Category forms of Local-Causality and Non-Signalling and their duals

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Abstract. Two fundamental (meta)physical principles – **NS** (Non-Signalling condition which states the impossibility to communicate by means of physical correlations) and **LC** (the principle of Local Causality which isolates classical correlations from those responsible for non-locality) are considered in the framework of category theory. The original form of these principles operates with properties of common probability distributions for outcomes of measurements implemented in two space-time regions. The suggested category form consists of some assertions about special commutative diagrams. To any common probability distribution in the matter of discourse, an arrow (morphism) in these diagrams is associated. In fact, **LC** turns into the condition of the arrow being able to factor through a definite standard arrow. **NS** looks like uniqueness of an arrow which makes commutative a special diagram incorporating the considered arrow associated to the distribution. By means of these diagrams dual forms of **NS** and **LC** are suggested.

1. Introduction

The theory that encompasses both quantum physics and gravity has not been formulated. This still anticipated ultimate construction must evidently retain the most fundamental notions of the present theoretical toolkit. These are few in number and duality conception is a paramount one. From the view-point of S.Majid [1] self-dual constructions are more promising as possible ingredients of the future theory. It is worth to consider this conjecture seriously and look for dual counterparts of known fundamental principles. Category theory suggests the most effective machinery in this search.

Presently, the category theory provides the most effective means for unification of various branches of mathematics [2]. Rapid growth of its physical applications [3] promises the future status of category methods among indispensable tools of theorists. In particular, perspectives in quantum gravity are connected with an important region of categories, theory of topoi [4].

The category theory is by its nature a science about arrows, morphisms, as deep generalizations of maps between sets. In the category context, various statements in mathematics and theoretical physics turn into assertions about existence, uniqueness or universality of some morphisms. These arrows appear as elements of commutative diagrams which play the role of equations. Profoundly a high level of abstraction can be reached within this approach revealing initially hidden relations between various theories and models. This stipulates the importance of the search of category forms of fundamental principles of theoretical physics. Moreover, given such a form, one can suggest its dual counterpart. This is important in the context of the present work.

Rapid progress of quantum information theory [5] gave rise to developed classification of correlations between spatially separated physical systems [6]. There are two important (meta)physical principles in this field. These are the **LC** principle (Local Causality) and the **NS** (Non-Signalling) principle that prohibits communications by means of correlations [7]. The **LC** and **NS** principles are of unequal status. In fact, **LC** stems from a pre-quantum world view, whereas the validity of **NS** is still immune to any doubt. Usually, these principles are formulated in terms of some specific properties of common probability distributions for outcomes of measurements implemented in two causally disconnected space-time regions. All meaningful distributions must evidently obey the **NS** principle. At the same time, the **LC** principle let one distinguish between distributions with classical correlations of outcomes from those demonstrating quantum and hypothetical 'hyper-quantum' correlations.

Our aim is two-fold. The first is to formulate **LC** and **NS** principles as some commutative diagrams. Secondly, these diagrams must be 'inverted' in a due way providing dual forms of **LC** and **NS**.

The following notions and notations will be used: the set $\mathfrak{A}$ enumerates types of possible measurement implemented in the first space-time region, A is the set of possible outcomes which is assumed unique for all measurements from $\mathfrak{A}$. Similar sets $\mathfrak{B}$ and B are introduced for the second space-time region. A common distribution $p(*,* | \mathfrak{a}, \mathfrak{b})$ for measurements $\mathfrak{a} \in \mathfrak{A}$ and $\mathfrak{b} \in \mathfrak{B}$ gives a probability $p(a, b|\mathfrak{a}, \mathfrak{b})$ of outcomes $a \in A$ and $b \in B$. This distribution meets the condition **NS** if its marginal distributions do not depend on the measurement type which outcomes the summation has been made over, i.e.

$$p_A(* |\mathfrak{a}) = \sum_{b \in B} p(*, b|\mathfrak{a}, \mathfrak{b}) \qquad \text{and} \qquad p_B(* |\mathfrak{b}) = \sum_{a \in A} p(a,* |\mathfrak{a}, \mathfrak{b}) \tag{1}$$

This prevents controlling the probability distribution on A by the choice of $\mathfrak{b} \in \mathfrak{B}$ and the same for B and $\mathfrak{a} \in \mathfrak{A}$. In the opposite case such a control would have provided signalling between the considered space-time regions by means of statistical correlations of outcomes.

A distribution $p(*,* |\mathfrak{a}, \mathfrak{b})$ on $A \times B$ obeys **LC** if there exist equally indexed sets of distributions $\{p_A^{(i)}(* |\mathfrak{a})\}_{i \in I}$ on A and $\{p_B^{(i)}(* |\mathfrak{b})\}_{i \in I}$ on B along with a distribution q_i on I so that

$$p(a, b|\mathfrak{a}, \mathfrak{b}) = \sum_{i \in I} q_i p_A^{(i)}(a|\mathfrak{a}) p_B^{(i)}(b|\mathfrak{b}) \tag{2}$$

for all $a \in A$ and $b \in B$. Distributions of such a type are known to keep Bell's inequalities [7] and correspond to classical-type correlations between subsystems in the considered space-time regions. The **NS** principle follows naturally from **LC**.

2. NS and LC diagrams

It is appropriate to elaborate the category form of **NS** first. In fact, with the set of distributions $p(*,* |\mathfrak{a}, \mathfrak{b})$ for various $\mathfrak{a} \in \mathfrak{A}$ and $\mathfrak{b} \in \mathfrak{B}$ one can associate the map (morphism)

$$p: \mathfrak{A} \times \mathfrak{B} \to \mathcal{D}(A \times B), \tag{3}$$

where $\mathcal{D}: Set \to Set$ is the monad [2] of distributions in the category of sets, i.e. the endofunctor which produces for any set X the set $\mathcal{D}(X)$ of all probability distributions on X. Monad is the triple $\langle \mathcal{D}, \mu: \mathcal{D}^2 \to \mathcal{D}, \eta: 1_{Set} \to \mathcal{D} \rangle$. Natural transformation μ maps weighted collections of distributions over a set into averaged distributions; unit η maps any element of a set into the distribution localized on the element. The functorial nature of $\mathcal{D}$ associates to any map of sets $f: X \to Y$ a map $\mathcal{D}f: \mathcal{D}(X) \to \mathcal{D}(Y)$ by the rule $\mathcal{D}f: p \mapsto \mathcal{D}f(p)$, where

$$[\mathcal{D}f(p)](y) = \sum_{f(x)=y} p(x). \tag{4}$$

Introducing the projection maps $\pi_{\mathfrak{A}}: \mathfrak{A} \times \mathfrak{B} \to \mathfrak{A}$ and $\pi_{\mathfrak{B}}: \mathfrak{A} \times \mathfrak{B} \to \mathfrak{B}$ and applying (4) for projections $\pi_A: A \times B \to A$ and $\pi_B: A \times B \to B$, we construct the diagram

$$\begin{array}{ccccc} \mathfrak{B} & \xleftarrow{\pi_{\mathfrak{B}}} & \mathfrak{A}\times\mathfrak{B} & \xrightarrow{\pi_{\mathfrak{A}}} & \mathfrak{A} \\ {\scriptstyle p_B}\downarrow & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p_A} \\ \mathcal{D}(B) & \xleftarrow{\mathcal{D}\pi_B} & \mathcal{D}(A\times B) & \xrightarrow{\mathcal{D}\pi_A} & \mathcal{D}(A) \end{array} \quad . \tag{5}$$

The existence of *unique* morphisms p_A and p_B (the uniqueness is denoted by dotted arrows), which makes the diagram (5) commutative, is equivalent to the condition (1) since the morphisms $\mathcal{D}\pi_A$ and $\mathcal{D}\pi_B$ symbolize evaluations of the marginal distributions.

The uniqueness of p_A and p_B can be restated attaining the consistency with the category form of LC below. Note that commutativity of the right part of (5) is equivalent to the uniqueness of the morphism $\mathcal{D}\pi_A \cdot p: \mathfrak{A}\times\mathfrak{B} \to \mathcal{D}(A)$ factoring through $\pi_{\mathfrak{A}}$. One can identify p and $\pi_{\mathfrak{A}}$ with morphisms from the terminal (single-element) object $1 = \{0\}$ of Set. Then the mentioned assertion of uniqueness turns to be equivalent to the uniqueness of the dotted morphism in the commutative diagram

$$\begin{array}{ccc} 1 & \xrightarrow{p} & Hom(\mathfrak{A}\times\mathfrak{B}, \mathcal{D}(A\times B)) \\ {\scriptstyle \pi_{\mathfrak{A}}}\downarrow & & \downarrow{\scriptstyle Hom(1_{\mathfrak{A}\times\mathfrak{B}}, \mathcal{D}\pi_A)} \\ Hom(\mathfrak{A}\times\mathfrak{B}, \mathfrak{A}) & \dashrightarrow & Hom(\mathfrak{A}\times\mathfrak{B}, \mathcal{D}(A)) \end{array} \tag{6}$$

Here, the dotted morphism maps any arrow $f: \mathfrak{A}\times\mathfrak{B} \to \mathfrak{A}$ into $p_A \cdot f: \mathfrak{A}\times\mathfrak{B} \to \mathcal{D}(A)$, where p_A is a morphism from (5). A similar diagram can be constructed from the left part of (5). Combining the both diagrams into a single one by means of the standard notation of maps product $\langle *,* \rangle$, one can reduce the assertion of validity of **NS** for p to the *uniqueness* of the morphism which makes commutative the diagram

$$\begin{array}{ccc} 1 & \xrightarrow{p} & Hom(\mathfrak{A}\times\mathfrak{B}, \mathcal{D}(A\times B)) \\ {\scriptstyle 1_{\mathfrak{A}\times\mathfrak{B}}}\downarrow & & \downarrow{\scriptstyle Hom(1_{\mathfrak{A}\times\mathfrak{B}}, \langle \mathcal{D}\pi_A, \mathcal{D}\pi_B \rangle)} \\ Hom(\mathfrak{A}\times\mathfrak{B}, \mathfrak{A}\times\mathfrak{B}) & \dashrightarrow & Hom(\mathfrak{A}\times\mathfrak{B}, \mathcal{D}(A)\times\mathcal{D}(B)) \end{array} \tag{7}$$

The equality $\langle \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}} \rangle = 1_{\mathfrak{A}\times\mathfrak{B}}$ is used here; the dotted morphism maps $h: \mathfrak{A}\times\mathfrak{B} \to \mathfrak{A}\times\mathfrak{B}$ into $(p_A \times p_B) \cdot h: \mathfrak{A}\times\mathfrak{B} \to \mathcal{D}(A)\times\mathcal{D}(B)$.

Turning to **LC**, we note that the construction in the right part of (2) lets one assert the existence of the natural morphism

$$r: \mathcal{D}[Hom(\mathfrak{A}, \mathcal{D}(A)) \times Hom(\mathfrak{B}, \mathcal{D}(B))] \to Hom(\mathfrak{A}\times\mathfrak{B}, \mathcal{D}(A\times B)) \tag{8}$$

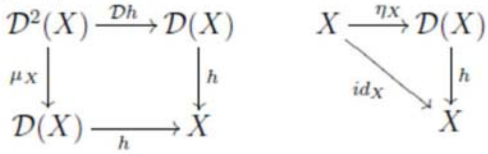
as a weighted sum of products of distributions. Hence, the presentability of p as the right part of (2) is equivalent to the existence of a morphism q from 1 into $\mathcal{D}[Hom(\mathfrak{A}, \mathcal{D}(A)) \times Hom(\mathfrak{B}, \mathcal{D}(B))]$ (i.e. the existence of an element of the last set) in the commutative diagram

$$\begin{array}{ccc} & & \mathcal{D}[Hom(\mathfrak{A}, \mathcal{D}(A)) \times Hom(\mathfrak{B}, \mathcal{D}(B))] \\ & \overset{q}{\nearrow} & \downarrow{\scriptstyle r} \\ 1 & \xrightarrow{p} & Hom(\mathfrak{A}\times\mathfrak{B}, \mathcal{D}(A\times B)) \end{array} \tag{9}$$

This diagram should be considered as a possibility to factor the morphism p through r. Had it been done for any p, the **LC** principle would have universally been valid. In fact, the lower part of the triangle diagram (9) coincides with the upper part of (7). Thus they can be combined into a single commutative diagram. It is worth to recall the logical implication **LC**⇒**NS**, i.e. (7) follows from (9).

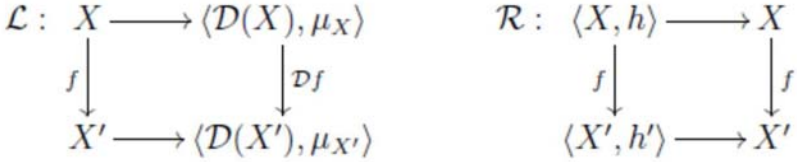
3. Duals of NS and LC

Quantum groups suggest the most convenient cite for studying mutually dual structures. Being compared, diagrams for these structures reveal substitution of objects in the corresponding vertices with co-objects and reversion and reversion of arrows. On the assumption of the same properties for duals of **NS** and **LC** one must suggest relevant co-objects. In the diagrams (5) and (9) sets stand in vertices. Some of them are modified by the $\mathcal{D}$-functor from the distribution monad $\langle \mathcal{D}, \mu, \eta \rangle$. We need its dual form. In the theorem by Eilenberg and Moore [2] a pair of adjoint functors $\mathcal{L}: Set \to Set^{\mathcal{D}}$ and $\mathcal{R}: Set^{\mathcal{D}} \to Set$ between categories of sets and $\mathcal{D}$-algebras. Any $\mathcal{D}$-algebra is a pair $\langle X, h: \mathcal{D}(X) \to X \rangle$ of a set X and a map h with properties



(10)

These insert convex structure into X. The convexity need not be mediated by linear structure in X. For instance, this is the case of $\mathcal{D}$-algebra $\langle \mathcal{D}(X), \mu_X \rangle$, where X is a set with no special structure. The functors $\mathcal{L}$ and $\mathcal{R}$ act as

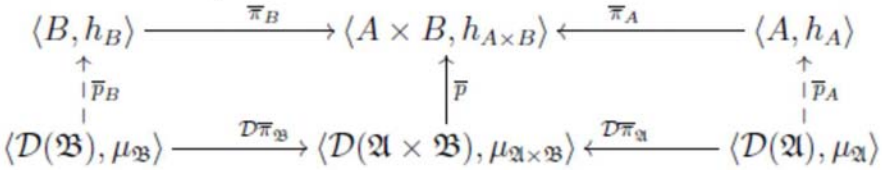


(11)

The adjunction quadruple $\langle \mathcal{L}, \mathcal{R}, \eta, \varepsilon \rangle$ (natural transformation $\varepsilon: \mathcal{L}\mathcal{R} \to 1_{Set^{\mathcal{D}}}$ is co-unit) gives rise to the distribution monad $\langle \mathcal{D}, \mu, \eta \rangle = \langle \mathcal{R}\mathcal{L}, \mathcal{R}\varepsilon\mathcal{L}, \eta \rangle$ in Set and to co-monad $\langle \overline{\mathcal{D}}, \overline{\mu}, \overline{\eta} \rangle = \langle \mathcal{L}\mathcal{R}, \mathcal{L}\eta\mathcal{R}, \varepsilon \rangle$ in $Set^{\mathcal{D}}$. By definition of the co-monad

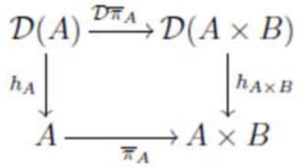
$$\overline{\mathcal{D}}(\langle X, h \rangle) = \langle \mathcal{D}(X), \mu_X \rangle. \tag{12}$$

Dual construction to (5) reverses arrows, places $\mathcal{D}$-algebras (convex sets) instead of ordinary sets and $\overline{\mathcal{D}}$-symbol instead of $\mathcal{D}$. By (12) we arrive at



(13)

Note that the sets A and B in this diagram must be convex. We need naturally constructed direct product of $\mathcal{D}$-algebras $\langle A, h_A \rangle$ and $\langle B, h_B \rangle$ (definition of $h_{A \times B}$ in terms of h_A and h_B) as well as horizontal morphisms of $\mathcal{D}$-algebras. Given an element $\{(a_i, b_i): p_i\}_{i \in I}$ in $\mathcal{D}(A \times B)$, one gets $\{a_i: p_i\}_{i \in I}$ in $\mathcal{D}(A)$ and $\{b_i: p_i\}_{i \in I}$ in $\mathcal{D}(B)$. These two are mapped to a in A and b in B by h_A and h_B respectively. The element (a, b) is interpreted as the image of $\{(a_i, b_i): p_i\}_{i \in I}$ by $h_{A \times B}$. Construction of $\overline{\pi}_A: \langle A, h_A \rangle \to \langle A \times B, h_{A \times B} \rangle$ is as follows. By assumption it must enter the commutative diagram



(14)

It is worth to specify the map $\mathcal{D}\overline{\pi}_A$ first. Assume

$$\mathcal{D}\overline{\pi}_A : \{a_i : p_i\}_{i\in I} \mapsto \{(a_i, b) : \tilde{p}_i\}_{i\in I, b\in B}. \tag{15}$$

Here $\tilde{p}_i$ is renormalized distribution homogeneous over B. Let $\overline{\pi}_A \doteq h_{A\times B} \cdot \mathcal{D}\overline{\pi}_A \cdot \eta_A$. The proof of its consistency is straightforward.

The dual of **NS** (13) can get the form similar to (7):

$$\begin{array}{ccc}
Hom(B, A\times B) & \dashrightarrow & Hom(\mathcal{D}(\mathfrak{B}), A\times B) \\
\uparrow{\scriptstyle \overline{\pi}_B} & & \uparrow{\scriptstyle Hom(\mathcal{D}\overline{\pi}_{\mathfrak{B}}, 1_{A\times B})} \\
1 & \xrightarrow{\overline{p}} & Hom(\mathcal{D}(\mathfrak{A}\times\mathfrak{B}), A\times B) \\
\downarrow{\scriptstyle \overline{\pi}_A} & & \downarrow{\scriptstyle Hom(\mathcal{D}\overline{\pi}_{\mathfrak{A}}, 1_{A\times B})} \\
Hom(A, A\times B) & \dashrightarrow & Hom(\mathcal{D}(\mathfrak{A}), A\times B)
\end{array} \tag{16}$$

For brevity, short notations are used here. Symbols of (convex) sets mean the corresponding $\mathcal{D}$-algebras. Hom-sets are collections of morphisms between $\mathcal{D}$-algebras. The arrow $\overline{p}: 1 \to Hom(\mathcal{D}(\mathfrak{A}\times\mathfrak{B}), A\times B)$ appears also in the following diagram that display the dual of **LC** principle:

$$\begin{array}{ccc}
 & & \mathcal{D}[Hom(\mathcal{D}(\mathfrak{A}), A)\times Hom(\mathcal{D}(\mathfrak{B}), B)] \\
 & \overset{\overline{q}}{\nearrow} & \downarrow{\scriptstyle \overline{r}} \\
1 & \xrightarrow{\overline{p}} & Hom(\mathcal{D}(\mathfrak{A}\times\mathfrak{B}), A\times B)
\end{array} \tag{17}$$

We direct arrows in (13) and (16) and all other diagrams from Gothic symbols labelling the sets of experimental settings to Latin symbols which stand for sets of outcomes. The last sets are convex and should be interpreted as collections of all possible mean outcome values. The arrow $\overline{r}$ has natural and unambiguous status in parallel with its counterpart in (9).

4. Discussion

It is worth to point the main distinction between the diagrams of **NS** and **LC** and their duals. The first pair place constraints onto maps from experimental settings $(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{A}\times\mathfrak{B}$ in causally disconnected regions to the set of probability distributions of outcomes. On the contrary, the duals of **NS** and **LC** deal with functions which maps distributions over experimental settings (incomplete knowledge of the last ones) onto mean outcome values.

The diagrams (16) and (17) encode some assertions about maps from $\mathcal{D}(\mathfrak{A}\times\mathfrak{B})$ to $A\times B$. The diagrams can still be interpreted in terms of conditional probability (3). That let one conclude upon a direct consideration that **LC**implies the validity of its dual form as well. Not the same for the principle **NS**. Relations between **NS**and its dual are worth of a separate close consideration.

The suggested approach is worth to be compared with that one from [8] which considers the **LC** principle and its violation from the view-point of sheaves [9]. The theory of sheaves uses actively category methods. Within the notations of the present work the main idea of [8] is the following. Pairs of measurements $\{\mathfrak{a}\in\mathfrak{A}, \mathfrak{b}\in\mathfrak{B}\}$ implemented in two space-time regions are interpreted as measurement *contexts*. To any context $\mathcal{C} = \{\mathfrak{a}, \mathfrak{b}\}$ the set of its *sections*, i.e. outcomes $\{a\in A, b\in B\}$, can be associated. This association $\mathcal{E}: \mathcal{C} \mapsto (A\times B)^{\mathcal{C}}$ may be considered as a contra-variant functor (presheaf) from the poset category of all subsets of $\mathfrak{A}\cup\mathfrak{B}$ ordered by inclusion into the category of sets. The functor $\mathcal{D}\cdot\mathcal{E}$ is also a presheaf. The elements of $\mathcal{D}\cdot\mathcal{E}(\mathcal{C})$, where $\mathcal{C} = \{\mathfrak{a}, \mathfrak{b}\}$, are the distributions $p(*, * \,|\mathfrak{a}, \mathfrak{b})$ from the present work. It is shown in [8] that the **LC** principle can be treated as the condition of the presheaf $\mathcal{D} \cdot \mathcal{E}$ being in fact a sheaf. The **NS** principle in the form (1) took part in this scheme as a selector of consistent collections of distributions. The phenomenon of quantum nonlocality, being a cause of **LC** violation, is a hurdle of turning $\mathcal{D} \cdot \mathcal{E}$ into a sheaf, i.e. there are consistent sets of distributions $p(*,* \,|\mathfrak{a},\mathfrak{b})$ for various possible contexts which can not be regarded as marginal distribution with respect to a global one $p(*,\ldots,* \,|\mathfrak{A} \cup \mathfrak{B})$.

The notions of sheaves are not used in the category-theoretic form of **NS** and **LC** suggested in the present work. Nevertheless, there is evidently a relation between the two forms of **LC** violation – impossibility for $\mathcal{D} \cdot \mathcal{E}$ to be a sheaf and impossibility to factor *any* morphism p through the standard morphism r in the diagram (9).

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