Direct CP violation in $K \to \pi\pi$ decay on the lattice with periodic boundary conditions

Masaaki Tomii

Physics Department, University of Connecticut, Storrs, CT 06269, USA

E-mail: masaaki.tomii@uconn.edu

Abstract. Following our recent publication on direct CP violation and the $\Delta I = 1/2$ rule in $K \to \pi\pi$ decay, which was made with G-parity boundary conditions, we revisit this problem with a conventional lattice setup employing periodic boundary conditions and two lattice spacings to check our previous result and to improve the precision. We show that the physical amplitude, which corresponds to an excited state in this case, can be obtained reliably with the Generalized Eigenvalue Problem (GEVP) method. Not only are periodic boundary conditions cheaper and allow the use of existing ensembles to take the continuum limit, but they provide a straightforward path to introduce electromagnetism and strong isospin symmetry breaking, which will be needed in the near future. In this article we show our preliminary results and discuss the prospect of the high-precision calculation of $K \to \pi\pi$ decay with periodic boundary conditions.

1. Introduction

It has been a long-time challenge for lattice community to calculate the $\Delta I = 1/2$ $K \to \pi\pi$ matrix elements and the direct CP violation parameter ε'. The RBC and UKQCD collaborations published the first paper [1], which was updated by Ref. [2]. This series of study was implemented with G-parity boundary conditions, with which π^+, π^- and π^0 satisfy anti-periodic boundary conditions and hence the two-pion state with the energy near $2m_\pi$ is forbidden. This property of G-parity boundary conditions makes it straightforward to extract on-shell kinematics of $K \to \pi\pi$ decays. The G-parity study is still carried out to take the continuum limit.

More lattice results with independent setups would indeed be desired. In this work we employ standard periodic boundary conditions (PBC), with which we may face challenge to extract on-shell kinematics of $K \to \pi\pi$ because of the presence of the $E_{\pi\pi} \approx 2m_\pi$ state in correlation functions. There are a few advantages of using PBC. First, we already have a number of lattice ensembles with PBC, while generating ensemble requires a lot of computer time. With those ensembles we can take the continuum limit as long as we are successful in calculating $K \to \pi\pi$ with PBC. In addition, PBC provides a simpler prescription to introduce the QED and isospin-breaking corrections, which are expected to significantly impact the value of ε'.

In this article we show our results on the 24^3 lattice ensemble with $2 + 1$-flavor Möbius domain wall fermions at the lattice cutoff $a^{-1} = 1.023$ GeV and mostly physical pion and kaon masses. We perform measurements with 258 configurations and all-to-all propagator technique [3]; With each configuration we calculate 2,000 low modes and the high-mode part is calculated with spin-color-time diluted random noise vectors and deflated CG algorithm.

2. Two-pion energies

In this section we explain how to create the two-pion state whose energy is near the kaon mass.

In finite volume the two-pion energy spectrum is discrete and depend on the volume size. In non-interacting two-pion center-of-mass system two-pion energies are expressed as $E = 2\sqrt{|\vec{p}_\pi|^2 + m_\pi}$ with the pion mass m_π and the an individual pion momentum $\vec{p}_\pi = 2\pi/L \times (n_x + \theta_x, n_y + \theta_y, n_z + \theta_z)$, where n_i is an integer that labels spatial momentum and θ_i is a parameter that depends on the boundary condition for the i direction. Typically $\theta_i = 0$ for periodic boundary condition and $\theta = 1/2$ for anti-periodic boundary condition. Since quark fields with periodic boundary conditions lead to periodic pions, the possible lowest energy in the limit of non-interacting two-pion system, which is the state one can extract on the lattice in the most straightforward way, is $2m_\pi$. The two-pion energy shifts because of interaction between the two pions are not significant and thus the picture described above is not exactly true but we can take rough picture; 1. two-pion energies are discrete, 2. the possible lowest two-pion energy in PBC is near m_π and 3. excited-state energies depend on the volume.

We employ the variational method with Generalize Eigenvalue Problem (GEVP) [4, 5], which is widely used for investigating contributions from multiple states to correlation functions. In this method we first calculate a matrix of euclidean two-point correlation functions:

$$C_{ij}(t) = \left\langle O_i(t) O_j(0)^\dagger \right\rangle - \langle O_i \rangle \left\langle O_j^\dagger \right\rangle, \tag{1}$$

where O_i stands for a two-pion interpolation operator. Since we calculate these correlation functions with finite time extension as well as in finite spatial volume, we need to subtract thermal effects, whose leading contribution is constant and therefore can be subtracted by $C_{ij,subt}(t) = C_{ij}(t) - C_{ij}(t + \delta_t)$, which behaves as

$$C_{ij,subt}(t) = \sum_n \left(1 - \mathrm{e}^{-E_n \delta_t}\right) A_{n,i} A^*_{n,j} \mathrm{e}^{-E_n t} + \ldots, \tag{2}$$

where n labels a two-pion state and $A_{n,i} = \langle 0 | O_i^I | \pi\pi(E_n) \rangle$. With givem $N \times N$ matrix of correlation functions we solve GEVP

$$C_{subt}(t) V_n(t, t_0) = \lambda_n(t, t_0) C_{subt}(t_0) V_n(t, t_0), \tag{3}$$

where $t_0 < t$, $V_n(t, t_0)$ is an generalized eigenvector that corresponds to the generalized eigenvalue $\lambda_n(t, t_0)$ The (generalized) eigenvalues provide the effective two-pion energies

$$E_n^{eff}(t, t_0) = -\frac{1}{t - t_0} \ln \lambda_n(t, t_0), \tag{4}$$

which are independent of t and t_0 where t_0 is large enough.

In this work we employ four two-pion operator $\pi\pi(000)$, $\pi\pi(001)$, $\pi\pi(011)$ and $\pi\pi(111)$, where three-digit number in the parentheses indicates the special momentum of a pion divided by its unit $2\pi/L$. In addition we introduce a σ operator, an iso-singlet scalar operator for the isospin-0 channel, which will be used for the $\Delta I = 1/2$ channel of $K \to \pi\pi$ decay.

Figure 1 shows results for the effective two-pion energies for the isospin-2 (left) and 0 (right) channels. While the latter channel couples with vacuum state and is noisy compared to other channel, the results indicate that we successfully extract the signal from the first few states.

Since we use the lattice ensembles we created for many other projects, the volume is not tuned to realize a two-pion energy that coincide with the kaon mass and obtain on-shell kinematics of $K \to \pi\pi$ decays. Nevertheless we find the energy of the first excited state for the isospin0 channel is close to the kaon mass and we would obtain a few percent off-shell kinematics from this two-pion state: $E_1 = 542(5)$ MeV and $m_K = 513.4(4)$ MeV. We plan to remove the systematic error on $K \to \pi\pi$ matrix elements and amplitudes due to the off-shell kinematics by performing an energy extrapolation.

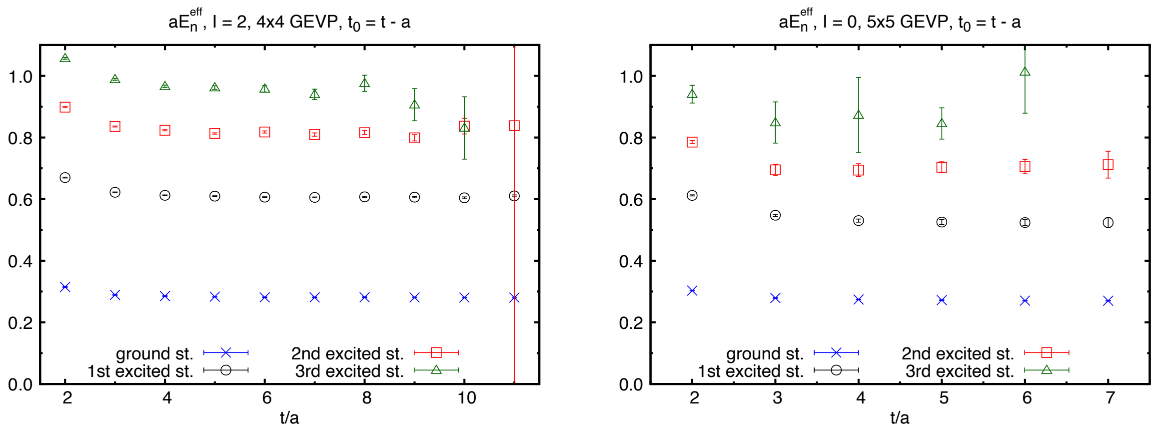

Figure 1. Effective $\pi\pi$ energies for $I = 2$ (left) and $I = 0$ (right) obtained from GEVP analysis. Here we solve GEVP using the four two-pion operators for $I = 2$ and additionally the σ operator for $I = 0$. The results are shown in lattice units.

3. $K \to \pi\pi$ matrix elements

To calculate the $K \to \pi\pi$ matrix elements we follow the procedure provided in Ref. [6]. With the eigenvectors obtained by solving GEVP in the previous section we calculate the $K \to \pi\pi$ three-point correlation functions with state-specific two-pion operators

$$\begin{aligned} C_{n,i}^{(t,t_0)}(t_1,t_2) &= \sum_a V_{n,a}(t,t_0)\left\langle O_a(t_1+t_2)Q_i(t_1)O^K(0)^\dagger\right\rangle \\ &= B_{n,i}M_{n,i}A_K^* \mathrm{e}^{-m_K t_1 - E_n t_2} + O\left(\mathrm{e}^{-m_{K^*}t_1 - E_n t_2}, \mathrm{e}^{-m_K t_1 - E_{N+1}t_2}\right), \end{aligned} \tag{5}$$

where Q_i is a $\Delta S = 1$ four-quark operator, O^K is a kaon interpolation operator, $B_{n,i} = \sqrt{1-\mathrm{e}^{-E_n\delta_t}}A_{n,i}$, $A_K = \langle 0|O^K|K(m_K)\rangle$ and

$$M_{n,i} = \langle \pi\pi(E_n)|Q_i|K\rangle\,. \tag{6}$$

We define the effective matrix elements by

$$M_{n,i}^{eff,(t,t_0)}(t_1,t_2) = C_{n,i}^{(t,t_0)}(t_1,t_2)R^K(t_1)R_n^{(t,t_0)}(t_2), \tag{7}$$

where

$$R^K(t_1) = \mathrm{e}^{m_K^{eff}(t_1)t_1/2}\left[C^K(t_1)\right]^{-1/2}, \tag{8}$$

$$R_n^{(t,t_0)}(t_2) = \left(1-\mathrm{e}^{-E_n^{eff}(t,t_0)\delta_t}\right)^{1/2}\mathrm{e}^{E_n^{eff}(t,t_0)t_2/2}\left[C_{n,subt}^{(t,t_0)}(t_2)\right]^{-1/2}. \tag{9}$$

Figure 2 shows the results for the effective matrix elements of the isospin-0 channel and with the four-quark operators Q_2 (left) and Q_6 (right). We perform correlated χ^2 fits to these effective matrix elements and the results are shown by the cyan band. In the plots we express the data point that are used for the χ^2 fit as filled colored points, while the other data points are expressed as unfilled gray points. There appear to be a plateau for each plot and $\chi^2/d.o.f.$ is reasonably small as indicated in the caption. These facts indicate that lattice calculation of $\Delta I = 1/2$ $K \to \pi\pi$ matrix elements is feasible and we will be able to provide proper values of $K \to \pi\pi$ amplitudes and ε'.

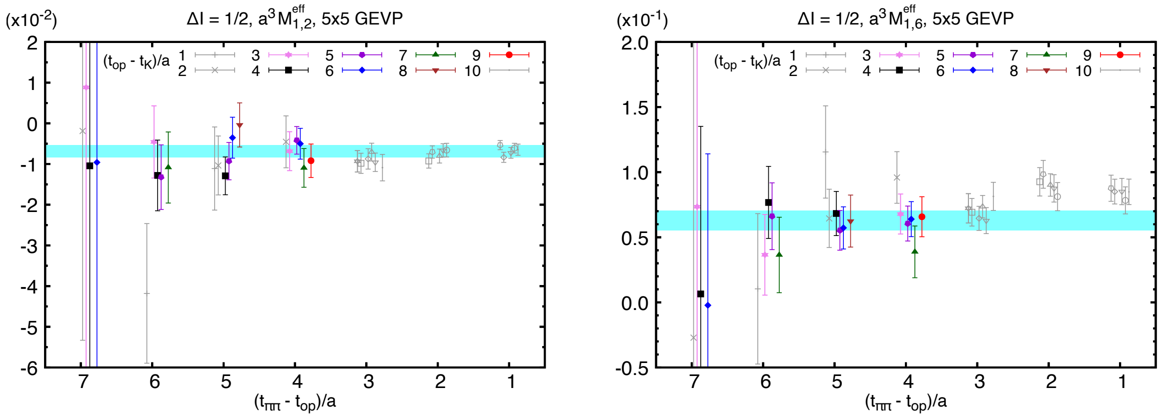

Figure 2. Effective matrix elements of $I = 0$ $K \to \pi\pi$ decay with near on-shell kinematics for Q_2 (left) and Q_6 (right) on the 24^3 ensemble. The band represents the result for constant fit with the fit range $t_{op} - t_K \geq 3$ and $t_{\pi\pi} - t_{op} \geq 4$ for both Q_2 and Q_6. Filled points are used fo r the fit, while open points are out of the fit range. The $\chi^2/d.o.f.$ is 0.5 for Q_2 and 0.4 for Q_6. The $\chi 2/d.o.f.$ is 0.50 for Q_2 and 0.37 for Q_6.

4. Summary

We demonstrate the feasibility of calculating $\Delta I = 1/2$ channel of $K \to \pi\pi$ decay on the lattice with PBC. The precision performance appear to be as good as or even better than the one we obtained in our G-parity calculation. With this prospect we proceed our calculation to obtain the continuum limit and further introduce QED and isospin-breaking effects in the not so distant future. Our current results are being summarized in a full paper that will be submitted soon.

References

[1] Z. Bai *et al.* [RBC and UKQCD], "Standard Model Prediction for Direct CP Violation in K→ππ Decay," Phys. Rev. Lett. **115**, no.21, 212001 (2015) doi:10.1103/PhysRevLett.115.212001 [arXiv:1505.07863 [hep-lat]].

[2] R. Abbott *et al.* [RBC and UKQCD], Phys. Rev. D **102**, no.5, 054509 (2020) doi:10.1103/PhysRevD.102.054509 [arXiv:2004.09440 [hep-lat]].

[3] J. Foley, K. Jimmy Juge, A. O'Cais, M. Peardon, S. M. Ryan and J. I. Skullerud, Comput. Phys. Commun. **172**, 145-162 (2005) doi:10.1016/j.cpc.2005.06.008 [arXiv:hep-lat/0505023 [hep-lat]].

[4] M. Luscher and U. Wolff, "How to Calculate the Elastic Scattering Matrix in Two-dimensional Quantum Field Theories by Numerical Simulation," Nucl. Phys. B **339**, 222-252 (1990) doi:10.1016/0550-3213(90)90540-T

[5] B. Blossier, M. Della Morte, G. von Hippel, T. Mendes and R. Sommer, JHEP **04**, 094 (2009) doi:10.1088/1126-6708/2009/04/094 [arXiv:0902.1265 [hep-lat]].

[6] J. Bulava, M. Donnellan and R. Sommer, JHEP **01**, 140 (2012) doi:10.1007/JHEP01(2012)140 [arXiv:1108.3774 [hep-lat]].