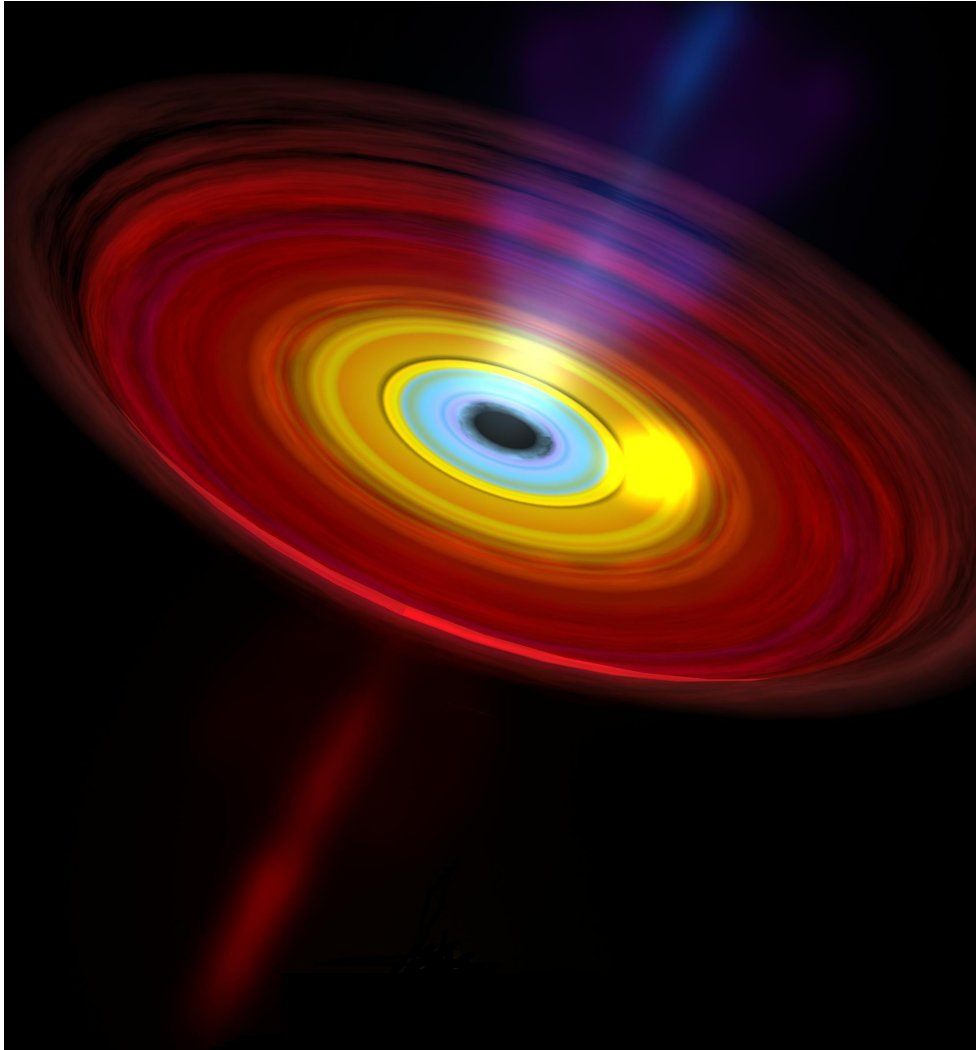


The Kerr-Metric: describing Rotating Black Holes and Geodesics



Bachelor Thesis
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Chapter 1

Introduction

In 1795 Laplace proposed, using Newton's theory of gravity, it was possible for very dense and massive objects to have an escape velocity larger than the speed of light. Not even light could escape from such an object: it would appear black. In 1915 Einstein published his famous theory of general relativity. This new theory predicted the possibility of such dark objects (called singularities: objects with infinite curvature due to an infinite density) from which light would not be able to escape: black holes.

Black holes are caused by singularities, points of infinite density in the spacetime. The spacetime around such a point is so strongly curved, that it exerts a very strong 'gravitational pull' such that everything nearby is drawn into the black hole. This 'pull' is so strong, not even light is able to escape from it: it will always 'fall' into the singularity. The area of spacetime for which light will always go to the singularity, is called the black hole. The singularities are real physical objects, like a book or a drain in a bath. But the black hole, the blackness around the singularity, is as tangible as a whirlpool of the water going into the drain: the black hole tells something about the curvature of the spacetime, as the whirlpool says something about the curvature of the water.

During the first halve of the twentieth century, black holes were mere 'thought experiments' of the theoretical physicists. In 1916 K. Schwarzschild gave the first solution for the Einstein equation of general relativity. His solution described the spacetime around a static massive object and was called the Schwarzschild-metric. Later, in 1963, R. Kerr discovered another solution: the Kerr-metric. The Kerr-metric gives the spacetime outside a massive rotating object. These two solutions describe the static and rotating black hole respectively.

In the second halve of the twentieth century, astronomers observed some strange phenomenon in the universe: very small objects that emitted jets of particles with very high energy. They proposed black holes to be the objects that were the sources of these jets. A black hole was no longer a theoretical

construct, but a real physical object.

Stars that have exhausted their thermonuclear fuel are no longer able to maintain their equilibrium with their inward gravitational force. The star will undergo a gravitational collapse. If the star is massive enough, its final state will be as a black hole. And because the star rotates before the collapse, the collapse will result in a spinning or rotating black hole. The spacetime around the final state of a very massive star is described by the Kerr-metric. [Begelman, 1995]

There are several motivations for studying black holes and the metrics that describe the spacetime around it. First of all, stellar mass black holes tell us something about the last stage in star evolution. They should have information about the final moments of the star his life.

Secondly there are (predictions of) very massive black holes in the center of the galaxies like our own galaxy. These black holes are important to the theory of cosmology as they are perhaps ‘seeds’ of galaxy formation. However, how they are formed is unclear, as well is the answer to the question whether they are properly described by the Kerr-metric. Supermassive black holes may have been created in the very early universe and tell us things of that era. They might even play a role in the debate of the dark matter, since black holes are very hard to observe and it is therefore hard to measure how many mass they contain in total and how large the portion of dark matter it has.

Further more, black holes are the extreme for gravitational theory since singularities are objects of infinite density and curvature. Therefore they form objects which probably have to be explained in terms of quantum gravity: large mass at small spacetime. Black holes may form testcases for this quantum gravity theory. [Begelman, 1995] [Rees, 2007]

This bachelor thesis is about the spacetime around a rotating black hole, as described by the Kerr-metric. But first it will treat three general types of black holes and several methods that are used for the detection of black holes in the universe. In chapter three, the case of a static black hole is explained: the Schwarzschild-metric. Here are the first important concepts of black holes discussed, for example the event horizon and time dilation. This third chapter is used for building a reference frame for the discussion of rotating black holes in chapter four and five. In the fourth and fifth chapter, the rotating black hole is discussed. Firstly the features of the black hole itself. And secondly the geodesics of test particles around the black hole. The last chapter provides a conclusion about rotating black holes and the geodesics around it.

Chapter 2

Black holes in general

2.1 How are black holes formed?

There are three general types of black holes: stellar black holes, primordial black holes and supermassive black holes. Each type of black hole has its own way of formation. These three types are discerned by their mass (and size). The more massive a particular source of a black hole is, the larger the curvature of spacetime, the larger the black hole.

2.1.1 Stellar black holes

A stellar black hole is a black hole that is formed at the end of the lifetime of a star. A star consists of nuclear fuel, mainly hydrogen. During its life, there is nuclear fusion which produces the energy a star radiates. This radiation causes an outward radiation pressure that is in equilibrium with the inward gravitational force of the star. After some time, the star is exhausted due to the fusion and the star is no longer hot enough for further nuclear fusion. The star can become a red giant to increase the temperature of the core. But the temperature will eventually not be high enough for further fusion.

The nuclear fusion process is halted and there is no radiation pressure any more. The star will undergo a gravitational collapse. This collapse may result in a supernova: the star is 'detonated' into a very large explosion in which large portions of its mass are blown away. After that, the core will recollapse.

Depending on the mass of the star at the moment of the final recollapse, the core can become one of the three following products: a white dwarf (mass less than 1,4 solar mass); a neutron star (mass between 1,4 and 3 solar mass); or a singularity, causing a black hole (mass greater than 3 solar mass). A black hole formed this way, is a stellar black hole. [Zeilik, 1998]

2.1.2 Primordial black holes

In the early universe, just after the Big Bang, the universe was very hot and dense (according to the Standard Model). Small quantum fluctuations in the density at that time are indicated by the galaxies at our present time: they are the products of inhomogeneities in the density during the early universe. Some regions might have been so dense, they would be sufficiently compressed by gravitation to overcome the velocities of the expansion and possible pressure forces from the inside. These very dense regions could further collapse to create a black hole: primordial black hole.

These primordial black holes may be less massive than stellar black holes: 10^{-5} g and upwards. As with all black holes, primordial black holes may have increased in mass since the early universe by the accretion of matter. But they may have lost mass by Hawking radiation as well. [Misner, 1973] [Carr, 1973]

2.1.3 Supermassive black holes

Supermassive black holes have masses 10^6 solar mass and upwards. They are found in the center of galaxies.

The formation of the supermassive black holes is less well understood than that of the stellar black hole. One idea on the formation is that they are formed by the collapse of the first generation of stars after the Big Bang, and have accreted matter over the time of millions of years. These supermassive black holes may have been the seeds for the next generation of galaxies. Interaction between galaxies, like mergers, could have resulted in the merging of both black holes at the centers. Another idea is that supermassive black holes are formed by the merging of clusters consisting of stellar black holes. [Rees, 2007] [web3]

2.2 How can black holes be observed?

There are several ways in which black holes can be observed in an indirect way: X-ray, spectral shift, gravitational lensing and flares. The only direct way to observe black holes is via gravitational waves. But this last method is not fully operational yet. Thus if a black hole is not near any matter, it will not be observable because all indirect methods use surrounding matter as an indicator for the presence of a black hole. Up till now, no black hole has been directly observed.

2.2.1 X-ray

Black holes can absorb material from the interstellar medium or a companion star. This process of absorbing material is called accretion. Because of the

possible angular momentum of the material, an accretion disk is formed around the black hole. The material in the accretion disk rotates around the black hole, the inner parts rotating faster than the outer parts. This causes friction. The friction has two effects: lowering the angular momentum of the material such that it can spiral into the black hole; and the material is heated up to higher temperature as it spirals into the black hole.

The material that falls in, into the gravitational field, is strongly compressed and heated up: it will radiate X-rays. The gravitational potential energy of the matter is converted into kinetic energy as it falls in, and because of the friction, kinetic energy is converted into heat and radiation. Because the infalling particles have very high velocities, they are relativistic and thus is the radiation they emit beamed. This beaming causes the jet-shape of the X-ray radiation.

The X-ray jets are caused by the gas that falls into the black hole, the black hole itself does not emit the radiation.

A supermassive black hole with an accretion disk which emits jets of X-ray are called quasars and they are found in active galactic nucleus (AGN's). A stellar black hole with X-ray jets is often located in a binary system in which the accretion disk is formed from the material of its companion star. These stellar black holes are called pulsars. [Begelman, 1995]

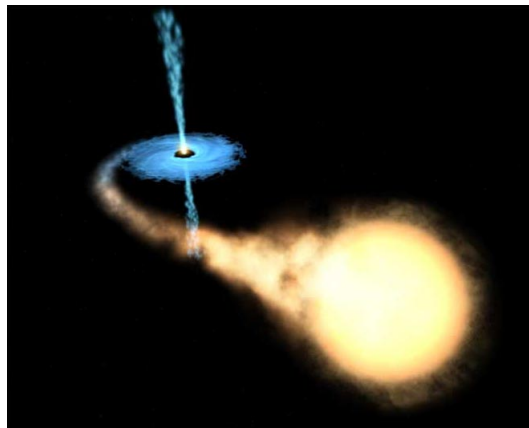


Figure 2.1: A stellar black hole with an accretion disk from its companion star. At the axis of rotation of the accretion disk there are two jets visible.[fig1]

2.2.2 Spectral shift

Black holes have a gravitational influence on their surrounding. In the case of a black hole in a binary star system or in the center of a galaxy, this gravitational influence can be measured.

In a binary system, with one visible star and an invisible object (like a black hole), it is possible to measure the mass of the invisible companion. Via the shift in the spectral lines of the star it is possible to determine the speed with which it orbits around its invisible partner. And then applying Newton's Law one can estimate the mass of the invisible partner. If the mass is larger than three solar mass, it is likely to be a black hole. If the mass is less, it could also be a neutron star. The black hole could also have X-ray jets as explained above.

At a galactic nucleus, the stars move in random directions, only responding to the total gravitational force from all the matter. All the stars have a certain average speed, depending on the total mass and the radius of the orbit of the star. This average speed varies differently with radius if there is a supermassive black hole at the center of the galaxy than when there is none: the stars move faster because of this large (invisible) gravitational pull. Furthermore, the shape of the orbits of the stars close to the black hole are more cigar-like shaped than for the case of no black hole. [Begelman, 1995]

2.2.3 Gravitational lensing

Gravitational lensing is the bending of a lightpath by a compact object. For example, if there is a galaxy far far away and between that galaxy and the Earth is a very compact and massive object, for example a cluster of galaxies or a black hole, then the path of the light from that galaxy is bent by the gravitational field of the compact object. The light is deflected and instead of the one galaxy, one sees several images of that galaxy positioned around the original galaxy. The number of images, and the possible distortions of the images, depend on the shape of the compact object. With gravitational lensing it is possible to determine the mass of the compact object. [web5]

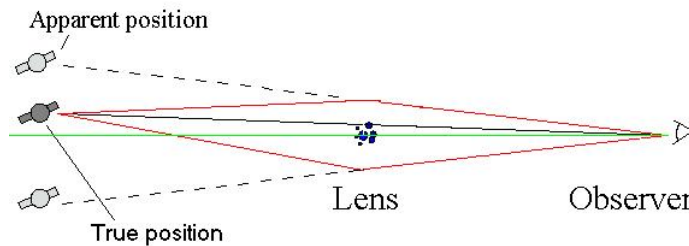


Figure 2.2: Overview of gravitational lensing: the red lines indicate the true light paths, the dotted lines are the light paths the observer would think the light has traveled. The lens can be a cluster of galaxies or (several) black holes. [web5]

2.2.4 Flares

Stars that are very close to supermassive black holes experience large tidal forces: the closer part of the star experiences a larger ‘gravitational pull’ than the outer part. If the tidal force is larger than the gravitational force that holds the star together, the tidal force tears the star apart. A part of the material from the star will fall into the black hole and give off radiation (using the same process as for the X-ray jets): the black hole will ‘flare up’.[Begelman, 1995]

2.2.5 Gravitational waves

If the gravitational field changes by the change in size or shape of a massive object, or by the acceleration of a massive object (provided the motion is not perfectly spherically like a spinning disk), the changes of the spacetime geometry are propagated by gravitational waves: ripples of spacetime. These waves can be regarded as radiation. The gravitational radiation is very weak compared to the electromagnetic radiation, and instead of dipole, it is quadrupole radiation. [web6]

An example of a changing gravitational field is a binary system of a neutron star and a black hole. Because they orbit around each other and they are both very massive, there are large changes in the gravitational field; they radiate strong gravitational waves. The system emits gravitational waves, therefore the neutron star and the black hole will lose energy, and eventually they will coalesce. Such a merging of a neutron star with a black hole will cause large changes in the spacetime as well, when the neutron star is absorbed by the black hole the gravitational field changes much.

Gravitational radiation, because it is very weak, even for the case of a binary system of a neutron star with a black hole, has not yet been directly observed. It has been indirectly proven in 1974 by R. Hulse and J. Taylor. There are large detectors being built with which one hopes to measure these gravitational waves (for example LISA). The detectors are large interferometers which should measure the ripples of the spacetime of the gravitational radiation. [Begelman, 1995]

2.2.6 Possible candidates

Three examples of black hole candidates are Cygnus X-1, A0620-00 and LMC X-3. These are all found in X-ray binaries. Lower mass limits of the first two are around 3,2 solar mass, and LMC X-3 has a mass of at least 7 solar masses. [web4]

Chapter 3

Static black holes

3.1 Introduction

Black holes can be fully described in terms of three parameters: mass, angular momentum and charge. Because of global conservation laws, these three properties are conserved during the collapse of the star. All other properties of the star that has collapsed to the black hole are lost during the collapse. This follows from the *A Black Hole Has no Hair* theorem. In addition to these parameters, there are four laws, derived from standard laws of physics, which describes the dynamics of a black hole in general. [Hawking, 1973]

The Schwarzschild black hole only has mass, it does not have an angular momentum or charge. It is a static black hole.

3.2 Schwarzschild metric

The Schwarzschild metric describes the spacetime curvature around static massive objects. Examples of such an object is a non-rotating star or a static black hole. In the derivation of the Schwarzschild metric, four assumptions were made: spacetime would be static, thus independent of coordinate time t ; spacetime is spherically symmetric; spacetime is empty, with the exception of the static massive object there are no other sources of curvature; and spacetime is asymptotically flat, the g_{tt} -component goes to c^2 , and the g_{rr} -component goes to 1 as r goes to infinity. [Foster, 2006]

The Schwarzschild metric is given by:

$$c^2 d\tau^2 = c^2 \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \quad (3.1)$$

with $m \equiv \frac{MG}{c^2}$. M is the mass of the massive object. For the remainder of this text, the following units for the speed of light $c = 1$ and the gravitational constant $G = 1$ will be used. As can be seen from the metric, the curvature is only radial dependent. In the figure below is an illustration of how a

three-dimensional spacetime would look like around a Schwarzschild black hole.

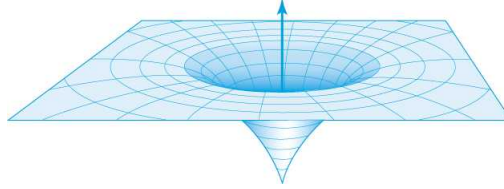


Figure 3.1: Three-dimensional spacetime around schwarzschild black hole: two spatial dimensions and one time dimension (indicated by the arrow pointing upwards) [fig2]

3.2.1 Curvature and time-dilation

As a consequence of the curvature, the physical distance between two (radial) coordinates is different from the distance between those two coordinates. The length of a radial line-element dR (choosing ϕ and θ constant) is given by:

$$dR \equiv (1 - 2m/r)^{-1/2} dr \quad (3.2)$$

The measured radial distance dR is larger than the radial coordinate distance dr . To measure a whole line, one needs to integrate the above expression. Coordinates are like street numbers: the distance between the 36th St. and 37th St. is not necessarily equal to the distance between 38th St. and 39th St. [Foster, 2006]

Furthermore, there is a time-dilation as well: for observers close to a schwarzschild black hole, time flows slower than for observers far away. For a static observer, which measures proptime $d\tau$, a time-interval is given by:

$$d\tau = (1 - 2m/r)^{1/2} dt \quad (3.3)$$

The time-dilation is an important cause of the gravitational redshift. Say you have a static emitter and a static receiver, both at different radial distances from the static black hole. The emitter emits signals to the observer, always using the same time-interval $d\tau_E$ between two signals. The receiver measures a time-interval $d\tau_R$ between two signals. The frequency of the signals measured by the emitter is $\nu_E \equiv n/\Delta\tau_E$, where n the number of signals is, emitted by the emitter, during a time-interval $\Delta\tau_E$. But the receiver measures a frequency of $\nu_R \equiv n/\Delta\tau_R$: he measures the same number of signals, but a different time-interval because he is further away from the black hole, and thus time flows faster for him. The redshift is given by:

$$\frac{\nu_R}{\nu_E} = \left[\frac{1 - 2m/r_E}{1 - 2m/r_R} \right]^{1/2} \quad (3.4)$$

If the receiver is closer to the black hole than the emitter, light is blueshifted. And if the emitter is closest to the black hole, the light (or any other signal) is redshifted. Notice that if the $r_E = 2m$ (at the event horizon), light is infinite redshifted.

3.3 Singularities

A singularity is a point in spacetime for which the curvature of the manifold goes to infinity. This is represented by a term in the metric going to infinity: it ‘blows up’. In other words, the curvature at that point is not well described by the metric. But a term in the metric can also ‘blow up’ due to (bad) choice of coordinates at that point, hence the difference between coordinate singularities and curvature singularities. Coordinate singularities can be removed by choosing a more fortunate coordinate system and curvature singularities can not be removed: they are real properties of the manifold. [Townsend, 1997]

The Schwarzschild metric has a coordinate singularity at $r = 2m$, then the g_{rr} term ‘blows up’. By changing to the Eddington-Finkelstein coordinates one can remove this singularity from the metric. The Eddington-Finkelstein coordinates are based on free falling photons (null geodesics), ingoing and outgoing. [Foster, 2006][Misner, 1976]

For the ingoing Eddington-Finkelstein coordinates, one needs to replace the coordinate time t by v :

$$v \equiv t + r + 2m \ln(r/2m - 1) \quad (3.5)$$

This leads to a Schwarzschild metric of:

$$d\tau^2 = \left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \quad (3.6)$$

There is no term that blows up at $r = 2m$. However, ingoing Eddington-Finkelstein coordinates are only able to describe ingoing photons into the static black hole. To give the geodesics of outgoing photons, one needs to use the outgoing Eddington-Finkelstein coordinates. To achieve this, one needs to replace the plus-sign of the $dvdr$ -term by a minus-sign. [Misner, 1973]

There is also a curvature singularity in the Schwarzschild metric, it is located at $r = 0$. The first term in the metric ‘blows up’ for that coordinate. The shape of the singularity is a point. This point has an infinite density.

It is possible for particles or photons to reach a curvature singularity. But once one has reached the singularity, it is impossible for it to move away from that point by extending their path in a continuous way. [Misner, 1973]

3.4 Event horizon and stationary surface limit

The first two terms in the Schwarzschild metric changes sign at $r = 2m$. The change of sign in the first term is the indication for the stationary surface limit, and the change in the second term indicates the event horizon. These surfaces are not a real physical surfaces in the sense that one is able to touch it, they are mathematical constructs, like the boundary between two countries.

3.4.1 Event horizon

An event horizon is a surface that can be considered as a one-way-membrane: it lets signals from the outside in, but it prevents signals from the inside to go to the outside. The event horizon is the boundary between where the curvature is strong enough and where it is not strong enough to prevent photons from within to escape to infinity. Photons emitted within the area of the event horizon are trapped forever within the black hole, photons emitted just outside the event horizon (emitted within the right direction) are able to reach infinity. The event horizon is a sphere-shaped surface around the black hole singularity (in three spatial dimensions).

Because signals from the inside of the event horizon can not pass the event horizon, observers at the outside are not able to see inside the event horizon. Thus they are unable to see the singularity of the rotating black hole. This is accordance with the theorem of ‘*Cosmic Censorship*’ [Hawking, 1974] which states that ‘naked singularities’ are forbidden.

The sphere-shaped surface of the rotating black hole is ‘generated’, given form, by photons that are forever trapped at the event horizon: they have no radial velocity. These photons (or horizon generators) were emitted precisely at the event horizon in radial outward direction and are not able to go beyond the event horizon (on both sides: inside and outside the event horizon) because of their tangential direction. These photons do not have end-points: they will be there for always. [Misner, 1973]

Say there is a photon that moves in a radial direction of a static black hole: $d\theta = d\phi = 0$. The metric is given by:

$$0 = \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 - \frac{2m}{r}\right)^{-1} dr^2 \quad (3.7)$$

Rewriting gives:

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2m}{r}\right) \quad (3.8)$$

In the limit of r going to infinity, the speed of the photon is the speed of light (the units are such that $c = 1$). However, at $r = 2m$, the speed of

the photon is zero: the event horizon. However, because of the coordinate singularity, other coordinates are needed for better analyses: ingoing or outgoing Eddington-Finkelstein coordinates.

For the case of the ingoing Eddington-Finkelstein coordinates, the null-geodesic of the radial photon is given by:

$$0 = \left(1 - \frac{2m}{r}\right) dv^2 - 2dvdr \quad (3.9)$$

Rearranging gives:

$$\frac{dv}{dr} \left[\left(1 - \frac{2m}{r}\right) \left(\frac{dv}{dr}\right) - 2 \right] = 0 \quad (3.10)$$

Solving this equation to dv/dr gives two solutions:

$$\frac{dv}{dr} = 0 \quad (3.11)$$

$$\frac{dv}{dr} = \frac{2}{(1 - 2m/r)} \quad (3.12)$$

Differentiating the expression for v (3.5) gives $dv/dr = dt/dr + \frac{1}{1-2m/r}$. Using this in combination with the first solution for dv/dr gives:

$$\frac{dt}{dr} = \frac{-1}{1 - 2m/r} \quad (3.13)$$

This gives the ingoing null geodesic, as it is negative in the region of $r > 2m$. Integrating the corresponding dv/dr gives $v = A$, where A is a constant.

The other solution gives the following function for $\frac{dt}{dr}$ and is integrated as follow:

$$\frac{dt}{dr} = \frac{1}{1 - 2m/r} \quad (3.14)$$

$$v = 2r + 4m \ln |r - 2m| + B \quad (3.15)$$

$$(3.16)$$

(B is a constant) This gives the outgoing null geodesic: it is positive for $r > 2m$. Notice the behaviour at $r = 2m$: the outgoing photon is not able to cross the $r = 2m$ boundary, where as, the ingoing photon can cross that boundary: ingoing photons can cross the event horizon, but outgoing photons can not cross it as they are trapped inside.

These geodesics can be pictured in a (two-dimensional) spacetime-diagram, see figure below (3.2). The axes are oblique to let it appear as in flat spacetime. [Foster, 2006]

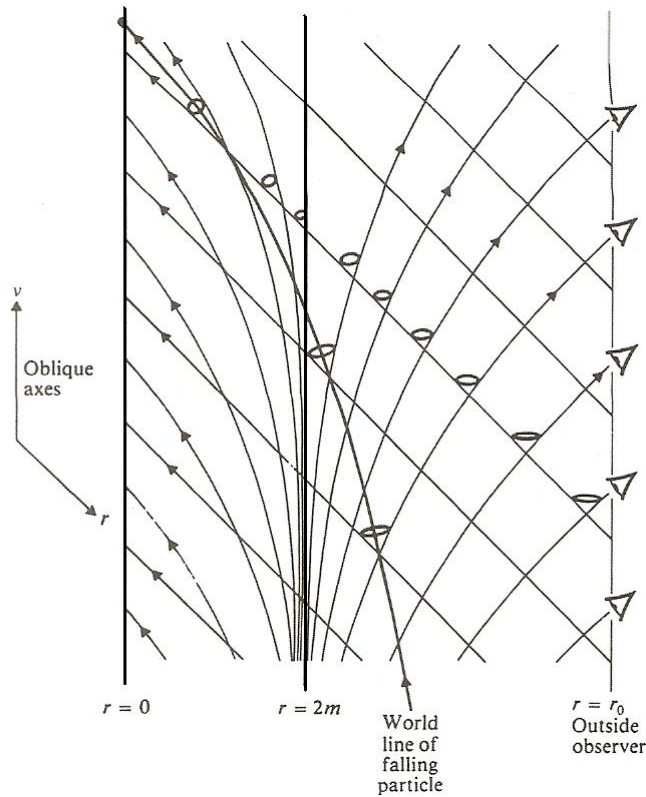


Figure 3.2: Eddington-Finkelstein spacetime-diagram. The small arrows indicate the direction of the outgoing photons. At $r = r_0$ an observer is located: he only observes photons emitted outside the event horizon. The ingoing photons are the straight lines. [Foster, 2006]

3.4.2 Stationary surface limit

A static observer is an observer that only moves in time: its spatial coordinates are constant (with respect to an inertial frame): $dt \neq 0$, $dr = d\theta = d\phi = 0$. This gives a line-element for a timelike observer of:

$$ds^2 = \left(1 - \frac{2m}{r}\right) dt^2 > 0 \quad (3.17)$$

For the case of a particle with mass, ds^2 needs to be time-like: larger than zero. However, at $r = 2m$, g_{tt} is equal to zero. Thus for the case of an observer that follows time-like geodesics, he can not be static at $r = 2m$, because dr , $d\theta$ or $d\phi$ can not be zero at that same point: the observer needs to move in spatial coordinates as well. And for $r < 2m$, the g_{tt} -term is negative: the time-like observer has to move in space within the event

horizon.

When an observer is not able to be static for $r < 2m$, $r = 2m$ is called a stationary limit surface: beyond that surface, one is not able to be stationary (or static).

3.5 Approaching a static black hole

Say there is an astronaut who is sent from a spaceship far away, into a schwarzschild black hole. The astronaut carries a flash light which he uses to create a light flash every second (using his own watch).

From the viewpoint of the spaceship, as the astronaut nears the black hole, the time-interval between two flashes increases and the astronaut slows down as he approaches the event horizon. This is caused by the time-dilation. And as the astronaut approaches the horizon, he appears to look more red: his light is red-shifted. Just before he looks to reach the event horizon, his speed appears to become zero and his light becomes infinite red-shifted: he will fade from view. The spaceship will never observe the astronaut crossing the event horizon.

From the viewpoint of the astronaut, he accelerates as he falls to the black hole. If he would look into space, every thing would seem to be normal. And he would flash his flashlight every second, as agreed with the persons on board the spaceship. As he crosses the event horizon, nothing special happens to him at that moment. What he would see inside the event horizon is not known to us.

The above story is not complete: it ignored the gravitational effects of the black hole on the astronaut. As the astronaut comes closer to the black hole, the gravitational force on his legs becomes substantially larger than the gravitational force on the upper part of his body (this is because the distance dependence of the gravitational force), providing his legs are closest to the black hole: the astronaut becomes stretched, the lower part more than the upper part of the body. As he nears, his legs become even more stretched as the difference in force increases. In the end his body will be stretched that far, that it is fatal to him. This point is reached outside the event horizon.[Hawking, 1988]

3.6 Geodesics

To discuss the geodesics of free test particles in the vicinity of a static black hole, one needs to derive expressions for their paths. The derivations for the geodesics form the first part of this section. After that, some types of motion are discussed for timelike and null-geodesics.

3.6.1 Derivations

The Lagrangian for a geodesic is given by

$$L = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (3.18)$$

where the dot is an indication for the derivative to some affine parameter λ . For the case of timelike-geodesics, λ can be set equal to the proptime τ .

The langrangian for the Schwarzschild-metric is given by:

$$L = \frac{1}{2} \left[\left(1 - \frac{2m}{r}\right) \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 - r^2 (\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2) \right] \quad (3.19)$$

The corresponding canonical momenta ($p_\mu = \frac{\partial L}{\partial \dot{x}^\mu}$) are given by:

$$p_t = \left(1 - \frac{2m}{r}\right) \dot{t} \quad (3.20)$$

$$p_r = - \left(1 - \frac{2m}{r}\right)^{-1} \dot{r} \quad (3.21)$$

$$p_\theta = -r^2 \dot{\theta} \quad (3.22)$$

$$p_\phi = - (r^2 \sin^2 \theta) \dot{\phi} \quad (3.23)$$

From the Euler-Lagrange equation

$$\frac{\partial L}{\partial x^\mu} - \frac{\partial}{\partial \lambda} \frac{\partial L}{\partial \dot{x}^\mu} = 0 \quad (3.24)$$

it follows that p_t and p_ϕ are constants of motion, because $\frac{\partial L}{\partial t} = \frac{\partial L}{\partial \phi} = 0$:

$$p_t = E = \left(1 - \frac{2m}{r}\right) \dot{t} \quad (3.25)$$

$$p_\phi = -L = -r^2 \dot{\phi} \quad (3.26)$$

with E as the energy of the particle at infinity, and L it's angular momentum.

The canonical momenta give rise to the following Hamiltonian:

$$H = p_t \dot{t} + p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L = L = cst \quad (3.27)$$

The equality between the Hamiltonian and Lagrange means there is no physical potential involved. And the constant of the equality can be choosen such that it is equal to 1/2 for timelike geodesics, and zero for null geodesics.

The Hamiltonian gives the following equation (in the equatorial plane $\theta = \pi/2$):

$$2L = \frac{E^2}{1 - 2m/r} - \frac{\dot{r}^2}{1 - 2m/r} - \frac{L^2}{r^2} = \delta \quad (3.28)$$

where δ is equal to -1 or 0 for respectively timelike or null geodesics. Every geodesic can be chosen such that it lies in one plane, and because of the spherical symmetry of the Schwarzschild-metric, every plane can be chosen as equatorial plane. Thus there is no loss of generality by choosing to be in the equatorial plane.

Furthermore, the Euler-Lagrange give an equation for the case of $\mu = 1$ (the radial coordinate):

$$\left(1 - \frac{2m}{r}\right)^{-1} \ddot{r} + \frac{m}{r^2} \dot{t}^2 - \left(1 - \frac{2m}{r}\right)^{-2} \frac{m}{r^2} \dot{r}^2 - r \dot{\phi}^2 = 0 \quad (3.29)$$

The equations (3.25), (3.26), (3.28) and (3.29) will be used throughout the next sections. [Chandrasekhar, 1983], [Foster, 2006]

3.6.2 Timelike geodesics

For the case of a timelike geodesic, equation (3.28) can be rewritten to

$$\left(\frac{dr}{d\tau}\right)^2 + \left(1 - \frac{2m}{r}\right) \left(\delta + \frac{L^2}{r^2}\right) = E^2 \quad (3.30)$$

Using the above equation with the expression (3.26) for the angular momentum with $u \equiv 1/r$ one gets

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{2m}{L^2}u - \frac{1 - E^2}{L^2} \quad (3.31)$$

The above equation determines the shape of the geodesic in one plane. And by using the expression (3.25) and (3.26) the solution can be completed. [Chandrasekhar, 1986]

Furthermore, it is possible to rewrite (3.30), using (3.25), to obtain an integral for the coordinate time [Misner, 1976]:

$$t = \int^r \frac{E dr}{(1 - 2m/r) [E^2 - (1 - 2m/r)(1 + L^2/r^2)]^{1/2}} \quad (3.32)$$

Vertical free-fall

A special case of a timelike geodesic is the vertical free-fall. In a free-fall, ϕ is constant ($\dot{\phi} = L = 0$). Using equation (3.30) one gets

$$\left(\frac{dr}{d\tau}\right)^2 + (1 - 2m/r) - E^2 = 0 \quad (3.33)$$

Further differentiation and then divide by \dot{r} gives:

$$2\dot{r}\ddot{r} + \frac{2m}{r^2}\dot{r} = 0 \quad (3.34)$$

$$\ddot{r} + \frac{m}{r^2} = 0 \quad (3.35)$$

which can be rewritten by using $m = GM/c^2$ to

$$\ddot{r} + \frac{GM}{r^2} = 0 \quad (3.36)$$

Which is the Newtonian equation for gravity. [Foster, 2006]

Effective potential

The second term in equation (3.30) can be interpreted as an effective potential

$$\left(\frac{dr}{d\tau}\right)^2 + V^2 = E^2 \quad (3.37)$$

$$V^2 = \left(1 - \frac{2m}{r}\right) \left(1 + \frac{L^2}{r^2}\right) \quad (3.38)$$

Drawing a graph of the potential as a function of r gives:

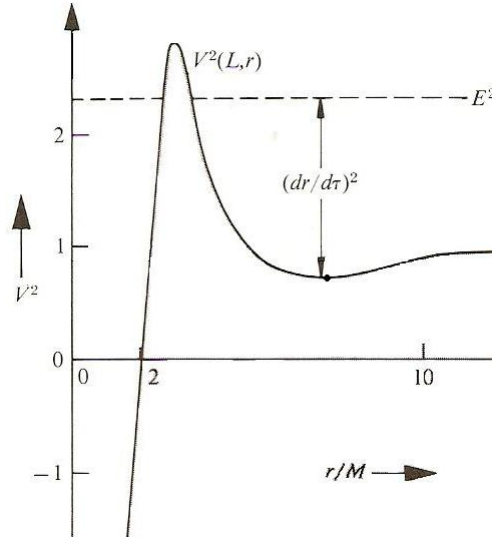


Figure 3.3: The effective potential. The dot indicates a minimum. [Misner, 1973]

The potential has an attractive part corresponding to the ‘gravitational force’ on the particle, but also a repulsive part caused by the conservation of angular momentum (similar to the centrifugal force). Different values of the angular momentum of the particle gives different effective potentials (see figure 3.4).

A particle with a effective potential as in figure 3.3 has for different values of its energy E different orbits. If his energy is equal to the effective

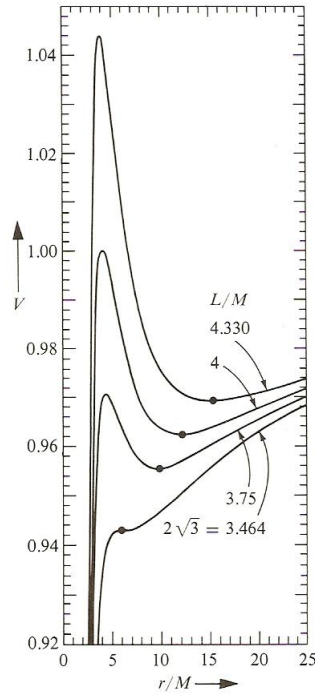


Figure 3.4: The effective potential for different values of the angular momentum L of the particle. [Misner, 1973]

potential energy at the minimum, then the particle follows a stable circular orbit. If the particle has a bit more energy, it will follow elliptical orbits, where the radii at which the energy equals the potential energy are the turning points: the radial velocity changes direction. For the case when the energy equals the potential energy at maximum, the particle moves along a so-called ‘knife-edge’ orbit: it is an unstable orbit and a small perturbation results in going to infinity or fall into the black hole. The last case, is when the energy E is larger than the maximum potential energy: the particle will directly go into the black hole. [Misner, 1973] [Taylor, 2000]

3.6.3 Null geodesics

For photons, $\delta = 0$, and equation (3.30) becomes:

$$\left(\frac{dr}{d\lambda}\right)^2 + \left(1 - \frac{2m}{r}\right) \frac{L^2}{r^2} = E^2 \quad (3.39)$$

Again, one can rewrite this to an equation which gives the shape in a plane, using $u = 1/r$ and equation (3.26):

$$\left(\frac{du}{d\phi}\right)^2 = 2mu^3 - u^2 + \frac{E^2}{L^2} \quad (3.40)$$

There are two special cases for photon geodesics that will be treated below: radial and circular geodesics.

Radial geodesic

A radial geodesic ($L = 0$) for a photon is described by

$$\frac{dr}{d\lambda} = \pm E \quad (3.41)$$

using equation (3.25) this becomes

$$\frac{dr}{d\lambda} = \pm \left(1 - \frac{2m}{r}\right) \quad (3.42)$$

Integrating it gives:

$$t = \pm \left[r + 2m \log \left(\frac{r}{2m} - 1 \right) \right] + C \quad (3.43)$$

where C is a constant. Integrating the expression (3.41) with respect to $d\lambda$ gives

$$r = \pm E\lambda + C \quad (3.44)$$

Meaning, a photon is able to cross the event horizon in its own affine parameter (3.44), but it would take an infinite amount of coordinate time t to reach the event horizon (3.43): an observer outside the event horizon will never observe the photon crossing the horizon. [Foster, 2006]

Circular orbit

For a circular orbit, $\dot{r} = \ddot{r} = 0$, one gets the following from equation (3.29):

$$\left(\frac{\dot{\phi}}{\dot{t}}\right)^2 = \frac{m}{r^3} \quad (3.45)$$

In combination with equations (3.39), (3.26) and (3.25) (giving $\left(\frac{\dot{\phi}}{\dot{t}}\right)^2 = \frac{1-2m/r}{r^2}$) this equality becomes:

$$\begin{aligned} \left(\frac{1-2m/r}{r^2}\right) &= \frac{m}{r^3} \\ r - 2m &= m \\ r &= 3m \end{aligned} \quad (3.46)$$

Giving a circular orbit for a photon at $r = 3m$. [Foster, 2006]

Chapter 4

Rotating black holes

4.1 Introduction

A rotating black hole has rotation in addition to the static black hole. The description of a rotating black hole uses two of the three parameters: mass and rotation. As such, a rotating black hole is not described by the Schwarzschild-metric but by another metric: the Kerr-metric. The Kerr-metric describes the spacetime around a rotating singularity.

Stellar black holes are caused by the collapse of stars. A star is a very massive, rotating but chargeless object. Because charges of opposite sign cancel, stars are neutral. Hence, the spacetime around a stellar black hole is described by the Kerr-metric.

Although the Kerr-metric gives the spacetime around rotating massive objects and the Schwarzschild-metric that of static massive objects, there are similarities between them. Both metrics are able to describe black holes that are caused by curvature singularities; they share a coordinate singularity at their event horizon; in both metrics two observers experience a time-dilation and curvature. Furthermore, the spacetime is both empty (with the exception of the one massive object) and asymptotically flat.

4.2 Kerr metric

The solution to the Einstein equation for a spinning, rotating massive object without charge, is given by the Kerr-metric. The object rotates around its z-axis (or its θ -axis). A rotating or spinning black hole without charge, is such an object.

The Kerr-metric can be described in different coordinate systems. The most common coordinate system used is the Boyer-Lindquist coordinate system. The advantage of this coordinate system is that it is written in spherical coordinates, which are easy to work with and some features are easily noticed. But the drawback of this coordinate system is that it has a

coordinate singularity at the event horizon (see next section). To ‘remove’ this coordinate singularity, one changes from coordinate system. An often used alternative is the Kerr coordinate system. Both coordinate systems are given below because both will be used in the discussion of the spacetime and the geodesics of particles around black holes.

4.2.1 Boyes-Lindquist coordinates

The Kerr metric in Boyes-Lindquist coordinates is given by [Foster, 2006]:

$$c^2 d\tau^2 = \left(1 - \frac{2mr}{\rho^2}\right) c^2 dt^2 + \frac{4mcr a \sin^2 \theta}{\rho^2} dt d\phi - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - \left[(r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^2 \theta}{\rho^2} \right] d\phi^2 \quad (4.1)$$

In which:

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta \quad (4.2)$$

$$\Delta \equiv r^2 + a^2 - 2mr \quad (4.3)$$

Furthermore: $m \equiv GM/c^2$ in which M is the mass of the rotating black hole and G the gravitational constant. The parameter a is the rotation parameter and is connected to the angular momentum J of the rotating black hole as follows:

$$a = J/Mc \quad (4.4)$$

The parameter a gives the direction and speed of the rotation. A positive value indicates a clockwise rotation of the object, a negative value indicate a counterclockwise rotation. The larger angular momentum, the larger the value for a , the faster the rotation. For the remainder of this, and the next chapters, special units are chosen such that $c = 1$ and $G = 1$. [Adler, 1967] [Foster, 2006]

In the special case of $J = 0$, a becomes zero and the Kerr-metric reduces to the Schwarzschild metric: the solution to the Einstein equations of a nonrotating mass. If $a^2 = m^2$ then the Kerr-metric describes the special case of the ‘Extreme Kerr geometry’. A rotation parameter $a^2 > m^2$ gives a ‘naked singularity’: a singularity that has no event horizon to hide it from sight. In this case the singularity would be observable. But that is in violation with physics because it would then be possible to observe a point of infinite density (‘*Cosmic censorship*’ theorem). As for now, naked singularities have never been observed. [Hawking, 1974]

The different terms in the metric are ‘responsible’ for different features of a rotating black hole. The first term is the g_{tt} -term and creates the

time dilation. If an observer would be completely (spatially) static, then $dr = d\theta = d\phi = 0$, then

$$d\tau^2 = \left(1 - \frac{2mr}{\rho^2}\right) dt^2 \quad (4.5)$$

giving $d\tau = dt\sqrt{1 - 2mr/\rho}$. Thus the closer the observer is to the black hole, the slower time would flow compared to an observer at infinity (he would measure $d\tau = dt$). Thus close to the rotating black hole, time flows less fast. Furthermore, the surface for which this metric term is zero, is the stationary limit surface.

The second term in the metric, the $g_{t\phi}$ -term, is the only off-diagonal component of the metric. This term creates the frame-dragging. Frame-dragging is the effect that spacetime appears to twist around the black hole. The geodesics of objects in the neighborhood of the rotating black hole are twisted around the black hole because of the frame-dragging. Beyond the stationary limit surface, the g_{tt} -term is negative, and to have timelike or null paths, this second-term needs to be positive: particles and photon need to rotate around the black hole. And for the case of $a = 0$ (no rotation), this term vanishes.

The third term, the g_{rr} -term, is the indicator of the event horizon. When this component is zero at the event horizon, particles and photons do not change their radial coordinate: they are not able to move further away from the black hole. They can not escape from the black hole.

4.2.2 Kerr coordinates

The Boyes-Lindquist being the spherical coordinate system of a rotating black hole, the Kerr coordinates being the coordinates that follow the path of a ‘radial’ infalling photon [Misner, 1976]:

$$d\tilde{V} = dt + \frac{r^2 + a^2}{\Delta} dr, \quad d\tilde{\phi} = d\phi + \frac{a}{\Delta} dr$$

By using these coordinates for the Kerr-metric one obtains the following metric [Novikov, 1989]:

$$\begin{aligned} ds^2 = & \left(1 - \frac{2mr}{\rho^2}\right) d\tilde{V}^2 - 2drd\tilde{V} - \rho^2 d\theta^2 \\ & - \rho^{-2} \left[(r^2 + a^2)^2 - \Delta a^2 \sin^2 \theta \right] \sin^2 \theta d\tilde{\phi}^2 + 2a \sin^2 \theta d\tilde{\phi} dr \\ & + \frac{4amr \sin^2 \theta}{\rho^2} d\tilde{\phi} d\tilde{V} \end{aligned} \quad (4.6)$$

with

$$\rho^2 \equiv r^2 + a^2 \cos^2 \theta \quad (4.7)$$

$$\Delta \equiv r^2 + a^2 - 2mr \quad (4.8)$$

(ρ^2 and Δ are the same as for the Boyes-Lindquist coordinates.) This form is similar to the ingoing Eddington-Finkelstein coordinates of the Schwarzschild-metric, both describe the ingoing, freely falling photon. The advantage of the Kerr-coordinates over the Boyes-Lindquist coordinates, is that it does not have a coordinate singularity at the event horizon. [Misner, 1973]

4.3 Singularities

The Kerr-metric in Boyes-Lindquist coordinates has a coordinate singularity at $r = m + \sqrt{m^2 - a^2}$ (the event horizon). For this value of r , $\Delta = 0$ and the coefficient of dr^2 goes to infinity. For a particle approaching the event horizon, the g_{rr} -term goes to infinity and the coordinate time needs to go to infinity for the particle to reach the event horizon. This creates an infinite twisting of the path of the particle around the black hole, because $d\theta > 0$. [Misner, 1973]

This coordinate singularity at the event horizon can be removed or avoided by choosing an other coordinate system, for example the Kerr coordinates.

A rotating black hole has a curvature singularity as well. This singularity has the shape of a ring. One can transform the Kerr-metric to the Kerr-Schild coordinates, and from these coordinates it follows that for $r = 0$, there is a ring given by $x^2 + y^2 = a^2$ (in which the x and y are Euclidian coordinates and a is the rotation parameter) which is the singularity. If there is no rotation, $a = 0$, the ring is just a point, as is the case for the Schwarzschild-metric. [Hawking, 1974]

4.4 Symmetries

As can be clearly seen from the Kerr-metric in Boyes-Lindquist coordinates: the metric terms are independent of coordinate time t and axial coordinate ϕ . Thus the solution is stationary and axial-symmetric: an observer on a worldline of constant θ and r , and with a uniform angular velocity sees a spacetime geometry that does not change while traveling along this worldline.

Because of these symmetries there are three coordinate transformations for which the Kerr-metric is invariant:

- $t' = t + cst$; r, θ, ϕ unchanged;
- $\phi' = \phi + cst$; t, r, θ , unchanged;
- $t' = -t$, $\phi' = -\phi$; r and θ unchanged.;

These symmetry properties are similar to the symmetry of an ordinary rotating spinner: if you take two photo's at two different moments, both pictures

will look the same. The same is true for the angle-symmetry. These three symmetry properties are general properties of homogenous spinning objects. Bear in mind that the Kerr-metric describes a black hole in an empty spacetime. As soon as one include a (non-test) particle in the spacetime (or metric), it loses its symmetry because one would be able to see the particle travel along its worldline and be able to discern the two pictures with the use of the position of the particle.

A tool for describing symmetries are Killing vectors. If you have a metric $g_{\mu\nu}$ on some coordinate system dx^a and that metric is independent of a coordinate x^k (for example $x^k = \phi$) such that

$$\frac{\partial g_{\mu\nu}}{\partial x^k} = 0 \quad (4.9)$$

for $k = a$, then the vector

$$\xi \equiv \left(\frac{\partial}{\partial x^k} \right) \quad (4.10)$$

is called the ‘Killing vector’. Killing vectors are coordinate independent as they are a properties of the spacetime itself. Killing vector ξ is an infinitesimal displacement which is length-conserved: a curve can be displaced in the direction of x^k by a shift of Δx^k , then the new curve has the same length as the original curve. If a geometry has a Killing vector, then the scalar product of the tangent vector of any geodesic with the Killing vector is constant:

$$p_k = \vec{p} \cdot \vec{\xi} = \text{constant} \quad (4.11)$$

Since the Kerr-metric has two symmetries, it has two Killing vectors associated with coordinate time t and axial coordinate ϕ :

$$\xi_t = \left(\frac{\partial}{\partial t} \right)_{r,\theta,\phi} \quad \text{and} \quad \xi_\phi = \left(\frac{\partial}{\partial \phi} \right)_{t,r,\theta} \quad (4.12)$$

The three scalar products of the Killing vectors gives three terms of the metric:

$$\vec{\xi}_t \cdot \vec{\xi}_t = g_{tt} \quad \vec{\xi}_t \cdot \vec{\xi}_\phi = g_{t\phi} \quad \vec{\xi}_\phi \cdot \vec{\xi}_\phi = g_{\phi\phi} \quad (4.13)$$

These equalities follow from the definition of the metric-terms. [Foster, 2006] [Misner, 1973]

4.5 Frame-dragging

A rotating black hole causes frame-dragging of the spacetime geometry. Frame-dragging is the twisting of spacetime around the black hole. It is like water in a bath that goes down the drain: before going through the drain it circles around the drain (because of the ‘frame-dragging’ of the water). Below is a figure in which the frame-dragging around a rotating black

hole is illustrated. It is as if you look on the xy -plane of a black hole that rotates clockwise around its z -axis. The radial lines spiral clockwise around the center of the black hole.

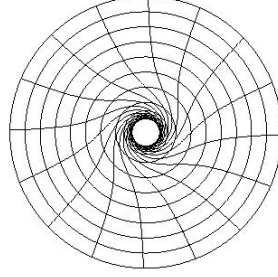


Figure 4.1: Frame-dragging of a rotating black hole, as viewed from above. [Web2]

Suppose somewhere is a rotating black hole. Then one could build a large steel frame around the rotating black hole, such that the frame is fixed in infinity at the distant stars. Thus in infinity, this steel frame is considered a lorentz frame. Near the black hole, gyroscopes are fixed onto the frame. From real experiments it follows that these gyroscopes rotate around the same axis as the rotation axis of the spinning black hole: the $g_{0\phi}$ -term causes a rotation of the gyroscopes with respect to the basis-vector $\left(\frac{\partial}{\partial\phi}\right)$, and since these basis-vectors are fixed to the frame, and thus fixed to the stars at ‘infinity’, these gyroscopes rotate with respect to these distant stars. Gyroscopes have the ability not to change direction due to external forces: thus they rotate due to the frame-dragging of the spacetime geometry. Thus the rotation of a black hole causes the twisting or frame-dragging of spacetime around the black hole itself. [Misner, 1973] [Sexl, 1979] [Foster, 2006]

The Lagrangian for a free test-particle following a geodesic is given by $L = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$, with the time-derivatives to proper-time τ . This gives:

$$L = \frac{1}{2}\left(1 - \frac{2mr}{\rho^2}\right)\dot{t}^2 + \frac{2mra\sin^2\theta}{\rho^2}\dot{t}\dot{\phi} - \frac{\rho^2}{2\Delta}\dot{r}^2 - \frac{\rho^2}{2}\dot{\theta}^2 - \frac{1}{2}\left[(r^2 + a^2)\sin^2\theta + \frac{2mra^2\sin^2\theta}{\rho^2}\right]\dot{\phi}^2 \quad (4.14)$$

Because L is not explicitly dependent on ϕ (ϕ is cyclic), the Euler-Lagrange equation with respect to the ϕ coordinate is:

$$\frac{\partial L}{\partial\dot{\phi}} = \text{constant} \quad (4.15)$$

$$\frac{2mra \sin^2 \theta}{\rho^2} \dot{t} - \left[(r^2 + a^2) \sin^2 \theta + \frac{2mra^2 \sin^2 \theta}{\rho^2} \right] \dot{\phi} = \text{constant} \quad (4.16)$$

Since the Kerr-metric describes a totally isolated rotating black hole, it is possible to choose the constant such that for r going to infinity, where $d\phi/d\tau = 0$ (no frame-dragging at infinity), the constant is equal to zero. Because of the properties of the Euler-Lagrange equations is the above equation valid for every point in spacetime and thus is the constant zero at the black hole as well. Bringing $\dot{\phi}$ to the other side and dividing by \dot{t} gives:

$$\begin{aligned} \frac{\dot{\phi}}{\dot{t}} &= \frac{d\phi}{dt} \\ &= \frac{2mra}{(r^2 + a^2)\rho^2 + 2mra^2 \sin^2 \theta} \end{aligned} \quad (4.17)$$

This is the observed angular velocity from far away, where $t \cong \tau$. Thus the observer which observes the free particle at the rotating black hole, would see it rotate around the black hole due to frame-dragging. Even if the particle has only radial velocity when is it far away from the black hole, when it comes closer tot the rotating black hole, it begins to rotate around the black hole. This is illustrated in the figure 4.5 below [Foster, 2006]

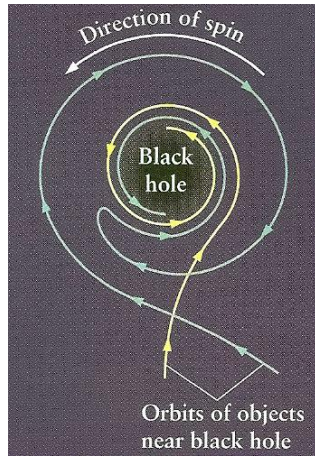


Figure 4.2: Orbits of two particles near a rotating black hole. The rotation of the black hole is counterclockwise. Because of the frame-dragging, both particles will eventually rotate with the direction of the rotation of the black hole, even though one particle first rotates in opposite direction.[Begelman, 1995]

The effect of frame-dragging increases with increasing a (a larger angular momentum) (in the domain of $0 < a^2 < m^2$) and decreases with increasing r .

4.5.1 Stationary observer

A stationary observer is an observer that moves along the Killing vectors: it moves, but it sees an unchanged spacetime in its neighborhood. There are no changes in the other direction components dr and $d\theta$. Following Misner (1973) it is possible to determine the possible values of the angular velocity of stationary observers. The angular velocity Ω relative to the rest frame of the distant stars is defined as:

$$\Omega \equiv \frac{d\phi}{dt} = \frac{d\phi/d\tau}{dt/d\tau} = \frac{u^\phi}{u^t} \quad (4.18)$$

where u^t and u^ϕ are two components of the 4-velocity. The 4-velocity of a stationary observer ($dr = d\theta = 0$) is, in terms of the Killing vectors, equal to:

$$\begin{aligned} \vec{u} &= u^t \frac{\partial}{\partial t} + u^\phi \frac{\partial}{\partial \phi} = u^t \left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \phi} \right) = \\ &= \frac{\xi_t + \Omega \xi_\phi}{|\xi_t + \Omega \xi_\phi|} = \frac{\xi_t + \Omega \xi_\phi}{(g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi})^{\frac{1}{2}}} \end{aligned} \quad (4.19)$$

An observer can not have every value of Ω : the 4-velocity must be within the future light cone since he follows timelike paths:

$$g_{tt} + 2\Omega g_{t\phi} + \Omega^2 g_{\phi\phi} < 0 \quad (4.20)$$

Solving this quadratic equation for Ω gives:

$$\Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}} \quad (4.21)$$

As can be seen from the above equation for Ω , there is an upperbound and a lowerbound on Ω . Defining a variable ω

$$\omega \equiv -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2mcr a}{(r^2 + a^2)\rho^2 + 2mra^2 \sin^2 \theta} = \frac{1}{2}(\Omega_{min} + \Omega_{max}) \quad (4.22)$$

with

$$\Omega_{min} = \omega - \sqrt{\omega^2 - g_{tt}/g_{\phi\phi}} \quad (4.23)$$

$$\Omega_{max} = \omega + \sqrt{\omega^2 - g_{tt}/g_{\phi\phi}} \quad (4.24)$$

$$\Omega_{min} < \Omega < \Omega_{max} \quad (4.25)$$

Notice that ω is the frame-dragging of the rotating black hole (see the expression for the angular velocity of the frame-dragging (4.17)).

Ω_{min} and Ω_{max} take the values of $-c/r$ and c/r respectively for far away from the black hole. Far away, ω (and ω^2) go to zero because of the r^4 dependence in the denominator. The fraction $g_{tt}/g_{\phi\phi}$ goes to c^2/r^2 since the denominator has a r^4 -term, but the nominator has a r^2 term in it. These limits are in agreement with special relativity: $r\Omega_{min} = -c$ and $r\Omega_{max} = c$.

For a decreasing radius, the minimum angular velocity Ω_{min} increases: the $g_{tt}/g_{\phi\phi}$ -term increases faster than the ω^2 . When g_{tt} reaches zero at the outer stationary limit surface, Ω_{min} becomes 0. As the radial distance increases even further, the range between Ω_{min} and Ω_{max} decreases and ω increases. At the event horizon $r = m + \sqrt{m^2 - a^2}$ the minimum and maximum angular velocities take the same value ($g_{tt}/g_{\phi\phi} = \omega^2$). Therefore there is no possibility there for an observer to be stationary since his angular velocity ought to be larger than Ω_{min} and smaller than Ω_{max} for the 4-velocity to lie within the future light-cone (see equation 4.25): thus timelike worldlines point inward to the event horizon.

4.6 Stationary limit surface

The stationary limit surface (also known as the ‘static limit’) is the outermost surface of a rotating black hole. It is the boundary between the two areas where observers can be static (outside the stationary limit surface) and where observers can no longer be static due to the strong frame-dragging (inside the stationary limit surface). In the region between the stationary surface limit and the event horizon, observers can be stationary.

The stationary limit surface is located at $r = m + \sqrt{m^2 - a^2 \cos^2 \theta}$. Inside the stationary limit surface, every observer, particle or photon rotates with the same direction as the rotation of the black hole.

The region between the stationary limit surface and the event horizon is called the ergosphere (the stationary limit surface itself is called the ergo-surface). This name originates from the possibility of the Penrose process inside the ergosphere. Particles in the ergosphere are still able to escape from the rotating black hole to infinity.

Because of the gravitational curvature of the spacetime geometry, the black hole causes gravitational redshift. Photons emitted by an emitter close to the static limit, send to an observer farther away, are redshifted.

4.6.1 Static Observers

A static observer is an observer that only moves in time: its spatial coordinates are constant (with respect to an inertial frame): $dt \neq 0$, $dr = d\theta = d\phi = 0$. This gives a line-element for a timelike observer of:

$$ds^2 = g_{tt}dt^2 = \left(1 - \frac{2mr}{\rho^2}\right) dt^2 > 0 \quad (4.26)$$

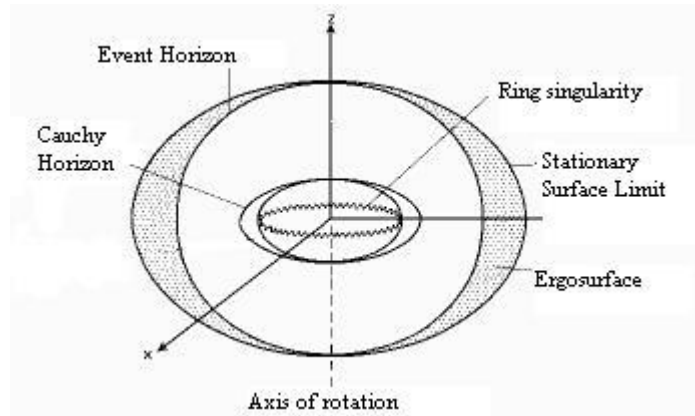


Figure 4.3: Overview of a rotating black hole. (Adapted from [fig3])

Thus in the case of a static observer, g_{tt} needs to be larger than zero (dt^2 is always positive). g_{tt} is zero if:

$$\begin{aligned} 1 - \frac{2mr}{\rho^2} &= 0 \\ \rho^2 - 2mr &= 0 \\ r^2 + a^2 \cos^2 \theta - 2mr &= 0 \end{aligned} \quad (4.27)$$

Solving this quadratic equation to r gives:

$$r = m \pm \sqrt{m^2 - a^2 \cos^2 \theta} \quad (4.28)$$

This gives two surfaces which depend on the mass m of the black hole and the rotational parameter a . The outer surface ($r = m + \sqrt{m^2 - a^2 \cos^2 \theta}$) is the static limit.

If a static observer goes through the ergosphere, then g_{tt} changes sign: the term becomes smaller than zero and timelike paths are only possible if $d\phi \neq 0$. Thus an observer inside the static limit needs to rotate to follow timelike paths. If the observer follows a geodesic, he would rotate with the same direction as the frame-dragging of the black hole.

Thus every particle and photon needs to rotate with the frame-dragging inside the ergosphere. Below is a figure which gives an illustration of the influence of the frame-dragging on the light cone. As you can see, the closer a light cone comes to the static limit, the more it tilts to the direction of the rotation of the black hole due to the frame-dragging. And inside in the ergosphere, light can no longer go in the opposite direction of the frame-dragging. [Misner, 1973]

Although $d\phi/dt$ needs to be larger than zero for timelike (or null) curves inside the ergosphere, dr/dt can have any sign in that region: particles can

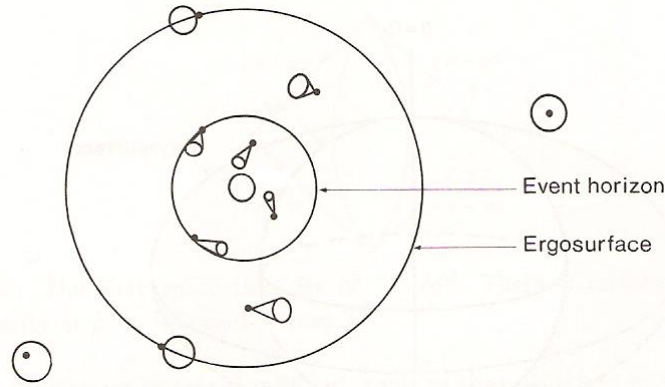


Figure 4.4: Lightcones near a rotating black hole. The view is from above: the small circles are the light cones as seen from above; the small dots within these circles are the tips (origins) of the cones. Inside the event horizon, light cones are so heavily tilted towards the singularity, light can not escape from the singularity; light cones in the ergosphere are tilted in the direction of the rotation and slightly to the singularity, but light can still go to infinity. [D'Eath, 1973]

go into the ergosphere from infinity, but they can also leave the ergosphere and go to infinity. [D'Eath, 1996]

4.6.2 Penrose process

The region between the static limit and the event horizon is called the ergosphere because the rotating black hole can do work on particles in this region ('ergo' is Greek for 'work'). This process of work done by the black hole on a particle is called the Penrose process.

A particle following a geodesic that enters the ergosphere under some specific circumstances can decay into two particles A and B inside the ergosphere. The ingoing particle has an energy E :

$$E = p \cdot \xi_t \quad (4.29)$$

which is equal to p^0 at infinity. This particle decays into two particles A and B, with energies E_A and E_B : $E = E_A + E_B$. The decay can be done in such a way that particle B goes through the event horizon into the black hole, and particle A escapes from the black hole to infinity. Because of (global) energy conservation

$$E_{blackhole,initial} + E = E_{blackhole,final} + E_A \quad (4.30)$$

Particle B, crossing the event horizon, has a negative energy because within the ergosphere, the sign of the killing vector ξ_t changes. The black

hole absorbs a negative energy. Particle A, that goes to infinity will gain that amount of energy because of energy conservation: $E_A > E$. [D'Eath, 1996] [Townsend, 1997]

4.6.3 Gravitational redshift

Outside the stationary limit surface it is possible for an observer to remain static with respect to the distant stars. Analogue to the derivation made in Foster (2006) for the redshift in the Schwarzschild-metric, it is possible to do the same derivation for the Kerr-metric, outside the stationary limit surface. The redshift in the Kerr-metric is given by:

$$\frac{\lambda_R}{\lambda_E} = \frac{1 - (2mr/\rho^2)_R}{1 - (2mr/\rho^2)_E} \quad (4.31)$$

This redshift formulae is only valid if both observer and emitter are outside the stationary limit surface since both need to be static for the derivation. Inside the stationary surface it is impossible to remain static due to the frame-dragging.

Because of different proper-times of the emitter and the observer (assuming they are at different distances to the rotating black hole), both measure a different frequency of the light one sends to the other. If the receiver is closest to the black hole, he observes the light to be blueshifted. If the emitter is closest to the black hole, the receiver observes the light to be redshifted. In the limit of the emitter going to the stationary limit surface, the light is redshifted to infinity.

The derivation of the redshift formula above makes explicit use of static emitters (and receivers). But particles at the stationary limit surface or inside the ergosphere can not remain static with respect to the distant stars (lorentz frame at infinity) because of the frame-dragging. And thus the equation for the redshift does not hold in this case. [Adler, 1976].

4.6.4 Different values for a and M

How does the stationary limit surface depend on the angular momentum parameter a , mass M and angle θ ? For $a = 0$ the ergosphere vanishes because there is no frame-dragging: the stationary limit surface will coalesce with the event horizon at $r = 2m$ (the inner stationary surface goes to $r = 0$). As a increases, the ergosphere becomes more flattened on top. If mass M increases, the angular momentum parameter a decreases: the ergosphere becomes more spherical by increment in the z -direction, and the inner stationary limit surface decreases. The ergosphere is an ellipsoid: because of the $\cos^2 \theta$ dependence it flattens at the z -axis as it coalesce with the event horizon at the z -axis. At the equatorial plane, the ergosphere has its maximum radius of $r = 2m$.

4.7 Event horizon

As stated in Section (3.4.1), an event horizon is a surface that can be considered as a one-way-membrane: it lets signals from the outside in, but it prevents signals from the inside to go to the outside. The curvature within the event horizon is that strong, that particles or photons can not escape from there to infinity. As well for the Kerr-metric as for the Schwarzschild-metric is the event horizon a sphere-shaped surface around the black hole singularity.

The horizon generators are the photons that have no-endpoints and will for always stay on the horizon. Whereas they follow straight lines for the Schwarzschild black hole, the null-geodesics are twisted for the Kerr black hole: they twist around the horizon, as the twists on a barber-pole (see figure 4.5). This twisting is caused by the frame-dragging: the photons are within the ergosphere and thus they can not be static. [Misner, 1973]

The Boyes-Lindquist coordinates have a coordinate singularity at the event horizon, therefore it is not an adequate coordinate system to describe the rotating black hole at that location. By choosing Kerr-coordinates, the properties of the event horizon can be made more clear.

4.7.1 Choice of coordinates

The line-element ds^2 of a photon in Boyes-Lindquist coordinates can be written as:

$$0 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{rr}dr^2 + g_{\theta\theta}d\theta^2 + g_{\phi\phi}d\phi^2 \quad (4.32)$$

Bringing the dr^2 -term to the other side and divide by dt^2 and g_{rr} gives:

$$\left(\frac{dr}{dt}\right)^2 = \frac{1}{g_{rr}} \left[g_{tt} + 2g_{t\phi} \frac{d\phi}{dt} + g_{\theta\theta} \left(\frac{d\theta}{dt}\right)^2 + g_{\phi\phi} \left(\frac{d\phi}{dt}\right)^2 \right] \quad (4.33)$$

with

$$\frac{1}{g_{rr}} = \frac{\Delta}{\rho^2} \quad (4.34)$$

If $\Delta = 0$ then the radial coordinate velocity becomes zero and thus is the photon unable to move radially any further:

$$\begin{aligned} \Delta &= 0 \\ r^2 + a^2 - 2mr &= 0 \\ r &= \frac{2m \pm 2\sqrt{m^2 - a^2}}{2} \\ r &= m \pm \sqrt{m^2 - a^2} \end{aligned} \quad (4.35)$$

$$r = m \left(1 \pm \sqrt{1 - a^2/m^2} \right) \quad (4.36)$$

Thus there are two surfaces of null geodesics that have no future endpoints. The inner surface is called the Cauchy horizon and the outer surface is the event horizon.

A closer inspection of the expression for the radial coordinate velocity at the event horizon (eqn. 4.33) gives that if a photon approaches the event horizon from both sides, it is halted at the horizon: a photon that comes from infinity and enters the event horizon, will never go further into the black hole! It will spiral an infinite time around the horizon, as the coordinate time t goes to infinity. But that is not what physical happens, this is caused by the coordinate singularity of the Boyes-Lindquist coordinates: the g_{rr} -term goes to infinity.

By changing the coordinates of the Kerr-metric to Kerr-coordinates, one is able to remove this coordinate singularity. It is possible to re-express the Kerr-metric (using Kerr coordinates) in a different way than in section 4.2.2:

$$\begin{aligned}
 ds^2 = & \frac{c^2 d\tilde{t}^2 r^2 \Delta}{(a^2 + r^2)^2 - a^2 \Delta} - \frac{4dr d\tilde{t} m r^3}{(a^2 + r^2)^2 - a^2 \Delta} \\
 & - \frac{dr^2 r^2 (2mr + r^2)}{(a^2 + r^2)^2 - a^2 \Delta} \\
 & - \left[d\tilde{\phi} - \frac{2adtmr}{(a^2 + r^2)^2 - a^2 \Delta} - \frac{adr(2mr + r^2)}{(a^2 + r^2)^2 - a^2 \Delta} \right]^2 \\
 & \times \frac{2amr}{\left((a^2 + r^2)^2 - a^2 \Delta \right)^2} \tag{4.37}
 \end{aligned}$$

(For checking, it is easiest to rewrite the above expression into equation (4.6).)

As one can see from this equation, if $\Delta = 0$ (thus at the event horizon), dr needs to become smaller than zero for timelike curves (which have $ds^2 > 0$): the first term is zero, and all other terms are positive because they are quadratic. [Thorne, 2005]

The above situation is illustrated in figure 4.5 below. Picture (a) and (b) correspond to the light cones one gets in Boyes-Lindquist coordinates. In the figure one is able to see the pinch-off of the light cones when they come nearer to the event horizon. At the horizon they permit only rotational movement: all photons become horizon generators. Figures (c) and (d) correspond to the case of the adapted Kerr-coordinates. As the light cones come closer to the horizon they do not pinch-off but they tilt over in the direction of the black hole. And at the event horizon, photons can only move further into the black hole or be a horizon generator.

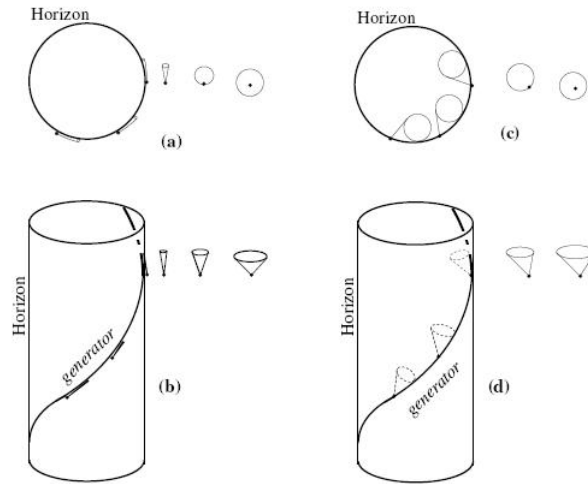


Figure 4.5: Lightcones near the event horizon: (a) and (b) are for Boyes-Lindquist coordinates, (c) and (d) for Kerr-coordinates. (b) and (d) are spacetime diagrams, and (a) and (c) are respectively their views from above. The light-cones in (a) and (b) are pinched-off, in (c) and (d) they tilt over towards the singularity. [Thorne, 2005]

4.7.2 Time-like vs. space-like

When one passes the static limit, the g_{tt} term changes sign. At the event horizon, the g_{rr} term changes sign as well. One can speculate what this means for the paths inside the event horizon, for example that the radial coordinate r corresponds to a time-parameter and the time-coordinate t corresponds to a spatial coordinate inside the event horizon.

But the inside of a real black hole is not properly described by the Kerr-metric. Because after the collapse, the inside does not tend to go to the Kerr-metric due to gravitational radiation [Novikov 1976].

4.7.3 Different values for a and M

A larger value for the rotation parameter a gives a decrease in the radius of the event horizon and thus means a stronger curvature: the spacetime manifold has the same curvature, but a shorter distance over which this curvature is spread. If $a = 0$ then the event horizon has $r = 2m$ and the Cauchy horizon has $r = 0$. A larger value for the mass-parameter m increases the radius of the outer horizon, but decreases the Cauchy horizon.

Chapter 5

Geodesics around a Kerr-black hole

This chapter is about the paths of free test particles outside a Kerr black hole. As these are freely falling particles, they are described by geodesics. To give these geodesics, one needs to find the expressions for the four-velocity u^μ . These expressions are derived in the first section of this chapter. After obtaining u^μ , two types of motion are discussed for different values of the parameters: radial r -motion and axial θ -motion.

5.1 Four constants of motion

To find expressions for the geodesics, one needs four constants of motion. Geodesics are often derived with the use of the Lagrangian L and the Euler-Lagrange equations, however, here we will use the Hamilton-Jacobi approach because this method will give us the fourth constant of motion. The derivation of the four constants of motion presented below, is analogue to the derivation given in [Carter, 1968] (and [Misner, 1973]).

The Hamiltonian is given by:

$$H(x^\mu, p_\mu) = p_\mu \dot{x}^\mu - L(x^\mu, \dot{x}^\mu) \quad (5.1)$$

with the dot indicating a derivative with respect to the affine parameter λ . For the case of a free particle, the Lagrangian L is

$$L(x^\mu, \dot{x}^\mu; t) = \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \quad (5.2)$$

The conjugate momenta p_μ is defined as

$$p_\mu \equiv \frac{\partial L}{\partial \dot{x}^\mu} = g_{\mu\nu} \dot{x}^\nu \quad (5.3)$$

The leads to the expression

$$H = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu - \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = L \quad (5.4)$$

The equality of the Hamiltonian to the Lagrange indicates there is no potential energy involved, as would be expected since this was the case for the Schwarzschild black hole as well.

Inverting the expression for the conjugate momentum above gives an equation for \dot{x}^μ in terms of the conjugate momentum:

$$\dot{x}^\mu = g^{\mu\nu}p_\nu \quad (5.5)$$

This gives the following expression for the Hamiltonian (using (5.3) and (5.5))

$$H = \frac{1}{2}g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}p_\mu\dot{x}^\mu = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu \quad (5.6)$$

To find the explicit equation for the Hamiltonian, one has to invert the metric $g_{\mu\nu}$ to $g^{\mu\nu}$. The expression for the inverse metric (in Kerr-coordinates) is given by the inverse of the line-element $ds = g_{\mu\nu}dx^\mu dx^\nu$:

$$\begin{aligned} \left(\frac{\partial}{\partial s}\right)^2 &= \rho^{-2}\left(\frac{\partial}{\partial\theta}\right)^2 + 2\rho^{-2}(r^2 + a^2)\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial\tilde{V}}\right) \\ &+ 2\rho^{-2}a\left(\frac{\partial}{\partial r}\right)\left(\frac{\partial}{\partial\tilde{\phi}}\right) + 2\rho^{-2}a\left(\frac{\partial}{\partial\tilde{V}}\right)\left(\frac{\partial}{\partial\tilde{\phi}}\right) \\ &+ \rho^{-2}a\sin^2\theta\left(\frac{\partial}{\partial\tilde{V}}\right)^2 + \rho^{-2}\sin^2\theta\left(\frac{\partial}{\partial\tilde{\phi}}\right)^2 + \rho^{-2}\Delta\left(\frac{\partial}{\partial r}\right)^2 \end{aligned} \quad (5.7)$$

Replacing $\frac{\partial}{\partial x^\mu}$ with p_μ one obtains the following equation for the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2}\rho^{-2}\left\{\Delta p_r^2 + 2\left[(r^2 + a^2)p_{\tilde{V}} + ap_{\tilde{\phi}}\right]p_r + p_\theta^2\right\} \\ &+ \frac{1}{2}\rho^{-2}\left\{\left[a\sin\theta p_{\tilde{V}} + \sin^{-1}\theta p_{\tilde{\phi}}\right]^2\right\} \end{aligned} \quad (5.8)$$

A constant of motion is defined as the p_μ for which $\frac{\partial H}{\partial x^\mu} = 0$. In the case of a rotating black hole, the Hamiltonian is not explicitly dependent on t and ϕ since the Kerr spacetime is symmetric in coordinate time and the axial coordinate. Therefore one is able to define these two constants of motion:

$$p_{\tilde{V}} = g_{\tilde{V}\nu}\dot{x}^\nu = E \quad (5.9)$$

$$p_{\tilde{\phi}} = g_{\tilde{\phi}\nu}\dot{x}^\nu = -L \quad (5.10)$$

Where E the energy of the test-particle at infinity is, and L the angular momentum around the symmetry axis. Thus the first two constants of motion are E and L

A third constant of motion follows from the relation $g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \delta$ where $\delta = 1$ for the case of a timelike geodesic, $\delta = 0$ for null geodesic and $\delta = -1$ for spacelike geodesics (this is the same as in the Schwarzschild case). It can be considered as a constant related to the rest mass of the particle. This leads to

$$H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = \frac{1}{2}\delta \quad (5.11)$$

So far, there are three constants of motion: energy E , angular momentum L and rest-mass δ . With the Hamilton-Jacobi method it is possible to obtain a fourth constant of motion, named after its 'inventor' Carter. The solution of the Hamilton-Jacobi method will be formulated in terms of all the constants of motion. [web1]

By the definition of the Hamilton-Jacobi method, the Hamilton-Jacobi equation is given by:

$$-\frac{\partial S}{\partial \lambda} = H = \frac{1}{2}g^{\mu\nu}p_\mu p_\nu = \frac{1}{2}\delta \quad (5.12)$$

Where S the Jacobi action is. If S is a solution to the Hamilton-Jacobi equation, then $\frac{\partial S}{\partial x_i} = p_i$, where p_i is a constant of motion. Assuming the case of a solution for S , consisting of variables that can be separated, S can be expressed in the constants of motion:

$$S = \frac{1}{2}\delta\lambda - E\tilde{V} + L\tilde{\phi} + S_\theta + S_r \quad (5.13)$$

In which S_θ is a function of θ and S_r a function of r .

Inserting the expression of S into the Hamiltonian by making the partial derivatives of S (5.8); using $p_i = \frac{\partial S}{\partial x_i}$ and multiplying everything with $2\rho^2$, gives:

$$\begin{aligned} -2\rho^2\frac{\partial S}{\partial \lambda} &= \Delta\left(\frac{\partial S_r}{\partial r}\right)^2 + 2[(r^2 + a^2)^2(-E) + aL]\left(\frac{\partial S_r}{\partial r}\right) \\ &+ \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 + [a\sin\theta(-E) + \sin^{-1}\theta L]^2 = \rho^2\delta^2 \end{aligned} \quad (5.14)$$

Rearranging gives:

$$\begin{aligned} \left(\frac{\partial S_\theta}{\partial \theta}\right)^2 &+ [a\sin\theta E - \sin^{-1}\theta L]^2 + a^2\delta^2\cos^2\theta = \\ &- \Delta\left(\frac{\partial S_r}{\partial r}\right)^2 + 2[(r^2 + a^2)^2E - aL]\left(\frac{\partial S_r}{\partial r}\right) - r^2\delta^2 \end{aligned} \quad (5.15)$$

Since both sides depend on different variables, for the expression to hold along the geodesic, both sides must be equal to the same constant. This constant is called the Carter constant K and is the fourth constant of motion. While using again the relation $p_i = \frac{\partial S}{\partial x_i}$, the two equations for K look like:

$$p_\theta^2 + (aE \sin \theta - L \sin^{-1} \theta)^2 + a^2 \delta^2 \cos^2 \theta = K \quad (5.16)$$

$$\Delta p_r^2 - 2[(r^2 + a^2)E - aL]p_r + \delta^2 r^2 = -K \quad (5.17)$$

Solving these two quadratic equations, gives two expressions for p_θ and p_r :

$$p_\theta = \sqrt{\Theta} \quad (5.18)$$

$$p_r = \Delta^{-1}(P + \sqrt{R}) \quad (5.19)$$

with

$$\Theta = K - (L - aE)^2 - \cos^2 \theta [a^2(\delta^2 - E^2) + L^2 \sin^{-2} \theta] \quad (5.20)$$

$$P = E(r^2 + a^2) - La \quad (5.21)$$

$$R = P^2 - \Delta(\delta^2 r^2 + K) \quad (5.22)$$

This then gives (using $\dot{x}^\mu = g^{\mu\nu} p_\nu$ and equations (5.8), (5.9), (5.10), (5.18), (5.19): the equations for the Hamiltonian and the constants of motion):

$$\rho^2 \dot{\theta} = \sqrt{\Theta} \quad (5.23)$$

$$\rho^2 \dot{r} = \sqrt{R} \quad (5.24)$$

$$\rho^2 \dot{V} = -a(aE \sin^2 \theta - L) + (r^2 + a^2)\Delta^{-1}[\sqrt{R} + P] \quad (5.25)$$

$$\rho^2 \dot{\phi} = -(aE - L \sin^{-2} \theta) + a\Delta^{-1}[\sqrt{R} + P] \quad (5.26)$$

which gives the four-velocity u^μ . The signs of square roots of Θ and R can be chosen independently, but one must be consistent in that choice.

The final solution (integrating equations (5.18) and (5.19) to θ and r respectively to obtain S_θ and S_r) for the Jacobi action is then given by

$$\begin{aligned} S &= \frac{1}{2}\delta\lambda - E\tilde{V} + L\tilde{\phi} + S_\theta + S_r \\ &= \frac{1}{2}\delta^2\lambda - E\tilde{V} + L\tilde{\phi} \\ &\quad + \int^\theta (\sqrt{\Theta})d\theta + \int^r \Delta^{-1}Pdr + \int^r \Delta^{-1}(\sqrt{R})dr \end{aligned} \quad (5.27)$$

Differentiating the Jacobi action with respect to the four constants of

motion (K, δ , E and L) gives respectively:

$$\int^{\theta} \frac{d\theta}{\sqrt{\Theta}} = \int^r \frac{dr}{\sqrt{R}} \quad (5.28)$$

$$\lambda = \int^{\theta} \frac{a^2 \cos^2 \theta}{\sqrt{\Theta}} d\theta + \int^r \frac{r^2}{\sqrt{R}} dr \quad (5.29)$$

$$\begin{aligned} \tilde{V} &= \int^{\theta} \frac{-a(aE \sin^2 \theta - L)}{\sqrt{\Theta}} d\theta \\ &\quad + \int^r \frac{r^2 + a^2}{\Delta} \left(1 + \frac{P}{\sqrt{R}}\right) dr \end{aligned} \quad (5.30)$$

$$\begin{aligned} \tilde{\phi} &= \int^{\theta} \frac{-(aE - L \sin^{-2} \theta)}{\sqrt{\Theta}} d\theta \\ &\quad + \int^r \frac{a}{\Delta} \left(1 + \frac{P}{\sqrt{R}}\right) dr \end{aligned} \quad (5.31)$$

Which are the first-integrals of motion. Again, the signs of the two squares can be chosen indepently.

As for a check on the results obtained, one can rewrite the first-integral for the coordinate time and compare it to the Schwarzschild first-integral for coordinate time (equation (3.32)) given below:

$$t = \int^r \frac{E dr}{(1 - 2m/r) [E^2 - (1 - 2m/r)(\delta + L^2/r^2)]^{1/2}}$$

By using the relation for the coordinate transformation from Boyes-Lindquist coordinate time to Kerr-coordinates (equation (4.6):

$$dt = d\tilde{V} - \frac{r^2 + a^2}{\Delta} dr$$

while setting the rotation parameter a to zero, one obtains the following:

$$\begin{aligned} t &= \int^r \frac{r^2}{\Delta} dr + \int^r \frac{(r^2)P}{\Delta\sqrt{R}} dr - \int^r \frac{r^2}{\Delta} dr \\ &= \int^r \frac{(r^2 + a^2)}{\Delta\sqrt{R}} dr \end{aligned} \quad (5.32)$$

the equation becomes

$$\begin{aligned} t &= \int^r \frac{r^2 P}{\Delta\sqrt{R}} dr \\ &= \int^r \frac{r^4 E dr}{r^2(1 - 2m/r) [E^2 r^4 - r^4(1 - 2m/r)(\delta^2 + K/r^2)]^{1/2}} \\ &= \int^r \frac{E dr}{(1 - 2m/r) [E^2 - (1 - 2m/r)(\delta^2 + K/r^2)]^{1/2}} \end{aligned} \quad (5.33)$$

This is equal to the expression for a massive test-particle of the Schwarzschild black hole. Notice that in this case, the Carter's constant K equals L^2 . However, the geometrical interpretation of the K is unclear.

Now the equations for the motion along geodesics are given it is possible to say some things about the geodesics.

5.2 θ -motion

The equation describing the θ -motion of particles is equation (5.23):

$$\rho^4 \dot{\theta}^2 = K - (L - aE)^2 - \cos^2 \theta [a^2(\delta^2 - E) + L^2 \sin^{-2} \theta] \quad (5.34)$$

Note that it is the $\dot{\theta}$ squared: a positive value indicate a positive or negative θ -velocity, but a negative $\dot{\theta}^2$ corresponds to an imaginary axial velocity.

It is convenient to write $\dot{\theta}^2$ as a function of u , where $u \equiv \cos \theta$. Then $u = 1$ corresponds to the z-axis ($\theta = 0$), $u = 0$ corresponds to the equatorial plane and $0 < u^2 < 1$ corresponds to $0 < \theta < \pi$ without the point $\theta = \pi/2$.

$$f(u) = \rho^4 \dot{\theta}^2 = Q - (Q + L^2 - a^2\Gamma) u^2 - a^2\Gamma u^4 \quad (5.35)$$

In which $Q \equiv K - (aE - L)^2$ and $\Gamma \equiv E^2 - \delta^2$. [Stewart, 1973]

5.2.1 Low energy particles

In this section, the discussed particles have low energies, around the order of one mass-energy.

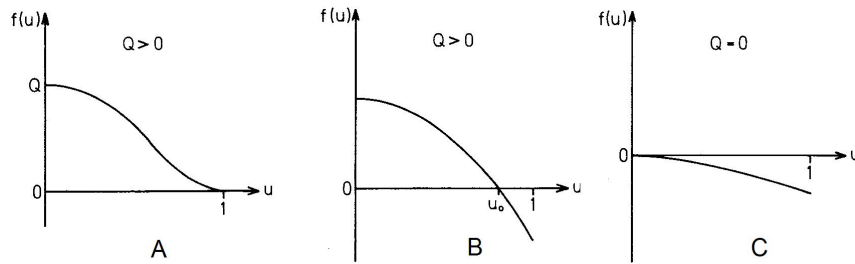


Figure 5.1: θ -motion for the case of low energy particles. Figure A corresponds to $L = 0$, B to $L > 0$ and C to $L^2 > a^2\Gamma$. [Stewart, 1973]

If a particle has a low energy and no angular momentum ($L = 0$), then the radial velocity is zero at the z-axis ($f(u) = 1$): a stationary point. (See the figure 5.1 A). The second derivative of θ is zero at $u = 1$, thus a particle can have an orbit at the z-axis. However, it is an unstable orbit for small θ -perturbations. This is the only case of θ -motion that a particle can have an orbit at the z-axis.

As the angular momentum increases, the stationary point will shift towards z-axis, away from the equatorial plane. But for these cases, the stationary point is not a point of an orbit. The black hole is pulling the particles towards the equatorial plane due to the frame-dragging. (See figure 5.1 B). The particle will follow an oscillatory motion that crosses the equatorial plane repeatedly, with θ lying in the range $\theta_0 \leq \theta \leq \pi - \theta_0$, where $u_0 = \cos \theta_0$, for which $f(u_0) = 0$.

A further increase in the angular momentum, decreases the value for $f(u_0) = Q$: the difference between E and L increases. The particle will cross the equatorial plane with a lower θ -velocity. If $Q = 0$ (and $L^2 > a^2\Gamma$), then $f(u_0) = 0$: the particle has a stable orbit at the equatorial plane, since $u = 0$ the only valid value for u is. In this case, there is no θ velocity. (See figure 5.1 C.)

If $L = \Gamma = 0$ in addition to $Q = 0$, then the particle can have any θ value as a constant value since $\dot{\theta} = 0$ for all u .

A larger value for the angular momentum would mean a further decrease in Q , Q becoming more negative. Then $f(u)$ is negative for every θ , which is not a physical possible situation. It is not possible for a particle to have such a large angular momentum and at the same time low energy. [Stewart, 1973] [Carter, 1968]

5.2.2 High-energy particles

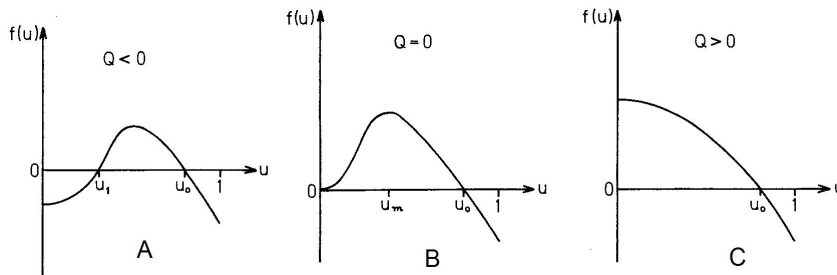


Figure 5.2: θ -motion for the case of high energy particles. Figure A corresponds to $L = 0$, B to $L^2 < a^2\Gamma$ and C to $L > 0$. [Stewart, 1973]

A high energy particle with a small angular momentum L , has a small Q : $Q < 0$. The particle will have an oscillatory motion between two angles θ_1 and θ_2 in case of the following inequality: $(Q + L^2 - a^2\Gamma)^2 + 4a^2\Gamma Q \leq 0$. This inequality must hold to have such a maximum. If the inequality does not hold, there is no real solution to the quadratic equation for which u -value $f(u) = 0$ and thus no possible physical situation.

The motion will not cross the equatorial plane. (See for reference, figure 5.2 A.) (Again, in the case of $L = 0$, it is able to cross the z-axis.) This

motion also has a stable orbit for the case $\theta_1 = \theta_2$, if the inequality is an equality.

As for particles with a larger angular momentum, the next special case is when $Q = 0$. For particles with high energies ($L^2 < a^2\Gamma$), this situation is different than for particles with low energies. The point u_1 has shifted towards the equatorial plane. And it is possible to have an unstable orbit at the equatorial plane. But the general motion is still oscillatory between $0 \leq u \leq u_0$. The maximum u_m is caused by the rotation of the black hole.

Further increase in the angular momentum would lead to a situation which is equal to the second case of the low energy particles (figure 5.1 B.): an oscillatory motion between the equatorial plane and θ_0 . [Stewart, 1973] [Carter, 1968]

5.3 r -motion

Just as one can do for the θ -motion, it is possible to rewrite the expression for $\rho^2\dot{r}^2$:

$$\begin{aligned} \rho^2\dot{r}^2 \equiv R(r) = & \Gamma r^4 + 2m\delta^2 r^3 + (a^2 E^2 - L^2 - a^2\delta^2 - Q)r^2 \\ & + 2m[(aE - L)^2 + Q]r - a^2Q \end{aligned} \quad (5.36)$$

Recall that $Q = K - (aE - L)^2$ and $\Gamma \equiv E^2 - \delta^2$. $R(r)$ is the square of radial velocity, and can therefore not be negative: a negative value for $R(r)$ gives a non-physical (imaginary) value for the radial velocity. Because $\Delta = 0$ at the event horizon r_+ , $R(r_+)$ needs to be equal or larger than zero at the event horizon (this follows easily from the original equation for $\rho^2\dot{r}^2$ (5.26)).

Some specific values for $R(r)$ are: $R(0) = -a^2Q$, and $R(r) \rightarrow \Gamma r^4$ as $r \rightarrow \infty$. A negative value for Γ corresponds to $E < \delta$. Such a particle does not have enough energy to go to infinity as it is bound by the black hole (E is not large enough compared to the 'effective potential' of the black hole).

There are four different cases which will be discussed, depending on the signs of Γ and Q . In this treatment, a distinction is being made between prograde orbits (L is positive and same direction as the rotation of the black hole) and retrograde orbits (L is negative and opposite to the direction or black hole rotation), because the frame-dragging forces the particle to rotate with the rotation of the black hole. This results that retrograde orbits further away than prograde orbits, will result in the particle being trapped into the black hole. Frame-dragging slows particles in retrogrades orbits down. [Chandrasekhar, 1983]

5.3.1 Case 1: $Q > 0, \Gamma > 0$

If the energy E is large enough, all the coefficients in equation (5.36) are non-negative, with the exception of the coefficient of r^0 : the particles are not able to reach the singularity, due to their low energy. [Carter, 1986]

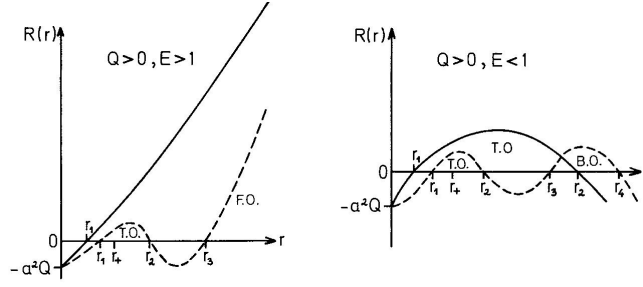


Figure 5.3: r -motion for the case of $Q > 0, \Gamma > 0$ on the left, and $Q > 0, \Gamma < 0$ on the right. F.O. means free orbit, T.O. trapped orbit, B.O. bound orbit. A solid line corresponds to retrograde orbits (large negative L), and a dotted line corresponds to prograde orbits (large positive L). [Stewart, 1973]

Since $R(0) < 0$ and $R(r_+) \geq 0$ there has to be a zero between those points, at $r = r_1$, in or on the horizon (see figure 5.3). Consider a particle moving from infinity inwards in the case of the solid line. Because $\dot{r}^2 > 0$, the particle will move inwards ($\dot{r} < 0$), cross the event horizon and move up to the point r_1 . Then it will reverse in motion, but it is not able to cross the event horizon again and it is trapped. (Trapped orbits are orbits which are partially within the event horizon: the particles with this orbit will become trapped into the black hole.) If a particle has $\dot{r} > 0$ (it moves away from the black hole), it is able to go to infinity.

Suppose L is increased to a large positive value (prograde orbits), leaving E and Q fixed. It follows that the shape of $R(r)$ will change to the dotted line in the graph. There are two additional zeros at $r_2 \cong 2m$ and at $r_3 > 2m$.

For the case of a dotted line, no particle is allowed in the range of $r_2 < r < r_3$. Any particle with $r < r_2$ is trapped inside: because of its prograde orbit, it is not able to resist the frame-dragging, and because of its close proximity it will cross the event horizon in the end. However, particles at $r > r_3$ move along a parabolic orbit around the black hole: they stay far away from the black hole to prevent being trapped by it. In the special case of $r_2 = r_3$, a particle at r_2 is in an unstable, spherical orbit around the black hole. This orbit is unstable for small r -perturbations.

5.3.2 Case 2: $Q > 0, \Gamma < 0$

As the energy of the particles of Case 1 decrease, they will eventually have energies of $E < \delta$. These particles have a lower velocity and are less able to escape the gravitational ‘pull’ from the black hole. Since $R(0) < 0$, $R(r_+) \geq 0$ and $R(r) \rightarrow -\infty$ as $r \rightarrow \infty$, there have to be at least two real

zeros. In figure 5.3 above the case for two zeros (solid line) is illustrated.

Particles with $r_1 \leq r_+ \leq r_2$ are trapped, other values for r are not allowed. If the particle has $\dot{r} > 0$, then he would follow an elliptical orbit largely within the ergosphere, but once inside the event horizon, it will not escape. In the case of $r_1 = r_2$ there is a special case where the particle orbits on the event horizon. This orbit is not stable, because small perturbations will cause the particle to cross the event horizon and disappear within it.

By changing the value for L to larger values, keeping E and Q fixed, one gets the dotted line. As in the previous case, particles with $r_1 \leq r \leq r_2$ are trapped. There are two additional zeros r_3 and r_4 . Particles in the range of $r_3 \leq r \leq r_4$ are bound and they will oscillate within that range and follow elliptical shaped orbits. The case of $r_3 = r_4$ gives a stable spherical orbit.

5.3.3 Case 3 and 4: $Q < 0$, $\Gamma > 0$ or $\Gamma < 0$

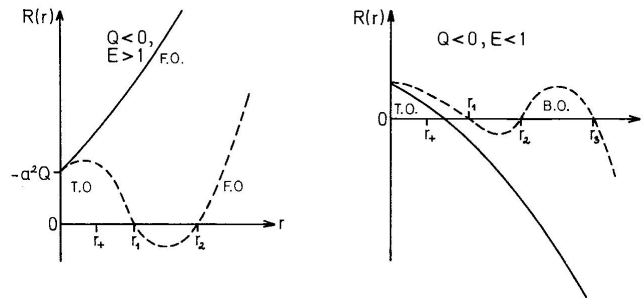


Figure 5.4: r -motion for the case of $Q < 0$, $\Gamma > 0$ on the left, and $Q < 0$, $\Gamma < 0$ on the right. F.O. means free orbis, T.O. trapped orbit, B.O. bound orbit. A solid line corresponds to retrograde orbits (large negative L), and a dotted line corresponds to prograde orbits (large positive L). [Stewart, 1973]

These two cases are rather similar to the first cases, but are about particles with a larger angular momentum (and energy), causing Q to become negative. In these cases there is a positive crossing with the $R(r)$ axis: trapped particles are able to reach $r = 0$. In both cases, there is a zero less than in the corresponding cases of positive Q .

5.4 Equatorial motion

For orbits in the equatorial plane, particles must have $Q = 0$ and $L^2 > a^2\Gamma$, because those particles are stable versus θ -perturbations. [Stewart, 1973]

For deriving the expression of the effective potential $V(r)$, similar as the effective potential for the Schwarzschild black hole (see section 3.6.2), one

should rewrite equations (5.23) and (5.24) to obtain respectively

$$K = \left(\rho^2\dot{\theta}\right)^2 + L^2 + a^2E^2 - 2aLE + a^2\delta^2\cos^2\theta - a^2E^2\cos^2\theta + L^2\cos^2/\sin\theta \quad (5.37)$$

$$\Delta K = -(\rho^2\dot{r})^2 + E^2(r^2 + a^2) + L^2a^2 - 2LEa(r^2 + a^2) - \Delta\delta^2r^2 \quad (5.38)$$

When these equations are made equal, it is possible to bring all terms to one side and come to an expression of the form

$$\alpha E^2 - 2\beta E + \gamma - r^4\dot{r}^2 = 0 \quad (5.39)$$

using $\theta = \pi/2$ for the equatorial plane ($\dot{\theta} = \ddot{\theta} = 0$)

$$\alpha = (r^2 + a^2)^2 - \Delta a^2 \sin^2\theta \quad (5.40)$$

$$\beta = 2mrLa \quad (5.41)$$

$$\gamma = L^2a^2 - \Delta\delta^2r^2 - \Delta L^2 \quad (5.42)$$

$$(5.43)$$

Solving equation (5.39) to find an equation for E leads to:

$$E = \frac{\beta + \sqrt{\beta^2 - \alpha\gamma + \alpha r^4\dot{r}^2}}{\alpha} \quad (5.44)$$

The allowed regions for a particle are those with $E \geq V(r)$, where $V(r)$ is the effective potential, the minimum allowed value of E at a radius r :

$$V(r) = \frac{\beta + \sqrt{\beta^2 - \alpha\gamma}}{\alpha} \quad (5.45)$$

The effective potential gives information about the radial motion. It depends on the angular momentum L and the radial distance r . If $E > V$ for every r and some L_0 , then all particles with energy E and angular momentum L_0 are able to just fall into the black hole. However, if the energy E_1 of a particle with L_1 is lower than some $V(r_1, L_1)$, then the particle is not able to come more closely to the black hole than r_1 , and depending on the shape of the effective potential, it could follow a circular, elliptical or parabolic orbit with r_1 as one turning point. [Misner, 1973]

Null-geodesics

Photons ($\delta = 0$) follow null-geodesics. Two plots are made for two different values for the angular momentum of a photon are in figure 5.5 below: $L = 2mE$ (prograde) and $L = -2mE$ (retrograde). As can be seen, these photons are able to follow circular orbits close to $r = 2m$ (stationary surface

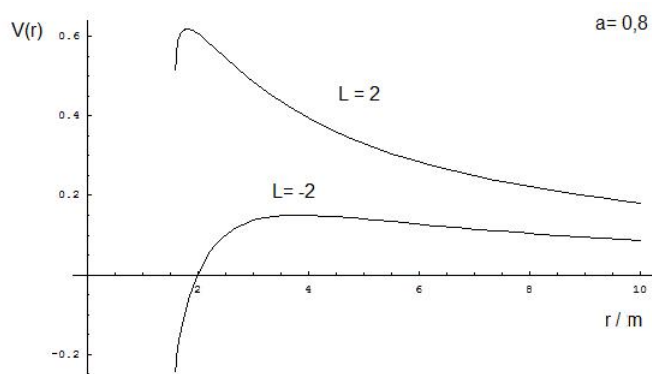


Figure 5.5: Two effective potentials for photons with different angular momentum: prograde ($L = 2mE$) and retrograde ($L = -2mE$).

limit) and $r = 4m$ respectively. Both are unstable against r -perturbations. [Misner, 1973] [Stewart, 1973]

The radius of the retrograde photon is larger because of the rotation of the black hole. Would this photon come nearer to the black hole, then the rotation of the spacetime geometry would force the photon to change its direction of rotation towards the direction of rotation of the black hole. As the photon loses its rotational velocity, it will fall into the black hole. (See figure 5.6.)

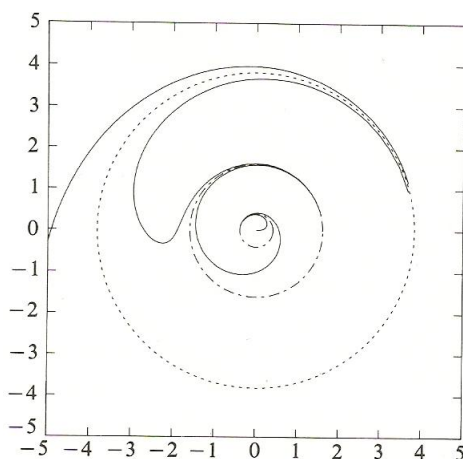


Figure 5.6: The orbit of a retrograde photon. [Chandrasekhar, 1983]

Timelike-geodesics

A particle with a test-mass has $\delta = 1$. A specific case is the innermost stable circular orbit for timelike particles at $r = m$ and $r = 9m$ (depending on their angular momentum). [Misner, 1973]

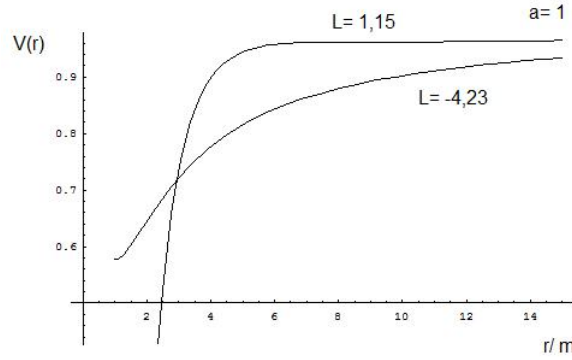


Figure 5.7: Two effective potentials for timelike geodesics in an Extreme Kerr geometry ($a = m$). As can be seen in the diagram, the potential of $L = 2m/\sqrt{3}$ has an stable orbit at $r = m$, and for $L = -22m/3\sqrt{3}$ it has a minimum at $r = 9m$.

The same reasoning about why the retrograde orbit for a photon has a larger radius can be applied to the retrograde orbit of a timelike particle.

Schwarzschild case

Setting the rotation parameter a in the expression for the effective potential to zero, gives the effective potential for the Schwarzschild black hole:

$$\alpha = r^4 \quad (5.46)$$

$$\beta = 0 \quad (5.47)$$

$$\gamma = -\Delta (\delta^2 r^2 + L^2) = -(r^2 - 2mr) (\delta^2 r^2 + L^2) \quad (5.48)$$

$$(5.49)$$

Giving:

$$\begin{aligned} V^2 &= \frac{-\alpha\gamma}{\alpha^2} = \frac{r^4 (r^2 - 2mr) (\delta^2 r^2 + L^2)}{r^8} \\ &= \frac{r^2 (1 - \frac{2m}{r}) r^2 (\frac{L^2}{r^2} + \delta^2)}{r^4} \\ &= \left(1 - \frac{2m}{r}\right) \left(\frac{L^2}{r^2} + \delta^2\right) \end{aligned} \quad (5.50)$$

Which is the same expression as equation (3.38) for timelike particles.

Chapter 6

Conclusion

A rotating black hole is described by the Kerr-metric. Using different coordinate systems, different properties or features of a rotating black hole can be described. One feature of such a (theoretical) black hole is its ring singularity. This ring can only be reached by particles following specific geodesics. Around this singularity is the Cauchy-horizon. With an increasing radial distance one reaches the event horizon and then the stationary limit surface. Once inside the event horizon, particles are not able to escape to the outside of the event horizon. Within the ergosphere it is impossible to be static due to the strong frame-dragging caused by the rotation of the black hole. Particles are forced to rotate with the direction of the rotation of the black hole. Outside the stationary limit surface it is possible to be static, but such a particle would not follow a geodesic (only at infinity is the frame-dragging zero).

The frame-dragging, or rotation of spacetime, has a very strong influence on the geodesics of particles (with and without mass), their orbits are for example twisted around the black hole; depend strongly on the sign of the angular momentum of the particle (whether it follows a retrograde or prograde orbit); and there are no stable θ -motions outside the equatorial plane. The possible geodesics put restraints on the energy and angular momentum of the particles in an accretion disk in the equatorial plane of the black hole: all particles rotate with the direction of the rotation of the black hole, otherwise all the retrograde particles would disrupt the disk; the particles should have energy larger than their rest-energy; and in combination with their angular momentum should satisfy the conditions for a stable orbit in the equatorial plane. Via collisions, the disk should be able to accrete particles which do initially not satisfy those conditions. The rotation of the black hole also influences the 'images' of a galaxy caused by the gravitational lensing: photons that pass by the black hole from the left are differently bent by the black hole than photons that pass by on the right. The image will be distorted because of the difference between retrograde and prograde orbits.

Although a rotating black hole has large influence on the surrounding matter, it has never been directly observed since it does not emit photons. But black holes are perhaps directly observed when large interferometers are build for the detection of gravitational waves. And that could lead to answers to questions about black holes, for example about supermassive and primordial black holes.

Chapter 7

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- fig2** http://tycho.bgsu.edu/~laird/cp_images/black_holes.html
- fig3** <http://www.644net.com/V1-5.files/image074.gif>

Cover image: a rotating black hole with an accretion disk and two jets. From: <http://esamultimedia.esa.int/images/spcs/xmm-newton/xmm20050110b.jpg>.