Fusion of Perturbed Defects in Conformal Field Theory

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Awarding institution:
King's College London

Title: Fusion of Perturbed Defects in Conformal Field Theory

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FUSION OF PERTURBED DEFECTS IN CONFORMAL FIELD THEORY

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SUBMITTED IN FULFILLMENT OF THE
REQUIREMENTS FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY IN MATHEMATICS

KING'S COLLEGE LONDON
JUNE 2011

KING'S COLLEGE LONDON
DEPARTMENT OF
MATHEMATICS

The undersigned hereby certify that they have read and recommend to the School of Natural and Mathematical Sciences for acceptance a thesis entitled "**Fusion of Perturbed Defects in Conformal Field Theory**" by **Dimitrios Manolopoulos** in fulfillment of the requirements for the degree of **Doctor of Philosophy in Mathematics**.

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KING'S COLLEGE LONDON

Date: **June 2011**

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Title: **Fusion of Perturbed Defects in Conformal Field Theory**

Department: **Mathematics**

Degree: **Ph.D.** Year: **2012**

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Στους γονείς μου
To my parents

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Abstract

The infinite-dimensional symmetry algebra of a conformal field theory (CFT), the Virasoro algebra, is generated by the holomorphic and anti-holomorphic part of the stress tensor. Besides such 'chiral symmetries' the CFT also has an integrable symmetry, that is, infinite families of commuting conserved charges. In this thesis a step towards combining these two symmetries into a single formalism is taken, by identifying integrable structures of a CFT through studying the representation category of the underlying chiral algebra. Then by introducing defects in the system, conserved charges can be constructed by perturbing certain conformal defects.

Starting from an abelian rigid braided monoidal category $\mathcal{C}$ one defines an abelian rigid monoidal category $\mathcal{C}_F$ which captures some aspects of perturbed conformal defects in two-dimensional CFT. Namely, for $\mathfrak{V}$ a rational vertex operator algebra one considers the charge-conjugation CFT constructed from $\mathfrak{V}$ (the Cardy case). Then $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$ and an object in $\mathcal{C}_F$ corresponds to a conformal defect condition together with a direction of perturbation. To each object in $\mathcal{C}_F$ one assigns a perturbed defect operator on the space of states of the CFT and then shows that the assignment factors through the Grothendieck ring of $\mathcal{C}_F$. This allows one to find functional relations between perturbed defect operators. Such relations are interesting because they contain information about the integrable structure of the CFT.

Acknowledgements

First and foremost, I would like to thank my supervisor, Ingo Runkel, for his excellent supervision and suggesting an interesting and tractable project. From the beginning of our collaboration it has been a pleasure to work alongside him. He always had the time to answer my questions and provide guidance whenever he felt that I didn't have the necessary background. Not only was he supportive with my work, but also when it came to personal matters, for this reason I think of him, not only as a supervisor, but as a friend.

I would also like to thank Andreas Recknagel and Gérard Watts for their full help and support when it came to applying for a Greek scholarship as well as for various issues during my PhD. I also gratefully acknowledge the 'Science and Technology Facilities Council' for their financial support, which enabled me to conduct the research presented in this thesis.

Furthermore, I would like to thank my family, my friends: Dionysis, Takis, and Apostolis as well as all my other friends here in London and in Greece, for their love and support throughout all these years. My flatmates, Nikos, Nasos, Fotis, Fanis and Zoi for making my stay here very enjoyable. Special thanks goes to my cousin Nadia for everything she did for me. I also thank my friend Pauline for checking my thesis for grammar and spelling errors.

Last but not least, I would like to thank Evi for her love, constant support and encouragement throughout the last year and a half of my PhD. She ensured that my focus remained solely on the thesis, and so it is to her that I owe its timely completion.

King's College London — Dimitrios Manolopoulos
February 28, 2012

Introduction and Summary

Conformal symmetry is a potent tool in the construction of two-dimensional conformal quantum field theories [BPZ84]. Their infinite-dimensional symmetry algebra, the Virasoro algebra, is generated by the modes of two conserved currents: the holomorphic and anti-holomorphic part of the stress tensor. Besides such 'chiral symmetries' obtained from conserved currents, in many examples the CFT also has an integrable symmetry, that is, infinite families of commuting conserved charges [BLZ96]. Present approaches to CFT tend to favour either the conformal or the integrable symmetry, and it seems worthwhile to eventually combine these two symmetries into a single formalism.

In this thesis, whose results have been published in a joint paper with Ingo Runkel in [MR09], we hope to take a step in this direction by continuing to develop the approach of [Ru08] which allows one to identify integrable structures of a CFT by studying the representation category of the chiral algebra. It is worth remarking that the idea to find questions about CFT that can be formulated on a purely categorical level, and that can then be investigated independent of whether there is an underlying CFT or not, has proved useful already in [FS03, FRS02-I] (the interested reader could consult [KR09] for a brief overview).

In [Ru08] families of conserved charges are constructed as perturbations of certain conformal defects. A conformal defect is a line of inhomogeneity on the world sheet of the CFT, that is, a line where the fields can have discontinuities or singularities. By putting a circular defect line on a cylinder we obtain the defect operator, a linear operator on the space of states. If one considers a particular class of conformal defects (so-called topological defects) and perturbs such a defect by a particular type of relevant defect field, one obtains a family of defect operators which still commute with $L_0 + \bar{L}_0$, the sum of the zero modes of the holomorphic and anti-holomorphic component of the stress tensor. Sometimes these perturbed defect operators obey functional relations. An example is provided by the non-unitary Lee-Yang CFT, the Virasoro minimal model of central charge $c = -22/5$. There, one obtains a family of operators $D(\lambda)$, $\lambda \in \mathbb{C}$, on the space of states of the model, which obey, for all $\lambda, \mu \in \mathbb{C}$,

$$[L_0 + \bar{L}_0, D(\lambda)] = 0 \quad , \quad [D(\lambda), D(\mu)] = 0 \quad , \quad D(e^{2\pi i/5}\lambda)\, D(e^{-2\pi i/5}\lambda) = \mathrm{id} + D(\lambda) \ . \tag{0.0.1}$$

The last relation above is closely linked to the description of the Lee-Yang model via the massless limit of factorising scattering and the thermodynamic Bethe Ansatz, see e.g. the review [DDT07]. This example illustrates that the functional relations obeyed by perturbed defect operators encode at least part of the integrable structure of the model. In fact, the defect operator in (0.0.1) (and more generally those for the $M_{2,2m+1}$ minimal models) can be understood as certain linear combinations of the chiral operators which were constructed in [BLZ96] to capture the integrable structure of these models.

This thesis presents a categorical structure that captures some aspects of perturbed defect operators, and in particular allows one to find functional relations such as the one in (0.0.1). We work in rational conformal field theory, so that the holomorphic fields of the model form a rational[1] vertex operator algebra $\mathfrak{V}$. We consider the 'Cardy case' CFT constructed from $\mathfrak{V}$, namely the CFT with charge-conjugation modular invariant - the thesis conclusions 6.3 contain a brief comment on how to extend the formalism to general rational CFTs. In the Cardy case the defects are

[1] By 'rational' we mean that the vertex operator algebra satisfies the conditions in [Hu05, Sect. 1].

labeled by representations of $\mathfrak{V}$. Denote $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$. The category describing the properties of perturbed defects is called $\mathcal{C}_F$. It is an enlargement of $\mathcal{C}$ which depends on a choice of object $F \in \mathcal{C}$. Roughly speaking, F is the representation of $\mathfrak{V}$ from which the perturbing field is taken, and the objects of $\mathcal{C}_F$ are pairs of an unperturbed defect together with a direction of perturbation.

Concretely, the objects in $\mathcal{C}_F$ are pairs (R, f) where $R \in \mathcal{C}$ and $f \colon F \otimes R \to R$ is a morphism in $\mathcal{C}$. The morphisms in $\mathcal{C}_F$ are those morphisms in $\mathcal{C}$ which make the obvious diagram commute (see Def. 4.1.1). If, in addition to being monoidal, the category $\mathcal{C}$ is also abelian rigid and braided (as it would be for $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$ with $\mathfrak{V}$ a rational vertex operator algebra), then $\mathcal{C}_F$ is an abelian rigid monoidal category (Thm. 4.3.2). In particular, the Grothendieck ring $K_0(\mathcal{C}_F)$ is well-defined. However, $\mathcal{C}_F$ is typically not braided. We will see in the example of the Lee-Yang model that there can be simple objects (U, f) and (V, g) in $\mathcal{C}_F$ such that $(U, f) \,\hat{\otimes}(V, g) \ncong (V, g) \,\hat{\otimes}(U, f)$, where $\hat{\otimes}$ denotes the tensor product in $\mathcal{C}_F$.

If $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$, we can assign a perturbed defect operator $D[(R, f)]$ to an object $(R, f) \in \mathcal{C}_F$, provided certain integrals and sums converge (see Sect. 5.2). Suppose that for two objects $(R, f), (S, g) \in \mathcal{C}_F$ the perturbed defect operators exist. Then the tensor product in $\mathcal{C}_F$ is compatible with composition of defect operators, $D[(R, f) \,\hat{\otimes}(S, g)] = D[(R, f)]D[(S, g)]$ (Thm. 5.2.1), and $D[(R, f)] = D[(S, g)]$ if (R, f) and (S, g) represent the same class in the Grothendieck ring $K_0(\mathcal{C}_F)$ (Cor. 5.2.2). Thus, identities of the form $[(A, a)] \cdot [(B, b)] = [(C_1, c_1)] + \cdots + [(C_n, c_n)]$ in $K_0(\mathcal{C}_F)$ will give rise to functional relations among the defect operators, such as the one quoted in (0.0.1) (see Chap. 6 for the Lee-Yang example).

The category $\mathcal{C}_F$ has similarities to categorical structures that appear in the treatment of defects in other contexts. In B-twisted $\mathcal{N} = 2$ supersymmetric Landau-Ginzburg models, boundary conditions [KL03, BHLS06, Laz05] and defects [BRo07] can be described by so-called matrix factorisations. There, one considers a category whose objects are pairs: a $\mathbb{Z}_2$-graded free module M over a polynomial ring and an odd morphism $f \colon M \to M$, so that $f \circ f$ takes a prescribed value. The morphisms of this category have to make the same diagram commute as those of $\mathcal{C}_F$. And as in $\mathcal{C}_F$, the module M can be interpreted as a defect in an unperturbed theory, and f as a perturbation. However, in the context of matrix factorisations one passes to a homotopy category, which is something we do not do for $\mathcal{C}_F$.

A more direct link comes from integrable lattice models. In one approach to these models, one uses the representation theory of a quantum affine algebra to construct families of commuting transfer matrices. The decomposition of tensor products of representations of the quantum affine algebra gives rise to functional relations among the transfer matrices [KNS94, RW, Ko03]. The category of finite-dimensional representations of a quantum affine algebra [CP91] shares a number of features with the category $\mathcal{C}_F$. For example, the tensor product of simple objects tends to be simple itself, except at specific points in the parameter space, where the tensor product is the middle term in a non-split exact sequence. To make the similarity a little more concrete, in App. A we point out that the evaluation representations of $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$ can be thought of as a full subcategory of $\mathcal{C}_F$ for appropriate $\mathcal{C}$ and F.

This thesis is organized as follows. In Chapter 1 we give a short review of CFT. In particular we review only those areas of CFT that will be relevant later on. In Chapter 2 we introduce the concept of defects and defect operators and we review some of their properties as well as their fusion rules. The machinery of topological field theory is discussed in Chapter 3, where we start by defining cobordisms and work our way up to define an n-dimensional TFT and at the end a 3-dimensional extended TFT. These first three chapters give the necessary background in order to describe the main results of the thesis, which are given in Chapters 4–6. More concretely, in Chapter 4 we introduce the category $\mathcal{C}_F$ and study its properties. In this section we make no reference to conformal field theory or vertex operator algebras. The relation of $\mathcal{C}_F$ to defect operators in conformal field theory is described in Chapter 4. There, we also show that the assignment of defect operators to objects in $\mathcal{C}_F$ factors through the Grothendieck ring of $\mathcal{C}_F$. In Chapter 6 we study the Lee-Yang Virasoro minimal model conformal field theory in some detail.

Chapter 1

Conformal Field Theory

In this Chapter we give a short introduction to Conformal Field Theory (CFT). However, it is beyond the scope of this thesis to present a full summary of CFT. Conformal field theory is a highly developed subject with many excellent reviews and textbooks available. A selection recommended by the author, in alphabetical order, is

[ASG89] An introduction by Alvarez-Gaume, Sierra and Gomez, written with an emphasis on the connection to knots and quantum groups.

[BYB] The book by Di Francesco, Mathieu and Sénéchal, which develops CFT from first principles. The treatment is self-contained, pedagogical, exhaustive and includes background material on QFT, statistical mechanics, Lie and affine Lie algebras.

[Ca08] Lectures given at Les Houches (2008) by John Cardy.

[Gab99] An overview of CFT centered on the role of the symmetry generating chiral algebra by Matthias Gaberdiel.

[Gin88] Lectures given at Les Houches (1988) by Paul Ginsparg.

In the following sections, an introduction is given to those areas of CFT that are most relevant to the current thesis. In some cases we just state the results since they are considered as standard in the literature and the readers may refer themselves to the recommendations mentioned above, for further details.

1.1 Conformal Invariance in Two Dimensions

Conformal field theories in two dimensions are Euclidean QFTs whose symmetry group contains, in addition to the Euclidean symmetries, local conformal transformations, i.e. transformations that preserve angles but not necessarily lengths. Indeed, in two dimensions there exists an infinite variety of coordinate transformations that, although not everywhere well defined, are locally conformal and they are holomorphic mappings from the complex plane to itself. The local conformal symmetry is of special importance in two dimensions since the corresponding symmetry algebra is infinite-dimensional (and in certain cases, e.g. Rational CFTs, see Sect. 1.6, organizes the Hilbert space of the quantum theory into finitely many representations). As a consequence, two-dimensional CFTs have an infinite number of conserved quantities, and are completely solvable by symmetry considerations alone.

Consider now a flat metric $g_{\mu\nu}$ on a space-time manifold $\mathscr{M}$.

Definition 1.1.1. A *conformal transformation* of the coordinates is an invertible mapping $x^\mu \mapsto x'^\mu$, that leaves the metric tensor invariant

$$g'_{\mu\nu}(x') = \Omega(x) g_{\mu\nu}(x) \ , \tag{1.1.1}$$

up to a scale factor $\Omega(x)$, called the *conformal factor*.

We will restrict ourselves to two dimensional Euclidean space with a metric $g_{\mu\nu} = \text{diag}(1,1)$. The set of all conformal transformations forms the *conformal group* which is isomorphic to $SO(3,1)$. For an infinitesimal transformation $x^\mu \mapsto x'^\mu = x^\mu + \epsilon^\mu(x)$ to be conformal, Definition 1.1.1 implies

$$\delta g_{\mu\nu} = \Omega(x) g_{\mu\nu} = 2\partial_{(\mu}\epsilon_{\nu)} \ . \tag{1.1.2}$$

The factor $\Omega(x)$ is determined by taking traces

$$\Omega(x) = \partial_\mu \epsilon^\mu \ . \tag{1.1.3}$$

Combining equations (1.1.2) and (1.1.3) we get

$$\partial_{(\mu}\epsilon_{\nu)} = \frac{1}{2}\partial_\rho \epsilon^\rho g_{\mu\nu} \ . \tag{1.1.4}$$

Equations (1.1.4) are the *Cauchy-Riemann equations* $\partial_1\epsilon_1 = \partial_2\epsilon_2$ and $\partial_1\epsilon_2 = -\partial_2\epsilon_1$. Therefore, if we identify the two dimensional Euclidean space with the complex plane we may write

$$\epsilon(z) = \epsilon^1 + i\epsilon^2 \ , \quad \bar{\epsilon}(\bar{z}) = \bar{\epsilon}^1 - i\bar{\epsilon}^2 \ , \tag{1.1.5}$$

in the complex coordinates $z = x + iy$ and $\bar{z} = x - iy$. The metric tensor in terms of $z, \bar{z}$ is given by

$$g_{\alpha\beta} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \ , \quad g^{\alpha\beta} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} \ , \tag{1.1.6}$$

where the indices α, β take the values z and $\bar{z}$, in that order. In this language, the holomorphic Cauchy-Riemann equations become

$$\partial_{\bar{z}} w(z, \bar{z}) = 0 \ , \tag{1.1.7}$$

whose solution is any holomorphic mapping $z \mapsto w(z) = z + \epsilon(z)$. Analytic functions automatically preserve angles and we see that there are infinitely many independent such transformations.

Remark 1.1.1. If we extend the Cartesian coordinates (x, y) to the complex plane, then the variables z and $\bar{z}$ are independent and $\bar{z}$ is not the complex conjugate of z, but rather a complex coordinate. However, it should be kept in mind that the physical space is the two-dimensional submanifold defined by $z^* = \bar{z}$.

Everything we have said up to now is purely local, we have not yet imposed any conditions for the conformal transformations to be everywhere well defined and invertible. Strictly speaking, in order to form a group, the mappings must be invertible and must map the whole plane to itself (more precisely the Riemann sphere). One, therefore, must distinguish *global conformal transformations*, which satisfy these requirements, from the local ones, which are not everywhere well defined. The group of conformal transformations on the Riemann sphere is finite dimensional and consists only of Möbius transformations

$$z \mapsto \frac{az+b}{cz+d} \ , \quad ad - bc = 1 \ , \tag{1.1.8}$$

where $a, b, c, d \in \mathbb{C}$. To each of these mappings we can associate the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \ . \tag{1.1.9}$$

We easily see that the composition of two maps corresponds to matrix multiplication and the condition $ad - bc = 1$ to $\det A = 1$. Therefore, the *global conformal group* in two dimensions is isomorphic to the Lie group $SL(2, \mathbb{C})/\mathbb{Z}_2$ and it is finite dimensional.

To the fields $\phi(z, \bar{z})$ in the theory we can associate a scaling dimension Δ and a spin s. Given such a field, we define the *holomorphic conformal dimension* h and its antiholomorphic counterpart $\bar{h}$ as

$$h = \frac{1}{2}(\Delta + s) \ , \quad \bar{h} = \frac{1}{2}(\Delta - s) \ . \tag{1.1.10}$$

Every conformal transformation $z = w(z)$ looks locally like a combined rescaling and rotation. The CFT will contain some fields, called *primary fields* which can only see this local behaviour, i.e. whose transformation properties depend only on the first derivative of w.

Definition 1.1.2. A field $\phi(z, \bar{z})$ that under *any local conformal transformations* $z \mapsto w(z), \bar{z} \mapsto \bar{w}(\bar{z})$, transforms as

$$\phi'(w, \bar{w}) = \left(\frac{dw}{dz}\right)^{-h} \left(\frac{d\bar{w}}{d\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}) \ , \tag{1.1.11}$$

is called a *primary field*. If $\phi(z, \bar{z})$, under *global conformal transformations*, transforms as in (1.1.11), then it is called a *quasi-primary field*.

The infinitesimal version of (1.1.11), under the conformal mapping $z \mapsto z + \epsilon(z)$ and $\bar{z} \mapsto \bar{z} + \bar{\epsilon}(\bar{z})$, is

$$\delta_{\epsilon,\bar{\epsilon}}\phi(z, \bar{z}) = \left(h\partial_z\epsilon + \epsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\epsilon} + \bar{\epsilon}\partial_{\bar{z}}\right) \phi(z, \bar{z}) \ . \tag{1.1.12}$$

1.2 The Stress Tensor and Ward Identities

We would now like to explore the consequences of conformal invariance for correlation functions in a fixed domain (usually the entire complex plane). It is necessary to consider transformations which are not conformal everywhere, i.e. local conformal transformations. This brings in the *energy-momentum tensor* (or *stress-energy tensor*). The name energy-momentum tensor refers to Minkowski space-time while the name stress-energy tensor refers to the elastic properties of materials. In a slight abuse of notation we will use both names. In a classical field theory it is defined as the Noether current which is conserved and symmetric, in response of the action[1] S to a general infinitesimal transformation $\epsilon^\mu(x)$,

$$\delta S = \int d^2x \; T^{\mu\nu} \partial_\mu \epsilon_\nu = \int d^2x \; T^{\mu\nu} \partial_{(\mu} \epsilon_{\nu)} \; . \tag{1.2.1}$$

This is valid even if the equations of motion are not satisfied. Then equations (1.1.2) and (1.1.3) imply that the corresponding variation of the action under an infinitesimal conformal transformation is

$$\delta S = \int d^2x \; T^\mu{}_\mu \Omega(x) = 0 \; , \tag{1.2.2}$$

where $\Omega(x) = \partial_\nu \epsilon^\nu$ is not an arbitrary function. The tracelessness of $T^{\mu\nu}$ then implies the invariance of the action under conformal transformations.

In complex coordinates $(z, \bar{z})$, the components of $T_{\alpha\beta}$ are

$$T_{zz} = \frac{1}{4}\left(T_{11} - 2iT_{21} - T_{22}\right) \; , \quad T_{\bar{z}\bar{z}} = \frac{1}{4}\left(T_{11} + 2iT_{21} - T_{22}\right) \; , \quad T_{z\bar{z}} = T_{\bar{z}z} = 0 \; . \tag{1.2.3}$$

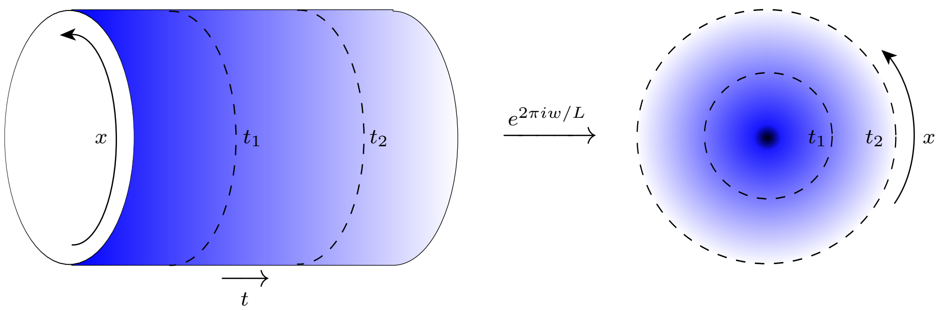
The conservation law $g^{\alpha\gamma}\partial_\gamma T_{\alpha\beta} = 0$, implies that

$$\bar{\partial} T = \partial \bar{T} = 0 \; , \tag{1.2.4}$$

where, $\bar{\partial} \equiv \partial_{\bar{z}}, \partial \equiv \partial_z$. Therefore, the energy-momentum tensor splits into a holomorphic and an antiholomorphic part and it is customary to write these parts as $T \equiv T(z) \equiv T_{zz}$ and $\bar{T} \equiv \bar{T}(\bar{z}) \equiv T_{\bar{z}\bar{z}}$, respectively.

[1] Even though we will never explicitly need an action in this thesis, it is sometimes useful to think that there is a path integral formulation of the theory.

We now use *radial quantization* on the complex plane. Consider an infinite cylinder of circumference L, with the time $t \in \mathbb{R}$, running along the "flat" direction of the cylinder and space being compactified with a coordinate $x \in [0, L]$, the points $(0, t)$ and (L, t) being identified. If we continue to Euclidean space, the cylinder is described by a single coordinate $w = x + it$ (or $\bar{w} = x - it$). We then "explode" the cylinder onto the complex plane (or rather, the Riemann sphere) via the mapping



(1.2.5)

The remote past ($t \to -\infty$) is situated at the origin $z = 0$, whereas the remote future ($t \to +\infty$) lies on the point at infinity on the Riemann sphere.

With the decomposition (1.2.4) of the energy-momentum tensor into holomorphic and antiholomorphic parts at hand, we can now define in radial quantization the *conserved charge*

$$Q = \frac{1}{2\pi i} \oint \left(dz\ T(z)\epsilon(z) + d\bar{z}\ \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) \right) \ , \tag{1.2.6}$$

from the *conserved current* $J^{\alpha}(z, \bar{z}) \equiv T^{\alpha\beta}(z, \bar{z})\epsilon_{\beta}(z, \bar{z}) = T(z)\epsilon(z) + \bar{T}(\bar{z})\bar{\epsilon}(\bar{z})$. The line integral is performed over some circle of fixed radius and our sign conventions are such that both the dz and the $d\bar{z}$ integrations are taken in the counter-clockwise sense (hence the symbol $\oint$). Note that (1.2.6) is a formal expression that cannot be evaluated until we specify what other fields lie inside the contour.

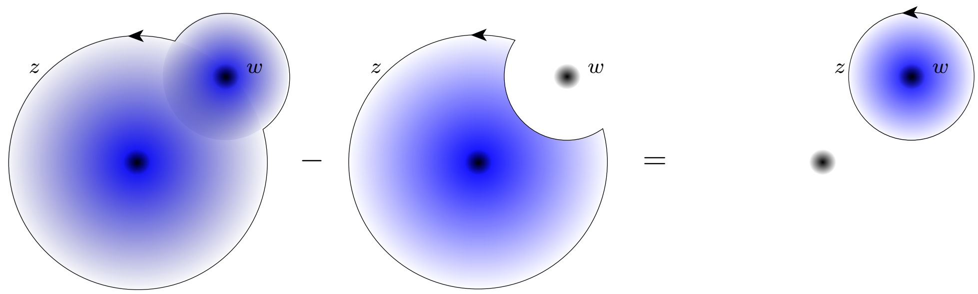
The variation of a primary field $\phi(w, \bar{w})$, is given by the *equal-time commutator* with the charge Q

$$\delta_{\epsilon,\bar{\epsilon}}\phi(w, \bar{w}) = [Q, \phi(w, \bar{w})] = \frac{1}{2\pi i} \oint \left[dz\ T(z)\epsilon(z) + d\bar{z}\ \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}), \phi(w, \bar{w}) \right] \ . \tag{1.2.7}$$

Now products of two operators $\mathcal{O}_1(z)\mathcal{O}_2(w)$, in Euclidean space quantization are only defined for $|z| > |w|$. Thus, we define the *radial-order operator*

$$\varrho\left(\mathcal{O}_1(z)\mathcal{O}_2(w)\right) := \begin{cases} \mathcal{O}_1(z)\mathcal{O}_2(w) \ , & \text{if } |z| > |w| \\ \mathcal{O}_2(w)\mathcal{O}_1(z) \ , & \text{if } |z| < |w| \end{cases} \ . \tag{1.2.8}$$

This allows us to define the meaning of the commutators in equation (1.2.7). Consider now the following pictorial equation



(1.2.9)

In this equation we have represented the contour integrations that we need to perform in order to evaluate the commutator in (1.2.7). We see that the difference combines into a single integration about a contour drawn tightly around the point w. Then we can write the equal-time commutator $[\mathcal{O}_1(z), \mathcal{O}_2(w)]$ as a contour integral around the point w, therefore (1.2.7) becomes

$$\begin{aligned} \delta_{\epsilon,\bar\epsilon}\phi(w,\bar w) &= \frac{1}{2\pi i}\left(\oint_{|z|>|w|} - \oint_{|z|<|w|}\right)\left\{dz\ \varrho\left(T(z)\phi(w,\bar w)\right)\epsilon(z) + d\bar z\ \varrho\left(\bar T(\bar z)\phi(w,\bar w)\right)\bar\epsilon(\bar z)\right\} \\ &= \frac{1}{2\pi i}\oint_w \left\{dz\ \varrho\left(T(z)\phi(w,\bar w)\right)\epsilon(z) + d\bar z\ \varrho\left(\bar T(\bar z)\phi(w,\bar w)\right)\bar\epsilon(\bar z)\right\} \\ &= \left(h\partial\epsilon + \epsilon\partial + \bar h\bar\partial\bar\epsilon + \bar\epsilon\bar\partial\right)\phi(w,\bar w) \ , \end{aligned} \tag{1.2.10}$$

where in the last line we have substituted the desired result, equation (1.1.12). Inserting the holomorphic and antiholomorphic parts of (1.2.10), separately in a correlator and using Cauchy's formula one can deduce the *conformal Ward identity*

$$\boxed{\begin{aligned} \langle T(z)\phi_1(w_1,\bar w_1)...\phi_n(w_n,\bar w_n)\rangle = \sum_{j=1}^{n}\left(\frac{h}{(z-w_j)^2} + \frac{1}{z-w_j}\partial_{w_j}\right) \\ \cdot\langle\phi_1(w_1,\bar w_1)...\phi_n(w_n,\bar w_n)\rangle + \text{reg}(z) \end{aligned}} \ , \tag{1.2.11}$$

where reg(z) is a regular function on the complex plane. A similar relation holds for $\bar{T}(\bar{z})$. In particular we see that the *operator product expansion* (OPE) of the stress tensor with a primary bulk field is

$$\boxed{T(z)\phi(w,\bar{w}) = \left(\frac{h}{(z-w)^2} + \frac{\partial}{z-w}\right)\phi(w,\bar{w}) + \text{reg}(z-w)} \ , \tag{1.2.12}$$

with a similar expression for $\bar{T}(\bar{z})$. The most general OPE for T (similarly for $\bar{T}$), consistent with associativity is

$$\boxed{T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{\partial}{z-w}T(w) + \text{reg}(z-w)} \ . \tag{1.2.13}$$

The constant c is called the *central charge* and fixes the properties of the CFT. The OPE of T with $\bar{T}$ has no poles. A consequence of (1.2.13) is the transformation behaviour of $T(z)$ under a conformal map $z \mapsto w(z)$

$$T(z) = \left(\frac{dw}{dz}\right)^2 T(w) + \frac{c}{12}\{w; z\} \ , \tag{1.2.14}$$

where $\{w; z\} := \frac{w'''(z)}{w'(z)} - \frac{3}{2}\left(\frac{w''(z)}{w'(z)}\right)^2$, is the *Schwarzian derivative*. Thus, we see that the energy momentum tensor is *not* a primary field. However, the Schwarzian derivative of (1.1.8) vanishes. This needs to be so, since $T(z)$ is a quasi-primary field.

In two dimensional CFTs, we can always take a basis of quasi-primary fields ϕ_i with fixed conformal weight. If we normalize their 2-point functions as

$$\langle \phi_i(z,\bar{z})\phi_j(w,\bar{w})\rangle = \frac{\delta_{ij}}{(z-w)^{2h_i}(\bar{z}-\bar{w})^{2\bar{h}_i}} \ , \tag{1.2.15}$$

then the OPE of two such fields will be of the form

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) \sim \sum_k C_{ij}{}^k (z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\phi_k(w,\bar{w}) \ , \tag{1.2.16}$$

where $C_{ij}{}^k$, are the *operator product coefficients* and are symmetric in i, j, k. In the following section we will see what is the exact form of the OPE $\phi_i(z,\bar{z})\phi_j(w,\bar{w})$, but in order to do that we need to know all the primary and descendant fields of the theory. This is done via the Hilbert space formulation of CFT.

1.3 Hilbert Space Formulation

A 2D CFT is determined by the following data:

♣ A *space of states*[2] $\mathscr{H}$, a $\mathbb{C}$-vector space, as well as, a *space of fields* $\mathscr{F}$, an $\mathcal{S}$-graded vector space $\mathscr{F} = \bigoplus_{\Delta\in\mathcal{S}} \mathscr{F}^{(\Delta)}$, with $\mathcal{S}$, the *spectrum*, a discrete subset of $\mathbb{R}$ and $0 < \dim \mathscr{F}^{(\Delta)} < \infty$.

♠ Its *correlation functions*, which are defined for collections of vectors in $\mathscr{F}$, together with an isomorphism $\iota\colon \mathscr{F} \to \mathscr{H}$, the *state-field correspondence*, in the sense that a field inserted at a point can be thought of as a state and vice versa.

As we have seen, two-dimensional CFTs contain an infinite variety of coordinate transformations that although not everywhere well defined, are locally conformal and they are holomorphic mappings from the complex plane to itself. The corresponding infinite-dimensional symmetry algebra of the CFT is related to a preferred subspace $\mathscr{F}_0$ of $\mathscr{F}$, that is characterised by the property that it only allows holomorphic dependance of the coordinates for the correlation functions, see [Gab99, Sect. 2.1] for example.

The correlation functions of the theory determine the OPE of the conformal fields, as one can see from (1.2.15) and (1.2.16) for example. In turn, the OPE of two conformal fields is given in terms of a sum of single fields as in (1.2.16). Thus, we see that the OPE defines a certain product on the fields via the operator product coefficients ${C_{ij}}^k$, which are the only non-trivial input in the OPE. It is, therefore, the operator product coefficients that force the product to involve the complex parameters z_i in a non-trivial way and hence, it does not directly define an algebra (in the appropriate sense); the resulting structure is a *vertex (operator) algebra*, $\mathfrak{V}$.

[2]May or may not be a Hilbert space, but it will be clear from the context. For example, as we will see later on, the space of states of the Lee-Yang model is not a Hilbert space, as the inner product is not positive-definite.

The OPE is associative and if we consider the case of two holomorphic fields $\phi_1, \phi_2 \in \mathscr{F}_0$, then the associativity of the OPE implies that the states in $\mathscr{F}_0$ form a *representation* of $\mathfrak{V}$. The same also holds for the vertex operator algebra associated to the anti-holomorphic fields and one can decompose the whole space $\mathscr{F}$ (or $\mathscr{H}$) as

$$\mathscr{H} = \bigoplus_{i,\bar{\jmath}\in\mathcal{I}} \left(R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}}\right)^{\oplus M_{i\bar{\jmath}}} \ , \tag{1.3.1}$$

where $\mathcal{I}$ denotes the set indexing the irreducible representations of $\mathfrak{V}$, $\{R_i \mid i \in \mathcal{I}\}$ the corresponding representations and $M_{i\bar{\jmath}} \in \mathbb{N}$ denotes the multiplicity with which the tensor product $R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}}$ occurs in $\mathscr{H}$.

Note that the operator formalism distinguishes a time direction from a space direction and thus we will work in radial quantization as we did in the previous section.

We must also assume the existence of a vacuum state $|0\rangle \in \mathscr{H}$ upon which the Hilbert space is constructed. In free field theories, the vacuum may be defined as the state annihilated by the positive frequency part of the field [BYB, Sect. 2.1 & 6.1.1].

Remark 1.3.1. To be precise we should call $|0\rangle$ the $\mathfrak{sl}(2)$-invariant vacuum, since e.g. for a non-unitary theory on a cylinder, it is not the state of lowest energy and thus not the real vacuum. It will always be clear from the context whether "vacuum" refers to the state of lowest energy or the $\mathfrak{sl}(2)$-invariant state $|0\rangle$. Moreover, the expressions, *correlation function*, *n-point function*, *amplitude* and *vacuum-expectation value* all refer to the (radially ordered) vacuum-expectation value $\langle 0| \dots |0\rangle$ with respect to the $\mathfrak{sl}(2)$-invariant vacuum.

For an interacting field $\phi \in \mathscr{F}$, we assume that the Hilbert space is the same as for a free field, except that the energy eigenstates are different [BYB, Sect. 6.1.1]. The timescales over which interactions happen are extremely short. The scattering (interaction) process takes place during a short time interval around some particular time t with $-\infty \ll t \ll +\infty$. Long before t, the incoming particles evolve independently and freely. In other words, we suppose that the interaction decreases (and eventually vanishes) as $t \to \pm\infty$ and that the asymptotic field $\phi_{\text{in}} \propto \lim_{t\to-\infty}\phi(x,t)$ is free. Within radial quantization, this asymptotic field reduces to a single operator, which upon acting on $|0\rangle$, creates a single asymptotic "in" state

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0}\phi(z,\bar{z})|0\rangle \ . \tag{1.3.2}$$

This is just the state-field correspondence, mentioned in ♠ above. Later on in this section, we will use this correspondence to lift the representation properties of the fields, onto the states. In this way, we will see that $\mathscr{H}$ can be decomposed into a direct sum of (highest weight) representations of the underlying symmetry algebra.

In this Hilbert space we must also define an inner product, which we do indirectly by defining an "out" state, together with the action of Hermitian conjugation on conformal fields. In radial quantization this can be done via the mapping $z \to 1/z^*$. This almost justifies the following definition of Hermitian conjugation on the real surface $\bar{z} = z^*$ (recall Remark 1.1.1).

$$\phi(z,\bar{z})^\dagger = \bar{z}^{-2h}z^{-2\bar{h}}\phi\left(\frac{1}{\bar{z}},\frac{1}{z}\right) \ , \tag{1.3.3}$$

where ϕ is a quasi-primary field of dimension $(h,\bar{h})$. Out-states then have the form

$$\begin{aligned}\langle\phi_{\text{out}}| &= \lim_{z,\bar{z}\to 0}\langle 0|\phi_{\text{in}}\left(\frac{1}{\bar{z}},\frac{1}{z}\right)\bar{z}^{-2h}z^{-2\bar{h}} \\ &= \lim_{z,\bar{z}\to 0}\langle 0|\phi_{\text{in}}(z,\bar{z})^\dagger \\ &= \lim_{z,\bar{z}\to 0}\left(\phi_{\text{in}}(z,\bar{z})|0\rangle\right)^\dagger \\ &= \left(|\phi_{\text{in}}\rangle\right)^\dagger \ .\end{aligned} \tag{1.3.4}$$

Then the inner product on $\mathscr{H}$ is

$$\begin{aligned}\langle\phi_{\text{out}}|\phi_{\text{in}}\rangle &= \lim_{\substack{z,\bar{z}\to 0\\ w,\bar{w}\to 0}}\langle 0|\phi(z,\bar{z})^\dagger\phi(w,\bar{w})|0\rangle \\ &= \lim_{\substack{z,\bar{z}\to 0\\ w,\bar{w}\to 0}}\bar{z}^{-2h}z^{-2\bar{h}}\left\langle 0\left|\phi\left(\frac{1}{\bar{z}},\frac{1}{z}\right)\phi(w,\bar{w})\right|0\right\rangle \\ &= \lim_{\zeta,\bar{\zeta}\to\infty}\bar{\zeta}^{2h}\zeta^{2\bar{h}}\langle 0|\phi(\bar{\zeta},\zeta)\phi(0,0)|0\rangle \ .\end{aligned} \tag{1.3.5}$$

Remark 1.3.2. Note, that according to the conformal two-point function

$$\left\langle\phi_1(\bar{\zeta},\zeta)\phi_2(0,0)\right\rangle = \frac{C_{12}}{\zeta^{2h}\bar{\zeta}^{2\bar{h}}} \ ,$$

the last expression in (1.3.5) is independent of ζ and this justifies the prefactors appearing in (1.3.3). If they were absent, the inner product $\langle\phi_{\text{out}}|\phi_{\text{in}}\rangle$ would not have been well defined as $\zeta\to\infty$. Note also that the passage from a vacuum expectation value to a correlator in the last equation is correct since the operators are already time-ordered within radial quantization. The first one is associated with $t\to\infty$ and the second one with $t\to-\infty$.

We can now define the action of the stress tensor T and its antiholomorphic counterpart $\bar{T}$ on the Hilbert space $\mathscr{H}$, via their mode expansion

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \ , \quad \bar{T}(\bar{z}) = \sum_{n\in\mathbb{Z}} \bar{z}^{-n-2}\bar{L}_n \ . \tag{1.3.6}$$

The exponent $-n-2$ in (1.3.6) is chosen so that for the scale change $z\to z/\lambda$, under which $T(z)\to\lambda^2 T(z/\lambda)$, we have $L_{-n}\to\lambda^n L_{-n}$. The operators $L_{-n},\bar{L}_{-n}$, thus have scaling dimension n. Equation (1.3.6) is formally inverted by the relations

$$L_n = \frac{1}{2\pi i}\oint dz\ z^{n+1}T(z) \ , \quad \bar{L}_n = \frac{1}{2\pi i}\oint d\bar{z}\ \bar{z}^{n+1}\bar{T}(\bar{z}) \ , \quad n\in\mathbb{Z} \ . \tag{1.3.7}$$

From (1.2.13) one can deduce that the modes fulfil the *Virasoro algebra*

$$\boxed{\begin{aligned} &[L_n,L_m] = (n-m)L_{n+m} + \tfrac{c}{12}\left(n^3-n\right)\delta_{n+m,0} \\ &\left[L_n,\bar{L}_m\right] = 0 \\ &\left[\bar{L}_n,\bar{L}_m\right] = (n-m)\bar{L}_{n+m} + \tfrac{c}{12}\left(n^3-n\right)\delta_{n+m,0} \end{aligned}} \ . \tag{1.3.8}$$

Note that the Virasoro algebra decomposes into holomorphic and antiholomorphic parts. These are denoted by **Vir** and $\overline{\textbf{Vir}}$, which are generated by the holomorphic and antiholomorphic modes respectively[3].

[3]Some times in the literature these are called *chiral* and *antichiral* or *left* and *right moving* parts.

In this thesis we will assume $c = \bar{c}$. In the case where $c = 0$ we retrieve the *Witt algebra*. One can identify $L_{-1} + \bar{L}_{-1}$ and $i\left(L_{-1} - \bar{L}_{-1}\right)$ as generators of translations, $L_0 + \bar{L}_0$ and $i\left(L_0 - \bar{L}_0\right)$ as generators of dilations and rotations respectively, while $L_1 + \bar{L}_1$ and $i\left(L_1 - \bar{L}_1\right)$ are generators of special conformal transformations.

The Virasoro algebra is infinite dimensional and it was originally discovered in the context of string theory [Vir70]. To see how one can obtain equations (1.3.8), one needs to employ a procedure for making contact between OPEs and commutators of operator modes. The commutator of two contour integrations $\left[\oint dz, \oint dw\right]$ is evaluated by first fixing w and deforming the difference between the two z integrations into a single z contour drawn tightly around the point w, as in (1.2.9). In evaluating the z contour integration, we may perform operator product expansions to identify the leading behavior as z approaches w. The w integration is then performed without further subtlety. For the modes of the stress-energy tensor, this procedure gives

$$\begin{aligned}
[L_n, L_m] &= \frac{1}{(2\pi i)^2}\left[\oint dz\ , \oint dw\right] z^{n+1}T(z)w^{n+1}T(w) \\
&= \frac{1}{(2\pi i)^2}\oint\oint dzdw\ z^{n+1}w^{n+1}\left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2}\right. \\
&\qquad\qquad\left. + \frac{\partial T(w)}{z-w} + \mathrm{reg}(z-w)\right) \\
&= \frac{1}{2\pi i}\oint dw\ \left(\tfrac{c}{12}(n+1)n(n-1)w^{n-2}w^{m+1}\right. \\
&\qquad\qquad \left. + 2(n+1)w^n w^{m+1}T(w) + w^{n+1}w^{m+1}\partial T(w)\right)\ .
\end{aligned} \tag{1.3.9}$$

Integrating the last term by parts and combining with the second term gives $(n-m)w^{n+m+1}T(w)$, so performing the w integration, produces the required result.

The vacuum state $|0\rangle \in \mathscr{H}$ must be invariant under global conformal transformations. This means that it must be annihilated by $L_{-1,0,1}$ and $\bar{L}_{-1,0,1}$. This, in turn, can be recovered from the condition that $T(z)|0\rangle$ and $\bar{T}(\bar{z})|0\rangle$ are well defined as $z, \bar{z} \to 0$, which implies

$$L_n|0\rangle = 0\ ,\quad \bar{L}_n|0\rangle = 0\ ,\quad n \geq -1\ . \tag{1.3.10}$$

From the state-field correspondence (1.3.2) we see that primary fields, when acting on the vacuum, create asymptotic states. Performing the corresponding contour integral with (1.2.12), we get the commutation relations

$$\begin{aligned}[L_n, \phi(w,\bar{w})] &= h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \\ \left[\bar{L}_n, \phi(w,\bar{w})\right] &= \bar{h}(n+1)\bar{w}^n\phi(w,\bar{w}) + \bar{w}^{n+1}\bar{\partial}\phi(w,\bar{w})\end{aligned} \, . \tag{1.3.11}$$

After applying these relations to the asymptotic state

$$|h,\bar{h}\rangle \equiv \phi(0,0)|0\rangle \ , \tag{1.3.12}$$

we take

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle \ , \quad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle \ . \tag{1.3.13}$$

Thus, $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian[4]. Similarly,

$$L_n|h,\bar{h}\rangle = \bar{L}_n|h,\bar{h}\rangle = 0 \ , \quad n \in \mathbb{N} \ . \tag{1.3.14}$$

The Hilbert space thus decomposes into *highest weight representations* of $\mathbf{Vir} \oplus \overline{\mathbf{Vir}}$ of the form (1.3.1). Each module is spanned by a *highest weight state* $|h,\bar{h}\rangle$ and an infinite set of *descendent states* of the form $L_{m_1}\ldots\bar{L}_{n_1}\ldots|h,\bar{h}\rangle$, with all $m,n<0$. Once we know the central charge c, of the theory and the conformal weights $\left(h,\bar{h}\right)$, of all primary fields, we can construct the Hilbert space. However, some care has to be taken in the construction of a basis, since not all products of L's and $\bar{L}$'s are linearly independent.

The inner product (1.3.5) of two highest weight states $|i\rangle$ and $|j\rangle$, simply is

$$\langle i|j\rangle = \delta_{ij} \ . \tag{1.3.15}$$

If we Hermitian conjugate T and $\bar{T}$ and restricting to the real surface $\bar{z}=z^*$, we get

$$L_n^\dagger = L_{-n} \ , \quad \bar{L}_n^\dagger = \bar{L}_{-n} \ , \tag{1.3.16}$$

This relation together with the Virasoro algebra and highest weight condition can be used to write the inner product of an arbitrary pair of fields in terms of the inner product of primary fields.

[4]As will be seen later, the Hamiltonian is proportional to $L_0 + \bar{L}_0 - \frac{c}{12}$.

Let $M(c,h)$ be a highest weight representation of **Vir**, then $M(c,h)$ has the decomposition [KR, Sect. 3.2]

$$M(c,h) = \bigoplus_{N\in\mathbb{N}} M(c,h)_{h+N} \ , \tag{1.3.17}$$

where $M(c,h)_{h+N}$ is the $(h+N)$-eigenspace of L_0, spanned by vectors of the form

$$L_{-k_1} \dots L_{-k_n} |h\rangle \ , \quad 1 \leq k_1 \leq \dots \leq k_n \ , \tag{1.3.18}$$

where $h+N = h+k_1+\dots+k_n$ is the L_0 eigenvalue of (1.3.18). The number $N \in \mathbb{N}$ is called the *level* of the state. Therefore, the operator L_0, acts as a grading operator on the **Vir**-module $M(c,h)$. The states (1.3.18) are called *descendants* of the asymptotic state $|h\rangle$. The number of states at level N is simply the number $p(N)$ of partitions of the integer N. $p(N)$ is given in terms of the generating function

$$\frac{1}{\varphi(q)} := \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{N=0}^{\infty} p(N) q^N \ , \tag{1.3.19}$$

where $p(0) \equiv 1$ and $\varphi(q)$ is Euler's function.

The subset of the full Hilbert space generated by the asymptotic state $|h\rangle$ and its descendants is closed under the action of the Virasoro generators and thus forms a **Vir**-module. If all the states of the form (1.3.18) in a highest weight representation of **Vir** are linearly independent, then this highest weight representation is called a *Verma module*, denoted by $V(c,h)$. Starting from a highest-weight state $|h\rangle \in V(c,h)$, one can build the set of states given in the following table.

Level	**Dimension**	**State**
0	h	$\|h\rangle$
1	$h+1$	$L_{-1}\|h\rangle$
2	$h+2$	$L_{-2}\|h\rangle,\ L_{-1}^2\|h\rangle$
3	$h+3$	$L_{-3}\|h\rangle,\ L_{-1}L_{-2}\|h\rangle,\ L_{-1}^3\|h\rangle$
4	$h+4$	$L_{-4}\|h\rangle,\ L_{-1}L_{-3}\|h\rangle,\ L_{-1}^2L_{-2}\|h\rangle,\ L_{-2}^2\|h\rangle,\ L_{-1}^4\|h\rangle$
$\vdots$	$\vdots$	$\vdots$
N	$h+N$	$p(N)$ states

Table 1.1: Basis for the Verma module $V(c,h)$.

The states in Table 1.1 form a basis for the Verma module. If among the vectors of a Verma module there exist states $|\chi\rangle$ which are also highest weight states, $L_n|\chi\rangle = 0$ for all $n \in \mathbb{N}$, then these states are called *null states* and are orthonormal to all the other states in the module. In particular for a null state we have $\langle\chi|\chi\rangle = 0$. We will call a null state, *singular*, if it is not the descendent of a null state. The fields $\phi \in \mathscr{F}$ that correspond to the states $|h\rangle \in \mathscr{H}$ in Table 1.1, arise from repeated OPEs of the primary field ϕ with $T(z)$, and constitute the *conformal family* $[\phi]$ of ϕ.

Now that we know all the states in the Hilbert space, and thus all the primary and descendent fields, we can write down the OPE of two primary fields. Let us consider (1.2.16) with ϕ_i and ϕ_j primary fields, and group together all the secondary fields belonging to the conformal family $[\phi_p]$ in the summation to write

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_{p,\{k,\bar{k}\}} \mathbf{C}_{ij}^{p\{k,\bar{k}\}} z^{\left(h_p-h_i-h_j+\sum_\ell k_\ell\right)} \bar{z}^{\left(\bar{h}_p-\bar{h}_i-\bar{h}_j+\sum_\ell \bar{k}_\ell\right)} \phi_p^{\{k,\bar{k}\}}(w,\bar{w}) \ . \tag{1.3.20}$$

Here, we have labeled the descendants $L_{-k_1}\dots L_{-k_n}\bar{L}_{-\bar{k}_1}\dots\bar{L}_{-\bar{k}_m}\phi_p$ of a primary field ϕ_p by $\phi_p^{\{k,\bar{k}\}}$, and we assume the normalization (1.2.15). Performing a conformal transformation on both sides of (1.3.20) and comparing terms, one can show that

$$\mathbf{C}_{ij}^{p\{k,\bar{k}\}} = C_{ij}{}^{p}\beta_{ij}^{p\{k\}}\bar{\beta}_{ij}^{p\{\bar{k}\}} \ , \tag{1.3.21}$$

where the constants $C_{ij}{}^{p}$ are the *structure constants*. They are the only nontrivial input in the OPE, in the sense that the structure constants are not directly fixed by the representation theory of the Virasoro algebra. The numbers $\beta, \bar{\beta}$ are determined by the representation properties of the fields and may be calculated in terms of the h's, $\bar{h}$'s and c, see [BPZ84] for detals.

We thus see that the data needed to specify a 2D CFT are given by $\left(h_i, \bar{h}_i, c\right)$ and the structure constants $C_{ij}{}^{p}$ between the primary fields. Everything else follows from the values of these parameters, which themselves cannot be determined by the conformal symmetry alone. However, as we will see in Sect. 3.5.2 one can construct a full CFT from its correlation functions, using the machinery of three dimensional topological field theory.

1.4 Modular Invariance

The CFT on the full complex plane we formulated up to now, decouples into holomorphic and antiholomorphic sectors. In fact, the two sectors may describe two distinct theories since they do not interfere. However this situation is very unphysical.

The decoupling exists only at the fixed point in parameter space (the conformally invariant point) and in the infinite plane geometry. One, therefore, can solve this problem by coupling the holomorphic and antiholomorphic sectors of the theory, through the geometry of space, on which the theory is defined. In this way, one imposes physical constraints on the holomorphic-antiholomorphic content of a CFT without leaving the fixed point. The infinite plane is topologically equivalent to the Riemann sphere, i.e. the Riemann surface of genus $g = 0$. One may study CFTs on Riemann surfaces of arbitrary genus g. The simplest non-spherical case is that of genus $g = 1$, i.e. a torus, which is equivalent to a plane with periodic boundary conditions, in two directions.

More precisely, consider the map (1.2.5), from the cylinder to the complex plane. We now want the inverse procedure, i.e. to go back to the infinite cylinder from which we can construct a torus of length R, by cutting a segment of the cylinder and by gluing the two boundaries of the segment together. In terms of CFT this means that one has to sum over *intermediate states*. The Hamiltonian and the momentum operators then propagate states along different directions of the torus and the spectrum of the theory is encoded in the *partition function*.

Consider now the map $z \mapsto w(z) = \frac{L}{2\pi i} \ln z$, from the complex plane parameterized by z, to the infinite cylinder of circumference L, parameterized by w. On the cylinder, time translations are movements in the imaginary direction, generated by the Hamiltonian

$$H_{\text{cyl}} = \int_0^L T_{tt}\ dx = -\oint \left(dw\ T(w) + d\bar{w}\ \bar{T}(\bar{w})\right)\ . \tag{1.4.1}$$

The transformation law (1.2.14) of the stress tensor, gives

$$T(w) = \left(\frac{2\pi i}{L}\right)^2 \left(z^2 T(z) - \frac{c}{24}\right)\ , \tag{1.4.2}$$

with a similar expression for $\bar{T}$. Then by changing integration variables, $dw = \frac{L}{2\pi i}\frac{1}{z}dz$, we obtain the action of H_{cyl} in the Hilbert space of the complex plane

$$\begin{aligned} H_{\text{cyl}} &= -\frac{2\pi i}{L}\oint \left\{dz\ \left(zT(z) - \frac{c}{24}\frac{1}{z}\right) + d\bar{z}\ \left(\bar{z}\bar{T}(\bar{z}) - \frac{c}{24}\frac{1}{\bar{z}}\right)\right\} \\ &= \frac{2\pi}{L}\left(L_0 + \bar{L}_0 - \frac{c}{12}\right) \end{aligned}\ . \tag{1.4.3}$$

In taking the last expression, we used (1.3.7) and the fact that $\oint dz\ \frac{1}{z} = 2\pi i$. As one can see, for the Hamiltonian (1.4.3) to be bounded from below, the Hilbert space must decompose into the direct sum of highest weight representations of $\mathbf{Vir} \oplus \overline{\mathbf{Vir}}$ as in (1.3.1).

The partition function of the theory on the torus is given as the trace over the whole space of states

$$Z(\tau,\bar{\tau}) = \text{Tr}_{\mathscr{H}}\left(e^{-RH_{\text{cyl}}}\right) = \text{Tr}_{\mathscr{H}}\left(q^{L_0 - \frac{c}{24}}\bar{q}^{\bar{L}_0 - \frac{c}{24}}\right)\ , \tag{1.4.4}$$

where $q = e^{2\pi i\tau}$ and $\tau = iR/L$, is called the *modular parameter* of the torus. Since the tori that are described by τ and $A\tau$, for $A \in SL(2,\mathbb{Z})/\mathbb{Z}_2$ are equivalent, the partition function is invariant under the *modular transformations*

$$\mathcal{T}\colon \tau \to \tau + 1 \ , \quad \mathcal{S}\colon \tau \to -\frac{1}{\tau} \ . \tag{1.4.5}$$

of the modular parameter τ. The group $SL(2,\mathbb{Z})/\mathbb{Z}_2$, is isomorphic to the *modular group* Γ, generated by (1.4.5). Note that the modular group will keep τ on the upper half plane. If we decompose the Hilbert space into representations of $\mathbf{Vir} \oplus \overline{\mathbf{Vir}}$ by (1.3.1), we can rewrite the partition function (1.4.4) as

$$Z(\tau,\bar{\tau}) = \sum_{i,\bar{\jmath}\in\mathcal{I}} M_{i\bar{\jmath}}\chi_i(\tau)\bar{\chi}_{\bar{\jmath}}(\bar{\tau}) \ , \tag{1.4.6}$$

where $M_{i\bar{\jmath}} \in \mathbb{N}_0$ is the multiplicity of occurrence of $R_i \otimes \bar{R}_{\bar{\jmath}}$ in $\mathscr{H}$ and

$$\chi_i(\tau) = \mathrm{Tr}_{R_i}\left(q^{L_0-\frac{c}{24}}\right) \ , \quad \bar{\chi}_{\bar{\jmath}}(\bar{\tau}) = \mathrm{Tr}_{\bar{R}_{\bar{\jmath}}}\left(\bar{q}^{\bar{L}_0-\frac{c}{24}}\right) \ , \tag{1.4.7}$$

are the *Virasoro characters* of the irreducible representations forming the Hilbert space of the theory. The characters transform into one another under the modular transformations (1.4.5) as

$$\chi_i(\tau+1) = \sum_{j\in\mathcal{I}} T_{ij}\chi_j(\tau) \ , \quad \chi_i\left(-\frac{1}{\tau}\right) = \sum_{j\in\mathcal{I}} S_{ij}\chi_j(\tau) \ , \tag{1.4.8}$$

where T and S are constant matrices, called the *modular matrices* and they are symmetric and unitary. Thus the torus partition function is modular invariant provided that

$$\sum_{i,\bar{\jmath}\in\mathcal{I}} S_{il}M_{i\bar{\jmath}}\bar{S}_{\bar{\jmath}\bar{k}} = \sum_{i,\bar{\jmath}\in\mathcal{I}} T_{il}M_{i\bar{\jmath}}\bar{T}_{\bar{\jmath}\bar{k}} = M_{l\bar{k}} \ . \tag{1.4.9}$$

The case where $M_{i\bar{\jmath}} = \delta_{i\bar{\jmath}}$, is called the *Cardy case.*

1.5 Fusion Algebra and the Verlinde Formula

The action of the Virasoro generators on the product of two primary fields, preserves the Virasoro algebra and endows the tensor product of the representations with the structure of a representation. This leads to a natural product on representations, called the *fusion product*, which constrains the fields that appear in the OPE. The consistency of the OPE (1.3.20) with the existence of null vectors leads to the *fusion algebra* of the CFT [Ca04]

$$R_i \otimes R_j = \sum_{k \in \mathcal{I}} \mathcal{N}_{ij}{}^k R_k \ , \tag{1.5.1}$$

where $\mathcal{N}_{ij}{}^k \in \mathbb{N}_0$ are the *fusion numbers*. This applies separately to the holomorphic and antiholomorphic sectors and determines how many copies of R_k occur in the fusion of R_i with R_j. The fusion algebra is commutative, associative and contains an identity given by the vacuum representation R_0.

Consistency of the CFT on the torus implies that the fusion numbers are given in terms of particular products of matrix elements of the modular matrix

$$\mathcal{N}_{ij}{}^k = \sum_{l \in \mathcal{I}} \frac{S_{il} S_{jl} \bar{S}_{kl}}{S_{0l}} \ . \tag{1.5.2}$$

This is the so called *Verlinde formula* [Ver88]. In this thesis, we will make the simplifying assumption that $\mathcal{N}_{ij}{}^k \in \{0, 1\}$. In full generality, the fusion numbers may be larger than one, but it is not so for the *Virasoro minimal models* (that we will study in Sect. 1.7). This reflects the absence of multiplicity greater than one in ordinary tensor products of representations of $\mathfrak{su}(2)$ [BYB, footnote p. 125].

1.6 Rational Conformal Field Theory

In Section 1.5, we gave an explicit relation between the modular transformation $\mathcal{S}$ of the characters and the fusion numbers $\mathcal{N}$ which proves to be a very general fact. This naturally leads to the concept of *rational conformal field theory* (RCFT). These are CFTs, whose Hilbert space contains only a finite number of irreducible highest weight representations, of the symmetry algebra. The term "rational" is because if there are only a finite number of primary fields then the conformal weights are all rational numbers [Va88, AM88]. RCFTs may contain an infinite number of Virasoro representations, but these can be reorganised into a finite set of irreducible representations corresponding to an *extended symmetry algebra*. Actually the only theories that contain only a finite number of Virasoro irreducible representations, are the Virasoro minimal models. For a condensed panoramic view of the development of two-dimensional RCFT in the last twenty-five years see [FRS10], or for a lightning review of RCFT see [GW03, Sect. 2].

Consider now a RCFT whose Hilbert space $\mathscr{H}$ decomposes into a finite number of irreducible representations

$$\mathscr{H} = \bigoplus_{i,j\in\mathcal{I}} (R_i \otimes \bar{R}_j)^{M_{ij}} \ , \tag{1.6.1}$$

of a chiral algebra $\mathfrak{V}$, such that $\mathbf{Vir} \subset \mathfrak{V}$. On the set $\mathcal{I}$, indexing the representations R_i, we assume there is the *charge conjugation* $(i^\vee)^\vee = i$, which preserves the conformal weights and the fusion numbers

$$h_i = h_{i^\vee} \ , \quad \mathcal{N}_{ij}{}^{k} = \mathcal{N}_{i^\vee j^\vee}{}^{k^\vee} \ . \tag{1.6.2}$$

From this we define the *charge conjugation matrix* as

$$C_{ij} = \delta_{ij^\vee} \ . \tag{1.6.3}$$

The charge conjugation matrix can be used to raise and lower indices (just like the metric tensor). The modular matrix satisfies

$$S^2 = C \ , \quad S_{ij^\vee} = \bar{S}_{ij} \ . \tag{1.6.4}$$

This requires that the characters, under modular transformations must transform as

$$\chi_i(q) = \sum_{j\in\mathcal{I}} S_{ij}\chi_j(\tilde{q}) \ , \quad \chi_i(\tilde{q}) = \sum_{j\in\mathcal{I}} S_{ij^\vee}\chi_j(q) = \sum_{j\in\mathcal{I}} \bar{S}_{ij}\chi_j(q) \ , \tag{1.6.5}$$

where $q = e^{2\pi i\tau}$ and $\tilde{q} = e^{-2\pi i/\tau}$. The fusion numbers also satisfy the following identities

$$\mathcal{N}_{ij}{}^{k} = \mathcal{N}_{ji}{}^{k} \ , \quad \sum_{k\in\mathcal{I}} \mathcal{N}_{ij}{}^{k}\mathcal{N}_{kl}{}^{r} = \sum_{k\in\mathcal{I}} \mathcal{N}_{il}{}^{k}\mathcal{N}_{kj}{}^{r} \ , \quad \mathcal{N}_{0i}{}^{j} = \delta_{ij} \ , \quad \mathcal{N}_{ij}{}^{0} = \delta_{ij^\vee} \ . \tag{1.6.6}$$

Note, that the commutativity and associativity of the fusion rules is reflected in the first and second identities respectively.

1.7 Minimal Models

A class of theories that are among the simplest CFTs are the *Virasoro minimal models*. These were introduced by Belavin, Polyakov and Zamolodchikov [BPZ84]. The minimal models are theories whose Hilbert space consists of a finite number of representations of the Virasoro algebra. The simplicity of minimal models allows for a complete solution and they were classified in [CIZ87]. The discovery of minimal models and their identification with statistical models at criticality, is the greatest application of conformal invariance so far. In this section we will discuss a selection of facts about minimal models and in the next subsection we will introduce the Lee-Yang model, which will be one of the main examples when we will discuss perturbed defects later on. The reader interested in more details on minimal models, may refer to the choice of texts mentioned in the beginning of Chapter 1.

The minimal models exist for specific values of the central charge for which the OPE of the fields closes even if the theory contains a finite number of primary fields. The allowed values for the central charge are given by

$$c = 13 - 6\left(t + t^{-1}\right) \ , \quad t \in \mathbb{Q}^{+} - \mathbb{N} - 1/\mathbb{N} \ . \tag{1.7.1}$$

In other words the central charge is parametrised by a rational number $t = p/q$ with $p, q \in \mathbb{Z}_{\geq 2}$ that have no common divisor. We denote these models by $M(p,q)$ and they are unitary if $p = q \pm 1$. Furthermore, the highest weight irreducible representations of **Vir** can be organised in a $(p-1)\times(q-1)$ table, called the *Kac table*. If $r \in \{1, ..., p-1\}$ and $s \in \{1, ..., q-1\}$, then the corresponding highest weights are given by

$$h_{r,s} = \frac{1}{4t}\left((r - st)^2 - (1-t)^2\right) \ . \tag{1.7.2}$$

Each representation with highest weight h and central charge c contributes to the Virasoro characters (1.4.7). These representations contain null states and in order to obtain irreducible representations one must quotient out these states, then the characters are given by

$$\chi_{r,s}(q) = \frac{q^{h_{r,s}-\frac{c}{24}}}{\varphi(q)} \sum_{n\in\mathbb{Z}} \left(q^{n(npq+qr-ps)} - q^{n(npq+qr+ps)+rs}\right) \ . \tag{1.7.3}$$

These under the modular transformation $\tau \mapsto -1/\tau$, transform into each other according to (1.6.5), with the modular matrix given by

$$S_{(r,s),(r',s')} = 2^{\frac{3}{2}}(pq)^{-\frac{1}{2}}(-1)^{1+rs'+sr'} \sin\left(\frac{\pi q r r'}{p}\right) \sin\left(\frac{\pi p s s'}{q}\right) \ . \tag{1.7.4}$$

Note that $S^2 = 1$.

1.7.1 The Lee-Yang Model

In this section, following [MR09, Sect. 4.1], we fix our conventions for the Lee-Yang model. This is the non unitary Virasoro minimal model $M(2,5)$ of central charge $c = -22/5$. The two irreducible highest weight representations of the Virasoro algebra that lie in the Kac table have highest weights $h_{(1,1)} = h_{(1,4)} = 0$ and $h_{(1,2)} = h_{(1,3)} = -1/5$. We will abbreviate $1 = (1,1)$ and $\phi = (1,2)$, and we will denote the corresponding representations by R_1 (for $h = 0$) and R_ϕ (for $h = -1/5$). The characters of R_1 and R_ϕ are (see e.g. [Na04])

$$\begin{aligned} \chi_1(\tau) = \mathrm{Tr}_{R_1} q^{L_0 - c/24} &= q^{11/60} \prod_{n\equiv_5 2,3} (1-q^n)^{-1} = q^{11/60}(1 + q^2 + q^3 + q^4 + \dots) \ , \\ \chi_\phi(\tau) = \mathrm{Tr}_{R_\phi} q^{L_0 - c/24} &= q^{-1/60} \prod_{n\equiv_5 1,4} (1-q^n)^{-1} = q^{-1/60}(1 + q + q^2 + q^3 + 2q^4 + \dots) \ , \end{aligned} \tag{1.7.5}$$

where $a \equiv_n b$ is a shorthand notation for $a \equiv b \pmod n$ and $q = e^{2\pi i \tau}$. The products are from $n = 1$ to infinity with the restriction modulo 5 as shown. Under the modular transformation $\tau \mapsto -1/\tau$ they transform as $\chi_i(-1/\tau) = \sum_{j\in\{1,\phi\}} S_{ij}\,\chi_j(\tau)$ with

$$S = \begin{pmatrix} S_{11} & S_{1\phi} \\ S_{\phi 1} & S_{\phi\phi} \end{pmatrix} = -\frac{1}{\left|\sqrt{d+2}\,\right|} \begin{pmatrix} 1 & d \\ d & -1 \end{pmatrix} \ , \tag{1.7.6}$$

where

$$d = \frac{1-\sqrt{5}}{2} = -0.618\ldots \ . \tag{1.7.7}$$

The space of states of the Lee-Yang model is

$$\mathscr{H} = R_1 \otimes_{\mathbb{C}} R_1 \ \oplus \ R_\phi \otimes_{\mathbb{C}} R_\phi \ . \tag{1.7.8}$$

The partition function

$$Z(\tau) = \mathrm{Tr}_{\mathscr{H}} \left(q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} \right) = |\chi_1(\tau)|^2 + |\chi_\phi(\tau)|^2 \ , \tag{1.7.9}$$

is modular invariant, as it should be.

Chapter 2

Defects and Defect Operators

In this chapter, we introduce the notion of *defect lines* (or simply *defects*). To each such defect, one can associate an operator in the space of states. These operators are called *defect operators* and they will play a very important role in this thesis. Before we go into further details it will be good at this point to mention a few reasons why defects are important.

When introducing a boundary into the system under consideration, two new ingredients appear: *conformal boundary conditions* and *boundary fields*, see for example [Ca89] and for a review [Ca04, Ca08]. Conformal boundary conditions describe a *universality class* of boundary critical behavior. A conformal defect is a universality class of critical behavior at a one dimensional junction of two such quantum systems. For example, the authors of [OA97] used the better understood boundary theory to deduce facts about defects. In particular, they found in the critical two dimensional Ising model with a defect line, the complete spectrum of boundary operators, exact two-point correlation functions and the universal term in the free energy of the defect line. It was also conjectured that all the possible universality classes of defect lines in the Ising model were found.

Defect lines can also describe duality symmetries, as in [FFRS04] for example, where it was shown that the fusion algebra of conformal defects of a 2D-CFT contains information about the internal symmetries of the theory and allows one to read off generalisations of Kramers-Wannier duality [KW41]. Furthermore, in [FGRS07] it was also shown that the isomorphism between two T-dual free boson CFTs can be described by the action of a *topological defect* (to be defined in Sect. 2.1), and hence that T-duality can be understood as a special type of order-disorder duality.

Defects, also appear in higher dimensional field theory. They provide more observables in gauge theories, 'surface operators', see for example [GuWi07, KT09, KaWi07]. Furthermore, defects provide an alternative point of view of orbifold theories, as well as a generalisation thereof [FFRS09].

Defects have also applications to quantum wires, domain walls in string theory, other works focused on general constructive methods or structural implications and finally, there are also articles which have originally been written in a different context but have implications for the study of defect systems. For those interested, see [QRW07] and references therein for a more extensive list of references.

2.1 Topological and Conformal Defects

A defect line is a line of inhomogeneity on the surface on which the CFT is defined, where fields can have discontinuities or singularities. A defect is characterised by a 'defect condition' in the same way a boundary of a system which is described by a boundary state $|b\rangle\rangle$, is characterised by a boundary condition. In particular, when the boundary is conformally invariant, the boundary state $|b\rangle\rangle$ must satisfy the condition $\left(L_n - \bar{L}_{-n}\right)|b\rangle\rangle = 0$.

To formulate the analogous ideas for defects, we first define what we call a defect operator. To do this, consider a CFT on a cylinder and denote by $\mathscr{H}$ the space of states on a circle. A defect line a that goes around the cylinder gives rise to a linear operator $D_a\colon \mathscr{H} \to \mathscr{H}$, called a defect operator.

Definition 2.1.1. A defect a is said to be *conformal* if the corresponding defect operator D_a obeys

$$\left[L_n - \bar{L}_{-n}, D_a\right] = 0 \ , \quad \forall n \in \mathbb{Z} \ . \tag{2.1.1}$$

A special class of solutions to this condition is provided by totally transmitting defects, also known as topological defects. Such defects were first investigated in the context of rational CFT in [PZ01a] and were termed *topological defects* in [BG04]:

Definition 2.1.2. A defect a is said to be *topological* if the corresponding defect operator D_a obeys

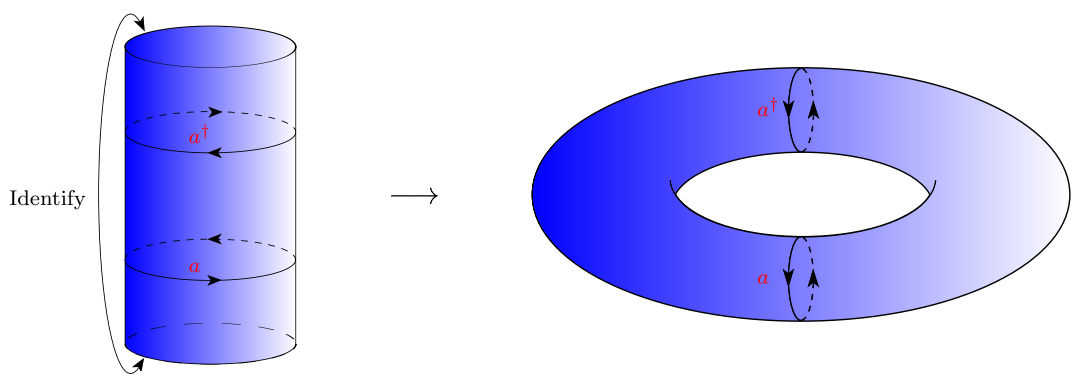
$$[L_n, D_a] = 0 = \left[\bar{L}_n, D_a\right] \ , \quad \forall n \in \mathbb{Z} \ . \tag{2.1.2}$$

This means that the holomorphic and anti-holomorphic components of the stress tensor are continuous across the defect line. As a consequence, the defect line is tensionless and can be deformed without affecting the value of correlators on the cylinder, as long as it is not taken across field insertions or other defect lines [BG04].

2.2 Petkova's and Zuber's Defect Lines

In Sect. 1.4, we saw that the partition function (1.4.4) of the theory on the torus is given as the trace over the whole space of states and is modular invariant. In this section, we consider the set of partition functions which result by inserting defects in our theory. These defects are compatible with conformal invariance in two dimensions and in particular they are topological defects, i.e. the corresponding defect operators satisfy (2.1.2). However, the partition function with defect insertions is not modular invariant, but one can find a consistency condition by using the modular transformation properties of the characters. In [PZ01a] the above procedure was used to show that the consistency equation gives a classification of defects and it was solved in particular cases. In this section we follow [PZ01a] to show that the fusion rules of two defects are just the fusion rules of the representations of the underlying chiral algebra.

Consider a RCFT with a chiral (vertex) algebra $\mathfrak{V}$ (the Virasoro algebra or one of its extensions). Furthermore, suppose this RCFT has a Hilbert space of the form (1.3.1). In order to study this CFT, we will use the techniques described in Sect. 1.4, but this time we want to insert one or more defects in our theory. Thus, consider a cylinder of circumference L and insert one or more defect lines along its non-contractible cycles. As an example, consider two defects with opposite orientation and along the imaginary direction of the cylinder. These are described by a and $a^\dagger$ respectively. That is, $a^\dagger$ is the same defect line as a, but with its orientation reversed relative to a. Then if we identify the two boundaries of the cylinder we take a torus:



To each defect line a we associate a defect operator $D_a\colon \mathscr{H} \to \mathscr{H}$. Then, as in the boundary case and Cardy's condition [Ca89], there are also a number of consistency conditions which must be satisfied by the operator D_a. To formulate these conditions, one first notes that as a consequence of (2.1.2), D_a is a sum of projectors,

$$P^{(i,\bar{\jmath};\alpha,\alpha')}\colon \left(R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}}\right)^{(\alpha')} \to \left(R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}}\right)^{(\alpha)} \ , \qquad (2.2.1)$$

intertwining the different copies of $R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}} \subset \mathscr{H}$, where $\alpha, \alpha' \in \{1, 2, ..., M_{i\bar{\jmath}}\}$ allow for repeated representations in the Hilbert space. If $\{|i, \mathbf{n}\rangle \otimes |\bar{\jmath}, \bar{\mathbf{n}}\rangle\}$ is an orthonormal basis of $R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{\jmath}}$ labelled by multi-indices $\mathbf{n}, \bar{\mathbf{n}}$, we may write

$$P^{(i,\bar{\jmath};\alpha,\alpha')} = \sum_{\mathbf{n},\bar{\mathbf{n}}} (|i, \mathbf{n}\rangle \otimes |\bar{\jmath}, \bar{\mathbf{n}}\rangle)^{(\alpha)} (\langle i, \mathbf{n}| \otimes \langle \bar{\jmath}, \bar{\mathbf{n}}|)^{(\alpha')} \ . \qquad (2.2.2)$$

There are $\sum_{i,\bar{\jmath}\in\mathcal{I}}|M_{i\bar{\jmath}}|^2$ many different projectors $P^{(i,\bar{\jmath};\alpha,\alpha')}$, because α,α' can take $M_{i\bar{\jmath}}$ different values. The projectors satisfy,

$$P^{(i,\bar{\imath};\alpha,\alpha')}P^{(j,\bar{\jmath};\beta,\beta')} = \delta_{ij}\delta_{\bar{\imath}\bar{\jmath}}\delta_{\alpha'\beta}P^{(i,\bar{\imath};\alpha\beta')} \ . \tag{2.2.3}$$

and they are required to be Hermitian,

$$\left(P^{(i,\bar{\jmath};\alpha,\alpha')}\right)^\dagger = P^{(i,\bar{\jmath};\alpha',\alpha)} \ . \tag{2.2.4}$$

This corresponds to interpreting the defect line $a^\dagger$ described by $D_a^\dagger$ as having opposite orientation relative to the line a described by D_a.

The most general linear combination of these operators is

$$D_a = \sum_{i,\bar{\jmath}\in\mathcal{I}} \frac{\Psi_a^{(i,\bar{\jmath};\alpha,\alpha')}}{\sqrt{S_{0i}S_{0\bar{\jmath}}}} P^{(i,\bar{\jmath};\alpha,\alpha')} \ , \tag{2.2.5}$$

where Ψ is an a priori arbitrary complex $n\times n$ matrix, $n=\sum_{i,\bar{\jmath}\in\mathcal{I}}(M_{i\bar{\jmath}})^2$. To understand this, recall from above that the P's form a basis of all intertwiners from $\mathscr{H}$ to $\mathscr{H}$ and there are $\sum_{i,\bar{\jmath}\in\mathcal{I}}|M_{i\bar{\jmath}}|^2$ such P's. The $\Psi_a^{(i,\bar{\jmath};\alpha,\alpha')}$ are a basis transformation, where $(i,\bar{\jmath};\alpha,\alpha')$ is thought of as one index taking n values and a as the other taking n values as well.

From now on we drop the indices α,α' for notational convenience.

A consistency condition is found by considering a pair of defect lines a and b, wrapping a canonical cycle on a torus. Using a Hamiltonian picture with time moving perpendicular to the defect lines, the torus partition function may be written as

$$Z_{a|b} := \mathrm{Tr}_{\mathscr{H}}\left(D_a^\dagger D_b \tilde{q}^{L_0-c/24}\tilde{\bar{q}}^{\bar{L}_0-c/24}\right) = \sum_{i,\bar{\jmath}\in\mathcal{I}} \frac{\left(\Psi_a^{(i,\bar{\jmath})}\right)^*\Psi_b^{(i,\bar{\jmath})}}{S_{0i}S_{0\bar{\jmath}}}\chi_i\left(\tilde{q}\right)\chi_{\bar{\jmath}}\left(\tilde{\bar{q}}\right) \ , \tag{2.2.6}$$

where $\tilde{q}=e^{2i\pi\tilde{\tau}}$. A second representation of the same partition function may be obtained by considering time running parallel to the defect lines. In this case, condition (2.1.2) for the defect line, ensures one may still construct two sets of generators L_n and $\bar{L}_n$ satisfying the Virasoro algebra (or more generally the chiral algebra $\mathfrak{V}$). Hence, the Hilbert space decomposes into irreducible representations,

$$\mathscr{H}_{a|b} = \bigoplus_{i,\bar{\jmath}\in\mathcal{I}}\left(R_i\otimes_{\mathbb{C}}\bar{R}_{\bar{\jmath}}\right)^{\oplus V_{i\bar{\jmath};a}{}^b} \ , \tag{2.2.7}$$

for some non-negative integers ${V_{i\bar{\jmath};a}}^b$. Thus, the partition function becomes,

$$Z_{a|b} := \mathrm{Tr}_{\mathscr{H}_{a|b}}\left(q^{L_0-c/24}\bar{q}^{\bar{L}_0-c/24}\right) = \sum_{i,\bar{\jmath}\in\mathcal{I}} {V_{i\bar{\jmath};a}}^b \chi_i(q)\chi_{\bar{\jmath}}(\bar{q}) \ . \tag{2.2.8}$$

One can equate these two expressions for the partition function, using the modular transformation properties of the characters, to obtain the *consistency equation*

$$\boxed{{V_{i\bar{\jmath};a}}^b = \sum_{k,\bar{l}\in\mathcal{I}} \frac{S_{ik}S_{\bar{\jmath}\bar{l}}}{S_{0k}S_{0\bar{l}}}\Psi_a^{(k,\bar{l})}\Psi_b^{*(k,\bar{l})}} \ . \tag{2.2.9}$$

Although the methods presented below can be applied to general RCFTs, here (and for the rest of this thesis) we only consider the Cardy case ($M_{i\bar{\jmath}} = \delta_{i\bar{\jmath}}$), for our purpose, so (1.3.1) becomes

$$\mathscr{H} = \bigoplus_{i\in\mathcal{I}} R_i \otimes_{\mathbb{C}} \bar{R}_i \ . \tag{2.2.10}$$

Because of (2.1.2) the defect operator D_a for $a \in \mathcal{I}$ will act as a multiple of the identity on each sector $R_i \otimes_{\mathbb{C}} \bar{R}_i$, and taking into account (2.2.5) it follows that

$$D_a \,|_{R_i\otimes_{\mathbb{C}}\bar{R}_i} = \frac{\Psi_a^{(i)}}{S_{0i}}\,\mathrm{id}_{R_i\otimes_{\mathbb{C}}\bar{R}_i} \ , \tag{2.2.11}$$

that is $P^{(i)} = \mathrm{id}_{R_i\otimes_{\mathbb{C}}\bar{R}_i}$ is the projector in $R_i\otimes_{\mathbb{C}}\bar{R}_i$. However, $D_0 := \mathbf{1} = \sum_{i\in\mathcal{I}} \mathrm{id}_{R_i\otimes_{\mathbb{C}}\bar{R}_i}$, for which $\Psi_0^{(i)} = S_{0i}$. This suggests the ansatz $\Psi_a^{(i)} = S_{ai}$, which satisfies the consistency equation (2.2.9) with

$${V_{ij;a}}^b = \sum_{k\in\mathcal{I}} \frac{S_{ik}S_{jk}}{S_{0k}S_{0k}} S_{ak}\bar{S}_{bk} = \sum_{c,k,l\in\mathcal{I}} \frac{S_{ik}S_{ak}\bar{S}_{ck}}{S_{0k}} \frac{S_{cl}S_{jl}\bar{S}_{bl}}{S_{0l}} = \sum_{c\in\mathcal{I}} {\mathcal{N}_{ia}}^c {\mathcal{N}_{cj}}^b \ . \tag{2.2.12}$$

and ${V_{ij;0}}^b = {\mathcal{N}_{ij}}^b$. Therefore, the resulting fusion rules of the defects are

$$D_a D_b = \sum_{i,j\in\mathcal{I}} \frac{S_{ai}S_{bj}}{S_{0i}S_{0j}} P^{(i)}P^{(j)} \overset{(1)}{=} \sum_{i\in\mathcal{I}} \frac{S_{ai}S_{bi}}{S_{0i}S_{0i}} P^{(i)} \overset{(2)}{=} \sum_{i,k,c\in\mathcal{I}} \frac{S_{ai}S_{bi}\bar{S}_{ci}}{S_{0i}} \frac{S_{ck}}{S_{0k}} P^{(k)} \overset{(3)}{=} \sum_{c\in\mathcal{I}} {\mathcal{N}_{ab}}^c D_c \ , \tag{2.2.13}$$

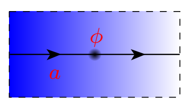
where in step (1) we used (2.2.3), in step (2) the fact that S is unitary and finally in (3) Verlinde's formula (1.5.2).

Therefore, we see that defects are labeled by irreducible representations of the chiral algebra $\mathfrak{V}$, and hence, their fusion rules are just the fusion rules for the representations of $\mathfrak{V}$. Thus, to summarise our results, we have

$$\boxed{D_a \mid_{R_i \otimes_{\mathbb{C}} \bar{R}_i} = \frac{S_{ai}}{S_{0i}} \, \mathrm{id}_{R_i \otimes_{\mathbb{C}} \bar{R}_i} \quad \text{and} \quad D_a D_b = \sum_{c \in \mathcal{I}} \mathcal{N}_{ab}{}^c D_c} \; . \tag{2.2.14}$$

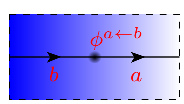
2.3 Defect Fields and Perturbed Defect Operators

Similar to boundary conditions and boundary fields, one can also consider defect fields which live on the defect line. For example, if we consider a defect field ϕ of weight $(h_\phi, 0)$, inserted on a defect line a on the cylinder, we have the following picture



(2.3.1)

which represents a segment of the cylinder. In addition to defect fields inserted on a defect line a, one can also consider *defect-changing fields* $\phi^{a \leftarrow b}$ which change a defect of type b to a defect of type a:



(2.3.2)

The space $\mathscr{H}^{a \leftarrow b}$ of defect changing fields decomposes into representations of $\mathfrak{V} \otimes_{\mathbb{C}} \bar{\mathfrak{V}}$ as

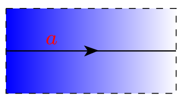
$$\mathscr{H}^{a \leftarrow b} = \bigoplus_{i,j \in \mathcal{I}} \left(R_i \otimes_{\mathbb{C}} \bar{R}_j \right)^{\oplus \left(\sum_{c \in \mathcal{I}} \mathcal{N}_{ij}{}^c \mathcal{N}_{ca}{}^b \right)} \; , \tag{2.3.3}$$

with the multiplicity given by (2.2.12). Then the space of bulk fields $\mathscr{H} = \bigoplus_{i, \bar{j} \in \mathcal{I}} R_i \otimes_{\mathbb{C}} \bar{R}_{\bar{j}}$, is the space of defect fields living on the invisible defect (labeled by R_0), so that $\mathscr{H} = \mathscr{H}^{0 \leftarrow 0}$ [Ru08].

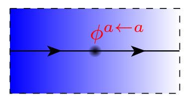
We will be interested in perturbing a defect of type a by a chiral defect field, that is by a field $\phi^{a \leftarrow a}$, of weight $(h, 0)$. The reason one perturbs with a chiral field of weight $(h,0)$ is that it can be shown that the defect commutes with the hamiltonian [Ru08]. Consider now a cylinder with a defect of type a placed on the circle wrapping the cylinder at $\mathbb{R}+z$, for $z \in \mathbb{C}$. Then the corresponding perturbed defect operator for D_a placed on the line $\mathbb{R}+z$, for $z \in \mathbb{C}$, is denoted by $D_a\left(\lambda\phi^{a\leftarrow a}\right)$ for some $\lambda \in \mathbb{C}$ and is obtained by inserting the exponential $\exp\left(\lambda \int_0^{2\pi} \phi^{a\leftarrow a}(x+z)\; dx\right)$ on the defect line. Explicitly,

$$D_a\left(\lambda\phi^{a\leftarrow a}; z\right) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \int_0^{2\pi} dx_1 \cdots dx_n \; D_a(x_1,\ldots,x_n;z)$$

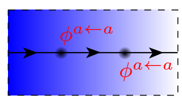
$$=$$



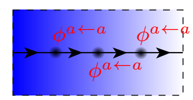
$$+\lambda \int_0^{2\pi} dx_1$$



$$+\frac{\lambda^2}{2}\int_0^{2\pi} dx_1 dx_2$$



$$+\frac{\lambda^3}{6}\int_0^{2\pi} dx_1 dx_2 dx_3$$



$$+\cdots \tag{2.3.4}$$

where the defect fields are inserted at the points $z+x_1,\ldots,z+x_n$ on the defect line.

Remark 2.3.1. To understand the physical meaning of the exponential, consider the path integral description of a field theory [BYB, Sect. 2.4.3] with a field ϕ and an action $S[\phi]$. Then consider the general unperturbed correlation function

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle_0 = \frac{1}{Z}\int D\phi\; \phi(x_1)\cdots\phi(x_n)e^{-S[\phi]}\;,$$

where Z is the vacuum functional. One can modify the action by terms localised on a line. For example, if ϕ lives on a cylinder of circumference L, one could replace $S[\phi]$ by $S[\phi]+\int_0^L V(\phi(x))\; dx$ for some potential V. More explicitly, the correlation function in the perturbed mode would be

$$\langle\phi(x_1)\cdots\phi(x_n)\rangle_{\text{pert}} = \frac{1}{Z_{\text{pert}}}\int D\phi\; \phi(x_1)\cdots\phi(x_n)e^{-\left(S[\phi]+\int_0^L V(\phi(x))\; dx\right)}\;.$$

Observe now that $\langle\phi(x_1)\cdots\phi(x_n)\rangle_{\text{pert}} = \langle\phi(x_1)\cdots\phi(x_n)e^{-\int_0^L V(\phi(x))\,dx}\rangle_0$. Thus the perturbed correlator is obtained by inserting $e^{-\int_0^L V(\phi(x))\,dx}$ into the unperturbed one. This is an example of a perturbed defect, where, however, the initial 'defect' is trivial.

Generally, one expects that if $\phi, \bar{\phi}$ has conformal weight $h_\phi, \bar{h}_{\bar{\phi}} < \frac{1}{2}$ then the multiple integrals in (2.3.4) converge, but we do not have a proof for that, see Sect. 5.2 for more details. Furthermore observe that

$$\frac{\partial}{\partial\bar{z}} D_a\left(\lambda\phi^{a\leftarrow a}; z\right) = 0\ , \tag{2.3.5}$$

since $\phi^{a\leftarrow a}$ is a chiral field; therefore $\frac{\partial}{\partial\bar{z}}$ annihilates each of the summands on the right hand side of (2.3.4). If one also performs a change of integration parameters in (2.3.5) we see that $D_a\left(\lambda\phi^{a\leftarrow a}; z\right) = D_a\left(\lambda\phi^{a\leftarrow a}; z + x\right)$. Combining these two observations it follows that

$$\frac{\partial}{\partial y} D_a\left(\lambda\phi^{a\leftarrow a}; iy\right) = 0\ , \quad y \in \mathbb{R}\ . \tag{2.3.6}$$

This means that one can move the perturbed defect along the cylinder without affecting the correlator under consideration, as long as the defect line does not cross any field insertions or other defect lines. Note also that $D_a\left(\lambda\phi^{a\leftarrow a}\right)$ still commutes with the anti-holomorphic modes of the chiral algebra and due to the simple decomposition (2.2.10) it has no choice but to also preserve the holomorphic representation. Thus it maps each sector $R_i \otimes_{\mathbb{C}} \bar{R}_i$ to itself.

2.4 Chirally Perturbed Topological Defects with $\mathfrak{su}(2)$-type Fusion Rules

In Sect. 2.2 we saw that topological defects inserted on a cylinder obey the fusion algebra of the representations of the underlying chiral algebra of the CFT. This algebra tells us how to fuse two or more of these defects and that the result will be a superposition of such defects. Furthermore, since the corresponding defect operators act as multiples of the identity on each sector of $R_i \otimes_{\mathbb{C}} \bar{R}_i$, then two such defect operators mutually commute[1]

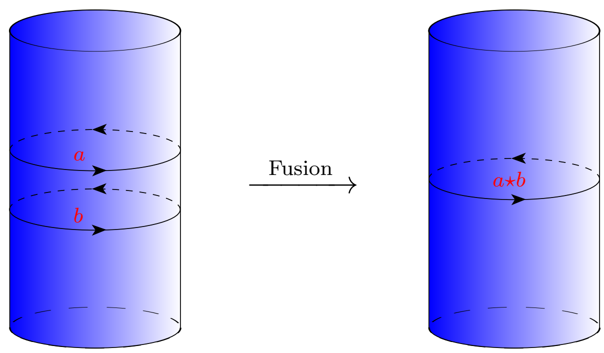
$$[D_a, D_b] = 0 \ , \quad \forall a, b \in \mathcal{I} \ . \tag{2.4.1}$$

Thus, one sees that the fusion algebra of defects is a commutative algebra. However, this is not always the case. If one switches on a perturbation in the system, then the algebra is in general not commutative, since, the corresponding *perturbed defect operators* do not act as multiples of the identity on each sector $R_i \otimes_{\mathbb{C}} \bar{R}_i$ anymore. However, as it was shown in [Ru08] there are special cases where there can be exceptions. It was also shown in [Ru08] that if the unperturbed defects satisfy $\mathfrak{su}(2)$ type fusion rules, then the operators associated to the perturbed defects obey functional relations known from the study of integrable models as T-systems. These functional relations are useful because, together with certain assumptions on their analytic properties, they can be solved in terms of a set of integral equations known as the thermodynamic Bethe ansatz [Za90], see [DDT07] for a review. This result can be used to explain the behaviour of the perturbed disc amplitudes, but it contains much more information than that since the defect operators act on all bulk states, not just on the ground state. In this section, we briefly review the results (and in most cases we just state them) of [Ru08] since they will play an important role in the work presented in this thesis.

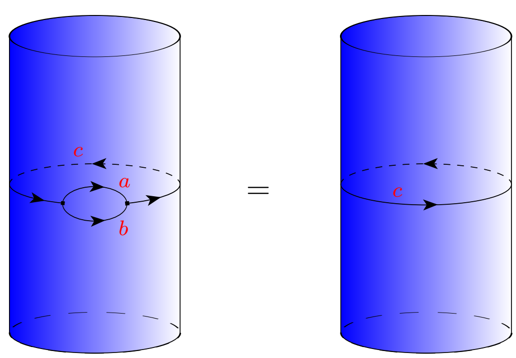
As we saw in Sect. 1.4 the hamiltonian of a CFT on a cylinder of circumference L is given by (1.4.3). Since the defects are topological, then $[H_{\text{cyl}}, D_a] = 0$ and therefore we can move a defect along the cylinder without affecting the correlator under consideration, as long as the defect line does not cross any fields or other defect lines. If two topological defects, a and b are inserted on adjacent loops on the cylinder, they can be fused into a single defect, denoted by $a \star b$, without encountering

[1] As an aside it is worth pointing out that all defects preserving $\mathfrak{V} \otimes_{\mathbb{C}} \bar{\mathfrak{V}}$ commute if and only if $M_{ij} \in \{0, 1\}$ for all entries of the modular matrix, specifying the decomposition of the space of bulk states cf. [Ru08, Sect. 2.5] and references therein.

a singularity.



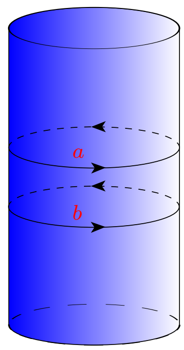
Defect lines can form junctions, for example when fusing two defects not along their entire length, but only along a segment. (A defect junction can alternatively be thought of as an insertion of a 'defect-joining field' of left/right conformal dimension 0.) The space of possible couplings joining two incoming defects a and b to an out-going defect c is ${\mathcal{N}_{ab}}^{c}$-dimensional [FFRS07]. The same holds when the roles of in- and out-going defects are reversed. In the nonzero coupling spaces (i.e. if ${\mathcal{N}_{ab}}^{c} = 1$) we choose, once and for all, basis elements such that



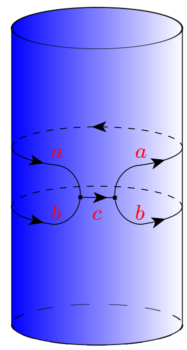
(2.4.2)

Equation (2.4.2) means that a 'defect bubble' without defect field insertions and which does not enclose any bulk fields can be omitted from the defect line. The identity (2.4.2) is valid locally on the surface under consideration in the sense that if the left hand side appears as part of a correlator, it can be replaced by the right hand side without affecting the value of the correlator.

Next, when fusing two defects along a segment one has the identity [FFRS07]

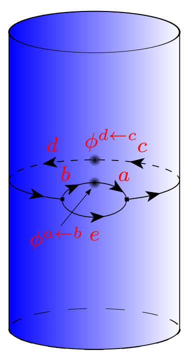


$$= \sum_{c \in \mathcal{I}} \qquad (2.4.3)$$

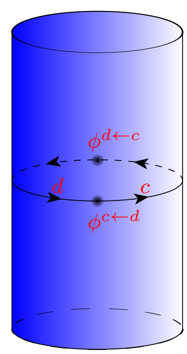


Here it is understood that the coupling of the defects a and b to c is zero if ${\mathcal{N}_{ab}}^{c} = 0$.

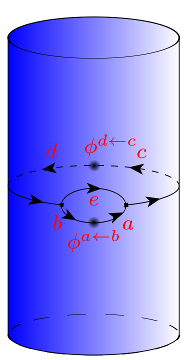
When collapsing a defect-bubble in the presence of defect fields, one finds the identities (see [Ru08, App. A.2])



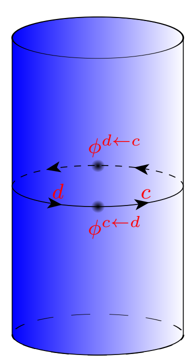
$$= \frac{\eta^{ab}}{\eta^{cd}} \mathsf{G}_{bc}^{(fae)d} \qquad (2.4.4)$$



as well as



$$= \frac{\eta^{ab}}{\eta^{cd}} \mathsf{G}_{bc}^{(fae)d} \frac{\mathsf{R}^{(be)d}}{\mathsf{R}^{(ae)c}}$$



(2.4.5)

where $\eta^{ab} \in \mathbb{C}$ describes the normalisation of $\phi^{a\leftarrow b}$, G is the inverse of the fusing matrix F and R is the braiding matrix. The F-matrix (as well as G) are the coefficients of a basis transformation in the category of representations of the VOA built on the vacuum representation R_0 (a so called ribbon category), while R interchanges two objects in the category, see Sect. 3.3 for a proper definition of $\mathsf{G}, \mathsf{F}, \mathsf{R}$. Note also that the superscript $f \in \mathcal{I}$ in G labels a fixed preferred representation and $\phi^{a\leftarrow b} \in R_f \otimes_{\mathbb{C}} \bar{R}_0 \subset \mathscr{H}^{a\leftarrow b}$.

On a superposition $a+b$ of defects, apart from perturbing by the defect fields $\phi^{a\leftarrow a}$ and $\phi^{b\leftarrow b}$ one can also perturb by the *defect changing fields* $\phi^{a\leftarrow b}$ and $\phi^{b\leftarrow a}$. The corresponding defect operator is

$$D_{a+b}\left(\lambda_{aa}\phi^{a\leftarrow a} + \lambda_{bb}\phi^{b\leftarrow b} + \lambda_{ab}\phi^{a\leftarrow b} + \lambda_{ba}\phi^{b\leftarrow a}\right) \ . \tag{2.4.6}$$

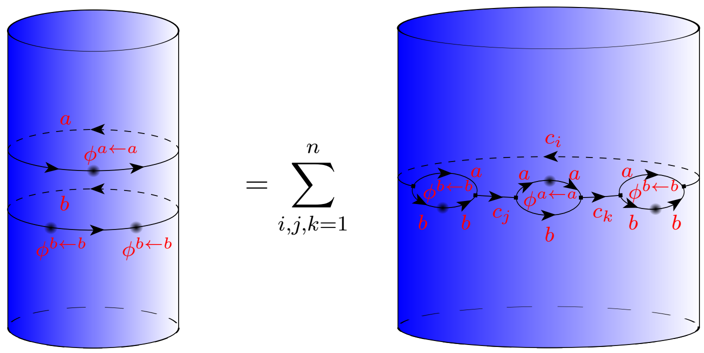
However, $\phi^{a\leftarrow b}(x)\phi^{a\leftarrow b}(y) = 0$ and $\phi^{a\leftarrow b}(x)\phi^{a\leftarrow a}(y) = 0$ and hence every $\phi^{a\leftarrow b}$ insertion must be paired with a $\phi^{b\leftarrow a}$ insertion, see (2.4.4) and (2.4.5) for example. In particular, if only $\phi^{a\leftarrow b}$ is involved in the perturbation, but not $\phi^{b\leftarrow a}$, no terms involving the defect changing field can contribute to the expansion of the exponential in the perturbed operator. Thus we have the identity

$$D_{a+b}\left(\lambda_{aa}\phi^{a\leftarrow a} + \lambda_{bb}\phi^{b\leftarrow b} + \lambda_{ab}\phi^{a\leftarrow b}\right) = D_{a+b}\left(\lambda_{aa}\phi^{a\leftarrow a} + \lambda_{bb}\phi^{b\leftarrow b}\right) \ . \tag{2.4.7}$$

Since the right hand side contains no contribution mixing the two defects, the perturbed operator is just the sum of the two individual perturbations,

$$D_{a+b}\left(\lambda_{aa}\phi^{a\leftarrow a} + \lambda_{bb}\phi^{b\leftarrow b} + \lambda_{ab}\phi^{a\leftarrow b}\right) = D_a\left(\lambda_{aa}\phi^{a\leftarrow a}\right) + D_b\left(\lambda_{bb}\phi^{b\leftarrow b}\right) \ . \tag{2.4.8}$$

Suppose now that the unperturbed defects a and b fuse to $a \star b = c_1 + \ldots + c_n$. To compute the result of the fusion in the perturbed case, we expand out the exponential generating the two perturbations and we use identities of the form



(2.4.9)

on each term. One can then apply (2.4.4) or (2.4.5) to collapse each of the defect bubbles to obtain the appropriate defect (changing) field. Altogether, for two chirally perturbed defects $D_a\left(\lambda\phi^{a\leftarrow a}\right)$ and $D_b\left(\mu\phi^{b\leftarrow b}\right)$ one gets the fusion

$$\boxed{D_a\left(\lambda\phi^{a\leftarrow a}\right)D_b\left(\mu\phi^{b\leftarrow b}\right) = D_{c_1+\ldots+c_n}\left(\sum_{i,j=1}^{n}\xi_{ij}\phi^{c_i\leftarrow c_j}\right)}\,, \tag{2.4.10}$$

where

$$\xi_{ij} = \lambda\frac{\eta^{aa}}{\eta^{c_ic_j}}\mathsf{G}_{ac_i}^{(fab)c_j} + \mu\frac{\eta^{bb}}{\eta^{c_ic_j}}\mathsf{G}_{ac_i}^{(fba)c_j}\frac{\mathsf{R}^{(ba)c_j}}{\mathsf{R}^{(ba)c_i}}\,, \tag{2.4.11}$$

[Ru08, Eqns. (2.20) and (2.21)].

Given for concreteness the minimal model $M(p,q)$, in the case of elementary topological defects, which are labeled by entries (r,s) in the Kac-table (modulo the usual $\mathbb{Z}_2$ identification), where $1 \leq r < p$ and $1 \leq s < q$, the fusion of these defects is given by the fusion of the corresponding irreducible representations of $\mathfrak{V}$.

In the case where the perturbing field has conformal weight less than $\frac{1}{2}$, it can be shown, that if one perturbs the defect by a chiral defect field ϕ of conformal weight $h_{1,3} = -1 + 2p/q$, in the subset of defects labeled by $(1,s)$, for $s = 2, \ldots, q-2$, the defect operators obey the functional relation

$$\boxed{D_{(1,s)}\left(\zeta\lambda\phi\right)D_{(1,s)}\left(\zeta^{-1}\lambda\phi\right) = \mathbf{1} + D_{(1,s-1)}\left(\lambda\phi\right)D_{(1,s+1)}\left(\lambda\phi\right)}\,, \tag{2.4.12}$$

for $\zeta = e^{i\pi p/q}$. It can also be shown that the perturbed defect operators mutually commute,

$$\boxed{\left[D_{(r,s)}\left(\lambda\phi\right), D_{(r',s')}\left(\mu\phi\right)\right] = 0}\,, \tag{2.4.13}$$

for all (r, s) and (r', s') in the Kac-table and for all $\lambda, \mu \in \mathbb{C}$.

Chapter 3

Topological Field Theory

Einstein's formulation of general relativity not only revolutionised our concepts of gravity, space and time but also linked geometry and physics. Since then the two are intertwined through string theory, supersymmetry and so on. In the new era of present developments, new links between mathematics and physics have emerged and one of those is between quantum physics and topology. One theory that does that is *topological field theory* (TFT). A TFT is a theory in which the output is unchanged under a variation of the metric on the background manifold, so that expectation values of observables must give rise to topological invariants of the manifold (e.g. Betti numbers).

Physical interest in TFTs comes mainly from the observation that they possess certain features one expects from a theory of quantum gravity [Cr93, Cr95, Sm03]. One of course can point out that there are local excitations in gravity, so it cannot be topological. However, TFTs serve as a toy model in which one can do calculations and gain experience before embarking on the quest for the full theory, which is expected to be much more complicated.

According to [At88] the best starting point is Witten's paper [Wi82] where he explained the geometric meaning of supersymmetry. Essentially what Witten showed is that QFTs should be viewed as the differential geometry of certain infinite-dimensional manifolds, including the associated analysis (e.g. Hodge theory) and topology (e.g.

Betti numbers).

According to [Law96] the first interesting TFT in three dimensions was introduced by Witten in 1989 [Wi89]. The partition function of the theory supplies invariants of 3-manifolds $\mathscr{M}$ in the form of a Feynman path integral. The data for this theory consists of an integer k, called the level, and a Lie group, G. From the same field theory, Witten also generated invariants of links embedded in $\mathscr{M}$, as the expectation value of a suitable observable known as a Wilson loop. In the simplest case when $\mathfrak{g} = \mathfrak{sl}(2)$ he showed that the partition function for a (framed) link in S^3 is just the value of the Jones polynomials for a suitable root of unity. In four dimensions, Witten [Wi88] produced a supersymmetric Lagrangian which formally reproduces the Donaldson theory [Do90]. Witten's formula can be understood as an infinite-dimensional analogue of the Gauss-Bonnet theorem (cf [At88]).

However, Witten's path integral approach, although it gives a topological invariant, has the disadvantage of not being defined rigorously, because it is unclear what measure one may put on the infinite dimensional space the path integral is over. Subsequently, Atiyah [At88] gave the axiomatic formulation of TFT, followed by Segal [Se89] and others (e.g. [RT91, Cr91, TV92]). According to Atiyah's axiomatic formulation (cf. [Ko, Intro.]) an n-dimensional TFT is a rule which to each closed oriented $n-1$-manifold X assigns a vector space $\mathcal{H}(X)$, and to each oriented n-manifold $\mathscr{M}$ with $\partial\mathscr{M} = X$ assigns a vector $Z(\mathscr{M})$ in $\mathcal{H}(X)$. This rule is subject to Atiyah's axioms which express that topologically equivalent manifolds have isomorphic associated vector spaces, and that disjoint unions of manifolds go to tensor products of vector spaces, etc. (see Def. 3.5.1).

Furthermore, one can formulate TFT in categorical terms. One can define the category of cobordisms $\mathbf{Cob}(n)$ (see Def. 3.2.1) whose objects are closed oriented $(n-1)$-manifolds X, and a morphism set $\mathrm{Hom}_{\mathbf{Cob}(n)}(X, X')$ whose elements are diffeomorphism classes (rel the boundary) of (compact topological closed) oriented n-manifolds $\mathscr{M}$, whose 'in-boundary' is X and whose 'out-boundary' is X' (see Def.

3.5.1). Composition of cobordisms is defined by gluing together the underlying manifolds along common boundary components. The operation of taking disjoint unions of manifolds and cobordisms gives this category monoidal structure. On the other hand, the category $\mathbf{Vect}_{\Bbbk}$ of vector spaces over a field $\Bbbk$, is monoidal under tensor products (see Sect. 3.3).

With this terminology one can define a TFT as a (symmetric) monoidal functor $(\mathcal{H}, Z)\colon \mathbf{Cob}(n) \to \mathbf{Vect}_{\Bbbk}$ (see Def. 3.5.1). This means roughly that, the closed manifolds represent space, while the cobordisms represent spacetime. The associated vector spaces are then the state spaces, and an operator associated to a spacetime is the time evolution operator (also called transition amplitude, or Feynman path integral). That the theory is topological means that the transition amplitudes do not depend on any additional structure on spacetime (like Riemannian metric or curvature), but only on the topology. In particular there is no time evolution along cylindrical spacetime. That disjoint union goes to tensor product expresses the principle in quantum mechanics that the state space of two independent systems is the tensor product of the two state spaces.

In the case of 2D TFT, the relations that hold in $\mathbf{Cob}(2)$ correspond precisely to the axioms of a commutative Frobenius algebra (this is due to Dijkgraaf [Di89]), i.e. there is a canonical equivalence of categories $2\mathbf{TFT} \simeq \mathbf{cFA}$ (see Thm. 3.5.3 [Ko, Thm. 3.3.2], see also [Ab96]).

In the context of rational 2D-CFT, it was shown in [RFFS05] that a (not necessarily commutative) symmetric special Frobenius algebra (in the braided monoidal category of representations of the vertex operator algebra giving the chiral symmetry) determines a CFT and that Morita-equivalent algebras give equivalent 2D CFTs. It is this latter relation that is further investigated in [RFFS05].

3.1 Cobordisms

In this section, following [Ko, Chap. 1] we discuss cobordisms in some detail. The notion of *cobordism* goes back to Pontryagin and Thorn [Th54] in 1954 (cf. [Ko, Chap. 1 Summary]). The name comes from the French word *bord* for boundary and the prefix 'co' has nothing to do with duality as it is used in categorical language. It simply means 'together'. Originally, a single manifold X was called *bordant* if it formed the boundary of some manifold $\mathscr{M}$, then two manifolds were called *cobordant* if together they formed the boundary of some manifold $\mathscr{M}$.

Before we start talking about cobordisms in more detail, it will be good at this point to introduce the concept of *in-* and *out-boundaries*. Let $\mathscr{M}$ be a n-manifold and X a closed submanifold of $\mathscr{M}$ of codimension 1 and assume both are oriented[1]. Suppose now X is a connected component of the boundary of $\mathscr{M}$; then it makes sense to ask whether the positive normal $\mathbf{n}$ points inwards or outwards compared to the induced orientation by $\mathscr{M}$ – locally the situation is that of a vector in $\mathbb{R}^n$ for which we ask whether it points in or out from the half-space $\mathbb{H}^n = \{x \in \mathbb{R}^n | \Lambda(x) \geq 0\}$, where $\Lambda \colon \mathbb{R}^n \to \mathbb{R}$ is a nonzero linear map.

Definition 3.1.1. Let X, Y and $\mathscr{M}$ be as above and let $\partial\mathscr{M} = \overline{X} \amalg Y$. Then X is called an *in-boundary* and Y an *out-boundary*. We write $X =: \partial_-\mathscr{M}$ and $Y =: \partial_+\mathscr{M}$.

Thus, the boundary of a manifold $\mathscr{M}$ is the union of various in- and out-boundaries. The in-boundary of $\mathscr{M}$ may be empty, and the out-boundary may also be empty. Note that if we reverse the orientation of both $\mathscr{M}$ and its boundary X, then the notion of what in-boundary or out-boundary are, is still the same.

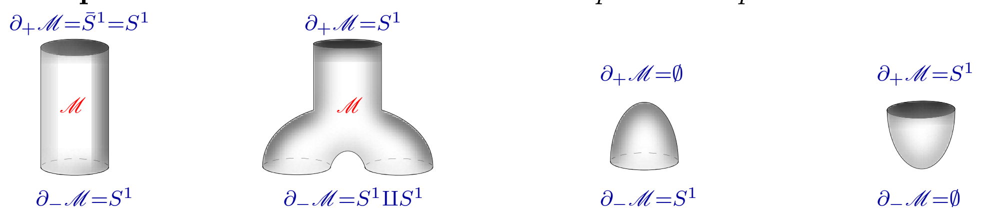
Definition 3.1.2. An *oriented cobordism* $\partial_-\mathscr{M} \xrightarrow{\mathscr{M}} \partial_+\mathscr{M}$ is a triple $(\mathscr{M}, \partial_-\mathscr{M}, \partial_+\mathscr{M})$ together with two orientation preserving diffeomorphisms[2] $\partial_-\mathscr{M} \to \mathscr{M} \leftarrow \partial_+\mathscr{M}$.

[1] In this chapter $X, Y, \ldots$ always denote $(n-1)$-manifolds, while $\mathscr{M}, \mathscr{N}, \ldots$ n-manifolds. Furthermore, the corresponding manifolds with opposite orientation are denoted by $\overline{X}, \overline{Y}, \ldots$ and $\overline{\mathscr{M}}, \overline{\mathscr{N}}, \ldots$ respectively.

[2] See Def. 3.1.3 for more details.

When a cobordism exists, $\partial_-\mathscr{M}$ and $\partial_+\mathscr{M}$ are said to be *cobordant*. $\partial_-\mathscr{M}$ and $\partial_+\mathscr{M}$ are also called the *bottom* and the *top* bases of the cobordism.

Example 3.1.1. *In two dimensions the simplest examples are:*



(i) cylinder (ii) pair of pants (iii) death of a circle (iv) birth of a circle

where the pictures are to be read from bottom to top.

A cobordism can be thought of as an interpolation between the two (boundary) manifolds. Another analogy is that of a history or a movie. In the context of string theory one may think of cobordism (ii) as the time propagation and merging of two closed strings.

One may also think of the cobordism as describing an evolution in time, say from time $t = 0$ to time $t = 1$. In other words we consider a smooth map from $\mathscr{M}$ to the unit interval $I = [0, 1]$ such that $\partial_-\mathscr{M}$ maps to 0 and $\partial_+\mathscr{M}$ maps to 1.

Remark 3.1.1. Cobordisms are *not* functions. An oriented cobordism is something that goes from one manifold $\partial_-\mathscr{M}$ to another manifold $\partial_+\mathscr{M}$. It makes no sense to ask what it does to a particular point of $\partial_-\mathscr{M}$. Note that we can have a cobordism from a nonempty manifold to $\emptyset$. This is not possible with functions of any kind.

Definition 3.1.3. Given two oriented cobordisms $(\mathscr{M}, \partial_-\mathscr{M}, \partial_+\mathscr{M})$ and $(\mathscr{M}', \partial_-\mathscr{M}, \partial_+\mathscr{M})$ from $\partial_-\mathscr{M}$ to $\partial_+\mathscr{M}$, together with the orientation preserving diffeomorphisms

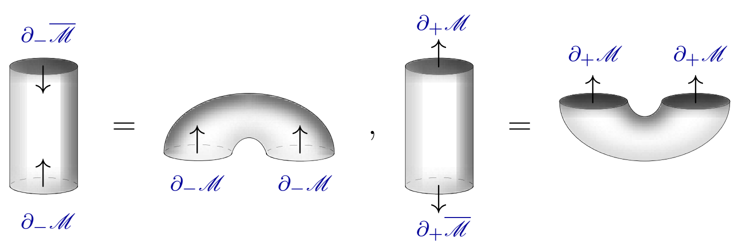
$$\begin{array}{ccc} & \mathscr{M}' & \\ \nearrow & & \nwarrow \\ \partial_-\mathscr{M} & & \partial_+\mathscr{M} \\ \searrow & & \swarrow \\ & \mathscr{M} & \end{array} \qquad (3.1.1)$$

we say they are *equivalent* if there is an orientation preserving diffeomorphism $\psi\colon \mathscr{M} \xrightarrow{\sim} \mathscr{M}'$ making this diagram commute:

$$\begin{array}{ccccc} & & \mathscr{M}' & & \\ & \nearrow & \uparrow & \nwarrow & \\ \partial_-\mathscr{M} & \circlearrowleft & \psi \,\big|\, \wr & \circlearrowleft & \partial_+\mathscr{M} \\ & \searrow & & \swarrow & \\ & & \mathscr{M} & & \end{array} \tag{3.1.2}$$

Note that the two triangles truly commute not just up to diffeomorphisms. Since $\partial_-\mathscr{M}$ and $\partial_+\mathscr{M}$ are submanifolds in $\mathscr{M}$ (and in $\mathscr{M}'$) situated on the boundary of $\mathscr{M}$ (and $\mathscr{M}'$) so that the maps $\partial_-\mathscr{M} \to \mathscr{M}$ and $\partial_-\mathscr{M} \to \mathscr{M}'$ are the corresponding embedding maps, then $\psi|_{\partial_-\mathscr{M}}$, induces the identity map on the boundaries. The same holds for $\partial_+\mathscr{M}$ [Ko, Def. 1.2.17].

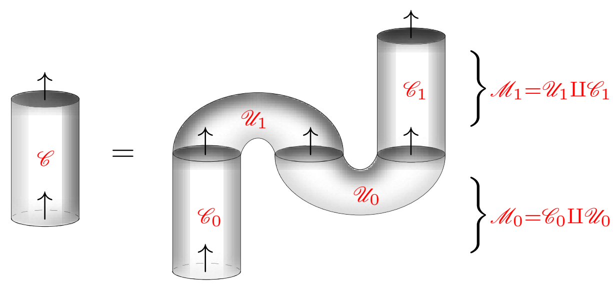
One can also consider cylinders with both boundaries being in or out-boundaries. In this case we have the following pictures respectively



(3.1.3)

The equal sign in (3.1.3) and in the pictures that will follow denotes that the cobordisms are equivalent. In the case at hand for example, it means that the cylinder can be drawn as a U-tube.

An important feature of cobordisms is that one can decompose them. In the movie analogy, this means that we take some intermediate frame (corresponding to time t) and regard it as a submanifold in $\mathscr{M}$ which splits it into two parts (not necessarily connected). For example, one can use (3.1.3) to decompose the cylinder as



(3.1.4)

This is called the *'snake decomposition'* of the cylinder (see [Ko, Sect. 1.2.21] for more details). Thus, we have found a decomposition of a cylinder into two cobordisms, $\mathscr{M}_0$ and $\mathscr{M}_1$, which are not cylinders.

3.2 The Category Cob(n)

The clearest formulation of bordisms[3] is in categorical terms (see for example [Lu09][Ko, Sect. 1.3 & 1.4.13]).

Definition 3.2.1. Let $n \in \mathbb{N}$. The category $\mathbf{Cob}(n)$ has as objects $(n-1)$-manifolds $X, Y, \ldots$ and a morphism $\mathscr{B} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(X, Y)$ is an equivalence class of bordisms from X to Y, that is, an oriented n-dimensional manifold $\mathscr{B}$ equipped with an orientation preserving diffeomorphism $\partial\mathscr{B} \cong \overline{X} \amalg Y$.

The identity morphism id_X is represented by the product bordism $\mathscr{B} = X \times I$, i.e. the cylinder.

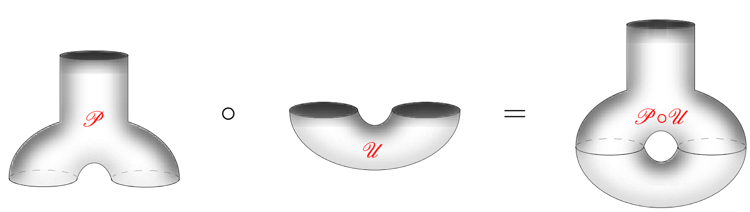
For a triple $X, X', X'' \in \mathbf{Cob}(n)$ and a pair of bordisms, $\mathscr{B} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(X, X')$ and $\mathscr{B}' \in \mathrm{Hom}_{\mathbf{Cob}(n)}(X', X'')$, composition of morphisms in $\mathbf{Cob}(n)$ is defined to be the morphism represented by the manifold $\mathscr{B}' \circ \mathscr{B} := \mathscr{B}' \amalg_{X'} \mathscr{B} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(X, X'')$,

[3] From now on we will abbreviate the word 'cobordism' by 'bordism' for short. That is, cobordism and bordism are the same thing.

i.e. we glue $\mathscr{B}'$ with $\mathscr{B}$ along the common boundary component X'. This completes the definition.

Remark 3.2.1. (i) The composition law for bordisms described in the above definition is potentially ambiguous, because we did not explain how to endow the manifold $\mathscr{B}' \circ \mathscr{B} := \mathscr{B}' \amalg_{X'} \mathscr{B}$ with a smooth structure. However, the composition law is well defined up to diffeomorphisms, but not up to a unique one. In other words, there is not a universal property. This problem is solved (at least in dimensions $n < 3$) if we consider *diffeomorphism classes of bordisms* instead of bordisms. In other words, the arrows are cobordism classes in the sense of Def. 3.1.3. So instead of speaking about *the* composition, one could speak only of *a* composition (for more information on this technical issue see [Ko, Sect. 1.3]). Such considerations lead to the notions of *higher-dimensional categories* where we have usual arrows (in dimension 1), arrows between arrows (dimension 2), and so on (see [Lu09] or [Pr09] for example). (ii) We regard two bordisms $\mathscr{B}$ and $\mathscr{B}'$ as defining the same morphism in $\mathbf{Cob}(n)$ if they are equivalent (see Def. 3.1.3). This equivalence extends to the evident diffeomorphism $\partial\mathscr{B} \cong \overline{X} \amalg Y \cong \partial\mathscr{B}'$ between their boundaries.

Example 3.2.1. *Consider the bordisms* $\mathscr{P} \in \mathrm{Hom}_{\mathbf{Cob}(2)}(S^1 \amalg S^1, S^1)$ *and* $\mathscr{U} \in \mathrm{Hom}_{\mathbf{Cob}(2)}(\emptyset, S^1 \amalg S^1)$*, then we can compose them as* $\mathscr{P} \circ \mathscr{U} = \mathscr{P} \amalg_{S^1 \amalg S^1} \mathscr{U} \in \mathrm{Hom}_{\mathbf{Cob}(2)}(\emptyset, S^1)$*. In terms of pictures:*



(3.2.1)

Following [Di97, Sect. 4], we have one extra operation that is not standard in categories. We also want to be able to glue two boundary components of a single irreducible manifold.

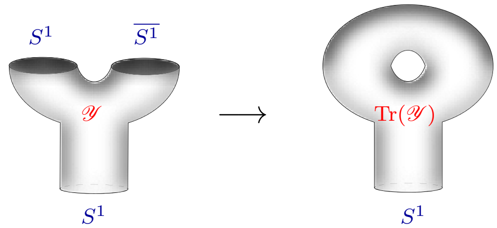
Definition 3.2.2. Let $\mathscr{M} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(\partial_{-}\mathscr{M}, \partial_{+}\mathscr{M})$. If two boundary components of $\mathscr{M}$ contain a common factor $X \in \mathbf{Cob}(n)$, we define the *partial trace*

$$\mathrm{Tr}_X \colon \mathscr{M} \to \mathrm{Tr}_X(\mathscr{M}) \ . \tag{3.2.2}$$

This is best explained in the following example.

Example 3.2.2. *For* $\mathscr{Y} \in \mathrm{Hom}_{\mathbf{Cob}(2)}\left(S^1, S^1 \amalg \overline{S^1}\right)$, *we get:*

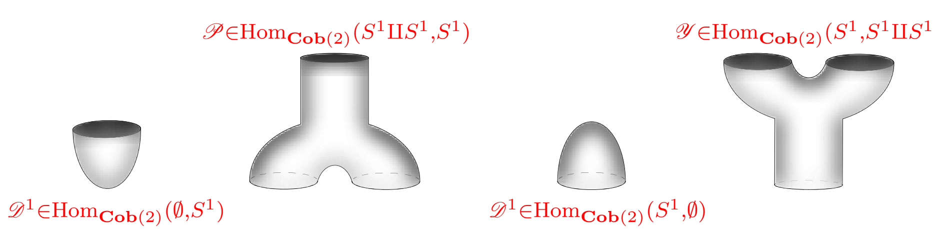
Tr_{S^1} :



$\in \mathrm{Hom}_{\mathbf{Cob}(2)}(S^1, \emptyset)$. (3.2.3)

In dimension two, 'everything is known' since surfaces are completely classified, so one can describe $\mathbf{Cob}(2)$ completely.

Proposition 3.2.3. *The category* $\mathbf{Cob}(2)$ *is generated under composition and disjoint union by* [Ko, Prop. 1.4.13]



(3.2.4)

As will be seen in Sect. 3.4, these are also the generators of a commutative Frobenius algebra and they are called: *unit*, *multiplication*, *co-unit* and *co-multiplication* respectively, subject to some relations.

3.3 Monoidal Structure and Graphical Calculus

The category structure describes how to connect bordisms in series; in other words, how to connect the output of one bordism to the input of another (composition of bordisms, see Example 3.2.1). It is also important to consider parallel couplings; that is, disjoint union of bordisms. This amounts to giving *symmetric monoidal structure* (cf. Def. 3.3.6) to the category $\mathbf{Cob}(n)$. In this section, we introduce all the categorical machinery that will be of interest for this thesis and which is necessary in order to make the connection with the CFT language and with defects presented in the previous chapters. For this reason, we just state the results that we need, for a detailed exposition the reader is referred to [McL, Chapters VII.1-3 & VIII] or [BK, Chapters 1-4]. We also introduce the graphical notation for morphisms in an Abelian monoidal category, following the conventions of [FRS02-I].

We start by recalling the definition of an Abelian category:

Definition 3.3.1. An abelian category $\mathcal{C}$ is an Ab-category[4] satisfying the following conditions:

(i) there is a zero object $\mathbf{0} \in \mathcal{C}$,

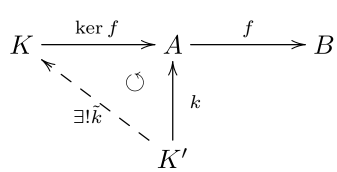
(ii) it has binary biproducts[5],

(iii) it has kernels and cokernels[6],

[4]That is, each Hom-set is an additive abelian group and composition is bilinear.

[5]For the definition of binary biproducts see equation (B.0.1).

[6]Let $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ be a morphism in $\mathcal{C}$. A *kernel* is a pair $(K, \ker f)$, where $K \in \mathcal{C}$ and $\ker f\colon K \to A$ such that the diagram commutes



This diagram describes the universal property of the kernel. Namely, for each $k\colon K' \to A$ such that $f \circ k = 0$, there exists a unique $\tilde{k}\colon K' \to K$ such that $\ker f \circ \tilde{k} = k$. The dual concept to that of kernel is that of a *cokernel*. In other words, for a cokernel consider the above diagram but with all its arrows reversed and in place of $\ker f$ put $\mathrm{cok}\, f$. Then the universal property of the cokernel

(iv) every monomorphism is a kernel and every epimorphism is a cokernel[7].

Note that the first two conditions insure that $\mathcal{C}$ is additive [McL, Sect. VIII. 3].

Definition 3.3.2. A monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}, \alpha, \lambda, \rho)$, consists of the following data: a category $\mathcal{C}$, a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, an identity object $\mathbb{1} \in \mathcal{C}$ and three natural isomorphisms α, λ, ρ. We require, the *associator*

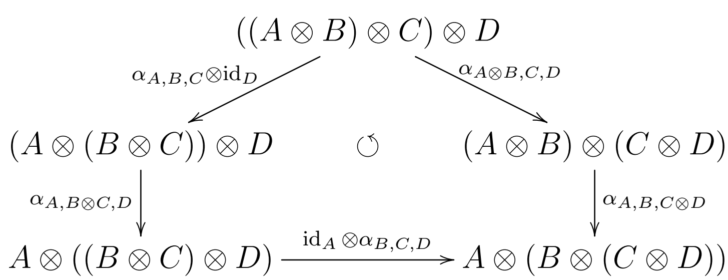
$$\alpha_{A,B,C}\colon (A \otimes B) \otimes C \xrightarrow{\sim} A \otimes (B \otimes C) \;, \tag{3.3.1}$$

and the two *unit isomorphisms*

$$\lambda_A\colon \mathbb{1} \otimes A \xrightarrow{\sim} A \text{ and } \rho_A\colon A \otimes \mathbb{1} \xrightarrow{\sim} A \;, \tag{3.3.2}$$

to be natural for all $A, B, C \in \mathcal{C}$ and to satisfy the following coherence conditions:

(i) **Pentagon axiom**: For any $A, B, C, D \in \mathcal{C}$, the diagram commutes



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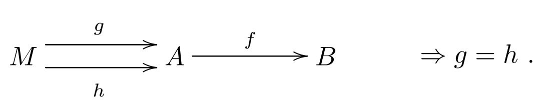
states that for each $k\colon A \to K'$ such that $k \circ f = 0$, there exists a unique $\tilde{k}\colon K \to K'$ such that $\operatorname{cok} f \circ \tilde{k} = k$.

[7] A *monomorphism* (also called *mono* or *monic*), is a morphism $f \in \operatorname{Hom}_{\mathcal{C}}(A, B)$, such that for all morphisms $g, h \in \operatorname{Hom}_{\mathcal{C}}(M, A)$



Namely, a monomorphism is a left-cancellative morphism, that is, an arrow $f\colon A \to B$ such that, for all morphisms $g, h\colon M \to A$, the equality $f \circ g = f \circ h$ implies $g = h$. The categorical dual of a monomorphism is an *epimorphism*. That is, a monomorphism in a category $\mathcal{C}$ is an epimorphism in the opposite category. Namely, consider the above diagram with all the arrows reversed, then an epimorphism is a right-cancellative morphism, that is, an arrow $f\colon B \to A$ such that, for all morphism $g, h\colon A \to M$ the equality $g \circ f = h \circ f$ implies $g = h$.

(ii) **Triangle axiom**: For any $A, B \in \mathcal{C}$, the diagram commutes

$$\begin{array}{ccc} (A \otimes \mathbb{1}) \otimes B & \xrightarrow{\alpha_{A,\mathbb{1},B}} & A \otimes (\mathbb{1} \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \circlearrowleft \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array}$$

Example 3.3.1. *The category* $\mathbf{Vect}_{\Bbbk}$ *of all vector spaces over a given field* $\Bbbk$*, with the usual tensor product* $\otimes_{\Bbbk}$ *of vector spaces, and with the one dimensional vector space* $\Bbbk$ *as unit, is a standard example of a monoidal category. Monoidal categories are often called* tensor categories.

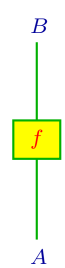
For the categories considered in this thesis we define a simple object as[8]

Definition 3.3.3. A *simple* (or irreducible) object A of an Abelian monoidal category $\mathcal{C}$, is an object for which $\mathrm{End}_{\mathcal{C}}(A) = \Bbbk\, \mathrm{id}_A$.

Definition 3.3.4. A *strict* monoidal category $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1})$ is a monoidal category $\mathcal{C}$, for which the isomorphisms α, λ, ρ are the identity morphisms.

A very useful way to represent morphisms in an Abelian monoidal category $\mathcal{C}$ is via graphs (cf. [FRS02-I, Sects. 2.1 & 2.2]), where lines stand for the identity morphisms. In this way, the identity morphism id_A and $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, are represented as

$\mathrm{id}_A \equiv$ 

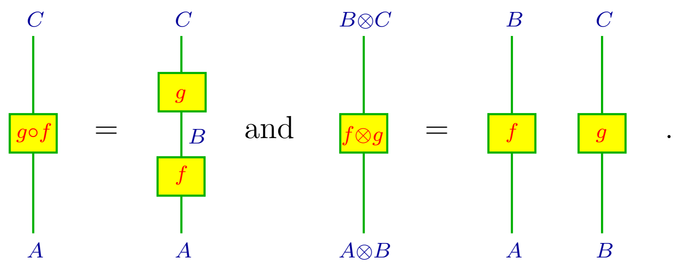
, $f \equiv$ 

. (3.3.3)

[8]In a general category, this defines an absolutely simple object, while simplicity of an object means that it does not possess a non-trivial proper subobject. In semisimple categories, absolutely simple implies simple, and in any abelian category over an algebraically closed ground field the two notions are equivalent cf. [FRS02-I, Footnote 3].

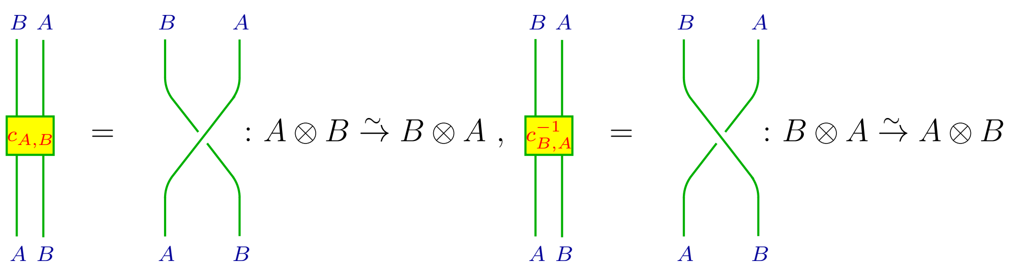
All the pictures are to be read from bottom to top. In particular, the tensor unit $\mathbb{1} \in \mathcal{C}$ is simple, thus $\mathrm{id}_{\mathbb{1}} = 1 \in \Bbbk$, therefore lines labeled by the tensor unit will be omitted, so that in the pictorial representation, morphisms in $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, A)$ or $\mathrm{Hom}_{\mathcal{C}}(A, \mathbb{1})$ emerge from and disappear into 'nothing', respectively. Furthermore, whenever we use the pictorial notation we silently pass to a strict version of $\mathcal{C}$. The non strict case follows by invoking coherence [McL, Sect. VII.2] and verifying that the α, ρ and λ sit in the required places.

In the pictorial notation, composition of two morphisms $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, C)$, amounts to concatenation of lines, while the tensor product to juxtaposition



(3.3.4)

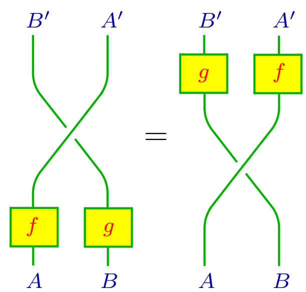
Definition 3.3.5. A *braided* monoidal category is a monoidal category $\mathcal{C}$, together with a family of braiding isomorphisms



(3.3.5)

natural in $A, B \in \mathcal{C}$, i.e. $c_{A',B'} \circ (f \otimes g) = (g \otimes f) \circ c_{A,B}$, for any morphisms $f \in$

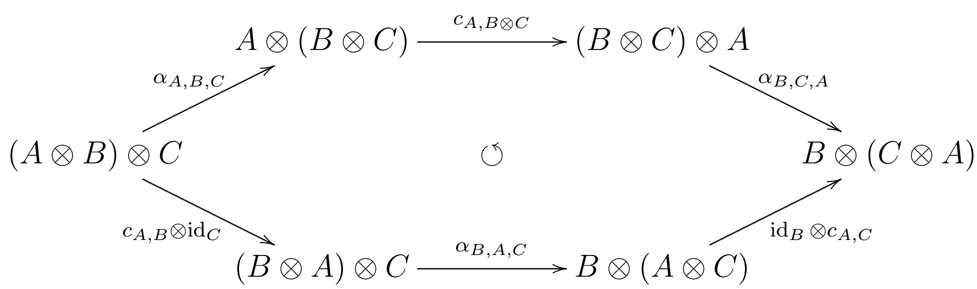
$\mathrm{Hom}_{\mathcal{C}}(A, A')$ and $g \in \mathrm{Hom}_{\mathcal{C}}(B, B')$. In terms of pictures:



(3.3.6)

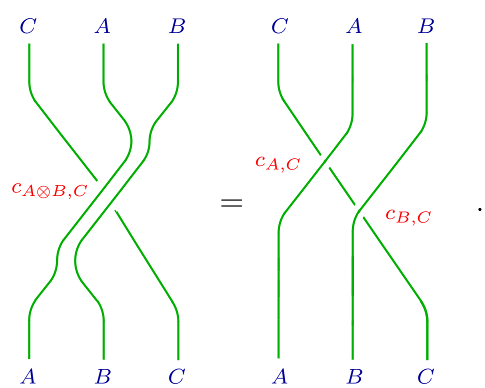
The braiding c and the associator α are required to be *tensorial*, i.e. to satisfy

Hexagon axioms: (a) For any $A, B, C \in \mathcal{C}$, the diagram commutes



(b) The same as in (a) but with c^{-1} in place of c and their arguments exchanged.

After passing to a strict category the above diagram is expressed in terms of pictures as:



(3.3.7)

Definition 3.3.6. A *symmetric* monoidal category is a braided monoidal category $\mathcal{C}$, such that, all the braiding isomorphisms satisfy $c^2 = \mathrm{id}$.

Example 3.3.2. *For each $n \in \mathbb{N}$, the category* $\mathbf{Cob}(n) = (\mathbf{Cob}(n), \amalg, \emptyset)$ *can be endowed with the structure of a symmetric monoidal category, where the tensor product is given by the disjoint union of manifolds* $\amalg \colon \mathbf{Cob}(n) \times \mathbf{Cob}(n) \to \mathbf{Cob}(n)$*. The unit object of* $\mathbf{Cob}(n)$ *is the empty set* $\emptyset$ *(of dimension $(n-1)$).*

Definition 3.3.7. Given a pair of symmetric monoidal categories $\mathcal{C} = (\mathcal{C}, \otimes, \mathbb{1}_{\mathcal{C}}, \alpha_{\mathcal{C}}, \lambda_{\mathcal{C}}, \rho_{\mathcal{C}})$ and $\mathcal{D} = (\mathcal{D}, \circledast, \mathbb{1}_{\mathcal{D}}, \alpha_{\mathcal{D}}, \lambda_{\mathcal{D}}, \rho_{\mathcal{D}})$, a *symmetric monoidal functor* from $\mathcal{C}$ to $\mathcal{D}$, is a triple (H, H^2, H^0), consisting of a functor $H \colon \mathcal{C} \to \mathcal{D}$, together with a natural transformation

$$H^2_{A,B} \colon H(A) \circledast H(B) \xrightarrow{\sim} H(A \otimes B) \ , \tag{3.3.8}$$

and an isomorphism $H^0 \colon \mathbb{1}_{\mathcal{D}} \xrightarrow{\sim} H(\mathbb{1}_{\mathcal{C}})$, such that the following diagrams commute,

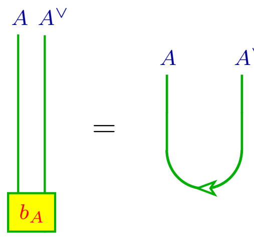
$$\begin{array}{ccccc}
(H(A) \circledast H(B)) \circledast H(C) & \xrightarrow{H^2_{A,B} \circledast \mathrm{id}_{H(C)}} & H(A \otimes B) \circledast H(C) & \xrightarrow{H^2_{A \otimes B, C}} & H((A \otimes B) \otimes C) \\
\Big\downarrow {\scriptstyle \alpha_{H(A),H(B),H(C)}} & & \circlearrowleft & & \Big\downarrow {\scriptstyle H(\alpha_{A,B,C})} \\
H(A) \circledast (H(B) \circledast H(C)) & \xrightarrow{\mathrm{id}_{H(A)} \circledast H^2_{B,C}} & H(A) \circledast H(B \otimes C) & \xrightarrow{H^2_{A, B \otimes C}} & H(A \otimes (B \otimes C))
\end{array}$$

$$\begin{array}{ccc}
\mathbb{1}_{\mathcal{D}} \circledast H(A) & \xrightarrow{\lambda_{H(A)}} & H(A) \\
\Big\downarrow {\scriptstyle H^0 \circledast \mathrm{id}_{H(A)}} & \circlearrowleft & \Big\downarrow {\scriptstyle H(\lambda_A)} \\
H(\mathbb{1}_{\mathcal{C}}) \circledast H(A) & \xrightarrow{H^2_{\mathbb{1}_{\mathcal{C}},A}} & H(\mathbb{1}_{\mathcal{C}} \otimes A)
\end{array}
\qquad
\begin{array}{ccc}
H(A) \circledast \mathbb{1}_{\mathcal{D}} & \xrightarrow{\rho_{H(A)}} & H(A) \\
\Big\downarrow {\scriptstyle \mathrm{id}_{H(A)} \circledast H^0} & \circlearrowleft & \Big\downarrow {\scriptstyle H(\rho_A)} \\
H(A) \circledast H(\mathbb{1}_{\mathcal{C}}) & \xrightarrow{H^2_{A,\mathbb{1}_{\mathcal{C}}}} & H(A \otimes \mathbb{1}_{\mathcal{C}})
\end{array}$$

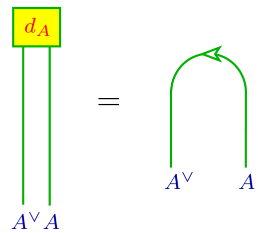
$$\begin{array}{ccc}
H(A) \circledast H(B) & \xrightarrow{c_{H(A),H(B)}} & H(B) \circledast H(A) \\
\Big\downarrow {\scriptstyle H^2_{A,B}} & \circlearrowleft & \Big\downarrow {\scriptstyle H^2_{B,A}} \\
Hs(A \otimes B) & \xrightarrow{H(c_{A,B})} & H(B \otimes A)
\end{array}$$

for all $A, B, C \in \mathcal{C}$. If $\mathcal{C}, \mathcal{D}$ are not symmetric, then H is *braided monoidal*.

Definition 3.3.8. A right-duality on a monoidal category $\mathcal{C}$, associates to every object $A \in \mathcal{C}$, another object $A^\vee \in \mathcal{C}$, called the right-dual object, together with morphisms



$: \mathbb{1} \to A \otimes A^\vee$,

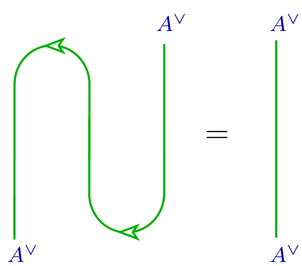


$: A^\vee \otimes A \to \mathbb{1}$, (3.3.9)

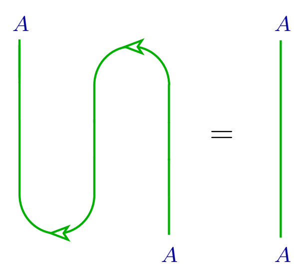
such that the following diagrams commute

$$\begin{array}{ccccccc} A^\vee & \xrightarrow{\rho^{-1}_{A^\vee}} & A^\vee \otimes \mathbb{1} & \xrightarrow{\mathrm{id}_{A^\vee} \otimes b_A} & A^\vee \otimes (A \otimes A^\vee) \\ {\scriptstyle \mathrm{id}_{A^\vee}}\downarrow & & \circlearrowleft & & \downarrow{\scriptstyle \alpha_{A^\vee,A,A^\vee}} \\ A^\vee & \xleftarrow[\lambda_{A^\vee}]{} & \mathbb{1} \otimes A^\vee & \xleftarrow[d_A \otimes \mathrm{id}_{A^\vee}]{} & (A^\vee \otimes A) \otimes A^\vee \end{array} \qquad \begin{array}{ccccccc} A & \xrightarrow{\lambda^{-1}_{A}} & \mathbb{1} \otimes A & \xrightarrow{b_A \otimes \mathrm{id}_A} & (A \otimes A^\vee) \otimes A \\ {\scriptstyle \mathrm{id}_{A}}\downarrow & & \circlearrowleft & & \downarrow{\scriptstyle \alpha^{-1}_{A,A^\vee,A}} \\ A & \xleftarrow[\rho_{A}]{} & A \otimes \mathbb{1} & \xleftarrow[\mathrm{id}_A \otimes d_A]{} & A \otimes (A^\vee \otimes A) \end{array}$$

for all $A \in \mathcal{C}$. These two commuting diagrams are called the *rigidity axioms* or *right duality axioms* and in terms of pictures they are given by

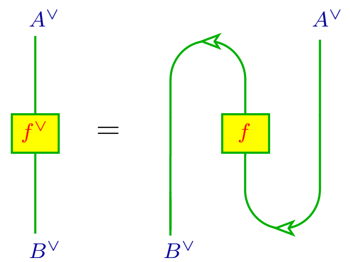


,



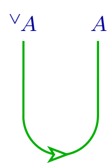
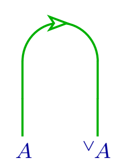
. (3.3.10)

Furthermore, a right duality, associates to every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, the morphism (see [BK, Lemma 2.1.6])



$$= \lambda_{A^\vee} \circ (d_B \otimes \mathrm{id}_{A^\vee}) \circ \alpha^{-1}_{B^\vee,B,A^\vee} \circ (\mathrm{id}_{B^\vee} \otimes f \otimes \mathrm{id}_{A^\vee}) \circ (\mathrm{id}_{B^\vee} \otimes b_A) \circ \rho_{B^\vee}$$
$$\in \mathrm{Hom}_{\mathcal{C}}(B^\vee, A^\vee) \ . \tag{3.3.11}$$

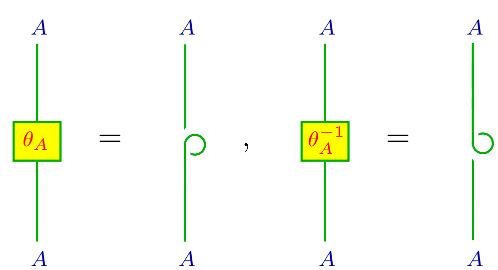
Similarly, a left-duality, associates to every object $A \in \mathcal{C}$ the left-dual object ${}^\vee A \in \mathcal{C}$ together with morphisms

$\tilde{b}_A \equiv$  $: \mathbb{1} \to {}^\vee A \otimes A \ , \quad \tilde{d}_A \equiv$  $: A \otimes {}^\vee A \to \mathbb{1} \ , \qquad (3.3.12)$

such that similar diagrams as in Def. 3.3.8 commute, and to every morphism $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ it associates the morphism ${}^\vee f \in \mathrm{Hom}_{\mathcal{C}}({}^\vee B, {}^\vee A)$.

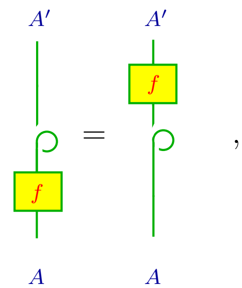
Definition 3.3.9. If every object in a monoidal category $\mathcal{C}$ has left and right duals, then $\mathcal{C}$ is called *rigid*.

Definition 3.3.10. A strict, braided monoidal category, is a *ribbon* category if it is rigid, and it comes together with a family of isomorphisms



$\in \mathrm{Hom}_{\mathcal{C}}(A, A) \ , \qquad (3.3.13)$

called the *twist*, one for each object $A \in \mathcal{C}$, and which is natural for any $f \in \mathrm{Hom}_{\mathcal{C}}(A, A')$

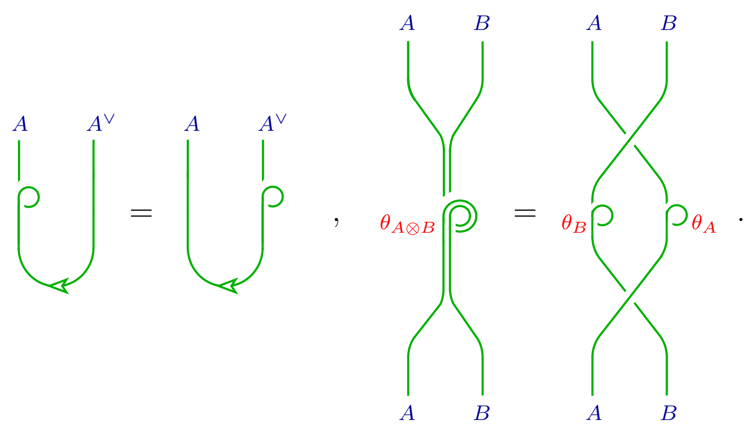


$$, \qquad (3.3.14)$$

and satisfies the *balancing axioms*:

$$\theta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}} = 1 \in \Bbbk \ , \quad \theta_{A^\vee} = (\theta_A)^\vee \ , \quad \theta_{A\otimes B} = c_{A,B} \circ (\theta_B \otimes \theta_A) \circ c_{B,A} \ . \qquad (3.3.15)$$

The last two can be expressed in terms of pictures as



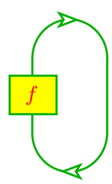
(3.3.16)

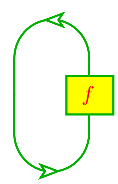
Remark 3.3.1. The reason one needs to impose the consistency conditions (3.3.6), (3.3.7), (3.3.10), (3.3.14) and (3.3.16) is that in a ribbon category, the morphisms are ribbons rather than lines[9] (hence the term) and the above axioms guarantee that the visualisation via ribbons is appropriate. For example, the picture drawn for the twist θ_A, cf. (3.3.13) is isotopic to the picture of id_A. If we had drawn a ribbon instead, then θ_A would no longer be isotopic to the trivial ribbon [BK, Sect. 2.3]. This means

[9]In this thesis, the interpretation with lines as ribbons is implicit. However, we talk a bit more about ribbons in Sect. 3.5.2 and we draw the first genuine ribbon graph in Sect. 5.1.

that the graphs one obtains by the composition of the duality, braid and twist, among themselves, share the properties of the corresponding glued ribbons [RT91].

Furthermore, a ribbon category comes equipped by definition with a left duality, defined on objects as ${}^{\vee}A := A^{\vee}$ and left duality morphisms (3.3.12). One can check that this left duality coincides with the right duality also on morphisms, i.e. ${}^{\vee}f = f^{\vee}$. Categories with coinciding left and right dualities are called *sovereign*. Thus, for example, the double dual $(A^{\vee})^{\vee}$ of an object $A \in \mathcal{C}$, is isomorphic (not equal in general) to A. Furthermore, since we have two dualities, one can define the left and right traces of $f \in \mathrm{End}_{\mathcal{C}}(A)$, via

$\mathrm{Tr}_{\mathrm{r}}(f) \equiv$ 

, $\mathrm{Tr}_{\ell}(f) \equiv$ 

$\in \mathrm{End}_{\Bbbk}(\mathbb{1}) = \Bbbk$, (3.3.17)

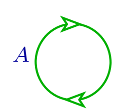
which are cyclic

$$\mathrm{Tr}_{\ell,r}(g \circ f) = \mathrm{Tr}_{\ell,r}(f \circ g) \ , \tag{3.3.18}$$

and obey

$$\mathrm{Tr}_{\ell,r}(f \otimes g) = \mathrm{Tr}_{\ell,r}(f)\mathrm{Tr}_{\ell,r}(g) \ . \tag{3.3.19}$$

In a ribbon category apart from the left and right dualities which coincide, the two notions of the trace coincide as well. Categories with this property are called *spherical* [BW96]. The trace of the identity morphism is the *quantum dimension*[10] of an object, which is additive under direct sums and multiplicative under tensor products

$\dim_q(A) := \mathrm{Tr}_q(\mathrm{id}_A) \equiv$ 

. (3.3.20)

[10]It is also called *categorical dimension* or *rank* [Maj, Sect. 9.3]. The word *quantum dimension* comes from the relation to quantum groups, see also the book by Kassel [Ka]. The reason is that it is a 'deformation' of the ordinary dimension. For example, the integer n gets replaced by $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$, see [Maj, Examples 9.3.6 & 9.3.7]. Thus 'quantised' is used as 'deformed away from its classical value by a parameter dependent deformation'.

Finally, in order to make contact with the CFT language developed in the first Chapter and therefore with defects, which were discussed in the second Chapter, we will need to recall the definition of the Grothendieck ring first.

Definition 3.3.11. The *Grothendieck group* $K_0(\mathcal{C})$ of an Abelian category $\mathcal{C}$ is the free abelian group generated by isomorphism classes (A) of objects $A \in \mathcal{C}$, quotiented by the subgroup generated by the relations $(A) = (K) + (C)$ for all short exact sequences $0 \to K \to A \to C \to 0$. We denote the equivalence class of (A) in $K_0(\mathcal{C})$ by $[A]$.

Definition 3.3.12. A functor $F\colon \mathcal{C} \to \mathcal{D}$ is said to be *right-exact* if for $A, B, C \in \mathcal{C}$, exactness of $A \to B \to C \to 0$ implies exactness of $F(A) \to F(B) \to F(C) \to 0$. A tensor product bifunctor is called right-exact if $X \otimes (-)$ and $(-) \otimes X$ are right-exact functors for all $X \in \mathcal{C}$. That is, if exactness of $A \to B \to C \to 0$ implies exactness of $X \otimes A \to X \otimes B \to X \otimes C \to 0$, similarly for $(-) \otimes X$.

Remark 3.3.2. If $\mathcal{C}$ is monoidal with exact[11] tensor product, then the Grothendieck group $K_0(\mathcal{C})$ carries a ring structure defined via $[A] \cdot [B] = [A \otimes B]$. In this case, $K_0(\mathcal{C})$ is called the *Grothendieck ring*.

We now make the connection of the above categorical constructions to the CFT language. First note that the representation category of a RCFT is a semisimple[12] monoidal category $\mathcal{C}$ with simple tensor unit. The simple objects of $\mathcal{C}$ are the irreducible representations of the chiral algebra $\mathfrak{V}$ and the morphisms in $\mathcal{C}$ are the $\mathfrak{V}$-intertwiners. The tensor product in $\mathcal{C}$ is the fusion tensor product of $\mathfrak{V}$-representations, with the tensor unit given by the vacuum representation, i.e. $R_0 = \mathbb{1}$.

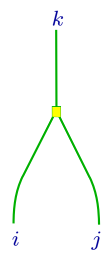
[11]The tensor product is said to be exact if it is both left and right exact, with the definition of left exactness being similar to that of right exactness.

[12]A *semisimple* category is characterised by the property that every object is the direct sum of finitely many simple objects.

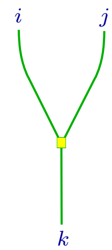
The isomorphism classes of (simple) objects, corresponds to the primary chiral vertex operators and the Grothendieck ring $K_0(\mathcal{C})$ is the fusion ring of the CFT. The duality in $\mathcal{C}$ encodes the existence of conjugate $\mathfrak{V}$-representations and the twist is given by the fractional part of the conformal weight, $\theta_A = \exp(-2\pi i h_A)\,\mathrm{id}_A$, for $A \in \mathcal{C}$, while the braiding accounts for the presence of braid group statistics in two dimensions, cf. [FRS02-I, Sect. 2.2] and references therein.

All the axioms of $\mathcal{C}$ can be viewed as formalisations of the properties of primary chiral vertex operators in RCFT. We can formulate these properties by fixing bases $\lambda_{ij}^k \in \mathrm{Hom}_{\mathcal{C}}(A_i \otimes A_j, A_k)$ and dual basis $\bar{\lambda}_k^{ij} \in \mathrm{Hom}_{\mathcal{C}}(A_k, A_i \otimes A_j)$, depicted as[13]

$\lambda_{ij}^k \equiv$



, $\bar{\lambda}_k^{ij} \equiv$



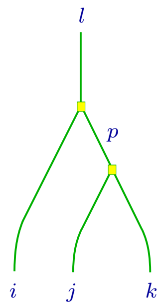
, (3.3.21)

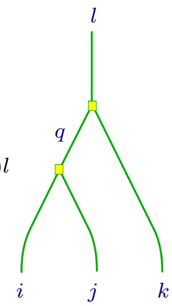
where the label $i \in \{0, 1, 2, \ldots, |\mathcal{I}| - 1\}$ in the pictures, is for notational simplicity, in place of the elements A_i of the family $\{A_i\}_{i \in \mathcal{I}}$ of objects in $\mathcal{C}$. Duality of the basis means that if we compose λ_{ij}^k with its dual $\bar{\lambda}_k^{ij}$ we get the identity, i.e. $\lambda_{ij}^k \circ \bar{\lambda}_k^{ij} = \mathrm{id}_k$. In terms of pictures this translates to placing the picture for λ_{ij}^k on top of the picture for $\bar{\lambda}_k^{ij}$ and then collapsing the bubble to obtain the identity morphism.

Once we have chosen a basis as above, there are two distinct bases for the morphism space $\mathrm{Hom}_{\mathcal{C}}(A_i \otimes A_j \otimes A_k, A_l)$, corresponding to its two decompositions $\bigoplus_{q \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(A_i \otimes A_j, A_q) \otimes \mathrm{Hom}_{\mathcal{C}}(A_q \otimes A_k, A_l)$ and $\bigoplus_{p \in \mathcal{I}} \mathrm{Hom}_{\mathcal{C}}(A_j \otimes A_k, A_p) \otimes \mathrm{Hom}_{\mathcal{C}}(A_i \otimes A_p, A_l)$, respectively. The coefficients of the basis transformation between the two are

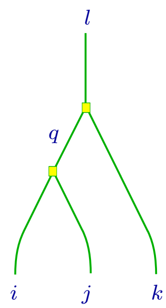
[13] Here we have assumed for simplicity that $\dim_{\Bbbk} \mathrm{Hom}_{\mathcal{C}}(A_i \otimes A_j, A_k) = \mathcal{N}_{ij}{}^k = 1$, so that the basis consists of only one non-zero vector.

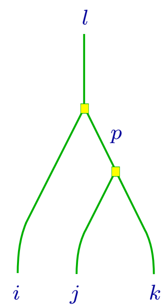
known as the *fusing matrices* F of $\mathcal{C}$. These are depicted as



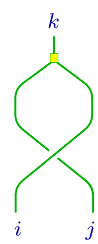
$$= \sum_{q\in\mathcal{I}} \mathsf{F}^{(ijk)l}_{p,q} \qquad (3.3.22)$$


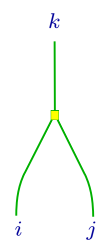
The inverse of F is denoted by G and is defined as



$$= \sum_{p\in\mathcal{I}} \mathsf{G}^{(ijk)l}_{q,p} \qquad (3.3.23)$$


Combining the braiding morphisms with the basis choice (3.3.21) we get the *braiding matrices* R:



$$= \mathsf{R}^{(ij)k} \qquad (3.3.24)$$


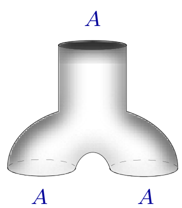
If we replace $c_{i,j}$ by its inverse $c^{-1}_{j,i}$ then the number we obtain is denoted by $\mathsf{R}^{-(ji)k}$. One easily checks that $\mathsf{R}^{(ij)k}\mathsf{R}^{-(ji)k} = 1$.

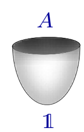
3.4 Frobenius Structure

In this section we follow [FRS02-I, Sect. 3.1]. We thus define a Frobenius algebra as an algebra object in a monoidal category $\mathcal{C}$. We start by recalling the definition of an algebra object in a monoidal category $\mathcal{C}$, but before we do that, note the following remark:

Remark 3.4.1. Note that the cobordism graphical notation we will use in this section and in 3.5.1, is usually reserved for commutative algebras, because the diagram with an exchange of the two arguments before the multiplication is homeomorphic to the one without the exchange. Therefore, the cobordism notation will be confusing if used for not necessarily commutative algebras. Here, we are only concerned with commutative Frobenius algebras, since these are in one to one correspondence (only in the category of vector spaces, not for general monoidal categories) with the 2 dimensional topological field theories that we will discuss in the next section.

Definition 3.4.1. A *commutative algebra object* (A, μ, η), in a braided monoidal category $\mathcal{C}$, is an object $A \in \mathcal{C}$ together with two morphisms

$\mu \equiv$ 

$: A \otimes A \to A \ , \quad \eta \equiv$ 

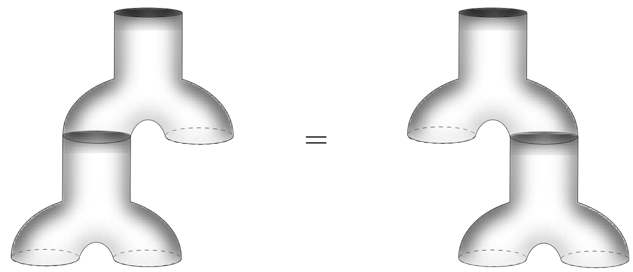
$: \mathbb{1} \to A \ ,$ (3.4.1)

called the *multiplication* and *unit* respectively, such that the diagrams commute

$$\begin{array}{ccccc} (A \otimes A) \otimes A & \xrightarrow{\alpha_{A,A,A}} & & & A \otimes (A \otimes A) \\ \downarrow{\scriptstyle \mu \otimes \mathrm{id}_A} & & \circlearrowleft & & \downarrow{\scriptstyle \mathrm{id}_A \otimes \mu} \\ A \otimes A & \xrightarrow{\mu} & A & \xleftarrow{\mu} & A \otimes A \end{array} \qquad A \otimes A \xrightarrow{c_{A,A}} A \otimes A \ (\circlearrowleft)$$

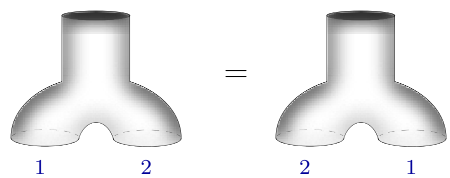
$$\begin{array}{ccccc} \mathbb{1} \otimes A & \xrightarrow{\eta \otimes \mathrm{id}_A} & A \otimes A & \xleftarrow{\mathrm{id}_A \otimes \eta} & A \otimes \mathbb{1} \\ & \searrow{\scriptstyle \lambda_A} \ \circlearrowleft & \downarrow{\scriptstyle \mu} & \circlearrowleft \ \swarrow{\scriptstyle \rho_A} & \\ & & A & & \end{array}$$

These two commutative diagrams can be expressed in terms of pictures (after passing to a strict category). For the first upper diagram we have



(3.4.2)

while for the first bottom diagram, which expresses the commutativity condition we have



(3.4.3)

and for the second diagram

(3.4.4)

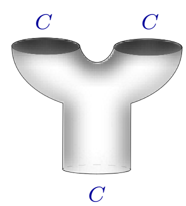
Note that we have suppressed the labels A and $\mathbb{1}$ on the algebra pictures and we will do that from now on, when this is unambiguous, that is, when we are dealing with morphisms involving only A and $\mathbb{1}$.

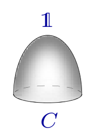
Example 3.4.1. *As an example of a commutative algebra object which can also be turned into a Frobenius algebra, consider the algebra of polynomials $\mathbb{C}[X]$ in one indeterminate X over the field $\mathbb{C}$, divided by the ideal $\langle X^d \rangle$, i.e. $A = \mathbb{C}[X]/\langle X^d \rangle$. This is a commutative algebra object in the category of finite dimensional complex vector spaces $\mathbf{Vect}_{\mathbb{C}}$. The reason one divides by the ideal $\langle X^d \rangle$ is to make the algebra object (viewed as a vector space over $\mathbb{C}$) finite dimensional, in order to be able to turn* *it into a Frobenius algebra. Thus in the case at hand* $\dim A = d$. *The tensor product bifunctor* $\otimes_{\mathbb{C}}\colon \mathbf{Vect}_{\mathbb{C}} \times \mathbf{Vect}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{C}}$ *is given by* $\cdot\colon A \times A \to A$. *More explicitly the multiplication is*

$$\left(\sum_{i=0}^{d-1} a_i X^i\right) \cdot \left(\sum_{j=0}^{d-1} b_j X^j\right) = \sum_{k=0}^{d-1} \left(\sum_{i+j=k} a_i b_j\right) X^k \ ,$$

where $a_i, b_j \in \mathbb{C}$ *and* $X^i \in A$. *The monoidal unit is given by the underlying field* $\mathbb{C}$.

Definition 3.4.2. A *commutative co-algebra object* (C, δ, ε), in a monoidal category $\mathcal{C}$, is an object $C \in \mathcal{C}$ together with two morphisms

$\delta \equiv$ 

$: C \to C \otimes C \ , \quad \varepsilon \equiv$ 

$: C \to \mathbb{1} \ ,$ (3.4.5)

called the *co-multiplication* and *co-unit* respectively, such that the diagrams commute

$$\begin{array}{ccccc}
(C \otimes C) \otimes C & \xleftarrow{\alpha^{-1}_{C,C,C}} & & & C \otimes (C \otimes C) \\
\uparrow{\scriptstyle \delta \otimes \mathrm{id}_C} & & \circlearrowleft & & \uparrow{\scriptstyle \mathrm{id}_C \otimes \delta} \\
C \otimes C & \xleftarrow{\ \delta\ } & C & \xrightarrow{\ \delta\ } & C \otimes C \\
& & \circlearrowleft & & \\
& \nwarrow & \xleftarrow{c_{C,C}} & \swarrow &
\end{array}
\qquad
\begin{array}{ccccc}
\mathbb{1} \otimes C & \xleftarrow{\varepsilon \otimes \mathrm{id}_C} & C \otimes C & \xrightarrow{\mathrm{id}_C \otimes \varepsilon} & C \otimes \mathbb{1} \\
& \nwarrow{\scriptstyle \lambda_C^{-1}} \ \circlearrowleft & \uparrow{\scriptstyle \delta} & \circlearrowleft \ \nearrow{\scriptstyle \rho_C^{-1}} & \\
& & C & &
\end{array}$$

In a similar way as in (3.4.2) and (3.4.4), one can express these two commutative diagrams in terms of pictures, but this time rotated 180°.

Definition 3.4.3. A *commutative associative Frobenius algebra*, in a braided monoidal category $\mathcal{C}$, is an object that is both an algebra and a co-algebra, i.e. an object

$(A, \mu, \eta, \delta, \varepsilon)$, for which the product and co-product are related by

 (3.4.6)

3.5 Topological Field Theory

In this section and its subsections, following [BK, Sect. 4.2], [Ko, Sect. 3.3], [Tu, Chap. III] and [FRS02-I, Sect. 2.4], we discuss in some detail topological field theory (TFT), which is the main subject of this chapter. When we say 'some detail' we mean only those aspects of TFT that will be of interest for this thesis. Also, when we say 'manifold' we mean a *compact topological closed oriented manifold with a boundary* (unless otherwise indicated) and all vector spaces considered are over a base field $\Bbbk$ of characteristic zero.

Roughly, a n-dimensional TFT (nD TFT), is a pair $(\mathcal{H}, Z)$, that to every closed oriented $n-1$-manifold X without a boundary, assigns a vector space $\mathcal{H}(X)$ and to every closed oriented n-manifold $\mathscr{M}$, assigns a vector $Z(\mathscr{M})$ in $\mathcal{H}(\partial\mathscr{M})$. This rule is subject to some axioms, due to Atiyah [At88], which express that topologically equivalent manifolds have isomorphic associated vector spaces and that disjoint union of manifolds goes to tensor products of vector spaces, etc. In other words, a TFT, is a symmetric monoidal functor $\mathcal{H}\colon \mathbf{Cob}(n) \to \mathbf{Vect}_\Bbbk$, from the category of cobordisms to the category of finite dimensional vector spaces. More concretely:

Definition 3.5.1. A n-dimensional TFT is a symmetric monoidal functor

$$(\mathcal{H}, Z)\colon \mathbf{Cob}(n) \to \mathbf{Vect}_\Bbbk \ , \tag{3.5.1}$$

where the first datum $\mathcal{H}$ gives the action on the objects, while the second datum Z, to every bordism assigns a linear map. In particular:

1. To $X \in \mathbf{Cob}(n)$, associates a finite-dimensional vector space $\mathcal{H}(X) \in \mathbf{Vect}_{\Bbbk}$.

2. To a bordism $\mathscr{M} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(\partial_-\mathscr{M}, \partial_+\mathscr{M})$, associates a linear map $Z(\mathscr{M}) \in \mathrm{Hom}_{\mathbf{Vect}_{\Bbbk}}(\mathcal{H}(\partial_-\mathscr{M}), \mathcal{H}(\partial_+\mathscr{M}))$, called the *operator invariant* of the bordism.

3. To any homeomorphism $f\colon X \to Y$, associates an isomorphism $f_\natural\colon \mathcal{H}(X) \xrightarrow{\sim} \mathcal{H}(Y)$.

4. Establishes functorial isomorphisms:

$$\mathcal{H}\left(\overline{X}\right) \cong \mathcal{H}(X)^\vee \ , \quad \mathcal{H}(\emptyset) \cong \Bbbk \ , \quad \mathcal{H}(X \amalg Y) \cong \mathcal{H}(X) \otimes_{\Bbbk} \mathcal{H}(Y) \ ,$$

which are compatible with each other and with the unit, braiding and associator isomorphisms as follows:

$\mathbf{Cob}(n)$	$\mathbf{Vect}_{\Bbbk}$
$\emptyset \cong \overline{\emptyset}$	$\Bbbk \cong \Bbbk^\vee$
$\overline{X \amalg Y} \cong \overline{Y} \amalg \overline{X}$	$(\mathcal{H}(X) \otimes_{\Bbbk} \mathcal{H}(Y))^\vee \cong \mathcal{H}(Y)^\vee \otimes_{\Bbbk} \mathcal{H}(X)^\vee$
$X \amalg \emptyset \cong X$	$\mathcal{H}(X) \otimes_{\Bbbk} \Bbbk \cong \mathcal{H}(X)$
$X \amalg Y \cong Y \amalg X$	$\mathcal{H}(X) \otimes_{\Bbbk} \mathcal{H}(Y) \cong \mathcal{H}(Y) \otimes_{\Bbbk} \mathcal{H}(X)$
$(X \amalg Y) \amalg Z \cong X \amalg (Y \amalg Z)$	$(\mathcal{H}(X) \otimes_{\Bbbk} \mathcal{H}(Y)) \otimes_{\Bbbk} \mathcal{H}(Z) \cong \mathcal{H}(X) \otimes_{\Bbbk} (\mathcal{H}(Y) \otimes_{\Bbbk} \mathcal{H}(Z))$

Table 3.1: Compatibility conditions for the functorial isomorphisms.

This data is required to satisfy the following axioms:

(A1) **Naturality:** Consider the bordisms $\mathscr{M} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(\partial_-\mathscr{M}, \partial_+\mathscr{M})$ and $\mathscr{N} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(\partial_-\mathscr{N}, \partial_+\mathscr{N})$ and let $f\colon \mathscr{M} \to \mathscr{N}$ be an orientation preserving homeomorphism, then the diagram

$$\begin{CD}
\mathcal{H}(\partial_-\mathscr{M}) @>{Z(\mathscr{M})}>> \mathcal{H}(\partial_+\mathscr{M}) \\
@V{f_\natural|_{\partial_-\mathscr{M}}}VV \circlearrowleft @VV{f_\natural|_{\partial_+\mathscr{M}}}V \\
\mathcal{H}(\partial_-\mathscr{N}) @>>{Z(\mathscr{N})}> \mathcal{H}(\partial_+\mathscr{N})
\end{CD}$$

commutes.

(A2) **Multiplicativity:** If $\mathscr{B} = \mathscr{M} \amalg \mathscr{N}$, then the third functorial isomorphism in 4. above, (or equivalently (3.3.8)), means that $Z(\mathscr{B}) = Z(\mathscr{M}) \otimes Z(\mathscr{N})$.

(A3) **Functoriality:** Consider the bordisms $\mathscr{M} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(\partial_-\mathscr{M}, \partial_+\mathscr{M})$ and $\mathscr{N} \in \mathrm{Hom}_{\mathbf{Cob}(n)}(\partial_-\mathscr{N}, \partial_+\mathscr{N})$ and let $f\colon \partial_+\mathscr{M} \to \partial_-\mathscr{N}$ be a homeomorphism and $\mathscr{B}$ the bordism obtained by the disjoint union of $\mathscr{M}$ and $\mathscr{N}$ using f, then

$$Z(\mathscr{B}) = kZ(\mathscr{N}) \circ f_\natural \circ Z(\mathscr{M}) \ ,$$

where $k \in \Bbbk$, is called the *anomaly* of the triple $(\mathscr{M}, \mathscr{N}, f)$.

(A3) **Normalization:** For any $X \in \mathbf{Cob}(n)$, we have

$$Z\left(X \times [0,1], X \times \{0\}, X \times \{1\}\right) = \mathrm{id}_{\mathcal{H}(X)} \ .$$

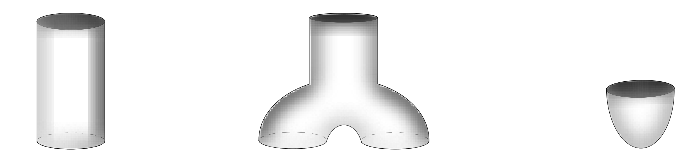
This completes the definition.

3.5.1 2D TFT

In this subsection we consider a 2D TFT, which can be viewed as a toy model of a TFT. The main result here will be that 2D TFTs are equivalent to commutative Frobenius algebras. In other words, there is a canonical equivalence between the category 2**TFT** of 2D TFTs and that of commutative Frobenius algebras **cFA**. This equivalence is proved (following [BK, Sect. 4.3]) in Theorem 3.5.3 below.

Before we state and prove the main theorem, the following two lemmas, which we state without a proof, will be useful [BK, Lemmas 4.3.2 & 4.3.3].

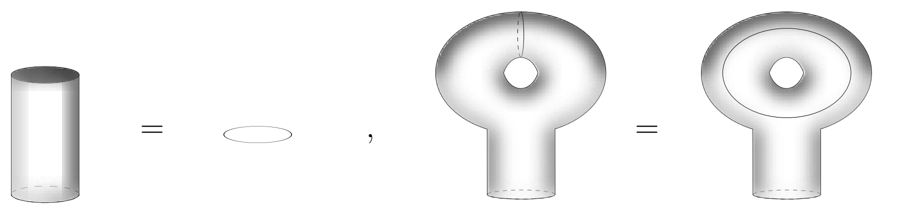
Lemma 3.5.1. *Every 2-manifold with a boundary can be cut into a union of*



(i) cylinders (ii) pair of pants (iii) discs

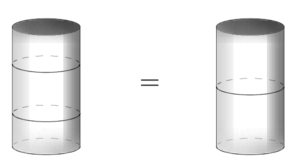
However, a 2-manifold $\mathscr{M}$ can be cut in several different ways and we will need to check when $Z(\mathscr{M})$ is well defined, thus we need the following result.

Lemma 3.5.2. *Any two ways to cut a 2-manifold $\mathscr{M}$ into cylinders, pair of pants and discs, can be related by isotopy of $\mathscr{M}$ and a sequence of 'simple moves', which are* (3.4.2)*,* (3.4.4)*, as well as*[14]



(3.5.2)

[14] *The idea is that one draws simple curves on the surface of the manifold and then cut the manifold into cylinders, pair of pants and discs as in Lemma 3.5.1. For example, the first set of pictures in* (3.5.2) *are obtained as follows: Consider the following pictures*



This says that if you have a cylinder with two circles inserted you can omit one of the circles. Then to obtain the required result observe that the circles cut the cylinder in question into three and two subcylinders respectively. Then if we remove the two outer subcylinders from both sides we recover the first set of pictures in (3.5.2)*. The second set of pictures in* (3.5.2) *simply says that the simple move in question is to interchange the two circles, i.e. the circle that lies around the arm can be interchanged with the circle that lies around the hole and vise versa. For more details see [HT80, Appendix].*

We now state the main theorem as promised:

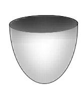
Theorem 3.5.3. *There is a canonical equivalence of categories* $2\mathbf{TFT} \simeq \mathbf{cFA}$. *In other words,*

(a) Every 2D TFT gives a Frobenius algebra.

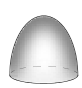
(b) Every Frobenius algebra gives a 2D TFT.

Proof. (a) By definition a 2D TFT is a symmetric monoidal functor $(\mathcal{H}, Z)\colon \mathbf{Cob}(2) \to \mathbf{Vect}_{\Bbbk}$. The generators of $\mathbf{Cob}(2)$ are given in Prop. 3.2.3, so we need to show that the TFT of these generators gives a commutative Frobenius algebra.

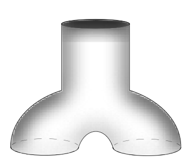
The only 1-dimensional, closed, connected manifold is the circle $S^1 \in \mathbf{Cob}(2)$ and $\overline{S^1} = S^1$. Let now $A := \mathcal{H}(S^1)$, be the vector space obtained $\mathcal{H}$. The disc $\mathscr{D}^1 \in \mathrm{Hom}_{\mathbf{Cob}(2)}(\emptyset, S^1)$ gives the linear map

$Z\Big($  $\Big)\colon \Bbbk \to A\ ,$

the unit $e \in A$. On the other hand we also have $\mathscr{D}^1 \in \mathrm{Hom}_{\mathbf{Cob}(2)}\left(\overline{S^1}, \emptyset\right)$, which gives

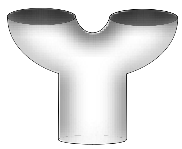
$Z\Big($  $\Big)\colon A \to \Bbbk\ ,$

the co-unit in A. The next generator is $\mathscr{P} \in \mathrm{Hom}_{\mathbf{Cob}(2)}(S^1 \amalg S^1, S^1)$, which gives

$Z\Big($  $\Big)\colon A \otimes_{\Bbbk} A \to A\ ,$

the multiplication $a \otimes b \mapsto ab$, where $a, b \in A$. Finally, $\mathscr{Y} \in \mathrm{Hom}_{\mathbf{Cob}(2)}(S^1, S^1 \amalg S^1)$

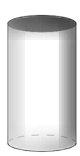
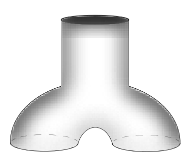
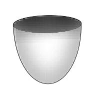
gives

$$Z\Bigg($$  $$\Bigg) : A \to A \otimes_{\Bbbk} A \ ,$$

the co-multiplication on A.

Now we have to show commutativity and associativity of multiplication. The commutativity follows from the fact that the flipping of the legs of a pair of pants is a homeomorphism and associativity follows from pictures (3.4.2) and the gluing axiom (A2). The unit property (3.4.4) is a consequence of the gluing and normalization axioms. This agrees with Def. 3.4.3, thus we proved (a).

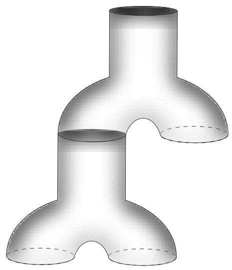
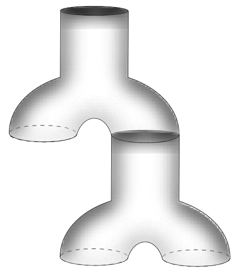
(b) Let A be a Frobenius algebra. To the circle we assign $\mathcal{H}(S^1) := A$. Now recall that the objects of $\mathbf{Cob}(2)$ are $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \ldots\}$ where $\mathbf{n}$ is the disjoint union of n circles, thus $\mathcal{H}(\mathbf{n}) := A^{\otimes n}$. For $f \colon S^1 \to S^1$ we let $f_\natural = \mathrm{id}$ and for $g \colon S^1 \to \overline{S^1}$ let $g_\natural \colon A \xrightarrow{\sim} A^\vee$ be the isomorphism given by the non-degenerate bilinear form $\mathrm{Tr}(ab)$, for $a, b \in A$. It is clear that[15]

$$Z\Bigg($$  $$\Bigg) \in A^\vee \otimes_{\Bbbk} A \ , \quad Z\Bigg($$  $$\Bigg) \in A^\vee \otimes_{\Bbbk} A^\vee \otimes_{\Bbbk} A \ , \quad Z\Bigg($$  $$\Bigg) \in A \ .$$

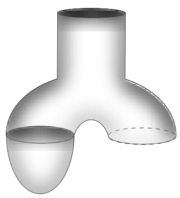
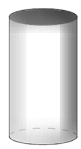
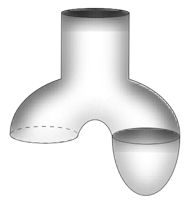
Using Lemma 3.5.1 we can extend this to any 2-manifold using axioms (A2) and (A3) from Def. 3.5.1, while using Lemma 3.5.2 we need to check that $Z(-)$ gives the same

[15]To see how we obtain the one for the cylinder $\mathscr{C}$ for example, note that $\partial\mathscr{C} = \overline{S^1} \amalg S^1$, then $Z(\mathscr{C}) \in \mathcal{H}(\partial\mathscr{C}) = \mathcal{H}\left(\overline{S^1} \amalg S^1\right) = \mathcal{H}\left(\overline{S^1}\right) \otimes_{\Bbbk} \mathcal{H}(S^1) = A^\vee \otimes_{\Bbbk} A$.

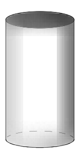
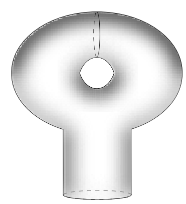
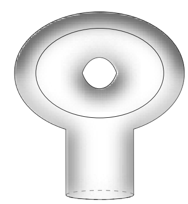
result on both sides of (3.4.2):

$$Z\Big($$  $$\Big) = Z\Big($$  $$\Big) \in A^\vee \otimes_{\Bbbk} A^\vee \otimes_{\Bbbk} A^\vee \otimes_{\Bbbk} A\ ,$$

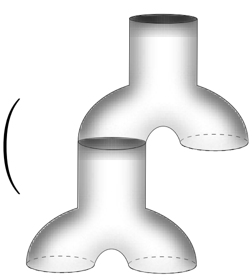
of (3.4.4):

$$Z\Big($$  $$\Big) = Z\Big($$  $$\Big) = Z\Big($$  $$\Big) \in A^\vee \otimes_{\Bbbk} A$$

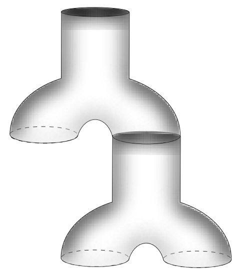
and of (3.5.2):

$$Z\Big($$  $$\Big) = Z\Big($$  $$\Big) \in A^\vee \otimes_{\Bbbk} A\ ,\quad Z\Big($$  $$\Big) = Z\Big($$  $$\Big) \in A\ .$$

We only show the first one; the rest can be shown in the same way. For the left hand side we have that

$$Z\Big($$  $$\Big) : (A \otimes A) \otimes A \to A$$

and for the right hand side

$Z\Big($  $\Big) : A \otimes (A \otimes A) \to A\ .$

Then associativity follows from pictures (3.4.2) and the gluing axiom (A2), hence they are equal. Therefore $Z(-)$ is well defined on any 2-manifold $\mathscr{M}$. □

3.5.2 3D TFT

In this subsection, we will introduce some of the machinery of 3D TFT which will play an important role in Chap. 5. There, we will relate the categorical constructions of Chap. 4 to defect operators and we will construct correlators of chiral defect fields, using the machinery of 3D TFT. In this subsection, however, we will not talk about defects. We will only introduce those areas of 3D TFT that are going to be relevant later on. For a more detailed exposition on 3D TFT the reader is referred to [FRS02-I, FRS04-II, FRS04-III, FRS05-IV, FFRS06-V, FFRS04, FFRS07, FFFS00, FFFS02], as well as to the books by Bakalov and Kirillov [BK, Sect. 4.4] and Turaev [Tu, Chap. IV], which we also follow here.

In Sect. 3.3 we saw that the objects, morphisms, the tensor product and so on, of a monoidal category $\mathcal{C}$ encodes some information about the chiral data[16] of the CFT. However, $\mathcal{C}$ contains strictly less information than the chiral data[17], but most of the important information of the CFT, such as, its field content, boundary conditions

[16] By chiral data we mean the representation theory of the chiral algebra and the conformal blocks of a RCFT.

[17] Roughly, as stated in [FRS02-I, Sect. 5], the category encodes only the monodromies of the conformal blocks, but not their functional dependence on the insertion points and the moduli of the world sheet or the information which state of a given representation of the chiral algebra is inserted. See also [FRS02-I, Foot. 22] for a more detailed explanation.

and defect lines, the OPE and the consistency of these data with factorisation, can be discussed at the level of $\mathcal{C}$ [FRS02-I, Sect. 5].

As we mentioned above, we will be interested in determining correlation functions of a RCFT. In order to do that, one needs to specify them as a particular element in the relevant space of conformal blocks. A very convenient characterisation of conformal blocks is via ribbon graphs in 3-manifolds. Then the coefficients in the expansion of a CFT correlator in terms of a chosen basis of conformal blocks are obtained as invariants of closed 3-manifolds with embedded ribbon graphs.

Let us briefly introduce the concept of a ribbon graph, following the conventions of [FRS02-I, Sect. 2.3]. A *ribbon graph* consists of the following data: an oriented 3-manifold $\mathscr{M}$, possibly with boundary, together with embedded ribbons and coupons. A *ribbon*, is an oriented rectangle $[-1/10, 1/10]\times[0, 1]$, together with an orientation for its *core* $\{0\}\times[0, 1]$. The *ends* of the ribbons are the two subsets $[-1/10, 1/10]\times\{0\}$ and $[-1/10, 1/10] \times \{1\}$. A *coupon*, is an oriented rectangle with two preferred opposite edges, called the top and bottom. The embeddings of ribbons and coupons into $\mathscr{M}$ are required to be injective. A ribbon minus its ends does not intersect any other coupon nor the boundary of $\mathscr{M}$, while its ends must lie, either on one of the preferred edges of some coupon or on $\partial\mathscr{M}$. Finally, the orientation of the ribbon and the coupon must agree whenever the ribbon ends on a coupon. The side of the ribbons and coupons that will face the reader will be drawn on a lighter colour that the back side and we will use open arrows to indicate their orientation.

One can use the machinery of Sect. 3.3 to assign numbers to ribbon graphs (e.g. in S^3, see [FRS02-I, Sect. 2.3] for example) and for this purpose, it is sufficient that $\mathcal{C}$ is a ribbon category. Consider for example the second graph in equation (3.3.17) and think of it as a ribbon graph in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. If we deform the ribbon so that it faces upwards as described above and arrange it in such a way that the bends, twists and crossing can be expressed as dualities, twists and braidings respectively, then we can assign to it an element in $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$ by reading the graph from bottom to top and interpreting it as concatenation of morphisms in $\mathcal{C}$. Thus, in the case at hand, the ribbon in (3.3.17) will correspond to the element $d_A \circ (\mathrm{id}_{U^\vee} \otimes f) \circ \tilde{b}_A \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$.

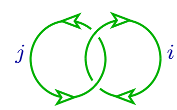
If in addition $\mathcal{C}$ is modular, this implies the highly non trivial result, due to Turaev [Tu], that it gives rise to a full extended 3D TFT, i.e. a 3D TFT with embedded ribbon graphs. Let us recall the definition of a modular tensor category.

Definition 3.5.2. A *modular tensor category* (MTC), is a semisimple, abelian, ribbon, $\Bbbk$-linear category $\mathcal{C}$, with absolutely simple tensor unit and with the additional properties:

(i) $\mathcal{C}$ has only a finite number of isomorphism classes of simple objects: $|\mathcal{I}| < \infty$.

(ii) The matrix $s = (s_{ij})_{i,j\in\mathcal{I}}$, is non-degenerate, where

$$s_{ij} := \mathrm{Tr}(c_{i,j}, c_{A_j,A_i}) \equiv \quad (3.5.3)$$



Remark 3.5.1. (i) The matrix s coincides, up to normalisation, with the modular S-matrix of the CFT via

$$s_{ij} = \frac{S_{ij}}{S_{00}} \; . \qquad (3.5.4)$$

Conversely, S_{00} and thereby s is recovered from the data of the MTC by requiring $S = S_{00} \cdot s$ to be unitary. In terms of s the quantum dimensions (3.3.20) are

$$\dim_q(A_i) = s_{i0} = \frac{S_{i0}}{S_{00}} \; . \qquad (3.5.5)$$

Note that (3.5.4) is a non-trivial statement. For minimal models and WZW models, (3.5.4) was first checked by explicitly computing both expressions for s and comparing them. A general argument in terms of conformal blocks was given in [MS90]. A general proof in terms of vertex operator algebras was given by Huang in [Hu04].

(ii) The axioms of a MTC can be best understood in the language of ribbons. This was done in Sect. 3.3 by using lines instead of ribbons (recall Rem. 3.3.1).

It was mentioned at the end of Sect. 1.3 that one can construct a full CFT via its correlation functions. In order to do that, an additional input is required. This input is a *symmetric special* Frobenius algebra object $(A, \mu, \eta, \delta, \varepsilon)$ in $\mathcal{C}$. For a definition of the terms symmetric special see [FRS02-I, Def. 3.4], since this is the last time we refer to these terms in this thesis. The correlators on an arbitrary world sheet X are expressed as specific elements in the vector spaces of conformal blocks[18] on the complex double $\widehat{X}$ (Def. 3.5.4) of X. Such an element is described by a ribbon graph in a 3-manifold $\mathscr{M}_X$, which is called the *connecting manifold* (Def. 3.5.5), such that $\partial\mathscr{M}_X = \widehat{X}$. In this thesis we will only consider orientable world sheets X. Now we define the notion of an *extended surface* or $\mathcal{C}$*-marked surface*.

Definition 3.5.3. An *extended surface* is a triple $(X, \gamma, A_\pm)$ with a lagrangian subspace $\mathcal{L}(X)$ of the first homology group $H_1(X, \mathbb{R})$[19], where X is an oriented compact surface, $\gamma \equiv (p_i, v_i)$ is a collection of finite disjoint arcs, i.e. a finite number of points $p_1, \ldots, p_n$ with a non-zero tangent vector v_i attached to every point p_i, labeled by pairs $A_\pm \equiv (A_i, \varepsilon_i)$ with $A_i \in \mathcal{C}$ and $\varepsilon_i \in \{\pm 1\}$.

Remark 3.5.2. Note that the world sheets themselves are not extended surfaces. For example world sheets can have defect lines or boundaries, but extended surfaces do not have such decorations. An example of an extended surface is the complex double $\widehat{X}$ produced from the world sheet X.

[18] To make the connection to the conclusions of Sect. 1.3 if we know the conformal blocks then the crossing symmetry [Gin88, Eqn. (3.32)] of the 4-point function yields a system of equations that determine the structure constants as well as the conformal weights $h, \bar{h}$, hence the full CFT.

[19] First note that for any oriented surface X, the real vector space $H_1(X, \mathbb{R})$ supports the intersection pairing $H_1(X, \mathbb{R}) \times H_1(X, \mathbb{R}) \to \mathbb{R}$, which is antisymmetric so that $H_1(X, \mathbb{R})$ is a symplectic vector space. In general, a *lagrangian subspace* of a symplectic vector space V, is a vector space $\mathcal{L}$ such that $V = \mathcal{L} \oplus \mathcal{L}^*$, where $\mathcal{L}, \mathcal{L}^*$ are both isotropic. Following [Tu, Sect. 3.3, 4.1 & 4.2], a natural source of lagrangian subspaces in the homologies of surfaces is provided by the theory of 3-manifolds. An oriented compact 3-manifold $\mathscr{M}$ gives rise to a lagrangian subspace in $H_1(\partial\mathscr{M}, \mathbb{R})$ which is the kernel of the inclusion homomorphism $H_1(\partial\mathscr{M}, \mathbb{R}) \to H_1(\mathscr{M}, \mathbb{R})$. The fact that this subspace is lagrangian is a well known corollary of the Poincaré duality [He].

The *opposite* of an extended surface is the triple $(-X, -\gamma, A^\vee_\pm)$, where $-\gamma \equiv (p_i, -v_i)$, $A^\vee_\pm = (A^\vee_i, -\varepsilon)$ and $-X$ is the same as X but with reversed orientation. For simplicity we will write X for an extended surface and $\overline{X}$ for its opposite.

Definition 3.5.4. The *complex double* $\widehat{X}$ of an extended surface X consists of two disconnected copies of X with opposite orientation $\widehat{X} \cong \overline{X} \amalg X$.

The next ingredient for defining a 3D TFT is the notion of an *extended cobordism.* It is defined similarly to Def. 3.1.2 but this time the 3-manifold $\mathscr{M}$ contains a ribbon graph, while the two boundaries $\partial_\pm \mathscr{M}$ are endowed with lagrangian subspaces $\mathcal{L}(\partial_- \mathscr{M}), \mathcal{L}(\partial_+ \mathscr{M})$ of their first homology groups $H_1(\partial_- \mathscr{M}, \mathbb{R})$ and $H_1(\partial_+ \mathscr{M}, \mathbb{R})$ respectively. Then turn $\partial_\pm \mathscr{M}$ into extended surfaces by taking as arcs the ends of ribbons, with orientation induced by the ribbons. When a ribbon ending on $\partial \mathscr{M}$ is labeled by $A_i \in \mathcal{C}$, then the corresponding arc is labeled by A_+ if the core of the ribbon points away from the surface and by $A^\vee_-$ otherwise. Then the triple $(\mathscr{M}, \partial_- \mathscr{M}, \partial_+ \mathscr{M})$ is an extended cobordism from $\partial_- \mathscr{M}$ to $\partial_+ \mathscr{M}$. We denote the category of extended 3-cobordisms by $\mathbf{Cob}_\mathcal{C}(3)$.

Definition 3.5.5. The *connecting 3-manifold* $\mathscr{M}_X$ consists of pairs (x, t), with $x \in \widehat{X}$ and $t \in [-1, 1]$, modulo the identification $(x, t) \sim (\sigma(x), -t)$, where σ is an orientation reversing map of order two. In other words

$$\mathscr{M}_X := (\widehat{X} \times [-1, 1])/\mathbb{Z}_2 \ , \tag{3.5.6}$$

where the group $\mathbb{Z}_2$ acts on $\widehat{X}$ by σ and on the interval $[-1, 1]$ by the sign flip $t \mapsto -t$.

Note that X is naturally embedded in $\mathscr{M}_X$, via the map

$$\begin{aligned} \iota \colon X &\to \mathscr{M}_X \\ x &\mapsto (x, 0) \end{aligned} \ . \tag{3.5.7}$$

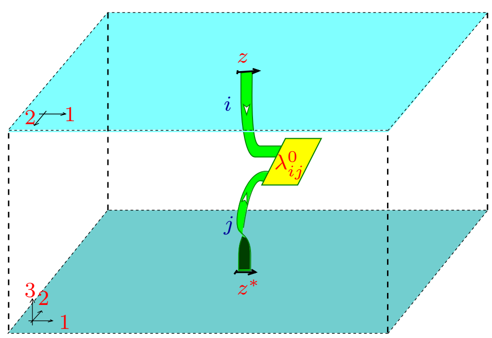
Thus the connecting manifold $\mathscr{M}_X$ can be regarded as a 'fattening' of the extended surface X. In this thesis, we will think of X as being embedded in $\mathscr{M}_X$ in this fashion.

Next, one needs to describe how to construct the ribbon graph in $\mathscr{M}_X$. We are not going to discuss this construction in detail but it is worth mentioning that this construction involves several choices, however the invariant associated to the graph is independent of all these choices. We are going to give some examples of ribbons embedded in $\mathscr{M}_X$, following [Ru10, Sect. 4.2]. The reader is referred to [FRS02-I, Sect. 5.1] for all the details of the construction.

In the following examples we will consider coupons with two incoming and one outgoing ribbons and vice versa. For incoming ribbons labelled R_i and R_j and the outgoing ribbon labelled R_k, the coupon is labelled by an element in $\mathrm{Hom}_{\mathcal{C}}(R_i \otimes R_j, R_k)$, i.e. an intertwiner from $R_i \times R_j$ to R_k. We pick basis and dual basis elements (3.3.21) and use them to label such coupons.

Example 3.5.4. *If we consider a bulk field $\Phi(z)$ on the Riemann sphere $X = \mathbb{C} \cup \{\infty\}$, then the relevant 3-manifold is $\mathscr{M}_X = X \times [-1, 1]$ and the corresponding ribbon graph is*

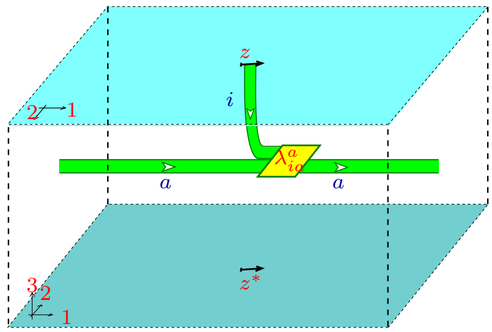
$\mathscr{M}_X =$



(3.5.8)

Example 3.5.5. *If we consider the Riemann sphere $X = \mathbb{C} \cup \{\infty\}$ with a defect of type a and a defect field $\phi^{a \leftarrow a}(z)$ insertion on that defect, then the relevant 3-manifold* *is* $\mathscr{M}_X = X \times [-1, 1]$ *and the corresponding ribbon graph is*

$$\mathscr{M}_X = \qquad (3.5.9)$$



Note that the linear map that the TFT associates to a bordism $\mathscr{M}$ does not change if we execute any of the modifications (3.3.20), (3.3.22), (3.3.23), (3.3.24) on a part of the embedded ribbon graph.

We have now gathered all the ingredients we need to define a 3D TFT. The definition is analogous to Def. 3.5.1, i.e. a *modular functor* $\mathrm{tft}_{\mathcal{C}} \colon \mathbf{Cob}_{\mathcal{C}}(3) \to \mathbf{Vect}_{\Bbbk}$ where now in addition to the given axioms, we require naturality in A_i [BK, Sect. 4.4]. The vector space $\mathcal{H}(\widehat{X})$ is the space of conformal blocks and a vector $Z(\mathscr{M}) \in \mathcal{H}(\widehat{X})$ is the correlator of the extended surface X, which we write as $\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_X)$.

Let us see this in an example following [FFFS02, Sect. 3.2]. Suppose X is closed with a given orientation, then $\widehat{X} = \overline{X} \amalg X$. Let X be endowed with n distinct disjoint arcs $\gamma_1, \dots, \gamma_n$, labeled by simple objects $i_1, \dots, i_n$. The connecting 3-manifold is $\mathscr{M}_X = X \times [-1, 1]$, with the embedded ribbon graph consisting of $\gamma_i \times [-1, 1]$, where γ_i runs over the marked arcs on X. To these data one associates an element $\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_X)$ on the space of conformal blocks $\mathcal{H}(\widehat{X})$, the correlator of X:

$$\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_X) = Z(X \times [-1, 1], \emptyset, \widehat{X}) \ . \qquad (3.5.10)$$

The next thing to do is to check if (3.5.10) obeys the modular and factorisation properties one expects for correlation functions. This is done in [FFFS02, Thms. 3.1 & 3.2].

Chapter 4

Category Theory for Perturbed Defects

In this and in the remaining chapters, we present the main results of this thesis which are published in the joint paper with Ingo Runkel [MR09].

In particular, in this chapter we present a categorical structure which captures some aspects of perturbed defect operators. Starting from a monoidal category $\mathcal{C}$, we then enlarge it to a category $\mathcal{C}_F$ whose objects are pairs (R, f), where $R \in \mathcal{C}$ and $f\colon F \otimes R \to R$ is a morphism in $\mathcal{C}$. Then we show that if $\mathcal{C}$ is abelian (cf. Def. 3.3.1), rigid (cf. Def. 3.3.9) and braided (cf. Def. 3.3.5) then $\mathcal{C}_F$ is an abelian rigid monoidal category (Thm. 4.3.2).

4.1 The Category $\mathcal{C}_F$

We start from a monoidal category $\mathcal{C}$ and enlarge it to a new category $\mathcal{C}_F$, depending on an object $F \in \mathcal{C}$. We then investigate how properties of $\mathcal{C}$ carry over to $\mathcal{C}_F$. In particular we will see that if $\mathcal{C}$ is braided and additive then we can define a monoidal structure on $\mathcal{C}_F$. The relation to perturbed defects is discussed in more detail in Chap. 5. The basic idea is that an object in $\mathcal{C}_F$ gives an unperturbed defect together with a direction for the perturbation by a defect field in the representation F.

Definition 4.1.1. Let $\mathcal{C}$ be a monoidal category and let $F \in \mathcal{C}$. The enlarged category $\mathcal{C}_F$ has as objects pairs $U_f \equiv (U, f)$, where $U \in \mathcal{C}$ and $f \colon F \otimes U \to U$. The morphisms $a \colon U_f \to V_g$ are all morphisms $a \colon U \to V$ in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{array}{ccc} F \otimes U & \xrightarrow{\mathrm{id}_F \otimes a} & F \otimes V \\ {\scriptstyle f}\downarrow & \circlearrowleft & \downarrow{\scriptstyle g} \\ U & \xrightarrow{\quad a \quad} & V \end{array}$$

The identity morphism id_{U_f} is id_U in $\mathcal{C}$, and the composition of morphisms is that of $\mathcal{C}$.

Remark 4.1.1. (i) The condition which singles out the subset of morphisms in $\mathcal{C}$ that belong to $\mathcal{C}_F$ is linear. Therefore, if $\mathcal{C}$ is an Ab-category, then so is $\mathcal{C}_F$. Similarly, if $\mathcal{C}$ is $\Bbbk$-linear for some field $\Bbbk$, then so is $\mathcal{C}_F$.

(ii) There is an action of the monoid (a monoid is an algebraic structure with a single associative binary operation and an identity element) $\mathrm{End}(F)^{\mathrm{op}}$ on $\mathcal{C}_F$[1]. Namely, for each $\varphi \in \mathrm{End}(F)$ we define the endofunctor $\mathcal{R}_\varphi$ of $\mathcal{C}_F$ on objects as $\mathcal{R}_\varphi(U_f) = (U, f \circ (\varphi \otimes \mathrm{id}_U))$ and on morphisms $U_f \xrightarrow{a} V_g$ as $\mathcal{R}_\varphi(a) = a$. We have $\mathcal{R}_\varphi \circ \mathcal{R}_\psi = \mathcal{R}_{\psi \circ \varphi}$ without the need for natural isomorphisms. This also shows that we have an action of $\mathrm{End}(F)^{\mathrm{op}}$ instead of $\mathrm{End}(F)$. If $\mathcal{C}$ is $\Bbbk$-linear, in this way, in particular, we obtain an action of $\Bbbk$ via $\lambda \mapsto \mathcal{R}_{\lambda \, \mathrm{id}_F}$.

(iii) If $\mathcal{C}$ is an Ab-category, we obtain an embedding I of $\mathcal{C}$ into $\mathcal{C}_F$. The functor $I \colon \mathcal{C} \to \mathcal{C}_F$ is defined via $I(U) = (U, 0)$ and $I(f) = f$; it is full[2] and faithful[3]. The

[1] Given an algebra A with multiplication $(a, b) \mapsto \mu(a, b)$, the opposite algebra has multiplication $(a, b) \mapsto \mu^{\mathrm{op}}(a, b) = \mu(b, a)$.

[2] A functor $F \colon \mathcal{C} \to \mathcal{D}$ is *full* when to every $A, B \in \mathcal{C}$ and to every $g \in \mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$, there is an $f \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, with $g = F(f)$.

[3] A functor $F \colon \mathcal{C} \to \mathcal{D}$ is *faithful* when to every pair of objects $A, B \in \mathcal{C}$ and to every pair of morphisms $f, g \in \mathrm{Hom}_{\mathcal{C}}(A, B)$, the equality $F(f) = F(g) \colon F(A) \to F(B)$ implies $f = g$.

forgetful[4] functor $\mathcal{C}_F \to \mathcal{C}$ is a left inverse for I.

(iv) One way to think of $\mathcal{C}_F$ is as a category of 'F-modules in $\mathcal{C}$', where the morphism $f\colon F \otimes U \to U$ in U_f is the 'action', and the morphisms of $\mathcal{C}_F$ intertwine this action. But F is not required to carry any additional structure, and so there is no restriction on the 'action' morphisms f.

(v) The category $\mathcal{C}_F$ can also be obtained as a (non-full) subcategory of the comma category $(F \otimes (-) \downarrow \mathrm{Id})$ (see [McL, Sect. II.6] for more details on comma categories). The objects of $(F \otimes (-) \downarrow \mathrm{Id})$ are triples (U, V, f) where $U, V \in \mathcal{C}$ and $f\colon F \otimes U \to V$. The morphisms $(U, V, f) \to (U', V', f')$ are pairs $(x\colon U \to U', y\colon V \to V')$ so that $y \circ f = f' \circ (\mathrm{id}_F \otimes x)$. The subcategory in question consists of all objects of the form (U, U, f) and all morphisms of the form (x, x).

(vi) The category of evaluation representations of the quantum affine algebra $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$ is a full subcategory of $\mathbf{Rep}\left(U_q(\mathfrak{sl}(2))\right)_F$, where F is $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$ understood as a $U_q\left(\mathfrak{sl}(2)\right)$-module. The details can be found in Appendix A. As briefly mentioned in the introduction, short exact sequences of representations of $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$ provide identities between transfer matrices for certain integrable lattice models. On the other hand, in Chap. 5 below we will see that short exact sequences in $\mathcal{C}_F$ give identities between certain defect operators in CFT. We hope that this similarity can be made more concrete in the future.

We will be interested in the Grothendieck group of $\mathcal{C}_F$, and to this end we need to know when $\mathcal{C}_F$ is abelian. The following theorem gives a sufficient condition. The proof is given in Appendix B.

Theorem 4.1.1. *If $\mathcal{C}$ is an abelian monoidal category with right-exact tensor product,*

[4] A functor which simply 'forgets' some or all of the structure of an algebraic object is called a *forgetful* functor or an *underlying* functor.

then $\mathcal{C}_F$ is abelian.

The following lemma will be useful; it is also proved in Appendix B.

Lemma 4.1.2. *Let $\mathcal{C}$ be as in Theorem 4.1.1 and $U_f \xrightarrow{a} V_g \xrightarrow{b} W_h$ be a complex in $\mathcal{C}_F$. Then $U_f \xrightarrow{a} V_g \xrightarrow{b} W_h$ is exact at V_g in $\mathcal{C}_F$ iff $U \xrightarrow{a} V \xrightarrow{b} W$ is exact at V in $\mathcal{C}$.*

4.2 Monoidal Structure on $\mathcal{C}_F$

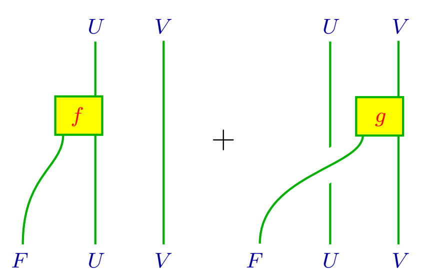
Let $\mathcal{C}$ be a braided monoidal Ab-category, see Def. 3.3.2 & Def. 3.3.1. The braiding and the abelian group structure on Hom-spaces allows us to define a tensor product $\hat{\otimes}$ on $\mathcal{C}_F$ as follows. On objects $U_f, V_g \in \mathcal{C}_F$ we set

$$U_f \,\hat{\otimes}\, V_g = (U \otimes V, T(f,g)) \ , \tag{4.2.1}$$

where $T(f,g)\colon F \otimes (U \otimes V) \to U \otimes V$ is defined as

$$T(f,g) = (f \otimes \mathrm{id}_V) \circ \alpha^{-1}_{F,U,V} + (\mathrm{id}_U \otimes g) \circ \alpha^{-1}_{U,F,V} \circ (c_{F,U} \otimes \mathrm{id}_V) \circ \alpha_{F,U,V} \ . \tag{4.2.2}$$

This definition, and some of the definitions and arguments below, are easier to understand upon replacing $\mathcal{C}$ by an equivalent strict category (cf. Def. 3.3.4) and using the graphical representation of morphisms in braided monoidal categories, see Sect. 3.3. For example, the graphical representation of (4.2.2) is

$T(f,g) =$  (4.2.3)

We will write $\mathbb{1}$ for the object $\mathbb{1}_0 \equiv (\mathbb{1}, 0)$ in $\mathcal{C}_F$. This will be the tensor unit for $\hat{\otimes}$.

Lemma 4.2.1. *The associator and unit isomorphisms of $\mathcal{C}$ are isomorphisms in $\mathcal{C}_F$ as follows: $\alpha_{U,V,W}\colon U_f \,\hat{\otimes}(V_g \,\hat{\otimes}\, W_h) \to (U_f \,\hat{\otimes}\, V_g) \,\hat{\otimes}\, W_h$, $\lambda_{U_f}\colon \mathbb{1} \,\hat{\otimes}\, U_f \to U_f$ and $\rho_{U_f}\colon U_f \,\hat{\otimes}\, \mathbb{1} \to U_f$.*

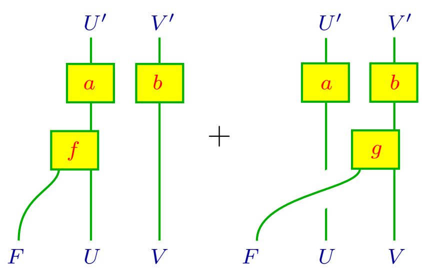
Proof. We have to show that

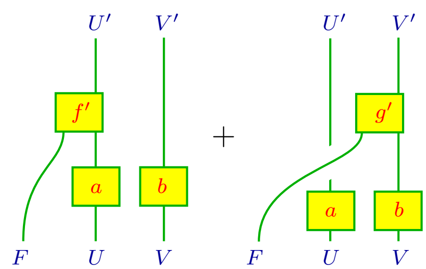
$$\alpha_{U,V,W} \colon \big(U \otimes (V \otimes W), T(f, T(g,h))\big) \to \big((U \otimes V) \otimes W, T(T(f,g),h)\big) \ ,$$
$$\lambda_U \colon \big(\mathbb{1} \otimes U, T(0,f)\big) \to (U,f) \ , \quad \rho_U \colon \big(U \otimes \mathbb{1}, T(f,0)\big) \to (U,f) \tag{4.2.4}$$

make the diagram in Def. 4.1.1 commute. These are all straightforward calculations. For example, $\rho_U \circ T(f,0) = \rho_U \circ (f \otimes \mathrm{id}_{\mathbb{1}}) \circ \alpha_{F,U,\mathbb{1}} = f \circ \rho_{F \otimes U} \circ \alpha_{F,U,\mathbb{1}} = f \circ (\mathrm{id}_F \otimes \rho_U)$. □

Lemma 4.2.2. *Let $a \colon U_f \to U'_{f'}$ and $b \colon V_g \to V'_{g'}$ be morphisms in $\mathcal{C}_F$. Then $a \otimes b \colon U \otimes V \to U' \otimes V'$ is also a morphism $U_f \,\hat{\otimes}\, V_g \to U'_{f'} \,\hat{\otimes}\, V'_{g'}$ in $\mathcal{C}_F$.*

Proof. We have to show that $(a \otimes b) \circ T(f,g) = T(f',g') \circ (\mathrm{id}_F \otimes (a \otimes b))$.

$(a \otimes b) \circ T(f,g) =$ 

$\overset{(1)}{=}$ 

$= T(f',g') \circ (\mathrm{id}_F \otimes (a \otimes b))$.

In step (1) in the first term we used the fact that $a \circ f = f' \circ (\mathrm{id}_F \otimes a)$, since a is a morphism in $\mathcal{C}_F$ (same for $b \circ g$ in the second term), while in the second term we used the naturality of the braiding (3.3.6). □

According to the previous lemma, on morphisms a, b we can define the tensor product to be the same as in $\mathcal{C}$,

$$a \,\hat{\otimes}\, b = a \otimes b \ . \tag{4.2.5}$$

One checks that $\hat{\otimes}$ is a bifunctor. Together with Lemma 4.2.1 this shows that $\mathcal{C}_F$ is a monoidal category.

Remark 4.2.1. (i) Even though $\mathcal{C}$ is braided, $\mathcal{C}_F$ is in general not. The reason is that $c_{U,V}$ is typically not a morphism in $\mathcal{C}_F$. Also, we actually demand too much when we require $\mathcal{C}$ to be braided, since all we use are the braiding isomorphisms where one of the arguments is given by F.

(ii) The functors $\mathcal{R}_\varphi$ defined in Remark 4.1.1 are strict monoidal functors. That is, $\mathcal{R}_\varphi\left(U_f \hat{\otimes} V_g\right) = \mathcal{R}_\varphi(U_f) \hat{\otimes} \mathcal{R}_\varphi(V_g)$ for objects and $\mathcal{R}_\varphi\left(a \hat{\otimes} b\right) = \mathcal{R}_\varphi(a) \hat{\otimes} \mathcal{R}_\varphi(b)$ for morphisms. This follows from $T(f \circ (\varphi \otimes \mathrm{id}_U), g \circ (\varphi \otimes \mathrm{id}_V)) = T(f,g) \circ (\varphi \otimes \mathrm{id}_{U \otimes V})$.

Theorem 4.2.3. *If $\mathcal{C}$ is an abelian braided monoidal category with right-exact tensor product, then $\mathcal{C}_F$ is an abelian monoidal category with right-exact tensor product. If the tensor product of $\mathcal{C}$ is exact, then so is that of $\mathcal{C}_F$.*

Proof. We have seen above that $\mathcal{C}_F$ is monoidal and in Theorem 4.1.1 that $\mathcal{C}_F$ is abelian. We will show that if $\otimes$ is right-exact, then the functor $X_x \hat{\otimes} (-)$ is right-exact. The arguments for $(-) \hat{\otimes} X_x$ and for 'exact' in place of 'right-exact' are analogous. Let $U_f \xrightarrow{a} V_g \xrightarrow{b} W_h \to 0$ be exact. Then $X \otimes U \xrightarrow{\mathrm{id}_X \otimes a} X \otimes V \xrightarrow{\mathrm{id}_X \otimes b} X \otimes W \to 0$ is exact in $\mathcal{C}$. By Lemma 4.1.2, $X_x \hat{\otimes} U_f \xrightarrow{\mathrm{id}_X \otimes a} X_x \hat{\otimes} V_g \xrightarrow{\mathrm{id}_X \otimes b} X_x \hat{\otimes} W_h \to 0$ is exact in $\mathcal{C}_F$. □

Corollary 4.2.4. *If $\mathcal{C}$ is an abelian braided monoidal category with exact tensor product, then $\mathcal{C}_F$ has a well-defined Grothendieck ring $K_0(\mathcal{C}_F)$.*

4.3 Duality On $\mathcal{C}_F$

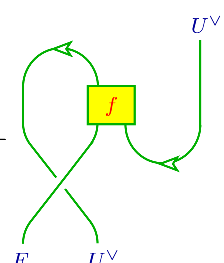
Suppose now that $\mathcal{C}$ is a braided monoidal Ab-category which has right duals (see Def. 3.3.8 for the precise definition of right duality and for the graphical representation, as well as for a definition of left duality). To a given object $U_f \in \mathcal{C}_F$ we assign the object

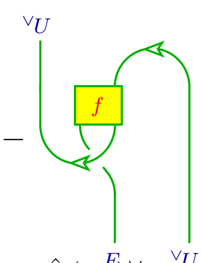
$$\begin{aligned}(U_f)^\vee &= (U^\vee, c(f)) \ ; \\ c(f) &= -\lambda_{U^\vee} \circ (d_U \otimes \mathrm{id}_{U^\vee}) \circ ((\mathrm{id}_{U^\vee} \otimes f) \otimes \mathrm{id}_{U^\vee}) \circ \left(\alpha^{-1}_{U^\vee,F,U} \otimes \mathrm{id}_{U^\vee}\right) \\ &\quad \circ \alpha_{U^\vee \otimes F,U,U^\vee} \circ (c_{F,U^\vee} \otimes b_U) \circ (\rho_{F\otimes U^\vee})^{-1} \ .\end{aligned} \tag{4.3.1}$$

If $\mathcal{C}$ has left duals, we define analogously

$$\begin{aligned}{}^\vee(U_f) &= ({}^\vee U, \tilde{c}(f)) \ ; \\ \tilde{c}(f) &= -\rho_{{}^\vee U} \circ \left(\mathrm{id}_{{}^\vee U} \otimes \tilde{d}_U\right) \circ \alpha^{-1}_{{}^\vee U,U,{}^\vee U} \circ \left(\left(\mathrm{id}_{{}^\vee U} \otimes \left(f \circ c^{-1}_{F,U}\right)\right) \otimes \mathrm{id}_{{}^\vee U}\right) \\ &\quad \circ \left(\alpha^{-1}_{{}^\vee U,U,F} \otimes \mathrm{id}_{{}^\vee U}\right) \circ \left(\left(\tilde{b}_U \otimes \mathrm{id}_F\right) \otimes \mathrm{id}_{{}^\vee U}\right) \circ \left(\lambda_F^{-1} \otimes \mathrm{id}_{{}^\vee U}\right) \ .\end{aligned} \tag{4.3.2}$$

Similarly, as in (4.2.2) it is helpful to pass to a strict category and write out the graphical representation of (4.3.1) and (4.3.2). This leads to the simple expressions

$c(f) = -$ 

, $\tilde{c}(f) = -$ 

. (4.3.3)

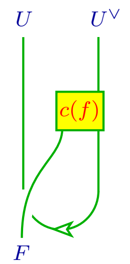
Lemma 4.3.1. *(i) If $\mathcal{C}$ has right duals, then $b_U : \mathbb{1} \to U_f \hat{\otimes} (U_f)^\vee$ and $d_U : (U_f)^\vee \hat{\otimes} U_f \to \mathbb{1}$ are morphisms in $\mathcal{C}_F$. (ii) If $\mathcal{C}$ has left duals, then $\tilde{b}_U : \mathbb{1} \to {}^\vee(U_f) \hat{\otimes} U_f$ and $\tilde{d}_U : U_f \hat{\otimes} {}^\vee(U_f) \to \mathbb{1}$ are morphisms in $\mathcal{C}_F$.*

Proof. The proof works similar in all four cases. Consider b_U as an example. The commuting diagram in Def. 4.1.1 boils down to the condition that the morphism $T(f, c(f)) \circ (\mathrm{id}_F \otimes b_U) : F \otimes \mathbb{1} \to U \otimes U^\vee$ has to be zero, i.e. that

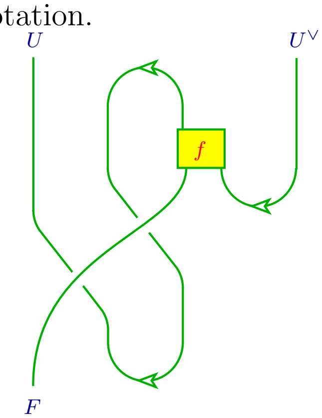
$$-\,T(0, c(f)) \circ (\mathrm{id}_F \otimes b_U) = T(f, 0) \circ (\mathrm{id}_F \otimes b_U) \ . \tag{4.3.4}$$

The calculation is best done using the graphical notation.

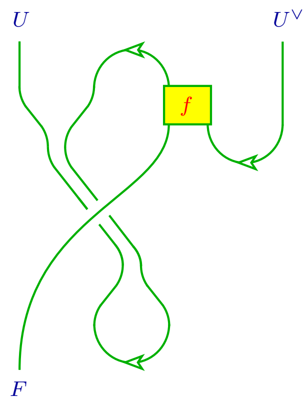
$$-T(0, c(f)) \circ (\mathrm{id}_F \otimes b_U) = -$$



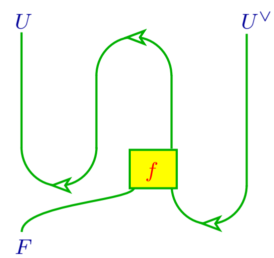
$$\overset{(1)}{=}$$



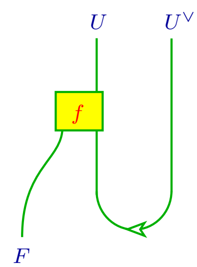
$$\overset{(2)}{=}$$



$$\overset{(3)}{=}$$



$$\overset{(4)}{=}$$



$$= T(f, 0) \circ (\mathrm{id}_F \otimes b_U) \ .$$

In step (1) we used equation (4.3.3) to substitute for $c(f)$, in step (2) we used the tensoriality property of the braiding (3.3.7), (3) uses the naturality property (3.3.6) of the braiding to pull b_U through $c_{F,U \otimes U^\vee}$ and the fact that $c_{F,\mathbb{1}} = \mathrm{id}_F$ and finally, in step (4) we used the right duality axioms (3.3.10). □

Theorem 4.3.2. *Let $\mathcal{C}$ be a braided monoidal Ab-category. If $\mathcal{C}$ has right and/or left duals, then so has $\mathcal{C}_F$. In particular, if $\mathcal{C}$ is rigid, so is $\mathcal{C}_F$.*

Remark 4.3.1. (i) Suppose $\mathcal{C}$ has left and right duals. Even if in $\mathcal{C}$ we were to have $U^\vee = {}^\vee U$, the same need not be true in $\mathcal{C}_F$ due to the distinct definitions of $c(f)$ and $\tilde{c}(f)$. Also, even if in $\mathcal{C}$ we were to have $(U^\vee)^\vee \cong U$, the same need not hold in $\mathcal{C}_F$. We will see this explicitly in the Lee-Yang example in Section 6.2.

(ii) Let $\mathcal{C}$ be as in Corollary 4.2.4. If $\mathcal{C}$ has right duals, then the existence of a right duality for $\mathcal{C}_F$ tells us that in $K_0(\mathcal{C}_F)$ we have $[(U_f)^\vee] \cdot [U_f] = [\mathbb{1}] + [W_h]$ and $[U_f] \cdot [(U_f)^\vee] = [\mathbb{1}] + [W'_{h'}]$ for some $W_h, W'_{h'} \in \mathcal{C}_F$. This will imply functional identities for perturbed defect operators via the relation described in Chap. 5. The same holds for left duals.

(iii) The functors $\mathcal{R}_\varphi$ defined in Remark 4.1.1 are compatible with these dualities in the sense that $\mathcal{R}_\varphi\left((U_f)^\vee\right) = \left(\mathcal{R}_\varphi(U_f)\right)^\vee$ and $\mathcal{R}_\varphi\left({}^\vee(U_f)\right) = {}^\vee\left(\mathcal{R}_\varphi(U_f)\right)$. This follows from $c\left(f \circ (\varphi \otimes \mathrm{id}_U)\right) = c(f) \circ (\varphi \otimes \mathrm{id}_{U^\vee})$ and $\tilde{c}\left(f \circ (\varphi \otimes \mathrm{id}_U)\right) = \tilde{c}(f) \circ (\varphi \otimes \mathrm{id}_{{}^\vee U})$.

Chapter 5

Relation to Defect Operators

As we have already seen in Chap. 2, defects are lines on the world sheet where the fields can be discontinuous or even singular. Suppose we are given a CFT that is well-defined on surfaces with defect lines, that is, it satisfies the axioms in [RS08, Sect. 3] (or at least a genus 0 version thereof). Recall that to a defect we can assign a linear operator D on the space of states $\mathscr{H}$ of the CFT. This operator can be extracted by wrapping the defect line around a short cylinder $[-\varepsilon, \varepsilon] \times S^1$, where we place two states u and v on the two boundary circles. The resulting amplitude, in the limit $\varepsilon \to 0$, is the pairing $\langle u, Dv \rangle$.

Working with fields rather than with states, the defect operator D is obtained as the correlator assigned to the Riemann sphere $\mathbb{C} \cup \{\infty\}$ with one in-going puncture at 0 and one out-going puncture at ∞, both with standard local coordinates, and a defect line placed on the unit circle S^1. By the state-field correspondence (1.3.2), the space of states $\mathscr{H}$ is at the same time the space of local bulk fields, so that again $D \colon \mathscr{H} \to \mathscr{H}$.

Here we are interested in topological defects (recall Def. 2.1.2). We will be also working in rational CFT, so that the chiral algebra of the CFT will be a rational vertex operator algebra $\mathfrak{V}$ (recall footnote 1 of the Introduction of the thesis). Denote by $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$ the category of (appropriate) representations of $\mathfrak{V}$. It is a semi-simple finite rigid braided monoidal category which is modular [HL94, Hu05]. We will not need many details about modular categories, but we note that $\mathcal{C}$ satisfies the conditions of Theorems 4.2.3 and 4.3.2.

Let us pick a set of representatives[1] $\{ R_i \,|\, i \in \mathcal{I} \}$ of the isomorphism classes of simple objects, so that $\mathbb{1} \equiv R_0 \equiv \mathfrak{V}$ is the monoidal unit. We restrict ourselves in this paper to the Cardy case constructed from $\mathfrak{V}$. The space of states of this model is

$$\mathscr{H} = \bigoplus_{i \in \mathcal{I}} R_i \otimes_{\mathbb{C}} R_i^{\vee} \ , \tag{5.0.1}$$

where $R_i^{\vee}$ denotes the contragredient representation to R_i. Also, we will only consider topological defects which are maximally symmetric in that they are compatible with the entire chiral symmetry $\mathfrak{V} \otimes_{\mathbb{C}} \mathfrak{V}$, i.e. (2.1.2) holds for the modes of all fields in $\mathfrak{V} \otimes_{\mathbb{C}} \mathfrak{V}$ not just for those of the stress tensor. As we saw in Sect. 2.2 according to [PZ01a, FRS02-I] the different maximally symmetric topological defects are labeled by representations of $\mathfrak{V}$, that is, objects $R \in \mathcal{C}$. We denote the defect operator of the defect labeled by $R \in \mathcal{C}$ by $D[R]$. The defect operator assigned to a simple object R_i is (recall (2.2.14))

$$D[R_i] = \sum_{j \in \mathcal{I}} \frac{S_{ij}}{S_{0j}} \, \mathrm{id}_{R_j \otimes_{\mathbb{C}} R_j^{\vee}} \ , \tag{5.0.2}$$

where by $\mathrm{id}_{R_j \otimes_{\mathbb{C}} R_j^{\vee}}$ we mean the projector to the direct summand $R_j \otimes_{\mathbb{C}} R_j^{\vee}$ of $\mathscr{H}$, and S is the modular matrix, i.e. the $|\mathcal{I}| {\times} |\mathcal{I}|$-matrix which describes the modular transformation of characters. If $R \cong \bigoplus_{i \in \mathcal{I}} (R_i)^{\oplus n_i}$ then $D[R] = \sum_{i \in \mathcal{I}} n_i \, D[R_i]$.

5.1 Correlators of Chiral Defect Fields

Recall that by a chiral defect field we mean a field that 'lives on the defect' and that has left/right conformal weight $(h, 0)$. The notion of defect fields is described for example in [FRS05-IV, Sect. 3.4] and [RS08, Sect. 3.2], see also Chap. 2 for a review.

[1] The notation R_i, where i is an index of a simple object, should not be confused with the notation R_f for objects of $\mathcal{C}_F$ (for some F), where $f\colon F \otimes R \to R$ is a morphism. The meaning of the index should be clear from the context, and in any case we will mostly use i, j, k for indices of simple objects and f, g, h, as well as c and x, for morphisms.

The defect fields have well-defined weights with respect to L_0 and $\bar{L}_0$ because we are considering topological defects, and those are transparent to the holomorphic and anti-holomorphic part of the stress tensor.

The space of chiral defect fields on a defect labeled by $R \in \mathcal{C}$ consists of all vectors $v \otimes_{\mathbb{C}} \Omega \in (R \otimes R^\vee) \otimes_{\mathbb{C}} \mathfrak{V}$, where $\Omega \in \mathfrak{V}$ is the vacuum vector of $\mathfrak{V}$, see [FRS05-IV, Eqn. (3.37)] and [PZ01a, PZ01b, FRS02-I]. Here, the tensor product $R \otimes R^\vee$ is the fusion tensor product in $\mathcal{C}$. Pick a representation $F \in \mathcal{C}$. A *chiral defect field in representation* F is specified by a vector $\phi \in F$ and a morphism $\tilde{f}\colon F \to R \otimes R^\vee$ in $\mathcal{C}$. Instead of $\tilde{f}$ we find it more convenient to give a morphism $f\colon F \otimes R \to R$.

We are going to define a defect operator for a defect labeled by a representation R with chiral defect fields ϕ inserted at mutually distinct points $e^{i\theta_1}, \dots, e^{i\theta_n}$ on the unit circle, where for each insertion we allow a different morphism $f_1, \dots, f_n$. We will denote this operator by

$$D[R; f_1, \dots, f_n; \theta_1, \dots, \theta_n]\colon \mathscr{H} \to \overline{\mathscr{H}} \ . \tag{5.1.1}$$

The operator D may have contributions in an infinite number of graded components of the target vector spaces. Hence, we have to pass to a completion of $\mathscr{H}$, namely to the direct product $\overline{\mathscr{H}}$ of the graded components of $\mathscr{H}$. We will later integrate over the variables θ_k, and the resulting operator commutes with the grading, so that we obtain an operator $\mathscr{H} \to \mathscr{H}$.

Let us restrict D to the sector $R_i \otimes_{\mathbb{C}} R_i^\vee$ of $\mathscr{H}$ and call the resulting operator D_i. Because the defect fields are all chiral, they do not affect the anti-holomorphic sector, and hence the image of D_i will lie entirely in the summand $\overline{R_i \otimes_{\mathbb{C}} R_i^\vee}$ of $\overline{\mathscr{H}}$. The operator D_i is an element of a suitable space of conformal blocks, namely of a tensor product (over $\mathbb{C}$) of two spaces of conformal blocks on the sphere, related to the two chiral halves of the CFT. On the first sphere $\mathbb{C} \cup \{\infty\}$ we have insertions of R_i at 0 and ∞, and of F at $e^{i\theta_1}, \dots, e^{i\theta_n}$. Insertions at ∞ will always be treated as out-going, the others as in-going. Because the defect fields are chiral, on the second sphere we just have insertions of $R_i^\vee$ at 0 and ∞. Altogether, the conformal block is an operator

$$\mathscr{C}[R; f_1, \dots, f_n; \theta_1, \dots, \theta_n]_i \colon R_i \otimes_{\mathbb{C}} R_i^{\vee} \otimes_{\mathbb{C}} F \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} F \longrightarrow \overline{R_i \otimes_{\mathbb{C}} R_i^{\vee}} \ . \tag{5.1.2}$$

It determines the defect operator D_i on a vector $u \otimes v \in R_i \otimes_{\mathbb{C}} R_i^{\vee} \subset \mathscr{H}$ via

$$D_i(u \otimes v) = \mathscr{C}[R; f_1, \dots, f_n; \theta_1, \dots, \theta_n]_i (u \otimes v \otimes \phi \otimes \cdots \otimes \phi) \ . \tag{5.1.3}$$

The vector space of conformal blocks from which (5.1.2) is taken is finite-dimensional, as is always the case in rational CFT, but its dimension can be quite high and will grow with the number n of insertions. We thus need an efficient method to specify elements in the space of conformal blocks. Such a method is provided by using three-dimensional topological field theory to describe correlators of rational CFT, see Sect. 3.5.2 for an introduction to 2D TFT and references therein, which treat defect lines and defect fields in detail.

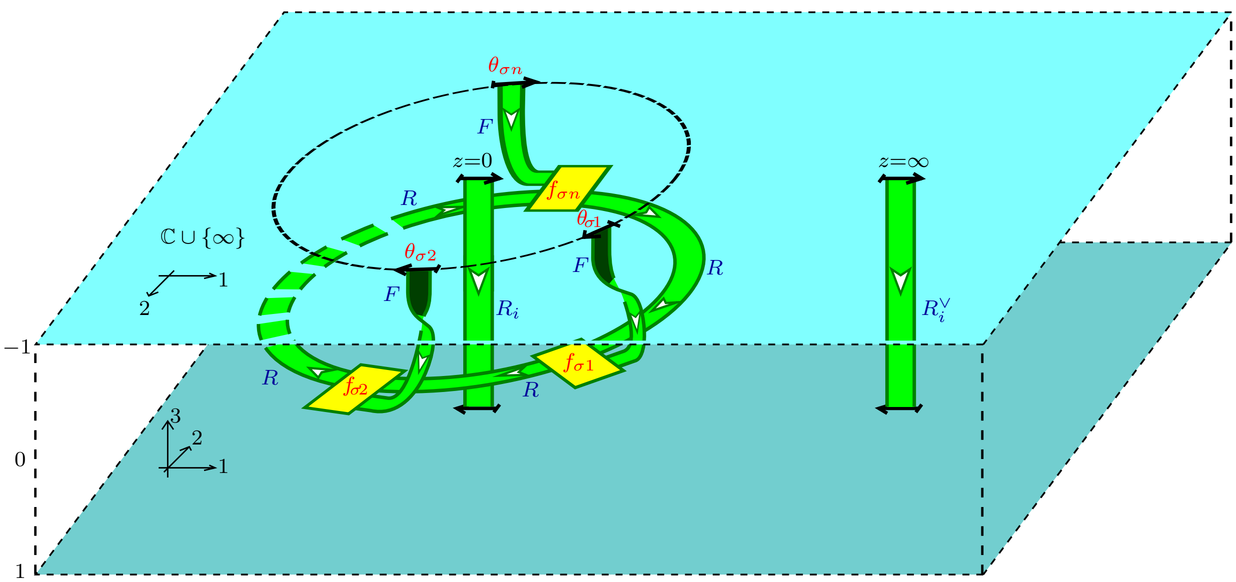
Recall from Sect. 3.5.2 that the 3D TFT assigns to a three-manifold $\mathscr{M}$ with embedded ribbon graph an element in the space of conformal blocks on the boundary surface $\partial\mathscr{M}$ of $\mathscr{M}$. If the 3D TFT is Chern-Simons theory for a gauge group G, the conformal blocks are those of the corresponding WZW model [Wi89, FK89], see also [Wi84] where the topological nature as well as the conditions under which the model becomes conformal are investigated. There is also a general construction, whereby the 3D TFT is defined by a modular category $\mathcal{C}$ (see Def. 3.5.2), which in turn is obtained from the representations of a rational vertex operator algebra [MS90, Hu05]. Let us denote this TFT as $\mathrm{tft}_{\mathcal{C}}$.

In the TFT approach to correlators of rational CFT, one starts from a world sheet X, possibly with boundary and defect lines, and with various field insertions, and constructs from this a three-manifold $\mathscr{M}_X$ with embedded ribbon graph. The boundary of $\mathscr{M}_X$ is the double $\widehat{X}$ of the surface X and the TFT assigns to $\mathscr{M}_X$ a conformal block in $\widehat{X}$, which we write as $\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_X)$. This is the correlator for the world sheet X.

Let us see how this works in the case at hand, where X is $\mathbb{C}\cup\{\infty\}$ with bulk fields in representation $R_i \otimes_{\mathbb{C}} R_i^{\vee}$ inserted at 0 and ∞, and with a defect line labeled R

placed on the unit circle on which defect fields in representation F are inserted at the points $e^{i\theta_1}, \dots, e^{i\theta_n}$. As X is oriented and has empty boundary, the three-manifold is simply $\mathscr{M}_X = X \times [-1, 1]$. Note that $\partial\mathscr{M}_X$ does indeed consist of two Riemann spheres, so that the TFT will determine an element in the tensor product of two spaces of conformal blocks on the sphere, as discussed above. It remains to construct the ribbon graph embedded in $\mathscr{M}_X$. To do this, we place a circular ribbon labeled by the representation R on the unit circle in the plane $X \times \{0\}$. This ribbon is connected to the marked points $e^{i\theta_k}$ on the boundary $X \times \{1\}$ of M_X with ribbons labeled by F. The junction of F and R is formed by the intertwiner $f_k \colon F \otimes R \to R$. For the bulk insertions at 0 and ∞ one places a vertical ribbon inside M_X connecting the marked points on the boundary components $X \times \{1\}$ and $X \times \{-1\}$. The resulting ribbon graph is

$$\mathscr{M}[R; f_1, \dots, f_n; \theta_1, \dots, \theta_n]_i =$$



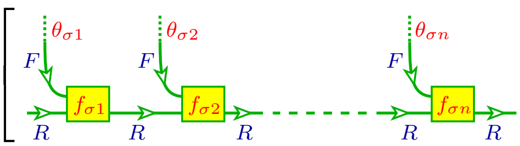
(5.1.4)

For the TFT conventions used here, see [FRS02-I, Sect. 2], and for more details on the construction of the ribbon graph consult [FRS05-IV, Sect. 3 & 4]. The orientation of the 'top' plane of $\mathscr{M}$ is obtained from that of $\mathscr{M}$ by taking the inward pointing normal. The arrows at the ends of the ribbons refer to a particular choice of local coordinates around the F-insertions, namely the local coordinate at $\exp(i\theta_{\sigma k})$ is given by $\zeta \mapsto -i(\exp(-i\theta_{\sigma k})\zeta - 1)$, so that $\exp(i\theta_{\sigma k})$ gets mapped to zero and the real axis of the local coordinate system is tangent to the defect circle. We do not demand that the $\theta_1, \dots, \theta_n$ are ordered. Instead we define $\sigma \in S_n$ to be the unique permutation of n elements for which $0 \le \theta_{\sigma 1} < \theta_{\sigma 2} \cdots < \theta_{\sigma n} < 2\pi$. Finally, the conformal block (5.1.2) is given by

$$\mathscr{C}[R; f_1, \dots, f_n; \theta_1, \dots, \theta_n]_i = \mathrm{tft}_{\mathcal{C}}\big(\mathscr{M}[R; f_1, \dots, f_n; \theta_1, \dots, \theta_n]_i\big) \ . \tag{5.1.5}$$

One can work out this conformal block in terms of intertwiners as in [FRS05-IV, Sect. 5], but we will not need such an explicit expression here. This conformal block in turn determines the defect operator (5.1.1) via $D = \bigoplus_i D_i$ with D_i given in (5.1.3).

The strength of the representation (5.1.5) lies in the fact that we can now use identities that hold within the 3D TFT, i.e. manipulations which change the ribbon graph inside $\mathscr{M}$ without modifying the value of $\mathrm{tft}_{\mathcal{C}}(\mathscr{M})$, to prove identities among conformal blocks. This will be used extensively in the proof of the next lemma. In fact, the manipulations below will only involve a neighbourhood of the circular ribbon in (5.1.4). For this reason, it is convenient to have a shorthand for (5.1.4) which only shows this region of $\mathscr{M}$. We will write

$$\mathscr{M}[R; f_1, \dots, f_n; \theta_1, \dots, \theta_n]_i = \mathscr{M}\Bigg[$$  $$\Bigg] \ . \tag{5.1.6}$$

Lemma 5.1.1. *(i) Let $0 \to K_h \to R_f \to C_c \to 0$ be an exact sequence in $\mathcal{C}_F$, and let $\theta_1, \dots, \theta_m \in [0, 2\pi)$ be mutually distinct. Then*

$$D[R; f, \dots, f; \theta_1, \dots, \theta_m] = D[K; h, \dots, h; \theta_1, \dots, \theta_m] + D[C; c, \dots, c; \theta_1, \dots, \theta_m] \tag{5.1.7}$$

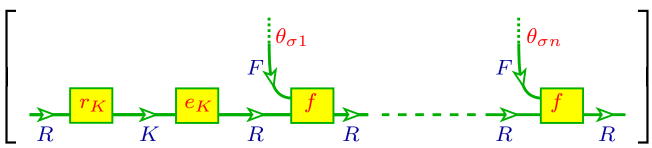
(ii) Let $R_f, S_g \in \mathcal{C}_F$, and let $\theta_1, \dots, \theta_m, \eta_1, \dots, \eta_n \in [0, 2\pi)$ be mutually distinct. Then

$$\begin{aligned} &\lim_{\varepsilon \to 0+} D[R; f, \dots, f; \theta_1, \dots, \theta_m] \ e^{\varepsilon(L_0 + \bar{L}_0)} \ D[S; g, \dots, g; \eta_1, \dots, \eta_n] \\ &\quad = D[R \otimes S; T(f,0), \dots, T(f,0), T(0,g), \dots, T(0,g); \theta_1, \dots, \theta_m, \eta_1, \dots, \eta_n] \end{aligned} \tag{5.1.8}$$

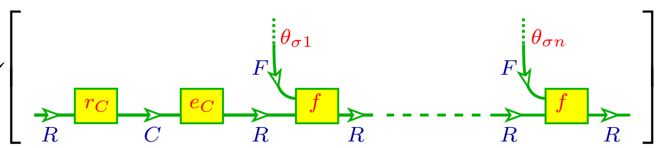
Proof. (i) Denote the morphisms in the exact sequence by $e_K\colon K_h \to R_f$ and $r_C\colon R_f \to C_c$. In the present situation, the category $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$ is modular, and thus in particular semi-simple. Therefore, in $\mathcal{C}$ the exact sequence $0 \to K \xrightarrow{e_K} R \xrightarrow{r_C} C \to 0$ splits, i.e. we can find $r_K\colon R \to K$ and $e_C\colon C \to R$ such that $r_K \circ e_K = \mathrm{id}_K$, $r_C \circ e_C = \mathrm{id}_C$, and $e_K \circ r_K + e_C \circ r_C = \mathrm{id}_R$. Using the decomposition of id_R we can write

$$\mathscr{C}[R; f, \ldots, f; \theta_1, \ldots, \theta_n]_i = \mathrm{tft}_{\mathcal{C}}(\mathscr{M}_K) + \mathrm{tft}_{\mathcal{C}}(\mathscr{M}_C) \tag{5.1.9}$$

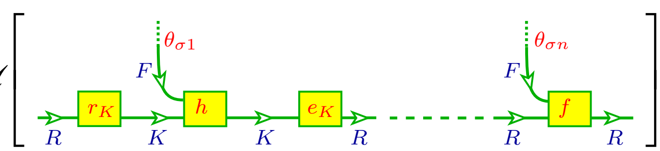
where

$\mathscr{M}_K = \mathscr{M}$  ,

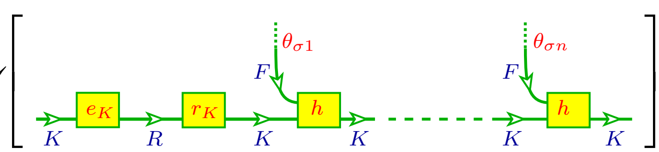
(5.1.10)

$\mathscr{M}_C = \mathscr{M}$  .

Since $e_K\colon K_h \to R_f$ is a morphism in $\mathcal{C}_F$, it satisfies the identity $e_K \circ h = f \circ (\mathrm{id}_F \otimes e_K)$. This can be used to move e_K past f, for example,

$\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_K) = \mathrm{tft}_{\mathcal{C}}\big(\mathscr{M}$  $\big)$. (5.1.11)

If one repeats this procedure and in this way takes e_K around the loop, one arrives at

$\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_K) = \mathrm{tft}_{\mathcal{C}}\big(\mathscr{M}$  $\big)$ (5.1.12)

$= \mathscr{C}[K; h, \ldots, h; \theta_1, \ldots, \theta_n]_i$.

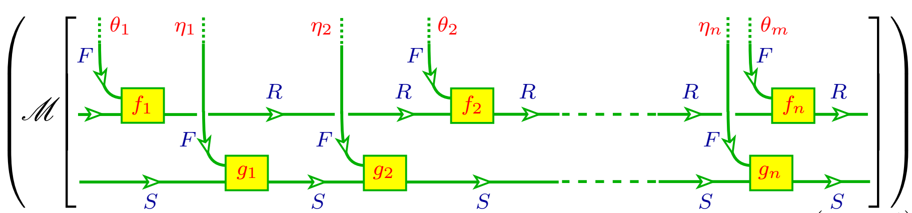
In the last step we used $r_K \circ e_K = \mathrm{id}_K$ and Equation (5.1.5). For $\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_C)$ one proceeds similarly, only that here $r_C\colon R_f \to C_c$ is the morphism in $\mathcal{C}_F$, and so one has to move r_C around the loop in the opposite sense. This results in

$$\mathrm{tft}_{\mathcal{C}}(\mathscr{M}_C) = \mathscr{C}[C; c, \dots, c; \theta_1, \dots, \theta_n]_i \ . \tag{5.1.13}$$

Combining (5.1.9), (5.1.12) and (5.1.13) establishes part (i) of the lemma.

(ii) Because the conformal block in (5.1.5) is a map from $R_i \otimes_{\mathbb{C}} R_i^\vee$ to the direct product $\overline{R_i \otimes_{\mathbb{C}} R_i^\vee}$ of the L_0,$\bar{L}_0$-eigenspaces in $R_i \otimes_{\mathbb{C}} R_i^\vee$, we have to take care that the composition is well-defined. This is ensured by the exponential in (5.1.8). Since the insertion points $e^{i\theta}$ of the intertwining operators (of the vertex operator algebra representations) are distinct, the limit $\varepsilon \to 0$ is well-defined. Let $\mathscr{C}_{\mathrm{lhs}}$ and $\mathscr{C}_{\mathrm{rhs}}$ be the conformal blocks obtained from the left and right hand side of (5.1.8), respectively. To see that $\mathscr{C}_{\mathrm{lhs}} = \mathscr{C}_{\mathrm{rhs}}$ we again use the 3D TFT. Let us look at a particular example of the ordering of the θ_k and η_k, say $\theta_1 < \eta_1 < \eta_2 < \theta_2 < \cdots < \eta_n < \theta_m$. The general case works along the same lines. Substituting the definitions, one finds that the three-manifold and ribbon graph for $\mathscr{C}_{\mathrm{rhs}}$ is

$$\mathscr{C}_{\mathrm{lhs}} = \mathscr{C}_{\mathrm{rhs}} = \mathrm{tft}_{\mathcal{C}}$$



(5.1.14)

To see that $\mathscr{C}_{\mathrm{lhs}}$ leads to the same result, one has to translate the composition of conformal blocks into a gluing of three-manifolds as in [FFFS02, Thm. 3.2]. Namely, one needs to cut out a cylinder around the R_i-ribbon at $z = 0$ of $D[R; \dots]$ and around the $R_i^\vee$-ribbon at $z = \infty$ of $D[S; \dots]$, and identify the resulting cylindrical boundaries. The resulting ribbon graph can be deformed to give (5.1.14). This establishes part

(ii) of the lemma. $\square$

5.2 Perturbed Topological Defects

The operator of the perturbed defect is defined via an exponentiated integral. That is, for an object $R_f \in \mathcal{C}_F$ we set[2]

$$D[R_f] = \sum_{n=0}^{\infty} \frac{1}{n!} D[R_f]^{(n)} \quad , \quad D[R_f]^{(n)} = \int_0^{2\pi} D[R; f, \ldots, f; \theta_1, \ldots, \theta_n] \, d\theta_1 \cdots d\theta_n \ . \tag{5.2.1}$$

Because of the permutation that orders the arguments in the definition (5.1.4), (5.1.5) and (5.1.3) of the defect operator, a path-ordering prescription is automatically imposed and does not need to be included explicitly in the integration regions for $D[R_f]^{(n)}$. The integrals in $D[R_f]^{(n)}$ and the infinite sum in $D[R_f]$ may or may not converge. Since we have no direct way to ensure convergence, we say that an object $R_f \in \mathcal{C}_F$ *has finite integrals* if $\varphi(D[R_f]^{(n)} v)$ exists for each $\varphi \in \mathscr{H}^*$, $v \in \mathscr{H}$, and $n \in \mathbb{N}_0$. Note that this is not a property of the category $\mathcal{C}_F$ alone, but instead also depends on the vertex operator algebra $\mathfrak{V}$ and the vector $\phi \in F$. As already mentioned in Sect. 2.3, generically one expects that if the element $\phi \in F$ has conformal weight $h_\phi < \frac{1}{2}$, then all $R_f \in \mathcal{C}_F$ have finite integrals (but we have no proof). Let $R_f \in \mathcal{C}_F$ have finite integrals. It is demonstrated in [Ru08, Sect. 2.2] that

$$\big[L_0, D[R_f]^{(n)}\big] = 0 \quad \text{and} \quad \big[\bar{L}_m, D[R_f]^{(n)}\big] = 0 \ , \quad \forall m \in \mathbb{Z}. \tag{5.2.2}$$

We will not discuss the convergence of the infinite sum in (5.2.1). Instead we will treat it as a formal power series in the following way. For $\zeta \in \mathbb{C}$ we have $D[R_{\zeta f}]^{(n)} = \zeta^n D[R_f]^{(n)}$. Now take ζ to be a formal parameter and let us define, by slight abuse

[2] Recall from below (5.1.5) that the local coordinate around the insertion of a defect field ϕ at $e^{i\theta}$ was chosen to be $\zeta \mapsto -i(e^{-i\theta}\zeta - 1)$. This choice makes (for example) $D[R; f; \theta]$ periodic under $\theta \rightsquigarrow \theta + 2\pi$. Had we instead chosen the standard local coordinates $\zeta \mapsto \zeta - p$ on the complex plane around a point p, $D[R; f; \theta]$ would have picked up the phase $e^{-2\pi i h_\phi}$.

of notation,

$$D[R_{\zeta f}] = \sum_{n=0}^{\infty} \frac{\zeta^n}{n!}\, D[R_f]^{(n)} \;\in \mathrm{End}(\mathscr{H})[\![\zeta]\!] \;. \tag{5.2.3}$$

Theorem 5.2.1. *Let ζ be a formal parameter.*

(i) Let $0 \to K_k \to R_f \to C_c \to 0$ be an exact sequence in $\mathcal{C}_F$, and let K_k, R_f, C_c have finite integrals. Then $D[R_{\zeta f}] = D[K_{\zeta k}] + D[C_{\zeta c}]$.

(ii) Let $R_f, S_g \in \mathcal{C}_F$ have finite integrals. Then $D[R_{\zeta f}]D[S_{\zeta g}] = D[(R\otimes S, \zeta T(f,g))]$.

Proof. Part (i) holds because by Lemma 5.1.1 (i) it already holds before integration. For part (ii) first note that the exponential in (5.1.8) is not necessary to make the composition $D[R_{\zeta f}]D[S_{\zeta g}]$ well-defined, because $D[R_{\zeta f}]$ commutes with $L_0+\bar{L}_0$ and we can write $D[R_{\zeta f}]D[S_{\zeta g}] = \lim_{\varepsilon\to 0} e^{-\varepsilon(L_0+\bar{L}_0)}D[R_{\zeta f}]e^{\varepsilon(L_0+\bar{L}_0)}D[S_{\zeta g}]$. We will therefore not write the limit in the equations below. Define operators A_n and B_n via

$$D[R_{\zeta f}]D[S_{\zeta g}] = \sum_{n\in\mathbb{N}} \frac{1}{n!}\zeta^n A_n \quad \text{and} \quad D[(R\otimes S, \zeta T(f,g))] = \sum_{n\in\mathbb{N}} \frac{1}{n!}\zeta^n B_n \;. \tag{5.2.4}$$

We have to show that $A_n = B_n$. Starting from A_n we find

$$\begin{aligned} A_n &= \sum_{m=0}^{n} \binom{n}{m} D[R_{\zeta f}]^{(m)} D[S_{\zeta g}]^{(n-m)} \\ &= \sum_{m=0}^{n} \binom{n}{m} \int D[R; f,\dots,f;\theta_1,\dots,\theta_m]\, D[S; g,\dots,g;\eta_1,\dots,\eta_{n-m}] \\ &= \sum_{m=0}^{n} \binom{n}{m} \int D[R\otimes S; T(f,0),\dots,T(f,0),T(0,g),\dots,T(0,g);\theta_1,\dots,\theta_m,\eta_1,\dots,\eta_{n-m}] \end{aligned} \tag{5.2.5}$$

where $\int \equiv \int_0^{2\pi} d\theta_1\cdots d\theta_m d\eta_1\cdots d\eta_{n-m}$ and in the last step we used Lemma 5.1.1 (ii). For B_n we get

$$B_n = \int_0^{2\pi} d\alpha_1\cdots d\alpha_n\; D[R\otimes S; T(f,g),\dots,T(f,g);\alpha_1,\dots,\alpha_n] \;. \tag{5.2.6}$$

To see that this is equal to the right hand side of (5.2.5) one first writes $T(f,g) = T(f,0) + T(0,g)$, then expands out the integrand into 2^n summands and groups together those with the same number of $T(f,0)$ and $T(0,g)$. The distinct ordering in each term can be absorbed into a change of integration variables as the angles α_k are all integrated from 0 to 2π. □

Theorem 5.2.1 implies the following corollary.

Corollary 5.2.2. *Let ζ be a formal parameter and let $R_f, S_g \in \mathcal{C}_F$ have finite integrals.*
(i) If $[R_f] = [S_g]$ in $K_0(\mathcal{C}_F)$, then $D[R_{\zeta f}] = D[S_{\zeta g}]$.
(ii) If $[R_f] \cdot [S_g] = [M_m]$ in $K_0(\mathcal{C}_F)$ then $D[R_{\zeta f}]D[S_{\zeta g}] = D[M_{\zeta m}]$.

Remark 5.2.1. (i) If all $R_f \in \mathcal{C}_F$ have finite integrals, then Corollary 5.2.2 says that the map $[R_f] \mapsto D[R_{\zeta f}]$ defines a ring homomorphism $K_0(\mathcal{C}_F) \to \mathrm{End}(\mathscr{H})[\![\zeta]\!]$. Since $D[R_{\zeta f}]$ commutes with L_0 and $\bar{L}_0$ (and in fact with all modes of the anti-holomorphic copy of the chiral algebra) the 'representation' of $K_0(\mathcal{C}_F)$ on $\mathscr{H}$ splits into an infinite direct sum of subrepresentations. One may then wonder why one should consider all of them together, rather than restricting one's attention to a given eigenspace. One reason to do this is that one expects $D[R_f]$ to have the following appealing behaviour under modular transformations. Let $Z[R_f](\tau) = \mathrm{Tr}_{\mathscr{H}} q^{L_0 - c/24}(q^*)^{\bar{L}_0 - c/24} D[R_f]$, where $q = \exp(2\pi i \tau)$, and let us assume that the infinite sum in $D[R_f]$ converges, and that the trace over $\mathscr{H}$ converges for τ in the upper half plane. The resulting power series in q and q^* will typically not have integral coefficients. But when expressed in terms of $\tilde{q} = \exp(-2\pi i/\tau)$ and $\tilde{q}^*$ we are counting the states that live on a circle intersected by the perturbed defect, and so we expect that

$$Z[R_f](\tau) = \sum_{(x,y)\in\mathbb{C}\times\mathbb{C}} n[R_f]_{x,y} \cdot \tilde{q}^{\,x}(\tilde{q}^*)^y \quad , \; n[R_f]_{x,y} \in \mathbb{N}_0 \; , \tag{5.2.7}$$

and $n[R_f]_{x,y} \neq 0$ only for countably many pairs. The infinite direct sum of subrepresentations on $\mathscr{H}$ has to conspire in a precise way in order to give rise to non-negative integer coefficients in the crossed channel.

(ii) The construction of perturbed topological defects and their relation to $\mathcal{C}_F$ applies also to perturbations of conformal boundary conditions. Of course, in this case the composition in Theorem 5.2.1 (ii) does not make sense, but Theorem 5.2.1 (i) remains valid. In the Cardy case, the discussion of perturbed boundary conditions is however subsumed in that of perturbed topological defects because (in the Cardy case) the boundary state of a perturbed boundary condition can always be written as $D[R_f]|\mathbb{1}\rangle\rangle$ for $|\mathbb{1}\rangle\rangle$ the Cardy boundary state [Ca89] associated to the vacuum representation of $\mathfrak{V}$. This follows from the 3D TFT formulation of boundary and defect correlators [FFFS02, FRS05-IV]. So in the Cardy case, treating perturbed conformal boundaries instead of perturbed topological defects amounts to forgetting the monoidal structure on $\mathcal{C}_F$.

Chapter 6

Lee Yang Model

In this chapter, we use the construction described in Chapters 4 & 5 to find functional relations, for perturbed defects, in the Lee-Yang model (introduced in Sect. 1.7.1). There, one obtains a family of operators $D(\lambda)$, $\lambda \in \mathbb{C}$, on the space of states of the model, which obey, for all $\lambda, \mu \in \mathbb{C}$,

$$[L_0 + \bar{L}_0, D(\lambda)] = 0 \ , \quad [D(\lambda), D(\mu)] = 0 \ , \quad D\left(e^{2\pi i/5}\lambda\right) D\left(e^{-2\pi i/5}\lambda\right) = \mathrm{id} + D(\lambda) \ . \tag{6.0.1}$$

The last relation above is closely linked to the description of the Lee-Yang model via the massless limit of factorising scattering and the thermodynamic Bethe Ansatz, see e.g. the review [DDT07]. This example illustrates that the functional relations obeyed by perturbed defect operators, encode at least part of the integrable structure of the model. In fact, the defect operator in (6.0.1) (and more generally those for the $M_{2,2m+1}$ minimal models) can be understood as certain linear combinations of the chiral operators which were constructed in [BLZ96] to capture the integrable structure of these models.

The two irreducible highest weight representations of the Virasoro algebra are denoted by R_1 (for $h = 0$) and R_ϕ (for $h = -1/5$). As already remarked in footnote 1, the notation R_1 and R_ϕ should not be confused with objects R_f of $\mathcal{C}_F$ (for some $\mathcal{C}$ and F); in any case we will never use 1 or ϕ to denote morphisms.

Let $\mathbf{Rep}(\mathfrak{V}_{2,5})$ be the category of all Virasoro representations at $c = -22/5$ which are isomorphic to finite direct sums of R_1 and R_ϕ. On $\mathbf{Rep}(\mathfrak{V}_{2,5})$ we have the fusion tensor product[1] with non-trivial fusion $R_\phi \otimes R_\phi \cong R_1 \oplus R_\phi$. The Grothendieck group of $\mathbf{Rep}(\mathfrak{V}_{2,5})$ is therefore isomorphic to $\mathbb{Z} \times \mathbb{Z}$ with generators $[R_1]$ and $[R_\phi]$. The product on $K_0(\mathbf{Rep}(\mathfrak{V}_{2,5}))$ has $[R_1]$ as multiplicative unit, and $[R_\phi]\cdot[R_\phi] = [R_1]+[R_\phi]$.

As described in Chap. 2 and in the introduction of Chap. 5, to each object $R \in \mathbf{Rep}(\mathfrak{V}_{2,5})$ we can associate a topological defect operator $D[R]\colon \mathscr{H} \to \mathscr{H}$ that commutes with the two copies of the Virasoro algebra. Since $D[R]$ depends only on $[R] \in K_0(\mathbf{Rep}(\mathfrak{V}_{2,5}))$, it is enough to give $D[R_1]$ and $D[R_\phi]$ as in (5.0.2),

$$D[R_1] = \mathrm{id}_{\mathscr{H}} \quad , \quad D[R_\phi] = d \cdot \mathrm{id}_{R_1 \otimes_{\mathbb{C}} R_1} - d^{-1} \cdot \mathrm{id}_{R_\phi \otimes_{\mathbb{C}} R_\phi} \ , \tag{6.0.2}$$

where d is as in (1.7.6). It is easy to check that indeed $D[R_\phi]D[R_\phi] = \mathrm{id} + D[R_\phi]$, as required by the corresponding relation in $K_0(\mathbf{Rep}(\mathfrak{V}_{2,5}))$.

We can now perturb the defect labeled R_ϕ by a chiral defect field with left/right conformal weights $(-\frac{1}{5}, 0)$ as described in Sect. 5.2. This amounts to considering the objects $R_\phi(\mu) \equiv (R_\phi, \mu \cdot \lambda_{(\phi\phi)\phi})$ in $\mathcal{C}_{R_\phi}$, where $\mu \in \mathbb{C}$ and $\lambda_{(\phi\phi)\phi}$ is a fixed non-zero morphism $R_\phi \otimes R_\phi \to R_\phi$. We then obtain a family of defect operators $D[R_\phi(\lambda)]$. In [Ru08] it was shown – assuming convergence – that these operators mutually commute,

$$\big[\, D[R_\phi(\lambda)]\, ,\, D[R_\phi(\mu)] \,\big] = 0 \qquad \text{for all } \lambda, \mu \in \mathbb{C} \ , \tag{6.0.3}$$

and that they satisfy the functional relation

$$D[R_\phi(e^{2\pi i/5}\lambda)]\, D[R_\phi(e^{-2\pi i/5}\lambda)] = \mathrm{id} + D[R_\phi(\lambda)] \qquad \text{for all } \lambda \in \mathbb{C} \ . \tag{6.0.4}$$

In the next section we recover this functional relation from studying the tensor product and exact sequences in the corresponding category $\mathcal{C}_F$.

[1] More precisely, $\mathfrak{V}_{2,5}$ is the Virasoro vertex operator algebra built on R_1. $\mathbf{Rep}(\mathfrak{V}_{2,5})$ is the category of admissible modules of $\mathfrak{V}_{2,5}$; this category is finite and semi-simple [Wa93, Def. 2.3 & Thm. 4.2] and forms a braided monoidal category [Hu95, Cor. 3.9].

6.1 The Category $\mathcal{C}_F$ for the Lee-Yang Model

The category $\mathbf{Rep}(\mathfrak{V}_{2,5})$ is equivalent (as a $\mathbb{C}$-linear braided monoidal category) to a category $\mathcal{V}$ defined as follows. The objects A of $\mathcal{V}$ are pairs $A = (A_1, A_\phi)$ of finite-dimensional complex vector spaces indexed by the labels $\{1, \phi\}$ used for simple objects in $\mathbf{Rep}(\mathfrak{V}_{2,5})$. A morphism $f\colon A \to B$ is a pair $f = (f_1, f_\phi)$ of linear maps, where $f_1\colon A_1 \to B_1$ and $f_\phi\colon A_\phi \to B_\phi$. This construction is described in more detail in Appendix C. The tensor product $\circledast$ of $\mathcal{V}$ is given on objects as

$$A \circledast B = \big(A_1 \otimes_{\mathbb{C}} B_1 \,\oplus\, A_\phi \otimes_{\mathbb{C}} B_\phi \,,\; A_1 \otimes_{\mathbb{C}} B_\phi \,\oplus\, A_\phi \otimes_{\mathbb{C}} B_1 \,\oplus\, A_\phi \otimes_{\mathbb{C}} B_\phi\big) \,. \quad (6.1.1)$$

The tensor product on morphisms and the non-trivial associator are described in Appendix C. The dual of an object $A \in \mathcal{V}$ is $A^\vee = (A_1^*, A_\phi^*)$, where A_1^* and A_ϕ^* are the dual vector spaces. The duality morphisms are given in Appendix C.

As representatives of the two isomorphism classes of simple objects we take $\mathbb{1} = (\mathbb{C}, 0)$ and $\Phi = (0, \mathbb{C})$. We are interested in the category $\mathcal{V}_F$ for $F = \Phi$. Note that $\Phi \circledast A = (A_\phi, A_1 \oplus A_\phi)$. Therefore, in an object $A_f \in \mathcal{V}_\Phi$, the morphism $f\colon \Phi \circledast A \to A$ has components $f_1\colon A_\phi \to A_1$ and $f_\phi\colon A_1 \oplus A_\phi \to A_\phi$. We will denote the two summands of f_ϕ as $f_{\phi 1}\colon A_1 \to A_\phi$ and $f_{\phi\phi}\colon A_\phi \to A_\phi$; for consistency of notation we will also denote $f_1 \equiv f_{1\phi}$. It is convenient to collect these three linear maps into a matrix

$$f \mathrel{\widehat{=}} \begin{array}{c} \\ A_1 \\ A_\phi \end{array} \!\!\begin{array}{c} \begin{matrix} A_1 & A_\phi \end{matrix} \\ \begin{pmatrix} 0 & f_{1\phi} \\ f_{\phi 1} & f_{\phi\phi} \end{pmatrix} \end{array} , \quad (6.1.2)$$

where we have also indicated the source and target vector spaces. We can now compute the dual of an object $A_f \in \mathcal{V}_\Phi$ according to (4.3.1). This is done in Appendix D with the simple result

$$(A_f)^\vee = (A^\vee, c(f)) \quad \text{with} \quad c(f) \mathrel{\widehat{=}} \begin{array}{c} \\ A_1^* \\ A_\phi^* \end{array} \!\!\begin{array}{c} \begin{matrix} A_1^* & A_\phi^* \end{matrix} \\ \begin{pmatrix} 0 & -d\zeta^2 f_{\phi 1}^* \\ -d^{-1} f_{1\phi}^* & -\zeta f_{\phi\phi}^* \end{pmatrix} \end{array} \quad \text{and} \quad \zeta = e^{-\pi i/5} \,. \quad (6.1.3)$$

The tensor product in $\mathcal{V}_\Phi$ is more lengthy. We have $A_f \hat{\circledast} B_g = (A \circledast B, T(f,g))$ where $T(f,g)\colon \Phi \circledast (A \circledast B) \to A \circledast B$. The source vector spaces of $T(f,g)$ are (we omit the '$\otimes_\mathbb{C}$')

$$\Phi \circledast (A \circledast B) = (A_1 B_\phi \oplus A_\phi B_1 \oplus A_\phi B_\phi, A_1 B_1 \oplus A_\phi B_\phi \oplus A_1 B_\phi \oplus A_\phi B_1 \oplus A_\phi B_\phi) \ . \quad (6.1.4)$$

In Appendix D we evaluate equation (4.2.2) for $T(f,g)$ in the category $\mathcal{V}_\Phi$. The result is best represented in a 5×5-matrix,

$$T(f,g) \mathrel{\widehat{=}} \begin{array}{c} \\ A_1 \otimes_\mathbb{C} B_1 \\ A_\phi \otimes_\mathbb{C} B_\phi \\ A_1 \otimes_\mathbb{C} B_\phi \\ A_\phi \otimes_\mathbb{C} B_1 \\ A_\phi \otimes_\mathbb{C} B_\phi \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 \otimes_\mathbb{C} B_1 & A_\phi \otimes_\mathbb{C} B_\phi & A_1 \otimes_\mathbb{C} B_\phi & A_\phi \otimes_\mathbb{C} B_1 & A_\phi \otimes_\mathbb{C} B_\phi \end{array} \\ \begin{pmatrix} 0 & 0 & \mathrm{id}_{A_1}\, g_{1\phi} & f_{1\phi}\, \mathrm{id}_{B_1} & 0 \\ 0 & 0 & f_{\phi 1}\, \mathrm{id}_{B_\phi} & \zeta^2\, \mathrm{id}_{A_\phi}\, g_{\phi 1} & f_{\phi\phi} + \zeta g_{\phi\phi} \\ \mathrm{id}_{A_1}\, g_{\phi 1} & \frac{1}{d} f_{1\phi}\, \mathrm{id}_{B_\phi} & \mathrm{id}_{A_1}\, g_{\phi\phi} & 0 & w f_{\phi 1}\, \mathrm{id}_{B_\phi} \\ f_{\phi 1}\, \mathrm{id}_{B_1} & \frac{1}{\zeta^2 d}\, \mathrm{id}_{A_\phi}\, g_{1\phi} & 0 & f_{\phi\phi}\, \mathrm{id}_{B_1} & \frac{w}{\zeta}\, \mathrm{id}_{A_\phi}\, g_{1\phi} \\ 0 & \frac{1}{wd}\left(f_{\phi\phi} + \frac{1}{\zeta} g_{\phi\phi}\right) & f_{\phi 1}\, \mathrm{id}_{B_\phi} & \zeta\, \mathrm{id}_{A_\phi}\, g_{\phi 1} & -\frac{1}{d}(f_{\phi\phi} + g_{\phi\phi}) \end{pmatrix} \end{array} . \quad (6.1.5)$$

Here ζ was given in (6.1.3), $w \in \mathbb{C}^\times$ is a normalisation constant (see Appendix D), and in the entries with sums we have omitted the identity maps. For example, $f_{\phi\phi} + \zeta g_{\phi\phi}$ stands for $f_{\phi\phi} \otimes_\mathbb{C} \mathrm{id}_{B_\phi} + \zeta\, \mathrm{id}_{A_\phi} \otimes_\mathbb{C} g_{\phi\phi}$.

6.2 Some Exact Sequences in $\mathcal{C}_F$

Two objects A_f and B_g in $\mathcal{V}_\Phi$ are isomorphic if and only if there exist isomorphisms $\gamma_1\colon A_1 \to B_1$ and $\gamma_\phi\colon A_\phi \to B_\phi$ such that

$$\begin{pmatrix} 0 & g_{1\phi} \\ g_{\phi 1} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} 0 & \gamma_1 \circ f_{1\phi} \circ \gamma_\phi^{-1} \\ \gamma_\phi \circ f_{\phi 1} \circ \gamma_1^{-1} & \gamma_\phi \circ f_{\phi\phi} \circ \gamma_\phi^{-1} \end{pmatrix} . \quad (6.2.1)$$

For $\lambda \in \mathbb{C}$ write $\Phi(\lambda) \equiv (\Phi, f(\lambda))$ with $f(\lambda)_1 = 0$ and $f(\lambda)_\phi = \lambda \cdot \mathrm{id}_\mathbb{C}$. In other words, $\Phi(\lambda) = \big((0,\mathbb{C}), (\lambda)\big)$. Then $\Phi(\lambda) \cong \Phi(\mu)$ if and only if $\lambda = \mu$. As another example,

$$\left((\mathbb{C},\mathbb{C}), \begin{pmatrix} 0 & a \\ b & c \end{pmatrix}\right) \cong \left((\mathbb{C},\mathbb{C}), \begin{pmatrix} 0 & a' \\ b' & c' \end{pmatrix}\right) \Leftrightarrow \begin{cases} ab = a'b' \ , \ c = c' \text{ and} \\ \mathrm{rk}(a) = \mathrm{rk}(a') \ , \ \mathrm{rk}(b) = \mathrm{rk}(b') \end{cases} , \quad (6.2.2)$$

where $\mathrm{rk}(a) \in \{0,1\}$ denotes the rank of the linear map $a \cdot \mathrm{id}_{\mathbb{C}}$.

For $\mathbb{1}$ and $\Phi(\lambda)$ there are no non-trivial exact sequences as the underlying objects in $\mathcal{V}$ are already simple. For $\big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&a\\b&c\end{smallmatrix}\big)\big)$ there are two exact sequences,

$$0 \to \Phi(\lambda) \to \big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&0\\b&\lambda\end{smallmatrix}\big)\big) \to \mathbb{1} \to 0 \quad , \quad 0 \to \mathbb{1} \to \big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&a\\0&\lambda\end{smallmatrix}\big)\big) \to \Phi(\lambda) \to 0 \ . \tag{6.2.3}$$

Let us explain how one arrives at the first one. One checks that there is a surjective morphism $\big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&a\\b&c\end{smallmatrix}\big)\big) \to \mathbb{1}$ in $\mathcal{V}_\Phi$ iff $(1,0)\big(\begin{smallmatrix}0&a\\b&c\end{smallmatrix}\big) = 0$, i.e. iff $a = 0$. To complete this to an exact sequence, we need an injective morphism $\Phi(\lambda) \to \big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&a\\b&c\end{smallmatrix}\big)\big)$. This exists iff $\big(\begin{smallmatrix}0&a\\b&c\end{smallmatrix}\big)\big(\begin{smallmatrix}0\\1\end{smallmatrix}\big) = \big(\begin{smallmatrix}0\\\lambda\end{smallmatrix}\big)$, i.e. iff $a = 0$ and $\lambda = c$. From (6.2.3) it follows that in $K_0(\mathcal{V}_\Phi)$ we have

$$\Big[\,\big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&0\\b&\lambda\end{smallmatrix}\big)\big)\,\Big] \;=\; [\mathbb{1}] + [\Phi(\lambda)] \;=\; \Big[\,\big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&a\\0&\lambda\end{smallmatrix}\big)\big)\,\Big] \ , \tag{6.2.4}$$

even though $\big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&0\\b&\lambda\end{smallmatrix}\big)\big)$ and $\big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix}0&a\\0&\lambda\end{smallmatrix}\big)\big)$ are not isomorphic unless $a = b = 0$.

Next let us look at the simplest non-trivial tensor product, $\Phi(\lambda)\hat{\circledast}\Phi(\mu)$. Formula (6.1.5) simplifies to

$$\Phi(\lambda)\hat{\circledast}\Phi(\mu) = \left((\mathbb{C},\mathbb{C}),\begin{pmatrix}0 & \lambda+\zeta\mu \\ \frac{1}{wd}(\lambda+\zeta^{-1}\mu) & -d^{-1}(\lambda+\mu)\end{pmatrix}\right) \ . \tag{6.2.5}$$

By comparing to (6.2.2) we see that $\Phi(\lambda)\hat{\circledast}\Phi(\mu) \cong \Phi(\mu)\hat{\circledast}\Phi(\lambda)$ iff either $\lambda = \mu = 0$ or $(\lambda+\zeta\mu)(\lambda+\zeta^{-1}\mu) \neq 0$. In particular, $\Phi(-\zeta\mu)\hat{\circledast}\Phi(\mu) \ncong \Phi(\mu)\hat{\circledast}\Phi(-\zeta\mu)$ unless $\mu = 0$. This shows that $\mathcal{V}_\Phi$ cannot be braided. The reducibility of $\Phi(\lambda)\hat{\circledast}\Phi(\mu)$ is summarised in three cases:

(i) if $\lambda \notin \{-\zeta\mu, -\zeta^{-1}\mu\}$ then $\Phi(\lambda)\hat{\circledast}\Phi(\mu)$ is irreducible,

(ii) if $\lambda = -\zeta\mu$ we have $0 \to \Phi(\zeta^{-2}\mu) \to \Phi(-\zeta\mu)\hat{\circledast}\Phi(\mu) \to \mathbb{1} \to 0$,

(iii) if $\lambda = -\zeta^{-1}\mu$ we have $0 \to \mathbb{1} \to \Phi(-\zeta^{-1}\mu)\hat{\circledast}\Phi(\mu) \to \Phi(\zeta^2\mu) \to 0$.

In $K_0(\mathcal{V}_\Phi)$ we therefore get the relations

$$[\Phi(\zeta^{-2}\lambda)]\cdot[\Phi(\zeta^2\lambda)] \overset{\text{(ii)}}{=} [\mathbb{1}] + [\Phi(\lambda)] \overset{\text{(iii)}}{=} [\Phi(\zeta^2\lambda)]\cdot[\Phi(\zeta^{-2}\lambda)] \ . \tag{6.2.6}$$

Combining with the case when $\Phi(\lambda)\hat{\circledast}\Phi(\mu)$ is irreducible we find that in $K_0(\mathcal{V}_\Phi)$ we have

$$[\Phi(\lambda)] \cdot [\Phi(\mu)] = [\Phi(\mu)] \cdot [\Phi(\lambda)] \quad \text{for all } \lambda, \mu \in \mathbb{C} \ . \tag{6.2.7}$$

In fact we could have obtained the reducibility in (ii) and (iii) above already from the existence of duals. Namely, by (6.1.3), $(\Phi(\lambda))^\vee = \Phi(-\zeta\lambda)$ and by Lemma 4.3.1 we have non-zero morphisms $b_\Phi \colon \mathbb{1} \to \Phi(\lambda)\Phi(-\zeta\lambda)$ and $d_\Phi \colon \Phi(-\zeta\lambda)\Phi(\lambda) \to \mathbb{1}$. Also note that taking the dual n times gives $\Phi(\lambda)^{\vee\cdots\vee} = \Phi((-\zeta)^n\lambda)$, and since $-\zeta$ is a 10th root of unity, the 10-fold dual is the first one that is again isomorphic to $\Phi(\lambda)$ (for $\lambda \neq 0$). This is different from e.g. fusion categories (which are by definition semi-simple [CE04, Def. 1.9]) where $V^{\vee\vee} \cong V$ for all simple objects V, see [CE04, Prop. 1.17].

To conclude our sample calculations in $\mathcal{V}_\Phi$ we point out that for a given $\big((\mathbb{C},\mathbb{C}), \left(\begin{smallmatrix} 0 & a \\ b & c \end{smallmatrix}\right)\big)$ at least one of the isomorphisms

$$\big((\mathbb{C},\mathbb{C}), \left(\begin{smallmatrix} 0 & a \\ b & c \end{smallmatrix}\right)\big) \cong \mathbb{1} \oplus \Phi(\lambda), \ \big((\mathbb{C},\mathbb{C}), \left(\begin{smallmatrix} 0 & a \\ b & c \end{smallmatrix}\right)\big) \cong \Phi(\lambda)\hat{\circledast}\Phi(\mu) \ , \tag{6.2.8}$$

holds for some $\lambda, \mu \in \mathbb{C}$. This is easy to check by comparing cases in (6.2.2) and (6.2.5).

6.3 Some Implications for Defect Flows

The relation (6.2.6) in $K_0(\mathcal{V}_\Phi)$ gives the functional relation (6.0.4) for the perturbed R_ϕ-defect in the Lee-Yang model. Let us point out one application of such functional relations, namely how they can give information about endpoints of renormalisation group flows. We use the notation for objects as in $\mathcal{V}_\Phi$, e.g. we write $D[\Phi(\lambda)]$ instead of $D[R_\phi(\lambda)]$.

We shall assume that $D[\Phi(\lambda)]$ is an operator-valued meromorphic function on $\mathbb{C}$, and that its asymptotics for $\lambda \to +\infty$ along the real axis is given by (compare to [BLZ96, Eqn. (62)] or [BLZ97, Eqn. (2.21)])

$$D[\Phi(\lambda)] \sim \exp(f\lambda^{1/(1-h_\phi)})D_\infty \ + \ \text{less singular terms} \ , \tag{6.3.1}$$

where $\lambda^{1/(1-h_\phi)} = \lambda^{5/6}$ has dimension of length, $f > 0$ is a free energy per unit length, and D_∞ is the operator describing the defect at the endpoint of the flow. We assume that this asymptotic behavior remains valid in the direction $\lambda = re^{i\theta}$, $r \to +\infty$, of the complex plane at least as long as the real part of $(e^{i\theta})^{5/6}$ remains positive, i.e. for $|\theta| < 3\pi/5$. This is a subtle point as in analogy with integrable models the asymptotics will be subject to Stokes' phenomenon, see e.g. [DDT07, App. D.1].

With these assumptions, we can substitute the asymptotic behavior (6.3.1) into the functional relation (6.0.4), which gives

$$\exp\big(f(\zeta^2\lambda)^{5/6} + f(\zeta^{-2}\lambda)^{5/6}\big) D_\infty D_\infty = \mathrm{id} + \exp(f\lambda^{5/6}) D_\infty \ . \tag{6.3.2}$$

As $f > 0$, the identity operator will be subleading, and since $(\zeta^2)^{5/6} + (\zeta^{-2})^{5/6} = 1$ the leading asymptotics demands that

$$D_\infty D_\infty = D_\infty \ . \tag{6.3.3}$$

Since D_∞ is the endpoint of a renormalisation group flow, we expect it to be a conformal defect, i.e. $[L_m + \bar{L}_{-m}, D_\infty] = 0$. On the other hand for every value of λ we have $\big[\bar{L}_m, D[\Phi(\lambda)]\big] = 0$, so that D_∞ is again a topological defect. Thus $D_\infty = m \cdot \mathrm{id} + n \cdot D_\phi$ for some $m, n \in \mathbb{N}$. This is consistent with (6.3.3) only for $D_\infty = \mathrm{id}$. We thus obtain the asymptotic behavior

$$D[\Phi(\lambda)] \xrightarrow{\lambda \to +\infty} \exp(f\lambda^{5/6})\,\mathrm{id} \ . \tag{6.3.4}$$

This is the expected result, because via the relation of perturbed defects and perturbed boundaries mentioned in Remark 5.2.1 (ii), the above flow agrees with the corresponding boundary flow obtained in [DPTW, Sect. 3]. It also agrees with the corresponding free field expression [BLZ97, Eqn. (2.21)].

This result allows us to make some statements about perturbations of the superposition of the 1- and ϕ-defect, i.e. the topological defect labeled by $R_1 \oplus R_\phi$. We can either perturb it by a defect field on the topological defect labeled R_ϕ alone, in which case we would get the operator $\mathrm{id} + D[\Phi(\lambda)]$ which flows to $D_\infty = \mathrm{id}$ as $\lambda \to +\infty$.

Or we can in addition perturb by defect-changing fields. In this case we can use the result (6.2.8), which tells us that we can write the perturbed defect as the composition $D[\Phi(\lambda)]D[\Phi(\mu)]$ for some λ, μ. Then, if the necessary λ, μ lie in the wedge of the complex plane where (6.3.4) is valid, we again have

$$D\Big[\big((\mathbb{C},\mathbb{C}),\big(\begin{smallmatrix} 0 & ra \\ rb & rc \end{smallmatrix}\big)\big)\Big] \xrightarrow{r\to+\infty} \exp(f' r^{5/6})\,\mathrm{id} \quad . \tag{6.3.5}$$

Conclusions

In conclusion, this thesis proposes an abelian rigid monoidal category $\mathcal{C}_F$, constructed from an abelian rigid braided monoidal category $\mathcal{C}$ and a choice of object $F \in \mathcal{C}$, that captures some of the properties of perturbed topological defects. To make the connection to defects, we set $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$, for $\mathfrak{V}$ a rational vertex operator algebra, and choose a $\mathfrak{V}$-module $F \in \mathcal{C}$ together with a vector $\phi \in F$. Then we consider the charge-conjugation CFT constructed from $\mathfrak{V}$ (the Cardy case). An object $U_f \in \mathcal{C}_F$ corresponds to an unperturbed topological defect labeled U and a perturbing field given by the chiral defect field defined via $\phi \in F$ and the morphism $f \colon F \otimes U \to U$. Assuming convergence of the multiple integrals and the infinite sum in (5.2.1), to U_f we can assign an operator $D[U_f]$ on the space of states $\mathscr{H} = \bigoplus_{i \in \mathcal{I}} R_i \otimes_{\mathbb{C}} R_i^\vee$ of the CFT. This operator describes the topological defect perturbed by the specified defect field. Again assuming convergence of all $D[\dots]$ involved, the main properties of the assignment $U_f \mapsto D[U_f]$ are

(i) $D[\mathbb{1}] = \mathrm{id}_{\mathscr{H}}$,

(ii) $D[U_{f=0}] = \sum_{i,j \in \mathcal{I}} \dim \mathrm{Hom}_{\mathcal{C}}(R_i, U)\, S_{ij}/S_{0j}\ \mathrm{id}_{R_j \otimes_{\mathbb{C}} R_j^\vee}$,

(iii) $\big[\, L_0\,,\, D[U_f] \big] = 0$ and $\big[\, \bar{L}_m\,,\, D[U_f] \big] = 0$ for $m \in \mathbb{Z}$,

(iv) if $0 \to K_h \to U_f \to C_g \to 0$ is an exact sequence, then $D[U_f] = D[K_h] + D[C_g]$,

(iv′) if $[U_f] = [V_g]$ in $K_0(\mathcal{C}_F)$, then $D[U_f] = D[V_g]$,

(v) $D[U_f \,\hat{\otimes}\, V_g] = D[U_f]D[V_g]$.

There is an anti-holomorphic counterpart of the construction in this paper, where one perturbs the topological defect by a defect field of dimension $(0, h)$. This generates another set of defect operators which commute with those introduced here.

The results presented in this thesis also leave a large number of question unanswered, and we hope to come back to some of these in the future:

1. In the Lee-Yang example it should be possible to describe the category $\mathcal{C}_F$ and its Grothendieck ring more explicitly. For example it would be interesting to know if $\mathcal{C}_F$ is generated by the $\Phi(\lambda)$ in the sense that every object of $\mathcal{C}_F$ is obtained by taking direct sums, tensor products, subobjects and quotients starting from $\Phi(\lambda)$. Note that at this stage we do not even know whether or not $\mathcal{C}_F$ is commutative in the Lee-Yang example.

2. Consider the case $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$ for a rational vertex operator algebra $\mathfrak{V}$ and let $U_f \in \mathcal{C}_F$ have finite integrals. Suppose the infinite sum $O(\zeta) = D[U_{\zeta f}]$ has a finite radius of convergence in ζ. One can then extend the domain of definition of $O(\zeta)$ by analytic continuation. To solve the functional relations it is most important to understand the global properties of $O(\zeta)$, in particular whether all functions $\varphi(O(\zeta)v)$ (for $\varphi \in \mathscr{H}^*$ and $v \in \mathscr{H}$) are entire functions on $\mathbb{C}$, and what their asymptotic behaviors are. It should be possible to address these questions with the methods reviewed and developed in [DDT07] and [IIKNS08].

3. The category $\mathcal{C}_F$ is designed specifically for the Cardy case. The formalism developed in [FRS02-I, FFRS07] allows one to extend this treatment to all rational CFTs with chiral symmetry $\mathfrak{V} \otimes_{\mathbb{C}} \mathfrak{V}$. The different CFTs with this symmetry are in one-to-one correspondence with Morita-classes of special symmetric Frobenius algebras A in $\mathcal{C} = \mathbf{Rep}(\mathfrak{V})$. Given such an algebra A, the category $\mathcal{C}_F$ has to be replaced by a category $\mathcal{C}(A)_F$ whose objects are pairs (B, f) where B is an A-A-bimodule[2] and $f\colon F \otimes^+ B \to B$ is an intertwiner of bimodules (see [FRS05-IV, Sect. 2.2] for the definition of $\otimes^+$). The details remain to be worked out. For $A = \mathbb{1}$ one recovers the Cardy case discussed in this paper.

4. It would be interesting to understand if the map $K_0(\mathcal{C}_F) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H})$ from the Grothendieck ring to defect operators is injective. The map $K_0(\mathcal{C}) \to \mathrm{End}_{\mathbb{C}}(\mathscr{H})$ taking the class $[R]$ of a representation of the rational vertex operator algebra $\mathfrak{V}$ to the topological defect $D[R]$ is known to be injective, and in fact a corresponding statement holds for symmetry-preserving topological defects in all rational CFTs with chiral symmetry $\mathfrak{V} \otimes_{\mathbb{C}} \mathfrak{V}$ [FRS08].

5. It would be good to investigate the properties of $\mathcal{C}_F$ in more examples. The evident ones are the Virasoro minimal models, the $SU(2)$-WZW model, the rational free boson, etc. Or, coming from the opposite side, one could use the fact that modular categories with three or less simple objects (and unitary modular categories with four or less simple objects) have been classified [RSW09], and study $\mathcal{C}_F$ for all $\mathcal{C}$ in that list and different choices of F. The proper treatment of supersymmetry in the present formalism also remains to be worked out.

6. One application of the perturbed defect operators is the investigation of boundary flows. As pointed out in Remark 5.2.1 (ii), in the Cardy case the boundary state of a perturbed conformal boundary condition can be written as $D[U_f]|\mathbb{1}\rangle\!\rangle$. However, for other modular invariants this need not be true. But, as in the unperturbed case [SFR06, Sect. 2], the category of perturbed boundary conditions will form a module category over the category of chirally perturbed defect lines. It would be interesting to investigate this situation in cases where the two categories are distinct (as abelian categories).

7. In general an object $U_f \in \mathcal{C}_F$ describes a topological defect perturbed by defect

[2] Briefly, for A, B algebra objects in $\mathcal{C}$, an A-B-bimodule M is a triple $(\dot{M}, \rho^A, \tilde{\rho}^B)$, where $\dot{M} \in \mathcal{C}$, $\rho^A \in \mathrm{Hom}_{\mathcal{C}}(A \otimes \dot{M}, \dot{M})$ and $\tilde{\rho}^B \in \mathrm{Hom}_{\mathcal{C}}(\dot{M} \otimes B, \dot{M})$, such that $(\dot{M}, \rho^A)$ is a left A-module and $(\dot{M}, \tilde{\rho}^B)$ is right B-module, such that the actions ρ^A and $\tilde{\rho}^B$ commute [FRS02-I]. If both algebras are the same, one sometimes uses the abbreviation A-bimodule instead of A-A-bimodule.

changing fields. Placed in front of the conformal boundary labeled by the vacuum representation $\mathbb{1} \in \mathcal{C}$ one obtains the boundary condition U perturbed by boundary changing fields. Such perturbations have been studied for unitary minimal models in [Gr01]. While our method is not directly applicable to unitary minimal models (the multiple integrals diverge in this case as $h_{1,3} \geq \frac{1}{2}$), one could still study it if the functional relations predict a similar flow pattern for the non-unitary models.

8. The relation to finite-dimensional representations of quantum affine algebras should be worked out beyond the remarks in App. A.

9. Baxter's Q-operator is a crucial tool in the solution of integrable lattice models. Such Q-operators have been obtained in chiral conformal field theory [FeS95, BLZ97, BLZ99], and in lattice models via the representation theory of quantum affine algebras [KNS94, RW, Ko03]. Recently they have also been studied in certain (discretised) non-rational conformal and massive field theories [BT09]. It would be good to translate these constructions and obtain Q-operators also in the present language.

Appendix A

Relation to Evaluation Representations of $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$

In this appendix we collect some preliminary remarks on the relation of a category of the form $\mathcal{C}_F$ and evaluation representations of the quantum affine algebra $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$. We follow the conventions of [CP91]. Let $q \in \mathbb{C}^\times$ be not a root of unity. The quantum group $U_q(\mathfrak{sl}(2))$ is generated by elements $e^\pm$, $K^{\pm 1}$ with relations

$$KK^{-1} = K^{-1}K = 1 \quad , \quad Ke^\pm K^{-1} = q^{\pm 2}e^\pm \quad , \quad [e^+, e^-] = \frac{K - K^{-1}}{q - q^{-1}} \ . \qquad \text{(A.0.1)}$$

The quantum group $U_q\left(\widehat{\mathfrak{sl}}(2)\right)$ is generated by elements $e_i^\pm$, $K_i^{\pm 1}$, $i = 0, 1$, with relations

$$K_iK_i^{-1} = K_i^{-1}K_i = 1 \quad , \quad K_ie_i^\pm K_i^{-1} = q^{\pm 2}e_i^\pm \quad , \quad [e_i^+, e_i^-] = \frac{K_i - K_i^{-1}}{q - q^{-1}} \ , \qquad \text{(A.0.2)}$$

as well as, for $i \neq j$,

$$\begin{gathered}
[K_0, K_1] = 0 \quad , \quad [e_0^\pm, e_1^\mp] = 0 \quad , \quad K_ie_j^\pm K_i^{-1} = q^{\mp 2}e_j^\pm \\
(e_i^\pm)^3 e_j^\pm - e_j^\pm (e_i^\pm)^3 = \frac{q^3 - q^{-3}}{q - q^{-1}}\Big((e_i^\pm)^2 e_j^\pm e_i^\pm - e_i^\pm e_j^\pm (e_i^\pm)^2\Big) \ .
\end{gathered} \qquad \text{(A.0.3)}$$

Let us abbreviate $U_q \equiv U_q(sl(2))$ and $\hat{U}_q \equiv U_q\left(\widehat{\mathfrak{sl}}(2)\right)$. There are infinitely many ways in which U_q is a subalgebra of $\hat{U}_q$. We will make use of the injective algebra homomorphism $\iota_1 \colon U_q \hookrightarrow \hat{U}_q$ given by (this is the case $i = 0$ in [CP91, Sect. 2.4])

$$\iota_1(K^{\pm 1}) = K_1^{\pm 1} \quad , \quad \iota_1(e^{\pm}) = e_1^{\pm} \ . \tag{A.0.4}$$

This turns $\hat{U}_q$ into an infinite-dimensional representation of U_q. Let $\mathcal{C}$ be the category of (not necessarily finite-dimensional) representations of U_q. The coproduct of U_q gives rise to a tensor product on $\mathcal{C}$ and the R-matrix of U_q to a braiding.

For each $a \in \mathbb{C}^{\times}$, there is a surjective algebra homomorphism $\mathrm{ev}_a \colon \hat{U}_q \to U_q$, described in [CP91, Sect. 4]. It has the property that $\mathrm{ev}_a \circ \iota_1 = \mathrm{id}_{U_q}$. An *evaluation representation* of $\hat{U}_q$ is a pull-back of a representation V of U_q via ev_a for some $a \in \mathbb{C}^{\times}$. We denote this representation of $\hat{U}_q$ by $V(a)$. Let $\mathcal{D}$ be the category of (not-necessarily finite-dimensional) evaluation representations of $\hat{U}_q$.

Theorem A.0.1. *$\mathcal{D}$ is a full subcategory of $\mathcal{C}_{\hat{U}_q}$.*

Proof. Define a map G from $\mathcal{D}$ to $\mathcal{C}_{\hat{U}_q}$ on objects by $G(V(a)) = (V, \mathrm{ev}_a \otimes_{U_q} \mathrm{id}_V)$, where we identified $U_q \otimes_{U_q} V \equiv V$. We will show that $f \colon V(a) \to W(b)$ is a morphism in $\mathcal{D}$ iff f is a morphism $G(V(a)) \to G(W(b))$ in $\mathcal{C}_{\hat{U}_q}$. Indeed, the condition for f to be an intertwiner $f \colon V(a) \to W(b)$ is that for all $u \in \hat{U}_q$ and $v \in V$ we have

$$\mathrm{ev}_b(u).f(v) = f(\mathrm{ev}_a(u).v) \ , \tag{A.0.5}$$

and the condition for f to be a morphism $(V, \mathrm{ev}_a \otimes_{U_q} \mathrm{id}_V) \to (W, \mathrm{ev}_b \otimes_{U_q} \mathrm{id}_W)$ is

$$(\mathrm{ev}_b \otimes_{U_q} \mathrm{id}_W) \circ (\mathrm{id}_{\hat{U}_q} \otimes_{U_q} f) = f \circ (\mathrm{ev}_a \otimes_{U_q} \mathrm{id}_V) \ . \tag{A.0.6}$$

If we evaluate this equality on $u \otimes_{U_q} v$ for $u \in \hat{U}_q$, $v \in V$, we obtain exactly (A.0.5). Thus we can define G on morphisms as $G(f) = f$. It is clear that G is compatible with composition, and that it is full. $\square$

Since $\mathcal{C}$ is abelian braided monoidal with exact tensor product, $\mathcal{C}_{\hat{U}_q}$ is abelian and monoidal by Theorem 4.2.3. Let $\left(\mathcal{C}_{\hat{U}_q}\right)_f$ be the full subcategory of $\mathcal{C}_{\hat{U}_q}$ formed by all (V, g) where V is a finite-dimensional representation of U_q. Note that $\left(\mathcal{C}_{\hat{U}_q}\right)_f$ is again an abelian monoidal category. Let $\mathbf{Rep}_f\left(\hat{U}_q\right)$ be the abelian monoidal category of all finite-dimensional representations of $\hat{U}_q$ of type (1,1) (as defined in [CP91, Sect. 3.2]). It would be interesting to understand the precise relation between $\left(\mathcal{C}_{\hat{U}_q}\right)_f$ and $\mathbf{Rep}_f\left(\hat{U}_q\right)$. For example, one might expect that $\mathbf{Rep}_f\left(\hat{U}_q\right)$ is a full subcategory of $\left(\mathcal{C}_{\hat{U}_q}\right)_f$.

As a first step towards this goal, one could use that all finite-dimensional irreducible representations of $\hat{U}_q$ of type (1,1) are isomorphic to tensor products of evaluation representations [CP91, Sect. 4.11]. However, to make use of this property one first has to establish that the tensor product of $\hat{U}_q$-representations is compatible with $\hat{\otimes}$ defined on $\left(\mathcal{C}_{\hat{U}_q}\right)_f$ via the tensor product and braiding on $\mathcal{C}$. We do not attempt this in the present paper but hope to return to this point in future work.

Appendix B

Proof of Theorem 4.1.1 and Lemma 4.1.2

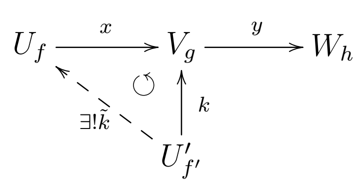
In this appendix, $\mathcal{C}$ satisfies the assumptions of Theorem 4.1.1. Namely, $\mathcal{C}$ is an abelian monoidal category with right-exact tensor product.

Lemma B.0.2. *Let $x\colon U_f \to V_g$ and $y\colon V_g \to W_h$ be morphisms in $\mathcal{C}_F$.*

(i) If $x\colon U \to V$ is a kernel of y in $\mathcal{C}$, then $x\colon U_f \to V_g$ is a kernel of y in $\mathcal{C}_F$.

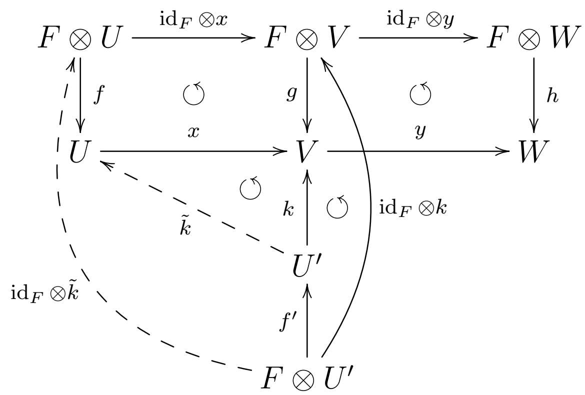
(ii) If $y\colon V \to W$ is a cokernel of x in $\mathcal{C}$, then $y\colon V_g \to W_h$ is a cokernel of x in $\mathcal{C}_F$.

Proof. (i) We need to show that x has the universal property of $\ker y$ in $\mathcal{C}_F$, that is, we need to show that there exists a unique $\tilde{k}\colon U'_{f'} \to U_f$, such that the diagram



commutes in $\mathcal{C}_F$. Since $x = \ker y$ in $\mathcal{C}$ we know that there exists a unique $\tilde{k}\colon U' \to U$

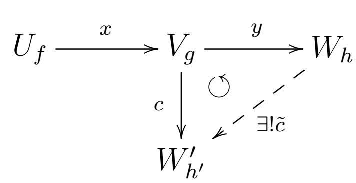
such that $k = x \circ \tilde{k}$. It remains to prove that $\tilde{k}$ is a morphism in $\mathcal{C}_F$, i.e. that $\tilde{k} \circ f' = f \circ (\mathrm{id}_F \otimes \tilde{k})$. To this end consider the following diagram in $\mathcal{C}$:



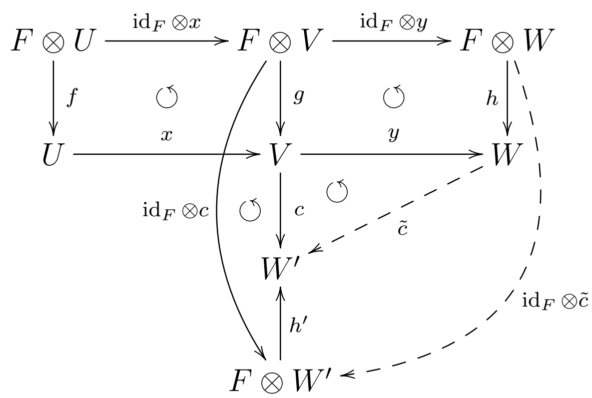
All the diagrams with ↻ commute, but the one with the two dashed arrows. To establish that also the latter commutes, since x is monic it is enough to show that $x \circ \tilde{k} \circ f' = x \circ f \circ (\mathrm{id}_F \otimes \tilde{k})$. Indeed,

$$x \circ f \circ (\mathrm{id}_F \otimes \tilde{k}) = g \circ (\mathrm{id}_F \otimes x) \circ (\mathrm{id}_F \otimes \tilde{k}) = g \circ (\mathrm{id}_F \otimes k) = k \circ f' = x \circ \tilde{c} \circ f' \ .$$

(ii) The proof works along the same lines as that of part (i), but, as opposed to part (i) here we need to use that the tensor product of $\mathcal{C}$ is right-exact. For this reason we spell out the details once more. We need to show that y has the universal property of $\operatorname{cok} x$ in $\mathcal{C}_F$, that is, we need to show that there exists a unique $\tilde{c}\colon W_h \to W'_{h'}$, such that the diagram



commutes in $\mathcal{C}_F$. Since $y = \operatorname{cok} x$ in $\mathcal{C}$ we know there exists a unique morphism $\tilde{c}\colon W \to W'$ in $\mathcal{C}$ such that $c = \tilde{c} \circ y$. It remains to show that $\tilde{c}\colon W_h \to W'_{h'}$ is a morphism in $\mathcal{C}_F$, i.e. that $\tilde{c} \circ h = h' \circ (\mathrm{id}_F \otimes \tilde{c})$. Consider the diagram:



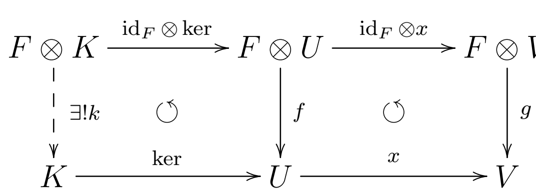
Since y is an epimorphism and the tensor product is right-exact, then $\mathrm{id}_F \otimes y$ is also an epimorphism. It is therefore enough to show that $\tilde{c} \circ h \circ (\mathrm{id}_F \otimes y) = h' \circ (\mathrm{id}_F \otimes \tilde{c}) \circ (\mathrm{id}_F \otimes y)$. Indeed,

$$h' \circ (\mathrm{id}_F \otimes \tilde{c}) \circ (\mathrm{id}_F \otimes y) = h' \circ (\mathrm{id}_F \otimes c) = c \circ g = \tilde{c} \circ y \circ g = \tilde{c} \circ h \circ (\mathrm{id}_F \otimes y) \ .$$

□

Lemma B.0.3. *$\mathcal{C}_F$ has kernels.*

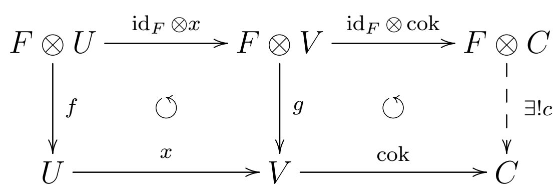
Proof. We are given $U_f, V_g \in \mathcal{C}_F$ and a morphism $x \colon U_f \to V_g$. Since $\mathcal{C}$ has kernels, there exists an object $K \in \mathcal{C}$ and a morphism $\ker \colon K \to U$ such that ker is a kernel of x in $\mathcal{C}$. We now wish to construct a morphism $k \colon F \otimes K \to K$ such that $\ker \colon K_k \to U_f$ is a morphism in $\mathcal{C}_F$. Consider the following diagram:



Note that $x \circ f \circ (\mathrm{id}_F \otimes \ker) = g \circ (\mathrm{id}_F \otimes (x \circ \ker)) = 0$. By the universal property of kernels in $\mathcal{C}$, there exists a unique morphism $k \colon F \otimes K \to K$ which makes the above diagram commute. Thus, ker: $K_k \to U_f$ is a morphism in $\mathcal{C}_F$. Since ker is a kernel of x in $\mathcal{C}$, by Lemma B.0.2 (i) ker is also a kernel of x in $\mathcal{C}_F$. □

Lemma B.0.4. *$\mathcal{C}_F$ has cokernels.*

Proof. The proof is similar to that for the existence of kernels, with the difference that for the existence of cokernels we need the tensor product of $\mathcal{C}$ to be right-exact. We are given a morphism $x\colon U_f \to V_g$. The morphism x has a cokernel $\mathrm{cok}\colon V \to C$ in $\mathcal{C}$. Consider the following diagram:



Since $\otimes$ is right-exact, $\mathrm{id}_F \otimes \mathrm{cok}$ is a cokernel of $\mathrm{id}_F \otimes x$. Note that $\mathrm{cok} \circ g \circ (\mathrm{id}_F \otimes x) = \mathrm{cok} \circ x \circ f = 0$. By the universal property of cokernels in $\mathcal{C}$, there exists a unique morphism $c\colon F \otimes C \to C$ which makes the above diagram commute. Thus, $\mathrm{cok}\colon V_g \to C_c$ is a morphism in $\mathcal{C}_F$. Since cok is a cokernel of x in $\mathcal{C}$, by Lemma B.0.2 (ii) it is also a cokernel of x in $\mathcal{C}_F$. □

The proof of Lemma B.0.3 shows that there exists a kernel for $x\colon U_f \to V_g$ of the form $\ker\colon K_h \to U_f$, with ker a kernel of x in $\mathcal{C}$. The proof of Lemma B.0.4 implies a similar statement for cokernels. Since kernels and cokernels are unique up to unique isomorphism, we get as a corollary the converse statement to Lemma B.0.2.

Corollary B.0.5. *Let $x\colon U_f \to V_g$ and $y\colon V_g \to W_h$ be morphisms in $\mathcal{C}_F$.*

(i) If $x\colon U_f \to V_g$ is a kernel of y in $\mathcal{C}_F$, then $x\colon U \to V$ is a kernel of y in $\mathcal{C}$.

(ii) If $y\colon V_g \to W_h$ is a cokernel of x in $\mathcal{C}_F$, then $y\colon V \to W$ is a cokernel of x in $\mathcal{C}$.

We have now gathered all the ingredients to prove Lemma 4.1.2.

Proof of Lemma 4.1.2. By Lemmas B.0.3 and B.0.4, $\mathcal{C}_F$ has kernels and cokernels. Let $\chi\colon K_k \to V_g$ be a kernel of $b\colon V_g \to W_h$ and let $\gamma\colon V_g \to C_c$ be a cokernel of $a\colon U_f \to V_g$. By Corollary B.0.5, also in $\mathcal{C}$ we have that χ is a kernel of $b\colon V \to W$ and γ is a cokernel of $a\colon U \to V$.

Suppose $U_f \xrightarrow{a} V_g \xrightarrow{b} W_h$ is exact at V_g in $\mathcal{C}_F$, i.e. χ is also a kernel for γ in $\mathcal{C}_F$. By Corollary B.0.5, χ is a kernel for γ in $\mathcal{C}$ and so $U \xrightarrow{a} V \xrightarrow{b} W$ is exact at V in $\mathcal{C}$. Conversely, if χ is a kernel for γ in $\mathcal{C}$, then by Lemma B.0.2 χ is also a kernel for γ in $\mathcal{C}_F$. Thus $U_f \xrightarrow{a} V_g \xrightarrow{b} W_h$ is exact at V_g in $\mathcal{C}_F$. □

Corollary B.0.6. *(to Lemma 4.1.2) Let $x\colon U_f \to V_g$ be a morphism in $\mathcal{C}_F$. Then x is monic in $\mathcal{C}_F$ iff it is monic in $\mathcal{C}$, and x is epi in $\mathcal{C}_F$ iff it is epi in $\mathcal{C}$.*

Lemma B.0.7. *$\mathcal{C}_F$ has binary biproducts.*

Proof. Let $U_f, V_g \in \mathcal{C}_F$ be given. Since $\mathcal{C}$ has binary biproducts, for $U, V \in \mathcal{C}$, there exists a $W \in \mathcal{C}$ and morphisms

$$U \underset{r_U}{\overset{e_U}{\rightleftarrows}} W \underset{r_V}{\overset{e_V}{\leftrightarrows}} V \tag{B.0.1}$$

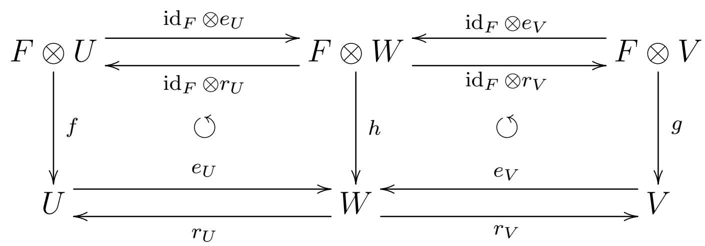
where e_A is the embedding map and r_A is the restriction map, such that

$$r_U \circ e_U = \mathrm{id}_U \ , \quad r_V \circ e_V = \mathrm{id}_V \ , \quad e_U \circ r_U + e_V \circ r_V = \mathrm{id}_W \ .$$

This implies $r_U \circ e_V = 0$ and $r_V \circ e_U = 0$. Define a morphism $h\colon F \otimes W \to W$ as

$$h = e_U \circ f \circ (\mathrm{id}_F \otimes r_U) + e_V \circ g \circ (\mathrm{id}_F \otimes r_V) \ .$$

We claim that (B.0.1) with U, W and V replaced by U_f, W_h and V_g, respectively, defines a binary biproduct in $\mathcal{C}_F$. To show these we need to check that the relevant four squares in



commute. For the first square one has

$$h \circ (\mathrm{id}_F \otimes e_U) = e_U \circ f \circ (\mathrm{id}_F \otimes (\underbrace{r_U \circ e_U}_{=\mathrm{id}_U})) + e_V \circ g \circ (\mathrm{id}_F \otimes (\underbrace{r_V \circ e_U}_{=0})) = e_U \circ f \ ,$$

and for the second one

$$r_U \circ h = r_U \circ e_U \circ f \circ (\mathrm{id}_F \otimes r_U) + r_U \circ e_V \circ g \circ (\mathrm{id}_F \otimes r_V) = f \circ (\mathrm{id}_F \otimes r_U) \ .$$

In a similar fashion one checks that also $h \circ (\mathrm{id}_F \otimes e_V) = e_V \circ g$ and $r_V \circ h = g \circ (\mathrm{id}_F \otimes r_V)$. □

Lemma B.0.8. *In $\mathcal{C}_F$ every monomorphism is a kernel and every epimorphism is a cokernel.*

Proof. First we show that every monomorphism is a kernel. We need to show that if $x\colon U_f \to V_g$ is mono in $\mathcal{C}_F$, there exists a W_h and $y\colon V_g \to W_h$ such that $x = \ker y$. Since $\mathcal{C}_F$ has cokernels we can choose $W_h = C_c$ and $y = \operatorname{cok} x$. Since by Corollary B.0.6 x is monic also in $\mathcal{C}$, we have $x = \ker(\operatorname{cok} x)$ in $\mathcal{C}$. Finally, by Lemma B.0.2 we get that $x = \ker(\operatorname{cok} x)$ also in $\mathcal{C}_F$. The proof that every epimorphism is a cokernel goes along the same lines. □

Proof of Theorem 4.1.1. Since $\mathcal{C}$ is an Ab-category, so is $\mathcal{C}_F$. As zero object in $\mathcal{C}_F$ we take $(\mathbf{0}, 0)$, where $\mathbf{0}$ is the zero object of $\mathcal{C}$ and $0 : F \otimes \mathbf{0} \to \mathbf{0}$ is the zero morphism. Furthermore, $\mathcal{C}_F$ has binary biproducts (Lemma B.0.7), has kernels and cokernels (Lemmas B.0.3 and B.0.4) and in $\mathcal{C}_F$ every monomorphism is a kernel and every epimorphism is a cokernel (Lemma B.0.8). Thus $\mathcal{C}_F$ is abelian. □

Appendix C

Finite Semisimple Monoidal Categories

Let $\Bbbk$ be a field. In this section we take $\mathcal{C}$ to be a $\Bbbk$-linear abelian semi-simple finite braided monoidal category, such that $\mathbb{1}$ is simple, and $\mathrm{End}_{\mathcal{C}}(U) = \Bbbk\, \mathrm{id}_U$ for all simple objects U. We also assume that $\mathcal{C}$ has right duals and that

$$\mathcal{C} \text{ is strict.}$$

Note that if we would add to this the data/conditions that $\mathcal{C}$ has compatible left-duals and a twist (so that $\mathcal{C}$ is ribbon), we would arrive at the definition of a premodular category [Br00]. Here we will content ourselves with right duals alone.

For explicit calculations in $\mathcal{C}_F$ it is useful to have a realisation of $\mathcal{C}$ in terms of vector spaces. One way to obtain such a realisation is as follows. Pick a set of representatives $\{U_i | i \in \mathcal{I}\}$ of the isomorphism classes of simple objects in $\mathcal{C}$ such that $U_0 = \mathbb{1}$. For each label $a \in \mathcal{I}$ define a label $\bar{a}$ via $U_{\bar{a}} \cong U_a^{\vee}$. Define the fusion rule coefficients ${\mathcal{N}_{ij}}^{k}$ as

$$ {\mathcal{N}_{ij}}^{k} = \dim_{\Bbbk} \left(\mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_j, U_k) \right) \ . \tag{C.0.1}$$

We restrict ourselves to the situation that

$$\mathcal{N}_{ij}{}^{k} \in \{0,1\} \ . \tag{C.0.2}$$

This is satisfied in the Lee-Yang model studied below, but also for other models such as the rational free boson or the $\widehat{\mathfrak{su}}(2)_k$-WZW model. Whenever $\mathcal{N}_{ij}{}^{k} = 1$ we pick basis vectors

$$\lambda_{(ij)k} \in \mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_j, U_k) \quad \text{such that} \quad \lambda_{(0i)i} = \lambda_{(i0)i} = \mathrm{id}_{U_i} \ . \tag{C.0.3}$$

The fusing matrices $\mathsf{F}^{(ijk)l}_{pq} \in \Bbbk$ are defined to implement the change of basis between two bases of $\mathrm{Hom}_{\mathcal{C}}(U_i \otimes U_j \otimes U_k, U_l)$ as in (3.3.22), and they obey the pentagon relation. See e.g. [FRS02-I, Sect. 2.2] for more details. The inverse matrices are denoted by $\mathsf{G}^{(ijk)l}_{pq}$, see (3.3.23),

$$\sum_{r \in \mathcal{I}} \mathsf{F}^{(ijk)l}_{pr} \mathsf{G}^{(ijk)l}_{rq} = \delta_{p,q} \ . \tag{C.0.4}$$

The braiding $c_{U,V}$ gives rise to the braid matrices $\mathsf{R}^{(ij)k} \in \Bbbk$, see (3.3.24)

With these ingredients, we define a $\Bbbk$-linear braided monoidal category $\mathcal{V} \equiv \mathcal{V}[\Bbbk, \mathcal{I}, 0 \in \mathcal{I}, \mathcal{N}, \mathsf{F}, \mathsf{R}]$. This definition will occupy the rest of this section. The objects of $\mathcal{V}$ are lists of finite-dimensional $\Bbbk$-vector spaces indexed by $\mathcal{I}$, $A = (A_i, i \in \mathcal{I})$, and the morphisms $f \colon A \to B$ are lists of linear maps $f = (f_i, i \in \mathcal{I})$ with $f_i \colon A_i \to B_i$.

There is an obvious functor $H \colon \mathcal{C} \to \mathcal{V}$ which acts on objects as $H(V) = (\mathrm{Hom}_{\mathcal{V}}(U_i, V), i \in \mathcal{I})$. For a morphism $f \colon V \to W$ we set $H(f) = (H(f)_i, i \in \mathcal{I})$, where $H(f)_i \colon \mathrm{Hom}_{\mathcal{V}}(U_i, V) \to \mathrm{Hom}_{\mathcal{V}}(U_i, W)$ is given by $\alpha \mapsto f \circ \alpha$. Since H is fully faithful and surjective we have:

Lemma C.0.9. *The functor $H \colon \mathcal{C} \to \mathcal{V}$ is an equivalence of $\Bbbk$-linear categories.*

We can now use H to transport the tensor product, braiding and duality from $\mathcal{C}$ to $\mathcal{V}$. Let us start with the tensor product in $\mathcal{V}$, which we denote by $\circledast$. For an object $A \in \mathcal{V}$ we denote by $(A)_i$ (or just A_i) the i^{th} component of the list A. We set

$$(A \circledast B)_i = \bigoplus_{j \in \mathcal{I}} \bigoplus_{\substack{k \in \mathcal{I} \\ \mathcal{N}_{jk}{}^{i} = 1}} A_j \otimes_{\Bbbk} B_k \ . \tag{C.0.5}$$

The direct summand $A_j \otimes_{\Bbbk} B_k$ can appear in several components $(A \circledast B)_i$. To index one specific direct summand, we introduce the notation $(A \circledast B)_{i(jk)}$ to mean

$$(A \circledast B)_{i(jk)} = A_j \otimes_{\Bbbk} B_k \subset (A \circledast B)_i \ . \tag{C.0.6}$$

This notation can be iterated. For example $(A \circledast (B \circledast C))_{i(jk(lm))}$ stands for the direct summand (we do not write out the associator and unit isomorphisms in the category of $\Bbbk$-vector spaces)

$$A_j \otimes_{\Bbbk} B_l \otimes_{\Bbbk} C_m \subset A_j \otimes_{\Bbbk} (B \circledast C)_k \subset (A \circledast (B \circledast C))_i \ . \tag{C.0.7}$$

while $((A \circledast B) \circledast C)_{i(j(kl)m)}$ stands for the direct summand

$$A_k \otimes_{\Bbbk} B_l \otimes_{\Bbbk} C_m \subset (A \circledast B)_j \otimes_{\Bbbk} C_m \subset ((A \circledast B) \circledast C)_i \ . \tag{C.0.8}$$

If $v \in A_j \otimes_{\Bbbk} B_k$, we denote by $(v)_{i(jk)}$ the element v in the direct summand $(A \circledast B)_{i(jk)} \subset (A \circledast B)_i$, etc.

On morphisms $f \colon A \to X$ and $g \colon B \to Y$ the tensor product is defined to have components $(f \circledast g)_i \colon (A \circledast B)_i \to (X \circledast Y)_i$, where, for $a \in A_j$ and $b \in B_k$,

$$\big(f \circledast g\big)_i\big((a \otimes_{\Bbbk} b)_{i(jk)}\big) = \big(\, f_j(a) \otimes_{\Bbbk} g_k(b)\,\big)_{i(jk)} \ \in X_j \otimes_{\Bbbk} Y_k \subset (X \circledast Y)_i \ . \tag{C.0.9}$$

The tensor unit $\mathbb{1} \in \mathcal{V}$ has components $\mathbb{1}_0 = \Bbbk$ and $\mathbb{1}_i = 0$ for $i \neq 0$. The unit isomorphisms of $\mathcal{V}$ are identities, but we find it useful to write them out to keep track of the indices of the direct summands,

$$\begin{array}{rcl} (\lambda_A)_i \colon (\mathbb{1} \circledast A)_i & \longrightarrow & A_i \\ (1 \otimes_{\Bbbk} a)_{i(0i)} & \longmapsto & (a)_i \end{array} \quad \text{and} \quad \begin{array}{rcl} (\rho_A)_i \colon (A \circledast \mathbb{1})_i & \longrightarrow & A_i \\ (a \otimes_{\Bbbk} 1)_{i(i0)} & \longmapsto & (a)_i \end{array} \ . \tag{C.0.10}$$

Finally, the associator has components $(\alpha_{A,B,C})_i \colon (A \circledast (B \circledast C))_i \to ((A \circledast B) \circledast C)_i$, where, for $v \in A_j \otimes_{\Bbbk} B_k \otimes_{\Bbbk} C_l$,

$$(\alpha_{A,B,C})_i\big(\, (v)_{i(jq(kl))}\,\big) = \sum_{p \in \mathcal{I}} \big(\, \mathsf{G}^{(jkl)i}_{pq}\, v\,\big)_{i(p(jk)l)} \ . \tag{C.0.11}$$

Its inverse is $(\alpha_{A,B,C})_i^{-1} \colon ((A \circledast B) \circledast C)_i \to (A \circledast (B \circledast C))_i$,

$$(\alpha^{-1}_{A,B,C})_i\big(\, (v)_{i(q(jk)l)}\,\big) = \sum_{p \in \mathcal{I}} \big(\, \mathsf{F}^{(jkl)i}_{pq}\, v\,\big)_{i(jp(kl))} \ . \tag{C.0.12}$$

We can now turn H into a monoidal functor (see Def. 3.3.7 excluding the last commuting diagram). To this end we need to specify natural transformations $H^2_{U,V}\colon H(U) \circledast H(V) \xrightarrow{\sim} H(U\otimes V)$ and an isomorphism $H^0\colon \mathbb{1}_{\mathcal{V}} \xrightarrow{\sim} H(\mathbb{1}_{\mathcal{C}})$. To describe $H^2_{U,V}$ we need the basis dual to $\lambda_{(ij)k}$, that is, elements $\bar{\lambda}_{(ij)k} \in \mathrm{Hom}_{\mathcal{V}}(U_k, U_i \otimes U_j)$ (see (3.3.21)) such that $\lambda_{(ij)k} \circ \bar{\lambda}_{(ij)k} = \mathrm{id}_{U_k}$. Note that $(H(U) \circledast H(V))_{i(jk)} = \mathrm{Hom}_{\mathcal{V}}(U_j, U) \otimes_{\mathbb{k}} \mathrm{Hom}_{\mathcal{V}}(U_k, V)$ and $H(U \otimes V)_i = \mathrm{Hom}_{\mathcal{V}}(U_i, U \otimes V)$. We set, for $u \in \mathrm{Hom}_{\mathcal{V}}(U_j, U)$ and $v \in \mathrm{Hom}_{\mathcal{V}}(U_k, V)$,

$$(H^2_{U,V})_i((u \otimes_{\mathbb{k}} v)_{i(jk)}) = ((u \otimes v) \circ y_{(jk)i})_i \ . \tag{C.0.13}$$

Finally, $(H^0)_i = 0$ for $i \neq 0$ and $(H^0)_0(1) = \mathrm{id}_{U_0} \in \mathrm{Hom}_{\mathcal{V}}(U_0, U_0)$.

Theorem C.0.10. *$(H, H^2, H^0)\colon \mathcal{C} \to \mathcal{V}$ is a monoidal functor.*

Proof. From Def. 3.3.7 we have to check that all (apart from the last) diagrams commute, for all $U, V, W \in \mathcal{C}$. More precisely, the following equalities of morphisms $H(U) \circledast (H(V) \circledast H(W)) \to H(U \otimes V \otimes W)$, $\mathbb{1}_{\mathcal{V}} \circledast H(U) \to H(U)$ and $H(U) \circledast \mathbb{1}_{\mathcal{V}} \to H(U)$, respectively, hold,

$$H^2_{U\otimes V,W} \circ (H^2_{U,V} \circledast \mathrm{id}_{H(W)}) \circ \alpha_{H(U),H(V),H(W)} = H^2_{U,V\otimes W} \circ (\mathrm{id}_{H(U)} \circledast H^2_{V,W}) \ ,$$

$$\lambda_{H(U)} = H^2_{H(\mathbb{1}),H(U)} \circ (H^0 \circledast \mathrm{id}_{H(U)}) \quad , \quad \rho_{H(U)} = H^2_{H(U),H(\mathbb{1})} \circ (\mathrm{id}_{H(U)} \circledast H^0) \ . \tag{C.0.14}$$

(Recall that $\mathcal{C}$ is strict.) The identities involving λ and ρ are most easy to check. For example, the ith component of two sides of the identity for λ are, for $u \in \mathrm{Hom}_{\mathcal{V}}(U_i, U)$,

$$(\lambda_{H(U)})_i((1 \otimes_{\mathbb{k}} u)_{i(0i)}) = (u)_i \qquad \text{and}$$

$$\begin{aligned}(H^2_{H(\mathbb{1}),H(U)})_i \circ (H^0 \circledast \mathrm{id}_{H(U)})_i((1 \otimes_{\mathbb{k}} u)_{i(0i)}) &= (H^2_{H(\mathbb{1}),H(U)})_i((\mathrm{id}_{U_0} \otimes_{\mathbb{k}} u)_{i(0i)}) \\ &= ((\mathrm{id}_{U_0} \otimes u) \circ \bar{\lambda}_{(0i)i})_i = (u)_i \ .\end{aligned} \tag{C.0.15}$$

To check the first condition in (C.0.14) we pick elements $u \in \mathrm{Hom}_{\mathcal{V}}(U_j, U)$, $v \in \mathrm{Hom}_{\mathcal{V}}(U_k, V)$, $w \in \mathrm{Hom}_{\mathcal{C}}(U_l, W)$ and evaluate both sides on the element $(u \otimes_{\Bbbk} v \otimes_{\Bbbk} w)_{i(jq(kl))}$. For the left hand side this gives

$$\begin{aligned}
&\big(H^2_{U\otimes V,W} \circ (H^2_{U,V} \circledast \mathrm{id}_{H(W)}) \circ \alpha_{H(U),H(V),H(W)}\big)_i \big((u \otimes_{\Bbbk} v \otimes_{\Bbbk} w)_{i(jq(kl))}\big) \\
&= \sum_{p\in\mathcal{I}} \big(H^2_{U\otimes V,W} \circ (H^2_{U,V} \circledast \mathrm{id}_{H(W)})\big)_i \big((\mathsf{G}^{(jkl)i}_{pq} \cdot u \otimes_{\Bbbk} v \otimes_{\Bbbk} w)_{i(p(jk)l)}\big) \\
&= \sum_{p\in\mathcal{I}} (H^2_{U\otimes V,W})_i \big((\mathsf{G}^{(jkl)i}_{pq} \cdot ((u \otimes v) \circ \bar{\lambda}_{(jk)p}) \otimes_{\Bbbk} w)_{i(pl)}\big) \\
&= \big(\sum_{p\in\mathcal{I}} \mathsf{G}^{(jkl)i}_{pq} \cdot (((u \otimes v) \circ \bar{\lambda}_{(jk)p}) \otimes w) \circ y_{(pl)i}\big)_i = \big((u \otimes v \otimes w) \circ (\mathrm{id}_{U_j} \otimes \bar{\lambda}_{(kl)q}) \circ \bar{\lambda}_{(jq)i}\big)_i \ .
\end{aligned} \tag{C.0.16}$$

For the right hand side we find

$$\begin{aligned}
&\big(H^2_{U,V\otimes W} \circ (\mathrm{id}_{H(U)} \circledast H^2_{V,W})\big)_i \big((u \otimes_{\Bbbk} v \otimes_{\Bbbk} w)_{i(jq(kl))}\big) \\
&= \big(H^2_{U,V\otimes W}\big)_i \big((u \otimes_{\Bbbk} [(v \otimes w) \circ \bar{\lambda}_{(kl)q}])_{i(jq)}\big) \\
&= \big((u \otimes [(v \otimes w) \circ \bar{\lambda}_{(kl)q}]) \circ \bar{\lambda}_{(jq)i}\big)_i = \big((u \otimes v \otimes w) \circ (\mathrm{id}_{U_j} \otimes \bar{\lambda}_{(kl)q}) \circ \bar{\lambda}_{(jq)i}\big)_i \ .
\end{aligned} \tag{C.0.17}$$

Thus H is indeed a monoidal functor. □

We define a braiding $c_{A,B} \colon A \circledast B \to B \circledast A$ on $\mathcal{V}$ by setting, for $a \in A_j$ and $b \in B_k$,

$$(c_{A,B})_i((a \otimes b)_{i(jk)}) = (\mathsf{R}^{(jk)i} b \otimes a)_{i(kj)} \ . \tag{C.0.18}$$

One verifies that $H(c_{U,V}) \circ H^2_{U,V} = H^2_{V,U} \circ c_{H(U),H(V)}$ so that H is a *braided monoidal* functor between $\mathcal{C}$ and $\mathcal{V}$.

It remains to define the right duality on $\mathcal{V}$. The components of the dual of an object are given by dual vector spaces, $(A^\vee)_k = A^*_{\bar{k}}$. We identify $\Bbbk^* = \Bbbk$ so that $\mathbb{1}^\vee = \mathbb{1}$. The duality morphisms $b_A \colon \mathbb{1} \to A \circledast A^\vee$ and $d_A \colon A^\vee \circledast A \to \mathbb{1}$ have components $(b_A)_i = 0 = (d_A)_i$ for $i \neq 0$. To describe the 0-component, we fix a basis $\{a_{i,\alpha}\}$ of each A_i, and denote by $\{a^*_{i,\alpha}\}$ the dual basis of A^*_i. Then

$$\begin{aligned} (b_A)_0 \colon (\mathbb{1})_0 &\longrightarrow (A \circledast A^\vee)_0 \\ (1)_0 &\longmapsto \sum_{k\in\mathcal{I}} \Big(\sum_\alpha a_{k,\alpha} \otimes_{\Bbbk} a^*_{k,\alpha} \Big)_{0(k\bar{k})} \end{aligned} \quad , \quad \begin{aligned} (d_A)_0 \colon (A^\vee \circledast A)_0 &\longrightarrow (\mathbb{1})_0 \\ (\varphi \otimes_{\Bbbk} a)_{0(\bar{k}k)} &\longmapsto \frac{\varphi(a)}{\mathsf{F}^{(k\bar{k}k)k}_{00}} \end{aligned} \quad . \tag{C.0.19}$$

As an exercise in the use of the nested index notation we demonstrate the first identity in (3.3.10) (or equivalently see the first commutative diagram above that). Let $a_{i,\alpha}$, $a^*_{i,\alpha}$ be as above. Then, for $\varphi \in A^*_{\bar{k}}$,

$$\begin{aligned} \big(\rho^{-1}_{A^\vee}\big)_k\big((\varphi)_k\big) &= (\varphi \otimes_{\Bbbk} 1)_{k(k0)} = \star_1 \\ \big(\,\mathrm{id}_{A^\vee} \circledast b_A\big)_k(\star_1) &= \sum_{l\in\mathcal{I}} \sum_\alpha \big((\varphi)_k \otimes_{\Bbbk} (a_{k,\alpha} \otimes_{\Bbbk} a^*_{k,\alpha})_{0(l\bar{l})} \big)_{k(k0)} \\ &= \sum_{l,\alpha} \big(\varphi \otimes_{\Bbbk} a_{k,\alpha} \otimes_{\Bbbk} a^*_{k,\alpha} \big)_{k(k0(l\bar{l}))} = \star_2 \\ \big(\alpha_{A^\vee,A,A^\vee}\big)_k(\star_2) &= \sum_{p\in\mathcal{I}} \sum_{l,\alpha} \big(\mathsf{G}^{(kl\bar{l})k}_{p0} \cdot \varphi \otimes_{\Bbbk} a_{k,\alpha} \otimes_{\Bbbk} a^*_{k,\alpha} \big)_{k(p(kl)\bar{l})} = \star_3 \\ \big(d_A \circledast id_{A^\vee}\big)_k(\star_3) &= \sum_{p,l,\alpha} \big(\mathsf{G}^{(kl\bar{l})k}_{p0} \cdot (d_A)_p((\varphi \otimes_{\Bbbk} a_{k,\alpha})_{p(kl)}) \otimes_{\Bbbk} (a^*_{k,\alpha})_{\bar{l}} \big)_{k(p\bar{l})} \\ &\overset{(a)}{=} \sum_\alpha \big(\mathsf{G}^{(k\bar{k}k)k}_{00} (\mathsf{F}^{(\bar{k}k\bar{k})\bar{k}}_{00})^{-1} \cdot \varphi(a_{k,\alpha}) \otimes_{\Bbbk} a^*_{k,\alpha} \big)_{k(0k)} \\ &\overset{(b)}{=} \big(1 \otimes_{\Bbbk} \varphi \big)_{k(0k)} = \star_4 \\ \big(\lambda_{A^\vee}\big)_k(\star_4) &= (\varphi)_k \ . \end{aligned} \tag{C.0.20}$$

In step (a) we used that $(d_A)_p$ is non-zero only for $p = 0$, and that in this case we are also forced to choose $l = \bar{k}$ (otherwise the direct summand $(\cdots)_{0(kl)}$ is empty). In step (b) the equality

$$\mathsf{F}^{(\bar{k}k\bar{k})\bar{k}}_{00} = \mathsf{G}^{(k\bar{k}k)k}_{00} \tag{C.0.21}$$

is used. This equality can be derived by using either F or G to simplify $(\lambda_{(\bar{k}k)0} \otimes \lambda_{(\bar{k}k)0}) \circ (\mathrm{id}_{U_{\bar{k}}} \otimes \bar{\lambda}_{(k\bar{k})0} \otimes \mathrm{id}_{U_k})$ to $\lambda_{(\bar{k}k)0}$ (which also shows that both are non-zero).

Remark C.0.1. (i) The above construction is a straightforward generalisation of the way one defines a (braided) monoidal category starting from a (abelian) group and a (abelian) three-cocycle, see [FRS04-III, Sect. 2] and references therein.

(ii) The construction is different from what one would do in Tannaka-Krein reconstruction for monoidal categories [Ha99]. There one constructs a fibre-functor from $\mathcal{C}$ to a category of R-R-bimodules for a certain ring R (isomorphic to $\Bbbk^{\oplus|\mathcal{I}|}$). However, this fibre-functor is typically neither an equivalence nor full.

Let $f\colon F \circledast A \to A$ and $g\colon F \circledast B \to B$ be morphisms in $\mathcal{V}$. We can now substitute the explicit structure morphisms (C.0.11), (C.0.12), (C.0.18) into the definition of $T(f,g)$ in Section 4.2. After a short calculation one finds, for $u \in F_j$, $a \in A_l$ and $b \in B_m$,

$$\begin{aligned} &T(f,g)_i\big(\,(u \otimes_\Bbbk a \otimes_\Bbbk b)_{i(jk(lm))}\,\big) \\ &\quad= \sum_{x,y\in\mathcal{I}} \Big(\delta_{y,m} \mathsf{G}^{(jlm)i}_{xk} (f)_x\big((u\otimes_\Bbbk a)_{x(jl)}\big) \otimes_\Bbbk (b)_y \\ &\qquad\qquad + \delta_{x,l} \frac{\mathsf{R}^{(jk)i}}{\mathsf{R}^{(jm)y}} \mathsf{F}^{(lmj)i}_{yk} (a)_x \otimes_\Bbbk (g)_y\big((u\otimes_\Bbbk b)_{y(jm)}\big)\Big)_{i(xy)} \;. \end{aligned} \tag{C.0.22}$$

When verifying this one needs to use the following two equivalent expressions for the B-matrix (see e.g. [FRS05-IV, Eqn. (5.46)]), one of which is [FRS05-IV, Eqn. (5.47)] and the other one appears in the calculation of $T(0,g)_i\big(\,(u \otimes_\Bbbk a \otimes_\Bbbk b)_{i(jk(lm))}\,\big)$,

$$\sum_p \mathsf{F}^{(ljm)i}_{yp}\, \mathsf{R}^{(jl)p}\, \mathsf{G}^{(jlm)i}_{pk} \;=\; \mathsf{B}^{(jlm)i}_{yk} \;=\; \frac{\mathsf{R}^{(jk)i}}{\mathsf{R}^{(jm)y}}\, \mathsf{F}^{(lmj)i}_{yk} \;. \tag{C.0.23}$$

For $c(f)$ the calculation is slightly longer, and one finds, for $u \in F_j$ and $\varphi \in A^*_{\bar k}$, and using (C.0.21) at an intermediate step,

$$\begin{aligned} &c(f)_i\big(\,(u \otimes_\Bbbk \varphi)_{i(jk)}\,\big) \\ &\quad= -\frac{\mathsf{F}^{(\bar\imath i\bar\imath)\bar\imath}_{00}}{\mathsf{F}^{(\bar k k\bar k)\bar k}_{00}}\, \mathsf{R}^{(jk)i}\, \mathsf{F}^{(kj\bar\imath)0}_{\bar k i} \sum_\alpha \varphi\big((f)_{\bar k}((u\otimes a_{\bar\imath,\alpha})_{\bar k(j\bar\imath)})\big)\cdot a^*_{\bar\imath,\alpha} \;\;\in\;\; (A^\vee)_i = A^*_{\bar\imath} \;. \end{aligned} \tag{C.0.24}$$

Appendix D

$T(f,g)$ and $c(f)$ for the Lee-Yang Model

The Lee-Yang model is the minimal model $M(2,5)$. The fusing matrices of minimal models are known from [DF85, FGP90]. We use the conventions of [Ru08, App. A.3]. The index set is $\mathcal{I} = \{1,\phi\}$ and the unit element is $1 \in \mathcal{I}$. The non-zero entries in the braiding matrix are, for $x \in \{1,\phi\}$

$$\mathsf{R}^{(\mathbb{1}x)x} = \mathsf{R}^{(x\mathbb{1})x} = 1 \quad , \quad \mathsf{R}^{(\phi\phi)1} = \zeta^2 \quad , \quad \mathsf{R}^{(\phi\phi)\phi} = \zeta \quad , \quad \text{where } \zeta = e^{-\pi i/5} \ . \tag{D.0.1}$$

The nonzero entries in the fusing matrices are, for $x,y,z \in \{1,\phi\}$

$$\mathsf{F}^{(1xy)z}_{zx} = \mathsf{F}^{(x1y)z}_{yx} = \mathsf{F}^{(xy1)z}_{yz} = \mathsf{F}^{(xyz)1}_{xz} = 1 \ ,$$

$$\mathsf{F}^{(\phi\phi\phi)\phi}_{11} = \frac{1}{d} \ , \ \ \mathsf{F}^{(\phi\phi\phi)\phi}_{1\phi} = w \ , \ \ \mathsf{F}^{(\phi\phi\phi)\phi}_{\phi 1} = \frac{1}{wd} \ , \ \ \mathsf{F}^{(\phi\phi\phi)\phi}_{\phi\phi} = \frac{-1}{d} \ \ \text{where } d = \frac{1-\sqrt{5}}{2} \ . \tag{D.0.2}$$

Here d is the quantum dimension of ϕ. The constant $w \in \mathbb{C}^\times$ depends on the choice of normalisation of the basis vectors $\lambda_{(\phi\phi)1}$ and $\lambda_{(\phi\phi)\phi}$. Different choices of w yield equivalent braided monoidal categories. There is a preferred choice related to the normalisation of the vertex operators, for which

$$w = \frac{\Gamma\big(\frac{1}{5}\big)\Gamma\big(\frac{6}{5}\big)}{\Gamma\big(\frac{3}{5}\big)\Gamma\big(\frac{4}{5}\big)} = 2.431\ldots \ , \tag{D.0.3}$$

but one may as well set w to 1. The inverse matrix of F is simply

$$\mathsf{G}^{(ijk)l}_{pq} = \mathsf{F}^{(kji)l}_{pq} \ . \tag{D.0.4}$$

Let us indicate how to obtain the explicit formulas quoted in Section 6.1. First of all, in terms of the notation (C.0.6) for the direct summands of $A \circledast B$, the individual components in (6.1.1) are, in the same order,

$$\begin{aligned} A \circledast B &= \big((A \circledast B)_1, (A \circledast B)_\phi\big) \\ &= \big(\ (A \circledast B)_{1(11)} \oplus (A \circledast B)_{1(\phi\phi)}\ ,\ (A \circledast B)_{\phi(1\phi)} \oplus (A \circledast B)_{\phi(\phi 1)} \oplus (A \circledast B)_{\phi(\phi\phi)}\ \big) \ . \end{aligned} \tag{D.0.5}$$

Consider a morphism $f \colon \Phi \circledast A \to A$. In terms of three linear maps in (6.1.2) the action of f on the individual summands of $\Phi \circledast A$ is as follows. For $1 \in \Phi_\phi = \mathbb{C}$, $a \in A_1$ and $b \in A_\phi$,

$$\begin{aligned} & & (f)_1\big(\,(1 \otimes_{\mathbb{C}} b)_{1(\phi\phi)}\,\big) = f_{1\phi}(b) \ , \\ (f)_\phi\big(\,(1 \otimes_{\mathbb{C}} a)_{\phi(\phi 1)}\,\big) = f_{\phi 1}(a) \ , & & (f)_\phi\big(\,(1 \otimes_{\mathbb{C}} b)_{\phi(\phi\phi)}\,\big) = f_{\phi\phi}(b) \ . \end{aligned} \tag{D.0.6}$$

To obtain the expression (6.1.3) for the dual of an object in $\mathcal{V}_\Phi$ we have to specialise (C.0.24) to the Lee-Yang model. For example, for $f \colon \Phi \circledast A \to A$ and $\varphi \in A^*_\phi$ one gets

$$\begin{aligned} c(f)_1\big(\,(1 \otimes_{\mathbb{C}} \varphi)_{1(\phi\phi)}\,\big) &= -\frac{\mathsf{F}^{(111)1}_{11}}{\mathsf{F}^{(\phi\phi\phi)\phi}_{11}}\, \mathsf{R}^{(\phi\phi)1}\, \mathsf{F}^{(\phi\phi 1)1}_{\phi 1} \sum_\alpha \varphi\big((f)_\phi((u \otimes a_{1,\alpha})_{\phi(\phi 1)})\big) \cdot a^*_{1,\alpha} \\ &= -d\zeta^2 \sum_\alpha \varphi\big(f_{\phi 1}(a_{1,\alpha})\big) \cdot a^*_{1,\alpha} = -d\zeta^2 f^*_{\phi 1}(\varphi) \ , \end{aligned} \tag{D.0.7}$$

which is the top right corner in (6.1.3). Expression (6.1.5) for the tensor product of two morphisms in $\mathcal{V}_\Phi$ is obtained from (C.0.22). Denote by $T^{i(xy)}_{i(\phi k(lm))}$ the linear map $T(f,g)_i$ restricted to $(\Phi \circledast (A \circledast B))_{i(\phi k(lm))}$ and projected to the summand $(A \circledast B)_{i(xy)}$,

$$T^{i(xy)}_{i(\phi k(lm))} = \delta_{y,m} \mathsf{F}^{(ml\phi)i}_{xk} f_{xl} \otimes_{\mathbb{C}} \mathrm{id}_{B_y} + \delta_{x,l} \frac{\mathsf{R}^{(\phi k)i}}{\mathsf{R}^{(\phi m)y}} \mathsf{F}^{(lm\phi)i}_{yk}\, \mathrm{id}_{A_x} \otimes_{\mathbb{C}} g_{ym} \tag{D.0.8}$$

In terms of these, the elements of the matrix (6.1.5) are

$$T(f,g) \mathrel{\widehat{=}} \begin{array}{c} \\ A_1{\otimes_{\mathbb{C}}}B_1 \\ A_\phi{\otimes_{\mathbb{C}}}B_\phi \\ A_1{\otimes_{\mathbb{C}}}B_\phi \\ A_\phi{\otimes_{\mathbb{C}}}B_1 \\ A_\phi{\otimes_{\mathbb{C}}}B_\phi \end{array} \begin{array}{c} \begin{array}{ccccc} A_1{\otimes_{\mathbb{C}}}B_1 & A_\phi{\otimes_{\mathbb{C}}}B_\phi & A_1{\otimes_{\mathbb{C}}}B_\phi & A_\phi{\otimes_{\mathbb{C}}}B_1 & A_\phi{\otimes_{\mathbb{C}}}B_\phi \end{array} \\ \left(\begin{array}{ccccc} 0 & 0 & T^{1(11)}_{1(\phi\phi(1\phi))} & T^{1(11)}_{1(\phi\phi(\phi 1))} & T^{1(11)}_{1(\phi\phi(\phi\phi))} \\ 0 & 0 & T^{1(\phi\phi)}_{1(\phi\phi(1\phi))} & \underline{T^{1(\phi\phi)}_{1(\phi\phi(\phi 1))}} & \underline{T^{1(\phi\phi)}_{1(\phi\phi(\phi\phi))}} \\ T^{\phi(1\phi)}_{\phi(\phi 1(11))} & T^{\phi(1\phi)}_{\phi(\phi 1(\phi\phi))} & T^{\phi(1\phi)}_{\phi(\phi\phi(1\phi))} & T^{\phi(1\phi)}_{\phi(\phi\phi(\phi 1))} & T^{\phi(1\phi)}_{\phi(\phi\phi(\phi\phi))} \\ T^{\phi(\phi 1)}_{\phi(\phi 1(11))} & T^{\phi(\phi 1)}_{\phi(\phi 1(\phi\phi))} & T^{\phi(\phi 1)}_{\phi(\phi\phi(1\phi))} & T^{\phi(\phi 1)}_{\phi(\phi\phi(\phi 1))} & T^{\phi(\phi 1)}_{\phi(\phi\phi(\phi\phi))} \\ T^{\phi(\phi\phi)}_{\phi(\phi 1(11))} & \underline{T^{\phi(\phi\phi)}_{\phi(\phi 1(\phi\phi))}} & T^{\phi(\phi\phi)}_{\phi(\phi\phi(1\phi))} & T^{\phi(\phi\phi)}_{\phi(\phi\phi(\phi 1))} & T^{\phi(\phi\phi)}_{\phi(\phi\phi(\phi\phi))} \end{array} \right) \end{array} . \qquad \text{(D.0.9)}$$

For example, the underlined entries are

$$\begin{aligned} T^{1(\phi\phi)}_{1(\phi\phi(\phi 1))} &= \zeta^2 \cdot \mathrm{id}_{A_\phi} \otimes_{\mathbb{C}} g_{\phi 1} \ , \\ T^{1(\phi\phi)}_{1(\phi\phi(\phi\phi))} &= f_{\phi\phi} \otimes_{\mathbb{C}} \ \mathrm{id}_{B_\phi} + \zeta \cdot \mathrm{id}_{A_\phi} \otimes_{\mathbb{C}} g_{\phi\phi} \ , \\ T^{\phi(\phi\phi)}_{\phi(\phi 1(\phi\phi))} &= \tfrac{1}{wd} \cdot f_{\phi\phi} \otimes_{\mathbb{C}} \ \mathrm{id}_{B_\phi} + \tfrac{1}{\zeta wd} \cdot \mathrm{id}_{A_\phi} \otimes_{\mathbb{C}} g_{\phi\phi} \ , \end{aligned} \qquad \text{(D.0.10)}$$

in agreement with (6.1.5).

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