Matrix Factorisations and Orbifold Equivalence in Landau Ginzburg Models

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Matrix Factorisations and Orbifold Equivalence in Topological Landau-Ginzburg Models

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Abstract

We have investigated matrix factorisations of polynomials corresponding to various Landau-Ginzburg models with $N = 2$ supersymmetry. These are non-conformal Lagrangian models with specific super-potentials and are thought to flow to a renormalisation group fixed point, which correspond to conformal field theories. Matrix factorisations can be used to construct BRST type operators which have a basis of states which correspond to the chiral primaries of the CFTs confirming the correspondence. We look at how these matrix factorisations can be created from exact sequences and put this into practice using the homological algebra package, Singular, to create exact sequences/free resolutions from a restricted list of ideals thereby producing a matrix factorisation factory whose only input is the potential. We managed to construct all ADE indecomposable matrix factorisations from simple ideals built from generators in the quotient ring. As a side result, this procedure required the development of a simple algorithm to identify isomorphic matrix factorisations. We also make some statements about invertibility of matrix elements and factors in order to discuss and where other Lagrangian, conformal theories, such as Liouville might fit in this correspondence. The main body of work concentrates on the nature of orbifold equivalence. This is an aspect of topological field theories with defects. We analyse the nature of the quantum dimension formula making some interesting discoveries which we use to refine a procedure to find such orbifold equivalences. This procedure was eventually successful, in theory only limited by computer power, and we review the current updated catalogue of orbifold equivalences and discuss the some implications of our findings and observations on such equivalences.

Acknowledgements

I would like to express my gratitude to my supervisor, Dr Andreas Recknagel, most of all for his infectious enthusiasm for our research. Whenever I found I had run out ideas or felt our research was not progressing, I would find a few hours with Andreas left me energised and eager to carry on. I would also like to thank him for his endless patience and the amount of time he has spent, almost always being available for discussions.

I would also like to thank my second supervisor Dr Gerard Watts for his support and advice throughout my time at King's College and in general to the mathematics department and all the people I have encountered in my time there and who have helped me. This includes staff and my fellow students. I must also mention Nils Carqueville who has provided valuable explanations and bespoke software and also Dr Ingo Runkel for valuable discussions about defects. My final thanks go to the STFC for funding me.

Being at King's has been a transformative experience and I don't think I will ever think the same way again. I am truly grateful that I now have the education to be able to comprehend and wonder at what is happening in modern theoretical physics, which really is one of the most exciting frontiers of human knowledge. As I go on from here, I'm sure I will retain my enthusiasm and passion.

Contents

1 Introduction

Two dimensional topological quantum field theories, or TQFTs, can be derived from corresponding conformal quantum field theories, and as a simplification of these theories may provide insights into string theory. Perturbative superstring theory is described by supersymmetric conformal field theories, CFTs, on the 2-d worldsheet. There are two main types of models. One can try to define these algebraically in terms of super-Virasoro algebras, or one can start from certain Lagrangian field theories, the Landau-Ginzburg, or LG, models. These are Lagrangian models with an interaction superpotential, a quasi-homogeneous (2.12) complex polynomial in several variables. There is strong evidence that such LG models flow to CFTs under RG flow, and one can match large classes of CFTs (e.g. minimal models) with a LG counterpart. In particular, the most important invariant of a CFT, its central charge, can be computed in terms of the (quasi-homogeneous) superpotential of the LG model (2.13).

Consequently there are two routes to a topological theory arising from these complementary descriptions of the same CFT. In the case of minimal models, there is a process called the *topological twist* (2.8), which starts with a simple re-definition of the stress energy tensor. One outcome of this process is the emergence of a nilpotent BRST operator, which can be used to define a cohomology. Its elements are the physical states of the topologically twisted CFT, they arise from the so-called *chiral primary* states (2.6) in the full CFT. If one starts from a LG model, no super-Virasoro generators are at hand, but one can still define a BRST operator and pass to its cohomology, obtaining a different description of physical states in the topological LG model. In practice, this description is a very simple one, basically due to the fact that the superpotential in the LG action is unaffected by renormalisation group flow (it is protected from corrections by supersymmetry). The set of physical states in the topological LG model is simply the Jacobi ring of the superpotential.

An interesting connection between these two approaches emerges when we consider TQFTs derived from boundary conformal theories, which in string theory represent open strings between *branes*. Boundary conditions derived from the preservation of $N = 2$ supersymmetry lead to a surprising connection between the topological branes and *matrix factorisations* of the superpotential. A matrix factorisation, or MF, can be seen most simply as a pair of commuting matrices whose product, in this context, is the superpotential multiplied by the identity matrix. The matrix factorisation of a potential, $W \in \mathbb{C}[x_1, x_2, ..., x_n]$ is constructed from commuting $N \times N$ matrices $E, J \in Mat_N(\mathbb{C}[x_1, x_2, ..., x_n])$, and appears as[1]

$$E.J \;=\; J.E \;=\; W\,\mathbb{I}_N \;. \tag{1.1}$$

We then define,

$$Q \;=\; \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} \;, \tag{1.2}$$

[1]See Appendix A for a derivation.

and

$$Q^2 \quad = \quad W\mathbb{I}_{2N} \ . \tag{1.3}$$

We will often refer to the rank $2N$ matrix Q as a rank N matrix factorisation.

The central unpublished conjecture in this field, due to Kontsevich, proposes that MFs represent branes in a topological boundary string model. There is overwhelming evidence to support this [9, 10, 48, 50, 58], essentially showing there is a correspondence between the cohomology or 'physical states' of a BRST operator constructed from certain MFs, and the boundary chiral primary states from minimal models, both subject to the same ADE classification and central charge. The nature of the ADE classification is very different in each case. For minimal models it comes from classification of the partition function under symmetry transformations [38]. For LG models it comes from the Arnold's classification of polynomials [1].

Category theory provides an axiomatic approach to topological field theory in terms of objects and morphisms and also underlies the surprising connection between the cohomology associated with MFs and boundary chiral primary states by equivalence of categories. The resulting formalism involves a number of interrelated physical and mathematical concepts, but some basic elements of the category theory description of topological LG models are quite intuitive. One starts out with a 2-dimensional worldsheet, the *bulk*, labeled by the superpotential of the LG defined there. If the worldsheet has a boundary, one attaches further labels to boundary components (boundary conditions, i.e. branes, i.e. MFs of the bulk potential). One can then also consider fault lines or *defects* bisecting the bulk, with different LG models, i.e. different potentials, defined on each of the bulk regions separated by the defect line. Again, one labels the defect line by addtional structure, which turns out to be a MF of the difference of the two potentials. One can view the LG potentials as objects in a category and defects, MFs of their difference, as morphisms between them. Moreover, one can show that elements in the cohomology between two MFs, which correspond to excitations living on the defect line, can again be viewed as morphisms between the MFs, so one actually obtains a bi-category, with two levels of morphisms. The LG potentials are objects, defects are 1-morphisms, and fields living on defects are 2-morphims.

It was within this more abstract categorial point of view that the general properties of a particularly interesting class of defects were discovered. These are defects which allow us to define a new classification of equivalence classes for LG potentials. This new equivalence is called *orbifold equivalence*. The choice of name will become clearer when we discuss simple examples below. One says that two LG potentials are orbifold equivalent if and only if there exists a defect with these special properties between them. The concrete definition of those special properties in terms of MFs will be given in Section 4. One interesting fact about orbifold equivalence classes is that they differ from the classification of singularities given by Arnold [1]. Two polynomials defining the same singularity need not be orbifold equivalent, and two orbifold equivalent polynomials need not define the same singularity. Similarly, it is not known how geometric equivalence of algebraic varieties and orbifold equivalence are related, see Section 5 on the elliptic curve for further remarks. On the other hand, if two potentials are orbifold equivalent, there is an equivalence between certain categories associated to the two potentials (see Sect 4 for a precise statement), which proves to be interesting from a category theory point of view, but also for the topological LG models associated to the two potentials, correlation functions for worldsheets with defects between the two models are related.

Orbifold equivalence, in relation to defects, is a notion of symmetry that has developed quite recently. Initially it was the description of group actions between regions described by the same superpotential [14, 15, 16], but recent work [26, 20, 21, 24, 29, 22, 23, 25] has extended the concept beyond this type of symmetry leading to surprising connections between otherwise unrelated polynomials/superpotentials with the same associated central charge (2.13) [15, 26]. The physical picture allows us to describe a worldsheet with two topological regions in terms of just one topological region with defects and field insertions [22, 29]. This is due to the topologically 'legal' moving of defects on the worldsheet without crossing. While the concept can be defined in a very general category-theoretic language, we focus on the most explicit setting, in terms of MFs, where orbifold equivalences arise from defects with special properties, namely they have non zero *quantum dimensions* (4.3). These are two complex numbers computed from a form of the topological correlator.

Examples are relatively difficult to construct, but we uncover some structural features that distinguish orbifold equivalences, most notably a finite Taylor expansion. We use those properties to devise a search algorithm and then present some new examples including Arnold singularities [1, 30]. After reviewing previous results and methods in this field we then outline the development of a more generalised approach. The general principle of our algorithm is an implementation of the weak form of Hilbert's Nullstellensatz theorem from Algebraic Geometry [45, 66], and in it's basic form it is only limited by computer speed and memory because of the complexity of the computations. It was in developing strategies to surmount this limitation that we have made discoveries concerning the structure of the correlator as a finite Taylor expansion resulting from observations concerning weight matrices, grading and mixed terms in such equivalences. We also made some interesting observations concerning coordinate transformations and the general landscape of such equivalence classes. Using this algorithm we have uncovered many more orbifold equivalences and the examples are listed. Only a selection of the examples are reproduced as the data is too lengthy and complex to list here but can be found at the webpage [100].

Not many concrete examples of orbifold equivalences were known, and the main aim of this thesis is to construct further examples, in more complicated situations than studied so far. We hope this thesis outlines the physical and mathematical concepts in enough detail and with a few interesting detours so as to be able to understand the context and criteria for orbifold equivalence and also the strategy which made it possible to find many new examples. The result is a more detailed picture of this equivalence, with a catalogue of examples which inevitably raises some new questions.

In order to understand how MFs are used to construct BRST operators we will give the simplest example of a computation of the cohomology in Sections 3.1 and 3.2. Such computations are best done using the computer algebra package, SINGULAR. It is instructive to see examples of the particular MFs, which are well studied in the literature and which confirm the correspondence between LG and minimal ADE models. The actual matrices and spectrum of these MFs are given explicitly for A and D model categories, and one exceptional example, E_6. Familiarity with these models gives context to the form of MFs we see when discussing orbifold equivalences between them. We also introduce two more well studied models, the elliptic curve and the quartic. These are different in nature to the ADE MFs in that there are continuous parameter spaces of non-isomorphic MFs, a feature which is also of interest later when we consider orbifold equivalences.

The category of MFs has many aspects which were relevant to our work. They are matrices and we discuss their algebra as regards tensor products, direct sums and equivalence. We also look at the form of the superpotential, it's properties as a quasi-homogeneous potential and touch on the classification and possible forms of these potentials. Understanding this helped us find more equivalences. There is also a short section on invertibility of both matrix factors and matrix elements. In connection with this, there is Appendix E where we discuss early work looking at Liouville type theories and integrability in the context of MFs, since these are also theories with a superpotential.

Another subject touched upon in the development of the background mathematics is that of MFs as exact sequences. An early, very simple, example of a matrix factorisation made its appearance in the Dirac equation, but they are also known in mathematics due to Eisenbud's discovery that free resolutions of modules over $\mathbb{C}[z]/\langle W\rangle$ become periodic [34]. MFs can be defined for general polynomials and are well studied in mathematics. We will look at the practical construction of all indecomposable objects, i.e. MFs in categories of MFs in connection with exact sequences and free resolutions. The homological algebra package, SINGULAR allowed us to put this into practice, constructing a matrix factorisation factory or MF factory whose only input is the potential. We managed to construct the full category of all ADE indecomposable MFs as exact sequences generated from different sets of simple ideals chosen from generators in the quotient ring. A side result of the development of this factory was a new and simple algorithm to identify isomorphic MFs. It is quite possible that this method can be extended to identifying direct sums of indecomposable MFs.

The new data produced has some obvious exceptions and we discuss the nature of orbifold equivalences in the light of these discoveries outlining features of the formulae used in connection with elliptic and quartic potentials which have a central charge $c = 3$. It is also worth mentioning here that we will often work in units of $\hat{c} = \frac{c}{3}$. Having managed to build up the catalogue of orbifold equivalences, the question arises if there are any other criteria for orbifold equivalence other than the matching of central charges? Some possible directions for future investigations along this line are indicated, which outline intriguing problems or questions. We also discuss the implications of the *fusion product* in connection with the quantum dimension formula under the assumption that certain undiscovered orbifold equivalences do exist between non-isomporphic MFs of the elliptic curve. Some of these issues raised could be solved with unlimited computing power.

This work has made extensive use of the computer algebra package SINGULAR [44], run on a MacBook Pro and similar laptops. The nature of calculations in SINGULAR requires the use of *Groebner* bases and *Buchberger's* algorithm (appendix B). These calculations can blow up unpredictably in terms of computer memory and this has been one of the main restrictions on computations, speed is the other. Because of this unpredictability it is not clear how current supercomputers would fare. Since we are computing with matrices these calculations also become much more complex (at least as N^4) as we go up with rank.

New material first appears at the end of section three where we use free resolutions to develop a simple algorithm to produce each ADE category of MFs, **MF factory**(W) and another to identify isomorphic MFs, **isomchk**(Q_1, Q_2). In section 4 we then have the main body of work, with some theorems and statements concerning the structure of the quantum dimension formula and we also look at the types of polynomials and coordinate transformations possible in connection with orbifold equivalences. We find orbifold equivalence for the remaining exceptional unimodal Arnold singularity pairs as well as some others, and outline the updated catalogue of MFs.

The material concerning orbifold equivalence is the subject of a paper, [81] and there is a webpage with more examples and some of the code we used at:
`https://nms.kcl.ac.uk/andreas.recknagel/oeq-page/`

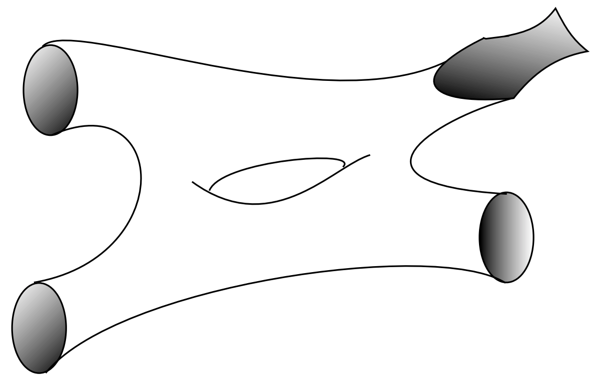

Figure 1: String worldsheet.

2 Topological strings

This section presents a tour of the ideas connecting both string theory, and also 2d Landau-Ginzburg theories, to 2d TQFTs in more detail. The meeting point of these theories is found in the boundary topological theory and we outline how MFs come into the picture on the LG side of the Landau-Ginzburg - conformal field theory correspondence.

2.1 Elements of topological field theory

In the standard model and quantum field theory there a map from spacetime to a number of different types of state spaces representing different particles with different internal symmetries. All particles are subject to the spacetime symmetries. Expectation values of observables are computed by a correlator, often described as a path integral. Although thought to be lacking in strict mathematical rigour it is a conceptual consequence of the double slit phenomena and an essential part of QFT.

String theory attempts to unite all the different possible state spaces, representing the different varieties of particles as different states of a fundamental 1-dimensional object called a string. A string traces out a *worldsheet* as it moves through space and time which can be described by two coordinates with the characteristics of space and time. The worldsheet carries a complex structure and so is a Riemann surface, and part of the overall symmetry is conformal symmetry on the plane. (The full symmetry of the worldsheet is actually diffeomorphism invariance). The model requires spacetime to be 10-dimensional and we need $N = 1$ spacetime supersymmetry for fermions which requires $N = 2$ supersymmetry on the worldsheet [8].

In string theory we also have correlators. The analog of the path integral in string theory becomes an integral over all possible surfaces [76] rather than all paths. One such surface is easily visualised as in fig 1. Instead of a 'picture' of lots of n-point functions for different varieties of particles, we now have a network of open and closed string worldsheets, although spacetime itself has become much more complicated with the addition of supersymmetry and extra compactified dimensions. The two dimensional Riemann surface now exists in a ten dimensional Calabi-Yau manifold. From a topological point of view the situation has become simpler and any assemblage of pipe and ribbons representing a worldsheet can be seen to consist of a few simple elementary pieces or building blocks.

It is a natural next step to think about two questions. First what is the implication of the topology of these surfaces to the computation of correlators? One answer to is given by a simplification of the state space of conformal field theory which leaves just the topologically invariant states. We can also aske if these basic building blocks can be represented in some sort of mathematical way and what can be discovered by doing such a thing? The answer to this is the axiomatisation of possible topological field theories using category theory and development of a correlator in this setting. .

2.2 N=2 CFTs and the topological twist

There is a process by which we can obtain a topological field theory by "topologically twisting" a two dimensional conformal field theory with $N = 2$, supersymmetry [69, 95]. Two-dimensional conformal field theories, 2d CFTs describes the field theories which can occur on a two-dimensional worldsheet. In the case of sigma models it is concerned with maps $\phi : \Sigma \to M$, where M is a Calabi-Yau or some other target manifold [38, 40] and Σ is a Riemann surface. For minimal models there is no action principle and these are not Lagrangian models. They are derived by defining the central charge and the state space algebra [8].

For minimal models with $c < 1$ the construction of a CFT can be done with just the definition of the central charge and highest weight states. Minimal models are also rational CFTs. Their central charge is given by

$$c \;=\; 1 - \frac{6(p' - p)^2}{pp'} \quad , \tag{2.1}$$

where p and p' are coprime integers and so form a discrete set [86]. We can take the tensor product of these minimal models to produce CFT's for higher values of c, as required by Gepner models [80]. The partition function for these models has modular transformation symmetry which can be classified by the ADE system for groups.

The Virasoro algebra results from the underlying conformal algebra, acting on states and operators alike. It represents the infinite spacetime symmetries of a Riemann surface. The algebra is given by a set of modes, corresponding to the Laurent expansion of functions on a complex surface. There are two sets $L_n, \bar{L}_n$ of modes with $n \in \mathbb{Z}$, called left-movers and right-movers obeying the Virasoro relations, The state space consists of *Verma* modules, constructed from highest weight states and descendent states, generated from these highest weight states. For a full account [8, 38, 40, 86, 80].

We will give a brief description of the algebra to show how $N = 2$ supersymmetry enters and enables us to define a BRST operator identifying topologically invariant states [69, 95, 94].

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \ , \\
[\bar{L}_m, \bar{L}_n] &= (m-n)\bar{L}_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \ , \\
[L_m, \bar{L}_n] &= 0 \ .
\end{aligned} \tag{2.2}$$

where c is the central charge. We will restrict the discussion to left-movers as the right-movers are completely analogous. The highest weight states $|h_i\rangle$ obey the relations

$$L_0|h_i\rangle = h_i|h_i\rangle \ , \qquad L_n|h_i\rangle = 0 \ \ , \ \ n > 0 \ . \tag{2.3}$$

There is a one to one correspondence between fields and states in CFT. Here we single out the fields ϕ_i which correspond to highest weight states and are called primary fields. As operators these act on the vacuum to produce highest weight states. $\phi_i|0\rangle = |h_i\rangle$.

For a viable string theory which could describe fermions we need supersymmetry on the worldsheet. This is the easiest way to eliminate unwanted tachyons from the space-time spectrum. Moreover, if one wants the space-time spectrum to be supersymmetric itself, the world-sheet theory should have N=2 supersymmetry. Then its Neveu-Schwarz sector describes space-time bosons, the Ramond sector space-time fermions, and the space-time supersymmetry arises from the spectral flow operator on the worldsheet. $N = 2$ superconformal models, with two generators G^+ , G^- provide the most fruitful and interesting models, giving rise to such topics as mirror symmetry and are necessary for type II string theories. The existence of two generators means that there is a $U(1)$ current, J. All these operators can be expanded into modes. The full super-conformal algebra for just the left movers is given by the Virasoro relations plus

$$\begin{aligned}
\{G_r^-, G_s^+\} &= 2L_{r+s} - (r-s)J_{r+s} - (r^2 - \frac{1}{4})\frac{c}{3}\delta_{r+s,0} \\
\{G_r^\pm, G_s^\pm\} &= 0 \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m,-n} \\
[J_m, G_s^\pm] &= \pm G_{m+s}^\pm \\
[L_m, G_s^\pm] &= (\frac{m}{2} - n)G_{m+s}^\pm \\
[L_m, J_n] &= -nJ_{m+n}
\end{aligned} \tag{2.4}$$

Note that in the NS (Neveu-Schwarz) sector the supersymmetry generator modes $G_r^\pm$ carry half integer indices, $r + \frac{1}{2}, \ s + \frac{1}{2} \in \mathbb{Z}$ whereas in the R (Ramon) sector the modes carry integer ones. If we take $r = \frac{1}{2}$ and $s = -\frac{1}{2}$ the first anti-commutator above gives us the relation

$$h \geq \frac{|q|}{2} \ , \tag{2.5}$$

for any highest weight (primary) state. h is the weight and q is the $U(1)$ charge.

We define *left chiral* states by $G^+_{-1/2}|\phi\rangle = 0$, and *left anti-chiral* states by $G^-_{-1/2}|\phi\rangle = 0$.

Chiral primary states are then defined by

$$G^-_{n+1/2}|\phi\rangle \ = \ G^+_{n+1/2}|\phi\rangle \ = \ 0 \ , \ n \geq 0 \ . \tag{2.6}$$

For chiral primary states we have equality

$$h = \frac{|q|}{2} \ . \tag{2.7}$$

We can make our theory topological by redefining these algebraic relations, which gives us a way of preserving only states which are not subject to spacetime transformations. Topological twisting [69, 95] is a redefinition of the stress energy tensor by the $U(1)$ current which leaves only the topologically invariant states. There are two types of twist, A and B, but we only need to look at the B-twist in order to understand the correspondence with topological states in LG models. The surviving states are the *chiral primary* states discussed above. First we make the B-twist transformations

$$L_n \to L_n - \frac{(n+1)}{2} J_n \ . \tag{2.8}$$

and

$$\begin{aligned} Q_r &= G^+_{r-\frac{1}{2}} \\ G_r &= G^-_{r+\frac{1}{2}} \end{aligned} \tag{2.9}$$

In the original CFT, both G^+ and G^- have spin $\frac{3}{2}$, but they have $U(1)$ charges $+1$ and -1. The new spin after the twist is [(old spin)+1/2×(old charge)] , so one does get a new-spin $= 2$ for one of them and new-spin $= 1$ for the other. We denote G^+ (formerly spin $\frac{3}{2}$ by G (now spin 2), and G^- (also formerly spin $\frac{3}{2}$) by Q (now spin 1) and the algebra becomes

$$
\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} \\
\{G_m, Q_n\} &= 2L_{m+n} + mJ_{m+n} + \frac{c}{6}m(m+1)\delta_{m+n,0} \\
[J_m, Q_n] &= Q_{m+n} \\
[L_m, Q_n] &= -nQ_{m+n} \\
[J_m, J_n] &= \frac{c}{3}m\delta_{m,-n} \\
[J_m, G_n] &= -G_{m+n} \\
[L_m, G_n] &= (m-n)G_{m+n} \\
[L_m, J_n] &= -nJ_{m+n} + \frac{c}{6}n(n+1)\delta_{m+n,0}\ .
\end{aligned} \tag{2.10}
$$

Now all indices are integer and there is no central charge in the Virasoro relations. The former central charge still makes an appearance in the other relations and is now called the background charge. The operator Q_0 has the property $Q_0^2 = 0$ and can be used as a BRST operator whose cohomology consists of the chiral primary states. The term BRST comes from QFT [38]. The cohomology consists of all the states in the kernel of Q_0. $|\phi\rangle : Q_0|\phi_k\rangle = 0$, but excludes states in the image of Q_0. $|\phi_i\rangle : |\phi_i\rangle = Q_0|\phi\rangle$. The cohomology consists of the chiral primary states and they form a *chiral ring* [69].

One can show that correlators of the physical states (those in the cohomology of Q_0) are independent of the insertion points – hence the theory is topological. This is because the new fields satisfy, $T^{top} = \{G(z), Q_0\}$, so the new energy-momentum tensor is itself Q_0 exact. We now have a simplified theory of topological states which are still related to the CFT from which they arose. It is important to note that Q here is an operator acting in the CFT and is not the same as the MF Q in eq(1.1).

2.3 Axiomatic TQFT

Categories are defined by sets of axioms which unite different collections of mathematical objects and the relationships between them. Two examples of categories are vector spaces and groups. In the former case, objects are given by the vector spaces themselves, while morphisms are linear maps; in the latter case, morphisms are given by group homomorphisms. It is, however, customary to call these standard examples of categories by the names of their objects.

We give the basic definition of a category and functors so as to be able to discuss the category of *cobordisms*, topological field theories and later MFs. The discussion of cobordisms allows us to look at topological closed string theory from a set of building blocks and then to define the possible relations between these.

Definition 2.1: A **category** consists of;

a set of objects O.

a set of morphisms, $hom(x, y)$ between objects for $x, y \in O$ and $f \in hom(x, y)$, $f : x \to y$.

These are subject to:

For any object there exists identity morphisms such that $\forall x \in O$, $\exists\, 1_x : x \to x$.

For any morphism $f : x \to y$, $1_x f = f 1_y$.

For $f, g \in hom(x, y)$ & $x, y, z \in O$ such that $f : x \to y$ & $g : y \to z$ the composite morphism $(fg) : x \to z$.

Composition of morphisms is associative. $f, g, h \in hom(x, y)$ & $w, x, y, z \in O$ such that $f : w \to x$, $g : x \to y$ & $h : y \to z$, $(fg)h = f(gh)$.

An isomorphism is a morphism with an inverse.

There are also *functors* between categories.

Definition 2.2: A **functor** $F : B \to C$, between categories B and C, consists of;

1) A function $F : O(B) \to O(C)$,

2) For any two objects in $O(B)$ a function $F(hom(x, y)) \to hom(F(x), F(y))$

3) F preserves identities and composition.

Specific categories have further requirements but all share the above axioms. By refining the definition we are in a position to describe cobordisms. A cobordism is an oriented n-manifold, M, whose boundary is the disjoint union of the two closed $(n-1)$ manifolds, Σ_1 , Σ_2. In the category of two-dimensional cobordisms $Bord_2$, the objects are S_1 oriented circles, which can be seen as carrying in- or out-states in a closed string scattering process, and any morphism between these represents the worldsheet between in- and out- states. The basic elements are shown in fig 2. [61]

An axiomatic TQFT was first proposed by Atiyah [5] and two-dimensional topological field theories have been studied extensively [61, 28, 29].

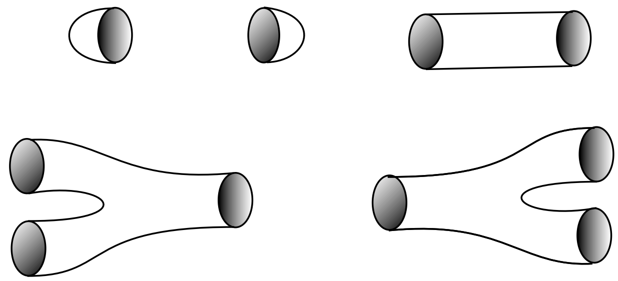

Figure 2: Cobordisms.

Definition 2.3: A two dimensional closed TQFT is a symmetric monoidal functor $F : Bord_2 \to Vect_{\mathbb{K}}$

This assigns a concrete state in a state space, an object in $Vect_{\mathbb{K}}$ to a type of cobordism. Monoidal refers to the fact that we can have a tensor product. The 'pair of pants' cobordism is the visual picture of this as we can see it is a morphism from $A \to A \otimes A$ for $A \in Vect_{\mathbb{K}}$.

Boundary TQFTs can also be represented by the category of MFs, and in this case the branes are the objects and the strings are morphisms between them. In the next section we shall see how the MF condition arises and after that the mathematics involved.

This formulation can be taken much further [22, 20, 21, 24], and is enhanced by another level of morphisms, (morphisms of morphisms) in the bi-category formulation, and these ideas have also been extended to higher dimensional topological field theories. We will only need enough category theory to provide the setting for the development of the formulation of orbifold equivalence and the quantum dimension formula, as we will see in the section on orbifold equivalences. The next step is to find a way to construct a field space and compute correlators and make a connection with the topological correlators from the CFT side.

2.4 LG models and matrix factorisations

$N = 2$ supersymmetric two dimensional conformal theories with boundaries provide a well studied collection of models for strings attached to branes [51] . $N = 2$ supersymmetry is necessary from a type II string theory perspective and it also the setting for the topological twist, see Section 2.2 (2.8). Preservation of $N = 2$ supersymmetry at a boundary is also important in LG models since the fixing of boundary conditions [9] provides the MF condition which is central to the category theory description of open topological strings between branes.

The general Landau-Ginzburg action can be written,

$$S_\Sigma \;=\; \int d^2x d^2\theta d^2\bar{\theta} K(\Phi,\bar{\Phi}) - \frac{1}{2}\int d^2x d^2\theta W(\Phi) - \frac{1}{2}\int d^2x d^2\bar{\theta}\bar{W}(\bar{\Phi}) \;. \qquad (2.11)$$

θ and $\bar{\theta}$ are the Grassmann supercoordinates on the worldsheet. The fields Φ and $\bar{\Phi}$ are chiral and anti-chiral super-fields. W is the super-potential. The first term, is the integral of a Kahler potential which can be set to $K(\Phi,\bar{\Phi}) \;=\; \Phi\bar{\Phi}$. The corresponding conformal theories arise in LG models when the renormalisation group flow goes to a conformal fixed point in the low energy limit.

An essential feature of LG models is the super-potential W, which is a *quasihomogeneous* polynomial in one or more complex variables, which are the superfields. For a quasi homogeneous polynomial $W \in \mathbb{C}[z]$, $\lambda \in \mathbb{C} \setminus 0$ and $d \in \mathbb{R}^+$

$$W(\lambda^{w_i} z_i) \;=\; \lambda^d W(z_i) \;. \qquad (2.12)$$

We call w_i the weights of the variables in W. Each x_i has weight w_i. This weight is referred to as the R-charge, a concept which can be extended to matrices as well as monomials and quasihomogeneous polynomials. In LG models we set the weight of W, (d), to 2 which can be achieved by a rescaling of the w_i. We can then compute the value of the central charge of the CFT the LG model must flow to, from the R-charges (weights) of the fields in the super-potential according to the formula [26, 71].

$$c \;=\; 3\sum_i (1-w_i) \;. \qquad (2.13)$$

These polynomials can be seen as the zero locus in a projective complex space and there is correspondence with LG models and non-linear sigma models in Calabi-Yau projective spaces [97, 71]. They have to have the property of being an isolated singularity and have been classified by Arnold[1]. The definition of non-degenerate isolated singularity W is

1) All partial derivatives are zero at the origin $\partial_{z_i} W|_{z=0} = 0 \;\forall i$.

2) The origin is the only such critical point.

These singularities or potentials have been classified according to the resolution of the singularity via a process called *blowing-up* [76]. This involves a re-parameterisation to two sets of variables describing the projective space and also by the introduction of a projective line or S_2 hence the term blow-up. Each step in the resolution introduces a new blow-up and the sequence of blow-up operations gives rise to a Dynkin diagram, which is then used to give these polynomials an ADE classification[1, 84, 76], (Appendix D).

$N = 2$ Landau-Ginzburg models in two dimensions are not conformal theories but there is strong evidence that they flow to infra-red conformal fixed points under renormalisation. Meanwhile the super-potential, W, is protected by supersymmetry during RG flow, hence it does not vary and so contains topological information on the RG fixed point.[2]

It is important that these MFs are not just postulated but that the MF condition (1.1) in terms of two fields E and J arises as a natural consequence of preservation of supersymmetry at a boundary [9]. Demanding the preservation of supersymmetry at a boundary produces constraints on these and gives the result that the super-potential W must be expressible as the product of two commuting matrix factors [9, 51]. The derivation of the MF condition from supersymmetry constraints on the boundary is outlined in appendix A.

If we consider our potential W to be a diagonal $W\mathbb{I}_{2N}$, for some rank R then this can be a MF. It is from these super-potentials that we can generate a MF and the corresponding Q. This allows us to compute cohomologies and to consider topological open/closed string theory with boundary conditions or *branes*. We can thus have two descriptions of a TFT, one from the CFT side and one from the MF side, and they can be compared.

From the CFT side, topological models were achieved by a transformation of the stress tensor which is called the *topological twist* [95, 69, 80, 51]. These models are topological because we can define a BRST operator which can project out the states which are independent of the metric and hence topological. In the case of MFs, a basis for the chiral primary states on the boundary is given by the cohomology of a BRST type operator, d_Q constructed from the MF, Q. The construction of the operators and their cohomology is covered in the next section. The super-potential essentially defines a topological field theory representing the vacuum structure of open strings between D-branes.

The MF condition leads to an operator, d_Q, whose cohomology has also been shown to match the boundary chiral primaries of the corresponding CFT. Evidence for this correspondence has been found in many cases [9, 11, 48, 50, 58, 60]. The evidence consists in matching the dimensions of the cohomological basis, and in some cases the R-charge and operator algebra with the boundary fields in the minimal models. The ADE classification appears on both sides of the correspondence between MFs and minimal models. On the side of MFs it is through classification of the potentials [1], and on the CFT minimal model side it is due to modular symmetries of the partition function of that model [38]. We note that not all Landau-Ginzburg potentials have an associated Dynkin diagram gen-

[2] It is far from obvious how such information can be obtained from the super-potential. The category theory identification between the two categories, the derived category of coherent sheaves, concerned with cohomologies on the CFT side and the category of MFs, was conjectured by Orlov, and Kontsevich suggested this provides the connection between the topologically invariant chiral primaries on the boundary from the CFT side and the cohomology of a BRST invariant operator, Q, constructed to square to the super-potential of the LG model, via the category of MFs.

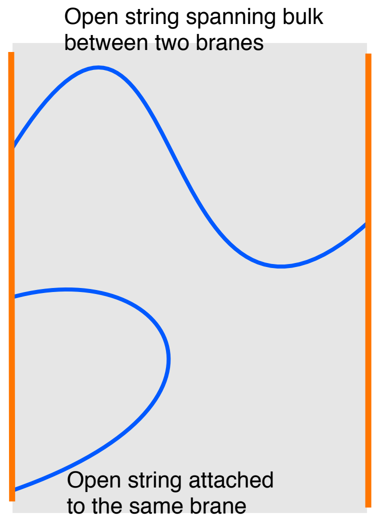

Figure 3: Open string picture

erated from the resolution of singularities. We will also look at some of these examples in the next section.

We can define the category $MF(W)$ for any potential W and consider the topological open string picture which is that of open strings attached to branes. See fig 3. We then have that the matrix factorisations (ie. the objects) are the branes (boundary conditions), and the elements of the cohomology (ie. the morphisms) are open string states (boundary fields). We denote the cohomology H^i_Q.[3] of d_Q. W is the potential governing the bulk fields existing between branes.

The chiral primaries in the bulk form a chiral ring, as the product of two chiral primaries is still a chiral primary. This is the Jacobi ring, $\mathcal{R} = \text{Jac}(W)$, satisfying the relation [69],

$$\text{Jac}(W) \quad = \quad \mathbb{C}[z]/(\partial_{z_1}W, \ \dots \ , \partial_{z_n}W) \ . \tag{2.14}$$

and bulk correlators for $\phi_1, \dots\phi_m \in \text{Jac}(W)$ are given by [69],

$$\langle \phi_1 \dots \phi_m \rangle_W \quad = \quad \text{res}_z \Big[\frac{\phi_1 \dots \phi_m}{(\partial_{z_1}W, \ \dots \ , \partial_{z_n}W)} \Big] \tag{2.15}$$

where n is the number of superfields. This is the basic correlator. Since it is computed as a residue we can see that only certain fields in the product with a certain *weight* or R-charge will contribute to a non-zero residue.

Additional structure has to appear if the worldsheet of the LG model has a boundary. The

[3] For construction see section 3.1. Note there are two cohomologies H^1_Q consisting of odd morphisms and H^0_Q consisting of even morphisms.

possible supersymmetry-preserving boundary conditions are the matrix factorisations Q of W [58, 9, 57, 48], and the degrees of freedom on the boundary are given by the cohomology H_Q which will also consist of rank $2N$ matrices. Correlation functions in topological LG theory are computed as residues of functions of several complex variables (see e.g. [43] for details), therefore in a model where the worldsheet has a boundary with boundary fields we must take the supertrace of operator products. For any given MF it has been shown that the correlator must take the form [58, 48];

$$\langle\, \phi\, \psi\, \rangle_Q^{\text{KapLi}} = \text{res}_z \left[\frac{\phi\, \text{str}\big(\partial_{z_1} Q \cdots \partial_{z_k} Q\ \psi\big)}{\partial_{z_1} W \cdots \partial_{z_k} W} \right] \tag{2.16}$$

for any bulk field $\phi \in \text{Jac}(W)$ and any boundary field $\psi \in H_Q$. The supertrace is defined using the $\mathbb{Z}_2$-grading matrix $\sigma = \text{diag}(1_N, -1_N)$, as $\text{str}(A) := \text{tr}(\sigma\, A)$. We notice the sub-index Q in this case. Although the super-potential is the same there can be different MFs of the same potential, moreover they may be of different rank. Different Q represent different branes.

The form (2.16) is often referred to as Kapustin-Li formula [58, 48]. A closely related expression will be used to define the *quantum dimension*, itself a correlator and property of the MFs we are interested in for the confirmation of orbifold equivalence.

The correlation functions above were first computed in physics, via localisation of path integrals for supersymmetric topological quantum field theories [46], but they have since been discussed in purely mathematical terms, notably in [73, 32]. It is clear that because the form is a residue, once we know the super-potential that any bulk field must be in the Jacobi ring but we will see that when working with graded MFs this form is also highly restrictive. We make the quick comment that all the indecomposable MFs in $MF(W_{ADE})$ i.e. for ADE classified potentials, are graded, as any MFs must be if they are from a category $MF(W)$ which only has discrete indecomposable objects. For a definition of *graded* see Section 4.2.

In order to better understand the basis on which the correspondence between minimal models in CFT and LG models stands, at least on the MF side, we have to see how the chiral primaries are computed. This also gives us a chance to review the ADE MFs and look at some further examples. The models which are ADE classified all have a finite set of indecomposable MFs. This means that all MFs of these potentials are either similarity transformations of these or of direct sums of these, or both.

3 Matrix factorisations

We have seen how MFs arise and their category theory setting, and the form of the correlation function for BRST invariant objects (which is constructed using derivatives of Q). There are several other mathematical aspects which are relevant to the later sections. This section is a collection of interesting, and for our discussion, relevant facts about MFs. Section 3 can be seen as a tool-kit, giving us the necessary technology for exploring orbifold equivalences which we do in Section 4.

MFs have their own interesting algebra of tensor products and direct sums which we will look at. We discuss *tachyon condensation* [18] and also briefly mention *deformations* and *obstructions* [31, 19]. Finally since every potential defines a category of MFs we look at the nature of potentials as quasi-homogenous polynomials and the types of polynomial possible for such potentials and give some background.

Firstly MFs are algebraic objects from which it is possible to construct a derivative and BRST type operator d_Q, which has a cohomology. The category of MFs for a particular potential, $MF(W)$, can include several MFs each with its cohomology and we give the definition of the cohomologies of a MF, before going on to look at the examples which have been shown to confirm the correspondence.

3.1 Cohomology of the d_Q operators

The matrix factorisation condition 1.2 can be used to construct a BRST type operator mapping boundary fermions to boundary bosons and visa versa. In conventional 4d QFT and string theory the cohomology of BRST operators, i.e. states that are closed but not exact give us the 'physical' states. In this case the cohomology of such operators gives us a basis for the chiral ring. i.e. the chiral primaries of the theory. Consider Q defined in (1.1),

$$Q \;=\; \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} , \tag{3.1}$$

with $Q^2 \;=\; W\mathbb{I}_{2N}$. Using Q, we can define the following operators by the transformations:

$$\begin{aligned} d_Q^0 \;&:\; M^B \to M^F \mid d_Q^0(B) \;=\; QB - BQ \\ d_Q^1 \;&:\; M^F \to M^B \mid d_Q^1(A) \;=\; QA + AQ \end{aligned} \tag{3.2}$$

where M^B and M^F are

$$M^B := \{\; all\ 2N \times 2N\ matrices\ of\ the\ form \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \}$$

$$M^F := \{ \text{ all } 2N \times 2N \text{ matrices of the form } \begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix} \} \tag{3.3}$$

It is easy to see that the operators d_Q^0 and d_Q^1 satisfy

$$d_Q^2 \ = \ d_Q^0.d_Q^1 \ = \ d_Q^1.d_Q^0 \ = \ 0 \ . \tag{3.4}$$

Consider $A \in M_F$, then

$$\begin{aligned} d_Q^0.d_Q^1 A \ &= \ [Q, \{Q, A\}] \\ &= \ Q^2 A + QAQ - QAQ - AQ^2 \\ &= \ [W\mathbb{I}_{2N}, A] \ = \ 0 \ . \end{aligned} \tag{3.5}$$

d_Q^2 acts similarly on M^B. This d_Q is called a twisted differential and we can look at the cohomologies.

We define the following cohomologies H^{even} and H^{odd} by,

$$\begin{aligned} H^{even} : \ &= \ Ker \ d_Q^0 / Im \ d_Q^1 \\ H^{odd} : \ &= \ Ker \ d_Q^1 / Im \ d_Q^0 \end{aligned} \tag{3.6}$$

H^{even} is the bosonic cohomology and H^{odd} represents the fermions. They are also referred to in the literature as H^0 and H^1 respectively. The operator d_Q^0 acts on states in the bosonic cohomology H^{even} by sending them to zero, but these states must not be in the Image of d_Q^1. Likewise states not in the image of d_Q^0 but in the *kernel* of d_Q^1 are in the fermionic cohomology H^{odd}. In the case of fermions/bosons we speak of odd/even morphisms resp. and we note Q itself is an odd morphism.

The cohomolgy in each case consists of rank $2N$ matrices. These can be expressed in a basis and we can compute the dimension of this basis.

3.2 $MF(W_{ADE})$ for minimal models

The evidence confirming the correspondence between boundary chiral primary states for minimal models and the cohomology of d_Q came from the study of MFs of the ADE polynomials. These are the LG models which can be compared to known conformal field theories.

On the CFT side of the correspondence, the partition function for minimal models has an ADE classification according to group symmetries, and on the LG/MF side the potential W, is also represented by an ADE classification, according to singularity theory. In the

LG case this is through the resolution of the singularity [1] via a process called *blowing-up* [76], which in turn generates a *Dynkin* or root diagram. An example of how the Dynkin diagram arises is given in Appendix D and further ADE examples are given in [76]. These seemingly unrelated structures, which share a classification, have been shown to have a correspondence through TFT and category theory. The ADE structure is present in both, but the identification of the MF condition with branes is not straightforward or obvious from the derivation of the ADE classification in each case.

The spectrum of boundary chiral primaries had been computed for the ADE, $N = 2$ superconformal models. The connection with corresponding states was then made with the cohomology H_Q of the BRST operator d_Q constructed from a matrix factorisation of the LG potential W. The match is made through the dimensions of the bases and R-charges for the representation of chiral primaries in each case [9, 10, 48, 50, 58]. In some cases the OPE structure was shown to match. The two theories contain the same information so for all intents and purposes we have two complementary descriptions of the same theory.

The ADE series of minimal models and the ADE polynomials correspond to a variety of possible potentials and the set of objects in any $MF(W)$ can look very different for different potentials, so we will give some important examples. The MFs of these models, for any specific potential, form a finite set of indecomposable MFs. Any other MFs are either direct sums, similarity transformations or both, of these basic MFs (indecomposable objects) for each MF category of type $MF(W_{ADE})$. This makes them easy to classify and describe.

For each LG model, one can consider two GSO[4] projections; at the level of TQFT, in particular of MFs, one projection is obtained by using the potential W, the other by adding a term z^2 to W (where z is an extra variable not present before). We will see that the possible MF for different GSO projections form different indecomposable sets. From the singularity theory point of view the potentials considered can be with or without an extra quadratic variable. For example the potentials $W = y^4 + x^3$ and $W = y^4 + x^3 \pm z^2$ are equivalently represented by E_6 Dynkin diagrams [84, 56]. They involve the same Dynkin diagram for the same resolution and have the same associated central charge (2.13). Sometimes in the literature this extra term is represented by the product of two extra complex variables, e.g. uv [84, 56]. We will see these potentials typically consist of up to three complex variables including the quadratic term as these have been classified but there is no limit and potentials involving more variables have been studied in connection with LG models [91], and are looked at in connection with orbifold equivalence.

For different GSO projections the number of indecomposable MFs in each case can be different. A good example is E_7. In the GSO projection with the added z^2 term there are seven indecomposable MFs. Without the extra term there are nine. The dimension

[4]GSO named after Ferdinando Gliozzi, Jol Scherk, and David I. Olive describes two possible projections preserving modular invariance in CFT resulting in two different string theories and in our case refers two the different MF categories with or without an extra quadratic term.

of the cohomological basis, or number of bosons and fermions are also different from each other, in some of the ADE models without the z^2 term [31].

We now present some examples and give the cohomological dimensions of these models and do a simple computation for the cohomology in the simplest case. This sort of computation is usually best left to Singular. We present A and D-models without the extra z^2 term and present the matrices for the simplest example, E_6 of the three exceptional singularities or E-models. We also give the dimensions of the cohomological basis in each example.

A models

$$A_{n-1}: \quad W \;=\; x^n \quad \text{or} \quad =\; x^n + z^2\ . \tag{3.7}$$

The A-model has only linear MF's for indecomposable MFs. We consider A-model potentials of the form $W = x^n,\ \ n \;=\; 2,\ 3,\ 4, ...$ This has $n-2$, non trivial factorisations of the form

$$J \;=\; x^{n-l} \qquad E \;=\; x^l \qquad l \;=\; 1,\ 2, ..., n-1 \tag{3.8}$$

$$Q \;=\; \begin{pmatrix} 0 & x^{n-l} \\ x^l & 0 \end{pmatrix} \tag{3.9}$$

We will use this simplest example to compute the cohomologies by hand

$$\begin{aligned} H^{even}: \;&=\; Ker\ d^0_Q / Im\ d^1_Q \\ H^{odd}: \;&=\; Ker\ d^1_Q / Im\ d^0_Q \end{aligned} \tag{3.10}$$

where

$$\begin{aligned} Ker\ d^0_Q \;&=\; B_1, B_2 \mid \quad B_1 \;=\; B_2 \\ Ker\ d^1_Q \;&=\; A_1, A_2 \mid \quad A_1 \;=\; -x^{(n-2l)} A_2 \\ Im\ d^0_Q \;&=\; \tilde{A}_1, \tilde{A}_2 \mid \ \tilde{A}_1 \;=\; x^{n-l} B_2 - B_1 x^{n-l}, \quad \tilde{A}_2 \;=\; x^l B_1 - B_2 x^l \\ Im\ d^1_Q \;&=\; \tilde{B}_1, \tilde{B}_2 \mid \quad \tilde{B}_1 \;=\; x^{n-l} A_2 + A_1 x^l \;=\; \tilde{B}_2 \end{aligned} \tag{3.11}$$

so

$$\begin{aligned} H^{even}: \;&=\; \begin{pmatrix} x^m & 0 \\ 0 & x^m \end{pmatrix}\ ,\ 0 \le m \le min[n-l, l] \\ H^{odd}: \;&=\; \begin{pmatrix} 0 & x^r \\ -x^{r+2l-n} & 0 \end{pmatrix}\ ,\ 0 \le\ r \le\ n-l\ . \end{aligned} \tag{3.12}$$

D models

$$D_n: \quad W \;=\; x^{n+1} + xy^2 \quad \text{or} \quad = \; x^{n+1} + xy^2 + z^2 \; . \tag{3.13}$$

A more interesting model for which the correspondence has been verified is he D-model which is another family of potentials depending on an integer n [11]. We will consider the potential W in two complex variables given by $W = x^{n+1} - xy^2$, $n \in \mathbb{N}$. There are two types of rank two MFs [11], each of these types is classified by another integer $l = 1, 2, ..., n-1$ as can be seen in the examples below.

First we have the $\mathcal{S}$-type MF,

$$ES_l = \begin{pmatrix} x^{n+1-l} & xy \\ -y & -x^l \end{pmatrix} , \qquad JS_l = \begin{pmatrix} x^l & xy \\ -y & -x^{n+1-l} \end{pmatrix} , \tag{3.14}$$

Then the $\mathcal{T}$-type,

$$ET_l = \begin{pmatrix} x^{n+1-l} & xy \\ -xy & -x^{l+1} \end{pmatrix} , \qquad JT_l = \begin{pmatrix} x^l & y \\ -y & -x^{n-l} \end{pmatrix} . \tag{3.15}$$

The dimensions of these can be computed using Singular, and are tabulated here

n	type	l=1	l=2	l=3	l=4	l=5	l=6	l=7
3	$\mathcal{S}$	(2,2)	(4,4)	(2,2)				
	$\mathcal{T}$	(4,2)	(4,2)	(2,0)				
4	$\mathcal{S}$	(2,2)	(4,4)	(4,4)	(2,2)			
	$\mathcal{T}$	(4,2)	(6,4)	(4,2)	(2,0)			
5	$\mathcal{S}$	(2,2)	(4,4)	(6,6)	(4,4)	(2,2)		
	$\mathcal{T}$	(4,2)	(6,4)	(6,4)	(4,2)	(2,0)		
6	$\mathcal{S}$	(2,2)	(4,4)	(6,6)	(6,6)	(4,4)	(2,2)	
	$\mathcal{T}$	(4,2)	(6,4)	(6,4)	(6,4)	(4,2)	(2,0)	
7	$\mathcal{S}$	(2,2)	(4,4)	(6,6)	(8,8)	(6,6)	(4,4)	(2,2)
	$\mathcal{T}$	(4,2)	(6,4)	(6,4)	(8,6)	(6,4)	(4,2)	(2,0)

where the pair (a, b) gives the bosonic and fermonic dimensions.

Then there is also the rank one $\mathcal{R}$-type MF

$$JR = x^{n-1} + y^2 \; , \qquad ER = x \tag{3.16}$$

This matrix factorisation has dimensions $(2, 0)$ for any value of n.

For n *even* we also have the rank one MFs

$$JR^+ = x^{n/2} + y \ , \qquad ER^+ = x(x^{n/2} - y) \tag{3.17}$$

and

$$JR^- = x(x^{n/2} + y) \ , \qquad ER^- = x^{n/2} - y \ . \tag{3.18}$$

The dimensions of the basis for bosonic and fermonic states in this case do vary with n.

$n =$	2	4	6	8	10
$\mathcal{R}^+$	(2,0)	(3,0)	(4,0)	(5,0)	(6,0)
$\mathcal{R}^-$	(2,0)	(3,0)	(4,0)	(5,0)	(6,0)

In summary the $\mathcal{S}$-type has equal (even) numbers of bosons and fermions, the $\mathcal{T}$-type always has two more bosons than fermions and the rank one MFs have no fermions. In addition $\mathcal{R}$-type always has 2 bosons for any value of n but the $\mathcal{R}^+$ and $\mathcal{R}^-$ have the number of bosons increasing with $= n/2 + 1$ and no fermions.

E models

The final three models for which the correspondence has been confirmed are the three exceptional models [60]. These are individual models and not part of a series as in the A and D case.

$$E_6: \quad W \ = \ x^3 + y^4 \quad \text{or} \quad = \ x^3 + y^4 + z^2 \ . \tag{3.19}$$

For the potential $W = y^4 + x^3 - z^2$, there are six basic MFs as outlined in [27]. We repeat them here. Note instead of E and J we use F and G to label the two matrix factors to avoid confusion with the label for the model E_6

.

$$F_1 = \begin{pmatrix} iz & 0 & x^2 & y^3 \\ 0 & -iz & y & -x \\ x & y^3 & iz & 0 \\ y & -x^2 & 0 & iz \end{pmatrix} \qquad G_1 = \begin{pmatrix} iz & 0 & x^2 & y^3 \\ 0 & -iz & y & -x \\ x & y^3 & iz & 0 \\ y & -x^2 & 0 & iz \end{pmatrix} \ .$$

dimension (4,4).

$$F_2 = \begin{pmatrix} z & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & z & 0 & 0 & 0 & x \\ 0 & 0 & z & -x & 0 & y \\ 0 & xy & -x^2 & z & y^3 & 0 \\ x & 0 & 0 & y & z & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & z \end{pmatrix} \quad G_2 = \begin{pmatrix} -z & -y^2 & xy & 0 & x^2 & 0 \\ -y^2 & -z & 0 & 0 & 0 & x \\ 0 & 0 & -z & -x & 0 & y \\ 0 & xy & -x^2 & -z & y^3 & 0 \\ x & 0 & 0 & y & -z & 0 \\ 0 & x^2 & y^3 & 0 & xy^2 & -z \end{pmatrix}$$

dimension (12,12).

$$F_3 = \begin{pmatrix} -y^2 - z & 0 & xy & x \\ -xy & y^2 - z & x^2 & 0 \\ 0 & x & -z & y \\ x^2 & -xy & y^3 & -z \end{pmatrix} \quad G_3 = \begin{pmatrix} -y^2 + z & 0 & xy & x \\ -xy & y^2 + z & x^2 & 0 \\ 0 & x & z & y \\ x^2 & -xy & y^3 & z \end{pmatrix} ,$$

dimension (6,6).

$$F_4 = \begin{pmatrix} -y^2 + z & 0 & xy & x \\ -xy & y^2 + z & x^2 & 0 \\ 0 & x & z & y \\ x^2 & -xy & y^3 & z \end{pmatrix} \quad G_4 = \begin{pmatrix} -y^2 - z & 0 & xy & x \\ -xy & y^2 - z & x^2 & 0 \\ 0 & x & -z & y \\ x^2 & -xy & y^3 & -z \end{pmatrix} .$$

dimension (6,6).

$$F_5 = \begin{pmatrix} -y^2 - z & x \\ x^2 & y^2 - z \end{pmatrix} \quad G_5 = \begin{pmatrix} -y^2 + z & x \\ x^2 & y^2 + z \end{pmatrix} .$$

dimension (2,2).

$$F_6 = \begin{pmatrix} -y^2 + z & x \\ x^2 & y^2 + z \end{pmatrix} \quad G_6 = \begin{pmatrix} -y^2 - z & x \\ x^2 & y^2 - z \end{pmatrix} . \tag{3.20}$$

dimension (2,2).

We have written these down only for E_6 , with $W_2 = x^3 + y^4 + z^2$. $W_1 = x^3 + y^4$ has the same number of indecomposable objects but it should be noted that for E_7 and for E_8 these numbers are not the same and that in general different GSO projections can have different numbers of indecomposable matrix factorisations each with its own particular cohomology. It is true for all potentials, not just the ADE series, that there are two forms, one with an additional term z^2, and one without, where z is an extra variable not necessarily occurring in W.

The other two E models have have also contributed as important examples of the correspondence [60, 27, 31]. We just give the polynomials here as they have a similar array of indecomposable objects (MFs of different rank) and morphisms (cohomologies).

$$E_7: \quad W = x^3 + xy^3 \quad \text{or} \quad = x^3 + xy^3 + z^2 . \tag{3.21}$$

$$E_8: \quad W = x^3 + y^5 \quad \text{or} \quad = x^3 + y^5 + z^2 . \tag{3.22}$$

When comparing or counting the MFs in a particular category we see that for some MFs the two matrix factors are isomorphisms of each other, for example F_1 and G_1 in the E_6 model above. In other cases they are not and then the two MFs represent a brane anti-brane pair as with (F_5, G_5) and (F_6, G_6).

3.3 Parameter families

The elliptic curve:

The elliptic curve or cubic curve most often studied in connection with MFs is given by

$$W \;=\; x^3 + y^3 + z^3 - dxyz \quad , \quad d \in \mathbb{C} \ . \tag{3.23}$$

The zero locus of this curve describes a torus and different values of the parameter d belong to geometrically inequivalent tori. There is a sixfold symmetry isomorphism expressed in terms of a J-invariant for the elliptic curve as a torus with complex parameter τ.[5] More on parameterisations of the elliptic curve can be found in §4.10. There have been several studies of the MFs for the elliptic curve [12, 42, 67], at rank 2 and rank 3 there are no discrete objects in $MF(W_d)$. Instead there are parameter families of MFs, or indecomposable objects. A fascinating aspect is that the parameter space is itself subject to the zero curve condition for the same elliptic curve, i.e. same d. Different points on this parameter space are in general non-isomorphic (see next section).

The elliptic curve potential is a quasi homogeneous polynomial with central charge $c = 3$ and all variables have the same weight. The parameterisation is the same for the two distinct types, at rank 2 and at rank 3 MF [12, 42]. The rank 2 MF is given by,

$$E \;=\; \begin{pmatrix} Q_1 & -Q_2 \\ L_2 & L_1 \end{pmatrix} \qquad\qquad J \;=\; \begin{pmatrix} L_1 & Q_2 \\ -L_2 & Q_1 \end{pmatrix} \ . \tag{3.24}$$

where linear terms are given by

$$\begin{aligned} L_1 \;&=\; \alpha_3 x_1 - \alpha_2 x_3 \\ L_2 &= -\alpha_3 x_2 + \alpha_1 x_3 \ , \end{aligned} \tag{3.25}$$

and quadratic terms are

$$\begin{aligned} Q_1 \;&=\; \frac{1}{\alpha_1\alpha_2\alpha_3}(\alpha_1\alpha_2 x_1^2 + \alpha_2^2 x_1 x_2 - \alpha_1^2 x_2^2 - \alpha_1\alpha_3 x_3^2) \\ Q_2 &= \frac{1}{\alpha_1\alpha_2\alpha_3}(\alpha_2^2 x_1^2 - \alpha_1^2 x_1 x_2 - \alpha_1\alpha_2 x_2^2 + \alpha_3^2 x_1 x_3) \ . \end{aligned} \tag{3.26}$$

The rank 3 MF has a much more 'symmetric' appearance.

$$E \;=\; \begin{pmatrix} \alpha_1 x_1 & \alpha_2 x_3 & \alpha_3 x_2 \\ \alpha_3 x_3 & \alpha_1 x_2 & \alpha_2 x_1 \\ \alpha_2 x_2 & \alpha_3 x_1 & \alpha_1 x_3 \end{pmatrix}$$

[5] The two periods of a torus can be seen as a grid on the complex plane described by one complex parameter, [8].

$$J = \begin{pmatrix} \frac{1}{\alpha_1}x_1^2 - \frac{\alpha_1}{\alpha_2\alpha_3}x_2x_3 & \frac{1}{\alpha_3}x_1^2 - \frac{\alpha_3}{\alpha_1\alpha_2}x_1x_2 & \frac{1}{\alpha_2}x_2^2 - \frac{\alpha_2}{\alpha_1\alpha_3}x_1x_3 \\ \frac{1}{\alpha_2}x_2^2 - \frac{\alpha_2}{\alpha_1\alpha_3}x_1x_3 & \frac{1}{\alpha_1}x_1^2 - \frac{\alpha_1}{\alpha_2\alpha_3}x_2x_3 & \frac{1}{\alpha_3}x_1^2 - \frac{\alpha_3}{\alpha_1\alpha_2}x_1x_2 \\ \frac{1}{\alpha_3}x_1^2 - \frac{\alpha_3}{\alpha_1\alpha_2}x_1x_2 & \frac{1}{\alpha_2}x_2^2 - \frac{\alpha_2}{\alpha_1\alpha_3}x_1x_3 & \frac{1}{\alpha_1}x_1^2 - \frac{\alpha_1}{\alpha_2\alpha_3}x_2x_3 \end{pmatrix} . \qquad (3.27)$$

In both cases the $\alpha' s$ have to satisfy the elliptic equation

$$\alpha_1^3 + \alpha_2^3 + \alpha_3^3 - d\alpha_1\alpha_2\alpha_3 = 0 \ . \qquad (3.28)$$

In the literature, the rank 3 MFs are refered to as long branes and the rank 2 as short branes [12, 42]. d is a parameter related to the standard modulus of the torus, the complex parameter τ. The coefficients for all different possible MFs lie on the surface of the same torus. The rank 3 MF has cohomological dimensions $(4, 4)$ and the rank 2 has dimension $(2, 2)$.

The quartic:

The quartic is given by

$$W = x^4 + y^4 \ . \qquad (3.29)$$

This potential has been investigated in [31]. It was part of a program of deformations which modelled a process to move between objects in $MF(W)$. An interesting result of this research was, in the case of the quartic, to turn up a one parameter family[6] of non-isomorphic MFs for the quartic curve. The parameterisation MFs are given by

$$Q(u) = \begin{pmatrix} 0 & 0 & y^2 & x - \omega u y \\ 0 & 0 & -x^3 - \omega u x^2 y & iu^2x^2 - y^2 \\ iu^2x^2 - y^2 & -x + \omega u y & 0 & 0 \\ x^3 + \omega u x^2 y & y^2 & 0 & 0 \end{pmatrix} . \qquad (3.30)$$

with $\omega = \epsilon^{\pi i/4}$.

This potential also has an associated central charge $c = 3$, just like the elliptic curve, and two variables of the same weight.

The cubic and quartic parameter families appear at $c = 3$ and all the known categories of MFs for lower values of central charge encountered have indecomposable objects. It would seem that either there are enough monomials of a low enough weight or c is big

[6] It seems as there are two different parameterisations but it has not been possible to show they are not parameterisations of isomorphic MFs at different parameter values.

enough. Another option is that these non-isomorphic parameter families could be a result of the variables in these matrix factorisations having the same weight.

In section 4 we do consider potentials for values of $c > 3$ in connection with orbifold equivalence but $MF(W)$ has not been studied in detail for these potentials and it may be interesting to ask if parameter families exist in this region of central charge, and for what sort of potentials. Sometimes we will work in units of $\hat{c} = c/3$.

3.4 Similarity transformations

The individual categories of ADE MFs were constructed from a finite number of indecomposable objects *up to isomorphism*. We describe equivalence of MFs here by considering similarity transformations. In the context of MFs, equivalence of two MFs (E, J) and $(\tilde{E}, \tilde{J})$, via row and column transformations, means that there exist two invertible matrices S and T such that

$$SJT^{-1} \;=\; \tilde{J} \;, \quad \text{and} \quad TES^{-1} \;=\; \tilde{E} \;. \tag{3.31}$$

MFs which satisfy these relations are called isomorphic. To show that the MFs after such row and column transformations are equivalent, first consider

$$Q \;=\; \begin{pmatrix} 0 & J \\ E & 0 \end{pmatrix} \;, \qquad E, J \in M_N(\mathbb{C}[\mathbf{x}]) \tag{3.32}$$

with

$$Q^2 \;=\; W(\mathbf{x})\mathbb{I}_{2N} \;. \tag{3.33}$$

If

$$U \;=\; \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} \;, \qquad S, T \in M_N(\mathbb{C}[\mathbf{x}]) \;, \text{ invertible} \tag{3.34}$$

then

$$\tilde{Q} \;=\; UQU^{-1} \tag{3.35}$$

and represents a factorisation of $W(\mathbf{x})$.

$$\tilde{Q}^2 \;=\; UQU^{-1}UQU^{-1} \;=\; UW(\mathbf{x})\mathbb{I}_{2N}U^{-1} \;=\; W(\mathbf{x})\mathbb{I}_{2N} \;=\; Q^2$$

Given the transformations, $M^B \to \tilde{M}^B \;=\; UM^BU^{-1}$, and $M^F \to \tilde{M}^F \;=\; UM^FU^{-1}$ we have

$$d^0_{\tilde{Q}}\tilde{M}^B \;=\; Ud^0_Q M^B U^{-1} \tag{3.36}$$

and

$$d^1_{\tilde{Q}}\tilde{M}^F \;=\; U d^0_Q M^F U^{-1}\ . \tag{3.37}$$

Since our cohomologies are expressed as subsets of the Kernel (closed forms which are not exact), we have $H^i_{d_{\tilde{Q}}} \;=\; UH^i_{d_Q}U^{-1},\ i \;=\; 0,1$.

So the two matrix factorisations are equivalent from the point of view of cohomologies and our new $\tilde{Q}$ is explicitly given by

$$\tilde{Q} \;=\; \begin{pmatrix} 0 & SJT^{-1} \\ TES^{-1} & 0 \end{pmatrix}\ , \qquad E, J \in M_N(\mathbb{C}[[\mathbf{x}]])\ . \tag{3.38}$$

We have shown equivalence of MFs by such similarity transformations.

Even though the order of E and J does not matter in the MF condition 1.1, as an MF the order of the pair (E, J) does matter. We say the pair (E, J) is the anti-brane to (J, E). If we have the case that E is isomorphic to J then there is no difference between the pairs (E, J) and (J, E) and the MF is its own anti-brane.

3.5 Direct sums

We can make direct sums of indecomposable objects from the same category $MF(W)$. This means we can only consider $Q_1 \oplus Q_2$ if both Q_1 and Q_2 are MFs of the same potential W. MFs of the same potential are interpreted as representing different branes and the cohomologies between direct sums represents not only open strings starting and ending on the same D brane but also strings stretched between branes. Given two different MFs representing two branes,

$$Q_1 \;=\; \begin{pmatrix} 0 & J_1 \\ E_1 & 0 \end{pmatrix}\ , \quad Q_2 \;=\; \begin{pmatrix} 0 & J_2 \\ E_2 & 0 \end{pmatrix}\ , \tag{3.39}$$

we construct the direct sum of these MFs, $Q_1 \oplus Q_2$, One possible arrangement is given by

$$Q \;=\; Q_1 \oplus Q_2 \;=\; \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} \;=\; \begin{pmatrix} 0 & J_1 & 0 & 0 \\ E_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & J_2 \\ 0 & 0 & E_2 & 0 \end{pmatrix} \tag{3.40}$$

The direct sum can be written in different ways. We note that each Q_i is 'fermionic'. Q_1 a $2n \times 2n$ matrix and Q_2 a $2m \times 2m$ matrix. For each vector space M_{2n} or M_{2m} on which each of these Q_i act we can define a matrix σ_i which gives us a $\mathbb{Z}_2$ grading for each of our vector spaces,

$$\sigma_i \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\ , \tag{3.41}$$

where each diagonal block represents a rank n or m identity matrix depending on i. 'Bosonic' elements commute with these σ_i and fermionic elements anti-commute. We can then define a total matrix σ which gives us a $\mathbb{Z}_2$ grading for M_{2n+2m} by

$$\sigma \;=\; \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \tag{3.42}$$

We can look at the action of the matrices $d^0_Q \;=\; d^0_{Q_1\oplus\, Q_2}$ and $d^1_Q \;=\; d^1_{Q_1\oplus Q_2}$.

$$d^0_Q \;=\; QM_b - M_bQ \qquad\qquad d^1_Q \;=\; QM_f + M_fQ\ . \tag{3.43}$$

In the above MFs there are matrix terms involving E_i and J_i with i the same, both either 1 or 2. These terms are the same as they would be for the individual MF, 1 or 2. There is nothing new in the cohomology here. They represent open strings starting and ending on the same brane.

$$d^0_{Q_1Q_1} \;=\; Q_1M_b - M_bQ_1 \qquad\qquad d^1_{Q_1Q_1} \;=\; Q_1M_f + M_fQ_1\ . \tag{3.44}$$

or

$$d^0_{Q_2Q_2} \;=\; Q_2M_b - M_bQ_2 \qquad\qquad d^1_{Q_2Q_2} \;=\; Q_2M_f + M_fQ_2\ . \tag{3.45}$$

There are also terms mixing E and J with different indices, i.e. from different branes and so representing strings streched between branes.

$$d^0_{Q_1Q_2} \;=\; Q_1M_b - M_bQ_2 \qquad\qquad d^1_{Q_1Q_2} \;=\; Q_1M_f + M_fQ_2\ . \tag{3.46}$$

We can consider another form for σ which allow us to see more easily which terms represent strings stretched between two different branes (two different matrix factorisations of the same potential). We can arrange our direct sum, via row and column transformations so it is in the form we are used to from the individual σ_i (3.41). Let

$$\sigma \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.47}$$

Here each entry in the above matrix is a $n+m$ square block. For this to work we must re-arrange our E_1, J_1, E_2, J_2 entries in Q (via row and column transformations). There are a number of different ways we can define a (fermonic) Q, for instance we can define Q as

$$Q = \begin{pmatrix} 0 & 0 & J_1 & 0 \\ 0 & 0 & 0 & J_2 \\ E_1 & 0 & 0 & 0 \\ 0 & E_2 & 0 & 0 \end{pmatrix} , \tag{3.48}$$

but we can also have an anti-diagonal arrangement,

$$Q = \begin{pmatrix} 0 & 0 & 0 & J_1 \\ 0 & 0 & J_2 & 0 \\ 0 & E_2 & 0 & 0 \\ E_1 & 0 & 0 & 0 \end{pmatrix} . \tag{3.49}$$

All of these options exist and are equivalent. The last two options are recognisable as components in a tensor product. and it is quite easy to see what is happening with this arrangement. Moreover, in this case the $\mathbb{Z}_2$ grading matrix, σ, for the direct sum has the same form as the individual σ_i and our Q 'looks' more consistent as it is an odd morphism. We can use one of these forms to show the result that the cohomology of a direct sum of a non-trivial MF with a trivial one is the same as for the original non-trivial MF. See Appendix C

3.6 Tensor products of matrix factorisations

The tensor product [89, 2] allows us to build MFs from smaller units, which are also MFs. If we consider a super potential which is expressed as a sum of terms, where each term can be a function of different variables. Consider a potential, W constructed from two factorisable terms, W_1 and W_2

$$W \;=\; W_1 + W_2 \;. \tag{3.50}$$

The two potentials W_1 and W_2 may depend on disjoint sets of variables, but need not. Then if $Q_1 = (E_1, J_1)$ is a rank N MF and $Q_2 = (E_2, J_2)$ is a rank M MF of potentials W_1 and W_2 respectively, we have a special tensor product construction $\hat{\otimes}$ which gives us [2, 89]

$$Q_{TP}(W_1 + W_2) \;=\; Q_1(W_1)\hat{\otimes}Q_2(W_2) \;, \tag{3.51}$$

where

$$E \;=\; \begin{pmatrix} E_1 \otimes \mathbb{I}_M & -\mathbb{I}_N \otimes J_2 \\ \mathbb{I}_N \otimes E_2 & J_1 \otimes \mathbb{I}_M \end{pmatrix} \qquad J \;=\; \begin{pmatrix} J_1 \otimes \mathbb{I}_M & \mathbb{I}_N \otimes J_2 \\ -\mathbb{I}_N \otimes E_2 & E_1 \otimes \mathbb{I}_M \end{pmatrix} \tag{3.52}$$

and we have

$$E.J \;=\; \left(\begin{array}{cc} E_1 J_1 \otimes \mathbb{I}_M + \mathbb{I}_N \otimes J_2 E_2 & 0 \\ 0 & \mathbb{I}_N \otimes J_2 E_2 + E_1 J_1 \otimes \mathbb{I}_M \end{array} \right) . \tag{3.53}$$

Note here that if E_1 and J_1 are $n \times n$ matrices and E_2 and J_2 are $m \times m$ matrices then each block in $E.J$ is an $nm \times nm$ block. This allows us to factorise any potential by just considering it as sums of any factorizable terms. Since all (for instance ADE) potentials contain a number of terms these tensor products always exist and are easily identified as part of any set of indecomposable MFs from the ADE series.

Tensor products also exist in CFT minimal models and so have to exist on the Landau-Ginzburg matrix factorisation side for it to be a faithful description. They must also be part of the monoidal structure of the category cobordisms and closed string TFT as well as $MF(W)$. In [82], dimensions of chiral rings and RR-charges[7] were the evidence presented inferring all permutation boundary states (including Recknagel-Schomerus branes) correspond to tensor products of so-called linear matrix factorisations although there are many more such matrix factorisations than known boundary states in the associated conformal field theory.

3.7 Tachyon condensation

In the present context *tachyon condensation* [11, 18, 27, 42, 48] is given by a very simple formula, involving two MFs (branes), Q_1 and Q_2 of a potential W as well as a fermion $\psi \in H^1_{Q_1,Q_2}$ (a state of an open string stretched between the two branes); from these ingredients, one can form another MF Q_ψ of W. We call this *cone* construction. In order to represent strings between branes we need to construct the direct sum (§3.5). In general when we view a direct sum as

$$Q = \left(\begin{array}{cc} Q_1 & 0 \\ 0 & Q_2 \end{array} \right) \tag{3.54}$$

we can simply add a member of the fermionic cohomology $\psi \in H^1_{Q_1,Q_2}$. In this way we construct what is called a cone,

$$Q_\psi = \left(\begin{array}{cc} Q_1 & \psi \\ 0 & Q_2 \end{array} \right) . \tag{3.55}$$

We see that $Q_\psi^2 = W\mathbb{I}_{n+m}$ and so Q_ψ is also a MF of the potential W. $Q_\psi \in MF(W)$.

[7] RR charges are not the same as R-charges, they are the D-brane charges on the CFT side

This is the concrete MF realisation of the cone construction, which plays a role in other triangulated categories, too, e.g. in the derived category of coherent sheaves, of which details in a LG-related context can be found [4] and [9, 18].

The term tachyon condensation goes back to older ideas of Sen's and others [85, 98], who realised that a system of two branes can become unstable and flow ("condense") to a new brane configuration if a (tachyonic) open string excitation is turned on (given a vacuum expectation value) between the branes. The most famous example of tachyon condensation is in the standard model of physics and is the Higgs mechanism: a non-zero vacuum expectation value for the Higgs field implies that the theory flows to a new vacuum (where the Higgs field is no longer tachyonic) – in that case, the new vacuum in particular breaks symmetries which were present before the tachyon condensation. The feature that survives in our present TFT setting is that one can create, via the cone construction, new MFs from old ones.

In all of the ADE examples there are a finite set of indecomposable objects. Triangulated categories have a shift functor which can shift between objects within the category. It was shown in [18, 27, 56] that for the ADE cases that the whole category could be generated from a single object and the fermionic cohomology members using the method of cone construction. In this case the cone construction is slightly different in that we consider the bosonic cohomology between brane and anti-brane, but this is completely equivalent to the fermionic cohomology of a brane-brane system.

The shift functor $T : Q \to Q'$ and multiplying Q by -1, takes us to the anti-brane by swapping the factors.

$$Q = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix} \qquad \text{and} \qquad Q' = \begin{pmatrix} 0 & -J \\ -E & 0 \end{pmatrix} . \tag{3.56}$$

It is straightforward to show that the fermionic cohomology between Q_1 and Q_2 is the same as the bosonic cohomology between Q_1 and Q'_2 i.e. $H^1_{Q_1,Q_2} = H^0_{Q_1,Q'_2}$.

In this way we construct a cone Q_ϕ.

$$Q_\phi = \begin{pmatrix} 0 & 0 & -E'_2 & 0 \\ 0 & 0 & \phi_1 & J_1 \\ -J_2 & 0 & 0 & 0 \\ \phi_2 & E_1 & 0 & 0 \end{pmatrix} \tag{3.57}$$

$$Q_1 = \begin{pmatrix} 0 & E_1 \\ J_1 & 0 \end{pmatrix} , \quad Q_2 = \begin{pmatrix} 0 & E_2 \\ J_2 & 0 \end{pmatrix} \quad Q'_2 = \begin{pmatrix} 0 & -J_2 \\ -E_2 & 0 \end{pmatrix}$$

$$\text{and} \quad \phi = \begin{pmatrix} \phi_1 & 0 \\ 0 & \phi_2 \end{pmatrix} , \quad \text{with } \phi \in H^0_{Q_1,Q'_2} \qquad . \tag{3.58}$$

For rank 2 MFs it can be shown the cone is not isomorphic to the direct sum we started from, but whether this is always the case is as yet unclear. If so we have Q_ϕ is still a valid MF. We have managed to move to another MF. If one is familiar with the particular category $MF(W)$ we might recognise, after similarity transformation, the new MF as an indecomposable object or direct sum of others. We say the brane tachyon system has condensed to a new brane system.

This procedure is described in a way that suggests some sort of algebraic dynamic to a topological theory hence the name, tachyon condensation. To say tachyon condensation describes the process of two branes with a string between them decaying into either one or more other branes is a point of view, as it could also be described as the opposite process of a brane system spontaneously transforming into a brane system plus tachyonic strings. It is an interesting algebraic relation involving objects and morphisms which can be used to generate new objects and is described by triangulated categories.

This has been looked at for the A and D-models in [48, 11], where, in the D-model for example, it was shown, by switching on suitable tachyons the following flows or condensates resulted

$$\mathcal{R}_0 \oplus \mathcal{R}_0^r \mapsto \mathcal{S}_1 \quad , \qquad \mathcal{S}_n \oplus \mathcal{R}_0 \mapsto \mathcal{T}_n \quad , \qquad \mathcal{T}_n \oplus \mathcal{R}_0^r \mapsto \mathcal{S}_n \quad . \tag{3.59}$$

These were seen to provide a consistency check with boundary RG flows and RR charges in the D-brane picture.

Tachyon condensation has been looked at for the ADE-models including E_6, E_7 and E_8 in [18], where the MFs of higher rank were obtained by condensates of tachyon-brane systems from lower ones.

In both the above cases higher rank MFs were obtained from lower but one would have to analyse every fermionic cohomology representative for every possible direct sum in a category (for a potential) to know the full interrelation of indecomposable objects under cone construction.

This process has also been looked at for other categories, $MF(W)$ where W is not a minimal model, specifically tachyon condensation on the cubic elliptic curve [12, 42]. The long brane, a rank 3 MF was produced from the direct sum of two short (rank 2) branes.

Deformations

Tachyon condensation via cone construction, is a special case of a more generalised way to obtain new MFs from old, in the sense that they are not isomorphic. This is by *deforming* a starting MF. The subject of deformations and obstructions is well studied [31, 19]. The basic formalism of deformations is to expand matrix elements by a deformation which is still a MF of the original potential.

$$Q' = Q + \delta Q \ : \ Q'^2 = Q^2 \ . \tag{3.60}$$

This idea will be come up again in connection with orbifold equivalence in §4.

3.8 Potentials

It is the super-potentials in Landau-Ginzburg models which carry the data we need. It is from them that MFs arise. These super-potentials are quasi-homogeneous polynomials and must be built out of three basic or atomic types. We list them here;

1) Fermat: x^a

2) Chain: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_n^{a_n}$

3) Loop: $x_1^{a_1}x_2 + x_2^{a_2}x_3 + ... + x_n^{a_n}x_1$.

All potentials must be built from these basic types by adding atomic polynomials with disjoint sets of variables [65]. We must also have the requirement that each x_i appears in only one atomic polynomial. For the most part we work with three variables but in later chapters we will look at such polynomials in four and six variables.

The above restrictions on the type of polynomial were to become useful in the search for orbifold equivalences. This is because when we were searching for orbifold equivalence at a certain weight, say a fermat A_N model and an E model, there may exist other chain C_m and loop L_m polynomials at the same weight which could well be in the same orbifold equivalence class. The D-model is interesting in this context as we can always construct the "transpose", the D^T-model with $\hat{c} = n/(n+1)$.

$$D_n : \ \ W \ = \ x^{n+1} + xy^2 \qquad \text{and} \qquad D_n^T : \ \ W \ = \ x^{n+1}y + y^2 \ . \tag{3.61}$$

Another useful aspect of these polynomials is that we can construct an exponent matrix from our polynomial, where the rows represent the monomials and the columns represent the variables. If we write $W \in \mathbb{C}[x_1, ..., x_n]$ in the form

$$W \ = \ \sum_{j=1}^{m} a_j \Pi_{i=1}^{n} x_i^{a_{ij}} \ . \tag{3.62}$$

The exponent matrix is then given by

$$E(W) = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ : & : & & : \\ \\ a_{m1} & a_{m2} & \ldots. & a_{mn} \end{pmatrix} \tag{3.63}$$

$$. \tag{3.64}$$

If this is a square matrix, i.e. if the number of monomials in the potential equals the number of variables one can show [65, 70] that the exponent matrix $E(W)$ of such a polynomial is an invertible matrix over the rationals. In this case the polynomial W is called invertible.

3.9 Invertibility of factors and matrix elements

In this subsection we will outline some results concerning invertibility of matrix elements and factors. These are important when it comes to identifying which MFs when searching for orbifold equivalences.

1) Firstly, to have a non trivial cohomology we must have both factors, E and J non-invertible.

2) If one of the summands in a direct sum of MFs (of the same potential W) has a trivial cohomology, then the direct sum has the same cohomology as that of the non trivial part (shown in appendix C) .

3) If one matrix element in E or J is invertible we can perform row and column transformations to obtain a matrix factorisation which is equivalent to the direct sum of a trivial MF and a non trivial MF. That is, we can use invertibility of a matrix element to reduce the rank of our MF [42].

Trivial cohomologies if E or J are invertible

We can show that if either E or J are invertible matrices then we have trivial cohomologies. Consider the action of d_Q on bosonic and fermionic matrices. As in eq (3.2) We can consider the cohomologies

$$H^{even} : = Ker\ d_Q^0 / Im\ d_Q^1$$

$$Ker\ d_Q^0 = \{(B_1, B_2)\ |JB_2 - B_1 J = 0\ ,\ \ EB_1 - B_2 E = 0\}$$

$$Im\ d_Q^1 \quad = \quad \{(\tilde{B}_1, \tilde{B}_2) \mid \tilde{B}_1 = JA_2 + A_1E\ , \quad \tilde{B}_2 = EA_1 + A_2J\}$$

We now suppose J is invertible. i.e. J^{-1} exists. Then, given any B_1 in the kernel of d_Q^0 and any A_1 in the pre-image of d_Q^1 we can define A_2 by $A_2 \ = \ J^{-1}B_1 - J^{-1}A_1E$. In other words,

$$\forall (B_1, B_2) \in Ker\ d_Q^0, \ \ \exists (A_1, A_2) : (B_1, B_2) = d_Q^1(A_1, A_2) \tag{3.65}$$

Since there is nothing in $Ker\ d_Q^0$ which cannot be expressed as the image of d_Q^1, H^{even} is trivial. Next we consider

$$H^{odd} : \ = \ Ker\ d_Q^1 / Im\ d_Q^0$$

$$Ker\ d_Q^1 \quad = \quad \{(A_1, A_2) \mid JA_2 + A_1E = 0\ , \quad EA_1 + A_2J = 0\}$$

$$Im\ d_Q^0 \quad = \quad \{(\tilde{A}_1, \tilde{A}_2) \mid \tilde{A}_1 = JB_2 - B_1J\ , \quad \tilde{A}_2 = EB_1 - B_2E\}$$

Again we suppose J^{-1} exists. This time, given any A_1 in the kernel of d_Q^1 and any B_1 in the pre-image of d_Q^0 we can define B_2 by $B_2 \ = \ J^{-1}A_1 - J^{-1}B_1J$ and this time we can make the statement,

$$\forall (A_1, A_2) \in Ker\ d_Q^1, \ \ \exists (B_1, B_2) : (A_1, A_2) = d_Q^1(B_1, B_2) \tag{3.66}$$

This time, since there is nothing in $ker\ d_Q^1$ which cannot be expressed as the image of d_Q^0, we have H^{odd} trivial. We could just have easily let E be invertible and got the same result. Thus we have shown that if either E or J are invertible then the MF has trivial cohomology.

Invertible matrix elements

Here we outline the argument in [42]. This shows that if our rank N MF contains one invertible matrix element, then the whole MF is equivalent to one of rank $N-1$ and a rank one trivial MF.

Consider the $n \times n$ factor E with one entry, $e_{ij} = \alpha$ invertible. i.e. $\frac{1}{\alpha}$ exists.

$$E \ = \ \begin{pmatrix} e_{11} \ldots & e_{1j} \ldots. & e_{1n} \\ \vdots & \vdots & \vdots \\ e_{i1} \ldots & e_{ij} \ldots. & e_{in} \\ \vdots & \vdots & \vdots \\ e_{n1} \ldots & e_{nj} \ldots. & e_{nn} \end{pmatrix} \tag{3.67}$$

We can now multiply the i'th row by $\frac{1}{\alpha}$ to give

$$E' \;=\; \begin{pmatrix} e_{11}\ ... & e_{1j}.... & e_{1n} \\ \vdots & \vdots & \vdots \\ e_{i1}/\alpha\ ... & 1.... & e_{in}/\alpha \\ \vdots & \vdots & \vdots \\ e_{n1}\ ... & e_{nj}.... & e_{nn} \end{pmatrix} \tag{3.68}$$

If we now consider the k'th column, we can replace it by,

$$e'_{lk} = e_{lk} - e_{ik}e_{lj} \tag{3.69}$$

here l runs from 1 to n. This gives us 0 in the e_{ik} position and we can do this for all the values of k except $k = j$.

To see how this works consider the 3×3 matrix,

$$\begin{pmatrix} a & d & f \\ b & 1 & g \\ c & e & h \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} a-bd & d & f-gd \\ 0 & 1 & 0 \\ c-be & e & h-ge \end{pmatrix} \tag{3.70}$$

Thus we can perform similar transformations for rows and columns to obtain,

$$\begin{pmatrix} e_{11}\ ... & e_{1j}.... & e_{1n} \\ \vdots & \vdots & \vdots \\ e_{i1}/\alpha\ ... & 1.... & e_{in}/\alpha \\ \vdots & \vdots & \vdots \\ e_{n1}\ ... & e_{nj}.... & e_{nn} \end{pmatrix} \quad \longrightarrow \quad \begin{pmatrix} e_{11}\ ... & 0.... & e_{1n} \\ \vdots & \vdots & \vdots \\ 0... & 1.... & 0 \\ \vdots & \vdots & \vdots \\ e_{n1}\ ... & 0.... & e_{nn} \end{pmatrix} \tag{3.71}$$

We can rearrange rows and columns to obtain the final form

$$E' \;=\; \begin{pmatrix} \tilde{E} & 0 \\ 0 & 1 \end{pmatrix} \;=\; TES^{-1} \tag{3.72}$$

Here $\tilde{E}$ is an $(n-1) \times (n-1)$ matrix. We note that we automatically have

$$SJT^{-1} \;=\; \begin{pmatrix} \tilde{J} & 0 \\ 0 & W \end{pmatrix} \tag{3.73}$$

since, $\tilde{Q}^2 \;=\; Q^2 \;=\; W$.

The outcome of this is the required result. This became significant in our search for orbifold equivalences (section 4), which required us to set a rank. If we found a solution of our equations at say rank N, and one of our matrix elements had a constant term, then we knew that matrix element was invertible and a whole row and column could be removed and so, we should look for a solution at rank $N-1$.

3.10 Exact sequences

MFs can also be seen from the perspective of exact sequences and as the result of infinite resolutions of modules over a quotient ring $\mathcal{R}_W = \mathbb{C}[x]/\langle W\rangle$ [66, 34]. We put theory into practice by playing with such resolutions which lead to some interesting results. Although the topic of MFs as exact sequences of modules would fit into the last section, we give it its own section as this led to some new work such as the generation of all indecomposable objects for the categories $MF(W_{ADE})$ from simple sets of ideals derived from the potential. In doing this we also found a new efficient algorithm for proving or disproving isomorphism between MFs suspected to be related by a similarity transformation.

MFs first occur in the mathematics literature with Eisenbud, as exact sequences of free modules as resolutions of Cohen-Macaulay modules [35, 34, 90, 66, 45]. We outline how these appear as resolutions and state the important result of periodicity of such resolutions.

Given a ring A, an open complex of A-modules is a sequence of modules and homomorphisms (E^i, d^i),

$$... \longrightarrow E^{i+1} \xrightarrow{d^{i+1}} E^i \xrightarrow{d^i} E^{i-1} \xrightarrow{d^{i-1}} ... \tag{3.74}$$

i is an integer and d^i maps E^i onto E^{i-1}. A module can simply be a set of generators of an ideal (arranged as a vector) or a matrix. For an exact sequence we have $d^i.d^{i+1} = 0$.

We describe E and J as a *twisted differential* we have $d^i.d^{i+1} = W$[58]. In this case if we start with a rank N matrix factor, E or J, we can resolve rank N modules over the quotient ring $R_W = \mathbb{C}[\mathbf{x}]/\langle W\rangle$, then we can see this periodicity of resolution of matrix factors because $EJ = W\mathbb{I}_N$.

$$... \longrightarrow R_W^N \xrightarrow{E} R_W^N \xrightarrow{J} R_W^N \xrightarrow{E} R_W^N \rightarrow .... \tag{3.75}$$

and our sequence is immediately periodic.

These sequences can result as a resolution of a general module. If we label the sequence (E, d) we can write it as a resolution of a module, M.

$$\longrightarrow E^n \longrightarrow E^{n-1} \longrightarrow ... \longrightarrow E^0 \longrightarrow M \longrightarrow 0 \ . \tag{3.76}$$

We abbreviate the sequence and write,

$$E_M \longrightarrow M \longrightarrow 0 \ . \tag{3.77}$$

The sequence E_M is a resolution of M, the starting module, and in a sense is generated from M as an exact sequence. Eisenbud showed that quite generally these resolutions become periodic. [34]. By choosing M we have a starting point for our resolution and these resolutions are MFs.

We started generating sequences using SINGULAR by trying to guess matrix factors, but we found we can also consider other starting points which are simple ideals described by a set of generators and in practice the majority of these resolutions still become periodic after only several steps and may become non trivial matrix factorisations.

3.10.1 Free resolutions in action

We used the Singular procedure, **mres** to construct these higher rank modules on the computer. It became clear that we could produce MFs from many different ideals, all consisting of just a few simple generators, since an ideal in a ring is a particular example of a module over the ring. These were factors of the individual terms in the potential or of a partial factorisation of the potential under consideration. A very general procedure was written because we wanted to produce MFs for potentials with more terms, for example the quintic $W = x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5$.

The next step was to see if we could construct all the indecomposable MFs from just the generators in the quotient ring. For instance for $W_{E_6} = y^4 + x^3 - z^2$, so we could use various combinations from the generators $\{x, x^2, y, y^2, y^3, (y^2 - z), (y^2 + z)\}$. In order to do this we needed a combinatorial process to sort through all the possibilities systematically and most crucially, we needed a way to identify the resulting modules. Rank was a indicator, as were the cohomological dimensions, but we needed more as we produced many *similar* MFs. In essence we were trying to produce a matrix factorisation *factory*.

Initially this MF factory had as its input, the potential W and an ideal $\langle I\rangle$ constructed from a set of generators. We managed to completely automate this process to try and produce good resolutions (trivial ones occurred) by trying every possible combination of generators constructed from monomials and polynomials of fixed weight less than 2, the weight of W, which was now the only input in our MF factory. This produced very many combinations and many of the ideals lead to isomorphic MFs which were not immediately recognisable as such. As we built up our list of matrix factorisations our procedure needed to know if we had already found that particular MF. Up to then the only method showing isomorphism was the method of double cones [18, 27], but this could only possibly confirm isomorphism and not in all cases. One method to trim down the list was to look at the dimensions of the cohomology but this was not complete either.

We managed to solve this problem which had a surprising solution outlined in the next section, and we managed to generate from nothing more than the input of the potential all the indecomposable sets of objects for $MF(W_{ADE})$. Thus by looking at all monomials/polynomials of weight w, with $w < 2 = |W|$ we could generate all indecomposable MFs of ADE polynomials, identify isomorphism and produce the complete set of indecomposable objects, from just the polynomial. In addition to developing an algorithm for identifying isomorphism, and now having a quick easy way to produce MFs for arbitrary potentials, this exploration gave some insight into the tensor product structure of some indecomposable MFs in these categories.

When running this procedure using SINGULAR on a laptop, the longest run to produce the whole set of indecomposable MFs was about three hours for W_{E_8}. For any given potential there are also only a finite number of possible ideals which are possible. There were no direct sums produced as a resultin any case, maybe because we cannot hope to generate direct sums by repeating generators as we will just be resolving the same ideal, but there might be other possibilities. Since only the complete set of indecomposable objects and nothing else was found it would be nice to relate this, maybe to a limit on the rank of indecomposable matrix factorisations for any given potential. Another simple investigation along this line would be to look at all possible tensor product constructions. We also did not try to create parameter families using this method, although any matrix factorisation which was created by this method of resolution could be a member of a parameter family at some special point.

3.10.2 Identifying isomorphisms

The essential element in tachyon condensation (section 3.7) is the use of a member of the fermionic cohomology of a matrix factorisation. The justification comes from triangulated categories in category theory and is outlined in [18] but the actual construction involves a direct sum of brane anti brane and a tachyonic string. The cohomology is key. Cone construction was also used in the method of *double cones* [20, 27] to show isomorphism between the known indecomposable matrix factorisations of any model and the corresponding cone constructions [27, 18]. This provided a sufficient but not a necessary criterion in that it could only confirm isomorphism not disprove it. In addition it would be complicated to implement. By considering the bosonic cohomology we were able to find a novel straightforward way to construct an algorithm which immediately tells us whether or not any two matrix factorisations are isomorphic or not.

Consider the bosonic cohomology under the action of the $d^0_{Q_1Q_2}$ operator as in equation 3.46.

$$d^0_{Q_1Q_2} \ : \ M^B \to M^F \mid d^0_Q(B) \ = \ Q_1B - BQ_2 \ . \tag{3.78}$$

The bosonic cohomology contains all closed (in the Kernel) but not exact forms (in the Image) and are computed in SINGULAR as a set $\{b_i\}$, of not necessarily linearly independent cohomology representatives $\{B_i\}$ which span the cohomology. For any B_i we have

$$Q_1B_i - B_iQ_2 \ = \ 0 \ . \tag{3.79}$$

We assume the Q_i in this case are indecomposable and that Q_2 arises from a similarity transformation of Q_1: $Q_2 \; = \; UQ_1U^{-1}$, where

$$U \;=\; \begin{pmatrix} S & 0 \\ 0 & T \end{pmatrix} , \qquad S, T \in \mathbb{C}[[\mathbf{x}]] \quad \text{invertible} . \tag{3.80}$$

The identity $\mathbb{I}$ is certainly in the cohomology of $d^0_{Q_1Q_1} = d^0_{Q_1}$. Therefore, for an element in the cohomology of $d^0_{Q_1Q_2}$ suppose we have some linear combination of the B_i say $B = \sum c_iB_i$ which is invertible, that is to say has a determinant which is expressible as a power series $\in \mathbb{C}[[\mathbf{x}]]$, and has a constant term. Then we have

$$\begin{aligned} Q_1B &= BQ_2 \; , \\ B^{-1}Q_1B &= Q_2 \; . \end{aligned} \tag{3.81}$$

This implies that Q_1 and Q_2 are isomorphic and $U = B$. We assume that the set of cohomology representatives $\{B_i\}$ is complete so that any member of the cohomology is expressible as a linear combination. Strictly speaking $H^0_{Q_1Q_2}$ defines the kernel condition but once we mod out the $Im(d^1_{Q_1Q_2})$, if anything is left it will include the invertible members of the cohomology.

We can use SINGULAR to test for this by creating a set of parameters u_i, then performing the sum over cohomology representatives using the procedure **MFcohom**[8] in the following way.

We use **MFcohom** to produce cohomology representatives B_i, and then introduce parameters u_i to form $B = \sum_i u_iB_i$. B depends on the u_i and the ring variables. We then set all ring variables to zero and take the determinant of this sum. If it is not identically zero, then the determinant is a polynomial expression in the u_i and we have an isomorphism because once again $Q_2 \; = \; B^{-1}Q_1B$. If the determinant is zero then there can be no isomorphism.

This is actualised in the procedure **isomchk** and has been used to verified isomorphism and non-isomorphism for many examples of indecomposable A,D and E series MFs, including many direct sums of E-models. It has also verified that selected different points in the elliptic curve parameterisation space represent non-isomorphic MFs and has also been tested for selected different parameterisations for the quartic.

One peculiarity about using the cohomology representatives which are produced by the routine called **MFcohom**, is that they do not form an irreducible set, for instance for direct sums and a linearly independent basis for the kernel, $Ker\ d^0_{Q_1Q_1}$ would be more useful for distinguishing if MFs were direct sums or not. This question is interesting because the cohomology in the case of a direct sum has a different basis from the case of an indecomposable MF if we consider

[8]Written by N. Carqueville

$$
\begin{aligned}
Q &= Q_1 \oplus Q_2 \\
&= \begin{pmatrix} Q_1 & 0 \\ 0 & Q_2 \end{pmatrix} .
\end{aligned} \tag{3.82}
$$

This MF has a bosonic cohomology, $H^0_{Q_1 \oplus Q_2}$ which includes the identity. When we have a direct sum, the cohomology representatives produced in SINGULAR appear in combination and can be further reducible. When written as above we have cohomology representatives which satisfy;

$$
b = b_1 \oplus b_2 . \tag{3.83}
$$

So for $Q = Q_1 \oplus Q_2$, a direct sum of these cohomology representatives will split in two parts, both of which will satisfy the kernel condition individually, i.e.

$$
(b_1 \oplus 0)(Q_1 \oplus Q_2) = (Q_1 \oplus Q_2)(b_1 \oplus 0) . \tag{3.84}
$$

and

$$
(0 \oplus b_2)(Q_1 \oplus Q_2) = (Q_1 \oplus Q_2)(0 \oplus b_2) . \tag{3.85}
$$

In the above case $b = b_1 \oplus b_2$ will be the identity over the direct sum of the spaces of Q_1 and Q_2 and b_1 and b_2 will be projectors over the subspaces,

$$
b_1 = \begin{pmatrix} \mathbb{I}_{N_1} & 0 \\ 0 & 0 \end{pmatrix} \qquad b_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbb{I}_{N_2} \end{pmatrix} \tag{3.86}
$$

We cannot have this decomposition of invertible cohomology representatives in the case of indecomposable MF as this would imply a contradiction An indecomposable MF will always have a complete identity matrix as a member of the bosonic cohomology. In the direct sum case it is true that

$$
\begin{aligned}
(b_1 \oplus 0)(Q_1 \oplus Q_2)(b_1 \oplus 0) &= (Q_1 \oplus 0) \\
(0 \oplus b_2)(Q_1 \oplus Q_2)(0 \oplus b_2) &= (0 \oplus Q_2) .
\end{aligned} \tag{3.87}
$$

If Q' is some direct sum $(Q_1 \oplus Q_2)$ of MFs then the identity is certainly in the bosonic cohomology of Q', but b_1 and b_2 may be totally unrecognisable as the the components of $\mathbb{I} \in H^0_{Q'}$.

Of course once we know all the indecomposable objects (maybe by using the resolution factory) for a potential W, we immediately know all the possible direct sums at any rank so we can always have a catalogue to compare against and then just use the **isomchk** algorithm. The ability to detect direct sums would be a useful tool in the search for orbifold equivalences as we outline in section 5.

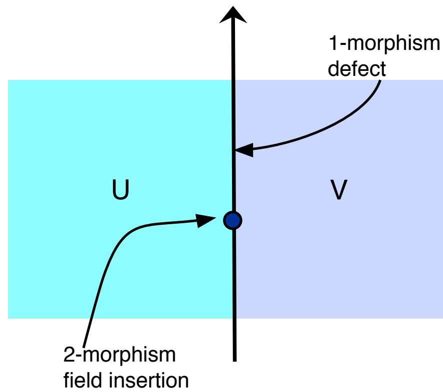

Figure 4: Defect TFT

4 Orbifold equivalence

4.1 2d-defect TQFT and quantum dimensions

A natural extension of the category theory description of a boundary TQFT, is to consider a model which can have defects [14]. A defect is simply a line dividing two regions with bulk potentials $U(x)$ and $V(y)$. Here and in the rest of the chapter we will assume x and y are sets of variables, sometimes called the left and right variables. In this setting our old boundary can be seen as a defect between a bulk and zero potential. The study of defects forms part of conformal field theory and of critical phenomena. These models are usually described by a bicategory or 2-category[18]. The essentially new elements are

- Objects are 2-dimensional regions defined by a potential.
- 1-morphisms are smooth defect lines between these regions. Note the defect line (fig.4) is a morphism from U to V.
- 2-morphisms are points on the smooth lines indicating field insertions.

In this theory, MFs of the potential $W = U(x) - V(y)$ represent the defect. Notice there is an arrow on the defect in fig.4, $\{x\}$ and $\{y\}$ are called the left and right variables. These diagrams are generally read from bottom to top if there is no arrow. The fields or points on the defect are represented by the cohomology of the MFs.

The simplest defect is the invisible defect or identity defect between two regions represented by the same potential, $U = V$. This is the identity defect I_V (see 4.16). One can construct further defects if the potential V admits a symmetry group G (the "orbifold group"), which is a finite subgroup of $\mathbb{C}[x]$-automorphisms which leaves V invariant. One can divide out the action of G on one side of the defect line and obtain a defect I_V^g.

For any defect one can compute two numbers, called the left and right quantum dimensions (see eq 4.3 below). More often than not, these quantum dimensions vanish, but for the identity defect and for I^g_V, they are non-zero.

The computation of quantum dimension can be extended to potentials which are not group symmetry orbifolds of each other. It can be shown (in the bi-category setting) that having a defect between U and V with non-zero quantum dimensions implies a close relation between the two MF categories of U and V. In view of the examples involving an orbifold group, this relation was called orbifold equivalence. Orbifold equivalences have been found for the ADE type potentials with matching central charge [24, 26]. We will see further examples after we see the definition of orbifold equivalence and quantum dimension.

There is a natural development from bulk correlators to boundary correlators [47, 59] and then to defect correlators and the computation of quantum dimensions. We have seen the form bulk correlators take in equation (2.15), and also the boundary correlator for boundary fields, (2.16) which we repeat here for convenience, given in terms of the differentials of the MFs, Q [47, 59]. For a potential W with n superfields

$$\langle\phi\psi\rangle_Q^{KapLi} \quad = \quad res_z\Big[\frac{\phi str(\partial_{z_1}Q, ..., \partial_{z_n}Q\psi)}{\partial_{z_1}W, ..., \partial_{z_n}W}\Big] \tag{4.1}$$

ϕ is any bulk field, ψ is a morphism from the cohomology of Q. The supertrace is simply the trace after multiplication with the $\mathbb{Z}_2$ grading operator $str(A) = tr(\sigma A)$

$$\sigma \quad = \quad \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & -\mathbb{I}_N \end{pmatrix} . \tag{4.2}$$

This formula (4.1) is often referred to as Kapustin-Li or Kap-Li correlator [47, 59] Such correlation functions were first computed in physics, via localisation, a technique for exactly computing path integrals in topologically twisted supersymmetric theories [47, 59] and relying on the BRST nature of Q.

These two LG potentials $U(x),\ V(y)$ are considered orbifold equivalent if they satisfy a condition that there must exist a particular MF, Q of the potential $W = U - V$ with non-zero quantum dimensions [15, 16, 26]. The left and right quantum dimensions are two different correlators, one computed in the region described only by $\{x\}$ coordinates and the other in the region described by the $\{y\}$ coordinates.

The left and right quantum dimensions are defined by

$$qd_L \quad = \quad (-1)^{\binom{m+1}{2}} res\Big[\frac{str(\partial_{x_1}Q...\partial_{x_m}Q.\partial_{y_1}Q...\partial_{y_n}Q)dx}{\partial_{x_1}U...\partial_{x_n}U}\Big]$$

$$qd_R \;=\; (-1)^{\binom{n+1}{2}} res\Big[\frac{str(\partial_{x_1}Q...\partial_{x_m}Q.\partial_{y_1}Q...\partial_{y_n}Q)dy}{\partial_{y_1}V...\partial_{y_m}V}\Big] \tag{4.3}$$

str means supertrace and the quantum dimensions are residues.

In the 2-category view of defects and by analogy with the construction of the Kapustin-Li correlator we can construct correlators for boundary and bulk fields. The fact that the formula for the correlator is a non-zero residue makes certain demands on the weight and form of the numerator or supertrace.

One can always construct the identity defect (4.16) with left and right quantum dimensions equal to 1. For the orbifold *group* symmetries, one can, for each $g \in G$, construct "twisted" identity defects I_V^g formed in the same manner as I_V in (4.16) except that each J_i is replaced with $J_i^g = x_i - g(y_i)$, and E_i replaced accordingly. Details are given in [14, 24, 15, 16], where it is also shown that the quantum dimensions 4.3 of I_V^g are given by $\det(g)^{\pm 1}$, see e.g. eq. (3.13) in [15].

In the 2-category view of defects and by analogy with the construction of the Kapustin-Li correlator we can construct correlators for boundary and bulk fields. The fact that the formula for the correlator is a non-zero residue makes certain demands on the weight and form of the numerator or supertrace. One consequence, which we will show, is that for *graded* MFs (next subsection) the central charge for both left and right potentials has to be equal.

4.2 Graded matrix factorisations

All the known MFs presented so far have quasi-homogeneous matrix entries, as do the MFs generated from exact sequences. Moreover they are graded. We will now define what being graded means and how it relates to R-charge and weight of variables. From the last subsection we know we have assigned weights to each variable such that the potential W is quasi homogeneous. We can therefore write.

$$W(\lambda^{|x_1|}x_1, ..., \lambda^{|x_k|}x_m) \;=\; \lambda^{|W|}W(x_1, ..., x_m)\ , \qquad \lambda \in \mathbb{C}^\times := \mathbb{C}\setminus\{0\}\ . \tag{4.4}$$

The weight of W (note this was previously called d) is set to 2, i.e. $|W| \;=\; 2$. The central charge $c(W)$ for a polynomial $W \in \mathbb{C}[x_1, ..., x_m]$ is then computed using the weights of the variables in the (quasi-homogeneous) polynomial according to the formula

$$c(V) = 3\sum_{i=1}^{m}(1 - |x_i|)\ , \tag{4.5}$$

and is a rational number. Sometimes it is more convenient to work with $\hat{c} = c/3$. From our study of invertibility of matrix elements in previous sections we know that $W \in \mathfrak{m}^2$

where $\mathfrak{m} = \langle x_1, ..., x_m \rangle$ is the maximal ideal of $\mathbb{C}[x]$. We call a rank N MF Q of a potential W graded if there exists a diagonal [9] matrix called the grading matrix of Q such that

$$U(\lambda)Q(\lambda^{|x_1|}x_1, ..., \lambda^{|x_k|}x_k)U^{-1}(\lambda) \quad = \quad \lambda Q(x_1, ..., x_k) \tag{4.6}$$

where

$$U(\lambda) = diag(\lambda^{g_1}, ..., \lambda^{g_{2N}}) \ . \tag{4.7}$$

We say that Q has R-charge 1 since the power of λ in front of Q is 1. This makes sense as we require $Q^2 = W\mathbb{I}_{2N}$, and the natural weight of the potential used in the LG Lagrangian is two (the correct R-charge from the point of view of supersymmetry). In this sense the R-charge of a graded matrix is analogous to the weight of a monomial (or polynomial).

It is easily seen that the grading relates to the weights of the matrix elements in the following way. If we denote the weight of Q_{ij} by w_{ij} we can read off from the equation above that

$$w_{ij} \quad = \quad g_j - g_i + 1 \ . \tag{4.8}$$

More generally, for a graded matrix of R-charge R this becomes

$$w'_{ij} \quad = \quad g_j - g_i + R \ . \tag{4.9}$$

One can instantly see that the *difference* in weights between any two columns or rows is the same for all matrix elements in those columns or rows. It is also clear that any even morphism with matrix elements along the leading diagonal must have the same weight as the R-charge of that matrix. This severely restricts the possibilities for the quantum dimension formula and value of central charge for the two potentials. Note that any matrix element of any weight can be a zero entry. These simple relations only hold with a diagonal U, and will be very useful in setting up a search algorithm outlined later.

We can use the notion of R-charges to provide a self-contained derivation of a statement that is well-known in the physics literature on topological Landau Ginzburg models, namely that the Kapustin-Li correlators have a background charge. Instead of employing arguments from an underlying twisted conformal field theory, this can be derived from properties of the residue alone. We can then use this to show that the two central charges for an orbifold equivalence must be equal.

[9] In [92] these matrices were not assumed diagonal. It has been conjectured by the author that all such grading matrices are diagonal and that all MFs with quasi homogeneous matrix elements are graded but so far, there is no analytic proof or counter examples.

Theorem 4.1: Given a Kapustin-Li correlator and a graded MF Q, the weights of bulk fields w_{bk} and the R-charges of boundary fields R_{bd} must add up to the central charge of the potential of the MF. $w_{bk} + R_{bd} = \hat{c}$, for the correlator to be non-zero.

Proof: Consider the Kapustin-Li correlator in the case Q is graded. Then the product of the derivatives of Q with respect to just the left variables x_i is

$$\Pi_{i=1}^{m} \partial_{x_i} Q = \partial_{x_1} Q ... \partial_{x_m} Q \ . \tag{4.10}$$

has R-charge $\hat{c}_1 = \sum_{i=1}^{m}(1 - |x_i|)$. The denominator has weight $m + \hat{c}_1$. This means the supertrace must be a polynomial in the variables x_i of weight

$$m + \hat{c}_1 - \sum_{i=1}^{m} |x_i| \quad = 2\hat{c}_1 \ , \tag{4.11}$$

otherwise the residue is zero since a residue must be an integrand with a single power of each variable in the denominator. Any residue is of the form

$$\mathcal{R} = res\Big[\frac{(\mathcal{B} str(\partial_{x_1} Q ... \partial_{x_m} Q.\ \mathcal{A}) dx}{\partial_{x_1} V ... \partial_{x_m} V}\Big] \tag{4.12}$$

where $\mathcal{B}$ is the product of bulk fields and $\mathcal{A}$ of boundary fields. However the residue is formulated the weights and R-charges are the crucial factor. If there are both types of fields we can see how they contribute to the residue. We simply add the weights of the bulk fields together with the R-charges of any boundary fields. This means that if Q is a graded MF of R-charge 1 the total weight of the numerator in the correllator must be $2\hat{c}$.

On occasion the denominator may not be a simple monomial but in that case we can make a transformation [66] allowing us to compute residues having more complicated polynomials in the denominator as the sum of a set of residues. This is possible as the Jacobi ring of V is a finite dimensional $\mathbb{C}$-vector space. For each $i = 1, ..., k$ there is a $\nu_i \in \mathbb{Z}_+$ and polynomials C_{ij} such that

$$z_i^{\nu_i} \quad = \quad \sum_j C_{ij}(z) \partial_{z_j} W(z) \ . \tag{4.13}$$

This implies that for any numerator f

$$res_z\Big[\frac{f}{\partial_{z_1} V ... \partial_{z_k} V}\Big] \quad = \quad res_z\Big[\frac{det(C) f}{z_1^{\nu_1} ... z_k^{\nu_k}}\Big] \tag{4.14}$$

This is done explicitly for the elliptic curve in section 5.3.

We note that it is always the case that the total number of variables plus fermionic (odd morphism) boundary fields must be even for the supertrace to exist.

Corrollary 4.2: If $U(x)$ and $V(y)$ are orbifold equivalent only if the total number of variables $n+m$ is even and only if the two potentials have the same central charge

To see this we just have to view the quantum dimension formula as a Kap-Li correlator. $\hat{c}(U) = \hat{c}(V)$.

Corrollary 4.3: $U(x)$ and $V(y)$ are orbifold equivalent and have a graded MF Q with non-zero quantum dimensions $q_L,\ q_R$, then these quantum dimensions must be constants.

Since, for the supertrace not to be zero, the number of variables for the two potentials constituting an orbifold equivalence must differ by an even number, we have the consequence that potentials cannot be orbifold equivalent to their alternate GSO projections. An advantage of these terms for constructing equivalences is that if we have two potentials with the same central charge but odd total number of variables we can always just add a z^2 term to one of the potential.

Another fact which we can deduce from the form of the quantum dimension formulae is that the quantum dimensions are invariant under similarity transformations of $Q' = SQS^{-1}$. This is an important point. If we have MFs confirming the same orbifold equivalence then we instantly know that the two MFs are non-isomorphic if their quantum dimensions are different. Unfortunately, having the same quantum dimension is not a sufficient condition for isomorphism. We will sometimes refer to such MFs as OEQs as they establish orbifold equivalence.

MFs which are graded allow for an analysis of the quantum dimension formulae. Being graded, having an R-charge and the weight of polynomials are all related. One more outcome is that one can show that if a potential has only a discrete unparameterised set of indecomposable MFs with quasi-homogenous entries, then they must be graded. This has implications for orbifold equivalences which must therefore also be graded if they become indecomposable objects in the limit $x = 0$ or $y = 0$.

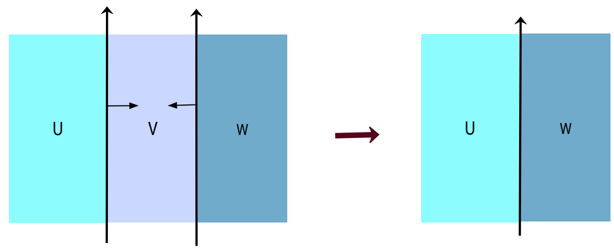

Figure 5: Fusion product

4.3 Fusion product

These models are topological and this allows us to move defects over regions, without crossing field insertions, and to create new defects by fusing old. By doing this we can have a special *fusion tensor product* [14, 22], which, through a complicated construction, and for pairs of OEQs, produces a MF which also confirms orbifold equivalence.

If we consider two defect lines which partition a worldsheet into three regions, with potentials $V_1(x), V_3(x')$ in the outer regions and $V_2(y)$ in the middle. These can be moved on top of each other, leaving a single defect between $V_1(x)$ and $V_3(x')$. This is shown in fig.5. In terms of MFs, the tensor product $Q_{12}(x,y)\hat{\otimes}Q_{23}(y,x')$ of two MFs $Q_{12}(x,y)$ of $V_1(x) - V_2(y)$ and $Q_{23}(y,x')$ of $V_2(y) - V_3(x')$ is a matrix factorisation of $V_1(x) - V_3(x')$. This has infinite rank over $C[x,x']$, but is equivalent to a finite-rank defect [63, 14] depending on x, x' only; extracting this finite rank defect yields a representative of the fusion product $Q_{12} \star Q_{23}$. The full construction involves a long process of taking the tensor product, replacing each y-monomial by the matrix that represents its multiplication on the Jacobi ring $\mathbb{C}[y]/(\partial_{y_i}V_2)$, then finding and splitting an idempotent morphism of the inflated tensor product [22], and row and column reduction.

The fusion product obtained this way also has invertible quantum dimensions if the factors in the product had invertible quantum dimensions. The new left (resp. right) quantum dimension of the fusion product are equal to the product of the left (resp. right) quantum dimensions of tensored orbifold equivalences. Therefore this fusion product gives us a way to obtain new orbifold equivalences in the sense that the new left resp. right quantum dimension of the new MF is a product of the two left resp. two right quantum dimensions of the Q_{12} and Q_{23}, and therefore the new MF is not isomorphic to either.

It is important to note that we cannot just set the middle variable, y to zero. We will see in the section on *mixed terms* (Section 4.8) that the standard tensor product of two MFs, one of $V(x)$ with one of $U(y)$ would have to have zero quantum dimension. In order to clarify some of the possibilities of tensor products of MFs of orbifold equivalences (OEQs) we list them here including the fusion tensor product. Q_1 and Q_2 are matrix factorisations with non zero quantum dimensions.

$$Q_1(U(x)-V(y))\hat{\otimes}Q_2(U'(x)-V'(y)) \qquad q_L = q_{L_1}\times q_{L_2}\ ,\ q_R = q_{R_1}\times q_{R_2}$$

$$Q_1(U(x)-V(y))\hat{\otimes}Q_2(U'(w)-V'(z)) \qquad q_L = q_{L_1}\times q_{L_2}\ ,\ q_R = q_{R_1}\times q_{R_2}$$

but

$$Q_1(U(x)-V(y))\ \hat{\otimes}\ Q_2(V(y)-V'(z)) \qquad q_L\ = q_R\ =\ 0\ , \tag{4.15}$$

and the fusion product

$$Q_1(U(x)-V(y))\ \star\ Q_2(V(y)-V'(z)) \qquad q_L = q_{L_1}\times q_{L_2}\ ,\ q_R = q_{R_1}\times q_{R_2}.$$

The last of these relations give a MF $Q(U(x)-W(x'))$ which proves orbifold equivalence $U \sim_{oeq} V'$.

Identity defect

Orbifold equivalence is an equivalence relation and therefore a potential should be equivalent to itself and we are able to construct a MF which confirms this. This is a MF representing a defect between two regions with the same bulk potential, $W(x,y) = V(x) - V(y)$. We can always create such a defect using an existing algorithm called the *identity defect* or id-defect. To construct it we set,

$$E_i := [V(x_1,...,x_i,y_{i+1},...y_n) - V(x_1,...,x_{i-1},y_i,...,y_n)]/(x_i - y_i) \tag{4.16}$$

and $J_i = x_i - y_i$, and we form a rank 1 MF Q_i from these. We can repeat then the process. The identity defect, $I_V := Q_1\hat{\otimes}...\hat{\otimes}Q_n$ is a MF of $W(x,y) = V(x) - V(y)$. It always has quantum dimensions ± 1. We can create direct sums (see below) from this indecomposable MF but this is the simplest defect.

We can imagine the special situation where we fuse two defects of a form, such that we are left with a defect between two bulk regions with the same potential. In this case $U(x) = V'(z)$. After such fusion we are left with a defect from a potential to itself. It is interesting to ask if this defect remembers anything of the middle potential. Another possibility would be if it is just isomorphic to the identity defect or id-defect, or a direct sum thereof. It turns out that this fusion defect is not simply an identity defect since we know if a fusion product is an indecomposable MF with $q_L \neq \pm 1$ then we do not have an ordinary indecomposable id-defect.

We have seen that as well as matching central charge, the total number of variables is even. Since $\sim_{oeq}$ is an equivalence relation there are further results, all of them have been proven in the literature [20, 21, 24], but that some are easy to see.

For instance, in the case of direct sums of OEQ MFs of the same potential the situation is straightforward. Since the quantum dimension is essentially a supertrace we simply add the (left resp. right) quantum dimensions of the individual MFs to get the (left resp. right) quantum dimension of the direct sum. We can summarise these facts along with the information on tensor products in a theorem:

Theorem 4.4

(a) If $V_1(x) \sim_{oeq} V_2(y)$ and $V_3(x') \sim_{oeq} V_4(y')$, then $V_1(x) + V_3(x') \sim_{oeq} V_2(y) + V_4(y')$. (Note that in this relation each potential depends on a separate set of variables.)

(b) $V_1 \sim_{oeq} V_1 + y_1^2 + y_2^2$ (Knörrer periodicity).

(c) The quantum dimensions do not change under similarity transformations, i.e. $q_L(Q) = q_L(U\,Q\,U^{-1})$ for any invertible even matrix U; analogously for $q_R(Q)$.

(d) The quantum dimensions are additive with respect to forming direct sums: if Q and $\widetilde{Q}$ are two MFs of $V_1 - V_2$, then $q_L(Q \oplus \widetilde{Q}) = q_L(Q) + q_L(\widetilde{Q})$, and analogously for $q_R(Q \oplus \widetilde{Q})$.

(e) Up to signs, the quantum dimensions are multiplicative with respect to fusion products $Q \star \widetilde{Q}$, and with respect to forming tensor products $Q_{12}(x, y) \otimes Q_{34}(x', y')$ (where Q_{12} factorises $V_1(x) - V_2(y)$ and Q_{34} factorises $V_3(x') - V_4(y')$, cf. item c).

(f) Passing to the adjoint defect interchanges left and right quantum dimensions: $q_L(Q^\dagger) = q_R(Q)$ and $q_R(Q^\dagger) = q_L(Q)$.

In the bicategory treatment of orbifold equivalences the bicategory is *pivotal* and it has adjoints: for each 1-morphism Q, i.e. each defect between $V_1(x)$ and $V_2(y)$, the right adjoint $Q^\dagger$ is a defect between $V_2(y)$ and $V_1(x)$, is given by

$$Q^\dagger(x, y) = \begin{pmatrix} 0 & J^T \\ -E^T & 0 \end{pmatrix} \tag{4.17}$$

The fusion product is denoted as $A(Q) := Q^\dagger \star Q$, sometimes called the "symmetry defect". The $Q^\dagger$ and Q represent the defects to be fused. This fusion product is a defect from V_2 to itself, and it can be shown [24] that

$$\text{hmf}^{\text{gr}}(V_1) \simeq \text{mod}(Q^\dagger \star Q) \tag{4.18}$$

where the right hand side denotes the category of modules over $A(Q)$, consisting of MFs of V_2 on which the defect $A(Q)$ acts via the fusion product and the left hand side is the category of graded MFs of V_1. This equivalence of categories is one of several relations existing between structures associated to V_1 and to V_2 as soon as the two potentials are orbifold equivalent.

Within the domain of LG models, orbifold equivalence leads to a "duality" of the two topological field theories: bulk correlators in the V_1-model can be computed as correlators in the V_2-model enriched by defect lines, the defect being $A(Q)$ – see e.g. [24] for a nice pictorial presentation of this fact.

Using the abstract framework of bi-categories, one can prove that these "dualities" and equivalences of categories follow because the defect $A(Q)$ possesses certain properties which are summarised by saying that this 1-morphism from V_2 to itself is a separable symmetric Frobenius algebra. A concise definition of this structure using diagrams is given in Sect. 3.1 of [15]. We will not copy this description here, but merely mention that the prime examples for such 1-morphisms are the symmetry defects A_G arising from the action of an orbifold group on a LG potential, see Sect 4.1. It is this power of abstraction that made it possible to realise that features one is familiar with from orbifold groups can persist without groups being involved.

4.4 OEQ examples

We have seen how to construct id-defects for any potential. This can be generalised to group orbifolds and *symmetry defects*. If the potential W is invariant under some finite group action, $x \mapsto g(x)$, $g \in G$, one can form defects I_W^g with non zero quantum dimensions by replacing, the matrix factor J_i with $J_W^g = x_i - g(y_i)$.

The concept of orbifold equivalences has been extended beyond the ideal of group orbifolds, [24, 26] such as some of the previously mentioned ADE series pairs. Not only are the MFs for these models all well known, and also have been shown to support the correspondence between LG and minimal models, but they provided the best possibility for orbifold equivalence due to the form of the A-model potential, as we shall see. Some of these potentials have the same central charge, and furthermore the A and D models are known to be orbifold equivalent but the E models were not so. For convenience we list them together.

$$W_{A_n} = x^{n+1} + z^2 \quad , \qquad W_{D_d} = x^{d-1} + xy^2 \ ,$$

$$W_{E_6} = x^3 + y^4 \quad , \qquad W_{E_7} = x^3 + xy^3 \quad , \qquad W_{E_8} = x^3 + y^5 \ .$$

with $d \geq 4$ and $n \geq 2$.

Whenever two of these potentials have the same central charge, they have been shown to be orbifold equivalent; the classes with more than one representative are $\{A_{d-1}, D_{d/2+1}\}$ for even d not equal to 12, 18 or 30, and $\{A_{11}, D_7, E_6\}$, $\{A_{17}, D_{10}, E_7\}$ and $\{A_{29}, D_{16}, E_8\}$. The $A \sim D$ orbifold equivalences are related to (simple current) orbifolds in the CFT context, but the $A \sim E$ orbifold equivalences do not arise from any group action and are examples of symmetries beyond groups [24, 26].

In order to form a picture of these special MFs we will now reproduce a concrete example of an equivalence [26]. We will denote variables from the *left* potential Uas x_i, and variables from the *right* potential V as y_i.

$A_{11} \sim_{oeq} E_6$ with $A_{11} = x_1^{12} + x_2^2$ and $E_6 = y_1^3 + y_2^4$,

$$Q(x, y) = \begin{pmatrix} 0 & E \\ J & 0 \end{pmatrix} \tag{4.19}$$

where the rank 2 matrices E, J are defined by

$$\begin{aligned} E_{1,1} &= y_2^2 - x_2 + \frac{1}{2} y_1 (sx_1)^2 + \frac{2t+1}{8} (sx_1)^6 \\ E_{1,2} &= -y_1 + y_2 (sx_1) + \frac{t+1}{4} (sx_1)^4 \\ E_{2,1} &= y_1^2 + y_1 y_2 (sx_1) + \frac{t}{4} y_1 (sx_1)^4 + \frac{2t+1}{4} y_2 (sx_1)^5 - \frac{9t+5}{48} (sx_1)^8 \\ E_{2,2} &= y_2^2 + x_2 + \frac{1}{2} y_1 (sx_1)^2 + \frac{2t+1}{8} (sx_1)^6 \end{aligned} \tag{4.20}$$

and $J = \text{adjugate(E)}$, and where $s, t \in \mathbb{C}$ satisfy $t^2 = 1/3$, $\quad s^{12} = -576(26t - 15)$.

The adjugate of a rank 2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is given by $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

The defect Q has quantum dimensions $q_L(Q) = s$, $q_R(Q) = 3(1 - t)/s$. This particular parameterisation is a result of the construction and starting point of this MF as outlined in [26]. The MF was found using a deformation of a simpler MF and then solving equations under the constraint of a non zero quantum dimension. A priori there may be more general parameterisations, and there may be a 'moduli' space of MFs which have a non zero quantum dimension. If the value of the quantum dimension depended on any parameters then the MFs would be non-isomorphic as isomorphic MFs have the same quantum dimension.

The method involves expanding matrix elements with all possible monomials of the same weight. In this case there was one outstanding variable of low weight. In $A_{11} \sim_{\text{oeq}} E_6$

this was x_1. The starting point MF was then found for the potential containing the other variables $V' = y_1^3 + y_2^4 - x_2^2$, and then each matrix element in that MF was expanded in all possible monomials containing the special low weight variable x_1. This left a minimal set of equations to be solved to satisfy the final MF conditions without eliminating the quantum dimension [26].

It is a striking feature that all the MFs found for the ADE equivalences in [26] have mixed terms. A careful explicit calculation of the supertrace numerator of the quantum dimension (same for left and right) for the above example showed it was exactly these mixed terms which contribute to a numerator which gives a valid residue. The numerator in the formulae is given by

$$\begin{aligned}\mathcal{N} &= str(\partial_{x_1}Q...\partial_{x_n}Q.\partial_{y_1}Q...\partial_{y_m}Q)\\ &= str(\partial_{x_1}Q\partial_{x_2}Q.\partial_{y_1}Qv\partial_{y_2}Q)\ ,\end{aligned} \tag{4.21}$$

since we are considering the case where each individual potential U and V has two variables.

$$tr\{\partial_{x_1}E\partial_{x_2}J\partial_{y_1}E\partial_{y_2}J - \partial_{x_1}J\partial_{x_2}E\partial_{y_1}Jv_{y_2}E\} \tag{4.22}$$

where

$$\begin{aligned}\partial_{x_1}E &= \begin{pmatrix} y_1s^2x_1 + \frac{12t+6}{8}s^6x^5 & y_2s + (t+1)s^4x^3 \\ y_1y_2s + y_1s^4x_1^3 + \frac{10t+5}{4}y_2s^5x_1^4 - \frac{9t+5}{6}s^8x_1^7 & y_1s^2x_1 + \frac{6t+3}{4}s^6x_1^5 \end{pmatrix}\ ,\\ \partial_{x_2}E &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\ ,\\ \partial_{y_1}E &= \begin{pmatrix} \frac{1}{2}y_1(sx_1)^2 & -1 \\ 2y_1 + y_2sx_1 + \frac{t}{4}y_1(sx_1)^4 & \frac{1}{2}y_1(sx_1)^2 \end{pmatrix}\ ,\\ \partial_{y_2}E &= \begin{pmatrix} 2y_2 & sx_1 \\ y_1sx_1 + \frac{2t+1}{4}(sx_1)^5 & 2y_2 \end{pmatrix}\ .\end{aligned} \tag{4.23}$$

In this case the differentials of J are just the adjugates of the differentials of E. Following the calculation through shows that the surviving parameters which contribute to the quantum dimension depend on just the terms with *mixed* variables from both $\{x\}$ and $\{y\}$. This observation led to an illuminating view of orbifold equivalence as a perturbation expansion which will be outlined later.

In this example of orbifold equivalences, the form of the quantum dimension has a parameterisation in terms of two complex numbers s and t. Both have irrational components.

This was implementable using Singular in the sense that we could construct the MF and compute cohomologies, only because in this particular case the parameters were expressible in terms of the **minpoly** function [40]. This feature of Singular allows for the definition of only one irrational for use in computations. Other examples had more complex parameterisations which were not implementable in this sense using Singular. The original MFs were constructed using Mathematica. This was one of the simplest 'working models' we could construct from [26]. We note that the cohomology for id-defects and this example $A_{11} \sim_{\text{oeq}} E_6$ contained no fermionic morphisms in H^1_Q.

The orbifold-equivalence classes for the ADE models all have central charge $c < 3,\ \hat{c} < 1$. There are also polynomials with $\hat{c} > 1$ and these are also classified by Dynkin diagrams and named accordingly [30]. Orbifold equivalence has also been shown for one pair, $W(E_{14}, Q_{10})$ of these Arnold's exceptional unimodal polynomials [74].

Computing the central charges of all polynomials in [30] leaves four possibilities for orbifold equivalence based also on the number of variables. These are $\{E_{13} \sim Z_{11}\}$, $\{E_{14} \sim Q_{10}\}$, $\{Z_{13} \sim Q_{11}\}$ and $\{W_{13} \sim S_{11}\}$ with respective central charges (c), $3\frac{1}{5},\ 3\frac{1}{4},\ 3\frac{1}{3}$ and $3\frac{3}{8}$.

In each case the left polynomial minus the right polynomial, $W(x, y) = U(x) - V(y)$ for the equivalence $\{U \sim V\}$ are

$$\begin{aligned}
W(E_{13}, Z_{11}) &= x_1^3 + x_1 x_2^5 - y_1^3 y_2 - y_2^5 \\
W(E_{14}, Q_{10}) &= x_1^3 + x_2^8 + x_3^2 - y_1^2 y_2 - y_3^3 - y_2^4 \\
W(Z_{13}, Q_{11}) &= x_1^3 x_2 + x_2^6 + x_3^2 - y_1^2 y_2 - y_3^3 - y_3 y_2^3 \\
W(W_{13}, S_{11}) &= x_1^4 + x_1 x_2^4 + x_3^2 - y_1^2 y_2 - y_3 y_2^2 - y_3^4\ .
\end{aligned} \tag{4.24}$$

The potential for the first of these correspondences has fewer variables as both E_{13} and Z_{11} potentials had squared terms. We note that the form of $E_{14} \sim_{OEQ} Q_{10}$ is slightly similar to those found for $E \sim_{OEQ} A$ for the ADE equivalences [26] in that there is one outstanding variable of low weight (or high power in the polynomial). Thus the method used in [26] of finding a starting MF for all variables except this low weight variable provides a viable basis for the search. This method is not so likely to work for the others. If we can find a MF for any of these potentials with non zero quantum dimensions then we have shown orbifold equivalence. By developing a more comprehensive algorithm we have finally managed to prove orbifold equivalence for the other three pairs [81].

One final observation is that in the already discovered case $W(E_{14}, Q_{10})$ [74] the potentials are sums of known potentials.

$$E_{14} \quad \sim \quad x_2^8 + x_3^2 \quad + \quad x_1^3 \quad = \quad A_7 \oplus A_2$$

$$Q_{10} \;\sim\; y_1^2 y_2 + y_2^4 \;\;+\;\; y_3^3 \;\;=\;\; D_5 \oplus A_2 \;. \tag{4.25}$$

This orbifold equivalence already follows from the $A \sim_{OEQ} D$ results of [24] and the fact that we have the identity defect Q_{id}, for $A_2 \sim_{OEQ} A_2$. To see this explicitly we can obtain a MF proving this orbifold equivalence by taking the tensor product of these MFs

$$Q(D_5 - A_7) \otimes Q_{id}(A_2 - A_2) \;\text{ to obtain }\; Q(E_{14} - Q_{10}) \;\text{ and }\; E_{14} \sim_{OEQ} Q_{10} \;. \tag{4.26}$$

This observation led us to consider the construction of orbifold equivalent matrix factorisations for other singularities, simply by building polynomials from basic, two or three variable building blocks.

Our final set of examples comes from the Arnold classification of exceptional unimodal polynomials allows for the possibility for five of these polynomials to be written in two forms, and in one case, U_{12} , three different forms. Equivalences have been found for E_{14}, Q_{12}, U_{12}, W_{12}. W_{13}, Z_{13} [75]. These have been termed auto-equivalences as they are equivalences between different forms of the same singularity and therefore they are not quite the same as the equivalences already discussed in existing examples. We will discuss these again when we look at re-parameterisations in section 4.10.

4.5 OEQs as an ideal membership problem

If one tries to generate examples of orbifold equivalences truly beyond simple singularities, one soon realises the approach taken in [26] is neither general nor systematic enough. In that work, the method employed to find expressions such as the one for $A_{11} \sim_{oeq} E_6$, was to set one of the variables x_i, y_j occurring in $W(x,y) = V_1(x) - V_2(y)$ to zero, to "pick" some matrix factorisation $\widetilde{Q}$ of the resulting potential $\tilde{W}$ and to expand each matrix element in all possible monomials of the same weight, under additional simplifying constraints such as $J = -\text{adjugate}(E)$, trying to obtain a matrix factorisation $Q(x,y)$ of the full $W(x,y)$. But as soon as one has to cope with a larger number of variables, or higher rank, one has to be very lucky to hit a good starting point $\widetilde{Q}$.

One strategy was to use the MF factory we had constructed to produce matrix factorisations of the orbifold equivalent potentials $W(x,y) = U(x) - V(y)$ and try and use the smallest of these as templates for a weight matrix.

In the examples $E \sim A$ [26] we can compute cohomology in only a few cases. Cases which are implementable in Singular interesting as it would confirm/refute a hypothesis that OEQs have an empty fermionic cohomology §5.1 and we could also look at the space of quantum dimensions to see if there are just discrete values of quantum dimension? The set of constraints are well defined for the all the $E \sim A$ and $D \sim A$ equivalences and less so for the auto-equivalences [75] but it is not clear in either case what the full parameter 'moduli space' is for orbifold equivalences. We suspect these may not be the most general solutions as some of the terms in matrix elements do not have coefficients. In the first examples some parameterisations may have been left out by fixing the starting point. In general, there are a set of nonlinear equations to solve and in an ideal situation we would know all possible parameterisations which produce an OEQ.

It is not essential to find 'concrete' MF to prove orbifold equivalence. Instead we can consider a *generalised* parameterised pair of matrices which, when considered as a possible MF must also have non zero quantum dimensions. Each matrix element of Q is a quasi homogeneous polynomial. We expand in all possible monomials of that weight for each of these matrix elements. In this expansion the big difference is every possible monomial is included and has a parameter in front of it.

The result of any such process is three items of data

(a) A MF which includes parameters.

(b) Two quantum dimensions, which are also expressed, in the graded examples, only in parameters and no variables.

(c) A set of constraints on the parameters.

We have to choose the rank of an MF and find a good starting point which has to be a weight matrix. We eventually found a way to construct viable weight matrices from scratch.

Given two potentials $V_1(x)$ and $V_2(y)$, with the same central charge and an even number of variables, we consider the total set of variables $z = (x, y)$. Then we consider all possible combinations of these variables to make monomials and arrange them by weight. We consider each entry Q_{ij} in Q, which will be of definite weight and expand it in all monomials of the same weight with coefficients a_i.

$$Q_{ij}(x,y) = \sum_{k\in\{w_{ij}\}} a(k)m_{ij}(k) \ , \tag{4.27}$$

where $\{w_{ij}\}$ represents the set of all monomials of a certain weight, which is the weight of that particular matrix element. So $|m_{ij}(k)| = |Q_{ij}|$ for all $k \in \{w_{ij}\}$. $m_{ij}(k)$ is the k'th monomial to be used up and does not start from 1 for each matrix element , k which just labels the a which are used up sequentially as every matrix element in Q is expanded and the set $k \in \{w_{ij}\}$ go from the range pertinent to that matrix element. As an example, if we consider the equivalence already looked at, $A_{11} \sim_{\text{oeq}} E_6$, the matrix E had four elements with weights which can be expressed as a weight matrix,

$$w(E) = \begin{pmatrix} 1 & 2/3 \\ 4/3 & 1 \end{pmatrix} . \tag{4.28}$$

We can start with the first matrix factor E from $EJ = W\mathbb{I}_n$, and the first matrix element to be parameterised would be E_{11}. The most general parameterisation of the first two matrix elements E_{11} and E_{12} would be

$$E_{11} = a(1)y_2^2 + a(2)x_2 + a(3)yx_1^2 + a(4)x_1^6 . \tag{4.29}$$

and

$$E_{12} = a(5)y_1 + a(6)y_2x_1 + a(7)x_1^4 \tag{4.30}$$

Every term is now parameterised. This way, given a weighting (or weight matrix), we can construct the most general odd morphism and then require it to satisfy the matrix factorisation condition $EJ = JE = W\mathbb{I}_n$. From this we have very many bilinear equations $f_r(a(p)) = 0$, in the coefficients $a(p)$, $r = 1, ..., N_e$ the number of bilinear equations and, $p = 1, ..., N_c$, the number of parameters used. Each individual monomial in every matrix element in

$$EJ - W\mathbb{I}_n = JE - W\mathbb{I}_n = 0 , \tag{4.31}$$

gives a bilinear equation in some of the $a(p)$.

We also require non-zero quantum dimensions $q_L(Q)$ and $q_R(Q)$. Fortunately these can be defined for any odd morphism. It does not have to be a matrix factorisation. To encode this property of the quantum dimension we introduce the auxiliary coefficients, $a_L, a_R \in \mathbb{C}$ and demand two extra conditions

$$a_L.q_L(Q) - 1 = 0 \quad \text{and} \quad a_R.q_R(Q) - 1 = 0 \tag{4.32}$$

We can see these are unsolvable for $q_L(Q),\ q_R(Q) = 0$ and so we obtain two equations, in just the coefficients for the non zero quantum dimension, as well as a large number of quadratic/bilinear equations in the coefficients. These coefficients now themselves become the variables in the equations to be solved, which can become a set of ideals. All we have to do is prove a solution to these equations exists by showing that 1 is not in the ideal, which can be checked by computing a standard basis of the ideal. This is an application of the weak form of *Hilbert's Nullstellensatz* [43, 45, 66]. This can all be programmed using the computer package Singular which implements variations of the *Buchberger* algorithm

[44] outlined in Appendix A, to express these ideals in the standard or *Groebner* basis [44]. In practice the only restrictions on this procedure are memory or time restrictions since the process of reducing the set of equations to the standard or *Groebner* basis should provide a solution although this involves an unpredictable but large increase in the number of ideals (appendix A).

In fact the most general and the first considered application of the weak form of Hilbert's Nullstellensatz to the general principle of finding solutions was to consider completely ungraded matrix factorisations. There the only input was the rank of the matrix factorisation and the list of permissible monomials, usually all possible monomials up to weight 2, the weight of the potential. Then every matrix element has all possible monomials, with different parameters.

To summarise, the basic algorithm is:

(1) Choose a rank N.

(2) Expand each matrix element in the weight matrix by every possible monomial in the set of ring variables which appear in W, using coefficients $a(i)$.

(3) Impose MF condition $E.J = J.E = W\mathbb{I}_n$ and obtain many quadratic equations in the $a(i)$ and also the quantum dimension equations which (in general) will not be quadratic and can be quite complicated.

(4) Put the set of generators of the ideal formed by all the equations into standard basis. If 1 is in the ideal then we know our set of equations is not solvable for such a weight matrix to give a MF which confirms orbifold equivalence.

Again with enough computer power this is completely solvable but in practice there are just too many calculations, hence the need for a good starting point or weight matrix. When restricting to graded matrix factorisations at a certain rank, deciding whether two given potentials, $V_1(x)$ and $V_2(y)$ have a matrix which confirms OEQ of given rank N exists is a less complex problem in the sense that there are only finitely many gradings $U(\lambda)$ that we need to try out at any rank. The number of equations and the number of coefficients are still typically very large (hundreds or thousands depending on rank and variable weights), and more often than not this is too much for the computing power available. The amount of memory used by the *Buchberger* algorithm (appendix A) is unpredictable because the number of generators can grow very quickly dependent on the starting problem. Experience with solving ideals confirms this.

Any method we can use to reduce the number of generators and finesse the process is desirable. This can be done at the beginning by construction of the weight matrices. Of course we have to choose the rank but by careful consideration of the weights we can set a minimum on the rank and then work our way upwards if nothing is found. The examples which have been found are all quite compact, in the sense that they have quite a low rank considering the number of variables. Since a lower rank means fewer equations we start with as low a rank as possible.

4.6 Orbifold equivalences as graded perturbations

The search for MFs which prove orbifold equivalence became a search for a way to exploit the structure of the quantum dimension formula in order to construct a more efficient algorithm. We will see that the quantum dimension formulae, viewed first as a Kapustin-Li correlator over either one of the potentials $U(x)$, $V(y)$ is expressible as a finite Taylor expansion in either the left or right variables, x or y.

When considering the graded matrix factorisations already found, we see the form is similar (but not identical) to those found in deformation theory [31], where the deformation directions are controlled by the space of boundary fermions, H^1_Q. Obstructions are controlled by H^0_Q, the bosonic cohomology. This gives the possibility to express our MF as an (in this case finite) expansion. Any graded defect between $V_1(x)$ and $V_2(y)$, can be viewed as a deformation of a MF of $V_1(x)$, with the variables y_j featuring as deformation parameters and $-V_2(x)\mathbb{I}_{2N}$ as obstruction term. Of course there are two views and we could see the matrix factorisation of $-V_2(y)$ as the one to be deformed.

Once we write our MF, $Q(x, y)$, as an expansion about x or $y = 0$, in the x or y variables, we can then analyse the properties of these graded matrix factorisations in more detail and try to develop a more efficient algorithm to construct them. The grading matrix and R-charge are key ingredients to this formulation. At the limits $x = 0$ or $y = 0$ one of our expressions for the quantum dimension is no longer valid. If one sets $y = 0$, one can of course not take a residue over the y-variables. The other quantum dimension is a Kapustin-Li correlator with boundary fermions. These boundary fermions then become the coefficients to first order in the Taylor expansion.

Boundary fermions

Let us just look at the case where all the y_j are set to zero. i.e. $y = 0$.

Let $Q(x, y)$ be a graded rank N matrix factorisation of $W(x, y) = V_1(x) - V_2(y)$ with $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, suppose $m + n$ even. We will assume $W \in \mathfrak{m}^2$, i.e.

that W has no linear terms.

We will abbreviate $Q_1(x) := Q(x,y)|_{y=0}$, and $F_j := \partial_{y_j} Q(x,y)|_{y=0}$ for $j = 1, \ldots, m$.

Then we have

(1) Q_1 is a MF of V_1, and F_j are fermions wrt. Q_1: $\{Q_1, F_j\} = \partial_{y_j} Q(x,y)^2|_{y=0} = \partial_{y_j} W\, \mathbf{1}|_{y=0} = 0$

(2) The F_j are, moreover, fermions of definite R-charge $R(F_j) = 1 - |y_j|$, i.e.

$$U(\lambda) F_j(\lambda^{|x_1|} x_1, ..., \lambda^{|x_k|} x_k) U(\lambda)^{-1} = \lambda^{1-|y_j|}\, F_j(x)$$

(The matrix $U(\lambda)$ provides a grading for $Q(x,y)$ and for $Q_1(x)$.)

(3) The left quantum dimension is a correlator of boundary fermions in the LG model with potential V_1 and boundary condition Q_1: $q_L(Q) = \langle\, F_1 \cdots F_m \,\rangle_{Q_1}^{\text{KapLi}}$.

To show this, we exploit the fact that $q_L(Q) \in \mathbb{C}[y]$ is actually a number, and so is independent of the y variables. Then $q_L(Q)$ must still exist in the limit $y = 0$ even though $q_R(Q)$ is no longer defined :

$$\begin{aligned} q_L(Q)(y) &= \operatorname{res}_x \left[\frac{\operatorname{str}\big(\partial_{x_1} Q \cdots \partial_{x_n} Q\ \partial_{y_1} Q \cdots \partial_{y_m} Q\big)}{\partial_{x_1} W \cdots \partial_{x_n} W} \right] \\ \Longrightarrow\ q_L(Q)|_{y=0} &= \operatorname{res}_x \left[\frac{\operatorname{str}\big(\partial_{x_1} Q_1 \cdots \partial_{x_n} Q_1\ F_1 \cdots F_m\big)}{\partial_{x_1} V_1 \cdots \partial_{x_n} V_1} \right] \\ &= \langle\, F_1 \cdots F_m \,\rangle_{Q_1}^{\text{KapLi}} \ . \end{aligned} \tag{4.33}$$

This realisation that the quantum dimension looks like a correlator with graded boundary fermions in either of the limits, x or $y \to 0$ gave rise to a new perspective on the formula.

We outline some immediate side results by using the properties of Kapustin-Li correlators and the fact that these all have the same grading matrix and well defined R-charges:

(1) $q_L(Q) = 0$ if any of the F_j is trivial in the Q_1-cohomology (i.e. if $F_j = [Q_1, A_j]$ for some A_j)

(2) $q_L(Q) = 0$ unless the R-charges of the F_j sum up to the "background charge" $\hat{c}(V_1) = \sum_i (1 - |x_i|)$ of the LG model with potential V_1; i.e. we need $\hat{c}(V_1) = \sum_j R(F_j) = \sum_j (1 - |y_j|) = \hat{c}(V_2)$.

Taylor expansion.

We can always write a Taylor expansion for Q about Q_1 or Q_2. The observation that

$$q_L(Q) = \langle\, F_1 \cdots F_m \,\rangle_{Q_1}^{\mathrm{KapLi}} \tag{4.34}$$

with

$$Q_1(x) := Q(x,y)|_{y=0} \text{ and } F_j := \partial_{y_j} Q(x,y)|_{y=0} \ , \tag{4.35}$$

i.e. that the fermions are first derivatives taken at $y = 0$, means we can view Q_1 and the F_j as zeroth and first order terms in the Taylor series or perturbation expansion of $Q(x, y)$. The fermions are first derivatives taken at $y = 0$ and so are the first order terms in the Taylor series. Note that this expansion has to have finitely many terms, as there can only be a finite number of monomials at any given weight[10]. There are also two different expansions, which must be equal because we can expand about $x = 0$ or $y = 0$. So we can write:

With $\quad Q_2(y) := Q(x,y)|_{x=0} \quad$ and $\quad \tilde{F}_i := \partial_{x_i} Q(x,y)|_{x=0} \quad$, we have

$$Q(x,y) = \sum_{k \geq 0} Q^{(k)} \ , \tag{4.36}$$

where

$$Q^{(0)} = Q_1(x) \ , \qquad Q^{(1)} = \sum_{j=1}^{m} y_j\, F_j(x) \ , \qquad Q^{(2)} = \sum_{j_1,j_2=1}^{m} y_{j_1} y_{j_2}\, Q^{(2)}_{j_1 j_2}(x) \quad \text{etc.}$$

$$q_R(Q) = \langle\, \tilde{F}_1 \cdots \tilde{F}_n \,\rangle_{Q_2}^{\mathrm{KapLi}} \quad . \tag{4.37}$$

The conditions from $Q(x,y)^2 = W(x,y)\mathbf{1}$ imply

$$\{Q_1, Q^{(k)}\} + \sum_{l=1}^{k-1} Q^{(k-l)} Q^{(l)} \;\; = \;\; -V_2^{(k)} \ . \tag{4.38}$$

We can see that this expansion terminates after finitely many steps, otherwise our matrix

[10] In [81] we give a more complex proof of the finiteness of the Taylor expansion

factorisation would have matrix elements with infinitely many terms of a certain weight which is impossible. Alternatively we can consider,

$M := Q^{(k)}_{j_1 \ldots j_k}(x)$ is an odd (wrt. $\mathbb{Z}_2$-grading) matrix with R-charge,

$r_M = 1 - |y_{j_1}| - \ldots - |y_{j_k}|$.

The weight of $M_{ss'}$ is $w(M_{ss'}) = g_{s'} - g_s + r_M$.

When r_M becomes too negative, $w(M_{ss'}) < 0$, hence $M_{ss'} = 0$.

Both of these aspects were to prove useful in refining the search but were also a novel way to view the quantum dimension formula.

4.7 Mixed terms

MFs with non-zero quantum dimension all show the feature of having terms with monomials containing variables from both the potentials $U(x)$ and $V(y)$. We will call these terms with variables from sets $\{x\}$ and $\{y\}$ *mixed* terms. Nevertheless the form of the potential consists of two functions of two separate sets of variables. It is clear from the above description that the numerator in the quantum dimension formula, must, in the limit, contain fermions, F_i, which, based on the arguments in Theorem 4.1, cannot just be constant matrices. Familiarity with a few orbifold equivalence examples suggests that there must also be mixed terms of the form, $x_i^n y_j^m$ or involving more variables, in the MF and the fermions, F_i are therefore not just constants, unless i refers to a quadratic term. We can show that there must be mixed terms in these MFs, $Q(x, y)$ unless $\hat{c}(V_2) = 0$.

Assume $Q(x, y) = Q_1(x) + Q_2(y)$ and therefore no matrix element has mixed terms. Then $\{Q_1(x), Q_2(y)\} = 0$ and also $\{\partial_{x_i} Q_1, F_j\} = 0$.

For simplicity, assume furthermore $V_2(y)$ contains no quadratic terms, i.e. $V_2 \in \mathfrak{m}^3$. Then

$$0 = -\partial_{y_{j_1}} \partial_{y_{j_2}} V_2(y)|_{y=0} = \{F_{j_1}, F_{j_2}\} + \{Q_2(y), \partial_{y_{j_1}} \partial_{y_{j_2}} Q_2(y)\}|_{y=0} \ . \tag{4.39}$$

The last term vanishes (since Q has no constant terms), so all the F_j anti-commute (and square to zero). Let $N := \sigma\, \partial_{x_1} Q_1 \cdots \partial_{x_n} Q_1\, F_1 \cdots F_m$, which is the argument of the trace in the residue formula for $q_L(Q)$. This matrix N itself is nilpotent,

$$N^2 = (\partial_{x_1} Q_1 \cdots \partial_{x_n} Q_1)^2 (F_1 \cdots F_m)^2 = 0 \quad , \tag{4.40}$$

hence $\mathrm{tr}(N) = 0$ and $q_L(Q) = 0$.

This in particular rules out tensor product branes: MFs of $W(x,y) = U(x) - V(y)$ which are constructed by tensor products as $Q(x,y) = Q_a(x)\hat{\otimes}Q_b(y)$ have zero quantum dimensions because there are no mixed terms. It is apparent that the non-zero quantum dimension depends crucially on these fermions. Note this formulation also applies if the limit $x = 0$ is taken instead. We can summarise this by writing

$$Q_{oeq}(x,y) \quad = \quad Q_1(x) + Q_{mixed}(x,y) + Q_2(y) \ , \tag{4.41}$$

where $Q_1(x) = Q_{oeq}(x,0)$ and $Q_2(y) = Q_{oeq}(0,y)$.

4.8 Weight split criterion

The perturbative expansion is a useful ingredient for an algorithmic search for MF which confirm orbifold equivalence or OEQs, but, as it stands we still need to select a suitable a grading (matrix) and a $Q_1(x)$ as a starting point. One option was to use the MF factory to produce MF from which we could copy viable weight matrices, but there is a certain compactness and parameterisation to the MFs already found which made this method arbitrary and on reflection, unlikely.

By considering what we know about the mixed terms and perturbative expansion we showed that the gradings (i.e. the weight matrices of $Q_1(x,y)$ and $Q(x,y)$) are subject to a highly selective criterion, which applies to any graded MF $Q(z)$ of any quasi-homogeneous potential $W(z)$, not just to defects. This is purely due to the fact that MFs are a pair and every term in the polynomial potential W in the equation $Q^2 = W\mathbb{I}_{2N}$ is produced from a pair of factors.

It will be more convenient for this discussion to rescale the variable weights such that all $|z_i|$ are natural numbers; so for the time being, the weight of $W(z)$ is given by some integer $|W| \in \mathbb{Z}_+$ instead of 2.

Before giving a general formulation, let us see how this pairing works in the concrete example of the $A_{11} \sim_{oeq} E_6$ orbifold equivalence found in [26] and which are already familiar with. Here, $V_{A_{11}}(x) = x_1^{12} + x_2^2$, $V_{E_6}(y) = y_1^3 + y_2^4$, and $W(z) = V_{A_{11}}(x) - V_{E_6}(y)$ with $z = (x,y)$. The variable weights are $|x_1| = 1, |x_2| = 6, |y_1| = 4, |y_2| = 3$ (after scaling up to integers, so that $|W| = 12$.

Any graded MF $EJ = JE = W\mathbb{I}_N$ must in particular contain (quasi-homogeneous) polynomials factorising the x_2^2-term from W – and such factors must occur in each row and each column of E and J. Up to constant pre-factors, these polynomials must be of the form $x_2 + f_{rs}$ for some f_{rs} having the same weight as x_2. So each row and each column of the weight matrices $w(E)$ and $w(J)$ must contain a 6. Likewise, the y_1^3-term has to be factorised, so each row and column of $w(E)$ and $w(J)$ has to contain a 4 (from a factor

$y_1^1 + \ldots$) or an 8 (from a factor $y_1^2 + \ldots$).

If we want to construct a rank $N = 2$ matrix factorisation of $W = V_{A_{11}} - V_{E_6}$, these two observations (together with the constraint that Q should be graded) fix the weight matrices completely, up to row and column permutations and up to swapping E and J:

$$w(E) \quad = \quad \begin{pmatrix} 6 & 4 \\ 8 & 6 \end{pmatrix} \tag{4.42}$$

which is indeed the weight matrix for the $A_{11} \sim_{oeq} E_6$ orbifold equivalence [26]. Thanks to the low rank and the small number of variables, it is fairly easy to arrive at a concrete Q once the above $w(E)$ is known.

In order to formulate the criterion in general, we need some notation. Let

$$W(z) = \sum_{\tau=1}^{T} m_\nu(z)$$

be the decomposition of the potential into monomial terms; each m_τ has weight D_W. For each $\tau = 1, \ldots, T$, let S_τ be the set of weights of possible non-trivial divisors of m_τ, i.e.

$$S_\tau = \Big\{ \ w \in \{1, \ldots, D_W\} \ : \ \exists f \in \mathbb{C}[z] \ \text{ s.th. } f \text{ divides } m_\tau \text{ and } f \text{ has weight } w \ \Big\}$$

Weight split criterion: If $Q(z)$ is a graded MF of $W(z)$ with weight matrix $w(Q)$, then each row and column of $w(Q)$ contains an element of S_τ for all $\tau = 1, \ldots, T$.

Let us look at two further examples to illustrate the usefulness of this criterion. For the two unimodal Arnold singularities $V_{E_{13}}(x) = x_2^3 + x_2 x_1^5$ and $V_{Z_{11}}(y) = y_1^3 y_2 + y_2^5$, the variable weights are $|x_1| = 2, |x_2| = 5, |y_1| = 4, |y_2| = 3$ (re-scaled so that W has weight 15). The best way to visualise this information is in a table where each column represents a term in the full potential.

x_2^3	$x_2 x_1^5$	$y_1^3 y_2$	y_2^5
x_2^2 , x_2	x_2 , x_1^5	y_1^3 , y_2	y_2^4 , y_2
	$x_2 x_1$, x_1^4	y_1^2 , $y_1 y_2$	y_2^3 , y_2^2
	$x_2 x_1^2$, x_1^3	y_1 , $y_1^2 y_2$	
	$x_2 x_1^3$, x_1^2		
	$x_2 x_1^4$, x_1^1		

x_2^3	$x_2 x_1^5$	$y_1^3 y_2$	y_2^5
10 , 5	5 , 10	12 , 3	12 , 3
	7 , 8	8 , 7	9 , 6
	9 , 6	4 , 11	
	11 , 4		
	13 , 2		

The terms in W admit weight splits $5 + 10$ and $5 + 10 = 7 + 8 = 9 + 6 = 11 + 4 = 13 + 2$ (from E_{13}) and $12 + 3 = 8 + 7 = 4 + 11$ and $3 + 12 = 6 + 9$ (from Z_{11}). In each row and each column of $w(E)$, there must be a 5 or a 10, and there must be one from the set $\{3, 4, 7, 8, 11, 12\}$.

One can just about fit the above weights into a rank 2 matrix $w(E)$ with entries $5, 12, 10, 3$, but this leads to zero quantum dimensions (the associated Q are tensor products and ruled out as orbifold equivalences, see section 4.7). One can see that there will not be any mixed terms as required for an orbifold equivalence

In any row column pairing coming from $E_{ij}J_{ji} \quad \forall j$, and fixed i, and therefore resulting in a diagonal element, all terms in the potential need to be possible so for each row or column the weight matrix must represent each term, or rather contain a member of a possible weight split for each term.

At rank 3, one can form 24 weight matrices $w(E)$ satisfying the weight split criterion, and one of those leads to an orbifold equivalence, see the next section. It is worth mentioning that the "successful" $w(E)$ is one where many entries are members of *both* the weight split list coming from E_{13} *and* the weight split list coming from Z_{11}; these offer the best opportunity for an "entanglement" of x and y variables. While this provided us with our best chance there are cases where there is no overlap between left and right weight splits, yet it still might be possible to create mixed terms by having matrix elements of the weights of just the mixed terms (see Quartic Elliptic, Section 5.3). While experience so far tells us that this is unlikely it has not been possible to rule out such situations. The weight split criteria provide us with a sensible method based on the evidence of already discovered orbifold equivalences.

Restricting the list of possible weight matrices and excluding certain ranks improved computability. How restrictive the weight split criterion can be becomes clear when one tries to construct an orbifold equivalence for the Arnold singularities Z_{13} and Q_{11}: here, one needs a rank 6 MF, and of about 2.7 million conceivable weight matrices $w(Q)$ only 60 pass the criterion.

4.9 Algorithmic search

In this chapter, we will present an algorithm based on the perturbative expansion introduced in section 4.6. First, we make some general remarks on the "computability" of orbifold equivalences and then we go on to outline a computer-implementable algorithm to deal with the problem.

The question whether there is a rank N orbifold equivalence Q between two given potentials $V_1(x)$ and $V_2(y)$ can be converted into an ideal membership problem and, for fixed N, can be decided by a finite computation.

To see this, let us write the matrix elements of Q as

$$Q_{rs} = \sum_{\vec{p}} a_{rs,\vec{p}}\, z^{\vec{p}} \quad \text{for} \quad r,s \in \{1,\dots,2N\} \tag{4.43}$$

where $z = (x_1,\dots,x_n,y_1,\dots,y_m)$ and where $\vec{p} \in \mathbb{Z}_+^{m+n}$ is a multi-index. The main "trick" now is to shift one's focus away from the variables z and work in a ring of polynomials in the $a_{rs,\vec{p}}$:

The requirement that Q is a rank N matrix factorisation of $W(z) = V_1(x) - V_2(y)$ imposes polynomial (in fact: bilinear) equations $f_\alpha^{\mathrm{MF}}(a) = 0$ on the coefficients $a_{rs,\vec{p}} \in \mathbb{C}$. ($\alpha$ labels the various bilinear equations, a collectively denotes all the coefficients.) These equations $f_\alpha^{\mathrm{MF}}(a) = 0$ exist for all monomials in the ring variable z, and if they correspond to matrix elements in Q^2 which are on the diagonal and monomials which occur in the potential then they contain constant factors.

The quantum dimensions can be computed, as the residue of a supertrace, whether or not Q is a matrix factorisation. For a graded Q, one obtains two polynomials (of degree $n+m$) in the $a_{rs,\vec{p}}$. The requirement that both quantum dimensions are non-zero is equivalent to the single equation

$$f^{\mathrm{qd}}(a) := q_L(Q) q_R(Q)\, a_{\mathrm{aux}} - 1 = 0$$

being solvable, where a_{aux} is an additional auxiliary coefficient.

Thus, the matrix Q is an orbifold equivalence between V_1 and V_2 if and only if the system

$$f_\alpha^{\mathrm{MF}} = 0 \,, \quad f^{\mathrm{qd}} = 0 \tag{4.44}$$

of polynomial equations in the coefficients $a_{rs,\vec{p}}$ and a_{aux} has a solution. By Hilbert's weak Nullstellensatz, this is the case iff

$$1 \notin \langle\, f_\alpha^{\mathrm{MF}}, f^{\mathrm{qd}} \,\rangle_{\mathbb{C}[a,a_{\mathrm{aux}}]} \,. \tag{4.45}$$

i.e. 1 must not be in the ideal spanned by f_α^{MF} and f^{qd}. To see this we must be in standard or groebner basis.

This type of ideal membership problem can be tackled rather efficiently with computer algebra systems like Singular. (Such systems are usually restricted to working over $\mathbb{Q}$, but for potentials V_1, V_2 with rational coefficients it is enough to study (4.45) over the rationals in order to prove or disprove existence of an orbifold equivalence with coefficients $a_{rs,\vec{p}}$ in the algebraic closure $\overline{\mathbb{Q}}$.)

Once a grading $U(\lambda)$, and hence a weight matrix for Q, has been chosen, it is easy to write down the most general homogeneous matrix elements Q_{rs} (4.43) that conform with this grading. Moreover, there is only a finite number of possible gradings $U(\lambda) = \mathrm{diag}(\lambda^{g_1},\dots,\lambda^{g_{2N}})$ for a given rank N.

To see this, recall that the weights of the Q-entries are given by $w(Q_{rs}) = g_s - g_r + 1$, and that we can fix $g_1 = 0$ without loss of generality, so in particular $w(Q_{1r}) = -g_r + 1$, and $w(Q_{r1}) = -g_r + 1$. (Here we set the weight of the potential to 2.)

Therefore, at least one of the g_r has to satisfy $-1 \leq g_r \leq 1$, otherwise the entire first row or column of Q would have to vanish (because the weights would all be negative), which would contradict the matrix factorisation conditions. We can repeat the argument for the g_r nearest to g_1 and find, overall, that $g_r \in [-2N, 2N]$ for all r.

Finally, Q_{rs} can be a non-zero polynomial in the x_i, y_j only if its weight $w(Q_{rs})$ is a sum of the (finitely many, rational) weights $|x_i|$, $|y_j|$, hence only finitely many choices g_r from the interval $[-2N, 2N]$ can lead to a graded rank N MF of $V_1(x) - V_2(y)$.

All in all, the question whether there exists a rank N orbifold equivalence between two given potentials V_1, V_2 can be settled in principle. It is reasonable to suppose that there is an upper bound $N_{\max}(V_1, V_2)$ such that, if no orbifold equivalence of rank $N < N_{\max}(V_1, V_2)$ exists, then none exists at all – but we have only circumstantial evidence: all known (indecomposable) examples have rank smaller than that of the nested tensor product MF obtained by factorising each monomial in $V_1 - V_2$; and packing a matrix factorisation "too loosely" risks making the supertrace inside the quantum dimensions vanish.

So much for the abstract question whether orbifold equivalence is a property that can be decided algorithmically at all. In order to search for concrete examples, we have devised an *algorithm* based on the perturbation expansion and the weight split criterion introduced in the previous sections:

(a) From the potentials $V_1(x)$, $V_2(y)$, compute the variable weights $|x_i|$, $|y_j|$.

(b) Choose a rank N.

(c) Exploiting the **weight split criterion**, compute all admissible gradings (i.e. weight matrices) if any for this rank.

(d) Choose, possibly iteratively, a weight matrix and form the most general MF $Q_1(x)$ of $V_1(x)$ with this weight matrix. (zeroth term in Taylor expansion)

(e) For each y_j, compute the space of fermions F_j of $Q_1(x)$ with R-charges $1 - |y_j|$. (1st order term)

(f) For any R-charge r_M that can occur in the **peturbative expansion** of $Q(x, y)$, determine the space of odd matrices with that R-charge. (higher terms)

(g) Compute $Q(x, y)$ using the **peturbative expansion**, and then also compute the quantum dimensions $q_L(Q)$ and $q_R(Q)$.

(h) Extract the conditions: $f_\alpha^{\mathrm{MF}}(a) = 0$ and $f^{\mathrm{qd}}(a) = 0$, on the coefficients appearing in $Q(x, y)$ and check whether this system of polynomial equations admits a solution by putting into standard/Groebner basis. Note all equations depend on unknown coefficients a, not the ring variables z.

Computer algebra systems such as Singular have in-built routines to perform the last step, employing (variants of) Buchberger's algorithm to compute a Groebner basis of the ideal spanned by f^{MF}, f^{qd}.

Already when forming $Q_1(x)$ with a given weight matrix, undetermined coefficients a can enter the game – but far fewer than would show up in the most general matrix $Q(x, y)$ with the same weight matrix, because one only uses the x-variables to form quasi-homogeneous entries: the perturbation expansion "organises" the computation to some extent from the outset. Nevertheless, even for harmless looking potentials V_1, V_2 one can easily end up with close to one thousand polynomial equations in one or two hundred unknowns $a_{rs,\vec{p}}$.

It is clear that these equivalences although they must have explicit, and possibly computable solutions, more generally have a parameter or moduli space of solutions. Again given no limits we could find the maximal set of ideals (most general equations) in all cases but due to restrictions on memory and run-time, in practice it is advisable and even necessary to make guesses for some of the coefficients $a_{rs,\vec{p}}$ occurring in $Q(x, y)$ or already in $Q_1(x)$, instead of trying to tackle the most general ansatz.

This could be a long and laborious process as one would never know if setting certain coefficients a to constants or zero, or even equating pairs, would preclude a solution down the line. Sometimes this would mean leaving the computer running for several hours or overnight before finding out there could be no solution with that set of 'guesses'. We have succeeded in automating most of the steps involved in making the equations tractable for Singular, some of the results are collected in the next subsection.

Finding an explicit solution for the coefficients a is of course desirable, but not necessary to prove orbifold equivalence between two potentials. It appears that Singular is not the optimal system for such tasks (even though it is very efficient in establishing solvability); feeding the resulting polynomial equations into Mathematica, say, might be more promising.

If one is content with existence statements, additional avenues are open: One could first employ numerical methods to find approximate solutions to the system of equations (4.44), then check whether any of them satisfies the criteria of the Kantorovich theorem or of Smale's α-theory [62, 83]. If so, one has proven (rigorously) that there is an exact solution in a neighbourhood of the numerical one. We did not take this route, but it might lead to a more efficient computational tool towards a classification of orbifold equivalent potentials.

4.10 New examples

We now present new examples of orbifold equivalences starting with all remaining pairs of unimodal Arnold singularities. We then add a series of equivalences obtained by simple transformations of variables. We do not reproduce the matrix factorisations, quantum dimensions and set of constraints here as these were most often very complicated. All MFs with non zero quantum dimension are catalogued and can all be reproduced using the procedures which can be found at [100].

Unimodal Arnold singularities

In each of the following cases, the potential $V_1(x)$ is orbifold equivalent to the potential $V_2(y)$:

(1) $E_{13} \sim_{oeq} Z_{11}$, rank 3
$V_1(x) = x_1^5\, x_2 + x_2^3$ and $V_2(y) = y_1^3\, y_2 + y_2^5$
$\hat{c} = \frac{16}{15}$.

(2) $Z_{13} \sim_{oeq} Q_{11}$, rank 6
$V_1(x) = x_1^6 + x_1\, x_2^3 + x_3^2$ and $V_2(y) = y_2\, y_3^3 + y_2^3 + y_1^2\, y_3$
$\hat{c} = \frac{10}{9}$.

(3) $S_{11} \sim_{oeq} W_{13}$, rank4
$V_1(x) = x_1^2\, x_3 + x_2\, x_3^2 + x_2^4$ and $V_2(y) = -y_1^2 + y_2^4 + y_2\, y_3^4$
$\hat{c} = \frac{9}{8}$.

Bimodal Arnold singularities

The only bimodal singularities with matching central charge are Q_{17} and W_{17} . We have not been able to find an orbifold equivalence for this pair so far, due to lack of memory when computing groebner basis ideals, but we **have** managed to find one for:

$$V_1^{17}(x) = x_1^{10} x_2 + x_2^3 \quad \sim_{oeq} \quad V_2^{17}(y) = y_1 y_2^7 + y_1^3 y_2. \text{ at rank 3}$$

These are a chain resp. a loop (or cycle) (see 3.8) [70, 65, 55], The central charge $\hat{c} = \frac{6}{5}$, is shared by the pair Q_{17} and W_{17} of bimodal Arnold singularities. Q_{17} and W_{17} remain to be found.

$$W_1^{17}(x) = x_1^5 x_3 + x_2 x_3^2 + x_2^2 \qquad \text{and} \qquad Q_1^{17}(x) = x_1^5 x_2 + x_2^3 + x_1 x_3^2 \ .$$

A simple tensor product

$A_5 \ \sim_{oeq} \ A_2 \times A_2$, rank 2

$V_1(x) = x_1^6 + x_2^2$ and $V_2(y) = y_1^3 + y_2^3$.

$\hat{c} = \frac{2}{3}$.

In contrast to $E_{14} \ \sim_{oeq} \ Q_{10}$, none of these cases can be traced back to known results on simple singularities. Lacking, therefore, any elegant abstract arguments, we can only establish these orbifold equivalences by finding explicit MFs Q of $V_1 - V_2$ with non-zero quantum dimensions.

In most cases, Q depends on coefficients a which are subject to solvable systems of polynomial equations. We list those matrices on the web-page [100], in the form of a Singular-executable text file. This page also provides a few small Singular routines to perform the necessary checks: extraction of the matrix factorisation conditions (bilinear equations on the a), computation of the quantum dimensions, computation of the Groebner basis for the ideal in (4.45). For the sake of completeness, and in order to give an impression of the complexity, the matrices and the polynomial equations are also reproduced in the appendix of the present paper.

In all of the five cases, the orbifold equivalence satisfies $q_L(Q)q_R(Q) \neq \pm 1$, hence these are true orbifold equivalences, not mere equivalences in the bicategory.

The web-page mentioned above also presents direct orbifold equivalences between D_7 and E_6, between D_{10} and E_7, and between D_{16} and E_8; that these simple singularities are orbifold equivalent follows already from the A-D and A-E results in [24, 26], what makes the direct D-E defects noteworthy is that they have at most rank 3. (The smallest orbifold equivalence between E_8 and A_{29} is of rank 4.)

Together with the straightforward $E_{14} \ \sim_{oeq} \ Q_{10}$ orbifold equivalence mentioned, the above list exhausts all orbifold equivalences among the (quasi-homogeneous) exceptional unimodal Arnold singularities: no other pairs with equal central charge exist among those fourteen potentials. The orbifold equivalent pairs are precisely the pairs that display “strange duality” (topological indices called Dolgachev and Gabrielov numbers are interchanged), see e.g. [84].

We should mention that the arguments one can use to treat the $E_{14} \sim_{oeq} Q_{10}$ case also show that orbifold equivalence does not respect the modality of a singularity. The exceptional unimodal Arnold singularity Q_{12} with is orbifold equivalent to the exceptional bimodal Arnold singularity E_{18} : the former is $D_6 \times A_2$, the latter $A_9 \times A_2$, and $D_6 \sim_{oeq} \ A_9$

due to the results of [24].

Unimodal $\sim$ bimodal singularities

$Q_{12} \sim_{oeq} E_{18}$

$V_{Q_{12}}(x) = x_1^5 + x_1\, x_2^2 + x_3^3$ and $V_{E_{18}}(y) = y_1^{10} + y_2^3 + y_3^2$:

$\hat{c} = \frac{2}{3}$.

By the same method, one can relate other exceptional Arnold singularities to sums of simple singularities; among the examples involving bimodal singularities are $Q_{16} \sim_{oeq} A_{13} \times A_2$ and $U_{16} \sim_{oeq} E_8 \times A_2 \sim_{oeq} A_5 \times A_4$.

Reparameterisations

One early experiment was to try and find orbifold equivalence for different elliptic curves but with the same J invariant by using the identity defect.

We wish to apply this to the three different parameterisations of the elliptic curve listed below which will be referred to as the fixed point or cubic, lambda or Weierstrass, and 'e' forms or representations.

$$\begin{aligned} V_c &= x^3 + y^3 + z^3 - dxyz \;, \\ V_\lambda &= -zy^2 - (x-z)(x-\lambda z)x \\ V_e &= -zy^2 + (x-e_1 z)(x-e_2 z)(x-e_3 z) \end{aligned} \tag{4.46}$$

The six permutations of the e_i in the V_e form are obviously the same equation as the brackets commute. There is a map from this form to the lambda form. This map involves a reparameterisation of the variables, which does depend on the order of the e_i. The set of transformations of both variables and the coefficients is,

$$\begin{aligned} \tilde{x} &= (e_2 - e_1)x + e_1 z \\ \tilde{y} &= (e_2 - e_1)^{3/2} y \\ \tilde{z} &= z \\ \lambda &= \frac{e_3 - e_1}{e_2 - e_1} \;. \end{aligned} \tag{4.47}$$

For different permutations of e_1, e_2, e_3 we have a different set of transformations. The six different lambda form a set of cross-ratios which can be expressed as

$$\lambda * = \{ \ \lambda \ , \ \lambda^{-1} \ , 1 - \lambda, \ (1-\lambda)^{-1}, \lambda(1-\lambda)^{-1}, \lambda^{-1}(1-\lambda)\} \tag{4.48}$$

Any potential, has six parameterisations for any J-invariant. All forms can be used to construct an identity defect MF. We were able to use Singular and the parameterisation for V_e to construct different potentials of the same J-invariant and then transfer to V_λ to compute orbifold equivalence. Using Singular parameters $a(1), \ a(2), \ a(3)$ we were able to reparameterise between different $\lambda*$.

$$e_1 = a(1) \ , \qquad e_2 = a(2)^2 + a(1) \ , \qquad e_3 = a(3)^2 + a(1) \ . \tag{4.49}$$

Then λ is given by

$$\lambda \ = \frac{(e_3 - e_1)}{(e_2 - e_1)} \ = \ \frac{a(3)^2}{a(2)^2} \tag{4.50}$$

And the potentials

$$\begin{aligned} V_\lambda &= -zy^2 + (x-z)(x-\lambda z)x \\ V_e &= -zy^2 + (x - e_1.z)(x-e_2 z)(x-e_3 z) \end{aligned} \tag{4.51}$$

First we created the identity defect from V_e to give $Q_e^{id}(x',y',z' : x'',y'',z'')$. Then we made the two sets of transformations.

$$\begin{aligned} x' &= (e_2 - e_1)x + e_1 y \ = \ a(2)^2 x + a(1)y \\ y' &= (e_3 - e_1)^{3/2} y \ = \ a(3)^3 y \end{aligned} \tag{4.52}$$

and

$$\begin{aligned} x'' &= (e_3 - e_1)x + e_1 y \ = \ a(3)^2 x + a(1)y \\ y'' &= (e_2 - e_1)^{3/2} y \ = \ a(2)^3 y \end{aligned} \tag{4.53}$$

to get two different values for the λ parameterisation into the MF Q_e^{id} and therefore show $V_{\lambda'} \sim_{oeq} V_\lambda$.

This of course was successful and gave us a non zero quantum dimension which was not ±1.i.e. $q_L = a(2)/a(3)$, $q_R = a(3)/a(2)$.

For completeness we write down two forms for the J invariant which characterises this sixfold symmetry of the elliptic. First [12],

$$\left(\frac{J(d)}{1728}\right)^{1/3} = -\frac{d(d^3+8)}{4(1-d^3)} \tag{4.54}$$

and shows that different values of d multiplied by the third root of unity also have the same J invariant.

Hartshorne p334 [45] gives an explanation and formula for J in terms of λ (which is invariant under $\lambda*$ transformations).

$$J(\lambda) = 2^8 \frac{(\lambda^2-\lambda+1)^3}{\lambda^2(\lambda-1)^2} \tag{4.55}$$

This result, and the results [74] on strangely dual orbifold equivalences suggests that there may be many other such equivalences. In fact a number of more or less expected orbifold equivalences, including infinite series, can be established via transformations of variables. This is encapsulated in the following lemma.

Lemma 4.5: Assume $Q(x,y)$ is an orbifold equivalence between $V_1(x)$ and $V_2(y)$, and assume that $y \mapsto y'$ is an invertible, weight-preserving transformation of variables. Then $Q(x,y')$ is an orbifold equivalence between $V_1(x)$ and $V_2(y')$ if the weights $|y_i|$ are pairwise different, or if $V_2(y) \in \mathfrak{m}^3$.

Proof:

First, focus on the variable transformation itself: We can assume, wlog., that the $y_1, \ldots, y_m$ are labeled by increasing weight, y_1 having the lowest weight. Then the transformation can be written as $y_j \mapsto y'_j = f_j(y) + \sum_{k \in I_j} A_{jk} y_k$ where $A_{jk} \in \mathbb{C}$, where $I_j = \{k : |y_k| = |y_j|\}$ and where f_j depends only on those y_l with $|y_l| < |y_j|$. As $y \mapsto y'$ preserves weights, f_j has no linear terms. The Jacobian $\mathfrak{J}$ of the transformation is lower block-diagonal and $\det(\mathfrak{J}) = \det(A)$, a non-zero constant.

Since $Q' := Q(x,y')$ is obviously a MF of $V_1(x) - V_2(y')$, we only need to study the quantum dimensions of Q'. The relation $q_R(Q') = \det(A)\, q_R(Q)$ results immediately from making a substitution of integration variables in the formula for the right quantum dimension.

The left quantum dimension of Q' can be expressed as a Kapustin-Li correlator (in the (V_1, Q_1) model) of the fermions $F'_j = \partial_{y'_j} Q'|_{y'=0} = \sum_{l=1}^{m} \frac{\partial y_l}{\partial y'_j}|_{y=0}\, F_l$. Here, we have already exploited $y' = 0 \Leftrightarrow y = 0$ to simplify, but the summation over l might still lead to linear combinations which are difficult to control. The extra assumptions on $V_2(y)$ avoid this: If all $|y_j|$ are pairwise different, then $\frac{\partial y_l}{\partial y'_j}|_{y=0} = b_j\,\delta_{j,l}$ for some non-zero constants b_j. If V_2 starts at order 3 or higher, the F_j anti-commute with each other inside the correlator:

$$0 = -\partial_{y_{j_1}}\partial_{y_{j_2}}V_2|_{y=0} = \{F_{j_1}, F_{j_2}\} + \{Q_1(x), \partial_{y_{j_1}}\partial_{y_{j_2}}Q(x,y)|_{y=0}\}\ ,$$

and the last term vanishes in the Q_1-cohomology, therefore does not contribute to the Kapustin-Li correlator. Hence, the correlator is totally anti-symmetric in the F_j, and the linear combination of correlators making up the left quantum dimension is simply $q_L(Q') = \det(A)^{-1}\,q_L(Q)$.

Applying this lemma to the identity defect of $V_1(x) - V_1(y)$, one can establish orbifold equivalences e.g. in the following cases:

(1) "Auto-equivalences" of unimodal Arnold singularities: different descriptions of the same singularity exist for U_{12}, Q_{12}, W_{12}, W_{13}, Z_{13} and E_{14}. The assumptions on the variable weights resp. structure of V_2 made in Lemma 4.4 hold for all these cases. These orbifold equivalences were already discussed in [75], and although the concrete formulas given there contain some errors, the general structure (Q being a nested tensor product of rank 4) coincides with what one obtains from the identity defect upon a weight-preserving transformation of variables.

The "auto-equivalence" between $V_{Q_{17}^T}$[11]$(x) = x_1^3x_2 + x_2^5x_3 + x_3^2$ and $V(y) = y_1^3y_2 + y_2^{10} + y_3^2$ is another such example, involving a bimodal Arnold singularity.

(2) Equivalences between quasi-homogeneous polynomials of Fermat, chain and loop (or cycle)(3.8) type at $\hat{c} < 1$:

$V_{A_{2n-1}}(x) = x_1^{2n} + x_2^2$ and $V_{D_{n+1}^T}(y) = y_1^n y_2 + y_2^2$

$V_{L_n}(x) = x_1^n x_2 + x_1 x_2^2$ and $V_{D_{2n}}(y) = y_1^{2n-1} + y_1 y_2^2$

$V_{C_n}(x) = x_1^2 x_2 + x_2^n x_3 + x_3^2$ and $V_{D_{2n+1}}(y) = y_1^{2n} + y_1 y_2^2 + y_3^2$

with $n \geq 2$ in all three pairs. Explicit orbifold equivalences for A-D^T were already given in [79].

(3) Cases involving non-trivial marginal bulk deformations, e.g.

at central charge $\hat{c} = \frac{10}{9}$, one finds an orbifold equivalence between the product $A_8 \times A_2$ of simple singularities, $V_{(A_8\times A_2)}(x) = x_1^9 + x_2^3$, and special deformations of Z_{13}^T, given by $V_{Z_{13}^T}(y) = y_1^6 y_2 + y_2^3 + \mu_2\, y_1^3 y_2^2$, if $\mu_2 = \pm\sqrt{3}$;

at central charge $\hat{c} = \frac{8}{7}$, the two deformed singularities $V_{E_{19}^T}(x) = x_1^3 x_2 + x_2^7 + \mu_1\, x_1 x_2^5$ and $V_2(y) = y_1 y_2^5 + y_1^3 y_2 + \mu_2\, y_1^2 y_2^3$ are orbifold equivalent as long as the two deformation parameters are related by $\mu_1 = \mu\,(\frac{1}{3}\mu_2^2 - 1)$ with $3\mu^3 = -\mu_2(\frac{2}{9}\mu_2^2 - 1)$.

[11] T means transpose and results from a transpose of the exponent matrix (3.63).

A first edition of an "OEQ catalogue", i.e. a list of polynomials sorted into orbifold equivalence classes based on the results of [24, 26] and our new findings, is available at the web-page [100].

Since, in all the examples listed after Lemma 4.5, we start from the identity defect, the orbifold equivalence resulting from the transformation of variables automatically satisfies $q_L(Q')\, q_R(Q') = 1$, so it is likely that they are "mere equivalences" in the bicategory $\mathcal{LG}$. (One way to verify this would be to compute and analyse the fusion product $(Q')^\dagger \star (Q'.)$ But Lemma 4.5 could also be applied to the orbifold equivalence between D_{n+1} and A_{2n-1}, say, to produce a defect with $q_L(Q')\, q_R(Q') = 2$ between D_{n+1} and D^T_{n+1}.

Furthermore, the potentials of type D^T_n, C_n (chain) and L_n (loop) listed in item (2) appear as separate entries in lists of quasi-homogeneous polynomials [65, 55], but not in lists of singularities (where more general types of transformations of variables are allowed to identify two singularities). The orbifold equivalences given in item (2) of Lemma 4.2 may not be surprising, but it is not clear to us whether there are abstract theorems guaranteeing that polynomials which are equivalent as singularities are (orbifold) equivalent in $\mathcal{LG}$. Such equivalences would certainly help in trying to prove orbifold equivalence by decomposing larger polynomials as we would know all the options for the available building blocks.

The full catalogue of known orbifold equivalences [100] has many more new orbifold equivalence classes. We also see that the known classes now contain chain and loop potentials as well as Fermat, and those containing D also have its transpose D^T. These are all potentials with $\hat{c} < 1$. Then there are many potentials with $\hat{c} > 1$ coming from the exceptional singularities [30], and it would be easy to build equivalences at higher values of $\hat{c}$, by tensoring known ones. The best way to catalogue these is by central charge stating all singularities of that value and arranging them into equivalence classes that way. One can ask does every value for central charge just define one equivalence class, and what is left that is really interesting to find using our algorithm or otherwise? The answer is there have been none found for $\hat{c} = 1$. Perhaps since we have such an abundance of equivalence classes based on potentials having the same central charge there is another question. Besides their interesting structure and being quite difficult to prove, what is there that is special about orbifold equivalences besides central charge? We discuss the possibility that some potentials for $\hat{c} = 1$ may not be orbifold equivalent in the next section.

From our development of the algorithm we have made some interesting observations and statements concerning the structure of the quantum dimension formula and orbifold equivalence allowing us to construct all possible weight matrices at any rank and also to rule out many on various grounds. For instance, we have to have that for any weight matrix to be be viable it has to be solvable for Q_1 and Q_2 individually. We also have to be able to construct the graded fermions. Necessity is the mother of invention and these discoveries would not have been made had we had unlimited computer power.

Most of these new equivalences were found using Singular on laptops which in spite of restriction on speed and especially memory, and with some effort proved equivalence in some not so obvious cases. Given that there are only a finite number of weight matrices at any rank we can search starting from the lowest possible rank (from the weight split criteria) and working our way upwards. The algorithm is foolproof so computer speed and especially memory are the only restrictions. There is good reason to think we do not need to go up in rank ad infinitum from our study of MFs as exact sequences generated, as resolutions of different ideals constructed from sets of generators from the quotient ring. Here there are also finite possibilities. Unfortunately without a rigorous theorem we would have to try and develop a procedure to detect direct sums by using the bosonic cohomology as discussed in subsection 3.11. If there was a maximal rank for any potential such procedures would detect nothing but direct sums at a certain rank and we could stop the search.

5 Further directions

In this section we look at a number of remaining questions and observations which were raised in the work done in the last section. The first thing one notices is that the MFs Q which provide orbifold equivalences and were implementable had empty fermionic cohomology. We ask is this a feature of orbifold equivalences? We also discuss the search for potentials of equal central charge which are not orbifold equivalent and compare some possible candidates, and explain why they might provide some counter-examples of potentials with matching central charge but where it is not possible to show orbifold equivalence. We outline the problem in detail for two of these examples and suggest a way of using supercomputers so that we might at least investigate the territory.

5.1 H^1_Q and orbifold equivalences

It has been observed that the fermionic cohomologies of some recently discovered MFs which prove orbifold equivalences are empty. Unfortunately the parameterisations of many of the MFs involved are subject to highly nonlinear constraints making it impossible to compute cohomologies in these cases using SINGULAR. If we consider the numerator in the quantum dimension formula (4.3), it is given by

$$\mathcal{N} \;=\; str(\partial_{x_1}Q...\partial_{x_m}Q.\partial_{y_1}Q...v_{y_n}Q)\ , \tag{5.1}$$

which can be rewritten as

$$tr\{\partial_{x_1}E\partial_{x_2}J\ ...\ \partial_{y_{n-1}}E\partial_{y_n}J - \partial_{x_1}J\partial_{x_2}E\ ...\ \partial_{y_{n-1}}J\partial_{y_n}E\}\ . \tag{5.2}$$

It is clear that a MF which proves orbifold equivalence cannot be its own anti-brane, so there are no constant (or invertible) fermions, but why there is an empty fermionic cohomology is a bigger question.

It is a reasonable question to ask if it is true that H^1_Q is trivial for all such MFs. Previously cone construction gave us a way to construct new indecomposable matrix factorisations not isomorphic to the initial matrix factorisation. We will look at some facts about cones in relation to this.

Isomorphism of parameterised cones

When a MF, Q has a non-trivial fermionic cohomology we can construct a cone based on the direct sum of two copies of such a Q. In the first instance we assume the cone exists, hoping to prove by contradiction that a MF which is the cone of the first Q must either have zero quantum dimension or be impossible to construct. This is in contradiction to the fact that the direct sum and hence cones must have a quantum dimension (which we will show) which is twice the quantum dimension of the original Q.

First we show that all parameterisations of a cone are isomorphic for a non zero parameter, u.

If we consider two cones parameterised by different, non-zero u_1 and u_2 and the same arbitrary element of the fermionic cohomology ψ of Q used to construct the cone from the direct sum,

$$Q_T = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix}, \tag{5.3}$$

we have

$$Q_1 = \begin{pmatrix} Q & u_1\psi \\ 0 & Q \end{pmatrix} \tag{5.4}$$

and,

$$Q_2 = \begin{pmatrix} Q & u_2\psi \\ 0 & Q \end{pmatrix}. \tag{5.5}$$

We assume Q_1 and Q_2 are isomorphic and show how we can construct an invertible element of the bosonic cohomology, $H^0_{Q_1Q_2}$, which then can be used to construct the similarity transformation between Q_1 and Q_2,

$$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}. \tag{5.6}$$

Then we have

$$Q_1B - BQ_2 = 0 \tag{5.7}$$

so

$$\begin{pmatrix} Q & u_1\psi \\ 0 & Q \end{pmatrix}\begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} - \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}\begin{pmatrix} Q & u_2\psi \\ 0 & Q \end{pmatrix} = 0 \tag{5.8}$$

$$\begin{pmatrix} QB_1 & u_1\psi B_2 \\ 0 & QB_2 \end{pmatrix} - \begin{pmatrix} B_1Q & B_1u_2\psi \\ 0 & B_2Q \end{pmatrix} = 0 \tag{5.9}$$

For this to work it must be true for all blocks so we must have

$$QB_1 = B_1 Q \quad \text{and} \quad QB_2 = B_2 Q \tag{5.10}$$

This would imply that both B_1 and B_2 are in the bosonic cohomology of Q. Since they are both in H^0_Q we can take them to be proportional to the identity, and adjust by a scale factor to account for the ratio $u_1 : u_2$ and then all we require is $[B, \psi] = 0$, where B' is the element of H^0_Q for isomorphism which is trivial as B is now the direct sum of two matrices proportional to the identity.

We let $B_1 = u_1.\mathbb{I}$ and $B_2 = u_2.\mathbb{I}$ and we have found our bosonic cohomology and isomorphism as long as u_i are **non-zero**, so we cannot use this construction to show isomorphism between a direct sum and its cone.

It is also interesting that any combination of ratios for B_1 and B_2 would be in the cohomology of the simple direct sum $Q_1 \oplus Q_2$ or any individual cone. That is to say, any linear combination of $B' \oplus 0$ and $0 \oplus B'$. The cohomology between the two parameterised cones, $C(u_1), C(u_2)$, fixes this ratio.

Cones and direct sums

We can still ask the question; Is a cone isomorphic to the direct sum it was derived from? Again consider if there exist invertible members of the bosonic cohomology which prove isomorphism (as in the **isomchk** algorithm). That is to say suppose we have two MFs. A direct sum Q and a cone Q_ψ and an invertible bosonic matrix A such that,

$$AQ' = Q'_\psi A \tag{5.11}$$

where A is a block matrix,

$$A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} , \tag{5.12}$$

and

$$Q' = \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} , \qquad Q'_\psi = \begin{pmatrix} Q & \psi \\ 0 & Q \end{pmatrix} . \tag{5.13}$$

We then have the following equations;

$$a_1 Q = Q a_1 + \psi a_3$$

$$a_2 Q = Q a_2 + \psi a_4$$

$$a_3 Q = Q a_3$$

$$a_4 Q = Q a_4 \tag{5.14}$$

Thus we have $a_3, a_4 \in Ker(d_Q^0)$.

Either a_3 and a_4 can certainly be chosen to be the invertible element in the bosonic cohomology of Q, but then we immediately have

$$\psi \;=\; a_1 a_3^{-1} Q - Q a_1 a_3^{-1} \qquad \text{or} \qquad a_2 a_4^{-1} Q = Q a_2 a_4^{-1} \;, \tag{5.15}$$

and hence ψ is in the image $Im(d_Q^0)$ and therefore not in the fermionic cohomology as originally stated.

If we choose a_3 or a_4 to be non-invertible then ψ is in the image $Im(d_Q^0)$ so both must be invertible.

We can not use the formula for the determinant of a block matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$

$$\det(M) \;=\; \det(A - BD^{-1}C)\det(D) \;, \tag{5.16}$$

Unfortunately having two non-invertible blocks does not mean that A is non-invertible. There are counter examples, but of course in this case these are not any block matrices but elements of the bosonic cohomology and perhaps this would make a difference.

If we could show A cannot be invertible then we would have a contradiction and we could make the statement that a true cone constructed from an element of the cohomology of d_Q^1, ($\psi \in H_Q^1$) is not isomorphic to the direct sum it is constructed from, however this is not the case at present and neither is this essential because we know the cone *can* be non-isomorphic to the direct sum.

Effects of a cone on the quantum dimension formulae

We can see that the existence of a cone would have no effect on the quantum dimension calculation, hence the non isomorphic cone and simple direct sum have the same quantum dimension. We can write the cone as $Q_c = Q' + \psi'$ where

$$Q' \;=\; \begin{pmatrix} Q & 0 \\ 0 & Q \end{pmatrix} \;, \qquad Q_c = \begin{pmatrix} Q & \psi \\ 0 & Q \end{pmatrix} \;, \qquad \psi' = \begin{pmatrix} 0 & \psi \\ 0 & 0 \end{pmatrix} \tag{5.17}$$

Then $\partial Q_c = \partial Q' + \partial \psi'$ and we can write products of differentials

$$\partial_x Q_c \partial_y Q_c \;=\; (\partial_x Q' + \partial_x \psi')(\partial_y Q' + \partial_y \psi')$$

$$= \partial_x Q' \partial_y Q' + \partial_x \psi' \partial_y Q' + \partial_x Q' \partial_y \psi' + \partial_x \psi' \partial_y \psi'$$

$$= \begin{pmatrix} \partial_x Q \partial_y Q & 0 \\ 0 & \partial_x Q \partial_y Q \end{pmatrix} + \begin{pmatrix} 0 & \partial_x \psi \; \partial_y Q \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \partial_x Q \partial_y \psi \\ 0 & 0 \end{pmatrix} \quad (5.18)$$

This implies that the addition of a nilpotent matrix such as ψ' makes no difference to the supertrace and hence quantum dimension of a direct sum. This would mean that if a MF with invertible quantum dimension and a non trivial fermionic cohomology existed, Q then there would exist non isomorphic MFs Q_ψ (cones and direct sum) which would have the same quantum dimension.

To summarise so far we have that:

i) All invertible parameterisations of a cone are isomorphic. And cones are not isomorphic to the null cone or simple direct sum.

ii) If a fermionic matrix did exist to construct a cone, it would make no difference to the quantum dimension of that cone or the direct sum it was derived from. This means that some non-isomorphic MFs have the same quantum dimensions.

Unfortunately, having different quantum dimensions is sufficient to show non-isomorphism but it is not necessary. Consider a direct sum of two MFs of an orbifold equivalence, one with zero quantum dimensions, Q_0 and one with non zero quantum dimensions, Q_{qd}. Then we could take the direct sum $Q' = Q_0 \oplus Q_{qd}$ and we would have Q_{qd} and Q' non isomorphic but with the same quantum dimension.

If we wish to prove the conjecture that these MFs always have trivial H^1_Q then we would have to show it directly from the formula for the numerator 5.1 or show that the same quantum dimensions are a sufficient and necessary criteria for different MFs of this sort at the *same rank* to be isomorphic. In either case both aspects are related and may provide a further insight into orbifold equivalence

5.2 Central charge $c = 3,\ \hat{c} = 1$

These models are very different from those for which orbifold equivalence has been found so far, $\hat{c} = 1$ seems to be a special value. Two potentials can only be orbifold equivalent (by a graded Q) if they have the same central charge. It is a natural question to ask, does equal central charge imply the potentials are orbifold equivalent? Ideally we would like to find counterexamples, and $\hat{c} = 1$ seems a good place to look for them for reasons that will be outlined. Unfortunately, we have nothing conclusive yet, but we present a partial analysis of two cases: one is elliptic-quartic, where we think orbifold equivalence is ruled out due to *incompatibility* of variable weights, there are no pairs monomials of the same weight and weight less than 2, in the left and right variables. The other case is to look at orbifold equivalence among geometrically inequivalent elliptic curves (section 3.3). Although neither of these questions have been solved, the problem is analysed and set out, with the possibility of verification by the perturbative algorithm of section 4 and more computing power.

5.3 Quartic $\sim$ Elliptic

The first pair of potentials we consider have no common variables with the same weight. We saw in section 4.7 that an orbifold equivalence requires a MFs with mixed terms. By considering the weights of all monomials we see that the introduction of mixed terms is a contrived affair, as there is no general guide from the weight criteria (6.8).

Let us begin by considering a graded MF $Q(x, y)$ of rank $2N$, of a potential W

$$W(x, y) = U(x) - V(y) \ , \quad x = \{x_i\} \ , \ y = \{y_i\} \ . \tag{5.19}$$

$$\begin{aligned} U &= y_1^3 + y_2^3 + y_3^3 - dy_1y_2y_3 \ , \\ V &= x_1^4 + x_2^4 + x_3^2 \ . \end{aligned} \tag{5.20}$$

Geometrically, the zero locus of the potential V is a torus, so there is a bi-rational equivalence [17] to the zero locus of U for a specific value of d. We can see this in the following way. Consider the quartic V, we *de-homogenise* by setting $x_2 = 1$.

$$V = x_1^4 + 1 + x_3^2 \tag{5.21}$$

Then the zero locus $V = 0$ becomes

$$x_1^4 + 1 + x_3^2 = 0 \quad \text{or} \quad (x_3 + x_1^2)(x_3 - x_1^2) = 1 \ . \tag{5.22}$$

We now make the substitutions $t = x_1^2 + x_3$ and $s = x_1 t$ then $V = 0$ becomes

$$s^2 = t^3 - t \ . \tag{5.23}$$

This is a de-homogenised version of the λ form of the elliptic [4.46] i.e.

$$U_\lambda = -zy^2 + w(w - z)(w - \lambda z) = 0 \tag{5.24}$$

with $z = 1$. In this case $\lambda = -1$. It is worth noting that we dehomogenise in different weighted projective spaces.

There is no general theorem, so far, which ensures that potentials with bi-rationallly equivalent zero loci are orbifold equivalent. Classification by orbifold equivalence might very well lead to different equivalence classes than classification by geometrical equivalence. This is the case with classification by equivalence of singularities. That the latter gives a different classification is already clear, the A-E case which are orbifold equivalent but different as singularities, or potentials differing by some z^2 term, which describe the same singularity but are not orbifold equivalent.

Using the formalism developed for the perturbative algorithm[12] we consider the limit $x = 0$. We can write

$$Q_y \;=\; Q(0,y) \;=\; F^c_i y_i + M^c_{jk} y_j y_k \ , \tag{5.25}$$

where F^c_i is the constant part of the matrix

$$F_i = \partial_{y_i} Q|_{y=0} \ . \tag{5.26}$$

and M^c_{jk} is the constant part of the matrix

$$M_{jk} = \partial_{y_j}\partial_{y_k} Q|_{y=0} \ . \tag{5.27}$$

We can also consider the limit $y = 0$ and write

$$Q_x \;=\; Q(x,0) \;=\; G^c_i x_i + N^c_{jk} x_j x_k + P^c_{abc} x_a x_b x_c \ , \tag{5.28}$$

where G^c_i is the constant part of the matrix

$$G_i = \partial_{x_i} Q|_{x=0} \tag{5.29}$$

and N^c_{jk} is the constant part of the matrix

$$N_{jk} = \partial_{x_j}\partial_{x_k} Q|_{x=0} \ \text{, for} \ \ j,k = 1,2 \quad \text{and} \quad \partial_{x_3} Q|_{x=0} \ \text{, for} \ x_3 \quad (z) \tag{5.30}$$

with P^c_{abc} defined analogously.

For such a graded matrix factorisation we notice the weights are such that there can be no quasi-homogenous polynomial matrix elements that contain pure y *and* pure x terms. If the matrix factorisation is to have non zero quantum dimensions we know there have to be 'mixed' terms. These mixed terms must also be of different weights and so distinct from the matrix elements which make up Q_x and Q_y.

We can summarise the possible mixed elements and weights of all possible matrix elements that appear in any expansion of Q. First we have the matrices with distinct matrix elements and the monomials they are associated with

[12]Note we slightly change notation from Q_1 to Q_x.

Q	F^c	M^c
G^c	A	B
N^c	C	D
P^c	E	H

	y	y^2
x	xy	xy^2
$x^2(orz)$	x^2y	x^2y^2
x^3	x^3y	x^3y^2

with weights and grading values

/6	4	8
3	7	11
6	10	14
9	13	17

Δg_{ij}	$-1/3$	$1/3$
$-1/2$	$1/6$	$5/6$
0	$4/6$	$8/6$
$1/2$	$7/6$	$11/6$

Generic anti-commutator relations

We assume we have the most general mixed terms.That is every weight from the above table is represented in our matrix factorisation. In this case they must form a large set of anti-commutator relations, from the MF condition of Q and the individual Q_x and Q_y. Since we know $Q^2 = Q_x^2 + Q_y^2$ we have

$$Q^2 - Q_x^2 - Q_y^2 \quad = \quad 0 \tag{5.31}$$

and

$$\{Q_x, Q_y\} + \{Q_E, Q_y\} + \{Q_x, Q_E\} + Q_E Q_E \quad = \quad 0 \tag{5.32}$$

where

$$Q_E = Axy + Bxy^2 + Cx^2y + Dx^2y^2 + Ex^3y + Hx^3y^2 \ . \tag{5.33}$$

Here the x and y carry no index and are just symbolic. It is the different weights that are important and variables can be added by weight and indices. It is to be understood for instance that we achieve the same weight for x_1^2 and x_2^2 as for x_3 and where x_3 can enter these equations.

$$\{Q_x, Q_y\} \quad = \quad \{G_i^c x_i + N_{jk}^c x_j x_k + P_{abc}^c x_a x_b x_c, F_i^c y_i + M_{jk}^c y_j y_k\} \tag{5.34}$$

We can drop the indices as they are easy to put back in using the tables above. Rewriting

$$\{Q_x, Q_y\} = \{G^c, F^c\}xy + \{N^c, F^c\}x^2y + \{P^c, F^c\}x^3y + \{G^c, M^c\}xy^2$$
$$+\{N^c, M^c\}x^2y^2 + \{P^c, M^c\}x^3y^2 \tag{5.35}$$

We also have

$$\{Q_E, Q_y\} = \{Axy + Bxy^2 + Cx^2y + Dx^2y^2 + Ex^3y + Hx^3y^2, F^cy + M^cy^2\}$$
$$= \{A, F^c\}xy^2 + (\{A, M^c\} + \{B, F^c\})xy^3 + (\{C, M^c\} + \{D, F^c\})x^2y^3$$
$$+\{B, M^c\}xy^4 + \{D, M^c\}x^2y^4 + (\{E, M^c\} + \{H, F^c\})x^3y^3$$
$$+\{H, M^c\}x^3y^4 + \{C, F^c\}x^2y^2 + \{E, F^c\}x^3y^2 \quad , \qquad (5.36)$$

$$\{Q_E, Q_x\} = \{Axy + Bxy^2 + Cx^2y + Dx^2y^2 + Ex^3y + Hx^3y^2, G^cx + N^cx^2 + P^cx^3\}$$
$$= \{A, G^c\}x^2y + (\{A, N^c\} + \{C, G^c\})x^3y + (\{A, P^c\} + \{C, N^c\} + \{E, G^c\})x^4y$$
$$\{B, G^c\}x^2y^2 + (\{B, N^c\} + \{D, G^c\})x^3y^2 + (\{B, P^c\} + \{D, N^c\} + \{H, G^c\})x^4y^2$$
$$+(\{C, P^c\} + \{E, N^c\})x^5y + (\{D, P^c\} + \{H, N^c\})x^5y^2$$
$$+\{E, P\}x^6y + \{H, P\}x^6y^2 \quad . \quad (5.37)$$

finally,

$$Q_E^2 = (Axy + Bxy^2 + Cx^2y + Dx^2y^2 + Ex^3y + Hx^3y^2)^2$$
$$= A^2x^2y^2 + \{A, B\}x^2y^3 + \{A, C\}x^3y^2 + B^2x^2y^4 + (\{A, D\} + \{C, B\})x^3y^3$$
$$+(\{A, E\} + C^2)x^4y^2 + (\{A, H\} + \{E, B\} + \{C, D\})x^4y^3 + \{B, H\} + D^2)x^4y^4$$
$$+(\{C, H\} + \{D, E\})x^5y^3 + \{D, H\}x^5y^4 + \{E, H\}x^6y^3 + \{C, E\}x^5y^2$$
$$+\{B, D\}x^3y^4 + E^2x^6y^2 + H^2x^6y^4 \ . \qquad (5.38)$$

Thus for each weight/type of monomial we have an equation. In order to show the complexity of the relations required to satisfy the mixed terms criteria of orbifold equivalence we present the full list of anti-commutators. This list could be narrowed down by just taking mixed terms of the form x_iy_k or some other restricted set.

$$xy \quad \{G^c, F^c\} = 0$$
$$xy^2 \quad \{A, F^c\} + \{G^c, M^c\} = 0$$
$$xy^3 \quad \{A, M^c\} + \{B, F^c\} = 0$$

$$\begin{aligned}
xy^4 \quad \{B, M^c\} &= 0 \\
x^2y \quad \{N^c, F^c\} + \{A, G^c\} &= 0 \\
x^2y^2 \quad \{N^c, M^c\} + \{C, F^c\} + \{B, G^c\} + A^2 &= 0 \\
x^2y^3 \quad \{C, M^c\} + \{D, F^c\} + \{A, B\} &= 0 \\
x^2y^4 \quad \{D, M^c\} + B^2 &= 0 \\
x^3y \quad \{P^c, F^c\} + \{A, N^c\} + \{C, G^c\} &= 0 \\
x^3y^2 \quad \{P^c, M^c\} + \{E, F^c\} + \{B, N^c\} + \{D, G^c\} + \{A, C\} &= 0 \\
x^3y^3 \quad \{A, D\} + \{C, B\} + \{E, M^c\} + \{H, F^c\} &= 0 \\
x^3y^4 \quad \{B, D\} + \{H, M^c\} &= 0 \\
x^4y \quad \{A, P^c\} + \{C, N^c\} + \{E, G^c\} &= 0 \\
x^4y^2 \quad \{B, P^c\} + \{D, N^c\} + \{H, G^c\} + \{A, E\} + C^2 &= 0 \\
x^4y^3 \quad \{A, H\} + \{E, B\} + \{C, D\} &= 0 \\
x^4y^4 \quad \{B, H\} + D^2 &= 0 \\
x^5y \quad \{C, P^c\} + \{E, N^c\} &= 0 \\
x^5y^2 \quad (\{D, P^c\} + \{H, N^c\} + \{C, E\} &= 0 \\
x^5y^3 \quad \{C, H\} + \{D, E\} &= 0 \\
x^5y^4 \quad \{D, H\} &= 0 \\
x^6y \quad \{E, P\} &= 0 \\
x^6y^2 \quad \{H, P\} + E^2 &= 0 \\
x^6y^3 \quad \{E, H\} &= 0 \\
x^6y^4 \quad H^2 &= 0
\end{aligned} \tag{5.39}$$

We stress again that these matrices carry implicit indices, for instance $\{E, P\}$ actually means,

$$\{E_{abcl}, P_{efg}\}x_a x_b x_c x_e x_f x_g y_l \tag{5.40}$$

This requires special consideration for x_3^2 terms.

Fermion equations

We might try and use the fermion equations to give us some more relations.

$$\partial_{x_i} Q^2|_{x=0} \;=\; \{Q_y, G_i\} \;\; , \qquad \partial_{y_i} Q^2|_{y=0} \;=\; \{Q_x, F_i\} \;\; . \tag{5.41}$$

The first gives

$$\{Q_y, G_i\} \;=\; \{F_l^c y_l + M_{mn}^c y_m y_n \;,\; G_i^c + Ay + By^2\} \;=\; 0 \;, \tag{5.42}$$

for $x_1,\; x_2$ and

$$\{Q_y, G_z\} \;=\; \{F_l^c y_l + M_{mn}^c y_m y_n \;,\; N_z^c + C_z y + D_z y^2\} \;=\; 0 \;, \tag{5.43}$$

equating monomials in either case gives nothing new.

Individual potentials

If we now just consider the square of the matrix factorisations of the separate individual potentials U or V.

$$\begin{aligned} Q_y^2 &= (F_l^c y_l + M_{mn}^c y_m y_n)^2 \;=\; U \;=\; y_1^3 + y_2^3 + y_3^3 = d y_1 y_2 y_3 \\ &= \{F^c, M^c\} \end{aligned} \tag{5.44}$$

we see that

$$\begin{aligned} \{F^c, F^c\} &= 0 \\ \{M^c, M^c\} &= 0 \end{aligned} \tag{5.45}$$

We have a similar situation for Q_y,

$$\begin{aligned} Q_y^2 &= (G_x^c + N^c x^2 + P_c x^3)^2 \;=\; V \;=\; x_1^4 + x_2^4 + z^2 \\ &= \{G^c, P^c\} + N^2 \end{aligned} \tag{5.46}$$

we see that

$$\begin{aligned} \{G^c, G^c\} &= 0 \\ \{P^c, P^c\} &= 0 \end{aligned} \tag{5.47}$$

Both of these give us some new anti-commutators, but no way has been found to solve these anti-commutators, or make use of them so far.

There is one observation we can make from this analysis. We note that it is always true, for any orbifold equivalence, that the first anti-commutator in the list, $\{F^c, G^c\} = 0$, for any pair of potentials. The lowest order elements supporting single powers of variables always anti-commute because their product (xy) cannot be compensated for by the product of a matrix element with a mixed term.

Supertrace of the numerator

At $\hat{c} = 1$ and with three variables in each potential the numerator in the quantum dimension formula has a special property. The numerator can be seen to be the product of two odd morphisms of R-charge equal to 1. This means the numerator has the same weight matrix in the limit $x = 0$ resp. $y = 0$ as Q_y resp. Q_x.

$$\begin{aligned} \mathcal{N}|_{x=0} &= \partial_{y_1} Q_y \partial_{y_2} Q_y \partial_{y_3} Q_y G_1 G_2 G_3 \\ &= (\mathsf{F}+\mathsf{M})_y . (\mathsf{F}'+\mathsf{M}')_x \quad . \end{aligned} \tag{5.48}$$

where $(\mathsf{F}+\mathsf{M})_y$ represents an odd morphism with single and quadratic powers of y derived from the product of the derivatives of Q_y w.r.t y, and $(\mathsf{F}'+\mathsf{M}')_x$ represents the product of the three fermions G_i. We write it in this suggestive form as it has the same weight matrix as Q_y.

We also have

$$\begin{aligned} \mathcal{N}|_{y=0} &= F_1 F_2 F_3 \partial_{x_1} Q_x \partial_{x_2} Q_x \partial_{x_3} Q_x \\ &= (\mathsf{G}'+\mathsf{N}'+\mathsf{P}')_y . (\mathsf{G}+\mathsf{M}+\mathsf{P})_x \quad . \end{aligned} \tag{5.49}$$

This splitting of the numerator into two odd morphisms with R-charge 1, is true when looking at possible elliptic $\sim$ elliptic equivalences or between any other polynomials with $\hat{c} = 1$ and an equal number of variables.

Tensor product of quartic and elliptic

MFs for the potential do exist (but they are tensor products, so with zero quantum dimensions). The grading matrices for these have matrix elements of the correct weight to admit these mixed terms.

Consider the MF of $V = x_1^4 + x_2^4 + z^2$. Let a, b be the polynomials,

$$a = x_1 - ix_2 \,, \qquad b = x_1^3 + ix_1^2 x_2 - x_1 x_2^2 + ix_2^3 \,. \tag{5.50}$$

Then the MF is,

$$E_q = \begin{pmatrix} a & z \\ -z & b \end{pmatrix} , \qquad J_q = \begin{pmatrix} b & -z \\ z & a \end{pmatrix} \tag{5.51}$$

with weight matrices,

$$W_{E_q} = \begin{pmatrix} 3 & 6 \\ 6 & 9 \end{pmatrix}, \qquad W_{J_q} = \begin{pmatrix} 9 & 6 \\ 6 & 3 \end{pmatrix} \tag{5.52}$$

expressed as multiples of one sixth.

For the elliptic we have $U = y_1^3 + y_3^3 + y_3^3 - dy_1y_2y_3$. We use the standard rank 2 'short branes'. These have the form of quadratic and linear polynomials. The rank 2 MF is given by

$$E = \begin{pmatrix} Q_1 & -Q_2 \\ L_2 & L_1 \end{pmatrix} \qquad J = \begin{pmatrix} L_1 & Q_2 \\ -L_2 & Q_1 \end{pmatrix} . \tag{5.53}$$

where linear terms are given by

$$L_1 = \alpha_3 x_1 - \alpha_2 x_3$$

$$L_2 = -\alpha_3 x_2 + \alpha_1 x_3 , \tag{5.54}$$

and quadratic terms are

$$Q_1 = \frac{1}{\alpha_1\alpha_2\alpha_3}(\alpha_1\alpha_2 x_1^2 + \alpha_2^2 x_1 x_2 - \alpha_1^2 x_2^2 - \alpha_1\alpha_3 x_3^2)$$

$$Q_2 = \frac{1}{\alpha_1\alpha_2\alpha_3}(\alpha_2^2 x_1^2 - \alpha_1^2 x_1 x_2 - \alpha_1\alpha_2 x_2^2 + \alpha_3^2 x_1 x_3) . \tag{5.55}$$

and have weight matrices

$$w(E_c) = \begin{pmatrix} 8 & 8 \\ 4 & 4 \end{pmatrix}, \qquad w(J_c) = \begin{pmatrix} 4 & 8 \\ 4 & 8 \end{pmatrix} \tag{5.56}$$

The tensor product construction gives us

$$E = \begin{pmatrix} E_q \otimes 1 & -1 \otimes J_c \\ 1 \otimes E_c & J_q \otimes 1 \end{pmatrix} \qquad J = \begin{pmatrix} J_q \otimes 1 & 1 \otimes J_c \\ -1 \otimes E_c & E_q \otimes 1 \end{pmatrix} \tag{5.57}$$

$$w(E) = \begin{pmatrix} 3 & 6 & 0 & 0 & 4 & 0 & 8 & 0 \\ 6 & 9 & 0 & 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 3 & 6 & 4 & 0 & 8 & 0 \\ 0 & 0 & 6 & 9 & 0 & 4 & 0 & 8 \\ 8 & 0 & 8 & 0 & 9 & 6 & 0 & 0 \\ 0 & 8 & 0 & 8 & 6 & 3 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 & 9 & 6 \\ 0 & 4 & 0 & 4 & 0 & 0 & 6 & 3 \end{pmatrix}, \qquad w(J) = \begin{pmatrix} 9 & 6 & 0 & 0 & 4 & 0 & 8 & 0 \\ 6 & 3 & 0 & 0 & 0 & 4 & 0 & 8 \\ 0 & 0 & 9 & 6 & 4 & 0 & 8 & 0 \\ 0 & 0 & 6 & 3 & 0 & 4 & 0 & 8 \\ 8 & 0 & 8 & 0 & 3 & 6 & 0 & 0 \\ 0 & 8 & 0 & 8 & 6 & 9 & 0 & 0 \\ 4 & 0 & 4 & 0 & 0 & 0 & 3 & 6 \\ 0 & 4 & 0 & 4 & 0 & 0 & 6 & 9 \end{pmatrix} \tag{5.58}$$

with grading matrix $g = \text{diag}\{0, 3, 0, 3, 5, 2, 1, -2, 3, 0, 3, 0, 2, 5, -2, 1\}$.

If we construct the weight matrix from the grading matrix we see the weights of many of the zero terms include weight 7 terms which correspond to the Axy and also there are weight 5 terms which have no expression but are the reflection in the anti diagonal of the weight 7 terms. The weight of the final polynomial is 12 or more precisely 12/6.

The full weight matrix for E is

$$w(E) \;=\; \begin{pmatrix} 3 & 6 & 3 & 6 & 4 & 1 & 8 & 5 \\ 6 & 9 & 6 & 9 & 7 & 4 & 11 & 8 \\ 3 & 6 & 3 & 6 & 4 & 1 & 8 & 5 \\ 6 & 9 & 6 & 9 & 7 & 4 & 11 & 8 \\ 8 & 11 & 8 & 11 & 9 & 6 & 13 & 10 \\ 5 & 8 & 5 & 8 & 6 & 3 & 10 & 7 \\ 4 & 7 & 4 & 7 & 5 & 2 & 9 & 6 \\ 1 & 4 & 1 & 4 & 2 & -1 & 6 & 3 \end{pmatrix} . \tag{5.59}$$

Similarity transformations and the grading matrix

We can, by means of row and column transformations, arrange our matrix factorisation so that the entries in E have weights that increase monotonically as we go from the top left matrix element to the top right, and the same for the top left to the bottom left. We can arrange the matrices by weight or Δg_{ij}.

element	Δg_{ij}	weight
G , x	-3,	3
F , y	-2	4
N , z or x^2	0	6
A , xy	1	7
M , y^2	2	8
P , x^3	3	9
C , x^2y	4	10
B , xy^2	5	11
E , x^3y	7	13
D , x^2y^2	8	14
H , x^3y^2	11	17

The mixed terms all lie at the end of the first row/column of E, with the exception of A. It seems reasonable that we start with only mixed terms of the form xy as represented by the weight 7 matrix, A. The first row and column of E give us all we need to know about the grading matrix. The first row gives us,

$$g_1 - \{g_{N+1} ... g_{i_1}\} \;=\; -1/2 \;, \quad -3$$

$$g_1 - \{g_{i_1} ... g_{i_2}\} \;=\; -1/3 \;, \quad -2$$

$$
\begin{aligned}
g_1 - \{g_{i_2}...g_{i_3}\} &= 0\ , \quad 0 \\
g_1 - \{g_{i_3}...g_{i_4}\} &= 1/6\ , \quad 1 \\
g_1 - \{g_{i_4}...g_{i_5}\} &= 1/3\ , \quad 2 \\
g_1 - \{g_{i_5}...g_{2N}\} &= 1/2\ , \quad 3
\end{aligned} \tag{5.60}
$$

where $N + 1 < i_1 < i_2 < i_3 < i_4 < i_5 < 2N$.

Similarly the first column gives us

$$
\begin{aligned}
\{g_1...g_{j_1}\} - g_{N+1} &= -1/2\ , \quad -3 \\
\{g_{j_1}...g_{j_2}\} - g_{N+1} &= -1/3\ , \quad -2 \\
\{g_{j_2}...g_{j_3}\} - g_{N+1} &= 0\ , \quad 0 \\
\{g_{j_3}...g_{j_4}\} - g_{N+1} &= 1/6\ , \quad 1 \\
\{g_{j_4}...g_{j_5}\} - g_{N+1} &= 1/3\ , \quad 2 \\
\{g_{j_5}...g_{N}\} - g_{N+1} &= 1/2\ , \quad 3
\end{aligned} \tag{5.61}
$$

where $1 < j_1 < j_2 < j_3 < j_4 < j_5 < N$.

We set $g_1 = 0$. This implies

$$
\begin{aligned}
\{g_{N+1}...g_{i_1}\} &= 3 \\
\{g_{i_1}...g_{i_2}\} &= 2 \\
\{g_{i_2}...g_{i_3}\} &= 0 \\
\{g_{i_3}...g_{i_4}\} &= -1 \\
\{g_{i_4}...g_{i_5}\} &= -2 \\
\{g_{i_5}...g_{2N}\} &= -3\ .
\end{aligned} \tag{5.62}
$$

And since $g_{N+1} = 3$, we have

$$
\begin{aligned}
\{g_1...g_{j_1}\} &= -1/2\ , \quad 0 \\
\{g_{j_1}...g_{j_2}\} &= -1/3\ , \quad 1
\end{aligned}
$$

$$
\begin{aligned}
\{g_{j_2}...g_{j_3}\} &= 0\ , \quad 3 \\
\{g_{j_3}...g_{j_4}\} &= 1/6\ , \quad 4 \\
\{g_{j_4}...g_{j_5}\} &= 1/3\ , \quad 5 \\
\{g_{j_5}...g_{N}\} &= 1/2\ , \quad 6
\end{aligned}
\tag{5.63}
$$

We can now write down a generalised grading matrix;

$$
g = \mathrm{diag}\{0, 1, 3, 4, 5, 6, 3, 2, 0, -1, -2, -3\}\ . \tag{5.64}
$$

The numbers represent a range of entries. We can now construct our weight matrix for the matrix factorisation Q .

$$
W_{ij} = g_i - g_j + 1\ . \tag{5.65}
$$

This assigns weights to the blocks of matrix elements. The following matrices are not a rank 6 but each number represents a possible block weight for a block of matrix elements. The possible weight matrix for E;

$$
w(E) = \begin{pmatrix} 3 & 4 & 6 & 7 & 8 & 9 \\ 4 & 5 & 7 & 8 & 9 & 10 \\ 6 & 7 & 9 & 10 & 11 & 12 \\ 7 & 8 & 10 & 11 & 12 & 13 \\ 8 & 9 & 11 & 12 & 13 & 14 \\ 9 & 10 & 12 & 13 & 14 & 15 \end{pmatrix} \tag{5.66}
$$

and J

$$
w(J) = \begin{pmatrix} 9 & 8 & 6 & 5 & 4 & 3 \\ 8 & 7 & 5 & 4 & 3 & 2 \\ 6 & 5 & 3 & 2 & 1 & 0 \\ 5 & 4 & 2 & 1 & 0 & -1 \\ 4 & 3 & 1 & 0 & -1 & -2 \\ 3 & 2 & 0 & -1 & -2 & -3 \end{pmatrix} . \tag{5.67}
$$

Most of these entries which are blocks of matrix elements cannot be filled. If we just look at weight 6 terms which would correspond to a single x_3 or x_i^2 , $i = 1, 2$ type term, there are not enough for this to be the weight matrix of a matrix factorisation. One can see by inspection that throwing in the higher weight matrices at the begining will not improve the situation.

This can be ordered using row and column transformations to give

$$w(E') \;=\; \begin{pmatrix} -1 & 1 & 1 & 2 & 3 & 4 & 4 & 6 \\ 1 & 3 & 3 & 4 & 5 & 6 & 6 & 8 \\ 1 & 3 & 3 & 4 & 5 & 6 & 6 & 8 \\ 2 & 4 & 4 & 5 & 6 & 7 & 7 & 9 \\ 3 & 5 & 5 & 6 & 7 & 8 & 8 & 10 \\ 4 & 6 & 6 & 7 & 8 & 9 & 9 & 11 \\ 4 & 6 & 6 & 7 & 8 & 9 & 9 & 11 \\ 6 & 8 & 8 & 9 & 10 & 11 & 11 & 13 \end{pmatrix} . \tag{5.68}$$

We can observe that every row/column must contain a weight 6, x_3 term, thereby implying that for every g_i , $i \in \{N+1, ..., 2N\}$ there must be an equal g_j , $j \in \{N+1, ..., 2N\}$. Thus the existence of a x_3 term means we only have to fix N of the g_i.

We can see from the above that there can only be a finite number of possible gradings at any rank and we know the minimum requirement in terms of pairs of factors for the elliptic is two terms, and the same goes for the quartic (with x_3^2). It should be straight forward to construct an algorithm which gives all the possible gradings systematically and at any rank.

The construction of any MF satisfying non-zero quantum dimensions would have to be very different from the previously discovered examples in that the mixed terms would be totally disjoint, in terms of matrix elements. This and the other peculiarities of this pairing might provide a basis for an analytic proof or systematic search which shows these are models with the same value for $\hat{c}$ but which are not orbifold equivalent. Even though the weight split criteria used before is inapplicable in this case, we have seen that given any rank, it would still be possible to compute all the possible gradings which would allow for the necessary terms for the elliptic-quartic pair to be a candidate for orbifold equivalence. This means it would be possible, in theory, to construct a systematic search for orbifold equivalences at every possible rank with every possible grading.

5.4 Elliptic $\sim$ Elliptic

This time we will just focus on the elliptic curve, which is a function of three complex variables and defined by one complex parameter, $d \in \mathbb{C}$. Here we write the potential as

$$\mathcal{E}(x, d) \;=\; x_1^3 + x_2^3 + x_3^3 - d x_1 x_2 x_3 \; . \tag{5.69}$$

Matrix elements and anti-commutator relations

Once again we start by assuming that we have a MF which has invertible quantum dimensions. Consider such a graded MF $Q(x,y)$ of rank N, of a potential

$$W(x,y) = \mathcal{E}(x,d_1) - \mathcal{E}(y,d_2) \ , \quad x = \{x_i\} \ , \ y = \{y_i\} \ . \tag{5.70}$$

We can write

$$\mathcal{E}(x,d_1) = Q_x^2 \ : Q_x \ = \ Q(x,0) \ = \ F_i^x x_i + M_{jk}^x x_j x_k \ , \tag{5.71}$$

and

$$\mathcal{E}(y,d_2) = Q_y^2 \ : Q_y \ = \ Q(0,y) \ = \ F_i^y y_i + M_{jk}^y y_j y_k \ , \tag{5.72}$$

where F_i^c is the constant matrix

$$F_i^x = \partial_{x_i} Q|_{x=0} \ . \tag{5.73}$$

and M_{jk}^x is the constant matrix

$$M_{jk}^x = \partial_{x_j}\partial_{x_k} Q|_{x=0} \ . \tag{5.74}$$

Likewise for the matrix coefficients in Q_y . If we assume we have the most general mixed terms we see these all have to have weight 4/3 and involve one variable from each set $x = \{x_i\} \ , \ y = \{y_i\}$.

Since we know $Q^2 = Q_x^2 + Q_y^2$,

$$Q^2 - Q_x^2 - Q_y^2 \ = \ 0 \ , \tag{5.75}$$

$$\{Q_x, Q_y\} + \{Q_E, Q_y\} + \{Q_x, Q_E\} + Q_E Q_E \ = \ 0 \ , \tag{5.76}$$

where the mixed terms all have weight 4/3 and Q_E must be of the form

$$Q_E = M_{ij}^E x_i y_j \ . \tag{5.77}$$

This leaves us with the following anti-commutator relations, suppressing indices;

$$\begin{aligned} xy \quad \{F_i^x, F_j^y\} &= 0 \\ x^2 y \quad \{M^E, F^x\} + \{F^y, M^x\} &= 0 \\ xy^2 \quad \{M^E, F^y\} + \{F^x, M^y\} &= 0 \end{aligned}$$

$$
\begin{aligned}
x^2y^2 \quad \{M^x, M^y\} + \{M^E, M^E\} &= 0 \\
x^3y \quad \{M^E, M^x\} &= 0 \\
xy^3 \quad \{M^E, M^y\} &= 0 \, .
\end{aligned} \tag{5.78}
$$

These matrices carry implicit indices, for instance, there are actually 9 different M^E_{ij} , one for each pairing from $\{x_i\}$ and $\{y_i\}$. We can also consider the individual potentials,

$$
\begin{aligned}
Q_y^2 &= (F^y_l y_l + M^y_{mn} y_m y_n)^2 = \mathcal{E}(y, d_2) = y_1^3 + y_2^3 + y_3^3 - d_2 y_1 y_2 y_3 \\
&= \{F^y, M^y\}(y^3)
\end{aligned} \tag{5.79}
$$

we see that

$$
\begin{aligned}
\{F^y, F^y\} &= 0 \\
\{M^y, M^y\} &= 0 \, ,
\end{aligned} \tag{5.80}
$$

and likewise for $\{F^x, F^x\}$ and $\{M^x, M^x\}$. Equations (1.11), (1.12) and (1.13) give us a more complete set of anti-commutators.

The supertrace of the numerator

For the elliptic curve we note again that the R-charge of the numerator must be 2 and that each triple $\mathcal{Q}_L = \partial_{x_1} Q \partial_{x_2} Q \partial_{x_3} Q$ and $\mathcal{Q}_R = \partial_{y_1} Q \partial_{y_2} Q \partial_{y_3} Q$ has R-charge 1, and so must have the same weight matrix and be similar in form to the original matrix factorisation. Next we consider the product at $y = 0$,

$$
\mathcal{N} = \partial_{x_1} Q_x \partial_{x_2} Q_x \partial_{x_3} Q_x \times \mathcal{Q}_R|_{y=0}
$$

where

$$
\mathcal{Q}_R|_{y=0} = \partial_{y_1}(Q_y + Q_E)|_{y=0} \times \partial_{y_2}(Q_y + Q_E)|_{y=0} \times \partial_{y_3}(Q_y + Q_E)|_{y=0} \, . \tag{5.81}
$$

First we consider the product $\mathcal{Q}_L = \partial_{x_1} Q_x \partial_{x_2} Q_x \partial_{x_3} Q_x$. We have

$$
\begin{aligned}
\partial_{x_1} Q_x &= F^x_1 + 2M^x_{11} x_1 + M^x_{12} x_2 + M^x_{13} x_3 \\
\partial_{x_2} Q_x &= F^x_2 + 2M^x_{22} x_2 + M^x_{21} x_1 + M^x_{23} x_3 \\
\partial_{x_3} Q_x &= F^x_3 + 2M^x_{33} x_3 + M^x_{31} x_1 + M^x_{32} x_2
\end{aligned} \tag{5.82}
$$

Then we consider the product of the fermions in the $Q_x^2 = U(x)$ theory, $\mathcal{Q}_R|_{y=0}$. In this case we have

$$
\partial_{y_1}(Q_y + Q_E)|_{y=0} = F^y_1 + M^E_{1'1} x_1 + M^E_{2'1} x_2 + M^E_{3'1} x_3
$$

$$\partial_{y_2}(Q_y + Q_E)|_{y=0} \;=\; F^y_2 + M^E_{1'2}x_1 + M^E_{2'2}x_2 + M^E_{3'2}x_3$$

$$\partial_{y_3}(Q_y + Q_E)|_{y=0} \;=\; F^y_3 + M^E_{1'3}x_1 + M^E_{2'3}x_2 + M^E_{3'3}x_3 \;. \qquad (5.83)$$

The dashed indices are from the left variables and note that the last set of three equations uses all nine of the possible mixed term matrices. The order of the indices is irrelevant for the M^x and M^y and specific for the M^E. We take the left index to describe the left variable x_i, and the right index to describe the right variable, y_i. As has been stated, the product of the three matrices in both $\mathcal{Q}_L$ and $\mathcal{Q}_R$ must combine to give a matrix of R-charge 1 in y. We can write this as

$$\partial_{x_1}Q_x\partial_{x_2}Q_x\partial_{x_3}Q_x \;=\; F_L x + M_L x^2 \;. \qquad (5.84)$$

This means the other factor in the numerator is also an odd-morphism of R-charge 1 and so can also be written in the form $F'_R y + M'_R y^2$. These matrices have matrix elements of two weights, $w_{ij} = \Delta g_{ij} + 1$, where $\Delta g_{ij} = g_i - g_j = -1/3$ or $1/3$. For $\mathcal{Q}_L$ the matrix F_L can only be made from the products of two F^x factors and an M^x factor such as $F^x_1 M^x_{12} F^x_3$, in this case contributing an x_1 monomial in front of the terms in F_L. Similarly the M_L term is made up from two M^x factors and an F^x factor such as $M^x_{12}F^x_2M^x_{33}$, here contributing an x_2x_3 monomial in front of the terms in M_L.

The situation is similar for $\mathcal{Q}_R$ except that the contributing terms are from the matrices of mixed variables, the M^E. Exactly the same matrix factorisation can be used to describe the potential $W' = V - U$ by simply multiplying Q by the $\mathbb{Z}_2$ grading matrix σ . Inspection of the numerator shows that the quantum dimensions are simply exchanged and there is no change in sign. The M^E must contribute in a similar way to both left and right quantum dimensions. The only thing which identifies the curves is the parameter $d \in \mathbb{C}$. Since the M^E act simply as factors in front of monomials they must act, in some way symmetrically, with respect to the $\{d_1\}$ and $\{d_2\}$.

The supertrace itself will therefore consist of the sum of terms along the leading diagonal of $\mathcal{N}$ and so will have a weight equal to the r-charge of $\mathcal{N}$ which is 2. This is required to have a residue in the quantum dimensions formula. This will also be important when we consider the denominator in the quantum dimension formula.

The denominator

The quantum dimension formula is a residue over a product of differentials of the left or right potential. We can just consider the left quantum dimension. The potential $\mathcal{E}(x, d_1) \;=\; x_1^3 + x_2^3 + x_3^3 - d_1 y_1 y_2 y_3$, therefore in the denominator we have

$$d_{x_1}\mathcal{E}\,..., d_{x_3}\mathcal{E} \;=\; (3x_1^2 - d_1x_2x_3)(3x_2^2 - d_1x_1x_3)(3x_3^2 - d_1x_1x_2)$$

$$= \quad (27-d_1^3)x_1^2x_2^2x_3^2 - 9d_1(x_1^3x_2^3 + x_2^3x_3^3 + x_1^3x_3^3) + 3d_1^2(x_1^4x_2x_3 + x_2^4x_1x_3 + x_3^4x_1x_2).$$

To be able to compute residues we have to make a transformation [66, 28]. This will give us a series of fractions with denominators of the same weight, 4, which are computable residues - i.e. monomials. Using a short Singular procedure we find we have to multiply our supertrace by the following sum of fractions to compute a sum of residues;

$$\begin{aligned}
&-d/[(3d^3-81)*x(1)^3*x(2)^3] \\
&+3d^2/[(d^6-54d^3+729)*x(1)^4*x(2)*x(3)] \\
&-d^2/[(9d^3-243)*x(1)*x(2)^4*x(3)] \\
&-1/[(d^3-27)*x(1)^2*x(2)^2*x(3)^2] \\
&-d/[(3d^3-81)*x(1)^3*x(3)^3] \\
&-d/[(3d^3-81)*x(2)^3*x(3)^3] \\
&-d^2/[(9d^3-243)*x(1)*x(2)*x(3)^4] \\
&-d^4)/[(3d^6-162d^3+2187)*x(3)^6]\ .
\end{aligned} \tag{5.85}$$

In the above d is the parameter in the elliptic curve $\mathcal{E}(x;d)$. Noting the inconsistency of the second term, from here we will assume terms two, three, four and seven in the above are of the correct form as these are the only relevant denominators for a weight 2 numerator to give a residue over all the left variables $\{x_i\}$.

Fusion tensor products

We assume we have a Q which proves orbifold equivalence for the elliptic case, with invertible quantum dimensions. We also make the further assumption that the matrix elements of F^x and M^x carry only information on the parameter d_1. Likewise for the right variables. F^y and M^y carry only information on the parameter d_2. We therefore conclude that the mixed matrix elements of M^E should carry information on both d_1 and d_2, otherwise there is one mixed term component of the MF which would fit for an OEQ for any two arbitrary elliptic curves. We can construct two orbifold equivalences, Q^α and Q^β such that:

$$(Q^\alpha)^2 \;=\; \mathcal{E}(x,d_1) - \mathcal{E}(y,d_2)$$

$$(Q^{\beta})^2 \;=\; \mathcal{E}(y, d_2) - \mathcal{E}(z, d_3) \; . \tag{5.86}$$

We can take the 'fusion' tensor product $Q'^{\gamma} \;=\; Q^{\alpha}\hat{\otimes}Q^{\beta}$, to obtain,

$$(Q'^{\gamma})^2 \;=\; \mathcal{E}(x, d_1) - \mathcal{E}(z, d_3) \; . \tag{5.87}$$

Note here that Q' is not necessarily the same matrix factorisation as Q. We know from taking the limit $y \to 0$, that the left quantum dimension is given by (up to a factor)

$$qd_L \;=\; \mathrm{res}\frac{\mathrm{str}(\mathcal{Q}_L(x, d1).\mathcal{Q}_R(x, d1, d2))}{\mathcal{D}} \tag{5.88}$$

where $\mathcal{D}$ is the denominator given by the second, third, fourth and seventh terms in (1.18). To be specific, for a pure x_1^3 term in the numerator the residue is multiplied by

$$\frac{3d_1^2}{(d_1^6 - 54d_1^3 + 729)} \tag{5.89}$$

and for x_2 and x_3

$$(-d_1^2)/9(d_1^3 - 27) \; . \tag{5.90}$$

For a term of the form $x_1x_2x_3$ the residue is multiplied by

$$-1/(d_1^3 - 27) \; . \tag{5.91}$$

These two types of monomial terms occur in the potential and are the only possibilities to give a non zero residue. We can therefore deduce that our left quantum dimension is of the form

$$\begin{aligned} qd_L \;=\; & -\frac{1}{(d_1^3 - 27)}\sum_l f_l'(d_1)g_l'(d_1, d_2) + \frac{(-d_1^2)}{9(d_1^3 - 27)}\sum_l f_l(d_1)g_l(d_1, d_2) \\ & + \frac{3d_1^2}{(d_1^6 - 54d_1^3 + 729)}\sum_l f_l''(d_1)g_l''(d_1, d_2) \; . \end{aligned} \tag{5.92}$$

One or two of the three above sums could be zero. Remember the first sum represents the product of matrix elements leading to a numerator of the form $x_1x_2x_3$ whereas the second sum corresponds to a cubic term in a single variable x_2 or x_3 and the third corresponds to a x_1^3 in the numerator. In each case the corresponding g_l are products of matrix elements involving products of constant term $F^y(d_2)$ and $M^E(d_1, d_2).x_i$ carrying the correct indices. The l in each sum is just an index for all the pairs of factors from the matrix multiplication $\mathcal{Q}_L(x, d1).\mathcal{Q}_R(x, d1, d2)$ occurring in the supertrace.

If we now consider the fusion product mentioned above, $Q'^\gamma = Q^\alpha \hat{\otimes} Q^\beta$, we have for the quantum dimensions the simple relation,

$$qd_L^\gamma = qd_L^\alpha qd_L^\beta \ . \tag{5.93}$$

This can be expressed as

$$\begin{aligned} qd_L^\gamma = & \Big[\frac{(-d_1^2)}{9(d_1^3-27)} \sum_l f_l(d_1) g_l(d_1, d_2) - \frac{1}{(d_1^3-27)} \sum_l f_l'(d_1) g_l'(d_1, d_2) \\ & + \frac{3d_1^2}{(d_1^6 - 54d_1^3 + 729)} \sum_l f_l''(d_1) g_l''(d_1, d_2)\Big] \\ \times & \Big[\frac{(-d_2^2)}{9(d_2^3-27)} \sum_l f_l(d_2) g_l(d_2, d_3) - \frac{1}{(d_2^3-27)} \sum_l f_l'(d_2) g_l'(d_2, d_3) \\ & + \frac{3d_2^2}{(d_2^6 - 54d_2^3 + 729)} \sum_l f_l''(d_2) g_l''(d_2, d_3)\Big] \ . \end{aligned} \tag{5.94}$$

This raises the question of whether or not the fusion tensor product, Q'^γ has a d_2 dependency. It is impossible for the d_2 dependency in any one of the terms in the first factor to cancel with the d_2 dependency of all the terms in the second factor. The other possibility is that in each term the d_2 factors in the sum cancel with the pre-factors.

One would expect the quantum dimensions to depend on the parameters d_i as there is nothing else distinctive about the potentials. In ADE cases the left and right quantum dimension involve a parameter and some kind of reciprocity in the algebraic relation between them. If there was such a parameter in the elliptic case, it would have to relate to both d's in the potential equation and if not then the quantum dimensions would be on an even footing. This sort of quantum dimension suggests an isomorphism which we know is not there. If the qd_L was some arbitrary constant then by symmetry of the construction, qd_R would have to be the same, as is the case with identity defects.

If this fusion product 'remembers' the d_2, and d_2 is still a parameter then we have a MF Q'^γ for any two elliptic orbifold equivalences which is at least a one parameter family since isomorphic orbifold equivalent MFs have the same quantum dimensions, and these would change with the parameter d_2.

An interesting question then is; if we have some parameterised orbifold equivalences for our middle variables, and these parameter(s) are remembered by the fusion tensor product, can we then carry on taking tensor products ad-infinitum?

Another related question is; since the final quantum dimension is always the product of quantum dimensions of the factors, once we have one OEQ MF can we carry on taking fusion tensor products to produce an infinite set of MFs for that orbifold equivalence?

Summary

1) Any d_2 dependency in the left quantum dimension comes from $\mathcal{Q}_R$. The g, g' and g'' in the sums above are from products of, or single matrix elements from M^E with those of F^y. We can write $\mathcal{Q}_R = F_R + M_R$. The matrix F_R can only be made from sums of the products of two F^y factors and an M^E factor such as $F_1^y M_{12}^E F_3^y$, in this case contributing an x_1 monomial in front of the factor in F_R. The M_R is made up from terms with two M^E factors and an F^y factor such as $M_{12}^E F_2^y M_{33}^E$, here contributing an $x_1 x_3$ monomial in front of terms in M_R.

2) The quantum dimensions will have to be a non-linear function involving some dependency on both parameters. This has implications for fusion tensor products.

3) The M^E have to be the same under interchange of d_1 and d_2 such that $M^E(d_1, d_2) = M^E(d_2, d_1)$, up to a sign. $(Q(-W) = \sigma Q(W))$. We have that $Q(y, x) = \sigma Q(x, y) = \sigma Q_x + \sigma Q_E(xy) + \sigma Q_y$. $Q(y, x)$ and $\sigma Q(x, y)$ are of the same rank and satisfy the same anti-commutation relations, but may not be the exact same MF as $Q \to Q_x$ as $y \to 0$, and Q_x may be different from Q_y, but all matrix elements will have the mixed terms up to a sign. Any term in a matrix element of Q_E can be written $M^E(d_1, d_2) x_i y_j$. This implies,

$$M^E(d_2, d_1) \quad = \quad \pm M^E(d_1, d_2) \tag{5.95}$$

depending if this element is in E or J.

4) If we consider the anti-commutators, which we repeat here, the situation is even more restrictive,

$$\begin{aligned} x^2y \quad \{M^E, F^x\} + \{F^y, M^x\} &= 0 \\ xy^2 \quad \{M^E, F^y\} + \{F^x, M^y\} &= 0 \\ x^2y^2 \quad \{M^x, M^y\} + \{M^E, M^E\} &= 0 \,. \end{aligned} \tag{5.96}$$

The fact that F^y has d_2 dependency and M^x has d_1 dependency implies that matrix elements in M^E satisfy

$$\sum_k M_{ik}^E(d_1, d_2) F_{kj}^x(d_1) + \sum_k F_{ik}^x(d_1) M_{kj}^E(d_1, d_2) = -\sum_k M_{ik}^x(d_1) F_{kj}^y(d_2) - \sum_k F_{ik}^y(d_2) M_{kj}^x(d_1) \ ,$$

(5.97)

for any i, j and for any x_a, x_b and y_c. Likewise for $(x \leftrightarrow y)$ and $(d_1 \leftrightarrow d_2)$ (from the second anti-commutator). The right hand side is a sum of products of factors with separate d_1 and d_2 dependency, so it should be the same for the left hand side, but the d_1 and d_2 dependency there is contained in single factors $M^E_{ik}(d_1, d_2)$. Can the r.h.s all be products of d_1 and d_2 factors and at the same time the M^E satisfy eq(5.95)?

The quantum dimensions, (5.94) are a complicated residue over several denominators, (5.85). If the quantum dimensions do not depend on the vales d_1, d_2, then the left and right quantum dimensions should be the same for all non-isomorphic pairs of MFs of the elliptic curve. If the values do appear in the quantum dimension then once there they can't be got rid of when taking fusion tensor products and we could create a hugely complex parameter space of non isomorphic orbifold equivalent MFs as they would have a parameter space of varying quantum dimensions.

5.5 Supercomputing

In these final examples the search for further equivalences (or lack of) could possibly be automated and carried out by super computers or parallel computing for ranks far higher than was possible on a laptop, which could not even solve the full weight matrix derived from the tensor product in the elliptic-quartic example. If we could search at increasing rank through every possible weight matrix we might or might not find an equivalence. It would be interesting to see how far we could go and still get a (negative) solution. In the case of $\hat{c} = 1$ we have seen that we can still compute all the weight matrices at a given rank for the elliptic-quartic pair even though the weight criterion do not apply because of the nature of the grading. It would be very encouraging if no results were found up to a relatively high rank in the search for an analytic proof that these potentials were not orbifold equivalent. In all the examples of orbifold equivalence found so far either Q_x or Q_y were indecomposable - or both. Given this fact and the experience with free resolutions and exact sequences we might, if we were using more computer power be able to detect a point were we are just getting direct sums of lower rank objects.

6 Conclusion

The early explorations into invertibility, specifically Liouville type theories and also exact sequences were a good contextual introduction to MFs in topological field theory, and also to the basic mathematics of MFs. The application of resolutions to generate MFs provided a practical way to learn about homological algebra and writing procedures in SINGULAR, and gave many insights into the structure of categories of MFs. It could be that there are still some statements to be made concerning the maximal rank of indecomposable MFs, the potential and quotient ring, from the point of view of resolutions. We also developed a simple algorithm for identifying isomorphic or non-isomorphic MFs.

The initial approach to MFs was from the point of view of the development from string theory to TFTs. The study of defects, as a natural mathematical extension to the notion of a boundary, quickly became the search for orbifold equivalences. These were rare mathematical objects to be sought after, and at first it seemed all the known examples were the only ones to be found. As we discovered more about the nature of the orbifold equivalence, we were able to prove theorems and establish criteria which meant we could implement an algorithm based on Hilbert's Nullstellensatz. We already knew the problem at any rank was essentially solvable with enough computer power but as our techniques were refined we suddenly managed to harvest many of these equivalences. The process became a game of solitaire to be played late into the night, often reaching dead ends after hours of work.

The first known OEQs were identity and symmetry defects at any value of central charge [14]. The other previously discovered OEQs were the ADE [26] and unimodal ($\hat{c} < 1$), and one of the Arnold exceptional singularity pairs ($\hat{c} > 1$) [30, 74, 75]. We have gone a long way to confirming and greatly expanding the catalogue and have made many useful observations and conjectures [81]. As we surveyed the now growing catalogue of orbifold equivalences we also started to observe the structure of these polynomials and learned about re-parameterisations in connection with OEQ MFs. We found that, in addition to our algorithm, re-parameterisations were also a valid tool and in conjunction with tensor products, we used these in constructing many new equivalences. We also found correspondences between the different Berglund-Huebsch-Krawitz mirror construction [55, 64, 65] of potentials with transposed exponent matrices. One possible avenue of exploration is to try to prove orbifold equivalence for "transposed potentials" in general and then look at the string spectra for LG models related in that way.

Eventually the novelty value in discovering these equivalences wore off (although the challenge was always worthy) and the question we had to ask was, could these always be found, or did these always exist when we had models of the same central charge? Many times we felt we were on the verge of showing that the pairs in Section 5 were counter examples because of the peculiarities of these polynomials at $\hat{c} = 1$. We did find OEQs between the chain $x_1x_2^2 + x_2x_3^2 + x_1^3$ or loop $x_1x_2^2 + x_2x_3^2 + x_3x_1^2$ and other polynomials at this value of $\hat{c}$, but it was always between polynomials with similar weight systems, either like the quartic or cubic. The potentials seem to fall into two camps. Although we have not solved this yet, we have seen no other models with the peculiarities these have and which were pointed out in the last section. It would be interesting and novel to find examples which we could show were not orbifold equivalent but have the same central charge. The case of geometrically inequivalent elliptic curves may provide such an example equal central charge but no orbifold equivalence. It would also be interesting to see if in the specific case of the elliptic $d = -1$ geometric bi-rational equivalence to the quartic is also an orbifold equivalence and if an MF can be found.

More computer power would certainly allow us to explore higher rank possibilities for both the suggested pairs and clean up any undiscovered equivalences, for instance W_{17} and Q_{17}, (assuming such a special MF exists). It would be interesting to see how high a rank we could explore at $\hat{c} = 1$ with these models and modern supercomputers. One nevertheless feels that proving there can be no orbifold equivalence with some of the $\hat{c} = 1$ pairs we looked at must be possible. If we had more computing power we could also find the closest thing to an explicit solution, for all known and future equivalences with a minimal set of parameters and constraints in a Groebner basis with no guesses for the coefficients.

I look forward to hearing of developments in this field and hope that a proof that no orbifold equivalence can exist for any of the pairs we looked at in section 5, is eventually found. Someone with access to more sophisticated computers or computing techniques might consider implementing the search as a computer oriented project, as the computing side was not fully explored and there are ways and means in existence of overcoming the memory restrictions we had.

If it turned out that $\hat{c} = 1$ was special that would be noteworthy. It is also special in CFT and $\hat{c}$ is *central* to the LG-CFT correspondence. Finding a pair where no orbifold equivalence could exist at some other value of central charge would be stranger but maybe these cases exist and are just difficult to show. The answer to this may lie in the polynomials themselves and the weights of the variables, which just as with the values of central charge in minimal models, are defined in terms of ratios. Both have a mysterious ADE classification and it is also tempting to think some other connection can be made on this basis. The outcome of our research is that we have some interesting results and are left with some more interesting questions.

Appendix A   The N=2 bulk action

Following the the development of Brunner et al [9] the general LG action can be written,

$$S_\Sigma \;=\; \int d^2x d^2\theta d^2\bar{\theta} K(\Phi,\bar{\Phi}) - \frac{1}{2}\int d^2x d^2\theta W(\Phi) - \frac{1}{2}\int d^2x d^2\bar{\theta}\bar{W}(\bar{\Phi}) \qquad \text{(A.1)}$$

The fields Φ and $\bar{\Phi}$ are chiral super fields. K is a Kahler metric and W is the super potential (Or more precisely $W+\bar{W}$). The first term, the D term is the integral of a potential which can be set to $K(\Phi,\bar{\Phi}) \;=\; \Phi\bar{\Phi}$, as we are only interested in the topological properties of the theory.

We are constructing an N=2 theory in (2,2) superspace. There are two bosonic coordinates (x^0, x^1) and four fermionic coordinates $(\theta^\pm, \bar{\theta}^\pm)$. We will work in light cone coordinate $x^+ = x^0 + x^1$ and $x^- = x^0 - x^1$. The fermionic coordinates satisfy $(\theta^\pm)^\dagger = \bar{\theta}^\pm$. There are four supercharges,

$$Q_\pm \;=\; \frac{\partial}{\partial\theta^\pm} + i\bar{\theta}^\pm\frac{\partial}{\partial x^\pm}\ , \qquad \bar{Q}_\pm \;=\; -\frac{\partial}{\partial\bar{\theta}^\pm} - i\theta^\pm\frac{\partial}{\partial x^\pm}\ . \qquad \text{(A.2)}$$

The covariant derivatives are then given by,

$$D_\pm \;=\; \frac{\partial}{\partial\theta^\pm} - i\bar{\theta}^\pm\frac{\partial}{\partial x^\pm}\ , \qquad \bar{Q}_\pm \;=\; -\frac{\partial}{\partial\bar{\theta}^\pm} + i\theta^\pm\frac{\partial}{\partial x^\pm}\ . \qquad \text{(A.3)}$$

These operators satisfy the algebra,

$$\{Q_\pm, \bar{Q}_\pm\} \;=\; -2i\partial_\pm\ , \qquad \{D_\pm, \bar{D}_\pm\} \;=\; 2i\partial_\pm \qquad \text{(A.4)}$$

chiral and anti-chiral superfields are defined by $\bar{D}_\pm\Phi \;=\; 0$ and $D_\pm\bar{\Phi} \;=\; 0$ and have the following expansions in terms of component fields,

$$\Phi(y^\pm,\theta^\pm) \;=\; \phi(y^\pm) + \theta^+\psi_+(y^\pm) + \theta^-\psi_- + \theta^+\theta^- F(y^\pm)\ . \qquad \text{(A.5)}$$

Here $y^\pm \;=\; x^\pm - i\theta^\pm\bar{\theta}^\pm$, and further expansion about $y^\pm = x^\pm$ gives,

$$\begin{aligned}\Phi \;=\;& \phi - i\theta^+\bar{\theta}^+\partial_+\phi - \theta^-\bar{\theta}^-\partial_-\phi - \theta^+\theta^-\bar{\theta}^-\bar{\theta}^+\partial_+\partial_-\phi \\ &+\theta^+\psi_+ - i\theta^+\theta^-\bar{\theta}^-\partial_-\psi_+ + \theta^-\psi_- - i\theta^-\theta^+\bar{\theta}^+\partial_+\psi_- + \theta^+\theta^- F\ . \qquad \text{(A.6)}\end{aligned}$$

The equations for anti-chiral superfields are analogous.

Variations under the operator, $\delta \;=\; \epsilon_+Q_- - \epsilon_-Q_+ - \bar{\epsilon}_+\bar{Q}_- + \bar{\epsilon}_-\bar{Q}_+$ are given by,

$$\begin{aligned}\delta\phi &= +\epsilon_+\psi_- - \epsilon_-\psi_+\ , & \delta\bar{\phi} &= +\bar{\epsilon}_+\bar{\psi}_- - \bar{\epsilon}_-\bar{\psi}_+ \\ \delta\psi_+ &= +2i\bar{\epsilon}_-\partial_+\phi + \epsilon_+F\ , & \delta\bar{\psi}_+ &= +2i\epsilon_-\partial_+\bar{\phi} + \bar{\epsilon}_+\bar{F} \\ \delta\psi_- &= +2i\bar{\epsilon}_+\partial_-\phi + \epsilon_-F\ , & \delta\bar{\psi}_- &= +2i\epsilon_+\partial_-\bar{\phi} + \bar{\epsilon}_-\bar{F}\ . \qquad \text{(A.7)}\end{aligned}$$

We can vary the action with $\bar{F}$ gives us the $\delta\bar{F}$ equation of motion $F = -\frac{1}{2}\bar{W}(\bar{\Phi})$. Then performing the Grassman integrals,

$$\frac{1}{2}\int_{\Sigma} d^2x d^2\theta W(\Phi)|_{\bar{\theta}^{\pm}=0} + c.c. \quad , \tag{A.8}$$

the bulk action can be written,

$$\begin{aligned} \mathcal{S}_{\Sigma} \; = \; & \int_{\Sigma} d^2x\{-\partial^{\mu}\bar{\phi}\partial_{\mu}\phi + \frac{i}{2}\bar{\psi}_-(\overleftrightarrow{\partial}_0 + \overleftrightarrow{\partial}_1)\psi_- + \frac{i}{2}\bar{\psi}_+(\overleftrightarrow{\partial}_0 - \overleftrightarrow{\partial}_1)\psi_+ \\ & -\frac{1}{4}|W'|^2 - \frac{1}{2}W''\psi_+\psi_- - \frac{1}{2}\bar{W}''\bar{\psi}_+\bar{\psi}_-\} \; . \end{aligned} \tag{A.9}$$

The boundary and B-type supersymmetry

Introduction of the boundary identifies holomorphic and anti-holomorphic parts of the algebra at the boundary, specifically we can write the N=2 boundary relations [93]

$$\begin{aligned} J(z)|_{x=0} &= \bar{J}(-\bar{z})|_{x=0} \\ T(z)|_{x=0} &= \bar{T}(-\bar{z})|_{x=0} \\ Q^{\pm}(z)|_{x=0} &= \bar{Q}^{\pm}(-\bar{z})|_{x=0} \end{aligned} \tag{A.10}$$

We will consider the B-type supersymmetry [9] with supercharge $Q = \bar{Q}_+ + \bar{Q}_-$. We can then re-define our fermions as $\eta \;=\; \psi_- + \psi_+$ and $\xi \;=\; \psi_- - \psi_+$. Now we have variations with $\delta \;=\; \epsilon\bar{Q} - \bar{\epsilon}Q$,

$$\begin{aligned} \delta\phi \; &= \epsilon\eta \; , & \delta\bar{\phi} \; &= -\bar{\epsilon}\bar{\eta} \\ \delta\eta \; &= \; -2i\bar{\epsilon}_+\partial_0\phi \; , & \delta\bar{\eta} \; &= \; 2i\epsilon_+\partial_0\bar{\phi} \\ \delta\xi \; &= \; +2i\bar{\epsilon}\partial_1\phi + \epsilon\bar{W}'(\bar{\phi}) \; , & d\bar{\xi}_+ \; &= \; +2i\epsilon\partial_1\bar{\phi} + \epsilon W'(\phi) \; . \end{aligned} \tag{A.11}$$

The superspace at the boundary can be spanned by coordinates $\theta^0 \;=\; \frac{1}{2}(\theta^- + \theta^+)$ and $\bar{\theta}^0 \;=\; \frac{1}{2}(\bar{\theta}^- + \bar{\theta}^+)$, giving the supercharges as,

$$\bar{Q} \;=\; \frac{\partial}{\partial\theta^0} + i\bar{\theta}^0\frac{\partial}{\partial x^0} \; , \qquad Q \;=\; -\frac{\partial}{\partial\bar{\theta}^0} - i\theta^0\frac{\partial}{\partial x^0} \; . \tag{A.12}$$

and covariant derivatives as,

$$\bar{D} \;=\; \frac{\partial}{\partial\theta^0} - i\bar{\theta}^0\frac{\partial}{\partial x^0} \; , \qquad D \;=\; -\frac{\partial}{\partial\bar{\theta}^0} + i\theta^0\frac{\partial}{\partial x^0} \; . \tag{A.13}$$

At the boundary we can re-arrange components to form two superfields. Firstly if we consider

$$\Phi'(y^0,\theta^0) \;=\; \phi(y^0)+\theta^0\eta(y^0)\;, \tag{A.14}$$

this is a chiral superfield satisfying $\bar{D}\Phi' = 0$, where $y^0 = x^0 - i\theta^0\bar{\theta}^0$. We also have a fermonic superfield,

$$\Theta'(y^0,\theta^0,\bar{\theta}^0) \;=\; \xi(y^0)-2\theta^0F(y^0)+2i\bar{\theta}^0[\partial_1\phi(y^0)+\theta^0\partial_1\eta(y^0)] \tag{A.15}$$

this is not chiral and satisfies the equation $D\Theta' = -2i\partial_1\Phi'$. We want to construct an invariant supersymmetric action. The first step is to set $W = 0$. The B-type Supersymmetry variation of the action (A.1) gives a surface term thet can be compensated for by

$$\mathcal{S}_{\partial\Sigma,\psi} \;=\; \frac{i}{2}\int_{\Sigma} dx^0\{\bar{\theta}\eta-\bar{\eta}\theta\}|_0^{\pi} \tag{A.16}$$

If we now turn on the super potential variation leaves the following surface term.

$$\delta(\mathcal{S}_{\Sigma}+\mathcal{S}_{\partial\Sigma,\psi}) \;=\; \frac{i}{2}\int_{\Sigma} dx^0\{\epsilon\bar{\eta}\bar{W}'+\bar{\epsilon}\eta W'\}|_0^{\pi} \tag{A.17}$$

The variations (A.11) only give us $\epsilon\eta$ or $\bar{\epsilon}\bar{\eta}$, not $\bar{\epsilon}\eta$ or $\epsilon\bar{\eta}$, so for an invariant action we introduce a second fermionic boundary superfield, Π with expansion,

$$\Pi \;=\; \pi(y^0)+\theta^0 l(y^0)-i\bar{\theta}^0[E(\phi+\theta^0\eta(y^0)E']\;. \tag{A.18}$$

This field satisfies $\bar{D}\Pi = E(\Phi')$ (and so is not chiral) and has the following variations.

$$\delta\pi \;=\epsilon l-\bar{\epsilon}E(\phi)\;, \qquad \delta\bar{\pi} \;=\bar{\epsilon}\bar{l}-\epsilon\bar{E}(\bar{\phi})$$

$$\delta l \;=-2i\bar{\partial}_0\pi l+\bar{\epsilon}E'(\phi)\;, \qquad \delta\bar{l} \;=-2i\partial_0\bar{\pi}l-\epsilon\bar{E}'(\bar{\phi}) \tag{A.19}$$

The additional terms for the action on the boundary read

$$\mathcal{S}_{\partial\Sigma} \;=\; -\frac{1}{2}\int dx^0 d^2\theta\bar{\Pi}\;\Pi|_0^{\pi}-\frac{i}{2}\int_{\partial\Sigma}dx^0 d\theta\Pi J(\Phi)_{\bar{\theta}=0}|_0^{\pi}\;+\;c.c.\;\;. \tag{A.20}$$

We have introduced a new field $J(\Phi)$ which can be seen to satisfy the equation of motion $l = -i\bar{J}(\bar{\Phi})$, giving the boundary action as,

$$\mathcal{S}_{\partial\Sigma} \;=\; \int dx^0\{i\bar{\pi}\partial_0\pi-\frac{1}{2}\bar{J}J-\frac{1}{2}\bar{E}E+\frac{i}{2}\pi\eta J'+\frac{i}{2}\bar{\pi}\bar{\eta}\bar{J}'-\frac{1}{2}\bar{\pi}\eta E'+\frac{1}{2}\pi\bar{\eta}\bar{E}'\}|_0^{\pi}\;. \tag{A.21}$$

From (A.19) we see that the variation of the boundary fermions become,

$$\delta\pi \;=\; -i\epsilon\bar{J}(\bar{\phi})-\bar{\epsilon}E(\phi)\;, \qquad \delta\bar{\pi} \;=i\bar{\epsilon}J(\phi)-\epsilon\bar{E}(\bar{\phi}) \tag{A.22}$$

The variation of this boundary action gives

$$\delta\mathcal{S}_{\partial\Sigma} \;=\; -\frac{i}{2}\int_{\partial\Sigma} dx^0\{\epsilon\bar{\eta}(\bar{E}\bar{J})' + \bar{\epsilon}\eta(EJ)'\} \tag{A.23}$$

Comparison with (A.17) shows that the whole action is invariant under supersymmetry when,

$$W \;=\; EJ + const.\ . \tag{A.24}$$

More precisely if we consider E, J and W as matrices we have our matrix factorisation condition as,

$$W - const. \;=\; EJ \;=\; JE\ . \tag{A.25}$$

Appendix B Buchberger's algorithm

In order to find if a set of ideals is solvable or not we are required to put them in a Groebner or standard basis. This process is done algorithmically in Singular and can be extremely unpredictable in terms of run time and memory use. This is because the algorithm vastly increases the number of generators.

Consider a set of generators g_i which define and ideal I. The generators are essentially polynomial equations $g_i = 0$ over a complex field. In most cases such as the computation of cohomology or resolutions over an ideal, the field consists of typically four to six ring variables. The algorithm for finding orbifold equivalences demands we solve many more equations as there are up to five hundred variables (formerly coefficients), and many equations $g_i = 0$. The algorithm depends on a given ordering of variables, which have to be specified in Singular. We outline Buchberger's algorithm in it's simplest form:

1) We first make a copy $F := I$.

2) For every $g_i \in F$ we denote the leading term l_i.

3) For every $g_i, g_j \in F$ we create a matrix $\{a_{ij}\}$ consisting of the least common multiple of l_i and l_j.

4) For every pair $g_i, g_j \in F$ define $S_{ij} = (a_{ij}/l_i)g_i - (a_{ij}/l_j)g_j$. This way we cancel leading terms but have a longer polynomial (ideal generator) which is eventually added to the list of generators.

5) Reduce* S_{ij} by I. If this is not zero then we add S_{ij} to I.

6) This process is then repeated from 1). We keep adding ideal generators and only when the process adds no more generators do we stop.

It is obvious that step 4) can give us many new generators of greater length. This often exhausted the computer memory and causing the software to 'crash'.

Appendix C Direct sums with a trivial MF

If we consider the direct sum $Q = Q_1 \oplus Q_2$, we can now show that in the special case where one of the Q_i is a trivial factorisation then the cohomology is the same as that of the non-trivial part. The trivial part is given by,

$$Q_1 = \begin{pmatrix} 0 & 1 \\ W(x) & 0 \end{pmatrix} . \tag{C.1}$$

We can arrange Q as,

$$Q = \begin{pmatrix} 0 & 0 & 0 & J \\ 0 & 0 & 1 & 0 \\ 0 & W & 0 & 0 \\ E & 0 & 0 & 0 \end{pmatrix} \tag{C.2}$$

The general forms for bosons ϕ and fermions ψ with $\sigma = \mathrm{diag}\{1, -1\}$ are given by

$$\phi = \begin{pmatrix} B_1' & \alpha^\phi & 0 & 0 \\ \beta^\phi & B_1 & 0 & 0 \\ 0 & 0 & B_2 & \gamma^\phi \\ 0 & 0 & \delta^\phi & B_2' \end{pmatrix}$$

$$\psi = \begin{pmatrix} 0 & 0 & \alpha^\psi & A_1' \\ 0 & 0 & A_1 & \beta^\psi \\ \gamma^\psi & A_2 & 0 & 0 \\ A_2' & \delta^\psi & 0 & 0 \end{pmatrix} \tag{C.3}$$

Action of d_Q on bosons:
Then $d_Q^0 : M_b \to M_f$ is given by

$$d_Q^0\phi = Q\phi - \phi Q$$

$$= \begin{pmatrix} 0 & 0 & J\delta^\phi - \alpha^\phi & JB_2' - B_1'J \\ 0 & 0 & B_2 - B_1 & \gamma^\phi - \beta^\phi J \\ W\beta^\phi - \gamma^\phi E & WB_1 - B_2W & 0 & 0 \\ EB_1' - B_2'E & E\alpha^\phi - \delta^\phi W & 0 & 0 \end{pmatrix} \tag{C.4}$$

We can see that the expressions for $(d_Q^0\phi)_{14}$ and $(d_Q^0\phi)_{41}$ are the same as those already obtained. The suffixes here denote blocks or sub matrices (not necessarily square) and not matrix elements, since a matrix factorisation of a direct sum naturally splits into sixteen sub matrices.

$$(d_Q^0\phi)_{13} = J\delta^\phi - \alpha^\phi \qquad (n \times n)(n \times 1) - (n \times 1)$$

$$(d_Q^0\phi)_{24} = \gamma^\phi - \beta^\phi J \qquad (1 \times n) - (1 \times n)(n \times n)$$

$$
\begin{array}{rcll}
(d_Q^0\phi)_{31} & = & W\beta^\phi - \gamma^\phi E & (1)(1\times n) - (1\times n)(n\times n) \\
(d_Q^0\phi)_{42} & = & E\alpha^\phi - \delta^\phi W & (n\times n)(n\times 1) - (n\times 1)(1)
\end{array} \tag{C.5}
$$

We can use (C.5) to write the relevant equations for the kernel of d_Q^1.

$$
\begin{array}{rcl}
Ker\ d_Q^0 \iff \alpha_i^\phi & = & \sum_j \delta_j^\phi J_{ji} \\
\gamma_i^\phi & = & \sum_j \beta_j^\phi J_{ij} \\
W\beta_i^\phi & = & \sum_j \gamma_j^\phi E_{ji} \\
\delta_i^\phi W & = & \sum_j \alpha_j^\phi E_{ij}
\end{array} \tag{C.6}
$$

Action of d_Q on fermions:

We can now consider $d_Q^1 : M_f \to M_b$.

$$
d_Q^1\psi = Q\psi + \psi Q
$$

$$
= \begin{pmatrix} JA_2' + A_1' & J\delta^\psi + \alpha^\psi W & 0 & 0 \\ \gamma^\psi + \beta^\psi E & A_2 + A_1 W & 0 & 0 \\ 0 & 0 & WA_1 + A_2 & W\beta^\psi + \gamma^\psi J \\ 0 & 0 & E\alpha^\psi + \delta^\psi & EA_1' + A_2' J \end{pmatrix} \tag{C.7}
$$

We can see that the expressions for $(d_Q^1\phi)_{14}$ and $(d_Q^1\phi)_{41}$ are again the same as those obtained from before. This time the new (matrix) equations are

$$
\begin{array}{rcll}
(d_Q^1\psi)_{12} & = & J\delta^\psi + \alpha^\psi W & (n\times n)(n\times 1) - (n\times 1)(1) \\
(d_Q^1\psi)_{21} & = & \gamma^\psi + \beta^\psi E & (1\times n) - (1\times n)(n\times n) \\
(d_Q^1\psi)_{34} & = & W\beta^\psi + \gamma^\psi J & (1)(1\times n) - (1\times n)(n\times n) \\
(d_Q^1\psi)_{43} & = & E\alpha^\psi + \delta^\psi & (n\times n)(n\times 1) - (n\times 1)
\end{array} \tag{C.8}
$$

We can use (C.8) to write the relevant equations for the kernel of d_Q^1.

$$
\begin{array}{rcl}
Ker\ d_Q^1 \iff W\alpha_i^\psi & = & -\sum_j \delta_j^\psi J_{ij} \\
\gamma_i^\psi & = & -\sum_j \beta_j^\psi E_{ji}
\end{array}
$$

$$
\begin{aligned}
W\beta_i^{\psi} &= -\sum_j \gamma_j^{\psi} J_{ji} \\
\delta_i^{\psi} &= -\sum_j \alpha_j^{\psi} E_{ij}
\end{aligned}
\tag{C.9}
$$

The cohomology H^1:

If we consider kernel and the specific entries in d_Q^1,

$$
\begin{aligned}
(Ker\ d_Q^1)_{\tilde{12}} &= (\alpha^{\psi}, \delta^{\psi}): \ J\delta^{\psi} + \alpha^{\psi} W &= 0 \\
(Ker\ d_Q^1)_{\tilde{43}} &= (\alpha^{\psi}, \delta^{\psi}): \quad E\alpha^{\psi} + \delta^{\psi} &= 0
\end{aligned}
\tag{C.10}
$$

are equivalent statements about α and δ. If we now look at the equivalent entries in the image of d_Q^0

$$
\begin{aligned}
(Im\ d_Q^0)_{12} &= \tilde{\alpha}^{\psi} = J\delta^{\phi} - \alpha^{\phi} \\
(Im\ d_Q^0)_{43} &= \tilde{\delta}^{\psi} = E\alpha^{\phi} - \delta^{\phi} W
\end{aligned}
\tag{C.11}
$$

Multiplying $\tilde{\delta}^{\psi}$by J and $\tilde{\alpha}^{\psi}$ by W and E we have

$$
\begin{aligned}
J\tilde{\delta}^{\psi} &= W\alpha^{\phi} - J\delta^{\phi} W \\
\tilde{\alpha}^{\psi} W &= J\delta^{\phi} W - \alpha^{\phi} W
\end{aligned}
\tag{C.12}
$$

and by comparing with (C.10) we can immediately see that (as far as $\tilde{\alpha}^{\psi}$ and $\tilde{\delta}^{\psi}$ are concerned) $Im\ d_Q^0 \subset Ker\ d_Q^1$. Moreover if we further consider (C.10) and (C.11) we see that there is no $(\alpha^{\psi}, \delta^{\psi})$ in $Ker\ d_Q^1$ that is not expresable in terms of $(\tilde{\alpha}^{\psi}, \tilde{\delta}^{\psi})$. This means that the cohomology $H^0 = Ker\ d_Q^1 / Im\ d_Q^0$ (again only as far as $\tilde{\alpha}^{\psi}$ and $\tilde{\delta}^{\psi}$ are concerned), is trivial.
For a complete treatment of H^0 we must also consider $(Ker\ d_Q^1)_{12}$ and $(Ker\ d_Q^1)_{12}$

$$
\begin{aligned}
(Ker\ d_Q^1)_{\tilde{21}} &= (\beta^{\psi}, \gamma^{\psi}): \quad \gamma^{\psi} + \beta^{\psi} E = 0 \\
(Ker\ d_Q^1)_{\tilde{34}} &= (\beta^{\psi}, \gamma^{\psi}): \quad W\beta^{\psi} + \gamma^{\psi} J = 0
\end{aligned}
\tag{C.13}
$$

Again these equations are equivalent. And again we look at the equivalent entries in the image of d_Q^0

$$
\begin{aligned}
(Im\ d_Q^0)_{21} &= \tilde{\beta}^{\psi} = \gamma^{\phi} - \beta^{\phi} J \\
(Im\ d_Q^0)_{14} &= \tilde{\gamma}^{\psi} = W\beta^{\phi} - \gamma^{\phi} E
\end{aligned}
\tag{C.14}
$$

Multiplying $\tilde{\beta}^{\psi}$by E on the rightand $\tilde{\alpha}^{\psi}$ by W and E we have

$$
\tilde{\beta}^{\psi} E = \gamma^{\phi} E - \beta^{\phi} W \tag{C.15}
$$

Again we see that any $\tilde{\beta}^{\psi}$ or $\tilde{\gamma}^{\psi}$ satisfies equations (C.13) so the Image of d^0_Q is in the kernel of d^1_Q. Moreover, any $\alpha^{\psi}, \beta^{\psi}, \gamma^{\psi}$ and δ^{ψ} are expressible as $\tilde{\alpha}^{\psi}, \tilde{\beta}^{\psi}, \tilde{\gamma}^{\psi}$ and $\tilde{\delta}^{\psi}$. So

$$\forall\ \ \alpha^{\psi}, \beta^{\psi}, \gamma^{\psi}, \delta^{\psi}\ \in\ Ker\ d^1_Q,\ \ \exists\ \ \tilde{\alpha}^{\psi}, \tilde{\beta}^{\psi}, \tilde{\gamma}^{\psi}, \tilde{\delta}^{\psi}\ \in\ Im\ d^0_Q \tag{C.16}$$

and the cohomology $H^1_Q\ =\ Ker\ d^1_Q/Im\ d^0_Q$ depends only on the cohomology of the E and J in the expression for Q in (C.2).

The cohomology H^0:

We can now consider the cohomology H^1. Again we can see that the expressions for $(d^0_Q\phi)_{14}$ and $(d^0_Q\phi)_{41}$ are again the same as those obtained before.
If we consider kernel and the specific entries in d^1_Q the new (matrix) equations are

$$\begin{aligned} (Ker\ d^0_Q)_{\tilde{1}\tilde{3}} &= (\alpha^{\psi}, \delta^{\psi}): & J\delta^{\phi} - \alpha^{\phi} &= 0 \\ (Ker\ d^0_Q)_{\tilde{4}\tilde{2}} &= (\alpha^{\psi}, \delta^{\psi}): & E\alpha^{\phi} - \delta^{\phi}W &= 0 \end{aligned} \tag{C.17}$$

are equivalent statements about α and δ. If we now look at the equivalent entries in the image of d^0_Q

$$\begin{aligned} (Im\ d^1_Q)_{12} &= \tilde{\alpha}^{\phi} = J\delta^{\psi} + \alpha^{\psi}W \\ (Im\ d^1_Q)_{43} &= \tilde{\delta}^{\phi} = E\alpha^{\psi} + \delta^{\psi} \end{aligned} \tag{C.18}$$

Multiplying $\tilde{\delta}^{\psi}$by J and $\tilde{\alpha}^{\psi}$ by W and E we have

$$\begin{aligned} J\tilde{\delta}^{\phi} &= W\alpha^{\psi} + J\delta^{\psi} \\ \tilde{\alpha}^{\phi} &= J\delta^{\psi} + \alpha^{\psi}W \end{aligned} \tag{C.19}$$

We see that $Im\ d^1_Q$ is in the $Ker\ d^0_Q$ as it should be. We also note that we can express any α^{ϕ} and δ^{ϕ} in $Ker\ d^0_Q$ as $\tilde{\alpha}^{\phi}$ and $\tilde{\delta}^{\phi}$ in the image of d^1_Q. Again this is only valid for $\tilde{\alpha}^{\phi}$ and $\tilde{\delta}^{\phi}$ and we need to check the entries $(Ker\ d^1_Q)_{21}$ and $(Ker\ d^1_Q)_{34}$

$$\begin{aligned} (Ker\ d^0_Q)_{\tilde{2}\tilde{1}} &= (\beta^{\phi}, \gamma^{\phi}): & \gamma^{\phi} - \beta^{\phi}J &= 0 \\ (Ker\ d^0_Q)_{\tilde{3}\tilde{4}} &= (\beta^{\phi}, \gamma^{\phi}): & W\beta^{\phi} - \gamma^{\phi}E &= 0 \end{aligned} \tag{C.20}$$

Again these equations are equivalent. We look at the equivalent entries in the image of d^0_Q

$$\begin{aligned} (Im\ d^1_Q)_{13} &= \tilde{\beta}^{\phi} = \gamma^{\psi} + \beta^{\psi}E \\ (Im\ d^1_Q)_{14} &= \tilde{\gamma}^{\psi} = W\beta^{\psi} + \gamma^{\psi}J \end{aligned} \tag{C.21}$$

Multiplying $\tilde{\beta}^\psi$by J on the right we have

$$\tilde{\beta}^\psi J \;\; = \;\; \gamma^\phi J + \beta^\phi W \qquad \text{(C.22)}$$

Again we see that any $\tilde{\beta}^\psi$ or $\tilde{\gamma}^\psi$ satisfies equations (C.20), so the image of d_Q^1 is in the kernel of d_Q^0. Moreover, any $\alpha^\psi, \beta^\psi, \gamma^\psi$ and δ^ψ are expressible in terms of $\tilde{\alpha}^\psi, \tilde{\beta}^\psi, \tilde{\gamma}^\psi$ and $\tilde{\delta}^\psi$.

We can therefore state that the direct sum of a non-trivial matrix factorisation with a trivial one has the same cohomology as that of the non-trivial matrix factorisation.

Appendix D Blow up of A_n

Here we follow the development of [84]. The ADE singularities describe complex surfaces embedded in $\mathbb{C}^3$. As such, we can write the A_n singularity as

$$xy \;=\; z^{n+1} \,. \tag{D.1}$$

First we resolve $A_1 : xy = z^2$. We do this by replacing the xyz space by two spaces. We want to blow up at $x = z = 0$. We choose two coordinate spaces $U_1 = (\tilde{x}, y, z)$ and $U_2(x, y, \tilde{z})$ and write

$$(x, y, z) = (z\tilde{x}, y, z) = (x, y, x\tilde{z}) \,. \tag{D.2}$$

The two spaces are glued at $\tilde{z}\tilde{x} = 1$ and $z = x\tilde{z}$. We can then see that our $A_1 : xy = z^2$ singularity becomes

$z(\tilde{x}y - z) = 0$ in the $U_1 \sim (\tilde{x}, y, z)$ space and

$x(y - x\tilde{z}^2) = 0$ in the $U_2 \sim (x, y, \tilde{z})$ space.

Now we have two smooth surfaces, $S_1 :\; y = x\tilde{z}^2$ and $S_2 :\; z = \tilde{x}y$. The two surfaces are glued together at,

$$\tilde{z}\tilde{x} \;=\; 1 \,, \qquad \text{and} \qquad x\tilde{z} = \tilde{x}y \,. \tag{D.3}$$

We now have a smooth surface and have resolved our singularity. Each surface is mapped onto the original singular surface by

$S_1 \to A_1 \;:\; (x, y, z) \;=\; (x, x\tilde{z}^2, x\tilde{z})$ and,

$S_2 \to A_1 \;:\; (x, y, z) \;=\; (\tilde{x}^2 y, y, \tilde{x}y)$.

Each surface is now parameterised by two variables, and the inverse image of the singular point at the origin is described by $x = 0$ in S_1 and $y = 0$ in S_2 and has co-ordinates $(\tilde{x}, \tilde{z})$ which satisfy $\tilde{x}, \tilde{z} = 1$, i.e. co-ordintates for a projective line $\mathbf{P^1}$.

If we now consider the A_{n-1} singularity $xy = z^n$, we can have the same parameterisations. This time our equation looks like

$\tilde{x}y = z^{n-1}$ in the $U_1 \sim (\tilde{x}, y, z)$ space, and

$x^{n-1}\tilde{z}^n = y$ in the $U_2 \sim (x, y, \tilde{z})$ space.

It is smooth in the $U_2 \sim (x, y, \tilde{z})$ space but we see we still have an A_{n-1} singularity in U_1 to resolve. We have again blown-up producing a projective line $\mathbf{P^1}$, but we have to repeat the process by blowing-up the $\tilde{x} - y$ plane at $\tilde{x} = z = 0$. After n iterations we have resolved our singularity and produced n surfaces S_i with coordinates: $(x_1, z_1) = (x, \tilde{x})$, $(x_2 = \tilde{z}, z_2)$, (x_3, z_3) , ... $(x_n, z_n = y)$. These are mapped to the A_n surface by $(x_i, z_i) \in S_i \mapsto (x, y, z)$ by

$$x = x_i^i z_i^{i-1} \quad , \qquad y = x_i^{n-i} z_i^{n+1-i} \quad , \qquad z = x_i z_i \quad . \tag{D.4}$$

The surfaces are glued together by $x_{i+1} z_i = 1$ and $x_i z_i = x_{i+1} z_{i+1}$ and the map onto the original A_n surface is isomorphic except at the pre-image of the singular point at $x = y = z = 0$, which we have blown up using $n - 1$ projective lines P_i each of which only intersect with P_{i-1} at $x_i = z_i = 0$. The intersection matrix of the components $\{P_1, ..., P_{n-1}\}$ is the same as the A_{n-1} Cartan matrix and the P_i can be seen as the nodes on a Dynkin diagram.

In this way we can classify some singularities through the way they are resolved using blow-ups thus creating a Dynkin diagram structure. For a fuller explanation and D and E examples see [84].

Appendix E Liouville theory

In light of the results on invertibility of matrix elements and matrix factors we can look at other conformal models, touching on integrability. We use (non) invertibility to show there can be no matrix factorisation of Liouville or Toda potentials. We also briefly look at the Lax pair formulation for obtaining the Toda equation of motion, and try and apply it to Landau -Ginzburg potentials. We then make a suggestion about the standing of these theories in relation to the CFT Landau-Ginzburg correspondence.

Liouville model and Landau-Ginsberg models involve very different types of super potential. LG models have quasi-homogeneous polynomials and Liouville theories have power series, but there is the possibility that this could be ignored as quasi homogeneity is strictly only relevant for the R symmetry and RG flow [71, 97], so why not consider a Lagrangian theory which is already conformal.

There are results computed for the characters of N=2 Liouville boundary states in the literature and both Liouville and Toda theories have information on the chiral primaries [54, 33]. The fact that a LG theory flows to a CFT and that LG potentials also contain all the topological information raises the question of whether or not these Lagrangian CFTs, i.e. Liouville and Toda theories also have such information encoded in their potentials in the same way.

The Liouville potential consists of an exponentiated superfield $W = e^x$. and potentials of this form are already conformal. Note that we are now working over the ring of power series. $W \in \mathbb{C}[[x]]$. Since the matrix factorisation condition allows us to add a constant to our potential [9], we can write the Liouville potential as

$$W \;=\; EJ \;=\; e^x - 1 \;. \tag{E.1}$$

W is of the form $xp(x)$ where $p(x)$ is an invertible power series. For any two matrix elements to contribute to the potential we must have $(J_1)_{ij}(E_1)_{ji} = xp(x)$ (no summation convention). This leaves no room for $(J_i)_{ij}$ and $(E_i)_{ji}$ to both be invertible. Thus any terms which contribute to a MF can be isolated as a trivial matrix factorisation in a direct sum and so have an empty cohomology.

Toda potentials are a similar case. We can use the same proof as used in the Liouville case to show there are no matrix factorisations possible of a general non affine N-component Toda potential. Suppose

$$W \;=\; EJ \;=\; \sum_{i=1}^{N} e^{xi} - N \tag{E.2}$$

If every element of E and J are non-invertible we can express all these elements as $\sum_{i=1}^{N} x_i p(x_i)$. We can now let $x_1 = x_2 =, ..., = x_N$ and we can then rewrite E and J as xE' and xJ'. Taking the determinant,

$$det(W\mathbb{I}_N) \;=\; det(xE'xJ') \tag{E.3}$$

W is of the form $xf(x)$ where $f(x)$ is invertible giving,

$$\begin{aligned} x^N(f(x))^N &= x^{2N} det(E'J') \\ f(x)^N &= x^N det(E'J') \end{aligned} \quad \text{(E.4)}$$

Since E' and J' are of the form $x^m Np(x)$ with $m \geq 0$ and with $p(x)$ invertible we have a contradiction as $f(x)$ is invertible.

It can be seen that a potential with linear term in a superfield prevents there from being a non-trivial cohomology. Liouville theory can be written down with a linear curvature term [99]. One way to remove linear term is to consider a specific curvature term with which cancels, Consider the Liouville potential with a linear curvature term as $W = e^x - -Rx - 1$ and we set $R = 1$. For Toda we would write

$$W(x_1, x_2, ..., x_n) = \sum_{=1}^{N} -e^{x_i} - \sum_{n=1}^{N} x_i - N \ . \quad \text{(E.5)}$$

Another way to remove the linear term is to consider affine Toda

$$W = \sum_{n=1}^{N} e^{x_i} + e^{-(\sum_{n=1}^{N} x_i)} - (N+1) \ . \quad \text{(E.6)}$$

There is a way to express affine Toda as the sum of squares of $Sinh$ functions and $e^{\phi} - 1$ type functions [78] and in this case tensor products could be constructed. We found an infinite cohomology for the specific curvature and for the affine case, but since these are no longer conformal theories it was not clear if they had a role to play and we did not pursue this further.

It is tempting to think we can get some sort of integrability condition from LG potentials since the MF condition is essentially a pair of commuting variables as is the Lax pair formulation of integrability [72]. If we consider the Lax pair formulation for Toda theories as outlined by Evans and Hollowood [37] we have a Lax pair defined in terms of the potential. The derivation of the equations of motion is from the zero curvature condition on the Lax pair and not from a Lagrangian. This process is outlined in appendices H and I for Liouvile and more general Toda theories. The Lax pair is given by,

$$\begin{aligned} A_\theta &= 2\beta D\Phi \mathbf{L_0} + \mathbf{G}_- \\ A_{\bar\theta} &= -e^{\beta\Phi}\mathbf{G}_+ \ . \end{aligned} \quad \text{(E.7)}$$

The zero curvature condition gives us the equations of motion. $F_{\theta\bar\theta} = 0$ is defined by,

$$F_{\theta\bar\theta} = DA_{\bar\theta} + \bar{D}A_\theta + \{A_\theta, A_{\bar\theta}\} = 0 \ . \quad \text{(E.8)}$$

The $\mathbf{L_0}$ and $\mathbf{G}_{+/-}$ are elements of OSp(1,2) as the algebra of the Lax representation. An equivalent derivation is given in by Mikhailov, M. Olshanetsky, A. Perelomov [72] using dummy variables λ and $1/\lambda$ instead of the OSp(1,2) generators.

We tried to emulate this process and construct a Lax pair using OSp(1,2) generators and apply this to the LG potential in the same way.

We can try to define a more general and hypothetical Lax pair as,

$$
\begin{aligned}
A_\theta &= \tilde{F}\mathbf{L_0} + E\mathbf{G}_- \\
A_\theta &= \tilde{H}\mathbf{L_0} + J\mathbf{G}_+
\end{aligned} \tag{E.9}
$$

Attempts to construct Lax pairs from more elements of the algebra of the LG potential via the constructions mentioned [37], [72] are also doomed to failure as these are constructions which specifically give Toda theories. There were other attempts at forming more general Lax pairs but in the end this formulation is interwoven with the integrability of Liouville and Toda theories and attempts to create such structure out of the factors from a matrix factorisation of the LG potential via the constructions mentioned [37], [72] seem doomed to failure as these are constructions which specifically give Toda theories.

We were not able to obtain cohomological basis from a Liouville or Toda theory without considering a precise (and arbitrary) linear curvature term or going to the affine model. In[99] The Liouville Lagrangian density is given as,

$$
\mathcal{L} = \frac{1}{4\pi} g^{ab}\partial_a\phi\partial_b\phi + QR + \phi\mu e^{2b\phi} \ . \tag{E.10}
$$

Q is a quantity called the background charge and $Q = b + 1/b$ [99]. The central charge of Liouville is then given by

$$
c_L = 1 + 6Q^2 \ . \tag{E.11}
$$

The value of central charge for minimal models satisfies $c_m < 1$ and Liouville theory, for the coupling constant b to be real, must have $c_L \geq 25$.[86]. This value for the central charge provides perhaps the best argument for viewing these theories in a different way. We can suggest that both minimal models and Liouville theory fall on the same side of the Landau-Ginzburg CFT correspondence and the chiral primaries found for Liouville and Toda theories must be the cohomology of MFs of potentials with central charge $c \geq 25$. Such potentials can be constructed from tensor products of linear A_n or other ADE matrix factorisations. Parameterised matrix factorisations (and hence cohomologies) appear at $c = 1$ so this may suggest that suitable polynomials for the representation of Liouville theory on the LG side would result in MFs which are parameterised.

The possibility has been discussed of making b complex [87] but this would not be of interest as these are not the theories already studied on the conformal side [33, 54], with which we would want to compare cohomologies of matrix factorisations. It could well be that MFs for such large values of central charge, or number of variables might have computability issues.

Appendix F Explicit defects

For the sake of completeness, we collect the orbifold equivalences that can serve to prove Theorem 4.1. The Singular-executable formats given on the web-page [100] should be of more practical use.

To save writing zeroes, we list matrices E and J only. Q is constructed from them as in (1.2). For fear of producing typos, we have refrained from attempts at simplifying the Singular output (except for the very easy case $A_5 \sim_{oeq} A_2 \times A_2$). The matrices spelled out in the following are the simplest ones we could find: what results from our Singular algorithm typically contains many more coefficients $a_{rs,\vec{p}}$, and we have chosen explicit values for some of them.

The orbifold equivalences are listed in the order they appear in Theorem 4.1.

(1) A rank 2 orbifold equivalence between A_5 and $A_2 \times A_2$:

$$E = \begin{pmatrix} x_1^2 - a_1(y_1+y_2) & x_2 + a_2x_1(y_1-y_2) \\ x_2 - a_2x_1(y_1-y_2) & -x_1^4 - 64a_1^8y_2^2 + 16a_1^5y_1y_2 - a_1x_1^2(y_1+y_2) - 4a_1^2y_1^2 \end{pmatrix}$$

$$J = \begin{pmatrix} x_1^4 + 64a_1^8y_2^2 - 16a_1^5y_1y_2 + a_1x_1^2(y_1+y_2) + 4a_1^2y_1^2 & x_2 + a_2x_1(y_1-y_2) \\ x_2 - a_2x_1(y_1-y_2) & -x_1^2 + a_1(y_1+y_2) \end{pmatrix}$$

where the coefficients have to satisfy

$$a_2^2 = 3a_1^2 \quad \text{and} \quad a_1^3 = \tfrac{1}{4} \ .$$

The quantum dimensions of Q are $q_L(Q) = -2a_1a_2$ and $q_R(Q) = -\frac{4}{3}a_2$. Since their product is 2, this is a "true orbifold equivalence", not an ordinary equivalence in the bi-category of Landau-Ginzburg potentials. On the other hand, since 2 is contained in any cyclotomic field, a group action might be the source of this orbifold equivalence.

(2) A rank 3 orbifold equivalence between E_{13} and Z_{11}:
The matrix elements E_{rs} and J_{rs} are given by

$E_{11} = -x_1^2 - y_1a_2$
$E_{12} = -x_1y_2a_3 - x_1y_2a_4 + x_2$
$E_{13} = -y_2a_4$
$E_{21} = x_1y_2a_3 - x_1y_2a_5 + x_2$
$E_{22} = y_2^2a_1^2a_4^2 + y_2^2a_1a_3a_4 + y_2^2a_1a_4^2 + y_2^2a_1a_4a_5 - x_1^3a_1 + y_2^2a_3^2 + y_2^2a_3a_4 - y_2^2a_3a_5 + y_2^2a_5^2 + x_1y_1a_1a_2 - x_1y_1a_1a_6 + x_1^3 - y_2^2a_7$
$E_{23} = x_1^2 + y_1a_6$
$E_{31} = -x_1^3a_1 - x_1^3 - y_2^2a_7$
$E_{32} = -x_1^2y_2a_1^2a_4 + 2y_1y_2a_1^2a_4a_2 - x_1^2y_2a_1a_3 - x_1^2y_2a_1a_4 - x_1^2y_2a_1a_5 + y_1y_2a_1a_3a_2 + 2y_1y_2a_1a_4a_2 + y_1y_2a_1a_5a_2 - y_1y_2a_1a_4a_6 - x_1^2y_2a_3 + x_1^2y_2a_5 + y_1y_2a_3a_2 - y_1y_2a_4a_6 - y_1y_2a_5a_6 + x_1x_2$
$E_{33} = x_1y_2a_5 + x_2$

$J_{11} = -x_1y_2^3a_1^2a_4^2a_5 - x_1^4y_2a_1^2a_4 - x_1y_2^3a_1a_3a_4a_5 - x_1y_2^3a_1a_4^2a_5 - x_1y_2^3a_1a_4a_5^2 + 2x_1^2y_1y_2a_1^2a_4a_2 - x_1^2y_1y_2a_1^2a_4a_6 + 2y_1^2y_2a_1^2a_4a_2a_6 - x_1^4y_2a_1a_3 - x_1^4y_2a_1a_4 - x_2y_2^2a_1^2a_4^2 - x_1y_2^3a_3^2a_5 - x_1y_2^3a_3a_4a_5 +$

$x_1y_2^3a_3a_5^2 - x_1y_2^3a_5^3 + x_1^2y_1y_2a_1a_3a_2 + 2x_1^2y_1y_2a_1a_4a_2 - x_1^2y_1y_2a_1a_3a_6 - 2x_1^2y_1y_2a_1a_4a_6 + y_1^2y_2a_1a_3a_2a_6 + 2y_1^2y_2a_1a_4a_2a_6 + y_1^2y_2a_1a_5a_2a_6 - y_1^2y_2a_1a_4a_6^2 - x_1^4y_2a_3 - x_2y_2^2a_1a_3a_4 - x_2y_2^2a_1a_4^2 - x_2y_2^2a_1a_4a_5 + x_1^2y_1y_2a_3a_2 - x_1^2y_1y_2a_3a_6 - x_1^2y_1y_2a_4a_6 + y_1^2y_2a_3a_2a_6 - y_1^2y_2a_4a_6^2 - y_1^2y_2a_5a_6^2 + x_1y_2^3a_5a_7 + x_1^3x_2a_1 - x_2y_2^2a_3^2 - x_2y_2^2a_3a_4 + x_2y_2^2a_3a_5 - x_2y_2^2a_5^2 - x_1x_2y_1a_1a_2 + x_1x_2y_1a_1a_6 + x_1x_2y_1a_6 + x_2y_2^2a_7$

$J_{12} = -x_1^2y_2^2a_1^2a_4^2 - y_1y_2^2a_1^2a_4^2a_2 - x_1^2y_2^2a_1a_3a_4 - x_1^2y_2^2a_1a_4^2 - x_1^2y_2^2a_1a_4a_5 - y_1y_2^2a_1a_3a_4a_2 - y_1y_2^2a_1a_4^2a_2 - y_1y_2^2a_1a_4a_5a_2 - x_1^2y_2^2a_3a_4 - x_1^2y_2^2a_3a_5 - y_1y_2^2a_3^2a_2 - y_1y_2^2a_3a_4a_2 + y_1y_2^2a_3a_5a_2 - y_1y_2^2a_5^2a_2 - x_1y_1^2a_1a_2^2 + x_1y_1^2a_1a_2a_6 + x_1y_1^2a_2a_6 + y_1y_2^2a_2a_7 - y_1y_2^2a_6a_7 - x_1x_2y_2a_3 + x_1x_2y_2a_5 + x_2^2$

$J_{13} = -y_2^3a_1^2a_4^3 - y_2^3a_1a_3a_4^2 - y_2^3a_1a_4^3 - y_2^3a_1a_4^2a_5 + x_1^3y_2a_1a_4 - y_2^3a_3^2a_4 - y_2^3a_3a_4^2 + y_2^3a_3a_4a_5 - y_2^3a_4a_5^2 - x_1y_1y_2a_1a_4a_2 + x_1y_1y_2a_1a_4a_6 + x_1^3y_2a_3 + x_1y_1y_2a_3a_6 + x_1y_1y_2a_4a_6 + y_2^3a_4a_7 - x_1^2x_2 - x_2y_1a_6$

$J_{21} = -3y_1y_2^2a_1^2a_4^2a_2 - 2y_1y_2^2a_1a_3a_4a_2 - 3y_1y_2^2a_1a_4^2a_2 - 2y_1y_2^2a_1a_4a_5a_2 + y_1y_2^2a_1a_4^2a_6 + x_1^5a_1 + x_1^2y_2^2a_3a_5 - x_1^2y_2^2a_5^2 - y_1y_2^2a_3^2a_2 - 2y_1y_2^2a_3a_4a_2 + y_1y_2^2a_3a_5a_2 - y_1y_2^2a_5^2a_2 - x_1y_1^2a_1a_2^2 + x_1^3y_1a_1a_6 + y_1y_2^2a_4^2a_6 + y_1y_2^2a_4a_5a_6 + x_1y_1^2a_1a_2a_6 + x_1^5 + x_1^3y_1a_6 + x_1y_1^2a_2a_6 + x_1^2y_2^2a_7 + y_1y_2^2a_2a_7 + x_1x_2y_2a_3 + x_2^2$

$J_{22} = x_1^3y_2a_1a_4 + x_1^3y_2a_4 + x_1^3y_2a_5 + x_1y_1y_2a_5a_2 + y_2^3a_4a_7 + x_1^2x_2 + x_2y_1a_2$

$J_{23} = x_1y_2^2a_3a_4 - x_1y_2^2a_4a_5 - x_1^4 - x_1^2y_1a_2 - x_1^2y_1a_6 - y_1^2a_2a_6 + x_2y_2a_4$

$J_{31} = -x_1^3y_2^2a_1^3a_4^2 - 2x_1^3y_2^2a_1^2a_4^2 - 2x_1^3y_2^2a_1^2a_4a_5 - 2x_1y_1y_2^2a_1^2a_3a_4a_2 + 3x_1y_1y_2^2a_1^2a_4^2a_2 + 2x_1y_1y_2^2a_1^2a_4a_5a_2 - y_2^4a_1^2a_4^2a_7 + x_1^6a_1^2 - x_1^3y_2^2a_1a_3a_4 - x_1^3y_2^2a_1a_4^2 + x_1^3y_2^2a_1a_3a_5 - 2x_1^3y_2^2a_1a_4a_5 - 2x_1^3y_2^2a_1a_5^2 - x_1^4y_1a_1^2a_2 - x_1y_1y_2^2a_1a_3^2a_2 + 3x_1y_1y_2^2a_1a_4^2a_2 + 4x_1y_1y_2^2a_1a_4a_5a_2 + x_1y_1y_2^2a_1a_5^2a_2 + x_1^4y_1a_1^2a_6 + x_1y_1y_2^2a_1a_3a_4a_6 - x_1y_1y_2^2a_1a_4^2a_6 - x_1y_1y_2^2a_1a_4a_5a_6 - y_2^4a_1a_3a_4a_7 - y_2^4a_1a_4^2a_7 - y_2^4a_1a_4a_5a_7 + x_1^2x_2y_2a_1^2a_4 - x_1^3y_2^2a_3a_4 - x_1^3y_2^2a_3a_5 - x_1^4y_1a_1a_2 - 2x_2y_1y_2a_1^2a_4a_2 + 2x_1y_1y_2^2a_3a_4a_2 + x_1y_1y_2^2a_5^2a_2 + x_1^2y_1^2a_1a_2^2 + x_1^4y_1a_1a_6 + x_1y_1y_2^2a_3a_4a_6 - x_1y_1y_2^2a_4^2a_6 + x_1y_1y_2^2a_3a_5a_6 - 2x_1y_1y_2^2a_4a_5a_6 - x_1y_1y_2^2a_5^2a_6 - x_1^2y_1^2a_1a_2a_6 + 2x_1^3y_2^2a_1a_7 - y_2^4a_3^2a_7 - y_2^4a_3a_4a_7 + y_2^4a_3a_5a_7 - y_2^4a_5^2a_7 - x_1y_1y_2^2a_1a_2a_7 + x_1y_1y_2^2a_1a_6a_7 - x_1^6 + x_1^2x_2y_2a_1a_3 + x_1^2x_2y_2a_1a_4 + x_1^2x_2y_2a_1a_5 - x_2y_1y_2a_1a_3a_2 - 2x_2y_1y_2a_1a_4a_2 - x_2y_1y_2a_1a_5a_2 + x_2y_1y_2a_1a_4a_6 - x_1^2y_1^2a_2a_6 - x_1y_1y_2^2a_2a_7 + x_1y_1y_2^2a_6a_7 + y_2^4a_7^2 - x_2y_1y_2a_3a_2 + x_2y_1y_2a_4a_6 + x_2y_1y_2a_5a_6 - x_1x_2^2$

$J_{32} = x_1^4y_2a_1^2a_4 - x_1^2y_1y_2a_1^2a_4a_2 - 2y_1^2y_2a_1^2a_4a_2^2 + x_1^4y_2a_1a_5 - x_1^2y_1y_2a_1a_4a_2 - y_1^2y_2a_1a_3a_2^2 - 2y_1^2y_2a_1a_4a_2^2 - y_1^2y_2a_1a_5a_2^2 + x_1^2y_1y_2a_1a_4a_6 + y_1^2y_2a_1a_4a_2a_6 - x_1^4y_2a_4 - x_1^4y_2a_5 - x_1^2y_1y_2a_5a_2 - y_1^2y_2a_3a_2^2 + x_1^2y_1y_2a_4a_6 + x_1^2y_1y_2a_5a_6 + y_1^2y_2a_4a_2a_6 + y_1^2y_2a_5a_2a_6 - x_1y_2^3a_3a_7 - x_1y_2^3a_4a_7 + x_1^3x_2a_1 - x_1x_2y_1a_2 + x_2y_2^2a_7$

$J_{33} = x_1^2y_2^2a_1^2a_4^2 - 2y_1y_2^2a_1^2a_4^2a_2 + x_1^2y_2^2a_1a_3a_4 + x_1^2y_2^2a_1a_4^2 + x_1^2y_2^2a_1a_4a_5 - y_1y_2^2a_1a_3a_4a_2 - 2y_1y_2^2a_1a_4^2a_2 - y_1y_2^2a_1a_4a_5a_2 + y_1y_2^2a_1a_4^2a_6 - x_1^5a_1 + x_1^2y_2^2a_4a_5 + x_1^2y_2^2a_5^2 - y_1y_2^2a_3a_4a_2 - x_1^3y_1a_1a_6 + y_1y_2^2a_4^2a_6 + y_1y_2^2a_4a_5a_6 + x_1^5 + x_1^3y_1a_2 + x_1y_1^2a_2a_6 - x_1^2y_2^2a_7 - y_1y_2^2a_6a_7 - x_1x_2y_2a_4 - x_1x_2y_2a_5 + x_2^2$

The seven coefficients $a_1, \ldots, a_7$ are subject to MF conditions which take the form of twelve algebraic equations $f_\alpha(a) = 0$ with

$f_1 = -(1/3)a_1a_3^2a_4a_6 - (1/3)a_1a_3a_4^2a_6 + (1/3)a_1a_3a_4a_5a_6 + (2/3)a_1a_4^2a_5a_6 + (2/3)a_1a_4a_5^2a_6 - (2/3)a_2a_3^3 - (1/3)a_2a_3^2a_4 + 2a_2a_3^2a_5 + (2/3)a_2a_3a_4a_5 - 2a_2a_3a_5^2 + (4/3)a_2a_5^3 - (1/3)a_3^2a_4a_6 - (1/3)a_3a_4^2a_6 + (2/3)a_4^2a_5a_6 + (4/3)a_4a_5^2a_6 + (2/3)a_2a_3a_7 - (4/3)a_2a_5a_7 - (5/3)a_3a_6a_7 + (1/3)a_5a_6a_7$

$f_2 = 2a_1^2a_3a_4^2a_6 - 4a_1^2a_4^2a_5a_6 + 2a_1a_3a_4^2a_6 - 4a_1a_4^2a_5a_6 - 2a_2a_3^3 + 6a_2a_3^2a_5 - 6a_2a_3a_5^2 + 4a_2a_5^3 - 2a_3a_4a_5a_6 + 4a_4a_5^2a_6 + 2a_2a_3a_7 - 4a_2a_5a_7 - 6a_3a_6a_7$

$f_3 = -7a_1^3a_2a_3a_4^2a_6 + 11a_1^3a_2a_4^2a_5a_6 - 4a_1^2a_2a_3^2a_4a_6 - 12a_1^2a_2a_3a_4^2a_6 - 2a_1^2a_2a_3a_4a_5a_6 + 15a_1^2a_2a_4^2a_5a_6 + 14a_1^2a_2a_4a_5^2a_6 - a_1^2a_4^3a_6^2 - a_1a_2a_3^3a_6 - 9a_1a_2a_3^2a_4a_6 - 8a_1a_2a_3a_4^2a_6 + 4a_1a_2a_3a_4a_5a_6 + 4a_1a_2a_4^2a_5a_6 + 7a_1a_2a_4a_5^2a_6 + 5a_1a_2a_5^3a_6 - 2a_1a_4^3a_6^2 - 2a_1a_4^2a_5a_6^2 - 12a_1^2a_2a_4a_6a_7 - a_2^2a_3^3 - 2a_2^2a_3^2a_4 + a_2^2a_3^2a_5 - a_2^2a_3a_5^2 - a_2a_3^3a_6 - 5a_2a_3^2a_4a_6 - 3a_2a_3a_4^2a_6 + a_2a_3^2a_5a_6 + a_2a_3a_4a_5a_6 - a_2a_3a_5^2a_6 + a_2a_5^3a_6 - a_4^3a_6^2 - 2a_4^2a_5a_6^2 - a_4a_5^2a_6^2 - 6a_1a_2a_3a_6a_7 - 7a_1a_2a_4a_6a_7 - 6a_1a_2a_5a_6a_7 + a_1a_4a_6^2a_7 + a_2^2a_3a_7 - a_2a_3a_6a_7 - a_2a_5a_6a_7 + a_4a_6^2a_7 + a_5a_6^2a_7$

$f_4 = -(5/2)a_1^3a_3a_4^2 + 2a_1^3a_4^2a_5 - 2a_1^2a_3^2a_4 - (9/2)a_1^2a_3a_4^2 + 2a_1^2a_3a_4a_5 + 3a_1^2a_4^2a_5 - 2a_1^2a_4a_5^2 - (3/2)a_1a_3^3 - 4a_1a_3^2a_4 - 2a_1a_3a_4^2 + (9/2)a_1a_3^2a_5 + (11/2)a_1a_3a_4a_5 + a_1a_4^2a_5 - (9/2)a_1a_3a_5^2 - 4a_1a_4a_5^2 + 6a_1^2a_4a_7 - (3/2)a_3^3 - 2a_3^2a_4 + (9/2)a_3^2a_5 + 3a_3a_4a_5 - (9/2)a_3a_5^2 - a_4a_5^2 + a_5^3 + 3a_1a_3a_7 + 6a_1a_4a_7 + 3a_1a_5a_7 + (7/2)a_3a_7 + a_4a_7 - a_5a_7$

$f_5 = 3a_1a_3^3a_4a_6 + a_1a_3^2a_4^2a_6 - 9a_1a_3^2a_4a_5a_6 + 8a_1a_3a_4a_5^2a_6 - 2a_1a_4^2a_5^2a_6 - 4a_1a_4a_5^3a_6 + 5a_2a_3^4 +$

$a_2a_3^3a_4 - 19a_2a_3^3a_5 - 2a_2a_3^2a_4a_5 + 25a_2a_3^2a_5^2 + a_2a_3a_4a_5^2 - 15a_2a_3a_5^3 - a_2a_4a_5^3 + 2a_2a_5^4 + 3a_3^3a_4a_6 + a_3^2a_4^2a_6 - 8a_3^2a_4a_5a_6 + 7a_3a_4a_5^2a_6 - 2a_4^2a_5^2a_6 - 5a_4a_5^3a_6 - 9a_1a_3a_4a_6a_7 + 4a_1a_4^2a_6a_7 + 9a_1a_4a_5a_6a_7 - 8a_2a_3^2a_7 + 2a_2a_3a_4a_7 + 11a_2a_3a_5a_7 + a_2a_4a_5a_7 + 11a_3^2a_6a_7 - 11a_3a_4a_6a_7 + 3a_4^2a_6a_7 - 11a_3a_5a_6a_7 + 11a_4a_5a_6a_7 + 3a_5^2a_6a_7 - 2a_2a_7^2 + 2a_6a_7^2$

$f_6 = 3a_1^2a_4^2a_6 + 2a_1a_3a_4a_6 + 5a_1a_4^2a_6 + 2a_1a_4a_5a_6 + a_2a_3^2 + 2a_2a_3a_4 - a_2a_3a_5 + a_2a_5^2 + 2a_3a_4a_6 + 2a_4^2a_6 + a_4a_5a_6 - a_2a_7 + a_6a_7$

$f_7 = a_1a_2 - a_1a_6 - a_6$

$f_8 = -2a_1^2a_2a_4a_6^2 - a_1a_2a_3a_6^2 - 3a_1a_2a_4a_6^2 - a_1a_2a_5a_6^2 - a_2^2a_3a_6 - a_2a_3a_6^2 - a_2a_4a_6^2 + 1$

$f_9 = a_1^2a_4^3a_7 + a_1a_3a_4^2a_7 + a_1a_4^3a_7 + a_1a_4^2a_5a_7 + a_3^2a_4a_7 + a_3a_4^2a_7 - a_3a_4a_5a_7 + a_4a_5^2a_7 - a_4a_7^2 + 1$

$f_{10} = 5a_1^2a_3a_4^2a_6 - a_1^2a_4^2a_5a_6 + 3a_1a_3^2a_4a_6 + 8a_1a_3a_4^2a_6 + 3a_1a_3a_4a_5a_6 - a_1a_4^2a_5a_6 + a_2a_3^3 + 3a_2a_3^2a_4 + a_2a_5^3 + 3a_3^2a_4a_6 + 3a_3a_4^2a_6 + a_3a_4a_5a_6 + a_4a_5^2a_6 - a_2a_3a_7 - a_2a_5a_7$

$f_{11} = -3a_1^2a_4a_6^2 - a_1a_3a_6^2 - 6a_1a_4a_6^2 - a_1a_5a_6^2 - a_2^2a_3 - a_2a_3a_6 - a_2a_4a_6 - a_3a_6^2 - 3a_4a_6^2$

$f_{12} = a_1^3a_4^3 + 2a_1^2a_4^3 + 3a_1^2a_4^2a_5 + a_1a_3a_4^2 + a_1a_4^3 + 3a_1a_4^2a_5 + 3a_1a_4a_5^2 + a_3a_4^2 + a_3a_4a_5 + a_4a_5^2 + a_5^3 - 2a_1a_4a_7 - a_3a_7 - a_4a_7 - a_5a_7$

These twelve equations are solvable, and the quantum dimensions, subject to the MF conditions, are given by

$q_L(Q) = a_1a_4a_6 + a_2a_3 + a_4a_6 + a_5a_6$

$q_R(Q) = (462a_1a_5a_6^2a_7^2 + 603a_1^3a_6^2 - 2002a_2^2a_3a_7^2 + 158a_2^2a_4a_7^2 - 853a_2^2a_5a_7^2 - 898a_2a_3a_6a_7^2 - 2784a_2a_4a_6a_7^2 - 136a_2a_5a_6a_7^2 + 214a_3a_6^2a_7^2 - 1294a_4a_6^2a_7^2 + 1111a_5a_6^2a_7^2 + 2646a_1^2a_6^2 - 261a_1a_6^2 - 291a_2^2 - 301a_2a_6 - 2095a_6^2)/764$

Note that these expressions result after reduction by the ideal spanned by the f_α, hence the quantum dimensions of this defect are non-zero numbers after inserting any special solution to the equations $f_\alpha(a) = 0$.

(3) A rank 6 orbifold equivalence between Z_{13} and Q_{11} is given by:

$E_{11} = 2y_3a_1^2a_2 + 2y_3a_1a_3$

$E_{12} = -(3/2)x_1^3a_1^3a_2^3 - x_1^3a_1^2a_3a_2^2 + (1/2)x_1^3a_1a_3^2a_2 + 2x_2y_3a_4a_1^2a_2 + 2x_2y_3a_4a_1a_3 - x_2y_3a_1 + x_1y_2a_3 + x_3$

$E_{13} = 0$

$E_{14} = -(3/8)x_1^2a_4a_1^3a_2^3 - (1/4)x_1^2a_4a_1^2a_3a_2^2 + (1/8)x_1^2a_4a_1a_3^2a_2 + (1/4)x_1^2a_3^2 - y_2$

$E_{15} = -x_2$

$E_{16} = 0$

$E_{21} = (3/2)x_1^3a_1^3a_2^3 + x_1^3a_1^2a_3a_2^2 - (1/2)x_1^3a_1a_3^2a_2 + x_2y_3a_1 - x_1y_2a_3 + x_3$

$E_{22} = (3/4)x_1^3x_2a_4a_1^3a_2^3 - x_1^2y_3^2a_1^4a_2^2 + (1/2)x_1^3x_2a_4a_1^2a_3a_2^2 - x_1^2y_3^2a_1^3a_3a_2 - (1/4)x_1^3x_2a_4a_1a_3^2a_2 - (1/2)x_1^3x_2a_3^2 + x_2^2y_3a_4a_1 + y_2y_3^2a_1^2 - x_1x_2y_2a_4a_3 + (1/2)x_1^2y_3^2a_2 - y_1^2a_5^2 + x_1x_2y_2 + x_2x_3a_4$

$E_{23} = (9/32)x_1^4a_4^2a_1^6a_2^6 + (3/8)x_1^4a_4^2a_1^5a_3a_2^5 - (1/16)x_1^4a_4^2a_1^4a_3^2a_2^4 - (1/8)x_1^4a_4^2a_1^3a_3^3a_2^3 + (1/32)x_1^4a_4^2a_1^2a_3^4a_2^2 + (3/8)x_1^2y_2a_4a_1^3a_2^3 + (1/4)x_1^2y_2a_4a_1^2a_3a_2^2 - (1/8)x_1^4a_3^4 - (1/8)x_1^2y_2a_4a_1a_3^2a_2 + x_1x_2y_3a_1^2a_2 + (3/4)x_1^2y_2a_3^2 + x_2y_1a_5 - y_2^2$

$E_{24} = 0$

$E_{25} = 0$

$E_{26} = (1/2)x_1^3y_3a_1^4a_2^3 - (3/8)x_1^2y_1a_4a_1^3a_5a_2^3 - (1/4)x_1^2y_1a_4a_1^2a_3a_5a_2^2 - (1/2)x_1^3y_3a_1^2a_3^2a_2 + (1/8)x_1^2y_1a_4a_1a_3^2a_5a_2 + (1/4)x_1^2y_1a_3^2a_5 + x_1y_2y_3a_1^2a_2 + x_2y_3^2a_1^2 + x_1x_2^2 - y_1y_2a_5$

$E_{31} = 0$

$E_{32} = (9/32)x_1^4a_4^2a_1^6a_2^6 + (3/8)x_1^4a_4^2a_1^5a_3a_2^5 - (1/16)x_1^4a_4^2a_1^4a_3^2a_2^4 - (1/8)x_1^4a_4^2a_1^3a_3^3a_2^3 + (1/32)x_1^4a_4^2a_1^2a_3^4a_2^2 + (3/8)x_1^2y_2a_4a_1^3a_2^3 + (1/4)x_1^2y_2a_4a_1^2a_3a_2^2 - (1/8)x_1^4a_3^4 - (1/8)x_1^2y_2a_4a_1a_3^2a_2 + x_1x_2y_3a_1^2a_2 + (3/4)x_1^2y_2a_3^2 - x_2y_1a_5 - y_2^2$

$E_{33} = -(3/4)x_1^2y_3a_4a_1^5a_2^4 - (1/2)x_1^2y_3a_4a_1^4a_3a_2^3 + (1/4)x_1^2y_3a_4a_1^3a_3^2a_2^2 + 3x_1^2y_3a_1^4a_2^3 + 2x_1^2y_3a_1^3a_3a_2^2 - (1/2)x_1^2y_3a_1^2a_3^2a_2 - 2y_2y_3a_1^2a_2 - 2y_2y_3a_1a_3 + x_2^2$
$E_{34} = (3/2)x_1^3a_1^3a_2^3 + x_1^3a_1^2a_3a_2^2 - (1/2)x_1^3a_1a_3^2a_2 + x_2y_3a_1 - x_1y_2a_3 + x_3$
$E_{35} = 0$
$E_{36} = -(3/8)x_1^2x_2a_4a_1^3a_2^3 + 2x_1y_3^2a_1^4a_2^2 - (1/4)x_1^2x_2a_4a_1^2a_3a_2^2 + 2x_1y_3^2a_1^3a_3a_2 + (1/8)x_1^2x_2a_4a_1a_3^2a_2 - 2y_1y_3a_1^2a_5a_2 + (1/4)x_1^2x_2a_3^2 - 2y_1y_3a_1a_3a_5 - x_2y_2$
$E_{41} = -(3/8)x_1^2a_4a_1^3a_2^3 - (1/4)x_1^2a_4a_1^2a_3a_2^2 + (1/8)x_1^2a_4a_1a_3^2a_2 + (1/4)x_1^2a_3^2 - y_2$
$E_{42} = -(3/8)x_1^2x_2a_4^2a_1^3a_2^3 - (1/4)x_1^2x_2a_4^2a_1^2a_3a_2^2 + (1/8)x_1^2x_2a_4^2a_1a_3^2a_2 + (1/4)x_1^2x_2a_4a_3^2 - x_2y_2a_4$
$E_{43} = -(3/2)x_1^3a_1^3a_2^3 - x_1^3a_1^2a_3a_2^2 + (1/2)x_1^3a_1a_3^2a_2 - x_2y_3a_1 + x_1y_2a_3 + x_3$
$E_{44} = -y_3^2a_1^2 - x_1x_2$
$E_{45} = x_1y_3a_1^2a_2 - y_1a_5$
$E_{46} = 0$
$E_{51} = -x_2$
$E_{52} = -x_2^2a_4$
$E_{53} = 0$
$E_{54} = x_1y_3a_1^2a_2 + y_1a_5$
$E_{55} = -x_1^2a_1^2a_2^2 + y_2$
$E_{56} = -(3/2)x_1^3a_1^3a_2^3 - x_1^3a_1^2a_3a_2^2 + (1/2)x_1^3a_1a_3^2a_2 - x_2y_3a_1 + x_1y_2a_3 + x_3$
$E_{61} = -x_1x_2a_3$
$E_{62} = (1/2)x_1^3y_3a_1^4a_2^3 + (3/8)x_1^2y_1a_4a_1^3a_5a_2^3 + (1/4)x_1^2y_1a_4a_1^2a_3a_5a_2^2 - (1/2)x_1^3y_3a_1^2a_3^2a_2 - (1/8)x_1^2y_1a_4a_1a_3^2a_5a_2 - (1/4)x_1^2y_1a_3^2a_5 + x_1y_2y_3a_1^2a_2 + x_2y_3^2a_1^2 - x_1x_2^2a_4a_3 + x_1x_2^2 + y_1y_2a_5$
$E_{63} = -(3/8)x_1^2x_2a_4a_1^3a_2^3 - (1/4)x_1^2x_2a_4a_1^2a_3a_2^2 + (1/8)x_1^2x_2a_4a_1a_3^2a_2 + 2y_1y_3a_1^2a_5a_2 + (1/4)x_1^2x_2a_3^2 + 2y_1y_3a_1a_3a_5 + x_1y_3^2a_2 - x_2y_2$
$E_{64} = x_1^2y_3a_1^2a_3a_2 + x_1y_1a_3a_5$
$E_{65} = (3/2)x_1^3a_1^3a_2^3 - (1/2)x_1^3a_1a_3^2a_2 + x_2y_3a_1 + x_3$
$E_{66} = x_1^4a_1^4a_2^4 + (3/2)x_1^4a_1^3a_3a_2^3 - (1/2)x_1^4a_1a_3^3a_2 + 2y_3^3a_1^4a_2 + 2y_3^3a_1^3a_3 + x_1^2y_2a_1^2a_2^2 + 2x_1x_2y_3a_1^2a_2 + x_1x_2y_3a_1a_3 + x_1x_3a_3 + y_2^2$

$J_{11} = -(3/4)x_1^3x_2a_4a_1^3a_2^3 + x_1^2y_3^2a_1^4a_2^2 - (1/2)x_1^3x_2a_4a_1^2a_3a_2^2 + x_1^2y_3^2a_1^3a_3a_2 + (1/4)x_1^3x_2a_4a_1a_3^2a_2 + (1/2)x_1^3x_2a_3^2 - x_2^2y_3a_4a_1 - y_2y_3^2a_1^2 + x_1x_2y_2a_4a_3 - (1/2)x_1^2y_3^2a_2 + y_1^2a_5^2 - x_1x_2y_2 - x_2x_3a_4$
$J_{12} = -(3/2)x_1^3a_1^3a_2^3 - x_1^3a_1^2a_3a_2^2 + (1/2)x_1^3a_1a_3^2a_2 + 2x_2y_3a_4a_1^2a_2 + 2x_2y_3a_4a_1a_3 - x_2y_3a_1 + x_1y_2a_3 + x_3$
$J_{13} = -(3/8)x_1^2x_2a_4^2a_1^3a_2^3 - (1/4)x_1^2x_2a_4^2a_1^2a_3a_2^2 + (1/8)x_1^2x_2a_4^2a_1a_3^2a_2 + (1/4)x_1^2x_2a_4a_3^2 - x_2y_2a_4$
$J_{14} = -(9/32)x_1^4a_4^2a_1^6a_2^6 - (3/8)x_1^4a_4^2a_1^5a_3a_2^5 + (1/16)x_1^4a_4^2a_1^4a_3^2a_2^4 + (1/8)x_1^4a_4^2a_1^3a_3^3a_2^3 - (1/32)x_1^4a_4^2a_1^2a_3^4a_2^2 - (3/8)x_1^2y_2a_4a_1^3a_2^3 - (1/4)x_1^2y_2a_4a_1^2a_3a_2^2 + (1/8)x_1^4a_3^4 + (1/8)x_1^2y_2a_4a_1a_3^2a_2 - x_1x_2y_3a_1^2a_2 - (3/4)x_1^2y_2a_3^2 - x_2y_1a_5 + y_2^2$
$J_{15} = -(1/2)x_1^3y_3a_1^4a_2^3 + (3/8)x_1^2y_1a_4a_1^3a_5a_2^3 + (1/4)x_1^2y_1a_4a_1^2a_3a_5a_2^2 + (1/2)x_1^3y_3a_1^2a_3^2a_2 - (1/8)x_1^2y_1a_4a_1a_3^2a_5a_2 - (1/4)x_1^2y_1a_3^2a_5 - x_1y_2y_3a_1^2a_2 - x_2y_3^2a_1^2 + x_1x_2^2a_4a_3 - x_1x_2^2 + y_1y_2a_5$
$J_{16} = -x_2^2a_4$
$J_{21} = (3/2)x_1^3a_1^3a_2^3 + x_1^3a_1^2a_3a_2^2 - (1/2)x_1^3a_1a_3^2a_2 + x_2y_3a_1 - x_1y_2a_3 + x_3$
$J_{22} = -2y_3a_1^2a_2 - 2y_3a_1a_3$
$J_{23} = (3/8)x_1^2a_4a_1^3a_2^3 + (1/4)x_1^2a_4a_1^2a_3a_2^2 - (1/8)x_1^2a_4a_1a_3^2a_2 - (1/4)x_1^2a_3^2 + y_2$
$J_{24} = 0$
$J_{25} = -x_1x_2a_3$
$J_{26} = x_2$
$J_{31} = 0$
$J_{32} = (3/8)x_1^2a_4a_1^3a_2^3 + (1/4)x_1^2a_4a_1^2a_3a_2^2 - (1/8)x_1^2a_4a_1a_3^2a_2 - (1/4)x_1^2a_3^2 + y_2$
$J_{33} = y_3^2a_1^2 + x_1x_2$

$J_{34} = (3/2)x_1^3a_1^3a_2^3 + x_1^3a_1^2a_3a_2^2 - (1/2)x_1^3a_1a_3^2a_2 + x_2y_3a_1 - x_1y_2a_3 + x_3$
$J_{35} = x_1^2y_3a_1^2a_3a_2 - x_1y_1a_3a_5$
$J_{36} = -x_1y_3a_1^2a_2 + y_1a_5$
$J_{41} = -(9/32)x_1^4a_4^2a_1^6a_2^6 - (3/8)x_1^4a_4^2a_1^5a_3a_2^5 + (1/16)x_1^4a_4^2a_1^4a_3^2a_2^4 + (1/8)x_1^4a_4^2a_1^3a_3^3a_2^3 - (1/32)x_1^4a_4^2a_1^2a_3^4a_2^2 - (3/8)x_1^2y_2a_4a_1^3a_2^3 - (1/4)x_1^2y_2a_4a_1^2a_3a_2^2 + (1/8)x_1^4a_3^4 + (1/8)x_1^2y_2a_4a_1a_3^2a_2 - x_1x_2y_3a_1^2a_2 - (3/4)x_1^2y_2a_3^2 + x_2y_1a_5 + y_2^2$
$J_{42} = 0$
$J_{43} = -(3/2)x_1^3a_1^3a_2^3 - x_1^3a_1^2a_3a_2^2 + (1/2)x_1^3a_1a_3^2a_2 - x_2y_3a_1 + x_1y_2a_3 + x_3$
$J_{44} = (3/4)x_1^2y_3a_4a_1^5a_2^4 + (1/2)x_1^2y_3a_4a_1^4a_3a_2^3 - (1/4)x_1^2y_3a_4a_1^3a_3^2a_2^2 - 3x_1^2y_3a_1^4a_2^3 - 2x_1^2y_3a_1^3a_3a_2^2 + (1/2)x_1^2y_3a_1^2a_3^2a_2 + 2y_2y_3a_1^2a_2 + 2y_2y_3a_1a_3 - x_2^2$
$J_{45} = (3/8)x_1^2x_2a_4a_1^3a_2^3 - 2x_1y_3^2a_1^4a_2^2 + (1/4)x_1^2x_2a_4a_1^2a_3a_2^2 - 2x_1y_3^2a_1^3a_3a_2 - (1/8)x_1^2x_2a_4a_1a_3^2a_2 + 2y_1y_3a_1^2a_5a_2 - (1/4)x_1^2x_2a_3^2 + 2y_1y_3a_1a_3a_5 + x_2y_2$
$J_{46} = 0$
$J_{51} = -(1/2)x_1^3y_3a_1^4a_2^3 - (3/8)x_1^2y_1a_4a_1^3a_5a_2^3 - (1/4)x_1^2y_1a_4a_1^2a_3a_5a_2^2 + (1/2)x_1^3y_3a_1^2a_3^2a_2 + (1/8)x_1^2y_1a_4a_1a_3^2a_5a_2 + (1/4)x_1^2y_1a_3^2a_5 - x_1y_2y_3a_1^2a_2 - x_2y_3^2a_1^2 - x_1x_2^2 - y_1y_2a_5$
$J_{52} = 0$
$J_{53} = 0$
$J_{54} = (3/8)x_1^2x_2a_4a_1^3a_2^3 + (1/4)x_1^2x_2a_4a_1^2a_3a_2^2 - (1/8)x_1^2x_2a_4a_1a_3^2a_2 - 2y_1y_3a_1^2a_5a_2 - (1/4)x_1^2x_2a_3^2 - 2y_1y_3a_1a_3a_5 - x_1y_3^2a_2 + x_2y_2$
$J_{55} = -x_1^4a_1^4a_2^4 - (3/2)x_1^4a_1^3a_3a_2^3 + (1/2)x_1^4a_1a_3^3a_2 - 2y_3^3a_1^4a_2 - 2y_3^3a_1^3a_3 - x_1^2y_2a_1^2a_2^2 - 2x_1x_2y_3a_1^2a_2 - x_1x_2y_3a_1a_3 - x_1x_3a_3 - y_2^2$
$J_{56} = -(3/2)x_1^3a_1^3a_2^3 - x_1^3a_1^2a_3a_2^2 + (1/2)x_1^3a_1a_3^2a_2 - x_2y_3a_1 + x_1y_2a_3 + x_3$
$J_{61} = 0$
$J_{62} = x_2$
$J_{63} = -x_1y_3a_1^2a_2 - y_1a_5$
$J_{64} = 0$
$J_{65} = (3/2)x_1^3a_1^3a_2^3 - (1/2)x_1^3a_1a_3^2a_2 + x_2y_3a_1 + x_3$
$J_{66} = x_1^2a_1^2a_2^2 - y_2$

The five coefficients $a_1, \ldots, a_5$ are subject to thirty-seven relatively simple conditions $f_\alpha(a) = 0$ with

$f_1 = a_1^2 + a_5^2$
$f_2 = -3a_1a_2a_4 + 52a_1a_3a_5^2 - 7a_3a_4 - 10$
$f_3 = -1839a_1a_2a_3 + 30a_2^2a_5^2 + 835a_3^2 + 72a_4^4$
$f_4 = 94888a_1a_4a_5^2 - 6675a_2^3 + 7504a_3a_4^3 + 41908a_4^2$
$f_5 = -159a_1a_2a_4 + 52a_3^2a_4^2 + 383a_3a_4 + 445$
$f_6 = 83a_1a_2a_5^2 + 14a_2a_3a_4^2 + 53a_2a_4 - 75a_3a_5^2$
$f_7 = 225a_1a_2a_4^2 + 2314a_1a_5^2 + 187a_3a_4^2 + 724a_4$
$f_8 = -36a_1a_2a_3 + 15a_2^2a_5^2 + 8a_3^3a_4 + 47a_3^2$
$f_9 = 2a_1a_3^2a_4 + 78a_1a_3 - 15a_2^2a_4^2 - 81a_2a_5^2$
$f_{10} = 10a_1a_2a_3a_4 + 13a_1a_2 + 6a_3^2a_4 + 29a_3$
$f_{11} = 33a_1a_2^2a_4 - 27a_2a_3a_4 - 72a_2 - 52a_3^2a_5^2$
$f_{12} = -145863a_2^3 + 47444a_3^4 - 31896a_3a_4^3 - 82080a_4^2$
$f_{13} = 1892a_1a_3^3 - 648a_1a_4^3 + 228a_2^2a_3a_4 - 2223a_2^2$
$f_{14} = 8a_1a_3a_4^2 + 44a_1a_4 + a_2^2a_3^2 - 92a_5^4$
$f_{15} = 1311a_1a_2a_3^2 + 342a_2^3a_4 - 211a_3^3 + 180a_4^3$
$f_{16} = -4926a_1a_2a_4^2 + 1157a_2^3a_3 + 102a_3a_4^2 + 7968a_4$
$f_{17} = 41a_2^4 - 132a_2a_4^2 - 408a_3a_4a_5^2 - 784a_5^2$

$f_{18} = 1002a_1a_2^3 - 12a_1a_4^2 - 469a_2^2a_3 + 1224a_4a_5^4$
$f_{19} = 7a_1a_3a_4 + 10a_1 - 3a_2a_4a_5^2 + 52a_3a_5^4$
$f_{20} = -2a_1a_3a_5^2 + 2a_2a_5^4 - 1$
$f_{21} = 67716a_1a_5^4 - 959a_2a_3^3 - 1584a_2a_4^3 + 23256a_4a_5^2$
$f_{22} = -71a_1a_2^2 + 116a_2a_3^2a_4 + 403a_2a_3 + 48a_4^3a_5^2$
$f_{23} = 3649a_2a_3^3 + 9999a_2a_4^3 + 33858a_3a_4^2a_5^2 + 55575a_4a_5^2$
$f_{24} = 654a_1a_3a_4^2 + 17604a_1a_4 + 1157a_2^2a_3^2 + 10350a_2a_4^2a_5^2$
$f_{25} = 283176a_1a_4^2a_5^2 - 170487a_2^3a_4 - 166964a_3^3 + 46476a_4^3$
$f_{26} = 123a_1a_2a_5^2 + 15a_2a_3a_4^2 + 106a_3^2a_4a_5^2 + 514a_3a_5^2$
$f_{27} = -6a_1a_3^2a_4 - 29a_1a_3 + 10a_2a_3a_4a_5^2 + 13a_2a_5^2$
$f_{28} = 3a_1a_2 + 3a_2^2a_4a_5^2 - a_3^2a_4 - 7a_3$
$f_{29} = 246a_1a_2a_4a_5^2 - 39a_2a_4^2 - 128a_3a_4a_5^2 - 623a_5^2$
$f_{30} = 33a_1a_2^2 + 36a_2a_3^2a_4 + 129a_2a_3 + 40a_3^3a_5^2$
$f_{31} = -22a_1a_3^3 - 24a_2^2a_3a_4 - 9a_2^2 + 54a_2a_3^2a_5^2$
$f_{32} = 2a_1a_2a_3^2 - 6a_2^3a_4 + 45a_2^2a_3a_5^2 + 3a_3^3$
$f_{33} = -40a_1a_2^2a_3 + 123a_2^3a_5^2 + 51a_2a_3^2 + 48a_4^2a_5^2$
$f_{34} = 9a_1a_2^2a_5^2 + 7a_1a_3^2 + 6a_2^2a_4 - 18a_2a_3a_5^2$
$f_{35} = -329a_1a_3a_4 + 89a_1 + 78a_2^2a_4^3 + 453a_2a_4a_5^2$
$f_{36} = -3211a_1a_2a_3 + 393a_2^3a_4^2 + 2637a_2^2a_5^2 + 306a_3^3a_4 + 1765a_3^2$
$f_{37} = -5112a_1a_4a_5^2 + 19a_2^3 + 612a_4^2 + 15008a_5^6$

The quantum dimensions of Q are:
$q_L(Q) = (24/13)a_2a_4a_5^3 - (4/13)a_1a_3a_4a_5 + (50/13)a_1a_5$
$q_R(Q) = -2a_1a_2a_5 - 2a_3a_5$.

(4) A rank 4 orbifold equivalence between S_{11} and W_{13}:

$E_{11} = x_1y_3 - y_1$
$E_{12} = -x_2y_3^2 + x_1^2 + x_2x_3$
$E_{13} = -x_2 + y_2$
$E_{14} = 0$
$E_{21} = -y_3^2 - x_3$
$E_{22} = -x_1y_3 - y_1$
$E_{23} = 0$
$E_{24} = -x_2 + y_2$
$E_{31} = x_2^3 + x_2^2y_2 + x_2y_2^2 + y_2^3 - x_3y_3^2$
$E_{32} = -x_1y_3^3 - y_1y_3^2$
$E_{33} = -x_1y_3 - y_1$
$E_{34} = y_2y_3^2 - x_1^2 - x_2x_3$
$E_{41} = 0$
$E_{42} = y_3^4 + x_2^3 + x_2^2y_2 + x_2y_2^2 + y_2^3$
$E_{43} = y_3^2 + x_3$
$E_{44} = x_1y_3 - y_1$

$J_{11} = -x_1y_3 - y_1$
$J_{12} = y_2y_3^2 - x_1^2 - x_2x_3$
$J_{13} = x_2 - y_2$
$J_{14} = 0$
$J_{21} = y_3^2 + x_3$

$J_{22} = x_1y_3 - y_1$
$J_{23} = 0$
$J_{24} = x_2 - y_2$
$J_{31} = -y_3^4 - x_2^3 - x_2^2y_2 - x_2y_2^2 - y_2^3$
$J_{32} = -x_1y_3^3 + y_1y_3^2$
$J_{33} = x_1y_3 - y_1$
$J_{34} = -x_2y_3^2 + x_1^2 + x_2x_3$
$J_{41} = 0$
$J_{42} = -x_2^3 - x_2^2y_2 - x_2y_2^2 - y_2^3 + x_3y_3^2$
$J_{43} = -y_3^2 - x_3$
$J_{44} = -x_1y_3 - y_1$

This rather simple Q does not depend on any coefficients, although more general orbifold equivalences between S_{11} and W_{13} can be found.

Its quantum dimensions are $q_L(Q) = -2$ and $q_R(Q) = -1$.

(5) A rank 3 orbifold equivalence between a chain and a loop at central charge $\hat{c} = \frac{6}{5}$:

$E_{11} = 2a_1x_1^4 + 2a_1x_1y_2^2$
$E_{12} = a_1x_1^3y_2 + a_1y_2^3 + y_1$
$E_{13} = a_1x_1^5 + a_1x_1^2y_2^2 + x_2$
$E_{21} = -2a_1x_1^3y_2 + -a_1y_2^3 + y_1$
$E_{22} = -a_1x_1^2y_2^2 + x_2$
$E_{23} = -a_1x_1^4y_2 + x_1y_1$
$E_{31} = -a_1x_1^5 + x_2$
$E_{32} = -x_1y_1$
$E_{33} = -y_1y_2$

$J_{11} = a_1x_1^5y_1y_2 + -a_1x_1^2y_1y_2^3 - x_1^2y_1^2 + x_2y_1y_2$
$J_{12} = a_1x_1^6y_1 + -a_1y_1y_2^4 + x_1x_2y_1 - y_1^2y_2$
$J_{13} = a_1^2 + 1x_1^{10} + a_1x_1^5x_2 + -a_1x_1y_1y_2^3 - x_1y_1^2 + x_2^2$
$J_{21} = -a_1^2x_1^9y_2 + a_1x_1^6y_1 + a_1x_1^4x_2y_2 + 2a_1x_1^3y_1y_2^2 + a_1y_1y_2^4 - x_1x_2y_1 - y_1^2y_2$
$J_{22} = x_1^{10} + -a_1^2x_1^7y_2^2 + 2a_1x_1^4y_1y_2 + a_1x_1^2x_2y_2^2 + 2a_1x_1y_1y_2^3 + x_2^2$
$J_{23} = a_1^2x_1^5y_2^3 + a_1^2x_1^2y_2^5 + a_1x_1^5y_1 + 2a_1x_1^3x_2y_2 + a_1x_1^2y_1y_2^2 + a_1x_2y_2^3 - x_2y_1$
$J_{31} = a_1^2 + 1x_1^{10} + a_1^2x_1^7y_2^2 + -a_1x_1^5x_2 + -2a_1x_1^4y_1y_2 + -a_1x_1^2x_2y_2^2 + -a_1x_1y_1y_2^3 + x_1y_1^2 + x_2^2$
$J_{32} = a_1^2x_1^8y_2 + a_1^2x_1^5y_2^3 + -a_1x_1^5y_1 + -a_1x_1^3x_2y_2 + -2a_1x_1^2y_1y_2^2 + -a_1x_2y_2^3 - x_2y_1$
$J_{33} = -a_1^2x_1^3y_2^4 + -a_1^2y_2^6 + -2a_1x_1^4x_2 + -a_1x_1^3y_1y_2 + -2a_1x_1x_2y_2^2 + y_1^2$

This contains only a single coefficient a_1 which has to satisfy $a_1^2 = -1$.

The quantum dimensions of this defect are $q_L(Q) = -2$ and $q_R(Q) = -3$.

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