2\right)^{2k-2} dz = 0,\\
&\operatorname*{Res}_{z=0} x^{2k-2}(z)\,\eta_0^2\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k+1}\left(1+z^2\right)^{2k-2} dz = -\frac{(2k-2)!}{((k-1)!)^2},\\
&\operatorname*{Res}_{z=0} x^{2k-1}(z)\,\eta_0^1\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k-1}\left(1+z^2\right)^{2k-1} dz = -\frac{(2k-1)!}{k!\,(k-1)!},\\
&\operatorname*{Res}_{z=0} x^{2k-1}(z)\,\eta_0^2\,dx(z) = -\operatorname*{Res}_{z=0} z^{-2k}\left(1+z^2\right)^{2k-1} dz = 0.
\end{aligned} \tag{5.2.30}$$

We see that the coefficients in (5.2.29) precisely coincide with the ones in (5.2.30). This implies that the x^{-1}-expansion of η_0^μ is indeed given by $\tilde\eta_0^\mu$.

By virtue of (5.2.26), we see that the x^{-1}-expansion of η_k^μ near $z=0$ is given by the following formula (for $k \geq 1$):

$$\begin{aligned}
\left(-\frac{d}{dx}\right)^k \eta_0^\mu &= \sum_{m=0}^{\infty} (\mathcal{S}_m)^\mu_\nu \left(-\delta_2^\nu\,(m+k)!\,\frac{1}{x^{m+k}}\right)\\
&= \sum_{m=d}^{\infty} (\mathcal{S}_{m-k})^\mu_\nu \left(-\delta_2^\nu\, m!\,\frac{1}{x^{m+1}}\right).
\end{aligned}$$

This coincides with the formula for $\tilde\eta_k^\mu$ for $k \geq 1$. Thus, we have proved that the x^{-1}-expansion of η_k^μ is given by $\tilde\eta_k^\mu$, which, in turn, implies that Equation (5.2.23) holds. This concludes the proof of the proposition.

5.2.3 Proof of Theorem 5.2.3

Recall Equation (5.2.14) for $\mathcal{W}_{g,n}$:

$$\mathcal{W}_{g,n}(z_1,\ldots,z_n) = \sum_{\vec d,\,\vec i} \langle \bar\tau_{d_1}(\tilde e_{i_1})\ldots\bar\tau_{d_n}(\tilde e_{i_n})\rangle_g \frac{W^{i_1}_{d_1}(z_1)}{2^{d_1}\sqrt2}\cdots\frac{W^{i_n}_{d_n}(z_n)}{2^{d_n}\sqrt2}.$$

Since we know how to integrate every $W^i_d(z)$, we simply define

$$F_{g,n}(z_1,\ldots,z_n) := \sum_{\vec d,\,\vec i} \langle \bar\tau_{d_1}(\tilde e_{i_1})\ldots\bar\tau_{d_n}(\tilde e_{i_n})\rangle_g \frac{\theta^{i_1}_{d_1}(z_1)}{2^{d_n}\sqrt2}\cdots\frac{\theta^{i_n}_{d_n}(z_n)}{2^{d_n}\sqrt2}. \tag{5.2.31}$$

Then from Proposition 5.2.5, we see that (5.2.5) and (5.2.6) are automatically satisfied. We also know from Proposition 5.2.5 that the x^{-1}-expansion of $F_{g,n}$ near $z_1=\cdots=z_n=0$ is given by

$$\begin{aligned}
&F_{g,n}(x_1,\ldots,x_n)\\
&\quad= \sum_{\vec d,\,\vec i} \langle \bar\tau_{d_1}(\tilde e_{i_1})\ldots\bar\tau_{d_n}(\tilde e_{i_n})\rangle_g \prod_{k=1}^{n} A^{i_k}_{\mu_k} \sum_{m=d}^{\infty} (\mathcal{S}_{m-d})^{\mu_k}_{\nu_k}\left(-\delta_1^{\nu_k}\delta_0^m\,\frac12 - \delta_2^{\nu_k}\, m!\,\frac{1}{x_k^{m+1}}\right).
\end{aligned}$$

Using (5.2.15) and (5.2.16), we find

$$\begin{aligned} &F_{g,n}(x_1,\dots,x_n) \\ &= \sum_{\vec{d},\vec{i}} \langle \tau_{d_1}(e_{\mu_1})\dots\tau_{d_n}(e_{\mu_n})\rangle_g \prod_{i=1}^{n} \left(-\delta_1^{\mu_i}\delta_0^{d_i}\,\frac{1}{2} - \delta_2^{\mu_i}\, d_i!\,\frac{1}{x^{d_i+1}}\right) \\ &= \left\langle \prod_{i=1}^{n}\left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x_i^{b+1}}\right)\right\rangle_{g,n}. \end{aligned} \tag{5.2.32}$$

The final condition (5.2.7) follows from the fact that $\langle\tau_0(1)^n\rangle_{g,n} = 0$ for all g and n in the stable range. This concludes the proof of the theorem.

5.3 The shift of variable simplification

Let us now turn our attention toward proving (5.1.7) of Theorem 5.1.1. In this section, as the first step, we establish a formula for the wave function $\Psi(x,\hbar)$ of (5.1.6) involving only the stationary Gromov-Witten invariants.

Our starting point is

$$\begin{aligned} \log\Psi(x,\hbar) = &\frac{1}{\hbar}S_0(x) + S_1(x) \\ &+ \sum_{g,d=0}^{\infty}\sum_{\substack{n=1 \\ 2g-2+n>0}}^{\infty} \frac{\hbar^{2g-2+n}}{n!}\left\langle\left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)^n\right\rangle_{g,n}^{d}. \end{aligned} \tag{5.3.1}$$

Using the string equation (5.3.6) and some earlier results in [32], we shall give an expression for $\log\Psi(x,\hbar)$ purely in terms of the stationary sector. More precisely, we prove the following lemma.

Lemma 5.3.1. *The function* $\log\Psi(x,\hbar)$ *is a solution to the following difference equation:*

$$\begin{aligned} \exp\left(-\frac{\hbar}{2}\frac{d}{dx}\right)\log\Psi(x,\hbar) = &\frac{1}{\hbar}(x - x\log x) \\ &+ \sum_{g,d=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\left\langle\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)^n\right\rangle_{g,n}^{d}. \end{aligned} \tag{5.3.2}$$

5.3.1 Expansion of S_0 and S_1

The functions $S_0(x)$ and $S_1(x)$ of (5.1.4) and (5.1.5) are derived from the first steps of the WKB method, that is, they are just imposed by the quantum spectral curve equation. In this subsection, we represent them in terms of the unstable $(0,1)$- and $(0,2)$-Gromov-Witten invariants.

First let us calculate these functions from the WKB approximation (5.1.13). After taking the semi-classical limit (5.1.9), we can calculate $S_1'(x)$ as follows:

$$\begin{aligned}
&e^{-\frac{1}{\hbar}S_0(x)-S_1(x)}(e^{\hbar\frac{d}{dx}}+e^{-\hbar\frac{d}{dx}}-x)e^{\frac{1}{\hbar}S_0(x)+S_1(x)}\\
&=e^{S_0'(x)+\hbar\left(\frac{1}{2}S_0''(x)+S_1'(x)\right)}e^{\hbar\frac{d}{dx}}+e^{-S_0'(x)+\hbar\left(\frac{1}{2}S_0''(x)-S_1'(x)\right)}e^{-\hbar\frac{d}{dx}}-x+O(\hbar^2)\\
&=e^{S_0'(x)}\left(1+\hbar\left(\frac{1}{2}S_0''(x)+S_1'(x)\right)\right)+e^{-S_0'(x)}\left(1+\hbar\left(\frac{1}{2}S_0''(x)-S_1'(x)\right)\right)\\
&\qquad -x+O(\hbar^2)\\
&=\hbar\left(\frac{S_0''(x)}{2}\left(e^{S_0'(x)}+e^{-S_0'(x)}\right)+S_1'(x)\left(e^{S_0'(x)}-e^{-S_0'(x)}\right)\right)+O(\hbar^2).
\end{aligned}$$

The coefficient of $\hbar$ must vanish, hence we can solve for $S_1'(x)$. Since

$$S_0''(x)=\frac{d}{dx}S_0'(x)=\frac{d}{dx}\log z=\frac{\frac{d}{dz}\log z}{x'(z)}=\frac{\frac{1}{z}}{1-\frac{1}{z^2}}=\frac{1}{z-\frac{1}{z}},$$

we find

$$S_1'(x)=-\frac{1}{2}\frac{1}{z-\frac{1}{z}}\frac{z+\frac{1}{z}}{z-\frac{1}{z}}=-\frac{1}{2}\frac{z(z^2+1)}{(z^2-1)^2}. \tag{5.3.3}$$

It is proved in [32, Equation (7.9) and Theorem 7.7] that

$$\begin{aligned}
\sum_{d=0}^{\infty}\left\langle\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x^{b+1}}\right)\right\rangle_{0,1}^{d}&=\sum_{d=1}^{\infty}\left\langle\left(-\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}}\right)\right\rangle_{0,1}^{d}\\
&=-2z+\left(z+\frac{1}{z}\right)\log\left(1+z^2\right),
\end{aligned}\tag{5.3.4}$$

and

$$\sum_{d=0}^{\infty}\left\langle\prod_{i=1}^{2}\left(-\sum_{b=0}^{\infty}\frac{b!\tau_b(\omega)}{x_i^{b+1}}\right)\right\rangle_{0,2}^{d}=-\log\left(1-z_1z_2\right). \tag{5.3.5}$$

The string equation for stationary Gromov-Witten invariants of $\mathbb{P}^1$ enables one to remove the puncture operator $\tau_0(1)$:

$$\left\langle\tau_0(1)\prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle_{g,n+1}^{d}=\sum_{\substack{j=1\\ b_j>0}}^{n}\left\langle\tau_{b_j-1}(\omega)\prod_{\substack{i=1\\ i\neq j}}^{n}\tau_{b_i}(\omega)\right\rangle_{g,n}^{d}. \tag{5.3.6}$$

Applying this to a single stationary insertion

$$\langle\tau_0(1)\tau_{b+1}(\omega)\rangle_{0,2}^{d}=\langle\tau_b(\omega)\rangle_{0,1}^{d}.$$

together with Equation (5.3.4), we calculate that

$$\begin{aligned}
&\sum_{d=1}^{\infty}\left\langle\left(-\frac{1}{2}\tau_0(1)\right)\left(-\frac{(2d-1)!\tau_{2d-1}(\omega)}{x^{2d}}\right)\right\rangle_{0,2}^{d}\\
&=\frac{1}{2}\frac{d}{dx}\left(-2z+\left(z+\frac{1}{z}\right)\log\left(1+z^2\right)\right)\\
&=\frac{1}{2}\log x+\frac{1}{2}\log z.
\end{aligned}\tag{5.3.7}$$

Note that the only condition we have for $S_0(x)$ is that $S_0'(x) = \log z$. Therefore, if we define

$$S_0(z) := F_{0,1}(z) = \int \mathcal{W}_{0,1}(z) = \int y(z)dx(z)$$

by formally applying (5.1.12) for $m = 0$, and impose the skew-symmetry condition (5.2.6) to the primitive $F_{0,1}(z)$, then from (5.3.4) we obtain

$$\begin{aligned} S_0(x) &= \frac{1}{z} - z + \left(z + \frac{1}{z}\right) \log z \\ &= (x - x\log x) + \sum_{d=1}^{\infty} \left\langle \left(-\frac{(2d-2)!\tau_{2d-2}(\omega)}{x^{2d-1}} \right) \right\rangle_{0,1}^{d} . \end{aligned} \tag{5.3.8}$$

The determination of $S_1(x)$ is trickier. If we formally apply (5.1.12) for $m = 1$, then we obtain

$$S_1(x) = -\frac{1}{2} F_{0,2}(z,z) \tag{5.3.9}$$

for the primitive

$$\begin{aligned} F_{0,2}(z_1,z_2) &= \int^{z_1}\int^{z_2} \mathcal{W}_{0,2}(z_1,z_2) \\ &= \int^{z_1}\int^{z_2} \left(\frac{dz_1dz_2}{(z_1-z_2)^2} - \frac{dx_1dx_2}{(x_1-x_2)^2} \right) \\ &= -\log(1-z_1z_2) + f(z_1) + f(z_2) + c. \end{aligned} \tag{5.3.10}$$

Here we are imposing the condition that $F_{0,2}(z_1,z_2)$ is a symmetric function. The fact that $F_{0,2}$ is a primitive of $\mathcal{W}_{0,2}$ does not determine the function $f(z)$. Therefore, we are free to choose $f(z)$ so that the differential equation (5.3.3) holds. Obviously, we need to choose $f(z) = \frac{1}{2}\log z$. In this way, using (5.3.5) and (5.3.7) as well, we obtain

$$\begin{aligned} S_1(x) &= -\frac{1}{2}\log\left(1-z^2\right) + \frac{1}{2}\log z \\ &= -\frac{1}{2}\log x + \frac{1}{2}\sum_{d=0}^{\infty} \left\langle \left(-\frac{\tau_0(1)}{2} - \sum_{b=0}^{\infty} \frac{b!\tau_b(\omega)}{x^{b+1}} \right)^2 \right\rangle_{0,2}^{d} . \end{aligned} \tag{5.3.11}$$

Remark 5.3.2. This adjustment of the choice of $S_1(x)$ also appears in the Hitchin fibration case of [31]. Still we have one degree of freedom for choosing a constant c of (5.3.10). It does not matter to the linear quantum curve equation (5.1.7), because the constant term c only affects on the overall constant factor of Ψ of (5.1.6).

5.3.2 A new formula for $\log \Psi$

We use Equations (5.3.8) and (5.3.11) to rewrite the formula (5.3.1) for $\log \Psi$ in the following way:

$$\log \Psi(x,\hbar) = \sum_{g,d=0}^{\infty} \sum_{n=1}^{\infty} \frac{\hbar^{2g-2+n}(-1)^n}{n!} \Theta_{g,n}^{d}, \tag{5.3.12}$$

where

$$\Theta^0_{0,1} := -x + x\log x + \frac{\hbar}{2}\log x + \sum_{k=2}^{\infty}\left\langle \tau_0(1)^k \tau_{k-2}(\omega)\right\rangle^0_{0,k+1} \frac{(-1)^k\hbar^k}{2^k k!}\frac{(k-2)!}{x^{k-1}} \quad (5.3.13)$$

and

$$\Theta^d_{g,n} := \sum_{k=0}^{\infty}\sum_{b_1,\dots,b_n=0}^{\infty}\left\langle \tau_0(1)^k \prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n+k} \frac{(-1)^k\hbar^k}{2^k k!}\frac{\prod_{i=1}^n b_i!}{x^{n+\sum_{i=1}^n b_i}}. \quad (5.3.14)$$

It is obvious that for dimensional reasons, $\Theta^0_{0,n} = 0$ for any $n \geq 2$. Lemma 5.3.1 is then a direct corollary to the following statement.

Lemma 5.3.3. *The quantities defined in* (5.3.13) *and* (5.3.14) *are given by*

$$\Theta^0_{0,1} = -\left(x + \frac{\hbar}{2}\right) + \left(x + \frac{\hbar}{2}\right)\log\left(x + \frac{\hbar}{2}\right); \quad (5.3.15)$$

$$\Theta^d_{g,n} = \sum_{b_1,\dots,b_n}\left\langle \prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n} \frac{\prod_{i=1}^n b_i!}{\left(x+\frac{\hbar}{2}\right)^{n+\sum_{i=1}^n b_i}}, \quad (5.3.16)$$

where in the second equation the sum is taken over all $b_1,\dots,b_n \geq 0$ *such that* $\sum_{i=1}^n b_i = 2g + 2d - 2$.

5.3.3 Proof of Lemma 5.3.3

The difference between the definitions (5.3.13)-(5.3.14) and the values (5.3.15)-(5.3.16) is simply the elimination of $\tau_0(1)$. Thus we prove Lemma 5.3.3 by using an iterated string equation:

$$\left\langle \tau_0(1)^k \prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n+k} = \sum_{\substack{j=1\\ b_j>0}}^{n}\left\langle \tau_0(1)^{k-1}\tau_{b_j-1}(\omega) \prod_{\substack{i=1\\ i\neq j}}^{n}\tau_{b_i}(\omega)\right\rangle^d_{g,n+k-1}, \quad (5.3.17)$$

where we assume $2g-2+n>1$ and $k>0$.

First, let us directly compute $\Theta^0_{0,1}$. Equation (5.3.17) implies that

$$\left\langle \tau_0(1)^k\tau_{k-2}(\omega)\right\rangle^0_{0,k+1} = \left\langle \tau_0(1)^{k-1}\tau_{k-3}(\omega)\right\rangle^0_{0,k} = \left\langle \tau_0(1)^2\tau_0(\omega)\right\rangle_{0,3} = 1. \quad (5.3.18)$$

Therefore,

$$\sum_{k=2}^{\infty}\left\langle \tau_0(1)^k\tau_{k-2}(\omega)\right\rangle^0_{0,k+1}\frac{(-1)^k\hbar^k}{2^k k!}\frac{(k-2)!}{x^{k-1}} = \sum_{k=2}^{\infty}\frac{(-1)^k\hbar^k}{2^k k!}\frac{(k-2)!}{x^{k-1}} = \left(x+\frac{\hbar}{2}\right)\log\left(\frac{x+\frac{\hbar}{2}}{x}\right) - \frac{\hbar}{2}.$$

This proves Equation (5.3.15).

The proof of Equation (5.3.16) goes as follows. Recall that $g + d > 0$ and $n > 0$. Equation (5.3.17) implies that any correlator $\left\langle \tau_0(1)^k \prod_{i=1}^n \tau_{b_i}(\omega) \right\rangle_{g,n+k}^d$ can be represented as a linear combination of the correlators $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^d$ with $\sum_{i=1}^n b_i = 2g + 2d - 2$. Moreover, for any $k \geq 0$ and $c_1, \ldots, c_n \geq 0$ such that $\sum_{i=1}^n c_i = k$, the coefficient of a particular correlator $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^d$ in $\left\langle \tau_0(1)^k \prod_{i=1}^n \tau_{b_i+c_i}(\omega) \right\rangle_{g,n+k}^d$ is equal to

$$\frac{k!}{c_1! \cdots c_n!}.$$

Therefore, the total coefficient of $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^d$ in $\Theta_{g,n}^d$ is equal to

$$\begin{aligned} &\sum_{k=0}^{\infty} \sum_{\substack{c_1,\ldots,c_n=0 \\ c_1+\cdots c_n=k}}^{\infty} \frac{(-1)^k \hbar^k}{2^k k!} \frac{\prod_{i=1}^n (b_i + c_i)!}{x^{n+\sum_{i=1}^n (b_i+c_i)}} \frac{k!}{c_1! \cdots c_n!} \\ &= \frac{\prod_{i=1}^n (b_i)!}{x^{n+\sum_{i=1}^n (b_i)}} \sum_{k=0}^{\infty} \left(\frac{-\hbar}{2x} \right)^k \sum_{\substack{c_1,\ldots,c_n \geq 0 \\ c_1+\cdots+c_n=k}} \prod_{i=1}^n \frac{(b_i + c_i)!}{b_i! c_i!}. \end{aligned} \tag{5.3.19}$$

On the other hand, expansion of the coefficient of $\langle \prod_{i=1}^n \tau_{b_i}(\omega) \rangle_{g,n}^d$ in Equation (5.3.16) is equal to

$$\begin{aligned} \frac{\prod_{i=1}^n (b_i)!}{\left(x + \frac{\hbar}{2}\right)^{n+\sum_{i=1}^n (b_i)}} &= \prod_{i=1}^n \frac{(b_i)!}{\left(x + \frac{\hbar}{2}\right)^{b_i+1}} \\ &= \prod_{i=1}^n \frac{(b_i)!}{(x)^{b_i+1}} \sum_{c_i=0}^{\infty} \left(\frac{-\hbar}{2x} \right)^k \frac{(b_i + c_i)!}{b_i! c_i!}. \end{aligned} \tag{5.3.20}$$

Since (5.3.19) and (5.3.20) are identical, we have proved Equation (5.3.16). This completes the proof of Lemma 5.3.1.

5.4 Reduction to the semi-infinite wedge formalism

In this Section we represent the formula for $\Psi(x, \hbar)$ in terms of the semi-infinite wedge formalism. We use the formula of Okounkov-Pandharipande [81] that relates the stationary sector of the Gromov-Witten invariants of $\mathbb{P}^1$ to the expectation values of the so-called $\mathcal{E}$-operators. In order to include the extra combinatorial factors that we have in the expansion of $\log \Psi(x, \hbar)$, we consider the $\mathcal{E}$-operators with values in formal differential operators.

5.4.1 Semi-infinite wedge formalism

In this subsection we recall very briefly some basic facts about the semi-infinite wedge formalism. For more details we refer to [33, 81, 86].

Let us consider a vector space $V := \bigoplus_{c=-\infty}^{\infty} V_c$, where V_c is spanned by the basis vectors $\underline{a_1} \wedge \underline{a_2} \wedge \underline{a_3} \wedge \cdots$ such that $a_i \in \mathbb{Z} + 1/2$, $i = 1, 2, \ldots$, $a_1 > a_2 > a_3 \ldots$, and for all but a finite number of terms we have $a_i = 1/2 - i + c$. We denote by ψ_k the operator $\underline{k}\wedge \colon V_c \to V_{c+1}$, and by ψ_k^* the operator $\partial/\partial \underline{k} \colon V_c \to V_{c-1}$. Both are odd operators, and they satisfy the graded commutation relation $[\psi_i, \psi_i^*] = 1$, with all other possible pairwise commutators equal to zero.

We denote by $: \psi_i \psi_j^* :$ the normally ordered product, that is, $: \psi_i \psi_j^* := \psi_i \psi_j^*$ for $j > 0$ and $: \psi_i \psi_j^* := -\psi_j^* \psi_i$ for $j < 0$. We introduce the operators $\mathcal{E}_n(z)$, $n \in \mathbb{Z}$ as

$$\mathcal{E}_n(z) := \sum_{k \in \mathbb{Z}+1/2} \exp\left(z\left(k - \frac{n}{2}\right)\right) : \psi_{k-r}\psi_k^* : + \frac{\delta_{n0}}{\zeta(z)}, \tag{5.4.1}$$

where $\zeta(z) = \exp(z/2) - \exp(-z/2)$. These operators satisfy the commutation relation $[\mathcal{E}_n(z), \mathcal{E}_m(w)] = \zeta(nw - mz)\mathcal{E}_{n+m}(z+w)$.

For any operator $\mathcal{A} = \mathcal{E}_{n_1}(z_1) \cdots \mathcal{E}_{n_m}(z_m)$ we denote by $\langle |\mathcal{A}| \rangle$ the coefficient of the vector $v_\emptyset := \underline{-1/2} \wedge \underline{-3/2} \wedge \underline{-5/2} \wedge \cdots$ in the basis expansion of $\mathcal{A}v_\emptyset$. If we want to compute a particular correlator $\langle |\mathcal{E}_{n_1}(z_1) \cdots \mathcal{E}_{n_m}(z_m)| \rangle$, first we use the commutation relation for the $\mathcal{E}$-operators, and then appeal to the simple fact that $\mathcal{E}_n(z)|\rangle = 0$ for $n > 0$, $\langle|\mathcal{E}_n(z) = 0$ for $n < 0$, and $\langle |\mathcal{E}_0(z_1) \cdots \mathcal{E}_0(z_n)| \rangle = 1/\left(\zeta(z_1) \cdots \zeta(z_n)\right)$. In this section we are mostly interested in correlators for the form

$$\langle |\mathcal{A}| \rangle = \left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(z_i) \mathcal{E}_{-1}(0)^d \right| \right\rangle. \tag{5.4.2}$$

For the purpose of establishing the results in [81], Okounkov and Pandharipande considered the *disconnected* version of Gromov-Witten invariants and Hurwitz numbers. The disconnectedness here means we allow disconnected domain curves mapped to $\mathbb{P}^1$. For example, they establish in [81, Proposition 3.1, Equation 3.4] a formula for disconnected stationary Gromov-Witten invariants of $\mathbb{P}^1$, which reads

$$\sum_{b_1, \ldots, b_n \geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega) \right\rangle^{\bullet d} \prod_{i=1}^n x_i^{b_i+1} = \frac{1}{(d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^n \mathcal{E}_0(x_i) \mathcal{E}_{-1}(0)^d \right| \right\rangle, \tag{5.4.3}$$

where $\langle \quad \rangle^\bullet$ denotes the disconnected Gromov-Witten invariant. Counting the number of disconnected domain curves and connected ones are related simply by talking the logarithm. Thus we have

$$\sum_{g=0}^{\infty} \sum_{b_1, \ldots, b_n \geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega) \right\rangle_{g,n}^d \prod_{i=1}^n x_i^{b_i+1} = \log \left(\sum_{b_1, \ldots, b_n \geq -2} \left\langle \prod_{i=1}^n \tau_{b_i}(\omega) \right\rangle^{\bullet d} \prod_{i=1}^n x_i^{b_i+1} \right).$$

This prompts us to introduce the *connected* correlator notation, corresponding to

(5.4.3), as follows:

$$\sum_{g=0}^{\infty}\sum_{b_1,\dots,b_n\geq -2}\left\langle\prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle_{g,n}^{d}\prod_{i=1}^{n}x_i^{b_i+1} = \frac{1}{(d!)^2}\left\langle\left|\mathcal{E}_1(0)^d\prod_{i=1}^{n}\mathcal{E}_0(x_i)\mathcal{E}_{-1}(0)^d\right|\right\rangle^{\circ}. \quad (5.4.4)$$

The connected correlator is also known as the *cumulant* in probability theory, which is calculate via the inclusion-exclusion formula. In general, for an operator $\mathcal{A}$ of (5.4.2), we denote by $\langle|\mathcal{A}|\rangle^{\circ}$ the contribution coming from the single operator of the form $\mathcal{E}_0(\sum_{i=1}^n z_i)$ in the end, after applying the commutation relation successively. Of course in terms of generating functions, this simply means we take the logarithm of the expression. See [33, Definition 2.12, Definition 2.14] for more detail.

5.4.2 A new formula for Ψ

Noticing that $\exp\left(\frac{\hbar}{2}\frac{d}{dx}\right)$ is an automorphism, from (5.3.2) we find

$$\log\Psi(x,\hbar) = \exp\left(\frac{\hbar}{2}\frac{d}{dx}\right)T(x),$$

where

$$T(x) := \sum_{g,d=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g-2+n}}{n!}\sum_{b_1,\dots,b_n=0}^{\infty}\left\langle\prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle_{g,n}^{d}\prod_{i=1}^{n}\left(-\frac{b_i!}{x^{b_i+1}}\right) + \frac{1}{\hbar}\langle\tau_{-2}(\omega)\rangle_{0,1}^{0}(x - x\log x).$$

Here we have used the convention of [81] that $\langle\tau_{-2}(\omega)\rangle_{0,1}^{0} = 1$ and $\tau_{-1}(\omega) = 0$. We are now ready to re-write the right-hand side in terms of expectation values of $\mathcal{E}$-operators. Corollary 5.4.2 of the following lemma is the main result of this section.

Lemma 5.4.1. *For any $d \geq 0$, $n \geq 1$, $(d,n) \neq (0,1)$, we have*

$$\sum_{g=0}^{\infty}\hbar^{2g-2+n}\sum_{b_1,\dots,b_n=0}^{\infty}\left\langle\prod_{i=1}^{n}\tau_{b_i}(\omega)\right\rangle_{g,n}^{d}\prod_{i=1}^{n}\left(-\frac{b_i!}{x_i^{b_i+1}}\right) = \frac{1}{(d!)^2\hbar^{2d}}\left\langle\left|\mathcal{E}_1(0)^d\prod_{i=1}^{n}\mathcal{E}_0\left(-\hbar\frac{\partial}{\partial x_i}\right)(\log x_i)\,\mathcal{E}_{-1}(0)^{-d}\right|\right\rangle^{\circ}. \quad (5.4.5)$$

For $d = 0$ and $n = 1$, we have

$$\frac{1}{\hbar}\langle\tau_{-2}(\omega)\rangle_{0,1}^{0}(x - x\log x) + \sum_{g=1}^{\infty}\hbar^{2g-1}\left\langle\prod_{i=1}^{n}\tau_{2g-2}(\omega)\right\rangle_{g,1}^{0}\left(-\frac{(2g-2)!}{x^{2g-1}}\right) = \left\langle\left|\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right|\right\rangle^{\circ}. \quad (5.4.6)$$

Here we denote by $\langle \quad \rangle^\circ$ the connected expectation value. This means that after the successive application of the commutation relation, all differential operators appear in one correlator. Of course for $d=0$, $n=1$, we have $\langle \mathcal{E}_0 \rangle^\circ = \langle \mathcal{E}_0 \rangle$. The following corollary is a straightforward application of Lemma 5.4.1.

Corollary 5.4.2. *We have the following expression for* $\log \Psi$*:*

$$\log \Psi(x,\hbar) = \sum_{d=0}^{\infty} \frac{1}{\hbar^{2d}(d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \sum_{n=1}^{\infty} \frac{\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)^n}{n!} \mathcal{E}_{-1}(0)^d \right| \right\rangle^\circ . \quad (5.4.7)$$

5.4.3 Proof of Lemma 5.4.1

The starting point of the proof is (5.4.4). Note that only negative b_i contribution comes from $\langle \tau_{-2}(\omega)\rangle_{0,1}^0 = 1$, which is the coefficient of x_i^{-1} in $\langle |\mathcal{E}_0(x_i)|\rangle^\circ$.

Let $A(x) = \sum_{i=-1}^{\infty} a_i x^i$ be an arbitrary Laurent series. Observe that

$$A\left(-\hbar\frac{d}{dx}\right)(\log x) = a_{-1}\left(\frac{x - x\log x}{\hbar}\right) + a_0 \log x - \sum_{i=1}^{\infty} a_i \frac{(i-1)!\hbar^i}{x^i}. \quad (5.4.8)$$

We can apply this observation to the correlator

$$\frac{1}{(d!)^2}\left\langle \left| \mathcal{E}_1(0)^d \prod_{i=1}^{n} \mathcal{E}_0(x_i)\,\mathcal{E}_{-1}(0)^{-d} \right| \right\rangle^\circ \quad (5.4.9)$$

and change $\mathcal{E}_0(x_i)$ to

$$\mathcal{E}_0\left(-\hbar\frac{\partial}{\partial x_i}\right)\log x_i.$$

If $(n,d) \neq (1,0)$, then we have a formal Laurent series in $x_1,\dots,x_n$, where the degree of each variable in each term is less than or equal to -1. Together with the computation of the degree of $\hbar$, which is $\sum_{i=1}^{n}(b_i+1) - 2d = 2g-2+n$, we establish Equation (5.4.5).

If $(n,d) = (1,0)$, then it is sufficient to observe that $\langle |\mathcal{E}_0(x)|\rangle^\circ = x^{-1} + O(x)$. Thus we have one additional term $(x - x\log x)/\hbar$ as in (5.4.8), which is exactly the first term in Equation (5.4.6).

This completes the proof of Lemma 5.4.1, and hence, Corollary 5.4.2.

5.5 Reduction to a combinatorial problem

The expression (5.4.7) of $\log \Psi$ in the form of the vacuum expectation value of the operator product allows us to convert the quantum curve equation (5.1.7) into a combinatorial formula.

Our starting point is the Ψ-function represented in the form

$$\Psi(x,\hbar) = 1 + \sum_{d=0}^{\infty} \frac{1}{\hbar^{2d}(d!)^2} \left\langle \left| \mathcal{E}_1(0)^d \sum_{n=1}^{\infty} \frac{1}{n!}\mathcal{A}(x)^n \mathcal{E}_{-1}(0)^d \right| \right\rangle^\star , \quad (5.5.1)$$

where

$$\begin{aligned}\mathcal{A}(x) &= \exp\left(\frac{\hbar}{2}\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x) \\ &= \sum_{k\in\mathbb{Z}+\frac{1}{2}} \exp\left(\left(-k+\frac{1}{2}\right)\hbar\frac{d}{dx}\right)(\log x) : \psi_k\psi_k^* : \\ &\quad + B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x - x\log x}{\hbar}\right).\end{aligned} \tag{5.5.2}$$

Here $B(t) := t/(e^t-1)$ in (5.5.2) is the generating series of the Bernoulli numbers, and the notation $\langle - \rangle^\star$ in (5.5.1) means that in the computation of this expectation value using the commutation relations, we never allow any $\mathcal{E}_1(0)$ and $\mathcal{E}_{-1}(0)$ to commute directly. We need this requirement since we exponentiate the series (5.4.7), which does not have terms without $\mathcal{E}_0$-operators. The goal of this section is to prove Corollary 5.5.2.

Lemma 5.5.1. *We have*

$$\exp\left(\frac{1}{\hbar^2}\right)\Psi(x,\hbar) = \exp\left(B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x-x\log x}{\hbar}\right)\right)X, \tag{5.5.3}$$

where $X := \sum_{d=0}^{\infty} X_d/\hbar^{2g}$*, and* X_d *is given by*

$$\begin{aligned} X_d &= \frac{1}{(d!)^2}\left\langle\left|\mathcal{E}_1(0)^d \exp\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \log\left(x - \left(k-\frac{1}{2}\right)\hbar\right) : \psi_k\psi_k^* :\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle \\ &= \sum_{\lambda\vdash d}\left(\frac{\dim\lambda}{d!}\right)^2 \prod_{i=1}^{\infty}\frac{x+(i-\lambda_i)\hbar}{x+i\hbar}.\end{aligned} \tag{5.5.4}$$

Corollary 5.5.2. *The quantum spectral curve equation*

$$\left[\exp\left(\hbar\frac{d}{dx}\right) + \exp\left(-\hbar\frac{d}{dx}\right) - x\right]\Psi(x,\hbar) = 0$$

is equivalent to the following equation for the function X:

$$\left[\frac{1}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right) + x\exp\left(-\hbar\frac{d}{dx}\right) - x\right]X = 0. \tag{5.5.5}$$

Proof of Lemma 5.5.1. Corollary 5.4.2 implies that

$$\begin{aligned}&\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d \exp\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle^\circ}{\hbar^{2d}(d!)^2} \\ &= \log\Psi(x,\hbar) + \frac{1}{\hbar^2} + 1.\end{aligned} \tag{5.5.6}$$

Indeed, we add terms with $n=0$, and it is easy to see that

$$\left\langle\left|\mathcal{E}_1(0)^d\mathcal{E}_{-1}(0)^d\right|\right\rangle^\circ = 0, \qquad d \geq 2,$$

and $\langle|\mathcal{E}_1(0)\mathcal{E}_{-1}(0)|\rangle^\circ = \langle|\mathrm{Id}|\rangle^\circ = 1$. Therefore,

$$\exp\left(\frac{1}{\hbar^2}\right)\Psi(x,\hbar) = \sum_{d=0}^{\infty} \frac{\left\langle \left|\mathcal{E}_1(0)^d \exp\left(\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\right)\mathcal{E}_{-1}(0)^d\right|\right\rangle}{\hbar^{2d}(d!)^2}. \tag{5.5.7}$$

From the definition of the operator $\mathcal{E}_0$, we have

$$\begin{aligned}
&\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\mathcal{E}_0\left(-\hbar\frac{d}{dx}\right)(\log x)\\
&= \exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\left(\sum_{k\in\mathbb{Z}+1/2}\log\left(x-k\hbar\right) : \psi_k\psi_k^* :\right)\\
&\quad + \exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)\frac{-\hbar\frac{d}{dx}}{\exp\left(-\frac{1}{2}\hbar\frac{d}{dx}\right)-\exp\left(\frac{1}{2}\hbar\frac{d}{dx}\right)}\left(\frac{x-x\log x}{\hbar}\right)\\
&= \sum_{k\in\mathbb{Z}+1/2}\log\left(x-\left(k-\frac{1}{2}\right)\hbar\right) : \psi_k\psi_k^* : + B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x-x\log x}{\hbar}\right).
\end{aligned} \tag{5.5.8}$$

Now define

$$A_1 = \sum_{k\in\mathbb{Z}+1/2}\log\left(x-\left(k-\frac{1}{2}\right)\hbar\right) : \psi_k\psi_k^* : \tag{5.5.9}$$

$$A_2 = B\left(-\hbar\frac{d}{dx}\right)\left(\frac{x-x\log x}{\hbar}\right). \tag{5.5.10}$$

Since A_1 and A_2 commute, we have $\exp(A_1+A_2) = \exp(A_2)\exp(A_1)$. Furthermore, since A_2 is a scalar operator, we have

$$\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d\exp(A_2)\exp(A_1)\mathcal{E}_{-1}(0)^d\right|\right\rangle}{\hbar^{2d}(d!)^2} = \exp(A_2)\sum_{d=0}^{\infty}\frac{\left\langle\left|\mathcal{E}_1(0)^d\exp(A_1)\mathcal{E}_{-1}(0)^d\right|\right\rangle}{\hbar^{2d}(d!)^2}.$$

This is exactly the right-hand side of Equation (5.5.3). □

Proof of Corollary 5.5.2. We just have to show that

$$\begin{aligned}
\exp(-A_2)\exp\left(\hbar\frac{d}{dx}\right)\exp(A_2) &= \frac{1}{x+\hbar}\exp\left(\hbar\frac{d}{dx}\right);\\
\exp(-A_2)\exp\left(-\hbar\frac{d}{dx}\right)\exp(A_2) &= x\exp\left(-\hbar\frac{d}{dx}\right);\\
\exp(-A_2)x\exp(A_2) &= x.
\end{aligned}$$

The last equality is tautological, and the first two are obtained by a straightforward computation. □

For completeness, let us also explain Equation (5.5.4). It is based on several standard facts about the semi-infinite wedge formalism. For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots)$ we associate a basis vector $v_\lambda \in V_0$ given by

$$\underline{\left(\lambda_1 - \frac{1}{2}\right)} \wedge \underline{\left(\lambda_2 - \frac{3}{2}\right)} \wedge \underline{\left(\lambda_3 - \frac{5}{2}\right)} \wedge \cdots . \tag{5.5.11}$$

Then, we have $\mathcal{E}_{-1}(0)^d v_\emptyset = \sum_{\lambda \vdash d} \dim \lambda \cdot v_\lambda$, $\left\langle \middle| \mathcal{E}_1(0)^d v_\lambda \right. = \dim \lambda$, and the fact that for any constants a_n, $n \in \mathbb{Z}+1/2$, v_λ is an eigenvector of the operator $\sum_{n\in\mathbb{Z}+1/2} a_n : \psi_n \psi_n^* :$ with the eigenvalue $\sum_{i=1}^\infty \left(a_{\lambda_i - i + 1/2} - a_{-i+1/2}\right)$. Therefore, v_λ is an eigenvector of the operator

$$A_1 = \exp\left(\sum_{k\in\mathbb{Z}+\frac{1}{2}} \log\left(x - \left(k - \frac{1}{2}\right)\hbar\right) : \psi_k \psi_k^* :\right) \tag{5.5.12}$$

with the eigenvalue

$$\exp\left(\sum_{i=1}^\infty \log\left(x + (i - \lambda_i)\hbar\right) - \log\left(x + i\hbar\right)\right) = \prod_{i=1}^\infty \frac{x + (i-\lambda_i)\hbar}{x + i\hbar}, \tag{5.5.13}$$

and the total weight of the vector v_λ in $\left\langle \middle| \mathcal{E}_1(0)^d A_1 \mathcal{E}_{-1}(0)^d \middle| \right\rangle$ is $(\dim \lambda)^2$. This implies Equation (5.5.4).

5.6 Key combinatorial argument

We have shown that the quantum curve equation (5.1.7) is equivalent to a combinatorial equation (5.5.5), which is indeed a first-order recursion equation for X_d of (5.5.4) with respect to the index d. In this section we prove (5.5.5).

Let $\lambda \vdash d$ be a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{\ell(\lambda)} > 0)$ of $d \geq 1$. We can always append it with $d - \ell(\lambda)$ zeros $\lambda_{\ell(\lambda)+1} := 0, \dots, \lambda_d := 0$ at the end so that we would have a partition of d of length d with non-negative parts. Throughout this section we use this convention that a partition of d has length d.

Consider the following sum over all partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_d)$ of $d \geq 1$:

$$X_d := \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \prod_{i=1}^d \frac{x + (i - \lambda_i)\hbar}{x + i\hbar}. \tag{5.6.1}$$

Here $H_\lambda := \prod_{ij} h_{ij}$, where h_{ij} is the hook length at the vertex (ij) of the corresponding Young diagram, so that $d!/\prod h_{ij}$ is the dimension of the irreducible representation corresponding to λ. Or equivalently, it is the number of the standard Young tableaux of this shape. We use the convention that $X_0 := 1$.

In this Section we prove the following key combinatorial lemma.

Lemma 5.6.1. *The series $X := \sum_{d=0}^\infty X_d/\hbar^{2g}$ satisfies the following equation:*

$$\left[\frac{1}{x+\hbar} \exp\left(\hbar \frac{d}{dx}\right) + x \exp\left(-\hbar \frac{d}{dx}\right) - x\right] X = 0. \tag{5.6.2}$$

Proof. In fact, (5.6.2) is a direct consequence of the following more refined statement.

Lemma 5.6.2. *For any $d \geq 1$ we have*

$$\frac{1}{x/\hbar+1}\exp\left(\hbar\frac{d}{dx}\right)X_{d-1}+\left[\frac{x}{\hbar}\exp\left(-\hbar\frac{d}{dx}\right)-\frac{x}{\hbar}\right]X_d=0. \tag{5.6.3}$$

Indeed, since $\left[x\exp\left(-h\frac{d}{dx}\right)-x\right]X_0=0$, the sum of Equation (5.6.3) for all $d \geq 1$ with coefficients $1/\hbar^{2d-1}$ yields Lemma 5.6.1. □

To prove Lemma 5.6.2, we need to recall some standard facts on the hook length formula as well as a recent result of Han [60].

5.6.1 Hook lengths and shifted parts of partition

We use the following result from [60]. For a partition $\lambda \vdash d$, $d \geq 1$, we define the so-called g-function:

$$g_\lambda(y) := \prod_{i=1}^{d}(y+\lambda_i-i). \tag{5.6.4}$$

For any $\lambda \vdash d$, $d \geq 1$, we denote by $\lambda \setminus 1$ the set of all partitions of $d-1$ that can be obtained from λ (or rather the corresponding Young diagram) by removing one corner of λ.

Lemma 5.6.3 (Han [60])**.** *For every partition λ we have*

$$\frac{1}{H_\lambda}\left(g_\lambda(y+1)-g_\lambda(y)\right)=\sum_{\mu\in\lambda\setminus 1}\frac{1}{H_\mu}g_\mu(y). \tag{5.6.5}$$

Here y is a formal variable.

We need the following corollary of this lemma.

Corollary 5.6.4. *For an integer $d \geq 1$ we have*

$$\sum_{\lambda\vdash d+1}\frac{1}{H_\lambda^2}\left(g_\lambda(y+1)-g_\lambda(y)\right)=\sum_{\mu\vdash d}\frac{1}{H_\mu^2}g_\mu(y). \tag{5.6.6}$$

Proof. We recall that for any $\mu \vdash d$, $d \geq 1$, we have:

$$\sum_{\substack{\lambda\vdash d+1\\ \lambda\setminus 1\ni\mu}}\frac{1}{H_\lambda}=\frac{1}{H_\mu}. \tag{5.6.7}$$

Therefore,

$$\begin{aligned}\sum_{\mu\vdash d}\frac{1}{H_\mu^2}g_\mu(y)&=\sum_{\mu\vdash d}\frac{1}{H_\mu}\sum_{\substack{\lambda\vdash d+1\\ \lambda\setminus 1\ni\mu}}\frac{1}{H_\lambda}g_\mu(y)\\&=\sum_{\lambda\vdash d+1}\frac{1}{H_\lambda}\sum_{\substack{\mu\vdash d\\ \mu\in\lambda\setminus 1}}\frac{1}{H_\lambda}g_\mu(y)\\&=\sum_{\lambda\vdash d+1}\frac{1}{H_\lambda^2}\left(g_\lambda(y+1)-g_\lambda(y)\right).\end{aligned}$$

□

5.6.2 Reformulation of Lemma 5.6.2 in terms of g-functions

We make the following substitution: $y := -x/\hbar$. Then we see that

$$X_d = \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y)}{\prod_{i=1}^d (y-i)}.$$

Moreover,

$$\begin{aligned} &\frac{1}{x/\hbar + 1} \exp\left(\hbar \frac{d}{dx}\right) X_{d-1} + \left[\frac{x}{\hbar} \exp\left(-\hbar \frac{d}{dx}\right) - \frac{x}{\hbar}\right] X_d \\ &= \frac{-1}{y-1} \exp\left(-\frac{d}{dy}\right) X_{d-1} + \left[-y \exp\left(\frac{d}{dy}\right) + y\right] X_d. \end{aligned} \tag{5.6.8}$$

Observe that

$$\begin{aligned} \frac{-1}{y-1} \exp\left(-\frac{d}{dy}\right) X_{d-1} &= -\sum_{\lambda \vdash d-1} \frac{1}{H_\lambda^2} \frac{g_\lambda(y-1)}{\prod_{i=1}^d (y-i)}; \\ -y \exp\left(\frac{d}{dy}\right) X_d &= (d-y) \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y+1)}{\prod_{i=1}^d (y-i)}; \\ y X_d &= y \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y)}{\prod_{i=1}^d (y-i)}. \end{aligned} \tag{5.6.9}$$

Using Corollary 5.6.4 we can rewrite the right hand side of Equation (5.6.9) as

$$\frac{-1}{y-1} \exp\left(-\frac{d}{dy}\right) X_{d-1} = \sum_{\lambda \vdash d} \frac{1}{H_\lambda^2} \frac{g_\lambda(y-1) - g_\lambda(y)}{\prod_{i=1}^d (y-i)}. \tag{5.6.10}$$

Therefore, the right hand side of Equation (5.6.8) is equal to

$$\frac{Y_d(y)}{\prod_{i=1}^d (y-i)}, \tag{5.6.11}$$

where

$$Y_d(y) := \sum_{\lambda \vdash d} \frac{(d-y) g_\lambda(y+1) + (y-1) g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2}. \tag{5.6.12}$$

Note that $Y_d(y)$ is a polynomial in y of degree $\leq d+1$, and Lemma 5.6.2 is equivalent to the following statement:

Lemma 5.6.5. *For any $d \geq 1$ we have $Y_d(y) \equiv 0$.*

5.6.3 Proof of Lemma 5.6.5

In this subsection we prove Lemma 5.6.5 and, therefore, Lemmas 5.6.2 and 5.6.1.

First of all, it is easy to check that for any $d \geq 1$ the polynomial $Y_d(y)$ has at least one root. Namely,

$$Y_d(d) = \sum_{\lambda \vdash d} \frac{(d-1) g_\lambda(d) + g_\lambda(d-1)}{H_\lambda^2} = 0. \tag{5.6.13}$$

Indeed, $g_\lambda(d)$ is not equal to zero only for $\lambda = (1, 1, \ldots, 1)$. In this case $g_\lambda(d) = d!$, $H_\lambda = d!$, and $(d-1)g_\lambda(d)/H_\lambda^2 = (d-1)/d!$. Notice that $g_\lambda(d-1)$ does not vanish only for $\lambda = (2, 1, 1, \ldots, 1, 0)$. In this case $g_\lambda(d-1) = -d \cdot (d-2)!$, $H_\lambda = d \cdot (d-2)!$, and $g_\lambda(d-1)/H_\lambda^2 = -(d-1)/d!$. Thus we see that always $Y_d(d) = 0$, establishing (5.6.13).

Now we proceed by induction. It is easy to check that $Y_1(y) \equiv 0$. Assume that we know that $Y_d(y) \equiv 0$. Corollary 5.6.4 then implies that

$$\begin{aligned}
Y_d(y) &= \sum_{\lambda \vdash d} \frac{(d-y)g_\lambda(y+1) + (y-1)g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2} \\
&= \sum_{\lambda \vdash d+1} \frac{(d-y)g_\lambda(y+2) + (2y-d-1)g_\lambda(y+1) + (2-y)g_\lambda(y) - g_\lambda(y-1)}{H_\lambda^2} \\
&= \sum_{\lambda \vdash d+1} \frac{((d+1)-(y+1))g_\lambda(y+2) + ((y+1)-1)g_\lambda(y+1) + g_\lambda(y)}{H_\lambda^2} \\
&\quad - \sum_{\lambda \vdash d+1} \frac{((d+1)-y)g_\lambda(y+1) + (y-1)g_\lambda(y) + g_\lambda(y-1)}{H_\lambda^2} \\
&= Y_{d+1}(y+1) - Y_{d+1}(y).
\end{aligned}$$

By assumption, we have $Y_d(y) \equiv 0$. Therefore, $Y_{d+1}(y+1) = Y_{d+1}(y)$ for any y. HenceY_{d+1} is constant. Since we have shown that $Y_{d+1}(d+1) = 0$, we conclude that $Y_{d+1} \equiv 0$.

This completes the proof of Lemmas 5.6.5, 5.6.2, and 5.6.1. Thus we have established the main theorem of this chapter.

6

Combinatorics of loop equations for branched covers of sphere

Abstract

We prove, in a purely combinatorial way, the spectral curve topological recursion for the problem of enumeration of bi-colored maps, which are dual objects to dessins d'enfant. Furthermore, we give a proof of the quantum spectral curve equation for this problem. Then we consider the generalized case of 4-colored maps and outline the idea of the proof of the corresponding spectral curve topological recursion.

6.1 Introduction

In this chapter we discuss the enumeration of *bi-colored maps*. They are decompositions of closed orientable two-dimensional surfaces into polygons of black and white color glued along their sides, considered as combinatorial objects. We count such decomposition of two-dimensional surfaces into a fixed set of polygons with some appropriate weights. This problem is then equivalent to enumeration of Belyi functions with fixed type of local monodromy data over its critical values (they are called *hypermaps* [69]), which is a special case of a more general Hurwitz problem.

Belyi functions are objects of principle importance in algebraic geometry; they allow to detect the algebraic curves defined over the field of algebraic numbers. There is a way to study them in terms of "dessins d'enfants", that is, some embedded graphs in two-dimensional surfaces, see [69] for a survey or [2] for some recent developments.

The local monodromy data of a Belyi function can be controlled by the choice of three partitions of the degree of the function. We consider a special generating function for enumeration of Belyi functions. Namely, we fix the length of the first partition to be n and we introduce some formal variables $x_1, \ldots, x_n$ to control the first partition as an n-point function; we introduce auxiliary parameters t_i, $i \geq 1$, in order to control the number of parts of length i in the second partition as a generating function; and we take the sum of all possible choices of the third partition so that the genus of the surface is equal to $g \geq 0$. This way we get some functions $W_n^{(g)}(x_1, \ldots, x_n)$ that also depend on formal parameters t_i, $i \geq 0$.

As soon as we get some meaningful combinatorial problem, where it is natural to arrange the answers into the generating functions of this type, it makes sense to check whether these functions $W_n^{(g)}(x_1,\ldots,x_n)$ can be reproduced via the *topological recursion* [50]. The theory of topological recursion has initially occurred as a way to solve a set of loop equations satisfied by the correlation functions of a particular class of matrix models [45, 16, 51, 17]. Then it has evolved to a more abstract and much more general mathematical theory that associates some functions $\mathcal{W}_n^{(g)}(x_1,\ldots,x_n)$ to some small input related to an algebraic curve called *spectral curve*, see [50]. The question is whether we can prove the topological recursion for the generating functions $W_n^{(g)}(x_1,\ldots,x_n)$ and, if yes, what would be the spectral curve in this case.

For an expert in matrix models the answer is obvious. Indeed, we go back to the original formulation in terms of bi-colored maps. It is a standard representation of correlation functions of a two matrix model, see a survey in [41] or more recent paper [5], and the topological recursion in this case is derived in [17]. However, the general question that one can pose there is whether there is any way to relate the topological recursion to the intrinsic combinatorics of bi-colored maps. There are two steps of derivation of the topological recursion in [17]. First, using skillfully chosen changes of variables in the matrix integral, one can define the *loop equations* for the correlation functions [43]. Then, via a sequence of formal computations, one can determine the spectral curve and prove the topological recursion.

The loop equations of a formal matrix model are equivalent to some combinatorial properties of bi-colored maps [93]. In this chapter, we exhibit these combinatorial relations deriving the loop equations directly from the intrinsic combinatorics of the bi-colored maps. This procedure can be generalized for deriving combinatorially the loop equations of an arbitrary formal matrix model. This allows us to give a new, purely combinatorial proof of the topological recursion for the functions $W_n^{(g)}(x_1,\ldots,x_n)$.

Let us stress that in [41, 5, 17] this problem of counting dessins d'enfant was addressed in matrix model approach. Here, by proving loop equations in a combinatorial way, we have a purely combinatorial approach to this problem.

Let us also note that although the above mentioned papers dealt with the same numbers (counting dessins d'enfant), different generating functions were considered. The link to the spectral curve topological recursion was not established there. Since spectral curve topological recursion arose in the context of matrix models, the step from a matrix model for a particular counting problem to the topological recursion is a well-known one, and in this particular case it follows from existing works. We stress, however, that we circumvent the matrix model approach and obtain a purely combinatorial proof for the spectral curve topological recursion.

A motivation for this chapter comes from a recent question posed by Do and Manescu in [26]. They considered the enumeration problem for a special case of our bi-colored maps, where all polygons of the white color have the same length a. In this case, they conjectured that this enumeration problem satisfies the topological recursion and proposed a particular spectral curve. So, as a special case of our result, we prove their conjecture, and it appears to be a purely combinatorial proof. Though similar problems were considered a lot recently [64, 6, 7], the question posed by Do and Manescu was not covered there.

There is a general principle that associates to a given spectral curve its quantisation, which is a differential operator called *quantum spectral curve* [59]. Conjecturally, this operator should annihilate the wave function, which is, roughly speaking, the exponent of the generating series of functions $\int^x \cdots \int^x W_n^{(g)}(x_1, \ldots, x_n) dx_1 \cdots dx_n$. We show that this general principle works in this case, namely, we derive the quantum spectral curve directly from the same combinatorics of loop equations. This generalizes the main result in [26] .

The combinatorics that we use in the analysis of bi-colored maps is in fact of a more general nature. The same idea of derivation of the loop equations can be used in more general settings. In particular, we outline the idea of how it would work for the enumeration of 4-colored maps, where the topological recursion was derived from the loop equations by Eynard in [44].

6.1.1 Organization of the chapter

In Section 6.2 we recall the definitions of hypermaps and discuss generating functions corresponding to hypermap enumeration problems.

In Section 6.3 we reformulate the definition of hypermaps in terms of bi-colored maps and, for use as a motivation for our combinatorial proof, recall the 2-matrix model which gives rise to enumeration of bi-colored maps.

In Section 6.4 we recall the form of the loop equations for the 2-matrix model and then we show that using purely combinatorial argument to prove the basic building blocks of loop equations, we can obtain a purely combinatorial proof of the spectral curve for the enumeration of bi-colored maps.

In Section 6.5 we review the problem of finding the quantum curve for enumeration of hypermaps.

In Section 6.6 we outline the idea of the proof of the spectral curve topological recursion for the even further generalization of our problem: the case of 4-colored maps, which corresponds to 4-matrix models.

6.2 Branched covers of $\mathbb{P}^1$

6.2.1 Definitions

We are interested in the enumeration of covers of $\mathbb{P}^1$ branched over three points. These covers are defined as follows.

Definition 6.2.1. Consider m positive integers $a_1, \ldots, a_m$ and n positive integers $b_1, \ldots, b_n$. We denote by $\mathcal{M}_{g,m,n}(a_1, \ldots, a_m | b_1, \ldots, b_n)$ the weighted count of branched covers of $\mathbb{P}^1$ by a genus g surface with $m + n$ marked points
$f\colon\ (\mathcal{S}; q_1, \ldots, q_m; p_1, \ldots, p_n) \to \mathbb{P}^1$ such that

- f is unramified over $\mathbb{P}^1 \backslash \{0, 1, \infty\}$;
- the preimage divisor $f^{-1}(\infty)$ is $a_1 q_1 + \ldots a_m q_m$;
- the preimage divisor $f^{-1}(1)$ is $b_1 p_1 + \ldots b_n p_n$;

Of course, a cover f can exist only if $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. In this case $d = b_1 + \cdots + b_n$ is called the degree of a cover.

These covers are counted up to isomorphisms preserving the marked points $p_1, \dots, p_n$ pointwise and covering the identity on $\mathbb{P}^1$. The weight of a cover is equal to the inverse order of its automorphism group.

Example 6.2.2. *In [26] the authors consider the case of*

$$\mathcal{M}_{g,d/a,n}(a, \dots, a|b_1, \dots, b_n),$$

and relate this enumeration problem to the existence of a quantum curve.

Since such a branched cover can be recovered just from its monodromy around 0, 1 and ∞, it is convenient to reformulate this enumeration problem in different terms.

Definition 6.2.3. Let us fix $d \geq 1$, $g \geq 0$, $m \geq 1$, and $n \geq 1$. A hypermap of type (g, m, n) is a triple of permutations $(\sigma_0, \sigma_1, \sigma_\infty) \in S_d^3$ such that

- $\sigma_0 \sigma_1 \sigma_\infty = Id$;
- σ_1 is composed of n cycles;
- σ_∞ is composed of m cycles.

A hypermap is called *connected* if the permutations σ_0, σ_1, σ_∞ generate a transitive subgroup of S_d. A hypermap is called *labelled* if the disjoint cycles of σ_1 are labelled from 1 to n.

Two hypermaps $(\sigma_0, \sigma_1, \sigma_\infty)$ and $(\tau_0, \tau_1, \tau_\infty)$ are equivalent if one can conjugate all the σ_i's to obtain the τ_i's. Two labelled hypermaps are equivalent if in addition the conjugation preserves the labelling.

By Riemann existence theorem, one has

Lemma 6.2.4. *The number $\mathcal{M}_{g,m,n}(a_1, \dots, a_m|b_1, \dots, b_n)$ is equal to the weighted count of labelled hypermaps of type (g, m, n) where the cycles of σ_∞ have lengths $a_1, \dots, a_m$ and the cycles of σ_1 have length $b_1, \dots, b_n$. Here the weight of a labelled hypermap is the inverse order of its automorphism group.*

6.2.2 Generating functions

In order to compute these numbers, it is very useful to collect them in generating functions. For this purpose, we define:

Definition 6.2.5. Let us fix integer $g \geq 0$ and $n \geq 1$ such that $2g - 2 + n > 0$. We also fix one more integer $a \geq 1$ that will be used to restrict the possible length of cycle in σ_∞.

The n-point correlation function is defined by

$$\Omega_{g,n}^{(a)}(x_1, \dots, x_n) := \sum_{m=0}^{\infty} \sum_{\substack{1 \leq a_1, \dots, a_m \leq a \\ 0 \leq b_1, \dots, b_n}} \mathcal{M}_{g,m,n}(a_1, \dots, a_m|b_1, \dots, b_n) \prod_{i=1}^{m} t_{a_i} \prod_{j=1}^{n} b_j x_i^{-b_j - 1}. \tag{6.2.1}$$

It is a function of the variables $x_1, \dots, x_n$ that depends on formal parameters $t_1, \dots, t_a$.

Remark 6.2.6. Note that the product

$$\mathcal{M}_{g,m,n}(a_1,\ldots,a_m|b_1,\ldots,b_n)\prod_{j=1}^{n} b_j$$

counts the same covers as in Definition 6.2.1, but with an additional choice, for each i, of one of the possible b_i preimages of a path from 1 to 0 starting at point p_i.

For later convenience in the definition of the quantum curve, we define the symmetric counterpart of the n-point correlation function by (for $(g,n)\neq(0,1)$)

$$\mathcal{F}^{(a)}_{g,n}(x) := \int^x \cdots \int^x \Omega^{(a)}_{g,n}(x_1,\ldots,x_n)dx_1\ldots dx_n \tag{6.2.2}$$

The special case $(g,n)=(0,1)$, as usual, includes a logarithmic term:

$$\mathcal{F}^{(a)}_{0,1}(x) := \log(x) + \int^x \Omega^{(a)}_{0,1}(x_1)dx_1 \tag{6.2.3}$$

Then we define the wave function by

$$Z^{(a)}(x,\hbar) := \exp\left[\sum_{g=0}^{\infty}\sum_{n=1}^{\infty}\frac{\hbar^{2g+n-2}}{n!}\mathcal{F}^{(a)}_{g,n}(x)\right]. \tag{6.2.4}$$

Remark 6.2.7. Note that in Do and Manescu's paper [26] a different definition of $F_{g,n}$ was used, differing by $(-1)^n$, which leads to a different definition of $Z^{(a)}$, and, in turn, to a slightly different quantum spectral curve equation. See more on this in Section 6.5.

6.3 Maps and matrix models

In the present section we discuss the definition of bi-colored maps and review certain matrix model results for the corresponding counting problem.

These matrix model results serve as a motivation for our combinatorial proof of the spectral curve topological recursion, which is given in the next section.

Namely, we recall known matrix integral formulas for the generating functions for bi-colored maps, and then we refer to the known proof of the spectral curve topological recursion corresponding to this matrix model. We note that this latter proof only uses the loop equations of the corresponding matrix model as its input. This allows us to give a new, purely combinatorial proof this spectral curve topological recursion, by proving the loop equations independently in a combinatorial way (in Section 6.4). This is the main result of this chapter.

6.3.1 Covers branched over 3 points and maps

There exists a natural graphical representation of hypermaps, given in [97].

Let us now describe how to associate a colored map[1] to any labelled hypermap.

Each independent cycle ρ_i in the decomposition of $\sigma_1 = \rho_1\rho_2 \dots \rho_n$ is represented by a black $|\rho_i|$-gon whose corners are cyclically ordered and labelled by the numbers composing ρ_i. We glue these black polygons by their corners following σ_0. Namely, for each disjoint cycle $\rho = (\alpha_1, \dots, \alpha_k)$ of σ_0, one attaches the corners of black faces labelled by $\alpha_1, \dots \alpha_k$ to a $2k$-valent vertex such that:

- Turning around the vertex, one encounters alternatively white and black sectors (k of each) separated by the edges adjacent to the vertex;
- when turning counterclockwise around the vertex starting from the corner labelled by α_1, the labels of the corner corresponding to the black sectors adjacent to the vertex form the sequence $\alpha_1, \alpha_2, \dots, \alpha_k$.

Example 6.3.1. *Let us give an example of a bi-colored map. Consider a hypermap corresponding to $d = 7$, $g = 0$,*

$$\begin{aligned} \sigma_0 &= (1, 5, 7)(4, 6), \\ \sigma_1 &= (1, 2, 3, 4)(5, 6, 7), \\ \sigma_\infty &= (1, 6, 3, 2)(4, 5) \end{aligned}$$

Then the corresponding bi-colored map can be seen in Figure 6.1.

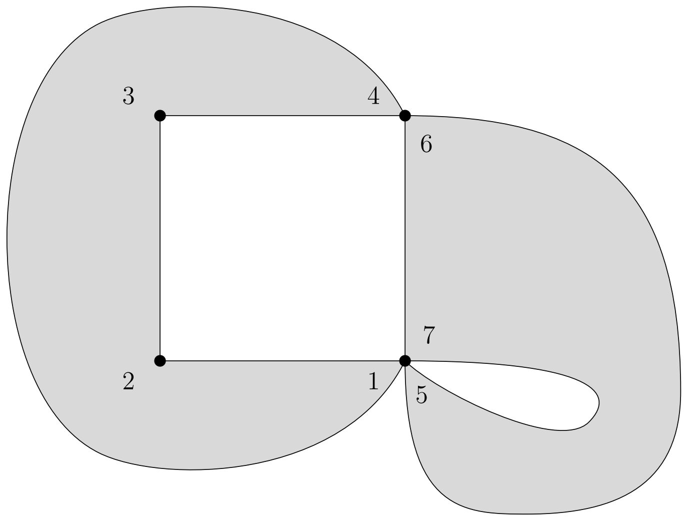


Figure 6.1: Bi-colored map

In this figure we see two black polygons corresponding to cycles $(1, 2, 3, 4)$ *and* $(5, 6, 7)$ *of* σ_1*; they are glued according to* σ_0*.*

[1] In the following, when referring to a map, we refer to a combinatorial object corresponding to a polygonalisation of a surface. These objects appear naturally in the literature in the context of random matrices and were introduced in physics as part of various attempts to quantize gravity in 2 dimensions and to approach string theory from a discrete point of view.

Let us fix $a \geq 1$. We denote by $G^{(a)}_{g,m,n}$ the set of bi-colored maps, where m is the number of white polygons, n is the number of black polygons, and g is the genus of the surface we get by gluing the polygons and a is the maximum perimeter of a white polygon. We assume that the black polygons are labelled, and we consider the maps up to combinatorial isomorphisms preserving this labelling. For a particular map $M \in G^{(a)}_{g,m,n}$ we denote by $\mathrm{Aut}(M)$ its automorphism group.

One can restate the problem of enumerating covers of $\mathbb{P}^1$ as counting bi-colored maps as follows.

Lemma 6.3.2. *The function $\Omega^{(a)}_{g,n}(x_1, \dots, x_n)$ is the generating function of bi-colored maps with an arbitrary number $m \geq 1$ of white faces whose perimeters are less or equal to a and n marked black faces with perimeters $b_1, \dots, b_n$. That is,*

$$\Omega^{(a)}_{g,n}(x_1, \dots, x_n) = \sum_{m=1}^{\infty} \sum_{M \in G^{(a)}_{g,m,n}} \frac{\prod_{i=1}^{a} t_i^{n_i(M)}}{|\mathrm{Aut}(M)|} \prod_{j=1}^{n} b_j(M) x_j^{-b_j(M)-1}. \tag{6.3.1}$$

Here by $n_i(M)$ we denote the number of white polygons of perimeter i in M, and $b_1(M), \dots, b_n(M)$ are the perimeters of the black polygons in M.

6.3.2 Matrix model and topological recursion

In the present subsection we recall the matrix model techniques of solving the problem of enumeration of bi-colored maps, which provide the motivation for our subsequent combinatorial proof of spectral curve topological recursion for this problem.

The enumeration of bi-colored maps is a classical problem of random matrix theory which is equivalent to the computation of formal matrix integrals. One can state this equivalence in the following way.

Lemma 6.3.3. *(see, e. g. [41]) Consider the partition function of a formal Hermitian two-matrix model*

$$\mathcal{Z}\left(\vec{t}^{(1)}, \vec{t}^{(2)}\right) := \int_{H_N}^{formal} dM_1\, dM_2\, e^{-N[\mathrm{Tr}(M_1 M_2) - \mathrm{Tr}\, V_1(M_1) - \mathrm{Tr}\, V_2(M_2)]} \tag{6.3.2}$$

where the potentials $V_i(x)$, $i = 1, 2$, are polynomials of degree d_i,

$$V_i(x) = \sum_{d=1}^{d_i} \frac{t_d^{(i)}}{d} x^d. \tag{6.3.3}$$

This partition function is a generating function of bi-colored maps, that is,

$$\mathcal{Z}\left(\vec{t}^{(1)}, \vec{t}^{(2)}\right) = \sum_{g,m,n=0}^{\infty} \sum_{M \in \mathcal{S}^{\bullet}_{g,m,n}} \frac{\prod_{i=1}^{d_1} \left[t_i^{(1)}\right]^{n_i^{(1)}(M)} \prod_{i=1}^{d_2} \left[t_i^{(2)}\right]^{n_i^{(2)}(M)}}{|\mathrm{Aut}(M)|} \tag{6.3.4}$$

where

- $\mathcal{S}^{\bullet}_{g,m,n}$ *is the set of bi-colored maps, possibly disconnected, of genus g composed of n black polygons and m white polygons glued by their edges, such that black polygons are glued only to white polygons and vice versa. Neither black nor white polygons are marked.*
- *By $n_i^{(1)}(M)$ (resp. $n_i^{(2)}(M)$) we denote the number of black (resp. white) polygons of perimeter i in M;*

It is also possible to enumerate connected maps with some specific marked faces by computing certain correlation functions of this formal matrix model.

Definition 6.3.4. For any set of words (non-commutative monomials) $\{f_i(x,y)\}_{i=1}^s$ in two variables, we define the correlator of the formal matrix model by

$$\left\langle \prod_{i=1}^{s} \operatorname{Tr} f_i(M_1,M_2) \right\rangle := \frac{\int_{H_N}^{formal} d\mu_N(M_1,M_2) \prod_{i=1}^{s} \operatorname{Tr} f_i(M_1,M_2)}{\mathcal{Z}\left(\vec{t}^{(1)},\vec{t}^{(2)}\right)},$$

where the measure of integration $\mu(M_1,M_2)$ is the same as before,

$$d\mu_N(M_1,M_2) := dM_1\, dM_2\, e^{-N[\operatorname{Tr}(M_1M_2)-\operatorname{Tr} V_1(M_1)-\operatorname{Tr} V_2(M_2)]}.$$

We denote by $\left\langle \prod_{i=1}^{s} \operatorname{Tr} f_i(M_1,M_2) \right\rangle_c$ its connected part.

In matrix models, one classically works with generating series of such correlators (named correlation functions) defined by

$$W_{k,l}(x_1,\dots,x_k;y_1,\dots,y_l) := \left\langle \prod_{i=1}^{k} \operatorname{Tr} \frac{1}{x_i-M_1} \prod_{j=1}^{l} \operatorname{Tr} \frac{1}{y_j-M_2} \right\rangle_c .$$

These correlation functions have to be understood as series expansions around $x_i, y_i \to \infty$:

$$W_{k,l}(x_1,\dots,x_k;y_1,\dots,y_l) := \sum_{\vec{n}\in\mathbb{N}^k} \sum_{\vec{m}\in\mathbb{N}^l} \left\langle \prod_{i=1}^{k} \frac{\operatorname{Tr} M_1^{n_i}}{x_i^{n_i+1}} \prod_{j=1}^{l} \frac{\operatorname{Tr} M_2^{m_j}}{y_j^{m_j+1}} \right\rangle_c . \tag{6.3.5}$$

These correlation functions admit a topological expansion, i. e. they can be written as

$$W_{k,l}(x_1,\dots,x_k;y_1,\dots,y_l) = \sum_{g=0}^{\infty} N^{2-2g-k-l} W_{k,l}^{(g)}(x_1,\dots,x_k;y_1,\dots,y_l)$$

where each of $W_{k,l}^{(g)}$ does not depend on N.

With this notation,

$$W_{k,l}^{(g)}(x_1,\dots,x_k;y_1,\dots,y_l) = \sum_{m,n=0}^{\infty} \sum_{\substack{\vec{\alpha}\in\mathbb{N}^k\\ \vec{\beta}\in\mathbb{N}^l}} \sum_{M\in\mathcal{S}^{\circ}_{g,m,n|\vec{\alpha},\vec{\beta}}} \frac{\prod_{i=1}^{d_1}\left[t_i^{(1)}\right]^{n_i^{(1)}(M)} \prod_{i=1}^{d_2}\left[t_i^{(2)}\right]^{n_i^{(2)}(M)}}{|\operatorname{Aut}(M)| \prod_{i=1}^{k} x_i^{\alpha_i+1} \prod_{j=1}^{l} y_j^{\beta_j+1}}, \tag{6.3.6}$$

where $\mathcal{S}^{\circ}_{g,m,n|\vec{\alpha},\vec{\beta}}$ is the set of connected bi-colored maps of genus g composed of n unmarked black faces, m unmarked white faces, k marked black faces of perimeters $\alpha_1, \dots, \alpha_k$, each having one marked edge, and l marked white faces of perimeter $\beta_1, \dots, \beta_n$, each having one marked edge too; black faces are only glued to white faces and vice versa, as above.

Such a model admits a spectral curve. This means that there exists a polynomial $P(x, y)$ of degree $d_1 - 1$ in x and $d_2 - 1$ in y such that the generating function for discs $W^{(0)}_{1,0}(x)$ satisfies an algebraic equation:

$$E_{2MM}(x, W^{(0)}_{1,0}(x)) = 0, \qquad x \in \mathbb{C},$$

where

$$E_{2MM}(x, y) = (V_1'(x) - y)(V_2'(y) - x) - P(x, y) + 1.$$

In [51, 17], it was proved that the correlation functions $W^{(g)}_{k,0}$ can be computed by topological recursion on this spectral curve.

First, let us recall the definition of the spectral curve topological recursion.

Definition 6.3.5. [50] Let C be an algebraic curve, x and y two functions on C and $\omega_{0,2}$ a bidifferential defined on $C \times C$. Denote ydx by $\omega_{0,1}$ and let a_i stand for zeroes of dx and $\sigma_i(z)$ stand for the deck transformation near a_i. Then *spectral curve topological recursion* defines n-multidifferentials $\omega_{g,n}$ by the following recursive formula

$$\omega_{g,n}(z_1, \dots, z_n) = \frac{1}{2} \sum_i \operatorname*{Res}_{z->a_i} \frac{\int_z^{\sigma_i(z)} \omega_{0,2}(\,\cdot\,, z_1)}{\omega_{0,1}(\sigma_i(z)) - \omega_{0,1}(z)} \Bigg[\omega_{g-1,n+1}(z, \sigma_i(z), z_2, \dots, z_n) + \sum_{\substack{g_1+g_1=g \\ I \sqcup J = \{2,\dots,n\}}}^{\text{stable}} \omega_{g_1,|I|+1}(z, z_I) \omega_{g_2,|J|+1}(\sigma_i(z), z_J) \Bigg], \quad (6.3.7)$$

"Stable" above the summation sign stands for taking the sum excluding the terms where $(g_1, |I|) = (0, 1)$ or $(g_2, |J|) = (0, 1)$.

Theorem 6.3.6. *[51, 17] The correlation functions of the 2-matrix models can be computed by the topological recursion procedure of [50] with the genus* 0 *spectral curve*

$$E_{2MM}(x, y) = (V_1'(x) - y)(V_2'(y) - x) - P(x, y) + 1$$

and the genus 0*, 2-point function defined by the bilinear differential*

$$\omega_{0,2}(z_1, z_2) = \frac{dz_1 dz_2}{(z_1 - z_2)^2}$$

for a global coordinate z *on the spectral curve.*

The proof of this theorem consists in three steps:

- First, find a set of equations satisfied by the correlation functions of the matrix model.
- Second, show that these equations admit a unique solution admitting a topological expansion.
- Third, exhibit a solution which immediately implies the topological recursion.

6.3.3 A matrix model for branched covers

Since the problem of enumerating branched covers can be rephrased in terms of bi-colored maps, one can find a matrix model representation for it.

Using the definition of the preceding section together with the hypermap representation of section 6.3.1, one immediately finds that

Lemma 6.3.7. *The correlation functions of the formal two matrix model with potentials $V_1(x) = 0$ and $V_2(x) = \sum_{i=1}^{a} \frac{t_i}{i} x^i$ coincide with the generating series of covers of $\mathbb{P}^1$ branched over 3 points defined in* (6.3.1)*, for $(g,k) \neq (0,1)$:*

$$W_{k,0}^{(g)}(x_1,\ldots,x_k) = \Omega_{g,k}^{(a)}(x_1,\ldots,x_k). \tag{6.3.8}$$

For $(g,k) = (0,1)$ we have

$$W_{1,0}^{(0)}(x_1) = \frac{1}{x} + \Omega_{0,1}^{(a)}(x_1). \tag{6.3.9}$$

Applying [50, 17], one can thus compute the generating series using topological recursion.

We have:

Theorem 6.3.8. *The generating series for hypermaps*

$$\omega_{g,k}(x_1,\ldots,x_k) = \Omega_{g,k}^{(a)}(x_1,\ldots,x_k) dx_1 \ldots dx_n \tag{6.3.10}$$

can be computed by topological recursion with a genus 0 spectral curve

$$E^{(a)}(x,y) = y \left(\sum_{i=1}^{a} t_i y^{i-1} - x \right) + 1 = 0 \tag{6.3.11}$$

and the genus 0 2-point function defined by the corresponding Bergmann kernel, i. e.

$$\omega_{0,2}(z_1,z_2) = \frac{dz_1 \otimes dz_2}{(z_1 - z_2)^2} \tag{6.3.12}$$

for a global coordinate z on the genus 0 spectral curve.

For brevity, we are not reproducing here the arguments from [50, 17], but we note that the only fact about the matrix model that is used in these arguments to prove the spectral curve topological recursion is the loop equations for the matrix model.

In the next section we prove these loop equations independently in a combinatorial way, and thus obtain a new, purely combinatorial, proof of Theorem 6.3.8, which is the main result of this chapter.

Remark 6.3.9. Theorem 6.3.8 in particular answers the question by Do and Manescu [26] considering such covers with only type a ramifications above 1. The spectral curve is indeed, like Do and Manescu suggested,

$$E^{(a)}(x,y) = y \left(y^{a-1} - x \right) + 1 = 0 \tag{6.3.13}$$

coinciding with the classical limit of their quantum curve.

Remark 6.3.10. Spectral curve (6.3.11) can be used as a superpotential to define a certain Frobenius manifold, and the $\omega_{g,n}$'s turn out to be generating functions for the correlators of the corresponding chomological field theory via the identification of [37] (see [35] for the details).

6.4 Loop equations and combinatorics

The proof of Theorem 6.3.8 in [50, 17] relies on the representation of our combinatorial objects in the form of a formal matrix integral. Actually, the only input from the formal matrix model is the existence of loop equations satisfied by the correlation functions of the model. These loop equations are of combinatorial nature and should reflect some cut-and-join procedure satisfied by the hypermaps being enumerated. However, a simple combinatorial interpretation of these precise 2-matrix model loop equations could not be found in the literature, even if some similar and probably equivalent equations have been derived combinatorially in some particular cases [9, 93]. In this section, we derive such an interpretation, allowing to bypass the necessity to use any integral (matrix model) representation and thus getting a completely combinatorial proof of the results of the preceding section.

Remark 6.4.1. While writing the paper, on which this chapter is based, we have been informed that such a direct derivation of the loop equations for the 2-matrix model is performed in chapter 8 [47] which is in preparation and whose preliminary version can be found online.

6.4.1 Loop equations

In order to produce the hierarchy of loop equations whose solution gives rise to the topological recursion, one combines two set of equations which can be written as follows:

- The first one corresponds to the change of variable

$$M_2 \to M_2 + \epsilon \frac{1}{x - M_1} \prod_{i=1}^{n} \mathrm{Tr} \frac{1}{x_i - M_1}$$

  in the formal matrix integral defining the partition function. To first order in ϵ, the compensation of the Jacobian (which is vanishing here) with the variation of the action gives rise to the equation:

$$\left\langle \mathrm{Tr} \left(\frac{M_1}{x - M_1} \right) \prod_{i=1}^{n} \mathrm{Tr} \frac{1}{x_i - M_1} \right\rangle = \left\langle \mathrm{Tr} \left(\frac{1}{x - M_1} V_2'(M_2) \right) \prod_{i=1}^{n} \mathrm{Tr} \frac{1}{x_i - M_1} \right\rangle \tag{6.4.1}$$

- The second one corresponds to the change of variable

$$M_1 \to M_1 + \epsilon \frac{1}{x - M_1} \frac{V_2'(y) - V_2'(M_2)}{y - M_2} \prod_{i=1}^{n} \mathrm{Tr} \frac{1}{x_i - M_1} \tag{6.4.2}$$

and reads

$$\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}M_2\right)\prod_{i=1}^{n}\mathrm{Tr}\,\frac{1}{x_i-M_1}\right\rangle \tag{6.4.3}$$
$$=\left\langle \mathrm{Tr}\left(\frac{V_1'(M_1)}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{i=1}^{n}\mathrm{Tr}\,\frac{1}{x_i-M_1}\right\rangle$$
$$+\frac{1}{N}\left\langle \mathrm{Tr}\left(\frac{1}{x-M_1}\right)\mathrm{Tr}\left(\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{i=1}^{n}\mathrm{Tr}\,\frac{1}{x_i-M_1}\right\rangle$$
$$+\frac{1}{N}\sum_{i=1}^{n}\left\langle \mathrm{Tr}\left(\frac{1}{(x_i-M_1)^2}\frac{1}{x-M_1}\frac{V_2'(y)-V_2'(M_2)}{y-M_2}\right)\prod_{j\neq i}\mathrm{Tr}\,\frac{1}{x_j-M_1}\right\rangle$$

Note that in these equations the correlators are not the connected ones, but they are generating functions of possibly disconnected maps of arbitrary genus.

6.4.2 Combinatorial interpretation

The loop equations (6.4.1), (6.4.3) make sense only in their $x, x_i, y \to \infty$ series expansions. These expansions generate a set of equations for the correlators of the matrix models which can be interpreted as relations between the number of bi-colored maps with different boundary conditions. In this section, we give a combinatorial derivation of these relations.

Definition of boundary conditions

In order to derive the loop equations, we have to deal with bi-colored maps with boundaries (or marked faces) of general type. A map with n boundaries is a map with n marked faces (polygons), each carrying a marked edge. The boundary conditions are defined as the colors of the marked faces.

However, in the following, we need to also introduce mixed-type boundary conditions described as follows.

A *bi-colored map with n mixed-type boundaries* is a map with n marked faces , where all unmarked faces are colored either black or white (as usual, black faces can border only white ones and vice versa). We do not assign color to the marked faces themselves; instead, we color their edges in the following sense. We say that each edge has two *flanks*, associated with two possible normal directions to the edge. Each of these two flanks for each edge is colored either black or white such that 1) for a given edge its two flanks are oppositely colored and 2) if a given edge belongs to an unmarked face, its flank in the direction of this face has the same color as the face. For convenience, for a given face $\mathcal{F}$ let us call the $\mathcal{F}$-facing flanks of the edges of $\mathcal{F}$ *inner* with respect to $\mathcal{F}$. Note that marked faces can be self-touching, and for an edge of a marked face $\mathcal{F}$ where self-touching occurs both flanks are inner with respect to $\mathcal{F}$. Also, in addition to everything described above, we mark one inner flank of exactly one of the edges of each marked face.

Slightly abusing the terminology, we call a marked face *black* if all inner flanks of its edges are black, and we call it *white* if all inner flanks of its edges are white.

The boundary conditions of a marked face are then given by the sequence of colors of the inner flanks of the edges of this face starting from the marked flank and going clockwise from it.

For a given bi-colored map with n mixed-type boundaries consider a set of n sequences of non-negative integers

$$S_i = b_{i,1}, a_{i,1}, b_{i,2}, a_{i,2} \dots b_{i,l_i}, a_{i,l_i}, \qquad i = 1, \dots n. \tag{6.4.4}$$

Here $b_{i,1}$ is the number of consecutive inner black flanks of the i-th marked face starting from the marked inner flank and going clockwise (it is equal to zero if the marked flank is white), $a_{i,1}$ is the number of the following consecutive white flanks, and so on.

We define $\mathcal{T}^{(g)}_{S_1,\dots,S_n}$ to be the number of connected bi-colored maps of genus g with n boundaries with the boundary conditions $S_1, \dots, S_n$.

Remark 6.4.2. In terms of correlators of a two matrix model, one can write

$$\mathcal{T}^{(g)}_{S_1,\dots,S_n} = N^{n+2g-2} \left\langle \prod_{i=1}^{n} \mathrm{Tr}\left(M_1^{b_{i,1}} M_2^{a_{i,1}} M_1^{b_{i,2}} M_2^{a_{i,2}} \dots M_1^{b_{i,l_i}} M_2^{a_{i,l_i}} \right) \right\rangle_c^g \tag{6.4.5}$$

where the superscript g means that we only consider the g'th term of the expansion in N^{-2} of this correlator.

Cut-and-join equations

With these definitions, we are ready to derive the loop equations (6.4.1) and (6.4.3).

Namely, we can generalize to the two matrix model the procedure developed by Tutte for the enumeration of maps [92] and then extensively developed in the study of formal random matrices. Let us consider a connected genus g map with $n+1$ boundaries with boundary conditions

$$S_0 = k+1, 0; \qquad S_i = k_i, 0, \quad i = 1, \dots, n. \tag{6.4.6}$$

This means that the inner flanks of all the edges of the marked faces are black. This map contributes to $\mathcal{T}^{(g)}_{k+1,0;k_1,0;\dots;k_n,0}$. Let us remove the edge with the marked inner flank from the boundary 0. Since one can only glue together faces of different colors, on the other side of this edge one can find only a white (unmarked) l-gon with $1 \le l \le d_2$. After removing the edge, let us mark in the resulting joint polygon the edge which is located clockwise from the origin of the removed edge (the origin of an edge is the vertex located on the counterclockwise side of the edge). We end up with a map that contributes to $\mathcal{T}^{(g)}_{0,l-1,k,0;k_1,0;\dots;k_n,0}$. This procedure is bijective between the sets considered. We take the sum over all possibilities, taking into account the weight of the edge and l-gon removed, and we see that

$$\mathcal{T}^{(g)}_{k+1,0;k_1,0;\dots;k_n,0} = \sum_{l=1}^{d_2} t_l^{(2)} \mathcal{T}^{(g)}_{0,l-1,k,0;k_1,0;\dots;k_n,0}. \tag{6.4.7}$$

Multiplying by $x^{-k-1} x^{-k_1-1} \dots x^{-k_n-1}$ and taking the sum over $k, k_1, \dots, k_n$, one recovers the loop equation (6.4.1).

This first equation produces mixed boundary condition out of homogeneous black conditions. Let us now proceed one step further and apply Tutte's method to the maps produced in this way.

Let us consider a map contributing to $\mathcal{T}^{(g)}_{0,l+1,k,0;k_1,0;\dots;k_n,0}$, i. e. a genus g connected map with boundary condition:

$$S_0 = 0, l+1, k, 0; \qquad S_i = k_i, 0, \quad i = 1, \dots, n. \tag{6.4.8}$$

Note that it follows from our definition that the marked inner flank of the 0-th marked face is white. When we remove the corresponding edge, we can produce different types of maps, namely, strictly one of the three following cases takes place.

- On the other side of the edge lies an unmarked black m-gon. We remove the edge and this gives a map that contributes to $\mathcal{T}^{(g)}_{0,l,k+m-1,0;k_1,0;\dots;k_n,0}$.
- The opposite flank of the edge is a black inner flank of the same marked face. Then two possible cases occur. The resulting surface can still be connected, giving rise to a map contributing to $\mathcal{T}^{(g-1)}_{m,0;0,l,k-m,0;k_1,0;\dots;k_n,0}$ for some $1 \le m \le k$, i. e. with one more boundary but a genus decreased by one. Or removing the edge with the marked flank can disconnect the map into two connected components giving contributions to $\mathcal{T}^{(h)}_{m,0;k_{\alpha_1},0;\dots;k_{\alpha_j},0}$ and $\mathcal{T}^{(g-h)}_{0,l,k-m,0;k_{\beta_1},0;\dots;k_{\beta_{n-j}},0}$ respectively, where $0 \le h \le g$ and $\{\alpha_1, \dots, \alpha_j\} \cup \{\beta_1, \dots, \beta_{n-j}\} = \{1, \dots, n\}$. This type of behavior can be thought of as a "cut" move.
- On the other side of the edge lies a marked black face with boundary condition $(k_i, 0)$. Removing the edge, one gets a contribution to $\mathcal{T}^{(g)}_{0,l,k+k_i-1,0;k_1,0;\dots;k_{i-1},0;k_i-m,0;k_{i+1},0\dots;k_n,0}$. This type of behavior can be thought of as a "join" move.

Once again, this procedure is bijective, if we take the sum over all cases. Taking into account the weight of the elements removed, we end up with an equation relating the number of bi-colored maps with different boundary conditions:

$$\begin{aligned}
&\mathcal{T}^{(g)}_{0,l+1,k,0;k_1,0;\dots;k_n,0} \\
&= \sum_{m=0}^{d_2} t^{(2)}_m \mathcal{T}^{(g)}_{0,l,k+m-1,0;k_1,0;\dots;k_n,0} \\
&+ \sum_{m=0}^{k} \mathcal{T}^{(g-1)}_{m,0;0,l,k-m,0;k_1,0;\dots;k_n,0} \\
&+ \sum_{m=0}^{k} \sum_{h=0}^{g} \sum_{\vec{\alpha}\cup\vec{\beta}=\{1,\dots,n\}} \mathcal{T}^{(h)}_{m,0;k_{\alpha_1},0;\dots;k_{\alpha_j},0} \mathcal{T}^{(g-h)}_{0,l,k-m,0;k_{\beta_1},0;\dots;k_{\beta_{n-j}},0} \\
&+ \sum_{i=1}^{n} \mathcal{T}^{(g)}_{0,l,k+k_i-1,0;k_1,0;\dots;k_{i-1},0;k_i-m,0;k_{i+1},0\dots;k_n,0}
\end{aligned} \tag{6.4.9}$$

where $\vec{\alpha} = \{\alpha_1, \dots, \alpha_j\}$ and $\vec{\beta} = \{\beta_1, \dots, \beta_{n-j}\}$. This equation is the genus g contribution to the expansion of the loop equation (6.4.3) when all its variables are large.

This concludes the fully combinatorial proof of the two matrix model's loop equations. The latter can be seen as some particular cut-and-join equations. One can now apply the procedure used in [17] for solving them (without having to introduce any matrix model consideration!) and derive the topological recursion for the generating functions of bi-colored maps with homogenous boundary conditions, which implies Theorem 6.3.8.

6.5 Quantum curve

In this section we prove a generalization of the theorem of Do and Manescu from [26] on the quantum spectral curve equation for enumeration of hypermaps.

Theorem 6.5.1. *The wave function $Z^{(a)}(x)$, defined in* (6.2.4)*, satisfies the ODE:*

$$\left(-\hbar x\frac{\partial}{\partial x}+1+\sum_{i=1}^{a} t_i\left(\hbar\frac{\partial}{\partial x}\right)^i\right)Z^{(a)}(x)=0 \tag{6.5.1}$$

Remark 6.5.2. The differential operator in the previous theorem is given by the naive quantization of the classical spectral curve (6.3.11), $y\leftrightarrow\hbar\frac{\partial}{\partial x}$. Note that in Do and Manescu's paper [26] a different definition of $Z^{(a)}$ was used, as noted above, and a different convention $y\leftrightarrow-\hbar\frac{\partial}{\partial x}$.

6.5.1 Wave functions

In the proof we use the notations coming from the formal matrix model formalism for simplicity, but as usual in the formal matrix model setup, they just represent well defined combinatorial objects which satisfy the loop equations derived in the preceding sections.

In what follows we identify N with $1/\hbar$.

From the definition of the wave function $Z^{(a)}$, given in formulas (6.2.2)-(6.2.4), from the identification between $W_{k,0}^{(g)}$ and $\Omega_{g,k}^{(a)}$ given by Equation (6.3.8) and from the definition of $W_{k,l}$ (6.3.5), we have

$$Z^{(a)}(x)=\exp\left(\frac{1}{\hbar}\log(x)+\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{b_1,\dots b_n=1}^{\infty}\frac{\left\langle\mathrm{Tr}(M_1^{b_1})\dots\mathrm{Tr}(M_1^{b_n})\right\rangle_c}{b_1\dots b_n x^{b_1}\dots x^{b_n}}\right). \tag{6.5.2}$$

The standard relation between connected and disconnected correlators imply

$$Z^{(a)}(x)=x^{1/\hbar}\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\sum_{b_1,\dots b_n=1}^{\infty}\frac{\left\langle\mathrm{Tr}(M_1^{b_1})\dots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_1\dots b_n x^{b_1}\dots x^{b_n}}. \tag{6.5.3}$$

In order to simplify the notation, we introduce functions $Z_n^r(x_1,\dots,x_n)$ and $Z^r(y,x)$ for integers $n\geq 1$, $r\geq 0$ (we call these functions non-principally-specialized wave functions).

Definition 6.5.3. The n-point wave function Z_n^r of level r is defined as

$$
\begin{aligned}
&Z_n^r(x_1,\ldots,x_n) := \\
&\log(x_1)\,(-1)^{n-1}\sum_{b_2\ldots b_n=1}^{\infty}\frac{\left\langle \mathrm{Tr}(M_2^r)\,\mathrm{Tr}(M_1^{b_2})\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_2\ldots b_n\; x_2^{b_2}\ldots x_n^{b_n}} \\
&+(-1)^n\sum_{b_1,b_2\ldots b_n=1}^{\infty}\frac{\left\langle \mathrm{Tr}(M_2^rM_1^{b_1})\,\mathrm{Tr}(M_1^{b_2})\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_1\ldots b_n\; x_1^{b_1}\ldots x_n^{b_n}},\quad r>0 \\
&Z_n^0(x_1,\ldots,x_n) := (-1)^n\sum_{b_1,b_2\ldots b_n=1}^{\infty}\frac{\left\langle \mathrm{Tr}(M_1^{b_1})\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle}{b_1\ldots b_n\; x_1^{b_1}\ldots x_n^{b_n}}
\end{aligned}
\tag{6.5.4}
$$

and the almost-fully principally-specialized wave function of level r is

$$Z^r(y,x)=\sum_{n=0}^{\infty}\frac{1}{n!}Z_n^r(y,x,\ldots x) \tag{6.5.5}$$

Note that with these definitions

$$Z^{(a)}(x)=x^{1/\hbar}Z^0(x,x) \tag{6.5.6}$$

6.5.2 Loop equations in terms of Z_n^r

Considering the coefficient in front of particular powers of $1/x$ and $1/x_i$'s in loop equations (6.4.1) and (6.4.3) we get the following equations relating particular formal matrix model correlators

$$\left\langle \mathrm{Tr}(M_1^{b_1})\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle=\sum_{i=1}^{a}t_i\left\langle \mathrm{Tr}\left(M_1^{b_1-1}M_2^{i-1}\right)\mathrm{Tr}(M_1^{b_2})\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle \tag{6.5.7}$$

$$
\begin{aligned}
&\left\langle \mathrm{Tr}\left(M_2^rM_1^{b_1}\right)\mathrm{Tr}(M_1^{b_2})\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle= \\
&\quad\hbar\sum_{j=2}^{n}b_j\left\langle \mathrm{Tr}\left(M_2^{r-1}M_1^{b_1+b_j-1}\right)\mathrm{Tr}(M_1^{b_2})\ldots\widehat{\mathrm{Tr}(M_1^{b_j})}\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle \\
&\quad+\hbar\sum_{p+q=b_1-1}\left\langle \mathrm{Tr}(M_2^{r-1}M_1^{p})\,\mathrm{Tr}(M_1^{q})\,\mathrm{Tr}(M_1^{b_2})\ldots\mathrm{Tr}(M_1^{b_n})\right\rangle,
\end{aligned}
$$

Here the hat above $\mathrm{Tr}(M_1^{b_j})$ means that it is excluded from the correlator.

Let us sum the above equations over all $b_1,\ldots,b_n$ from 1 to ∞ with the coefficient

$$\frac{(-1)^n}{x_1^{b_1}b_2\ldots b_n x_2^{b_2}\ldots x_n^{b_n}}.$$

(note the absence of the $1/b_1$ factor). We get:

Lemma 6.5.4. *Loop equations, written in terms of Z_n^r, read*

$$(-x_1\frac{\partial}{\partial x_1})Z_n^0(x_1,\dots,x_n) = \sum_{i=1}^{a} t_i(-\frac{\partial}{\partial x_1})Z_n^{i-1}(x_1,\dots,x_n) - \frac{1}{\hbar x_1}Z_{n-1}^0(x_2,\dots,x_n), \tag{6.5.8}$$

$$\begin{aligned}
&\frac{1}{\hbar}(-x_1\frac{\partial}{\partial x_1})Z_n^1(x_1,\dots,x_n) = \\
&-\sum_{j=2}^{n}\left[(-\frac{\partial}{\partial x_j})Z_{n-1}^0(x_j,x_2,\dots\widehat{x_j}\dots x_n) + \frac{1}{\hbar x_j}Z_{n-2}^0(x_2,\dots\widehat{x_j}\dots x_n)\right] \\
&+\frac{2}{\hbar}(-\frac{\partial}{\partial x_1})Z_n^0(x_1,\dots,x_n) - \frac{1}{\hbar^2 x_1}Z_{n-1}^0(x_2,\dots,x_n) \\
&- x_1\frac{\partial^2}{\partial u_1\partial u_2}\Big|_{u_1=u_2=x_1} Z_{n+1}^0(u_1,u_2,x_2,\dots,x_n) \\
&-\sum_{j=2}^{n}\frac{1}{(x_1-x_j)}\left[x_1\frac{\partial}{\partial x_1}Z_{n-1}^0(x_1,\dots\widehat{x_j}\dots x_n) - x_j\frac{\partial}{\partial x_j}Z_{n-1}^0(x_j,x_2,\dots\widehat{x_j}\dots x_n)\right],
\end{aligned} \tag{6.5.9}$$

and, for all $r>1$,

$$\begin{aligned}
&\frac{1}{\hbar}(-x_1\frac{\partial}{\partial x_1})Z_n^r(x_1,\dots,x_n) = \\
&-\sum_{j=2}^{n}(-\frac{\partial}{\partial x_j})Z_{n-1}^{r-1}(x_j,x_2,\dots\widehat{x_j}\dots x_n) + \frac{1}{\hbar}(-\frac{\partial}{\partial x_1})Z_n^{r-1}(x_1,\dots,x_n) \\
&- x_1\frac{\partial^2}{\partial u_1\partial u_2}\Big|_{u_1=u_2=x_1} Z_{n+1}^{r-1}(u_1,u_2,x_2,\dots,x_n) \\
&-\sum_{j=2}^{n}\frac{1}{(x_1-x_j)}\left[x_1\frac{\partial}{\partial x_1}Z_{n-1}^{r-1}(x_1,x_2,\dots\widehat{x_j}\dots x_n) - x_j\frac{\partial}{\partial x_j}Z_{n-1}^{r-1}(x_j,x_2,\dots\widehat{x_j}\dots x_n)\right].
\end{aligned} \tag{6.5.10}$$

6.5.3 Symmetrization of loop equations

Last step to obtain quantum curve equation is to put all equations (6.5.8)–(6.5.10) into principal specialization: put all x_i's equal to x.

The following obvious statement plays a crucial role in the induction:

Lemma 6.5.5. *Let $f(x_1|x_2,\dots,x_n)$ be a symmetric function in the variables $x_2,\dots,x_n$ (so, x_1 is treated specially here). Then we have the following formula for the derivative in the principal specialization.*

$$\frac{\partial}{\partial x}f(x|x,\dots,x) = \frac{\partial}{\partial u}\Big|_{u=x}f(u|x,\dots,x) + (n-1)\frac{\partial}{\partial u}\Big|_{u=x}f(x|u,x,\dots,x). \tag{6.5.11}$$

In particular, if $f(x_1,x_2,\dots,x_n) = \frac{\partial}{\partial x_1}g(x_1,x_2,\dots,x_n)$, then

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{\partial}{\partial y}\Big|_{y=x}g(y|x,\dots,x) &= \frac{\partial^2}{\partial u^2}\Big|_{u=x}g(u|x,\dots,x) \\
&\quad + (n-1)\frac{\partial^2}{\partial u_1\partial u_2}\Big|_{u_1=x,u_2=x}g(u_1|u_2,x,\dots,x).
\end{aligned} \tag{6.5.12}$$

Since Z^0_n is symmetric in all its arguments, the first equation of (6.5.8) is equivalent to

$$\begin{aligned}
&\frac{1}{n}(-x\frac{\partial}{\partial x})Z^0_n(x,\dots,x) = \\
&-t_1\frac{1}{\hbar x}Z^0_{n-1}(x,\dots,x) - \frac{1}{n}\frac{\partial}{\partial x}Z^0_n(x,\dots,x) \\
&+\sum_{i=2}^{a} t_i(-\frac{\partial}{\partial y})\Big|_{y=x} Z^{i-1}_n(y,x,\dots,x)
\end{aligned} \tag{6.5.13}$$

We multiply this by $\frac{1}{(n-1)!}$ and take the sum over $n \geq 0$. We have:

$$\begin{aligned}
&(-x\frac{\partial}{\partial x})Z^0(x,\dots,x) = \\
&-t_1\left(\frac{\partial}{\partial x}+\frac{1}{\hbar x}\right)Z^0(x,\dots,x) \\
&+\sum_{i=2}^{a} t_i \sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z^{i-1}_n(y,x,\dots,x).
\end{aligned} \tag{6.5.14}$$

Then, the existence of a quantum curve equation relies on two observations:

Lemma 6.5.6. *We have:*

$$\begin{aligned}
i>1:\ &\sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z^{i}_n(y,x,\dots,x) \\
&=\left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right)\sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z^{i-1}_n(y,x,\dots,x) \\
i=1:\ &\sum_{n=0}^{\infty}\frac{1}{(n-1)!}(-\frac{\partial}{\partial y})\Big|_{y=x} Z^{i}_n(y,x,\dots,x) \\
&=\hbar\left[-\frac{\partial^2}{\partial x^2}-\frac{2}{\hbar x}\frac{\partial}{\partial x}-\frac{1/\hbar(1/\hbar-1)}{x^2}\right]Z^0 \\
&=-\frac{1}{\hbar}\left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right)^2 Z^0
\end{aligned} \tag{6.5.15}$$

Proof. These equations are direct corollaries of Equations (6.5.8), we just have to put them into principal specialization and apply Lemma 6.5.5. □

We combine Equation (6.5.14) and Lemma (6.5.6), and we obtain the following equation:

$$(-\hbar x\frac{\partial}{\partial x})Z^0 = -\sum_{i=1}^{a} t_i\left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right)^i Z^0. \tag{6.5.16}$$

which, with help of commutation relation

$$x^{1/\hbar}\left(\frac{1}{x}+\hbar\frac{\partial}{\partial x}\right) = \hbar\frac{\partial}{\partial x}\circ x^{1/\hbar}, \tag{6.5.17}$$

leads directly to the statement of Theorem 6.5.1.

6.6 4-colored maps and 4-matrix models

It turns out that the ideas above can be applied not only to bi-colored maps (which correspond to the 2-matrix model case), but also to 4-colored maps. In the current section we outline the idea of the proof of the spectral curve topological recursion for the enumeration of 4-colored maps.

4-colored maps arise as a natural generalization of bi-colored maps. Instead of considering partitions of surfaces into black and white polygons, we consider partitions into polygons of four colors c_1, c_2, c_3, c_4, such that polygons of color c_1 are glued only to polygons of color c_2, polygons of color c_2 are glued only to polygons of colors c_1 and c_3, polygons of color c_3 are only glued to those of color c_2 and c_4 and finally polygons of color c_4 are only glued to polygons of color c_3. This can be represented in terms of the following color incidency matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{6.6.1}$$

Applying considerations similar to the ones in the above sections, it's easy to see that the problem of enumeration of such 4-colored maps is governed by a 4-matrix model with the interaction part of the potential being equal to

$$-N \operatorname{Tr}(M_1 M_2 - M_1 M_4 + M_3 M_4), \tag{6.6.2}$$

since the inverse of the above incidency matrix is equal to

$$\begin{pmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \end{pmatrix} \tag{6.6.3}$$

We see that in the 4-colored maps case, after a renumeration of matrices and a certain change of signs, this still gives us the matrix model for a chain of matrices (which is no longer true for, e.g., 6-colored maps). Fortunately, the case of matrix model for a chain of matrices was studied by Eynard in [44], and the master loop equation obtained there gives rise to the spectral curve topological recursion for this problem. Again, it's easy to see in the analogous way to what was discussed in the previous sections that the individual building blocks of loop equations can be proved to hold by purely combinatorial means.

Bibliography

[1] D. Abramovich and A. Vistoli, "Compactifying the space of stable maps," *Journal of the American Mathematical Society* **15** no. 1, (2002) 27–75, arXiv:math/9908167.

[2] N. M. Adrianov, N. Y. Amburg, V. A. Dremov, Y. A. Levitskaya, E. M. Kreines, Y. Y. Kochetkov, V. F. Nasretdinova, and G. B. Shabat, "Catalog of dessins d'enfants with ≤ 4 edges," *Journal of Mathematical Sciences* **158** (2009) 22–80, arXiv:0710.2658.

[3] M. Aganagic, R. Dijkgraaf, A. Klemm, M. Mariño, and C. Vafa, "Topological strings and integrable hierarchies," *Comm. Math. Phys.* **261** (2006) 451–516, arXiv:hep-th/0312085.

[4] A. Alexandrov, A. Morozov, and A. Mironov, "Partition functions of matrix models: First special functions of string theory," *International Journal of Modern Physics A* **19** no. 24, (2004) 4127–4163, arXiv:hep-th/0310113.

[5] A. Alexandrov, A. Mironov, A. Morozov, and S. Natanzon, "On KP-integrable Hurwitz functions," *JHEP* **80** (2014) , arXiv:1405.1395.

[6] J. Ambjorn and L. Chekhov, "The matrix model for dessins d'enfants," *l'Institut Henri Poincare D* **1** no. 3, (2014) 337–361, arXiv:1404.4240.

[7] J. Ambjørn and L. Chekhov, "A matrix model for hypergeometric Hurwitz numbers," *Theoretical and Mathematical Physics* **181** no. 3, (2014) 1486–1498, arXiv:1409.3553.

[8] J. E. Andersen, L. O. Chekhov, P. Norbury, and R. C. Penner, "Models of discretized moduli spaces, cohomological field theories, and gaussian means," *Journal of Geometry and Physics* **98** (2015) 312–339, arXiv:1501.05867.

[9] O. Bernardi and M. Bousquet-Mélou, "Counting colored planar maps: algebraicity results," *Journal of Combinatorial Theory B* **101** (2011) 315–377, arXiv:0909.1695.

[10] V. Bouchard and B. Eynard, "Private communication,".

[11] V. Bouchard and B. Eynard, "Think globally, compute locally," *JHEP* **02** (2013) 143, arXiv:1211.2302 [math-ph].

[12] V. Bouchard and B. Eynard, "Reconstructing WKB from topological recursion," arXiv:1606.04498 [math-ph].

[13] V. Bouchard and M. Mariño, "Hurwitz numbers, matrix models and enumerative geometry," *Proc. Sympos. Pure Math.* **78** (2008) 263–283, arXiv:0709.1458.

[14] V. Bouchard, D. H. Serrano, X. Liu, and M. Mulase, "Mirror symmetry for orbifold Hurwitz numbers," *Journal of Differential Geometry* **98** no. 3, (2014) 375–423, arXiv:1301.4871.

[15] A. Buryak and S. Shadrin, "A remark on deformations of Hurwitz Frobenius manifolds," *Lett. Math. Phys.* **93** no. 3, (2010) 243–252, `arXiv:0912.5114`.

[16] L. Chekhov and B. Eynard, "Hermitean matrix model free energy: Feynman graph technique for all genera," *JHEP* **14** (2006) , `arXiv:hep-th/0504116`.

[17] L. Chekhov, B. Eynard, and N. Orantin, "Free energy topological expansion for the 2-matrix model," *JHEP* **53** (2006) , `arXiv:math-ph/0603003`.

[18] W. Chen and Y. Ruan, "Orbifold Gromov-Witten theory," in *Orbifolds in mathematics and physics*, vol. 310. 2002. `arXiv:math.AG/0103156`.

[19] A. Chiodo, "Towards an enumerative geometry of the moduli space of twisted curves and r-th roots," *Compositio Mathematica* **144** no. 06, (2008) 1461–1496, `arXiv:math/0607324`.

[20] A. Chiodo and Y. Ruan, "Landau-Ginzburg/Calabi-Yau correspondence for quintic three-folds via symplectic transformations," *Inventiones mathematicae* **182** no. 1, (2010) 117–165, `arXiv:0812.4660`.

[21] A. Chiodo, D. Zvonkine, *et al.*, "Twisted r-spin potential and Givental's quantization," *Advances in Theoretical and Mathematical Physics* **13** no. 5, (2009) 1335–1369, `arXiv:0711.0339`.

[22] C. Cordova, B. Heidenreich, A. Popolitov, and S. Shakirov, "Orbifolds and exact solutions of strongly-coupled matrix models," `arXiv:1611.03142`.

[23] R. Dijkgraaf, L. Hollands, and P. Sułkowski, "Quantum curves and $\mathcal{D}$-modules," *JHEP* **47** (2009) , `arXiv:0810.4157`.

[24] R. Dijkgraaf, L. Hollands, P. Sułkowski, and C. Vafa, "Supersymmetric gauge theories, intersecting branes and free fermions," *Journal of High Energy Physics* **2008** no. 02, (2008) 106, `arXiv:0709.4446`.

[25] N. Do, O. Leigh, and P. Norbury, "Orbifold Hurwitz numbers and Eynard-Orantin invariants," `arXiv:1212.6850`.

[26] N. Do and D. Manescu, "Quantum curves for the enumeration of ribbon graphs and hypermaps," *Communications in Number Theory and Physics* **8** no. 4, (2014) 677–701, `arXiv:1312.6869`.

[27] B. Dubrovin, "Geometry of 2d topological field theories," in *Integrable systems and quantum groups*, pp. 120–348. Springer, 1996. `arXiv:hep-th/9407018`.

[28] B. Dubrovin, "Painlevé transcendents in two-dimensional topological field theory," in *The Painlevé property: one century later*, pp. 287–412. Springer, 1999. `arXiv:math/9803107`.

[29] B. Dubrovin, "On almost duality for Frobenius manifolds," *Translations of the American Mathematical Society-Series 2* **212** (2004) 75–132, `arXiv:math/0307374`.

[30] B. Dubrovin and Y. Zhang, "Bihamiltonian hierarchies in 2d topological field theory at one-loop approximation," *Communications in mathematical physics* **198** no. 2, (1998) 311–361, arXiv:hep-th/9712232.

[31] O. Dumitrescu and M. Mulase, "Quantum curves for Hitchin fibrations and the Eynard–Orantin theory," *Letters in Mathematical Physics* **104** (2014) 635–671, arXiv:1310.6022.

[32] O. Dumitrescu, M. Mulase, B. Safnuk, and A. Sorkin, "The spectral curve of the Eynard–Orantin recursion via the Laplace transform," in *Algebraic and Geometric Aspects of Integrable Systems and Random Matrices*, pp. 263–315. American Mathematical Society, 2013. arXiv:1202.1159.

[33] P. Dunin-Barkowski, M. Kazarian, N. Orantin, S. Shadrin, and L. Spitz, "Polynomiality of Hurwitz numbers, Bouchard–Mariño conjecture, and a new proof of the ELSV formula," *Advances in Mathematics* **279** (2015) 67–103, arXiv:1307.4729.

[34] P. Dunin-Barkowski, D. Lewanski, A. Popolitov, and S. Shadrin, "Polynomiality of orbifold Hurwitz numbers, spectral curve, and a new proof of the Johnson–Pandharipande–Tseng formula," *Journal of the London Mathematical Society* (2015) , arXiv:1504.07440.

[35] P. Dunin-Barkowski, P. Norbury, N. Orantin, A. Popolitov, and S. Shadrin, "Dubrovin's superpotential as a global spectral curve," *Journal of the Institute of Mathematics of Jussieu* (2017) , arXiv:1509.06954.

[36] P. Dunin-Barkowski, N. Orantin, A. Popolitov, and S. Shadrin, "Combinatorics of loop equations for branched covers of sphere," *Int. Math. Res. Not.* (to appear) , arXiv:1412.1698.

[37] P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, "Identification of the Givental formula with the spectral curve topological recursion procedure," *Communications in Mathematical Physics* **328** (2014) 669–700, arXiv:1211.4021.

[38] P. Dunin-Barkowski, S. Shadrin, and L. Spitz, "Givental graphs and inversion symmetry," *Letters in Mathematical Physics* **103** (2013) 533–557, arXiv:1201.4930.

[39] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, "Hurwitz numbers and intersections on moduli spaces of curves," *Inventiones mathematicae* **146** (2001) 297–327, arXiv:math/0004096.

[40] B. Eynard, "A short overview of the "topological recursion"," arXiv:1412.3286.

[41] B. Eynard, "Formal matrix integrals and combinatorics of maps," arXiv:math-ph/0611087.

[42] B. Eynard, "Intersection numbers of spectral curves," arXiv:1104.0176 [math-ph].

[43] B. Eynard, "Large N expansion of the 2-matrix model," *JHEP* **51** (2003) , `arXiv:hep-th/0210047`.

[44] B. Eynard, "Master loop equations, free energy and correlations for the chain of matrices," *JHEP* **18** (2003) , `arXiv:hep-th/0309036`.

[45] B. Eynard, "Topological expansion for the 1-hermitean matrix model correlation functions," *JHEP* **24** (2004) , `arXiv:hep-th/0407261`.

[46] B. Eynard, "Invariants of spectral curves and intersection theory of moduli spaces of complex curves," *Communications in Number Theory and Physics* **8** no. 3, (2014) 541–588, `arXiv:1110.2949`.

[47] B. Eynard, "Counting surfaces: Combinatorics, matrix models and algebraic geometry," *Progress in Mathematical Physics* **114** (2016) .

[48] B. Eynard and M. Mariño, "A holomorphic and background independent partition function for matrix models and topological strings," *J. Geom. Phys.* **61** (2011) 1181–1202, `arXiv:0810.4273`.

[49] B. Eynard, M. Mulase, and B. Safnuk, "The Laplace transform of the cut-and-join equation and the Bouchard-Marino conjecture on Hurwitz numbers," `arXiv:0907.5224`.

[50] B. Eynard and N. Orantin, "Invariants of algebraic curves and topological expansion," *Communications in Number Theory and Physics* **1** no. 2, 347–452, `arXiv:math-ph/0702045`.

[51] B. Eynard and N. Orantin, "Topological expansion of the 2-matrix model correlation functions: diagrammatic rules for a residue formula," *JHEP* **34** (2005) , `arXiv:math-ph/0504058`.

[52] B. Eynard and N. Orantin, "Algebraic methods in random matrices and enumerative geometry," *Topical review J. Phys. A: Math. Theor.* **42** (2009) , `arXiv:0811.3531`.

[53] C. Faber, S. Shadrin, and D. Zvonkine, "Tautological relations and the r-spin Witten conjecture," *Ann. Sci. Éc. Norm. Supér.* **43** no. 4, (2010) 621–658, `arXiv:math/0612510`.

[54] B. Fang, C.-C. M. Liu, and Z. Zong, "The Eynard-Orantin recursion and equivariant mirror symmetry for the projective line," *Geom. Topol.* (to appear) , `arXiv:1411.3557`.

[55] G. Frobenius and L. Stickelberger, "Über die Differentiation der elliptischen Functionen nach den Perioden und Invarianten," *J. Reine Angew. Math.* **92** (1882) 311–327.

[56] A. Givental, "Gromov-Witten invariants and quantization of quadratic hamiltonians," *Mosc. Math. J* **1** no. 4, (2001) 551–568, `arXiv:math/0108100`.

[57] A. Givental, "Semisimple Frobenius structures at higher genus," *International mathematics research notices* **2001** no. 23, (2001) 1265–1286, `arXiv:math/0008067`.

[58] A. Givental, "A_{n-1} singularities and n-KdV hierarchies," *Mosc. Math. J* **3** no. 2, (2003) 475–505, `arXiv:math/0209205`.

[59] S. Gukov and P. Sułkowski, "A-polynomial, B-model, and quantization," *JHEP* **70** (2012) , `arXiv:1108.0002`.

[60] G.-N. Han, "Hook lengths and shifted parts of partitions," *Ramanujan J.* **23** (2010) 127–135, `arXiv:0807.1801`.

[61] L. Hollands, *Topological strings and quantum curves*. Ph.D. Thesis, University of Amsterdam, 2009. `arXiv:0911.3413`.

[62] F. Janda, R. Pandharipande, A. Pixton, and D. Zvonkine, "Double ramification cycles on the moduli spaces of curves," `arXiv:1602.04705`.

[63] P. Johnson, R. Pandharipande, and H.-H. Tseng, "Abelian Hurwitz-Hodge integrals," *Michigan Mathematical Journal* **60** no. 1, (2011) 171–198, `arXiv:0803.0499`.

[64] M. Kazarian and P. Zograf, "Virasoro constraints and topological recursion for Grothendieck's dessin counting," *Letters in Mathematical Physics* **105** no. 8, (2015) 1057–1084, `arXiv:1406.5976`.

[65] A. Kokotov and D. Korotkin, "Tau-functions on Hurwitz spaces," *Mathematical Physics, Analysis and Geometry* **7** no. 1, (2004) 47–96, `arXiv:math-ph/0202034`.

[66] A. Kokotov and D. Korotkin, "A new hierarchy of integrable systems associated to Hurwitz spaces," *Philosophical Transactions of the Royal Society of London A* **366** no. 1867, (2008) 1055–1088, `arXiv:math-ph/0112051`.

[67] M. Kontsevich and Y. Manin, "Gromov-Witten classes, quantum cohomology, and enumerative geometry," *Communications in Mathematical Physics* **164** no. 3, (1994) 525–562, `arXiv:hep-th/9402147`.

[68] I. Krichever, "Algebraic-geometric n-orthogonal curvilinear coordinate systems and solutions of the associativity equations," *Functional Analysis and Its Applications* **31** no. 1, (1997) 25–39, `arXiv:hep-th/9611158`.

[69] S. Lando and A. Zvonkin, *Graphs on Surfaces and Their Applications*, vol. 141. Springer-Verlag, 2004.

[70] D. Lewanski, A. Popolitov, S. Shadrin, and D. Zvonkine, "Chiodo formulas for the r-th roots and topological recursion," *Letters in Mathematical Physics* (2016) 1–19, `arXiv:1504.07439`.

[71] Y. Manin, *Frobenius manifolds, quantum cohomology, and moduli spaces*, vol. 47. American Mathematical Soc., 1996.

[72] T. Milanov, "The Eynard–Orantin recursion for the total ancestor potential," *Duke Mathematical Journal* **163** no. 9, (2014) 1795–1824, arXiv:1211.5847.

[73] T. Milanov, "The Eynard-Orantin recursion for simple singularities," *Commun. Num. Theor. Phys.* **09** (2015) 707–739, arXiv:1501.03677 [math.AG].

[74] A. Mironov and A. Morozov, "Check-operators and quantum spectral curves," arXiv:1701.03057.

[75] A. Y. Morozov, "String theory: What is it?," *Soviet Physics Uspekhi* **35** no. 8, (1992) 671.

[76] A. Y. Morozov, "Integrability and matrix models," *Physics-Uspekhi* **37** no. 1, (1994) 1–55, arXiv:hep-th/9303139.

[77] M. Mulase, S. Shadrin, and L. Spitz, "The spectral curve and the Schrödinger equation of double Hurwitz numbers and higher spin structures," arXiv:1301.5580.

[78] D. Mumford, "Towards an enumerative geometry of the moduli space of curves," in *Arithmetic and geometry, Vol.II*, vol. 36, pp. 271–328. 1983.

[79] P. Norbury, "Counting lattice points in the moduli space of curves," *Mathematical Research Letters* **17** (2010) 467–481, arXiv:0801.4590.

[80] P. Norbury and N. Scott, "Gromov–Witten invariants of $\mathbf{P}^1$ and Eynard–Orantin invariants," *Geometry and Topology* **18** (2014) 1865–1910, arXiv:1106.1337.

[81] A. Okounkov and R. Pandharipande, "Gromov-Witten theory, Hurwitz theory, and completed cycles," *Annals of mathematics* **163** no. 2, (2006) 517–560, arXiv:math/0204305.

[82] R. Pandharipande, A. Pixton, and D. Zvonkine, "Relations on $\overline{M}_{g,n}$ via 3-spin structures," *Journal of the American Mathematical Society* **28** no. 1, (2015) 279–309, arXiv:1303.1043.

[83] H. E. Rauch, "Weierstrass points, branch points, and moduli of riemann surfaces," *Communications on Pure and Applied Mathematics* **12** no. 3, (1959) 543–560.

[84] K. Saito, "Period mapping associated to a primitive form," *Publications of the Research Institute for Mathematical Sciences* **19** no. 3, (1983) 1231–1264.

[85] S. Shadrin, "BCOV theory via Givental group action on cohomological fields theories," *Mosc. Math. J* **9** no. 2, (2009) 411–429, arXiv:0810.0725.

[86] S. Shadrin, L. Spitz, and D. Zvonkine, "On double Hurwitz numbers with completed cycles," *Journal of the London Mathematical Society* **86** no. 2, (2012) 407–432, arXiv:1103.3120.

[87] S. Shadrin, L. Spitz, and D. Zvonkine, "Equivalence of ELSV and Bouchard–Mariño conjectures for r-spin Hurwitz numbers," *Mathematische Annalen* **361** no. 3-4, (2015) 611–645, arXiv:1306.6226.

[88] V. Shramchenko, "Deformations of Frobenius structures on Hurwitz spaces," *International Mathematics Research Notices* **2005** no. 6, (2005) 339–387, arXiv:math-ph/0408026.

[89] V. Shramchenko, "Real doubles of Hurwitz Frobenius manifolds," *Communications in mathematical physics* **256** no. 3, (2005) 635–680, arXiv:math-ph/0402015.

[90] V. Shramchenko, "Riemann-Hilbert problem associated to Frobenius manifold structures on Hurwitz spaces: irregular singularity," *Duke Mathematical Journal* **144** no. 1, (2008) 1–52, arXiv:0707.0696.

[91] C. Teleman, "The structure of 2d semi-simple field theories," *Inventiones mathematicae* **188** no. 3, (2012) 525–588, arXiv:0712.0160.

[92] W. T. Tutte, "On the enumeration of planar maps," *Bull. Amer. Math. Soc.* **74** (1968) 64–74.

[93] W. T. Tutte, "Chromatic sums revisited," *Aequationes Math.* **50** (1995) 95–134.

[94] R. Vakil, "The moduli space of curves and Gromov–Witten theory," in *Enumerative invariants in algebraic geometry and string theory*, pp. 143–198. Springer, 2008. arXiv:math/0602347.

[95] E. Witten, "Algebraic geometry associated with matrix models of two-dimensional gravity," *Topological methods in modern mathematics (Stony Brook, NY, 1991)* (1993) 235–269.

[96] J. Zhou, "Frobenius manifolds, spectral curves, and integrable hierarchies," arXiv:1512.05466.

[97] A. Zvonkin, "Megamaps: Construction and examples," in *Discrete Mathematics and Theoretical Computer Science, Conference edition: "Discrete Models: Combinatorics, Computation and Geometry"*, pp. 329–339. July 2001.

[98] D. Zvonkine, "An introduction to moduli spaces of curves and its intersection theory," http://anton.zorich.free.fr/Roscoff2012/IntroMgn.pdf.

[99] D. Zvonkine, "A preliminary text on the r-ELSV formula,".

Summary

In my thesis I consider interplay between several different structures in mathematical physics. These structures are used to solve a large class of problems in enumerative algebraic geometry and combinatorics in a universal way. The problems can range from counting certain one-dimensional drawings on two-dimensional surfaces to counting maps of certain type from a two-dimensional surface to some higher-dimensional space. The structures that we study in this thesis allow to encode the solutions to this type of enumerative and combinatorial problems in some general compact form.

In one approach the solutions to the enumerative problems are encoded in a complex algebraic curve with certain functions on it. From this initial small set of data one can reconstruct the full solution with the help of a recursive procedure that is absolutely universal and does not depend on a particular problem.

In another approach the solutions to the enumerative problems are encoded as certain integrals over some complicated spaces that parametrize different complex structures on two-dimensional surfaces. This reveals that the solutions to the enumerative problems reflect the geometric properties of the space of complex structures, also in a universal way.

These two approaches turn out to be related in many different ways. In this thesis their relation is studied in the framework of an advanced differential geometric structure called Frobenius manifold.

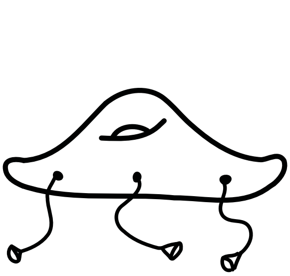
Samenvatting

Dit proefschrift gaat over het samenspel van verschillende structuren in de mathematische fysica. Deze structuren worden gebruikt om een grote klasse van problemen in enumeratieve algebraïsche meetkunde en combinatoriek op te lossen. De problemen kunnen variëren van het tellen van bepaalde eendimensionale tekeningen op oppervlakken tot het tellen van een bepaald type afbeeldingen van oppervlak naar een hogerdimensionale ruimte. De structuren die worden onderzocht in dit proefschrift laten de oplossingen van dergelijke enumeratieve en combinatorische problemen coderen in een aantal algemene compacte vorm.

In één benadering worden de oplossingen voor de enumeratieve problemen gecodeerd in een complex-algebraïsche kromme met daarop bepaalde functies. Uit deze kleine begindata kan men de volledige oplossing reconstrueren met behulp van een recursieve procedure die volledig universeel is en niet afhankelijk is van een bepaald probleem.

In één andere benadering worden de oplossingen voor de enumeratieve problemen gecodeerd als bepaalde integralen over een aantal ingewikkelde ruimten die verschillende complexe structuren op oppervlakken parametriseren. Hieruit blijkt dat de oplossingen voor de enumeratieve problemen de meetkundige eigenschappen van de ruimte van complexe structuren weerspiegelen, ook op een universele manier.

Beide benaderingen blijken op verschillende manieren samen te hangen. In dit proefschrift wordt het verband onderzocht in het kader van een geavanceerde differentiaalmeetkundige structuur genaamd Frobeniusvariëteit.



2017