Modèles topologiques de type cohomologique en théorie quantique des champs.

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Topological models of cohomological type in quantum field theories.

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1 Introduction

It is a widespread opinion that topology first appeared with the famous problem of the seven bridges of Königsberg:

The city of Königsberg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other and the mainland by seven bridges. The problem was to find a walk through the city that would cross each bridge once and only once. The islands could not be reached by any route other than the bridges, and every bridge must have been crossed completely every time; one could not walk halfway onto the bridge and then turn around and later cross the other half from the other side. The walk need not start and end at the same spot.

Although it is Leonhard Euler in 1736 who showed that the problem has no solution, one can nonetheless consider Carl Friedrich Gauss (1777-1855) as the true father of topology, with a special mention of the integral which determines the induction coefficient of two electric current loops and which now bears his name. In fact some recent researches suggest that it is from astronomical considerations, rather than electromagnetic, that Gauss discovered his integral formula for linking numbers. The excellent article [1] provides more details on this story. Gauss integral computing linking numbers (which are integers) is invariant under continuous deformations of the ambient manifold into which the links are immersed: it is an ambient isotopic invariant. This is where topology intervenes in this context.

Since then, topology has been largely developed and has become an independent part of mathematics. Topology has recently made a remarkable comeback in physics in the context of Quantum Field Theory[1]. This might be a little surprising at first sight, since a Quantum Field Theory (QFT) is fundamentally based on locality while topology is more concerned about global aspects of manifolds. We shall return later to the interplay between local and global aspects of QFT's.

In this review we are going to present two examples where QFT provides information about the topological nature of the space on which they are defined. The common thread in these two examples is that they are both based on cohomology. Let us point out that the QFT's we will deal with are all Euclidean.

The fact that cohomology plays a role in topology should not be too surprising deal with objects which are invariant under continuous deformations of the manifold on which they are defined. It is perhaps more surprising that cohomology can be used in physics (and more specifically in quantum theories). For instance, Deligne-Beilinson cohomology that will appear at length in this report is a basic element of Geometric Quantization. This sounds as an echo to our previous questioning about locality.

We will start by recalling some basic facts concerning cohomology. Then, we will show how equivariant cohomology, combined with QFT, gives rise to a large number of topological invariants, such as those of Donaldson and Mumford. Our second example will be part of

[1] Actually an extension of the Gauss integral, coined as "helicity" by H. Mofatt in 1969 [44], has been introduced in 1958 by L. Woltjer [43] in his study of magnetic fields of the Crab Nebula, thus referring to Gauss initial point of view in a sort of epistemological loop. Since then helicity has played an important role in astrophysics, solar physics and plasma physics

the epistemological loop mentioned in the previous footnote: we will show how Chern-Simons abelian theories allow to compute link invariants, actually those expressed in terms of the Gauss integral, and how the construction can be generalised when the manifold has torsion and/or has a dimension larger than three. The last part of this report will be devoted to proposal for future works.

2 A fly over Equivariant and Deligne-Beilinson cohomologies

In this section we will give a quick presentation of the two cohomologies we will use in the sequel. There will be almost no proof, but mostly references to specialized text books such as [4, 2]. We will first consider equivariant cohomology and then Deligne-Beilinson cohomology. We will assume that the reader is relatively familiar with the standard cohomologies such as de Rham or Čech, as well as the singular cohomology. Similar knowledge will be assumed for the corresponding homologies as well as Poincaré duality.

2.1 Equivariant Cohomology: Weil, Kalkman and Cartan schemes

Equivariant cohomology is based on the quite natural following idea: if one considers a principal bundle with total space P, base space B and typical fibre a Lie group G, is it possible to determine the cohomology of B from that of P? One knows that $B \cong P/G$, where the quotient is made for the right action on G on P. The interest of the construction that answers the previous question is that if M is a manifold on which a Lie group G is acting, then, even if the quotient M/G is not a manifold, one can define a cohomology for the action of G on M from that of M. We will show that there are different ways to present Equivariant Cohomology. These ways will be referred to as different *schemes*. The first scheme we will introduce is the one of Weil which is from our point of view the most natural. Cartan scheme is a bit more subtle, and will be introduced at the end. As to Kalkman scheme, besides the fact it provides an elegant way to relate Weil and Cartan schemes, it also proves a very powerful tool in applications of Equivariant Cohomology within the framework we will present later.

Let us consider à principal fibre bundle $\xi = (P, B, G, \pi)$ with total space a smooth manifold P, base space a smooth manifold B, typical fibre a Lie group G acting (transitively) to the right on P and with projection $\pi : P \to B$. We will denote by m_B the dimension of B and by m_P the dimension of P. One introduces the exterior derivative d_P on P which acts on $\Omega^*(P) = \bigoplus \Omega^k(P)$, the space of smooth differential forms on P: $d_P|_k : \Omega^k(P) \to \Omega^{k+1}(P)$. This provides $\Omega^*(P)$ with the structure of a differential complex on which closed forms are defined as $d_P\omega = 0$ and exact forms as $\omega = d_P\eta$. De Rham cohomology groups are then defined on P according to:

$$H^k_{dR} = \frac{Ker(d_P|_k)}{Im(d_P|_{k-1})} \,. \tag{2.1}$$

Associated with the right action of G on P, one introduces two more differential operators: the interior derivative (or contraction) and the Lie derivative. Since the typical fibre of our principal bundle ξ is a Lie group, then the fibres of P, *i.e.* the sub-sets $\pi^{-1}(\{x\})$, where $x \in B$, can be identified with the orbits of G in P. One then defines the vertical direction in the tangent space at $p \in P$ as the set of tangent vectors at p which are sent to zero by $(d\pi)_p$, the tangent mapping at p associated with π:

$$V_pP = \{\Upsilon_p \in T_pP / (d\pi)_p(\Upsilon_p) = 0\} = Ker((d\pi)_p) \,. \tag{2.2}$$

Since the typical fiber of ξ is the Lie group G, the vertical space V_pP can be canonically identified with the tangent space of G at $g \in G$: $V_pP \simeq T_pG$. Thanks to the adjoint action of G, T_gG can be canonically sent to T_eG, the tangent space of G at the identity of G, which is nothing but the Lie algebra $\mathcal{G}$ of G. Hence, one gets: $V_pP \sim \mathcal{G}$. For any $\omega \in \Omega^1(P)$ and $\lambda \in \mathcal{G}$, on defines **the interior derivative** of ω along λ at p by:

$$\forall p \in P, \quad i_p(\lambda)\omega = \omega(\widetilde{\lambda}_p)\,, \tag{2.3}$$

where $\widetilde{\lambda}_p \in V_pP$ is the tangent vector canonically associated with λ, via the previously mentioned isomorphism $V_pP \sim \mathcal{G}$.

Let us recall that differential forms are dualizing vector fields, which gives a meaning to the RHS of (2.3), thus providing a 0-form (*i.e.* a function) on P. The definition of the interior derivative straightforwardly extends to $\Omega^k(P)$:

$$i_p : \mathcal{G} \times \Omega^k(P) \to \Omega^{k-1}(P)\,. \tag{2.4}$$

This mapping is bilinear and is also called the contraction on $\Omega(P)$. For antisymmetry reasons, the interior derivative satisfies:

$$\forall p \in P, \forall \lambda \in \mathcal{G}, \quad i_p(\lambda)i_p(\lambda) = 0\,. \tag{2.5}$$

Whereas d_P increases by one the degree of forms, i_p decreases it by one.

The last derivative to be introduced reads:

$$l_p = i_p \circ d_P + d_P \circ i_p = \{i_p, d_P\}\,. \tag{2.6}$$

This derivative retains the degree of forms on which it acts and is called **the Lie derivative**. It corresponds to the idea of an infinitesimal action of G (and so of an action of $\mathcal{G}$) on $\Omega(P)$. Thus, one obtains a bilinear mapping:

$$l_p : \mathcal{G} \times \Omega^k(P) \to \Omega^k(P)\,. \tag{2.7}$$

that commutes in an obvious way with d_P. One also has:

$$\forall \lambda_1, \lambda_2 \in \mathcal{G}, \quad [l_p(\lambda_1), l_p(\lambda_2)] = l_p([\lambda_1, \lambda_2])\,, \tag{2.8}$$

as well as:

$$\forall \lambda_1, \lambda_2 \in \mathcal{G}, \quad [l_p(\lambda_1), i_p(\lambda_2)] = i_p([\lambda_1, \lambda_2])\,. \tag{2.9}$$

A k-form $\omega \in \Omega^k(P)$ is said to be **horizontal** if:

$$\forall \lambda \in \mathcal{G}, \quad i_p(\lambda)\omega = 0\,. \tag{2.10}$$

and **invariant** if:

$$\forall \lambda \in \mathcal{G}, \quad l_p(\lambda)\omega = 0\,. \tag{2.11}$$

A form which is both horizontal and invariant will be called **basic**. The spaces of basic k-forms on the total space P is denoted $\Omega^k_G(P)$. One can show the following important result:

$$\Omega^k_G(P) \cong \Omega^k(B)\,, \tag{2.12}$$

which means that basic forms on the total space P of the principal bundle χ can be canonically identified with forms on the base space B of χ. This is where the terminology "basic" comes from. More precisely, for any $\omega \in \Omega^k_G(P)$ the canonically associated form $\widetilde{\omega} \in \Omega^k(B)$ is such that: $\pi^*\widetilde{\omega} = \omega$. Proving the existence of $\widetilde{\omega}$ is the difficult part in the demonstration of (2.12). Uniqueness is more obvious: if $\pi^*\widetilde{\omega_1} = \pi^*\widetilde{\omega_2} = \omega$, then $\pi^*(\widetilde{\omega_1} - \widetilde{\omega_2}) = 0$. But the only form whose pull-back via π^* is the zero form is the zero form itself. Conversely, to any form $\widetilde{\omega} \in \Omega^k(B)$ is canonically associated the k-form $\pi^*\widetilde{\omega} \in \Omega^k(P)$ such that:

$$\forall \lambda \in \mathcal{G}, \quad i_p(\lambda)(\pi^*\widetilde{\omega}) = \widetilde{\omega}(\pi_*\widetilde{\lambda}_p) = \widetilde{\omega}(0) = 0\,, \tag{2.13}$$

and:

$$\forall \lambda \in \mathcal{G}, \quad l_p(\lambda)(\pi^*\widetilde{\omega}) = i_p(\lambda)(\pi^*(d_P\widetilde{\omega})) = (d_B\widetilde{\omega}))(\pi_*\widetilde{\lambda}_p) = 0\,, \tag{2.14}$$

where d_B denotes the exterior derivative on B. These last two equations prove that $\pi^*\widetilde{\omega}$ is basic. Let us remark that if $\widetilde{\omega} \in \Omega^k(B)$ is closed, then so is $\pi^*\widetilde{\omega}$ since $d_P \circ \pi^* = \pi^* \circ d_B$. Conversely, if $\omega \in \Omega^k_G(P)$ is closed, then so is the canonically associated form $\widetilde{\omega} \in \Omega^k(B)$. Indeed, if $\omega \in \Omega^k_G(P)$ was defining a form $\widetilde{\omega}$ on B which is not closed, since $\pi^*\widetilde{\omega} = \omega$, then one would necessarily infer that ω also is not closed, hence a contradiction. Consequently, if one denotes $H^k_{dR,G}(P)$ the space of cohomology classes build from $\Omega^k_G(P)$ and $\Omega^{k-1}_G(P)$, one deduces that:

$$H^k_{dR,G}(P) \cong H^k_{dR}(B)\,. \tag{2.15}$$

More generally, if P is a smooth manifold and G a Lie group acting to the right on P, then the spaces $H^k_{dR,G}(P)$, build from basic elements of P, as just exposed, are called the **basic cohomology** groups of P for the right action of G. From (2.15), this cohomology coincides with the cohomology of P/G when this quotient is a manifold.

The next ingredient is provided by $\mathcal{W}(G)$, the Weil algebra of G. It is the differential graded algebra generated by two $\mathcal{G}$-valued objects: the "connection" θ, of degree 1, and its "curvature" Θ, of degree 2, such that:

$$d_{\mathcal{W}}\theta = \Theta - \frac{1}{2}[\theta, \theta]\,, \tag{2.16}$$

where $d_{\mathcal{W}}$ is the exterior derivative of $\mathcal{W}(G)$. One has the Bianchi identity:

$$d_{\mathcal{W}}\Theta = -[\theta, \Theta]\,. \tag{2.17}$$

This differential graded algebra is a way to describe connections (and their curvatures) on principal bundles without the need of specifying either a base space nor the total space of the bundle. Only the group is taken into consideration. It plays a fundamental role in the theory of classifying spaces. The Weil algebra can be endowed with an interior derivative, $i_{\mathcal{W}}$, and a Lie derivatives, $l_{\mathcal{W}}$, for which one has, for any $\lambda \in \mathcal{G}$:

$$i_{\mathcal{W}}(\lambda)\theta = \lambda \quad , \quad l_{\mathcal{W}}(\lambda)\theta = -[\lambda, \theta] \,. \tag{2.18}$$

and

$$i_{\mathcal{W}}(\lambda)\Theta = 0 \quad , \quad l_{\mathcal{W}}(\lambda)\Theta = -[\lambda, \Theta] \,. \tag{2.19}$$

It can be shown, for instance using a homotopy operator, that the cohomology of $\mathcal{W}(G)$ is trivial [3]. This is mainely due to the algebraic nature of this space. In fact, in the action of $i_{\mathcal{W}}$ and $l_{\mathcal{W}}$ all references to points of a base space have disappeared.

Let us now consider a manifold $\mathcal{M}$ on which a Lie group G acts to the right. The Lie algebra of G is denoted $\mathcal{G}$. As explained before, one can then construct the basic cohomology of $\mathcal{M}$, $H_{dR,G}(\mathcal{M})$. Let the graded algebra $\Omega(\mathcal{M}) \otimes \mathcal{W}(G)$ be provided with the natural operations $d_{\mathcal{M}} + d_{\mathcal{W}}$, $i_{\mathcal{M}} + i_{\mathcal{W}}$ and $l_{\mathcal{M}} + l_{\mathcal{W}}$, which turns it into a graded differential algebra. The elements of $\Omega(\mathcal{M}) \otimes \mathcal{W}(G)$ annihilated by $(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda)$ and $(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda)$ for any $\lambda \in \mathcal{G}$, are called **equivariant cochains**. Equivariant cochains annihilated by $d_{\mathcal{M}} + d_{\mathcal{W}}$ are called **equivariant cocycles**, and equivariant cochains which can be written as the $d_{\mathcal{M}} + d_{\mathcal{W}}$ of some other equivariant cochains are called **equivariant coboundaries**. This generates a cohomology called **Weil scheme of Equivariant Cohomology**. The mapping:

$$\zeta \mapsto exp\{-i_{\mathcal{M}}(\theta)\}\zeta \,. \tag{2.20}$$

is an isomorphism of the differential algebra $\Omega(\mathcal{M}) \otimes \mathcal{W}(G)$ for which:

$$\begin{array}{rcl} d_{\mathcal{M}} + d_{\mathcal{W}} & \rightarrow & D_K \equiv d_{\mathcal{M}} + d_{\mathcal{W}} + l_{\mathcal{M}}(\theta) - i_{\mathcal{M}}(\Theta) \\ i_{\mathcal{M}} + i_{\mathcal{W}} & \rightarrow & I_K \equiv i_{\mathcal{W}} \\ l_{\mathcal{M}} + l_{\mathcal{W}} & \rightarrow & L_K \equiv l_{\mathcal{M}} + l_{\mathcal{W}} \,. \end{array} \tag{2.21}$$

It can easily be checked that D_K, I_K and L_K are respectively exterior, interior and Lie derivatives, and that the equivariant cohomology these derivatives define is isomorphic to the one of Weil scheme. The isomorphism is provided by a canonical extension of (2.20) to appropriate spaces. The version of Equivariant Cohomology thus obtained will be called **Kalkman scheme of Equivariant Cohomology**.

Finally, from Kalkman scheme, if one sets $\theta \equiv 0$, then $D_K^2|_{\theta=0}$ reduces to zero on *invariant* cochains and not on the whole differential algebra. This gives rise to the so-called **Cartan scheme of Equivariant Cohomology**. It is in this scheme that E. Witten adopted in [5] when he showed the interest of Equivariant Cohomology for the computation of observables in some topological models. We will come back to this later.

Although popular, Cartan's scheme is not well adapted to explicit computations. Hence, we will prefer to use Weil or even Kalkman schemes, the relevance of which will appear in the examples of the next section.

Let us now have a look at the second type of cohomology we will use in the topological models we want to present here.

2.2 Deligne-Beilinson Cohomology

Once again, there exist several, but equivalent, ways to present Deligne-Beilinson Cohomology [16, 17, 18, 19]. For instance, it can be defined as the cohomology of a cone. This seems to be more appropriate when dealing within algebraic geometry and in particular the theory of regulators [17, 18]. For smooth manifolds it can be constructed more explicitly introducing cochains, a differential and then cocycles on some Čech-de Rham bi-complex [18]. This smooth Deligne-Beilinson Cohomology can also be seen as a realization of Cheeger-Simons Differential Characters [20, 21], Harvey-Lawson "Sparks" [22] or Singer-Hopkins Differential Cohomology [23]. The advantage of the Čech-de Rham explicit method is that it provides expressions which are quite convenient for physicists [24]. On the other hand, using Differential Characters or Sparks turns out to be particularly well-adapted when dealing with abelian Chern-Simons theories as it will appear later. In the sequel DB will stand for "Deligne-Beilinson", and we will only consider smooth, closed (*i.e.* compact and without boundary) manifolds.

If one chooses to use the Čech-de Rham bi-complex, DB Cohomology on a manifold M can be constructed as follow. Let $\mathcal{U} = (U_i)_{i\in I\subset\mathbb{N}}$ be a good cover of M, *i.e.* a cover such that any non empty intersection of elements of $\mathcal{U}$ is contractible. This is equivalent to say that all non empty intersections have no homology, nor cohomology (except in degree zero of course). A **DB cochain** is defined as a collection

$$\omega^{[p]} = (\omega^{(0,p)}, \omega^{(1,p-1)}, ..., \omega^{(p,0)}, n^{(p+1,-1)})\,, \tag{2.22}$$

where the $\omega^{(k,p-k)}$'s are $\Omega^{p-k}(U_i)$-valued Čech k-cochains (with $\Omega^{p-k}(U_i)$ the space of $(p-k)$-forms on $(U_i)\in\mathcal{U}$), whereas the last term, $n^{(p+1,-1)}$, is an **integral** Čech $(p+1)$-cochain.

One provides the set of DB cochains with a differential, denoted D and defined by:

$$D = \delta + \widetilde{d}\,, \tag{2.23}$$

where δ is the Čech differential, $\widetilde{d} = \pm d_M$ depending on whether the exterior derivative d_M is taken on objects having even or odd Čech degree [2], and $\widetilde{d} \equiv 0$ on p-forms. When acting on "pure" Čech cochains (denoted $\omega^{(p,-1)}$), $\widetilde{d}$ has to be understood (up to sign) as the injection of numbers into (constant) functions. More explicitly:

$$D\omega^{[p]} = (0, \delta\omega^{(0,p)} + \widetilde{d}\omega^{(1,p-1)}, ..., \delta\omega^{(p,0)} + \widetilde{d}n^{(p+1,-1)}, \delta n^{(p+1,-1)})\,. \tag{2.24}$$

The differential D is a truncation on p-forms of the Čech-de Rham differential used to prove that real Čech cohomology and de Rham cohomology are isomorphic. We can check without

[2] This alternation of sign ensures that $D^2 = 0$

any difficulty that (2.24) leads to $D^2 = 0$. One will say that a DB cochain $\omega^{[p]}$ is a **DB cocycle** if:

$$D\omega^{[p]} = 0\,, \tag{2.25}$$

which locally (*i.e.* with respect of the $\omega^{(k,p-k)}$'s) reads:

$$\left\{\begin{array}{ccc} \delta\omega^{(0,p)} + \widetilde{d}\omega^{(1,p-1)} & = & 0 \\ \delta\omega^{(1,p-1)} + \widetilde{d}\omega^{(2,p-2)} & = & 0 \\ & \vdots & \\ \delta\omega^{(p-1,1)} + \widetilde{d}\omega^{(p,0)} & = & 0 \\ \delta\omega^{(p,0)} + \widetilde{d}n^{(p+1,-1)} & = & 0 \\ \delta n^{(p+1,-1)} & = & 0\,. \end{array}\right. \tag{2.26}$$

These equations are often referred to as the "descent" equations of the DB cocycle $\omega^{[p]}$. Not surprisingly, one says that a DB cocycle is a **DB coboundary** whenever:

$$\exists\eta^{[p-1]} \;/\quad \omega^{[p]} = D\eta^{[p-1]}\,, \tag{2.27}$$

which locally reads:

$$\left\{\begin{array}{ccc} \omega^{(0,p)} & = & \widetilde{d}\eta^{(0,p-1)} \\ \omega^{(1,p-1)} & = & \delta\eta^{(0,p-1)} + \widetilde{d}\eta^{(1,p-2)} \\ & \vdots & \\ \omega^{(p-1,1)} & = & \delta\eta^{(p-2,1)} + \widetilde{d}\eta^{(p-1,0)} \\ \omega^{(p,0)} & = & \delta\eta^{(p-1,0)} + \widetilde{d}n^{(p,-1)} \\ n^{(p+1,-1)} & = & \delta m^{(p,-1)}\,. \end{array}\right. \tag{2.28}$$

One finally defines the pth DB Cohomology space as the quotient of the space of DB p-cocycles by the space of DB p-coboundaries, as usual in cohomology theory. One denotes $[\omega^{[p]}]$, or simply $[\omega]$, a DB class of degree p, and one says that the DB cocycles associated with this class are its representative for the good cover $\mathcal{U}$, whereas equations (2.28) identify the ambiguities on the representatives of a given DB class. The resulting cohomology spaces are independent of the good cover and are only depending on the manifold M. This is actually also true within Čech cohomology when one uses good covers. They are denoted:

$$H^p_D(M,\mathbb{Z})\,. \tag{2.29}$$

To be a more precise, one should define $H^p_D(M,\mathbb{Z})$ as the inductive limit over refined good covers of M of the previously build DB cohomology spaces.
Let us point out that our choice of degree for the DB classes is the one coming from Sparks and Differential Characters, whereas in DB theory one uses a degree shifted by minus one.

One of the first results that which shows the interest of Deligne-Beilinson Cohomology is the following: $H^1_D(M,\mathbb{Z})$ canonically identifies with the space of equivalence classes of

$U(1)$-principal bundles with connections, the classification being made with respect to $U(1)$-isomorphisms.

Let us first show that a DB class $[\omega]$ of degree p defines in a natural way a closed $(p+1)$-form with integral periods on M, *i.e.* a curvature when $p = 1$. Indeed, (2.26) implies that $\delta d_M \omega^{(0,p)} = 0$ and therefore that there exists a form $\omega^{(-1,p+1)}$ on M whose restrictions in the opens of the good cover $\mathcal{U}$ coincide with the local expressions of $d_M \omega^{(0,p)}$. Thus, the descent equations (2.26) can be seen as standard Čech-de Rham descent equations for the (global) form $\omega^{(-1,p+1)}$. These descent equations end (by construction) with an integral Čech cocycle $n^{(p+1,-1)}$. Therefore, $\omega^{(-1,p+1)}$ turns out to be a closed form with integral periods on M. Let us remark that $\omega^{(-1,p+1)}$ is closed simply because its restrictions are exact. Note that the Čech-de Rham descents of a closed form with integral periods generically end with a real Čech cocycles. However this cocycle can always be turned into an integer cocycle. The Čech cohomology class associated with this integer cocycle can only determine a free cohomology class but not a torsion class. The interpretation of $H^p_D(M, \mathbb{Z})$ as a classifying space can be made in terms of abelian (bundle) Gerbes together with their connections. This gives a "geometric" meaning to the DB construction [25].

Let us now try to give a more precise description of the spaces $H^p_D(M, \mathbb{Z})$. To begin with, let us note that from the descent equations (2.26), and the "ambiguity" equations (2.28), any DB class $[\omega]$ defines an integral Čech cohomology class $[n]$, thus generating a map:

$$\delta_2 : H^p_D(M, \mathbb{Z}) \to \check{H}^{p+1}(M, \mathbb{Z}) \,, \tag{2.30}$$

where $\check{H}^{p+1}(M, \mathbb{Z})$ denotes the p-th Čech cohomology group of M. This map is surjective as the descent equations easily show. One can equivalently say that the following sequence:

$$H^p_D(M, \mathbb{Z}) \stackrel{\delta_2}{\to} \check{H}^{p+1}(M, \mathbb{Z}) \to 0 \,. \tag{2.31}$$

is exact.

Equations (2.28) which organize the ambiguities of (2.26) for a given DB class $[\omega]$ imply that DB classes with the same image under δ_2 may differ by an element of $\Omega^p(M)/\Omega^p_{\mathbb{Z}}(M)$, the quotient of the space of p-forms on M by the subspace of closed p-forms with integral periods. This provides an extension to the left of (2.31) into the following exact sequence:

$$0 \to \Omega^p_{\mathbb{Z}}(M) \stackrel{i}{\to} \Omega^p(M) \to H^p_D(M, \mathbb{Z}) \stackrel{\delta_2}{\to} \check{H}^{p+1}(M, \mathbb{Z}) \to 0 \,. \tag{2.32}$$

From this last exact sequence we infer that $H^p_D(M, \mathbb{Z})$ is an affine bundle with (discrete) base space $\check{H}^{p+1}(M, \mathbb{Z})$, and translation group $\Omega^p(M)/\Omega^p_{\mathbb{Z}}(M)$. This structure plays a crucial role when Deligne-Beilinson Cohomology is used within the abelian Chern-Simons framework in order to compute link invariants.

If on the other hand one rather starts from the other edge of the descent equations defining the DB class $[\omega]$, that is to say if one first consider the generalised curvature $\omega^{(-1,p+1)}$ canonically defined by $[\omega]$, then another surjective mapping naturally emerges:

$$\delta_1 : H^p_D(M, \mathbb{Z}) \to \Omega^{p+1}_{\mathbb{Z}}(M) \,. \tag{2.33}$$

There follows another exact sequence:

$$H_D^p(M,\mathbb{Z}) \stackrel{\delta_1}{\to} \Omega_{\mathbb{Z}}^{p+1}(M) \to 0\,, \tag{2.34}$$

which can be completed into the following exact sequence:

$$0 \to \check{H}^p(M,\mathbb{R}/\mathbb{Z}) \to H_D^p(M,\mathbb{Z}) \stackrel{\delta_1}{\to} \Omega_{\mathbb{Z}}^{p+1}(M) \to 0\,. \tag{2.35}$$

where $\check{H}^p(M,\mathbb{R}/\mathbb{Z})$ denotes the p-th $\mathbb{R}/\mathbb{Z}$-valued Čech cohomology group of M. This last exact sequence shows that $H_D^p(M,\mathbb{Z})$ is also an affine bundle over $\Omega_{\mathbb{Z}}^{p+1}(M)$ whose translation group is now $\check{H}^p(M,\mathbb{R}/\mathbb{Z})$. Let us point out that whereas it has a discrete basis with respect to (2.32), $H_D^p(M,\mathbb{Z})$ has a continuous (although potentially disconnected) basis with respect to (2.35).

In the particular case where $p = m = dimM$ the exact sequence (2.32) becomes:

$$0 \to \frac{\Omega_{\mathbb{Z}}^m(M)}{\Omega^m(M)} \to H_D^m(M,\mathbb{Z}) \stackrel{\delta_2}{\to} \check{H}^{m+1}(M,\mathbb{Z}) = 0\,. \tag{2.36}$$

A choice of a normalized volume form on M allows to prove that $\Omega_{\mathbb{Z}}^m(M)/\Omega^m(M) \simeq \mathbb{R}/\mathbb{Z}$, thus leading to:

$$H_D^m(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}\,. \tag{2.37}$$

One can chose the zero class of $H_D^m(M,\mathbb{Z})$ as origin on the unique fiber of $H_D^m(M,\mathbb{Z})$, which makes the isomorphism (2.37) canonical.

Čech-de Rham descents of closed forms with integral periods on M cannot by themselves generate alone Deligne-Beilinson spaces. For instance, as we already mentioned, such descents can only generate the free part of the Čech cohomology groups. This is due to the fact that Čech-de Rham deals with forms and then only provide an isomorphism for real cohomologies thus eliminating torsion. In fact its the truncation ($\widetilde{d} \equiv 0$ on p-formes) which gives access via ambiguities to torsion. One then finds torsion either in the Čech cohomology group at the end of the exact sequence (2.32), or in the $\mathbb{R}/\mathbb{Z}$-valued Čech cohomology at the start of the exact sequence (2.35). In other words, torsion is either contained in the base space of $H_D^p(M,\mathbb{Z})$ when one deals with (2.32)), or in the translation group of $H_D^p(M,\mathbb{Z})$ when one deals with (2.35).

Beside their affine bundle structure, DB spaces enjoys a natural $\mathbb{Z}$-module structure: the sum of two DB classes of degree p is a DB class of degree p, and so is any integral combination of DB classes of degree p. There is also a natural pairing on Deligne-Beilinson Cohomology spaces:

$$*_D : H_D^p(M,\mathbb{Z}) \times H_D^p(M,\mathbb{Z}) \to H_D^{p+q+1}(M,\mathbb{Z})\,. \tag{2.38}$$

This pairing, called the **DB product**, is graded commutative:

$$[\omega^{[p]}] *_D [\omega^{[q]}] = (-1)^{(p+1)(q+1)}[\omega^{[q]}] *_D [\omega^{[p]}]\,. \tag{2.39}$$

Let us give an example by considering two DB classes of degree 1, hence defining two inequivalent classes of $U(1)$-bundles with connections. Let $(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)})$ and $(\widetilde{A}^{(0,1)}, \widetilde{\Lambda}^{(1,0)}, \widetilde{n}^{(2,-1)})$

be a representative for each of these classes, with respect to a good cover $\mathcal{U}$ of M. The DB class resulting from the DB product of the two classes of these DB cocycles can be represented by the following DB 3-cocycle:

$$(A^{(0,1)}\check{\wedge}d\widetilde{A}^{(0,1)}, \Lambda^{(1,0)}\check{\wedge}d\widetilde{A}^{(0,1)}, n^{(2,-1)}\check{\wedge}\widetilde{A}^{(0,1)}, n^{(2,-1)}\check{\wedge}\widetilde{\Lambda}^{(1,0)}, n^{(2,-1)}\cup\widetilde{n}^{(2,-1)})\,, \tag{2.40}$$

where $\check{\wedge}$ denotes the combination of the exterior product $\wedge$ with the cup product $\cup$. This combination is the one used in the standard Čech-de Rham machinery. When $p+q+1=m=dimM$, using (2.37), one gets:

$$*_D : H^p_D(M,\mathbb{Z}) \times H^q_D(M,\mathbb{Z}) \to H^m_D(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}\,. \tag{2.41}$$

We will see later on that (2.41) defines an $\mathbb{R}/\mathbb{Z}$-duality between $H^p_D(M,\mathbb{Z})$ and $H^{(m-p-1)}_D(M,\mathbb{Z})$.

Remembering the link between DB classes of degree 1 and $U(1)$-connections, one is naturally led to wonder whether integration of a DB class over a cycle on M is well defined. This would be something which generalizes $U(1)$ holonomies. It is well-known that the integral $\oint_z A$, of an abelian gauge field (*i.e.* a $U(1)$-connection) A over a 1-cycle z of M, does not define a real (or complex) number but rather an element of $\mathbb{R}/\mathbb{Z}$. From a physical point of view this is nothing but the Aharonov-Bohm[3] effect [8]. This can be seen for instance when the cycle is trivial ($z=bc$) since then one can locally write:

$$\oint_z A = \oint_{bc} A = \int_c dA = \int_c F(A)\,, \tag{2.42}$$

with $F(A)=dA$ the curvature of A. Yet, the chain c, whose boundary is z, is not unique, and if $\widetilde{c}$ is another such chain then there exists a 2-cycle Σ on M such that $\widetilde{c}=c+\Sigma$. Accordingly:

$$\oint_z A = \int_{\widetilde{c}} F(A) = \int_{c+\Sigma} F(A) = \int_c F(A) + \int_\Sigma F(A)\,. \tag{2.43}$$

Since, up to a normalization factor 2π, $F(A)$ has integral periods, one concludes that $\oint_z A$ is defined modulo integers. There would remain to show that $\oint_z A$ can be extended to any 1-cycles on M. This will be achieved by defining the integral of a general DB classes of degree p on p-cycles of M. We have already mentioned that the Čech-de Rham description of DB classes (in term of their representatives) provide explicit formulas. Therefore, finding an expression for $\oint_{z_p}[\omega^{[p]}]$ might be easier using this explicit description. Fortunately, there is a descent for (singular) p-cycles of M into Čech p-cycles associated with good covers of M. Here is the essence of this construction details of which can be found in the classic article from A. Weil ([7]).

Let $\mathcal{U}$ be a good cover of M and z_p be a (singular) p-cycle on M. One says that z_p is a $\mathcal{U}$-cycle if there exist a family $c_{(0,p)}$ of (singular) p-chains indexed by I, such that:

$$z_p = \partial c_{(0,p)} \equiv \sum_{i\in I} c^i_{(0,p)}\,. \tag{2.44}$$

[3]In fact, first discovered by W. Ehrenberg and R. E. Siday in 1949 [6]

For a given good cover, a generic singular p-cycle z_p of M is not necessarily a $\mathcal{U}$-cycle. However, it is always possible to refine $\mathcal{U}$ into a good cover $\mathcal{V}$ in such a way that z_p turns out to be a $\mathcal{V}$-cycle. It will only remain to show that the whole construction is independent of the good cover. For a p-cycle z_p, one show that there exists a collection:

$$z_{[p]} = \left(c_{(0,p)}, c_{(1,p-1)}, ..., c_{(p,0)}, \zeta_{(p,-1)}\right), \tag{2.45}$$

where the $c_{(k,p-k)}$'s are Čech chains taking their values in the set of singular chains of M, and $\zeta_{(p,-1)}$ is an integral Čech cycle. Furthermore, this collection verifies the following homological descent:

$$\left\{\begin{array}{ccc} bc_{(0,p)} & = & \partial c_{(1,p-1)} \\ bc_{(1,p-1)} & = & \partial c_{(2,p-2)} \\ & \vdots & \\ bc_{(p-1,1)} & = & \partial c_{(p,0)} \\ b_0 c_{(p,0)} & = & \zeta_{(p,-1)} \,. \end{array}\right. \tag{2.46}$$

In these equations, b denotes the boundary operation on singular chains and ∂ the boundary operation on Čech chains. This extends definition (2.44). These two operations can be seen as dualizing de Rham[4] d for b, and Čech δ for ∂. As for b_0, it is the operation which associates to any singular 0-cycle its integer coefficients obtained by decomposing it over a base of points (which are 0-cycles). Let us note that the descent equations (2.46) imply that $\zeta_{(p,-1)}$ is an **integral** Čech cycle, and $\partial c_{(0,p)} = z_p$ an **integral** singular cycle. This construction allows in particular to show that singular and Čech homologies are isomorphic.
Let us consider an example: z is a 1-cycle on M such that the good cover of M induces a good cover of z made of three open sets. We write $\mathcal{U}|_z = (V_1, V_2, V_3)$, with $V_i = U_i \cap z$ and $V_{123} = V_1 \cap V_2 \cap V_3 = \emptyset$. Not every cycle and good cover are such. But for a given cycle it is always possible to find a good cover that meets our requirements. We could say that $\mathcal{U}|_z$ is an "excellent" cover of z. One decomposes z with respect to $\mathcal{U}|_z$ by considering three integer 1-chains, let say c_1, c_2 and c_3, such that: $c_1 + c_2 + c_3 = z$, and whose boundaries are contained into the intersections of $\mathcal{U}|_z$, that is to say: $bc_1 = x_{12} - x_{31}$, $bc_2 = x_{23} - x_{12}$ and $bc_3 = x_{31} - x_{23}$, with $x_{ij} \in U_{ij} = U_i \cap U_j$. The Čech 1-chain so generated is given by: $\zeta_{12} = 1 = -\zeta_{21}$ in U_{12}, $\zeta_{23} = 1 = -\zeta_{32}$ in U_{23}, and $\zeta_{31} = 1 = -\zeta_{13}$ in U_{31}. These integers are nothing but the weights of the points x_{12}, x_{23} and x_{31} seen as basic 0-cycles in the various U_{ij} of $\mathcal{U}|_z$. Although by construction $U_{123} = U_1 \cap U_2 \cap U_3 = \emptyset$ on z, in M one might have $U_{123} = U_1 \cap U_2 \cap U_3 \neq \emptyset$. If this happens this means that the cycle z was actually a boundary, and this implies that there exists a Čech 2-chain τ_{ijk} such that $\zeta_{ij} = \sum_k \tau_{ijk}$. Conversely if $U_{123} = U_1 \cap U_2 \cap U_3 \neq \emptyset$ also holds in M, then the cycle is not a boundary and the same applies to ζ_{ij}. As a final point, let us noticed that from the point of view of z provided with the excellent cover $\mathcal{U}|_z$ the cycle ζ_{ij} is not trivial because there is no possibility to construct τ_{ijk} since $U_{123} = U_1 \cap U_2 \cap U_3 = \emptyset$ on z. This amounts to compute the first homology group of S^1 which is well-known to be non trivial.

[4]Actually b is dualizing the singular coboundary operation so that one needs to first send forms into singular cochains, which is done by integration, in order to see a duality between d and b

One immediately notices that beside their last term, all the terms of (2.22) and (2.45) have degrees that allow integration. Therefore, it seems natural to set, for any cycle z_p

$$\oint_{z_p} [\omega^{[p]}] = \sum_{k=0}^{p} \int_{c_{(k,p-k)}} \omega^{(k,p-k)} \mod \mathbb{Z}\,, \tag{2.47}$$

where integrals have to be understood with also a summation over all the Čech indices appearing there, in such a way that the final result has no Čech indices. Equality (2.47) is defined modulo integers, that is to say, the integral is an element of $\mathbb{R}/\mathbb{Z}$, as expected. When the cycle z_p is a boundary, *i.e.* $z_p = bc_{p+1}$, one immediately sees that (2.47) yields:

$$\oint_{z_p} [\omega^{[p]}] = \int_{c_{p+1}} F^{p+1} = \mod \mathbb{Z}\,, \tag{2.48}$$

where F^{p+1} is the closed form with integral period associated with (*i.e.* the curvature of) $[\omega^{[p]}]$.

When z_p is a torsion cycle, that is to say z_p is not a boundary but that there an integer m and an *integral* chain c_{p+1} such that $m.z_p = bc_{p+1}$, if one denotes $\zeta_{(p,-1)}$ an integral Čech cycle associated (by the Weil descent) with z_p and $\theta_{(p+1,-1)}$ an integral Čech chain such that $m.\zeta_{(p,-1)} = \partial\theta_{(p+1,-1)}$ (and so associated with c_{p+1}), then it is easy to show that (2.47) gives:

$$\oint_{z_p} [\omega^{[p]}] = \frac{1}{m}\left[\int_{c_{p+1}} F^{p+1} - \langle \omega^{(p+1,-1)}, \theta_{(p+1,-1)}\rangle\right] \mod \mathbb{Z}\,, \tag{2.49}$$

where $\langle\ ,\ \rangle$ denotes the duality operation between integral Čech chains and cochains. Hence, the couple $(F^{p+1}, [\omega^{(p-1,-1)}])$, associated with a DB class $[\omega^{[p]}]$, completely determines this DB class on the torsion group of degree p of M [21]. Finally, one can check that the expression defining the integration of the DB $[\omega^{[p]}]$ over the cycle z_p is independent of the chosen representative of $[\omega^{[p]}]$ as well as of the descent of z_p, and of the good cover. In fact, by checking all these independencies one also proves that integration is actually performed in $\mathbb{R}/\mathbb{Z}$ rather than in $\mathbb{R}$.

We have now just established an $\mathbb{R}/\mathbb{Z}$-duality between cycles on M and DB spaces. This is exactly how Cheeger and Simons have introduced Differential Characters [20] which are a different but equivalent way to see Deligne-Beilinson Cohomology.

Denoting $H^p_D(M,\mathbb{Z})^\star \equiv Hom(H^p_D(M,\mathbb{Z}),\mathbb{R}/\mathbb{Z})$, the Pontrjagin dual of $H^p_D(M,\mathbb{Z})$ and $Z_p(M)$ the space of singular p-cycles on M, one has:

$$Z_p(M) \subset Hom(H^p_D(M,\mathbb{Z}),\mathbb{R}/\mathbb{Z})\,. \tag{2.50}$$

This reminds us of another inclusion: singular chains seen as de Rham currents on M, the latter being dual to forms on M.

By combining the DB product $*_D$, integration and result (2.37), one deduces that there is another inclusion (or rather an injection):

$$H^{m-p-1}_D(M,\mathbb{Z}) \subset Hom(H^p_D(M,\mathbb{Z}),\mathbb{R}/\mathbb{Z})\,, \tag{2.51}$$

which is analogous to the inclusion of $\Omega^p(M)$ into the space of de Rham $(m-p)$-currents on M. From this point of view, the elements of $H^p_D(M,\mathbb{Z})^\star$ can be seen as distributional DB classes.

Eventually, one can show that any singular p-cycle z on M defines a canonical distributional DB $(m-p-1)$-class such that:

$$\oint_{z_p} [\omega^{[p]}] = \int_M \Omega^{[p+1]} *_D [\eta_z] \mod \mathbb{Z}\,, \tag{2.52}$$

for any $[\omega^{[p]}] \in H^p_D(M,\mathbb{Z})$ (see [24] for details). To compute the RHS of (2.52) using the integration formula (2.48), one has to decompose M itself according to Weil method thus obtaining something equivalent to (2.45). This decomposition, that we call a "polyhedral" decomposition of M, is quite standard. It can be used for instance to define the integration of a top form on M via the corresponding Čech objects.

One can also rewrite formula (2.48) using a partition of unity with compact support and subordinated to the good cover $\mathcal{U}$ of M. This avoids the use of a Weil decomposition of M (and of any cycle z in M). It also avoids problems occurring when dealing with distributional classes and products of such objects which might appear in (2.52). Once more we refer the reader to [24] for details. This is also very similar to the usual case where one can either use partition of unity or chains to define integration of forms.

Let us have a look at the dual spaces $Hom(H^p_D(M,\mathbb{Z}),\mathbb{R}/\mathbb{Z})$. If one applies Pontrjagin duality to the exact sequences (2.32) and (2.35), one deduces that these spaces are themselves terms of exact sequences. More specifically one finds that:

$$0 \to Hom(\Omega^{p+1}_{\mathbb{Z}}(M),\mathbb{R}/\mathbb{Z}) \to H^p_D(M,\mathbb{Z})^\star \to \check{H}^{m-p}(M,\mathbb{Z}) \to 0\,, \tag{2.53}$$

and:

$$0 \to \check{H}^{m-p-1}(M,\mathbb{R}/\mathbb{Z}) \to H^p_D(M,\mathbb{Z})^\star \to Hom(\Omega^p(M)/\Omega^p_{\mathbb{Z}}(M),\mathbb{R}/\mathbb{Z}) \to 0\,. \tag{2.54}$$

To establish these two exact sequences one uses:

$$Hom(\check{H}^p(M,\mathbb{R}/\mathbb{Z}),\mathbb{R}/\mathbb{Z}) \cong \check{H}^{m-p}(M,\mathbb{Z})\,. \tag{2.55}$$

Let us have a closer look at the case $m=3$ and $p=1$ since it will be met later on. The exact sequences into which $H^1_D(M,\mathbb{Z})$ and $H^1_D(M,\mathbb{Z})^\star$ are embedded read:

$$0 \to \frac{\Omega^1_{\mathbb{Z}}(M)}{\Omega^1(M)} \;\to\; H^1_D(M,\mathbb{Z}) \to \check{H}^2(M,\mathbb{Z}) \to 0\,, \tag{2.56}$$

$$0 \to \check{H}^1(M,\mathbb{R}/\mathbb{Z}) \;\to\; H^1_D(M,\mathbb{Z}) \to \Omega^2_{\mathbb{Z}}(M) \to 0\,, \tag{2.57}$$

and:

$$0 \to Hom(\Omega^2_{\mathbb{Z}}(M),\mathbb{R}/\mathbb{Z}) \;\to\; H^1_D(M,\mathbb{Z})^\star \to \check{H}^2(M,\mathbb{Z}) \to 0\,, \tag{2.58}$$

$$0 \to \check{H}^1(M,\mathbb{R}/\mathbb{Z}) \;\to\; H^1_D(M,\mathbb{Z})^\star \to Hom(\frac{\Omega^1(M)}{\Omega^1_{\mathbb{Z}}(M)},\mathbb{R}/\mathbb{Z}) \to 0\,, \tag{2.59}$$

and the canonical injection (2.51) gives:

$$H^1_D(M,\mathbb{Z}) \subset H^1_D(M,\mathbb{Z})^\star \,. \tag{2.60}$$

This sheds some new light on the previous exact sequences since we have the following injections:

$$\Omega^2_{\mathbb{Z}}(M) \subset Hom(\frac{\Omega^1(M)}{\Omega^1_{\mathbb{Z}}(M)}, \mathbb{R}/\mathbb{Z}) \qquad , \qquad \frac{\Omega^1_{\mathbb{Z}}(M)}{\Omega^1(M)} \subset Hom(\Omega^2_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z}) \,, \tag{2.61}$$

which confirms the inclusion (2.60), using the "5 lemma".

We will see the full interest of inclusion (2.60) in the study and computation of link invariants within the framework of abelian Chern-Simons theories.

In order to conclude, and as announced at the beginning of this mathematical section, let us point out that there exists another way to introduce Deligne-Beilinson Cohomology. More precisely, S-S. Chern and J. Simons have shown that for any (compact) Lie group G there exist natural objects which trivialise (either locally on M or globally on a G-bundle over M) symmetric invariant polynomials in curvature forms [45]. The objects thus obtained are not forms on M, but Cheeger-Simons Differential Characters [20, 21, 19]. Note that this is done in a non abelian context, although at the end everything is abelian. More recently Harvey and Lawson have proposed an alternative description in terms of Sparks [22]. In this approach a closed form with integral periods is connected with a rectifiable current (one representing a cycle) on M. The class of the closed form would then be a Poincaré dual of the class of the cycle (or the current representing it). All these points of view are equivalent at the level of smooth manifolds and they all provide the same set of exact sequences as well as the same Pontrjagin dual spaces. The same holds true with Hopkins-Singer Differential Cohomology as shown by J. Simons and D. Sullivan [46].

3 Topological models

We are going to present two families of topological models strongly based on the two cohomologies previously introduced. We will only expose the main results obtained in the original articles. These articles can be found at the end of this review.

First, topological Yang-Mills in 3 dimensions, topological gravity in 2 and then 4 dimensions will be dealt with. For these models Equivariant Cohomology will be used, thus allowing to identify some observables of the corresponding theories which will appear as topological invariants: those of Donaldson in the case of Yang-Mills and of Mumford in the case of 2D gravity.

In a second step we will focus on the theory of links in a three dimensional space and more specifically on the role played by the abelian Chern-Simons field theory in the determination of link invariants. This is where Deligne-Beilinson Cohomology will appear as a very powerful tool, allowing to get links invariants in a purely geometrical way, without the use of any surgery technique, to which it provides an alternative.

Let us amusingly note that the use of Equivariant Cohomology will concern even-dimensional manifolds whereas the use of Beilinson-Deligne Cohomology will concern odd-dimensional ones.

3.1 Equivariant Observables

As in section 2.1 but with slightly different notations let us consider a manifold $\mathcal{M}$ on which a Lie group G is acting to the right. Let $\xi = (P, \mathcal{M}, H, \pi)$ be a principal bundle over $\mathcal{M}$ with structure group a compact Lie group H. In general, H has **nothing** to do with G. The Lie algebra of H is denoted $\mathcal{H}$, and $\mathcal{G}$ denotes the one of G . By construction H acts to the right on the total space P in such a way that $\mathcal{M} \cong P/H$. As done before, all derivatives will be indexed by the space on which they are defined.

Let Γ be a G-invariant H-connection on ξ:

$$\forall\lambda \in \mathcal{G}, l_P(\lambda)\Gamma = 0\,. \tag{3.62}$$

The pull-back $\widehat{\Gamma}$ of Γ on $\Omega^*(P)\otimes\mathcal{W}(G)$ is a 1-form on P and a 0-form in $\mathcal{W}(G)$. Consequently:

$$\forall\lambda \in \mathcal{G}, i_{\mathcal{W}}(\lambda)\widehat{\Gamma} = 0\,. \tag{3.63}$$

In $\Omega^*(P)\otimes\mathcal{W}(G)$, the equivariant curvature of $\widehat{\Gamma}$ is defined by (denoting ω and Ω the generators of $\mathcal{W}(G)$)

$$R^{eq}_K(\widehat{\Gamma}, \omega, \Omega) = D_K\widehat{\Gamma} + \frac{1}{2}[\widehat{\Gamma}, \widehat{\Gamma}]\,. \tag{3.64}$$

Therefore, if $\mathcal{I}_H$ is an H-invariant symmetric polynomial on $\mathcal{H}$ one can consider the H-characteristic class $\mathcal{I}^{eq}_{H,K}(\widehat{\Gamma}, \omega, \Omega) \equiv \mathcal{I}_H(R^{eq}_K(\widehat{\Gamma}, \omega, \Omega))$. It is well-defined on $\mathcal{M}$ and satisfies:

$$\begin{aligned} D_K\mathcal{I}^{eq}_{H,K}(\widehat{\Gamma}, \omega, \Omega) &= 0 \\ I_K(\lambda)\mathcal{I}^{eq}_{H,K}(\widehat{\Gamma}, \omega, \Omega) &= 0 \\ L_K(\lambda)\mathcal{I}^{eq}_{H,K}(\widehat{\Gamma}, \omega, \Omega) &= 0\,, \end{aligned} \tag{3.65}$$

where all derivatives appearing here were defined by (2.21). We are in Kalkman scheme of the Equivariant Cohomology, as recalled by the index K. This shows why this scheme is so interesting: it allows to construct cohomology classes in a very simple way, only relying on constraint (3.63) whereas in Weil scheme one should have to consider a quantity $\widetilde{\Gamma}$ such that $(i_P + i_{\mathcal{W}})(\lambda)\widetilde{\Gamma} = 0$. Of course, once an Equivariant Cohomology class has been identified in Kalkman scheme, one can switch to Weil scheme by the use of (2.20), thus getting:

$$\mathcal{I}^{eq}_{H,K}(\widehat{\Gamma}, \omega, \Omega) \to \mathcal{I}^{eq}_{H,W}(\widehat{\Gamma}, \omega, \Omega) = \mathcal{I}_H(R^{eq}_W(\widehat{\Gamma}, \omega, \Omega))\,. \tag{3.66}$$

with

$$\begin{aligned} R^{eq}_W(\widehat{\Gamma}, \omega, \Omega) &= \exp\{-i_P(\omega)\} R^{eq}_K(\widehat{\Gamma}, \omega, \Omega) \\ &= (d_P + d_{\mathcal{W}})(\widehat{\Gamma} + i_P(\omega)\widehat{\Gamma}) + \frac{1}{2}[\widehat{\Gamma} + i_P(\omega)\widehat{\Gamma}, \widehat{\Gamma} + i_P(\omega)\widehat{\Gamma}]\,. \end{aligned} \tag{3.67}$$

When $\mathcal{M}$ can be endowed with a G-connection θ, with curvature Θ, one can respectively replace ω and Ω by θ and Θ in $\mathcal{I}^{eq}_{H,K}(\widehat{\Gamma}, \omega, \Omega)$ in such a way that equations (3.65) become:

$$\begin{aligned} d_{\mathcal{M}}\mathcal{I}^{eq}_{H,W}(\widehat{\Gamma}, \theta, \Theta) &= 0 \\ i_{\mathcal{M}}(\lambda)\mathcal{I}^{eq}_{H,W}(\widehat{\Gamma}, \theta, \Theta) &= 0 \\ l_{\mathcal{M}}(\lambda)\mathcal{I}^{eq}_{H,W}(\widehat{\Gamma}, \theta, \Theta) &= 0\,, \end{aligned} \tag{3.68}$$

for any $\lambda \in \mathcal{G}$. The cohomology classes thus obtained are remarkably independent of $\widehat{\Gamma}$ and θ. Once the basic forms $\mathcal{I}^{eq}_{H,W}(\widehat{\Gamma}, \theta, \Theta)$ have been obtained, one knows that they uniquely define forms on $\mathcal{M}/G$ which can be integrated over cycles on this space, thus obtaining G-invariant quantities. We will now take some example to make all this clearer. But before this let us make some remarks. We do no need any action (or lagrangian) in order to determine equivariant observables. We only need the structure equations of the topological model. In fact, it is through these structure equations that the topological model will be identified as such. They typically read:

$$\begin{aligned} s^{top}\phi &= \psi + L^{top}(\omega)\phi \\ s^{top}\psi &= -L^{top}(\Omega)\phi + L^{top}(\omega)\psi \\ s^{top}\omega &= \Omega - \frac{1}{2}[\omega, \omega] \\ s^{top}\Omega &= [\Omega, \omega]\,, \end{aligned} \tag{3.69}$$

where ϕ is the fundamental field of the theory (a connection, a metric, etc.). The expression of the BRST operator s^{top} will depend on the chosen scheme (Kalkman or Weil) in which the topological model is described.

3.1.1 Topological Yang-Mills in 3 dimensions and Donaldson invariants

In this topological model one considers a four dimensional manifold B^4 seen as a euclidian version of a space-time manifold. Then let $\xi = (P, B^4, H, \pi)$ be a principal bundle over B^4

where H is a compact Lie group (most of the time $H = SU(N)$ or $SO(N)$). The manifold $\mathcal{M}$ of the previous paragraph is now taken as $\mathcal{A}$, the affine space of H-connections on ξ, while G is the group of vertical automorphisms of P. Therefore, and unlike the general case, there is a link between H and G since this last group appears as the gauge group of the structure group H of ξ. As before, $\mathcal{H}$ and $\mathcal{G}$ will denote the Lie algebras of H and G, respectively. Furthermore, and also unlike the general case, there is a natural action (or lagrangian) associated with this topological model, which is:

$$S_{YM^4_{top}} = \int_{B^4} Tr[F \wedge F]\,, \tag{3.70}$$

where F is the curvature of a H-connection on B^4 (or equivalently on ξ). Up to some normalisation factor, the topological lagrangian $Tr[F \wedge F]$ identifies with the second Chern class of the bundle ξ. Accordingly, the action takes its values in $\mathbb{Z}$.

The Weil algebra $\mathcal{W}(G)$ can be nicely realised with the use of a copy $\widetilde{\mathcal{A}}$ of $\mathcal{A}$ endowed with a connection $\widetilde{\omega}$ and its curvature $\widetilde{\Omega}$ both playing the role of generators of $\mathcal{W}(G)$. The fundamental fields of this model are therefore: $a \in \mathcal{A}$, $d_{\mathcal{A}}a$, $\widetilde{\omega}$ and $\widetilde{\Omega}$. The structure equations read:

$$\begin{aligned} s^{top}a &= \psi + l_{\mathcal{A}}(\widetilde{\omega})a = \psi + L^{top}(\widetilde{\omega})a = \psi - \nabla_a\widetilde{\omega} \\ s^{top}\psi &= -L^{top}(\widetilde{\Omega})a + L^{top}(\widetilde{\omega})\psi = -\nabla_a\widetilde{\Omega} + [\psi,\widetilde{\omega}] \\ s^{top}\widetilde{\omega} &= \widetilde{\Omega} - \frac{1}{2}[\widetilde{\omega},\widetilde{\omega}] \\ s^{top}\widetilde{\Omega} &= [\widetilde{\Omega},\widetilde{\omega}]\,, \end{aligned} \tag{3.71}$$

where:

$$\begin{aligned} s^{top} &= d_{\widetilde{\mathcal{A}}} + d_{\mathcal{A}} + l_{\mathcal{A}}(\widetilde{\omega}) - i_{\mathcal{A}}(\widetilde{\Omega}) \\ \psi &= d_{\mathcal{A}}a \;=\; \psi_K\,, \end{aligned} \tag{3.72}$$

in Kalkman scheme, and:

$$\begin{aligned} s^{top} &= d_{\widetilde{\mathcal{A}}} + d_{\mathcal{A}} \\ \psi &= d_{\mathcal{A}}a - l_{\mathcal{A}}(\widetilde{\omega})a \;=\; \psi_W\,, \end{aligned} \tag{3.73}$$

in Weil scheme. In both schemes $L^{top} = l_{\widetilde{\mathcal{A}}} + l_{\mathcal{A}}$. In (3.72) (resp. (3.73)) one recognizes Kalkman (resp. Weil) differential. To go from one scheme to the other we naturally use the equivalent of (2.20):

$$\psi_K = \exp\{-i_{\mathcal{A}}(\widetilde{\omega})\}\psi_W\,. \tag{3.74}$$

For any $\lambda \in \mathcal{G}$

$$I^{top}(\lambda)\,\phi = \left| \begin{array}{ll} \lambda & \text{if } \phi = \tilde{\omega} \\ 0 & \text{otherwise}\,, \end{array} \right. \tag{3.75}$$

where $I^{top} = i_{\widetilde{\mathcal{A}}}$ (resp. $I^{top} = i_{\widetilde{\mathcal{A}}} + i_{\mathcal{A}}$) in Kalkman (resp. Weil) scheme. Note that equations in (3.71) are of BRST type and that their "form" is independent of the chosen scheme.

In order to obtain observables, we now consider $\mathcal{M} = \mathcal{A} \times B^4$, together with the principal bundle over $\mathcal{M}$ whose total space is $\mathcal{Q} = \mathcal{A} \times P$ and structure group H, the same as P. Let us recall that on the other hand G acts to the right on $\mathcal{A}$ and to the left on P. Therefore, G acts accordingly on $\mathcal{Q}$. We construct a G-invariant H-connection $\widehat{\Gamma}$ on $\mathcal{Q}$ in the following way: for any $a \in \mathcal{A}$ one considers the H-bundle with total space P over B^4 endowed with the H-connection a. This gives rise to a collection $\widehat{a}$ of H-connections such that:

$$\forall (a,p) \in \mathcal{Q}\,, \quad \widehat{a}(a,p) = a(p)\,. \tag{3.76}$$

The H-connection thus generated $\mathcal{Q}$ can be extended to $\widetilde{\mathcal{A}} \times \mathcal{Q}$. We refer the reader to [33] for details. The fundamental vector field associated with the action of $\lambda \in \mathcal{G}$ takes the following form:

$$\widetilde{\lambda} = l_P(\widetilde{\lambda}_P) a_\mu \frac{\delta}{\delta a_\mu} - \widetilde{\lambda}_P^\alpha e_\alpha\,, \tag{3.77}$$

where $\widetilde{\lambda}_P$ is the fundamental vector field associated with the action of λ on P, and e_α is the fundamental vector field associated with a basis $(T_\alpha)_\alpha$ of $\mathcal{H}$. Since $\widehat{a}$ does not depend on $\widetilde{\mathcal{A}}$, we have:

$$\forall \lambda \in \mathcal{G}\,, \quad (i_{\widetilde{\mathcal{A}}} + i_{\mathcal{Q}})(\lambda)\widehat{a} = -i_P(\lambda)\widehat{a} = -\overline{\lambda}\,, \tag{3.78}$$

where the sign in front of $\overline{\lambda}$ comes from the change of the right action to a left action of $\mathcal{G}$ on P. the $\mathcal{H}$-valued function $\overline{\lambda}$ on $\mathcal{Q}$ is defined by:

$$\forall (a,p) \in \mathcal{Q}\,, \quad \overline{\lambda}(a,p) = \lambda(p)\,. \tag{3.79}$$

Let us recall that for $\lambda \in \mathcal{G}$: $\lambda(p) \in \mathcal{H}$ for any $p \in P$.

In the same way we have:

$$\begin{aligned} \forall \lambda \in \mathcal{G}\,, \quad (l_{\widetilde{\mathcal{A}}} + l_{\mathcal{Q}})(\lambda)\widehat{a} &= l_{\widetilde{\mathcal{A}}}\left(l_P(\widetilde{\lambda}_P) a_\mu \frac{\delta}{\delta a_\mu}\right)\widehat{a} - l_P(\widetilde{\lambda}_P)\widehat{a} \\ &= l_P(\widetilde{\lambda}_P)\widehat{a} - l_P(\widetilde{\lambda}_P)\widehat{a} \\ &= 0\,. \end{aligned} \tag{3.80}$$

The connection $\widehat{a}$ is clearly G-invariant, so one can apply the general construction of section 3.1. The equivariant curvature thus obtained reads:

$$F_K^{eq}(\widehat{a}, \widetilde{\omega}, \widetilde{\Omega}) = D_K \widehat{a} + \frac{1}{2}[\widehat{a}, \widehat{a}]_H\,, \tag{3.81}$$

where

$$D_K = d_{\widetilde{\mathcal{A}}} + (d_{\mathcal{A}} + d_P) + (l_{\mathcal{A}} + l_P)(\widetilde{\omega}) - (i_{\mathcal{A}} + i_P)(\widetilde{\Omega})\,. \tag{3.82}$$

Finally, taking into account the G-invariance of $\widehat{a}$, one concludes that the equivariant curvature in Kalkman scheme can be written:

$$F_K^{eq}(\widehat{a},\widetilde{\omega},\widetilde{\Omega}) = \widehat{F}(\widehat{a}) + d_{\mathcal{A}}\widehat{a} + i_P(\widetilde{\Omega})\widehat{a} = \widehat{F}(\widehat{a}) + \widetilde{\psi}_K + \widehat{\widetilde{\Omega}}\,, \tag{3.83}$$

with $\widehat{F}(\widehat{a}) = d_P\widehat{a} + 1/2[\widehat{a},\widehat{a}]_H$.

Let us leave aside computational details needed to go to Weil scheme, and let us directly consider an H-invariant symmetric polynomial I_H which generates the (automatically) equivariant form: $I_{H,W}^{eq}(\widehat{a},\widetilde{\omega},\widetilde{\Omega}) = I_H(F_K^{eq}(\widehat{a},\widetilde{\omega},\widetilde{\Omega}))$. One can eventually substitute to $\widetilde{\omega}$ and $\widetilde{\Omega}$ a connection ω and its curvature Ω, defined on $\mathcal{A}$, in such a way that any reference to $\widetilde{A}$ disappears. Equivariant constraints reduce to:

$$\begin{aligned} (d_{\mathcal{A}} + d_{B^4})\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega) &= 0 \\ (i_{\mathcal{A}} + i_{B^4})(\lambda)\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega) &= 0 \\ (l_{\mathcal{A}} + l_{B^4})(\lambda)\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega) &= 0\,, \end{aligned} \tag{3.84}$$

for any $\lambda \in \mathcal{G}$. The fact that P has been replaced by B^4 in (3.84) is due to the nature of I_H. Indeed, $\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega)$ satisfies even stronger constraints than those of the Weil scheme. Indeed, we have:

$$\forall\varepsilon \in \mathcal{G}\,,\quad i_{\mathcal{A}}(\varepsilon)\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega) = 0 = i_P(\varepsilon)\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega) = i_{B^4}(\varepsilon)\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega)\,, \tag{3.85}$$

which implies that $\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega)$ is a well-defined form on $\mathcal{M} = \mathcal{A}\times B^4$. We can decompose this form according to:

$$\mathcal{I}_{H,W}^{eq}(\widehat{a},\omega,\Omega) = \sum_{k=0}^{2n} \mathcal{I}^{(k,2n-k)}\,, \tag{3.86}$$

where each term $\mathcal{I}^{(k,2n-k)}$ is a k-form on $\mathcal{A}$ and a $(2n-k)$-form on B^4, such that the following recursive relations $(\forall\lambda)$:

$$\begin{aligned} d_{\mathcal{A}}\mathcal{I}^{(k-1,2n-k+1)} + d_{B^4}\mathcal{I}^{(k,2n-k)} &= 0 \\ i_{\mathcal{A}}(\lambda)\mathcal{I}^{(k+1,2n-k-1)} - i_{B^4}(\lambda)\mathcal{I}^{(k,2n-k)} &= 0 \\ l_{\mathcal{A}}(\lambda)\mathcal{I}^{(k,2n-k)} - l_{B^4}(\lambda)\mathcal{I}^{(k,2n-k)} &= 0\,, \end{aligned} \tag{3.87}$$

are fulfilled. Finally, one "eliminates" B^4 by integrating $\mathcal{I}^{(k,2n-k)}$ over a $(2n-k)$-cycle γ_{2n-k} in B^4, thus providing:

$$\mathcal{O}^k = \int_{\gamma_{2n-k}} \mathcal{I}^{(k,2n-k)}\,. \tag{3.88}$$

From (3.87), these quantities verify:

$$d_{\mathcal{A}}\mathcal{O}^k = 0 \quad,\quad i_{\mathcal{A}}\mathcal{O}^k = 0 \quad,\quad l_{\mathcal{A}}\mathcal{O}^k = 0\,. \tag{3.89}$$

Thus, we have obtained basic cohomology classes. It can be shown that these classes coincide with the 4-dimensional Donaldson invariants.

We have presented here a somewhat detailed construction so that the reader could become more familiar with notations and techniques used. In the forthcoming examples we will only give the broad lines, referring the reader to the original articles for details.

3.1.2 Topological Gravity in 2 dimensions and Mumford invariants

In this second example, we replace the previous manifold B^4 by a closed (*i.e.* compact and without boundary) Riemann surface Σ_g^0 of genus $g > 1$. To avoid confusion we will denote Σ_g the smooth 2-dimensional manifold on which this Riemann surface Σ_g^0 is build. With respect to $\mathcal{C}^\infty$ structures, closed surfaces are classified by their genus. However for a given genus there are many inequivalent (with respect to conformal equivalence) Riemann surfaces. The space of admissible conformal structures on Σ_g identifies with $\mathcal{B}(\Sigma_g^0)$, the space of Beltrami differentials on Σ_g^0, this Riemann surface being seen as an origin. Thus, this identification is not canonical since it depends on the origin Σ_g^0. Nevertheless, changing Σ_g^0 gives an isomorphic representation of the space of admissible conformal structures on Σ_g. Let us remind quickly how $\mathcal{B}(\Sigma_g^0)$ is built. If $\{U_\alpha, (z_\alpha, \overline{z}_\alpha)\}_{\alpha\in I}$ is a complex atlas that defines the conformal structure of $\mathcal{B}(\Sigma_g^0)$, then any other admissible conformal structure on Σ_g is given by some complex coordinates $(Z_\alpha^{(\mu)}, \overline{Z}_\alpha^{(\mu)})_{\alpha\in I}$ satisfying the Beltrami equation:

$$\left(\partial_{\overline{z}_\alpha} - \mu_{\overline{z}_\alpha}^{z_\alpha}\partial_{z_\alpha}\right) Z_\alpha^{(\mu)} = 0\,. \tag{3.90}$$

The collection made of the $\mu_{\overline{z}_\alpha}^{z_\alpha}$'s appearing in this equation defines a vector field valued 1-form on Σ_g^0: $\mu = \mu_{\overline{z}_\alpha}^{z_\alpha} d\overline{z}_\alpha \otimes \partial_{z_\alpha}$, named a Beltrami differential on Σ_g^0. The set of Beltrami differentials on Σ_g^0 will be denoted by $\mathcal{B}(\Sigma_g^0)$. It can be seen as the space of generators of admissible conformal structures on Σ_g starting from the one of Σ_g^0. However, $\mathcal{B}(\Sigma_g^0)$ does not provide a one-to-one identification of admissible conformal structures on Σ_g.

Let μ_1 and μ_2 be two Beltrami differentials on Σ_g^0. If $(Z_\alpha^{(\mu_1)}, \overline{Z}_\alpha^{(\mu_1)})_{\alpha\in I}$ and $(Z_\alpha^{(\mu_2)}, \overline{Z}_\alpha^{(\mu_2)})_{\alpha\in I}$ are the two conformal structures they define via (3.90), let us denote $\Sigma_g^{\mu_1}$ and $\Sigma_g^{\mu_2}$ the Riemann surfaces thus generated. Note that there exists diffeomorphisms $\phi_{\mu_1} : \Sigma_g^0 \to \Sigma_g^{\mu_1}$ and $\phi_{\mu_2} : \Sigma_g^0 \to \Sigma_g^{\mu_2}$ since these surfaces have the same genus (they all "come from" Σ_g). Now, if there exist $\varphi \in Diff_0(\Sigma_g^0)$ (the connected component to the identity of the group of diffeomorphisms of Σ_g^0) and a **conformal** map $\Phi_{\mu_1\mu_2} : \Sigma_g^{\mu_1} \to \Sigma_g^{\mu_2}$, one says that μ_1 and μ_2 are **conformally equivalent**. The space generated by this equivalence relation between Beltrami differentials is called the Teichmüller space of Σ_g^0. Formally:

$$\mathcal{T}(\Sigma_g^0) = \frac{\mathcal{B}(\Sigma_g^0)}{Diff_0(\Sigma_g^0)}\,. \tag{3.91}$$

As for $\mathcal{B}(\Sigma_g^0)$, for a fixed genus all Teichmüller spaces are isomorphic, even if they depend on Σ_g^0. The quotient defining Teichmüller spaces is built from the natural action of the infinite dimensional Lie group $Diff_0(\Sigma_g^0)$ on $\mathcal{B}(\Sigma_g^0)$. This quite obviously suggests what to do in order to construct equivariant observables. Let us simply say that to obtain them, one will have to follow the same reasoning as in the previous example: one introduces a copy of $\mathcal{B}(\Sigma_g^0)$, provided with a connection $\widetilde{\omega}$ and its curvature $\widetilde{\Omega}$, the fundamental fields for this model being μ, $\widetilde{\omega}$ and $\widetilde{\Omega}$. The action of $Diff_0(\Sigma_g^0)$ on these fields is then given by the following structure equations:

$$\begin{aligned}
s^{top}\mu &= \nu + L^{top}(\widetilde{\omega})\mu = \nu - l_{\mathcal{B}}(\widetilde{\omega})\mu = \nu - \overline{\partial}_\mu - \{\mu, \widetilde{\omega}_\mu\} = \nu - \overline{D}_\mu \widetilde{\omega}_\mu \\
s^{top}\nu &= -L^{top}(\widetilde{\Omega})\mu + L^{top}(\widetilde{\omega})\nu = \overline{D}_\mu \widetilde{\Omega}_\mu - \{\nu, \widetilde{\omega}_\mu\} \\
s^{top}\widetilde{\omega}_\mu &= \widetilde{\Omega}_\mu 1\frac{1}{2}\{\widetilde{\omega}_\mu, \widetilde{\omega}_\mu\} \\
s^{top}\widetilde{\Omega}_\mu &= -\{\widetilde{\Omega}_\mu, \widetilde{\omega}_\mu\}\,,
\end{aligned} \tag{3.92}$$

where $\widetilde{\omega}_\mu = \left(\widetilde{\omega}^z + \mu_{\overline{z}}^{z}\widetilde{\omega}^{\overline{z}}\right)$ and $\widetilde{\Omega}_\mu = \left(\widetilde{\omega}^z + \mu_{\overline{z}}^{z}\widetilde{\Omega}^{\overline{z}} + \nu_{\overline{z}}^{z}\widetilde{\omega}^{\overline{z}}\right)$. We can already remark the similarity between (3.92) and (3.71). Of course, s^{top} and ν have an expression which depends on the chosen scheme.

Let us recall that when $g > 1$, the Gauss curvature of the corresponding surfaces can always be normalized to -1. One considers the trivial bundle $\mathcal{M} = \mathcal{B}(\Sigma_g^0) \times \Sigma_g$ endowed with the complex structure defined by μ et Z^μ, where Z^μ shortly denotes the coordinates $(Z_\alpha^{(\mu)}, \overline{Z}_\alpha^{(\mu)})_{\alpha\in I}$ previously met. Thus, over $\mu \in \mathcal{B}(\Sigma_g^0)$ one finds the Riemann surface Σ_g^μ and over $\mu = 0$ the "original" Riemann surface Σ_g^0. For any $\mu \in \mathcal{B}(\Sigma_g^0)$ one considers the holomorphic tangent bundle of Σ_g^μ, thus obtaining a family $T_\mu^{(1,0)}(\Sigma_g)$. One then goes to the associated $GL(1,\mathbb{C})$-principal bundle, $\mathcal{P}T_\mu^{(1,0)}(\Sigma_g)$, which plays the role of the space $\mathcal{Q}$ met in Topological Yang-Mills. A set of holomorphic coordinates on $\mathcal{P}T_\mu^{(1,0)}(\Sigma_g)$ is then locally given by μ, Z^μ and $E^{Z^\mu} \in GL(1,\mathbb{C})$.

For any $\mu \in \mathcal{B}(\Sigma_g^0)$ one provides Σ_g^μ with the metric $ds_\mu^2 = \rho_{Z^\mu\overline{Z}^\mu} dZ^\mu d\overline{Z}^\mu$ where $\rho_{Z^\mu\overline{Z}^\mu}$ satisfies:

$$\partial_{Z^\mu}\partial_{\overline{Z}^\mu} \ln(\rho_{Z^\mu\overline{Z}^\mu}) = \rho_{Z^\mu\overline{Z}^\mu}\,. \tag{3.93}$$

This is nothing but saying that the Gauss curvature ofe Σ_g is -1, as already mentioned. A $Diff_0(\Sigma_g^0)$-invariant $GL(1,\mathbb{C})$-connection quite naturally shows off:

$$\widehat{\Gamma} = \mathcal{D}\ln(\rho_{Z^\mu\overline{Z}^\mu}) + D\ln(E^{Z^\mu})\,, \tag{3.94}$$

with $\mathcal{D}$ the type $(1,0)$ of the total differentialacting on $\mathcal{P}T_\mu^{(1,0)}(\Sigma_g)$, $D = \mathcal{D} + \overline{\mathcal{D}}$, and $D\ln(E^{Z^\mu})$ is the Maurer-Cartan form on $GL(1,\mathbb{C})$. Once more we refer the reader to the original articles for all the details.

Once the connection $\widehat{\Gamma}$ has been identified, one can apply in extenso the general method. This leads to the introduction of the Kalkman equivariant curvature of $\widehat{\Gamma}$, before switching to Weil scheme. After having eliminated the copy of $\mathcal{B}(\Sigma_g^0)$, replacing $\widetilde{\omega}$ and $\widetilde{\Omega}$ by a connection θ with curvature Θ on $\mathcal{B}(\Sigma_g^0)$, the Weil equivariant curvature decomposes according to:

$$R_W^{eq}(\widehat{\Gamma}, \theta, \Theta) = R^{(2,0)} + R^{(1,1)} + R^{(0,2)}\,, \tag{3.95}$$

where the first index denotes a form degree on $\mathcal{B}(\Sigma_g^0)$ and the second a form degree on Σ_g^0. One then generates equivariant observables of the topological model by taking powers of $R \equiv R_W^{eq}$, that is to say:

$$\begin{aligned} R^n &= (R^{(2,0)})^n + n(R^{(2,0)})^{n-1}R^{(1,1)} + \left(n(R^{(2,0)})^{n-1}R^{(0,2)} + \frac{n(n-1)}{2}(R^{(2,0)})^{n-1}(R^{(1,1)})^2\right) \\ &= \mathcal{O}^{(2n,0)} + \mathcal{O}^{(2n-1,1)} + \mathcal{O}^{(2n-2,2)} \,. \end{aligned} \tag{3.96}$$

It is the dimension of Σ_g^0 that produces the truncation in degree. There is a fundamental difference with Topological Yang-Mills: here the gauge group is $Diff_0(\Sigma_g^0)$, and it does act on Σ_g^0, whereas the gauge group $G = Aut(P)$ reduces to the identity on B^4, the base space of P. Accordingly, if one integrates the different terms occurring in (3.96) on cycles of Σ_g^0 with the hope to get $Diff_0(\Sigma_g^0)$-invariants, one is immediately face with the non invariance of 0-cycles and 1-cycles under the action of $Diff_0(\Sigma_g^0)$. Only Σ_g^0 itself (and its multiples) is $Diff_0(\Sigma_g^0)$-invariant. This reduces the topological invariants of 2D Gravity to:

$$\mathcal{O}^{2n-2} = \int_{\Sigma_g^0} \mathcal{O}^{(2n-2,2)} \,. \tag{3.97}$$

These observables coincide with Mumford invariants.

3.1.3 Topological Gravity in 4 dimensions

In the previous example we have chosen to use Beltrami differentials as fundamental fields of the topological 2D Gravity. This was natural because we were dealing with Riemann surfaces. Yet, we also use metrics on these Riemann surfaces during the procedure leading to the equivariant observables. This suggest another way to treat 2D Gravity, based on metrics from the beginning. In fact there is a description of Teichmüller spaces in term of metrics given by:

$$\mathcal{T}(\Sigma_g) = \frac{\mathfrak{Met}(\Sigma_g)}{Diff_0(\Sigma_g) \ltimes \mathfrak{Weyl}(\Sigma_g)} \,. \tag{3.98}$$

In (3.98), $\mathfrak{Met}(\Sigma_g)$ denotes the space of metrics on Σ_g, $\mathfrak{Weyl}(\Sigma_g)$ the group of Weyl transformations on $\mathfrak{Met}(\Sigma_g)$, and $\ltimes$ the semi-direct product corresponding to the obvious action of $Diff_0(\Sigma_g)$ on $\mathfrak{Weyl}(\Sigma_g)$. Instead of presenting the construction and the computations leading to the equivariant observables in this metric approach, we will rather show how metrics can be used to provide topological invariants in the framework of (euclidian) 4D Gravity.

Let B^4 be closed four dimensional manifold. The topological model is now defined by the following structure equations:

$$\begin{aligned} s^{top}g &= \psi + L^{top}(\widetilde{\omega})g \\ s^{top}\psi &= -L^{top}(\widetilde{\Omega})g + L^{top}(\widetilde{\omega})\psi \\ s^{top}\widetilde{\omega} &= \widetilde{\Omega} - \frac{1}{2}[\widetilde{\omega},\widetilde{\omega}] \\ s^{top}\widetilde{\Omega} &= [\widetilde{\Omega},\widetilde{\omega}] \,, \end{aligned} \tag{3.99}$$

where $g \in \mathfrak{Met}(B^4)$, $\widetilde{\omega}$ is a connection on a copy of $\mathfrak{Weyl}(\Sigma_g)$ with curvature $\widetilde{\Omega}$. As already done, one considers the fiber bundle $\mathcal{Q} = \mathfrak{Met}((B^4) \times \mathcal{R}((B^4)$, $\mathcal{R}((B^4)$ being the canonical frame bundle of B^4. This last bundle is well-known to be a principal bundle over B^4 with structure group $GL(4,\mathbb{R})$. The required $Diff_0(B^4)$-invariant $GL(4,\mathbb{R})$-connection for the construction is the equivalent of 3.94), that is to say:

$$\widehat{\Gamma} = \Gamma_g^{LC} + \frac{1}{2} g^{-1} \delta g \,, \tag{3.100}$$

with Γ_g^{LC} the Levi-Civita connection associated with g, and δ the exterior derivative on $\mathfrak{Met}(B^4)$.

With $\widehat{\Gamma}$ one gets the equivariant curvature within Kalkman scheme, then switch to Weil scheme thus obtaining the equivariant curvature $R_W^{eq}(\widehat{\Gamma}, \widetilde{\omega}, \widetilde{\Omega})$.

The last step consists in identifying $GL(4,\mathbb{R})$-invariant symmetric polynomials which turn out to define Euler and Pontrjagin class of B^4. Actually, the Pontrjagin class is sufficient. Finally one eliminates the copy of $\mathfrak{Met}(B^4)$ by replacing $\widetilde{\omega}$ and $\widetilde{\Omega}$ by θ and Θ, a connection and its curvature on $\mathfrak{Met}(B^4)$. Eventually, one gets:

$$\begin{aligned} E_W^{eq} &= Q^{(4,0)} + Q^{(3,1)} + Q^{(2,2)} + Q^{(1,3)} + Q^{(0,4)} \\ P_W^{eq} &= G^{(4,0)} + G^{(3,1)} + G^{(2,2)} + G^{(1,3)} + G^{(0,4)} \,, \end{aligned} \tag{3.101}$$

where the first index is a form degree on $\mathfrak{Met}(B^4)$ and the second on B^4. The relevant observables are obtained from $(E_W^{eq})^m (P_W^{eq})^n$ once one truncates by the dimension of B^4, that is to say:

$$(E_W^{eq})^m (P_W^{eq})^n = V^{(4m+4n,0)} + V^{(4m+4n-1,1)} + V^{(4m+4n-2,2)} + V^{(4m+4n-3,3)} + V^{(4m+4n-4,4)} \,. \tag{3.102}$$

Detailed expressions for $V^{(4m+4n-k,k)}$ can be found in the original article [34]. As in the two dimensional case, the gauge group $Diff_0(B^4)$ acts non trivially on B^4, which implies that the only invariant cycle is B^4 itself (and multiples of it). This gives for equivariant observables of this topological model:

$$\mathcal{V}^{2n-2} = \int_{B^4} V^{(4m-4n-4,4)} \,. \tag{3.103}$$

The techniques presented below can be applied to any even dimensional closed manifold. In the 2-dimensional case one can wonder whether the equivariant observables obtained using Beltrami differentials are the same as those obtained from metrics. To our knowledge, there is no answer to that last question.

3.1.4 Representatives of the Thom Class of a vector bundle

As a last example of the use of Equivariant Cohomology, let us show how it provides other interesting mathematical quantities.

An important role is played by the so-called Thom class a of vector bundle. One can see such a class as the Poincaré dual of the zero section of some vector bundle. A famous representative of the Thom class is provided by the Mathaï-Quillen form. However, the construction we have previously used to get manifold invariants, based on Equivariant Cohomology, can be applied in the context of vector bundles thus generating a large family of new representatives of the Thom class of vector bundles. We will give a feeling of this without going into details, which can be found in the original article [4], as usual.

Let $\zeta(E, M, V, \pi)$ be a vector bundle over a smooth closed m-dimensional manifold M, with total space E, typical fiber a linear space V of dimension $n = 2k$, and of projection π. One usually says that ζ is a rank n vector bundle over M. We denote by Ω^n_{rdv} the space of n-forms on E with fast decrease along the fibers of ζ. Working with these forms prevents divergencies at infinity when integrating along the fibers. The associated (de Rham) cohomology space is denoted H^n_{rdv}. The Thom class of ζ (or equivalently of E) is the cohomology class $\mathcal{T}(E) \in H^n_{rdv}$ for which any of its representative τ^n_E satisfies:

$$\int_V \tau^n_E = 1\,. \tag{3.104}$$

This means that the integral of τ^n_E along the fibers of ζ gives rise to the constant function 1 on M. Note that in (3.104) the integration is done over V whereas τ^n_E lives on E, so it has to be understood as a simple notation standing for "integration along the fibers".

Our aim is then to generate representative of $\mathcal{T}(E)$ with the use of Equivariant Cohomology. To achieve this purpose, let us first provide V with a hermitian product $(\ ,\ )_V$ from which one selects an orthogonal basis of V, $\mathfrak{B} = \{\vec{e}_p\}_{p=1,\dots,n}$:

$$(\vec{e}_p, \vec{e}_q) = \delta_{pq}\,. \tag{3.105}$$

One decomposes any vector of V according to:

$$\vec{v} = \sum_{p=1}^{n} v^p \vec{e}_p = \sum_{p=1}^{n} (\vec{e}_p, \vec{v})\vec{e}_p\,. \tag{3.106}$$

Such a decomposition provides a coordinates system $(v^p)_{p=1,\dots,n}$ on V, subordinated to $\mathfrak{B}$, turning V into a smooth manifold. Let V^* be the (algebraic) dual of V. We provide V^* with the dual basis $\mathfrak{B}^* = \{\vec{e}^*_p\}_{p=1,\dots,n}$ as well as with the hermitian product $(\ ,\ )_{V^*}$ that dualises $(\ ,\ )_V$ in such a way that: $\vec{e}^*_p(\vec{e}_q) = (\vec{e}_p, \vec{e}_q)_V = (\vec{e}^*_p, \vec{e}^*_q)_{V^*} = \delta_{pq}$. Finally, one introduces the "coordinates" $(\varpi_p)_{p=1,\dots,n}$ for the Grassmann algebra ΛV^*, together with the derivatives δ, I and L, dual to those of V

We already mentioned that the Thom class of a vector bundle $\zeta(E, M, V, \pi)$ can be seen as the Poincaré dual of $s_0(E)$, the image of M in E by the zero section s_0 of ζ. With our notations, an obvious representative of this Poincaré dual is provided by the Dirac distribution according to:

$$\delta(\vec{v})dv^1 \wedge dv^2 \wedge\ \dots \wedge dv^n\,. \tag{3.107}$$

One can write this current as a Fourier transform and then use the topological BRST operator s^{top}, whose expression depends on the scheme one wants to work in. One obtains:

$$U = \frac{1}{(2\pi)^n} \int dbd\varpi \exp\{is^{top}(\varpi.\vec{v}) + i(\varpi, b)_{V^*}\}\,, \tag{3.108}$$

which is an equivariant cohomology class.

In order to get new representatives, one considers the tangent bundle TV and then the associated frame bundle $R(V)$. This last one is canonically a $GL(n,\mathbb{R})$-principal bundle over V. One endows $R(V)$ with local coordinates coming from those of V and some of $GL(n,\mathbb{R})$. The last ingredient of the construction is provided by the isometry group of $(\ ,\ )_V$: $SO(n)$, and by $\xi(P, M, SO(n), \overline{\pi})$, a $SO(n)$-principal bundle over M. The total bundle used to generate equivariant observables is then $\mathcal{Q} = \xi \times R(V)$ (or $P \times R(V)$). Let us note that, as in the case of the topological Yang-Mills model, there is a relation between the various Lie groups appearing in the construction since $SO(n) \subset GL(n,\mathbb{R})$. One must not confuse them.

In a now standard way, one endows $\mathcal{Q}$ with the $SO(n)$-invariant $GL(n,\mathbb{R})$-connection defined by:

$$\widehat{\Gamma} = b^{-1}(\Gamma^{LC}_g)b + b^{-1}d_R b\,, \tag{3.109}$$

where Γ^{LC}_g is the Levi-Civita connection defined by the metric g, itself defined by:

$$ds^2(\vec{v}) = e^{\varphi}((dv^p)^2 + \sigma(v^p dv^p)^2)\,, \tag{3.110}$$

where φ and σ are functions only depending on $(\vec{v},\vec{v})_V$. The metric g is the canonical $SO(n)$-invariant metric on V, therefore Γ^{LC}_g and $\widehat{\Gamma}$ are $SO(n)$-invariant, the latter being the lift on $\mathcal{Q}$ of the former. Eventually, one constructs equivariant curvatures first in Kalkman scheme (the most "natural"), then switch it to Weil scheme to get R^{eq}_W. This is injected into a symmetric $GL(n,\mathbb{R})$-invariant polynomial, which, as in the gravitational case, generates the Euler class:

$$E^{eq}_W = \frac{\epsilon^{\mu_1\rho_1\ldots\mu_d\rho_d}}{\sqrt{g}} g_{\rho_1\nu_1}\ldots g_{\rho_d\nu_d}(R^{eq}_W)^{\nu_1}_{\mu_1} \wedge \ldots \wedge (R^{eq}_W)^{\nu_d}_{\mu_d}\,. \tag{3.111}$$

Of course (3.111) depends on g that itself depends on φ and σ. Accordingly these two functions appear as parameters for a whole family of representatives of the Thom class $\mathcal{T}(E)$. In the original article where all this is detailed ([4]), it was shown that for $n = 2$ (*i.e.* $d = 1$) the Mathaï-Quillen representative belongs to the equivariant family.

This example concludes our presentation of the use of Equivariant Cohomology in some topological models. Of course this use is not systematic what ever the topological model is. Also, although the procedure used provides topological observables, we do not know if one can obtain them all this way. We send the reader to the end of this review for open questions.

3.2 Chern-Simons and links invariants

The link between Chern-Simons theories and invariants polynomials of knots started with a set of remarks made by E. Witten in [13].

Explicit perturbative computations for $SU(N)$ were originally performed by E. Guadagnini, M. Martellini and M. Mintchev in a Euclidean Quantum Field Theory framework ([28]). Beside the fact it is, by essence, perturbative, this approach uses a Euclidean metric in $\mathbb{R}^3$ all over the computations, although at the end one obtains isotopic invariants. To our knowledge, all computations already done are in agreement with the expansion of the corresponding polynomial. However all these computations are perturbative. Despite some attempts to define Quantum Field Theory over closed manifolds (see for instance [47]), nothing conclusive seems to have been achieved in the case of a Chern-Simons theory. As we will see, the true mathematical nature of this theory might be one of the reasons for this.

We will present an alternative point of view based on Deligne-Beilinson Cohomology. Historically, we studied the use of this Cohomology Theory within Quantum Field Theory in a totally independent way in [24]. It is only after the general considerations presented in this earlier work that it appeared that Chern-Simons could be a very good playground to apply DB Cohomology techniques. It was rather a surprise to see how this idea has proven so successful. Unfortunately the price to pay is to be in the abelian framework of the Chern-Simons theory. But we still think that there are many more benefits than drawbacks in using Deligne-Beilinson cohomology: non-perturbative treatment, all $(4l+3)$-dimensional closed manifolds treated, torsion taken into account, quantisation of all charges (k for the space or q for the loops), and some more not yet investigated similar properties usually obtained from surgery.

In this introductory section M will denote a smooth closed manifold of dimension 3. The general case of a closed smooth manifold of dimension $4l+3$ will be discussed in the last section. Also the reader is referred to the original articles [9, 35, 32] for details.

Let us consider $[\omega] \in H^1_D(M, \mathbb{Z})$, and write:

$$cs_1([\omega]) = [\omega] *_D [\omega] \in H^3_D(M, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}\,, \tag{3.112}$$

the DB square of this class. If one uses the Čech-de Rham technique to get representatives of DB classes, $cs_1([\omega])$ is made of 5 "components" the first of which is $\omega^{(0,1)} \wedge d\omega^{(0,1)}$, where $\omega^{(0,1)}$ is the highest component of some representative of $[\omega]$. One immediately identifies this highest component of $[\omega] *_D [\omega]$ with the abelian Chern-Simons lagrangian. Actually, and as we already mentioned in the mathematical introduction, any Chern-Simons lagrangian can be seen as a local representative of a DB class which is canonically associated with a second Chern class, even in the non-abelian case.

From now on, $cs_1([\omega])$ will be considered as the fundamental lagrangian of the abelian Chern-Simons theory. The level k Chern-Simons theory is described by the lagrangian:

$$cs_k([\omega]) = k cs_1([\omega]) = k[\omega] *_D [\omega]\,. \tag{3.113}$$

From a field theoretic point of view, k should rather be named the *coupling constant* of the theory.

If we want to interpret $cs_k([\omega])$ as a DB cohomology class, then:

$$cs_k([\omega]) \in H^3_D(M, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z} \Leftrightarrow k \in \mathbb{Z}\,, \tag{3.114}$$

thus implying a quantisation of the coupling constant k. This perfectly agrees with the fact that the Chern class associated with $cs_k([\omega])$ is integral if and only if $k \in \mathbb{Z}$.

The action of the abelian Chern-Simons theory is just defined as the integral of the lagrangian, that is to say:

$$CS_k([\omega]) = \int_M cs_k([\omega]) = k \int_M [\omega] *_D [\omega] \,. \tag{3.115}$$

One has $CS_k([\omega]) \in \mathbb{R}/\mathbb{Z} \Leftrightarrow k \in \mathbb{Z}$, which coincides with the previous quantization constraint (3.114).

If one considers the (formal) Chern-Simons functional measure:

$$d\mu_k([\omega]) \equiv \mathcal{D}[\omega] \exp\{2i\pi CS_k([\omega])\} \,, \tag{3.116}$$

then due to the exponential one deduces that quantisation of k is a necessary and sufficient condition for this exponential to be well-defined. From now on we will assume that:

$$k \in \mathbb{Z} \,. \tag{3.117}$$

The functional measure (3.116) enjoys the following important property:

$$d\mu_k([\omega] + \overline{\alpha}) = d\mu_k([\omega]) \times \exp\left\{2i\pi k \int_M (2[\omega] *_D \overline{\alpha} + \overline{\alpha} *_D \overline{\alpha})\right\} \,, \tag{3.118}$$

called the Cameron-Martin like (or simply Cameron-Martin) property. This property is typical of gaussian measures and of their functional generalizations. In fact, due to the presence of i in the exponential, the functional measure (3.116) is not a gaussian measure but rather quadratic, which is actually enough to ensure (3.118). The Cameron-Martin property will be a keystone of our construction. As a final remark, let us recall that the Lebesgue measure $\mathcal{D}$ used in (3.116) is formal since there is no infinite-dimensional analogue of Lebesgue measure.

Condition (3.118) is unfortunately not sufficient to define a functional measure since such a measure depends on the space on which we try to define it. In a first step, let us assume that this space is $H^1_D(M,\mathbb{Z})$, the space of classes of $U(1)$-connections on M. From section 2.2 we know that this space is an affine bundle over $\check{H}^2(M,\mathbb{Z})$. It is generically made of two parts: a free part and a torsion part, both being discrete. The translation group of the fibers of $H^1_D(M,\mathbb{Z})$ is $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. Integration over an affine space can be defined via integration over the underlying linear space. Therefore, our functional measure will have to be defined as some extension of a functional measure on $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. More precisely, for each base point $\kappa \in \check{H}^2(M,\mathbb{Z})$ one fixes an origin $[\eta]_\kappa$ on the fiber over κ, what amounts to define a global (discrete) section of $H^1_D(M,\mathbb{Z})$. Then, one decomposes any element of $H^1_D(M,\mathbb{Z})$ according to this section, let us say:

$$[\omega] = [\eta_\kappa] + \overline{\alpha} \,, \tag{3.119}$$

where $\kappa = \delta_2([\omega])$ (*cf.* (2.31)) and $\overline{\alpha} \in \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. The functional measure (3.116) takes the more precise form:

$$d\mu_k([\omega]) \equiv \sum_{\kappa} \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M ([\eta_\kappa] + \overline{\alpha}) *_D ([\eta_\kappa] + \overline{\alpha})\right\}, \tag{3.120}$$

On the fiber over $\kappa = 0$ there exists a canonical origin which is the zero connection (*i.e.* the connection defined by the exterior derivative d_M and associated with the trivial bundle $M \times U(1)$). However, such a canonical choice does not exist on the other fibers[5] of $H^1_D(M, \mathbb{Z})$. On the other hand, if one wants the loops (*i.e.* cycles) to be among the fields of the theory one is lead to extend our functional measure to a distributional version of $H^1_D(M, \mathbb{Z})$, which corresponds to the usual setting of a Quantum Field Theory into which fields are (tempered) distributions. But we have seen that the Pontrjagin dual of $H^1_D(M, \mathbb{Z})$ is such a distributional extension. Consequently, we also want to extend the Chern-Simons functional measure to $H^1_D(M, \mathbb{Z})^*$ (or a sub space of it which would correspond to the configuration space of the quantum theory). Fortunately, $H^1_D(M, \mathbb{Z})^*$ is also an affine bundle over $\check{H}^2(M, \mathbb{Z})$ which contains $H^1_D(M, \mathbb{Z})$ and $Z_1(M)$ (see (2.51)). Similarly, the translation group of this affine bundle, $Hom(\Omega^2_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$, contains $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$ (see 2.61). Therefore an extension of $d\mu_k([\omega])$ corresponds to an extension of the measure on $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$ to a measure on $Hom(\Omega^2_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$. On the other hand, we have already noticed that $H^1_D(M, \mathbb{Z})^*$ also contains $Z_1(M)$. Hence, cycles can be used as "natural" origins on the fibers of $H^1_D(M, \mathbb{Z})^*$. However, there is an infinite number of such cycles and no canonical way to pick one, unlike for the zero cycle which lies on the fiber over the zero class in $\check{H}^2(M, \mathbb{Z})$.

Let us write $[\gamma]$ the DB class canonically defined by a cycle $\gamma \in Z_1(M)$. For each $\kappa \in \check{H}^2(M, \mathbb{Z})$ one picks up, once and for all, a fundamental cycle (or loop) γ_κ. This is possible thanks to Poincaré duality: $\check{H}^2(M, \mathbb{Z}) \simeq \check{H}_1(M, \mathbb{Z})$. One then chooses as origin over κ the DB class $[\gamma_\kappa]$ associated with γ_κ.

We can be more precise: the space $\check{H}^2(M, \mathbb{Z})$ can be decomposed into its free part and its torsion part. The free part of $\check{H}^2(M, \mathbb{Z})$ is of the form $\mathbb{Z}^N$ for some positive integer N. If $\{\vec{\kappa}_{(j)}\}_{j=1,\ldots,N}$ denotes the canonical basis of $\mathbb{Z}^N$, the previous construction associates to each basis vector $\vec{\kappa}_{(j)}$ a fundamental cycle $\gamma_{\vec{\kappa}_{(j)}}$, as well as its DB class $[\gamma_{\vec{\kappa}_{(j)}}]$. The same holds true for the torsion part of $\check{H}^2(M, \mathbb{Z})$ except that there exist some integers m_a ($a = 1, \ldots, \widetilde{N}$) such that $m_a \vec{\kappa}_a = 0$, with $\vec{\kappa}_a$ forming a basis of the torsion sector. Let us denote $\gamma_{\vec{\kappa}_a}$ the fundamental (torsion) cycles chosen as origins over κ_a, and $[\gamma_{\vec{\kappa}_a}]$ its DB class.

Thanks to this, the Chern-Simons functional measure over $H^1_D(M, \mathbb{Z})^*$ can be itself decomposed according to:

$$d\mu_k([\omega]) \equiv \sum_{j=1}^{N} \sum_{\vec{\kappa}_{(j)} \in \mathbb{Z}} \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M \left([\gamma_{\vec{\kappa}_{(j)}}] + \overline{\alpha}\right) *_D \left([\gamma_{\vec{\kappa}_{(j)}}] + \overline{\alpha}\right)\right\} + \sum_{a=1}^{\widetilde{N}} \sum_{\vec{\kappa}_a=1}^{m_a-1} \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M ([\gamma_{\vec{\kappa}_a}] + \overline{\alpha}) *_D ([\gamma_{\vec{\kappa}_a}] + \overline{\alpha})\right\}. \tag{3.121}$$

[5] In fact, there also exists such a canonical choice on torsion fibers [30].

Of course the problem of finding a functional measure on the $\overline{\alpha}$'s remains, and will not be investigated further. From now on we assume that such a measure satisfying the Cameron-Martin property can be found, either on the full space $Hom(\Omega^2_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$ or on a convenient subspace.

A first problem that is faced when dealing with (3.121) is the presence of products $[\gamma_{\vec{\kappa}}] *_D [\gamma_{\vec{\kappa}}]$. If one assumes the previous decomposition for $[\gamma_{\vec{\kappa}}]$ only the following fundamental products will actually occur: $[\gamma_{\vec{\kappa}_{(i)}}] *_D [\gamma_{\vec{\kappa}_{(j)}}]$, and $[\gamma_{\vec{\kappa}_a}] *_D [\gamma_{\vec{\kappa}_b}]$. When including the Wilson lines we will also meet mixed products like $[\gamma_{\vec{\kappa}_{(i)}}] *_D [\gamma_{\vec{\kappa}_a}]$. Such products can be ill-defined as are generally products of distributions, and some regularisation may be necessary for (3.121) to become meaningful. If one has chosen the fundamental cycles defining the $\gamma_{\vec{\kappa}_{(i)}}$'s and $\gamma_{\vec{\kappa}_a}$'s in such a way that they have no self-intersection and no intersection with each anothers, then the regularisation of all the previous fundamental products can be done in the zero DB class. Beside the fundamental cycles generating the homology (and hence by Poincaré duality the cohomology) of M, one can also encounter DB products of DB classes of homologically trivial cycles. For products between cycles with no intersection, the result is actually free of ambiguities and divergencies and it coincide with the linking number. When the product is the DB square of a trivial cycle, the standard regularisation by framing can be applied. Note that in both cases this regularisation is finer than the previous one but in term of DB classes it still correspond to a zero regularisation. In the case of a mixed product (containing a trivial cycle and a non trivial one) zero regularisation still holds since the linking is defined up to an integer in such cases, *i.e.* it is zero in $\mathbb{R}/\mathbb{Z}$. What is remarkable is that only these fundamental regularisation are necessary to compute link invariants. Of course other products of distributions appear in the theory (think about the Chern-Simons action itself), but what ever regularisation is chosen for them, as long as we apply the zero one for fundamental cycles, the result does not depending on these regularizations.

Let γ be a loop on M, and $[\gamma]$ its DB representative class. As already noticed, for any $[\omega] \in H^1_D(M, \mathbb{Z})$ one has:

$$\oint_\gamma [\omega] = \int_M [\omega] *_D [\gamma] \mod \mathbb{Z} \,. \tag{3.122}$$

Let us assume that $[\gamma] = q[\gamma_I]$, where $[\gamma_I]$ is one of the $[\gamma_{\vec{\kappa}_{(j)}}]$ or of the $[\gamma_{\vec{\kappa}_a}]$, and q a real number. We can write:

$$\oint_\gamma [\omega] = q \int_M [\omega] *_D [\gamma_I] \mod \mathbb{Z} \,. \tag{3.123}$$

In Chern-Simons theory q is called the **charge** (or **colour**) of the loop γ. From the point of view of Deligne-Beilinson Cohomology, this charge has to be quantised for $q[\gamma_I]$ to be a DB class. This quantisation ensure that:

$$\exp\{\oint_\gamma [\omega]\} = \exp\{q \int_M [\omega] *_D [\gamma_I]\} \,, \tag{3.124}$$

is well-defined for any DB class $[\omega]$. This quantity is usually called the holonomy of γ along $[\omega]$. Thus we have established that:

$$q \in \mathbb{Z}\,. \tag{3.125}$$

For a loop decomposed as:

$$[\gamma] = q[\gamma_I] + \overline{\beta_c}\,, \tag{3.126}$$

with $\overline{\beta_c} \in Hom(\Omega^2_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$, if we also decompose $[\omega]$ according to $[\omega] = [\gamma_J] + \overline{\alpha}$, we obtain:

$$\exp\left\{2i\pi \oint_\gamma [\omega]\right\} = \exp\left\{2i\pi \int_M ([\gamma_J] + \overline{\alpha}) *_D (q[\gamma_I] + \overline{\beta_c})\right\}\,. \tag{3.127}$$

Zero regularisation gives a meaning to products like $[\gamma_J] *_D [\gamma_I]$. Therefore, zero regularisation also applies to Wilson loops.

We eventually introduce the expectation value of a Wilson loop:

$$\langle W(\gamma)\rangle_k = \sum_{\vec{\kappa}} \int \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M ([\gamma_{\vec{\kappa}}] + \overline{\alpha}) *_D ([\gamma_{\vec{\kappa}}] + \overline{\alpha})\right\} \times \tag{3.128}$$
$$\times \exp\left\{2i\pi \int_M ([\gamma_{\vec{\kappa}}] + \overline{\alpha}) *_D (q[\gamma_I] + \overline{\beta_c})\right\}\,,$$

where $\vec{\kappa}$ stands for a basis vector of $\check{H}^2(M, \mathbb{Z})$ (either free or torsion). A more precise decomposition can be obtained from (3.121).

We are now going to explain how to obtain links invariants using (3.128). We will start with the case of closed manifolds without torsion such as S^3 and $S^1 \times \Sigma_g$ where Σ_g is a closed surface of genus g. Then we will present a quite simple (but non trivial) case with torsion: $SO(3) \simeq \mathbb{R}P^3$. Lastly, we will show how this approach naturally extends to higher-dimensional closed manifolds.

3.2.1 Links invariants on torsionless 3-dimensional closed manifolds

Let us start with the simplest case of S^3. Here $\check{H}^2(S^3, \mathbb{Z}) = 0$ and therefore $H^1_D(S^3, \mathbb{Z})^* \simeq Hom(\Omega^2_{\mathbb{Z}}(S^3), \mathbb{R}/\mathbb{Z})$. This non-canonical isomorphism is made "canonical" by choosing the zero connection as origin. The affine bundle $H^1_D(S^3, \mathbb{Z})^*$ is made of only one fiber. The Chern-Simons functional measure reduces to:

$$d\mu_k(\overline{\alpha}) = \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M \overline{\alpha} *_D \overline{\alpha}\right\}\,. \tag{3.129}$$

with $\overline{\alpha} \in Hom(\Omega^2_{\mathbb{Z}}(S^3), \mathbb{R}/\mathbb{Z})$. On the other hand, Poincaré duality implies that $\check{H}_1(S^3, \mathbb{Z}) = 0$, which means that any 1-cycle (or loop) in S^3 is trivial (*i.e.* contractible). Consequently, the canonical DB representative of a cycle $\gamma = bc$ is simply generated by β_c, the de Rham current of the 2-chain c. This reads:

$$[\gamma] = [0] + \overline{\beta_c} = \overline{\beta_c}\,. \tag{3.130}$$

For a given cycle γ, the chain c is not unique but two such chains differ by a 2-cycle. The collection of chains bounding γ define an element of $Hom(\Omega^2_{\mathbb{Z}}(S^3), \mathbb{R}/\mathbb{Z})$ since the integral of an element $\Omega^2_{\mathbb{Z}}(S^3)$ over any 2-cycle is by definition an integer. The Wilson line of a loop $\gamma = bc$ thus simplifies to:

$$W(\gamma) = \exp\left\{2i\pi \oint_\gamma \overline{\alpha}\right\} = \exp\left\{2i\pi \int_M \overline{\alpha} *_D \overline{\beta_c}\right\} ,$$

and its expectation value reads:

$$\langle W(\gamma)\rangle_k = \int \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M \overline{\alpha} *_D \overline{\alpha}\right\} \times \exp\left\{2i\pi \int_M \overline{\alpha} *_D \overline{\beta_c}\right\} . \tag{3.131}$$

Let us assume that the loop γ holds charge $q \in \mathbb{Z}$, that is to say $\gamma = q\gamma_0$ for some fundamental loop (*i.e.* an embedding of S^1 in S^3). Since $\gamma_0 = bc_0$, then:

$$\langle W(\gamma)\rangle_k = \int \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M \overline{\alpha} *_D \overline{\alpha}\right\} \times \exp\left\{2i\pi q \int_M \overline{\alpha} *_D \overline{\beta}_{c_0}\right\} . \tag{3.132}$$

One shows that by setting:

$$\overline{\alpha} \mapsto \overline{\chi} = \overline{\alpha} + q\frac{\overline{\beta}_{c_0}}{2k} , \tag{3.133}$$

then (3.132) turns into:

$$\langle W(\gamma)\rangle_k = \exp\left\{-2i\pi k q^2 \int_M \frac{\overline{\beta}_{c_0}}{2k} *_D \frac{\overline{\beta}_{c_0}}{2k}\right\} \times \int \mathcal{D}\overline{\chi} \exp\left\{2i\pi k \int_M \overline{\chi} *_D \overline{\chi}\right\} . \tag{3.134}$$

Assuming that the functional has been normalised, one finally gets:

$$\langle W(\gamma)\rangle_k = \exp\left\{-2i\pi k q^2 \int_M \frac{\overline{\beta}_{c_0}}{2k} *_D \frac{\overline{\beta}_{c_0}}{2k}\right\} . \tag{3.135}$$

But $\overline{\beta}_{c_0}$ is the de Rham current of the 2-chain c_0. Therefore, the DB square in the exponential gives:

$$\int_M \frac{\overline{\beta}_{c_0}}{2k} *_D \frac{\overline{\beta}_{c_0}}{2k} = \int_M \frac{\overline{\beta}_{c_0} \wedge d\overline{\beta}_{c_0}}{4k^2} \mod \mathbb{Z} , \tag{3.136}$$

in such a way that:

$$\langle W(\gamma)\rangle_k = \exp\left\{-2i\pi \frac{q^2}{4k} c_0 \pitchfork \gamma_0\right\} . \tag{3.137}$$

This expression is obviously ill-defined since there appears the self-linking of γ_0: $L_{\gamma_0} = c_0 \pitchfork \gamma_0$. Nevertheless, using regularisation by "framing", one deduces that:

$$\langle W(\gamma)\rangle_k = \exp\left\{-2i\pi \frac{q^2}{4k} c_0 \pitchfork \gamma_0^f\right\} . \tag{3.138}$$

where γ_0^f is a chosen framing of γ_0. Now, if L is a generic **link** that decomposes as:

$$L = \sum_i q_i \gamma_0^i \,, \tag{3.139}$$

with $\gamma_0^i = bc_0^i$ some fundamental framed loops and q_i some (integral) charges, one has:

$$\langle W(L) \rangle_k = \exp\left\{ -2i\pi \frac{1}{4k} \sum_{i,j=1}^{n} q_i L_{(L)}^{ij} q_j \right\} \,. \tag{3.140}$$

where the linking matrix $(L_{(\gamma)}^{ij})$ of the link L has been introduced. It is defined by:

$$L_{(L)}^{ij} = c_0^i \pitchfork \gamma_0^j \,, \tag{3.141}$$

with the framing convention when $i = j$. Expression (3.140) is the one of abelian link invariants on S^3.

One could have noticed that the expectation value of the Wilson line $W(\gamma)$ satisfies a $2k$-nilpotency property (also named $2k$-periodicity). For details on this property see [9].

Let us now consider the less trivial case $M = S^1 \times \Sigma_g$. In this example the use of Deligne-Beilinson will appear more acutely than in the case of S^3.

The first step is to try to write the Chern-Simons measure (3.121) in an explicit way, taking into account the precise homology (or cohomology) of $M = S^1 \times \Sigma_g$. It is well-known, if not obvious, that:

$$\check{H}^2(M, \mathbb{Z}) = \check{H}^2(S^1 \times \Sigma_g, \mathbb{Z}) = \check{H}_1(S^1 \times \Sigma_g, \mathbb{Z}) \simeq \mathbb{Z}^{2g+1} \,. \tag{3.142}$$

Hence as done in the general section, one introduces the canonical basis $\{\vec{n}_{(i)}\}_{i=0,\ldots,2g}$, in such a way that any $\vec{n} \in \mathbb{Z}^{2g+1}$ decomposes according to:

$$\vec{n} = \sum_{i=0}^{2g} n^i \vec{n}_{(i)} \,. \tag{3.143}$$

One then chooses a representative for each $\vec{n}_{(i)} \in \mathbb{Z}^{2g+1}$, let say the fundamental loops γ_i, to which correspond DB classes $[\gamma_i]$. A very convenient choice of section s of $H_D^1(M, \mathbb{Z})^*$ is then given by:

$$\begin{aligned} s : \mathbb{Z}^{2g+1} &\to H_D^1(M, \mathbb{Z})^* \\ \vec{n} &\mapsto s(\vec{n}) = \sum_{j=0}^{2g} n^j [\gamma_j] \equiv [\gamma_{\vec{n}}] \,. \end{aligned} \tag{3.144}$$

According to this section, any DB class $[\omega] \in H_D^1(M, \mathbb{Z})^*$ decomposes as:

$$[\omega] = [\gamma_{\vec{n}}] + \overline{\alpha} \,, \tag{3.145}$$

where $\vec{n} = \delta_2([\omega])$ (see 2.30) and $\overline{\alpha} \in Hom(\Omega^2_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z}) \supset \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$.

Our functional measure takes the form:

$$d\mu_k([\omega]) \equiv \sum_{\vec{n}\in\mathbb{Z}^{2g+1}} \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M ([\gamma_{\vec{n}}] + \overline{\alpha}) *_D ([\gamma_{\vec{n}}] + \overline{\alpha})\right\} . \tag{3.146}$$

Once more, products of de Rham currents occur implying regularisations in particular in the DB squares $[\gamma_{\vec{n}}] * [\gamma_{\vec{n}}]$, .

When $i \neq j$, products $[\gamma_i] * [\gamma_j]$ are regularised by $[0]$. For homologically trivial cycles this is exactly as obvious as in the $M = S^3$ case since the corresponding DB product are given by the linking of γ_i with γ_j. This is related to a general property of linking on manifolds (see for instance [36]). When the loops $[\gamma_i]$ and $[\gamma_j]$ are not trivial, as already mentioned their linking is not a uniquely-defined integer. Yet, setting $[\gamma_{\vec{n}}] * [\gamma_{\vec{n}}] = [0]$ remains a consistent choice of regularisation for this product.

Thus, our choice of taking fundamental cycles as origins on fibers of $H^1_D(M, \mathbb{Z})^*$ leads to a somewhat natural regularisation of products $[\gamma_i] * [\gamma_i]$. For DB squares, one uses "framing" and set $[\gamma_i] * [\gamma_i^f] = [0]$. Let us point out that we do NOT say that "framing" defines self-linking of non trivial loops. We just say that it provides a regularisation into the zero DB class for products like $[\gamma_i] * [\gamma_i^f]$. And it is the only one required. In other words, we do not need a definite expression for the self-linking but we only need to know it is an (undefined) integer which ensures the consistency of the zero regularisation (*i.e.* regularisation into the zero DB class). We could also obtain the same result by (homotopically) "smoothing" the various cycles and then taking the limit when the smoothed forms go to the initial currents.

Let γ be a fundamental loop in M. Then, for any $[\omega] \in H^1_D(M, \mathbb{Z})^*$ we have:

$$\oint_\gamma [\omega] = \int_M [\omega] *_D [\gamma] \mod \mathbb{Z} . \tag{3.147}$$

Using the discrete section (3.144), one gets:

$$\oint_\gamma [\omega] = \int_M ([\gamma_{\vec{n}}] + \overline{\alpha}) *_D ([\gamma_{\vec{q}}] + \overline{\beta_c}) \mod \mathbb{Z} . \tag{3.148}$$

where $\vec{n} = \delta_2[\omega]$, $\vec{q} = \delta_2[\gamma]$, and $\overline{\beta_c}$ is generated by the de Rham current of a 2-chain c such that: $\gamma = \sum_i q_i\gamma_i$ (these have been previously defined). One finally obtains for the expectation value of the Wilson loop of γ:

$$\langle W(\gamma)\rangle_k = \sum_{\vec{n}\in\mathbb{Z}^{2g+1}} \int \mathcal{D}\overline{\alpha} \exp\left\{2i\pi k \int_M ([\gamma_{\vec{n}}] + \overline{\alpha}) *_D ([\gamma_{\vec{n}}] + \overline{\alpha})\right\} \times \tag{3.149}$$

$$\times \exp\left\{2i\pi \int_M ([\gamma_{\vec{n}}] + \overline{\alpha}) *_D ([\gamma_{\vec{q}}] + \overline{\beta_c})\right\} .$$

Unlike the trivial case $M = S^3$, in (3.149) we cannot get rid of $[\gamma_{\vec{n}}]$ and $[\gamma_{\vec{q}}]$, nor can we perform the shift (3.133) on $[\gamma_{\vec{q}}] + \overline{\beta_c}$. Indeed, we cannot divide $[\gamma_{\vec{q}}]$ by $2k$. However, the shift:

$$\overline{\alpha} \mapsto \overline{\chi} = \overline{\alpha} + q\frac{\overline{\beta}_{c_0}}{2k} , \tag{3.150}$$

made on $Hom(\Omega^2_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$ remains possible. In fact, one can notice that the functional measure (3.146) has the quite hidden following invariance [9, 32]:

$$d\mu_k\left([\omega] + \overline{\left(\frac{m\beta_\sigma}{2k}\right)}\right) = d\mu_k([\omega])\,, \tag{3.151}$$

for any current β_σ corresponding to a 2-cycle (like a closed a surface) σ on M. Thus, if $(\sigma_j)_J$ denotes a set of generators $\check{H}_2(S^1 \times \Sigma_g, \mathbb{Z})$, and if β_j are the associated 1-currents, one sets for any loop γ:

$$N_j(\gamma) \equiv \oint_\gamma \beta_j \equiv \gamma \pitchfork \sigma_j \in \mathbb{Z}\,. \tag{3.152}$$

Note that thanks to Poincaré and Hom dualities, there are as many σ_j than generators of $\check{H}_2(S^1 \times \Sigma_g, \mathbb{Z})$.

Under a shift by $\overline{(m\beta_j/2k)}$ the expectation value of the Wilson loop of γ changes according to:

$$\begin{aligned} \langle W(\gamma)\rangle_k &= \langle W(\gamma)\rangle_k \cdot \frac{1}{2k}\sum_{m=0}^{2k-1} \exp\left\{2i\pi\frac{m}{2k}\oint_\gamma \beta_j\right\} \\ &= \langle W(\gamma)\rangle_k \cdot \frac{1}{2k}\sum_{m=0}^{2k-1} \exp\left\{2i\pi\frac{m}{2k}N_j(\gamma)\right\}\,. \end{aligned} \tag{3.153}$$

But:

$$\frac{1}{2k}\sum_{m=0}^{2k-1} \exp\left\{2i\pi\frac{m}{2k}N_j(\gamma)\right\} = \left\{\begin{array}{ll} 1 & \text{if } N_j\,(\gamma) \equiv 0 \bmod 2k \\ 0 & \text{otherwise.}\end{array}\right. . \tag{3.154}$$

Consequently:

$$(N_j(\gamma) \neq 0 \bmod 2k) \Rightarrow (\langle W(\gamma)\rangle_k = 0)\,. \tag{3.155}$$

On the other hand:

$$(\forall j,\ N_j(\gamma) = 0 \bmod 2k) \Rightarrow (\gamma \in [0] \in \check{H}_1(S^1 \times \Sigma_g, \mathbb{Z}))\,. \tag{3.156}$$

This shows that only homologically trivial links give rise to a non trivial expectation value of their Wilson line, and in this case:

$$\langle W(\gamma)\rangle_k = \exp\left\{-2i\pi\frac{1}{4k}\sum_{i,j=1}^{n} q_i L^{ij}_{(\gamma)} q_j\right\}\,, \tag{3.157}$$

just as in the $M = S^3$ case.

Let us point out that the infinite sum in (3.149) has been truncated into a sum from 0 to $(2k-1)$ (see equations (3.153)). This was achieved thanks to $2k$-nilpotency. This truncation explains why one also speaks about a $2k$-periodicity. Since the nilpotency property induces an obvious degeneracy of the infinite sum, the truncation appears as a simple (re)normalisation of the measure. We would like to recall that there is NO way to obtain all these results from the field theoretic point of view, simply because we do not have a version of QFT on a generic closed manifold. What is usually done is to compute perturbatively (or not) the links invariants of a given Chern-Simons theory over $\mathbb{R}^3$, then use an argument stating that links invariants on S^3 are those of $\mathbb{R}^3$, and finally use surgery to deduce invariants on any closed 3-manifold.

We can now have a look at a case with torsion, namely $\mathbb{R}P^3$. Let us note that its a purely torsion case because there is no free 1-cycle on $\mathbb{R}P^3$, except the trivial ones of course.

3.2.2 A case with torsion: $\mathbb{R}P^3$

The simplest closed 3-manifold with torsion is without any contest $\mathbb{R}P^3 \simeq SO(3) \simeq SU(2)/\mathbb{Z}_2$. Its first homology group is $\check{H}_1(\mathbb{R}P^3,\mathbb{Z}) = \mathbb{Z}_2 \equiv \{\overline{0},\overline{1}\}$ (with $2\times\overline{1}=\overline{0}$), and by Poincaré duality one has:

$$\check{H}^2(\mathbb{R}P^3,\mathbb{Z}) \simeq \check{H}_1(\mathbb{R}P^3,\mathbb{Z}) = \mathbb{Z}_2 \,. \tag{3.158}$$

Consequently, $H^1_D(\mathbb{R}P^3,\mathbb{Z})^* \supset H^1_D(\mathbb{R}P^3,\mathbb{Z})$ is an affine bundle over $\mathbb{Z}_2 = \{\overline{0},\overline{1}\}$. As usual, the fiber over the class $\overline{0}$ contains the zero DB class (*i.e.* the zero connection), and over $\overline{1}$ one picks up a torsion cycle τ as origin. Its DB class is denoted $[\tau]$. By construction one has:

$$2\times[\tau] = [0] + \overline{\beta_c} = \overline{\beta_c} \,, \tag{3.159}$$

which corresponds to the homological identify $2\times\tau = bc$.

The Chern-Simons functional measure is now made of two terms :

$$d\mu_k([\omega]) = \mathcal{D}\overline{\alpha}\exp\left\{2i\pi k\int_M \overline{\alpha}*_D\overline{\alpha}\right\} + \mathcal{D}\overline{\alpha}\exp\left\{2i\pi k\int_M ([\tau]+\overline{\alpha})*_D([\tau]+\overline{\alpha})\right\} \,. \tag{3.160}$$

for the same reasons as in the torsionless case, the level k (or coupling constant or space charge) is quantised:

$$k\in\mathbb{Z} \,. \tag{3.161}$$

Yet, due to the presence of $[\tau]*_D[\tau]$ in (3.160), if one tries to regularise this DB square in the zero class as we did in the torsionless case, one is faced with the definition of linking of torsion cycles. For torsion cycles of order 2, like τ, their linking is in general a half-integer (see [35]). This prevents from regularising $[\tau]*_D[\tau]$ into the zero DB class, and makes the Chern-Simons measure ill-defined except if one assumes that $k=2l$ since in this case the linking will always be an integer and so $[\tau]*_D[\tau]$ can rightfully be regularised into [0]. Hence from now on we will assume that:

$$k = 2l \;, l\in\mathbb{Z} \,, \tag{3.162}$$

in such a way that:

$$k[\tau] *_D [\tau] = 2l[\tau] *_D [\tau] = l(2[\tau] *_D [\tau])\,, \tag{3.163}$$

For Wilson lines there are two different cases to consider. Either the cycle is trivial, $\gamma = bc$, or it is of order two: $2\gamma = bc'$.

First, let us assume that $\gamma = bc$, and therefore that $[\gamma] = \overline{\beta_c} \in Hom(\Omega^2_{\mathbb{Z}}(\mathbb{R}P^3), \mathbb{R}/\mathbb{Z})$, $\overline{\beta_c}$ being the de Rham 1-current of the 2-chain c. One can straightforwardly apply the computation made in the $M = S^3$ case, and mainely based on the shift:

$$\overline{\alpha} \mapsto \overline{\chi} = \overline{\alpha} + \frac{\overline{\beta_c}}{2k}\,, \tag{3.164}$$

or:

$$\overline{\alpha} \mapsto \overline{\chi} = \overline{\alpha} + q\frac{\overline{\beta_c}}{2k}\,, \tag{3.165}$$

when γ holds charge q. As usual, details can be found in the original article [35]. Thus, for a trivial link we get, without any surprise:

$$\langle W(L) \rangle_k = \exp\left\{ -2i\pi \frac{1}{4k} \sum_{i,j=1}^{n} q_i L^{ij}_{(\gamma)} q_j \right\}\,. \tag{3.166}$$

Now, let us assume that γ has torsion: $2\gamma = bc$ whereas $\gamma \neq bc'$. We have denoted by τ the fundamental torsion cycle whose DB class $[\tau]$ plays the role of origin of the fiber of $H^1_D(\mathbb{R}P^3, \mathbb{Z})^*$ over $\overline{1}$ (the torsion fiber). Accordingly, there exists a 2-chain y such that $\gamma = \tau + by$, and the expectation value of the Wilson line of $\gamma = \tau + by$ holding charge q reads:

$$\begin{aligned} \langle W(\gamma) \rangle_k &= \int \mathcal{D}\overline{\alpha} \exp\left\{ 2i\pi k \int_M \overline{\alpha} *_D \overline{\alpha} \right\} \cdot \exp\left\{ 2i\pi q \int_M \overline{\alpha} *_D \left([\tau] + \overline{\beta_y}\right) \right\} + \\ &+ \exp\left\{ 2i\pi \int_M ([\tau] + \overline{\alpha}) *_D ([\tau] + \overline{\beta_c}) \right\} \cdot \exp\left\{ 2i\pi q \int_M ([\tau] + \overline{\alpha}) *_D \left([\tau] + \overline{\beta_y}\right) \right\}. \end{aligned} \tag{3.167}$$

Once again, the consistency of the usual regularisation by "framing" for the product $[\tau] *_D [\tau]$ is questioned, and once more one must add an hypothesis for this regularisation to work. One can ask the charge to satisfy:

$$q = 2m\ , m \in \mathbb{Z}\,. \tag{3.168}$$

The factor 2 occurring in this charge constraint allows the zero regularisation of the term $[\tau] *_D [\tau]$ in (3.167) to be consistent. Furthermore, performing the standard shift:

$$\overline{\alpha} \mapsto \overline{\chi} = \overline{\alpha} + \frac{\overline{\beta_{u+2y}}}{2k}\,, \tag{3.169}$$

where u is a 2-chain such that $2\tau = bu$, one obtains:

$$\langle W(\gamma)\rangle_k = \exp\left\{-2i\pi\frac{m^2}{4k}(u+2y)\pitchfork\gamma^f\right\}\,, \tag{3.170}$$

with γ^f a framing of γ. The intersection appearing in (3.170) is a well-defined integer and if one put back the charge into this expression one gets the usual result:

$$\langle W(\gamma)\rangle_k = \exp\left\{-2i\pi\frac{q^2}{4k}\frac{c\pitchfork\gamma^f}{2}\right\}\,, \tag{3.171}$$

where c is a 2-chain such that $2\gamma = bc$.

This terminates our treatment of link invariants from the abelian Chern-Simons theory on closed 3-manifolds. We can now see how to generalise this construction to higher-dimensional closed manifolds. This is done in the last section that follows.

3.2.3 Higher-dimensional cases

Let M be a smooth m-dimensional closed manifold. Let us try to find a Chern-Simons lagrangian for this manifold. From the physical point of view one would be tempted to consider a p-form A, generalising the idea of $U(1)$-connections, and to naively write:

$$L = A\wedge dA \tag{3.172}$$

But we know from the start that this is not the right direction to take if we want to deal with connections rather than forms. Accordingly, it seems better to consider DB classes $[\omega]$ whose DB square $[\omega]*_D[\omega]$ will define our Chern-Simons lagrangian. If $[\omega]\in H_D^p(M,\mathbb{Z})$ then $[\omega]*_D[\omega]\in H_D^{2p+1}(M,\mathbb{Z})$. Consequently, one must have:

$$m = 2p+1\,, \tag{3.173}$$

for the integral over M of $[\omega]*_D[\omega]$ to be well-defined. This constrains M to be **odd-dimensional**.

Since the DB product is a graded product (see (2.39)), one also has:

$$[\omega]*_D[\omega] = (-1)^{(p+1)(p+1)}[\omega]*_D[\omega]\,. \tag{3.174}$$

Thus a necessary condition for our Chern-Simons lagrangian not to be trivial is that:

$$(p+1)(p+1) = 2l\,,\ \ l\in\mathbb{Z}\,, \tag{3.175}$$

Constraints (3.173) and (3.175) combine to give:

$$m = 4n+3\,, \tag{3.176}$$

The case of 3-dimensional manifold is a special case.

Actually (3.176) also agrees with the linking theory. Indeed, on a manifold of dimension $4n+1$ the intersection of a trivial $(2n)$-cycle with any transversal $(2n+1)$-chain is zero. But this intersection is a way to define the linking between two trivial $(2n)$-cycles in a $4n+1$-dimensional closed manifold (see [36]).

Eventually, the level k Chern-Simons action in dimension $m = 4n+3$ is chosen as:

$$CS_k([\omega]) = \int_M cs_k([\omega]) = k \int_M [\omega] *_D [\omega] \, , \tag{3.177}$$

with $[\omega] \in H_D^{2n+1}(M, \mathbb{Z})$. For the same reason than in the three-dimensional case, the coupling constant has to be quantised:

$$k \in \mathbb{Z} \, , \tag{3.178}$$

and one introduces the Pontrjagin dual $H_D^{2p+1}(M, \mathbb{Z})^* \supset Z_{2n+1}(M)$ as the quantum configuration space. From (2.32) it is an affine bundle over $\check{H}^{2n+2}(M, \mathbb{Z})$ whose translation group on the fibers is $Hom(\Omega^{2n+2}(M), \mathbb{R}/\mathbb{Z}) \supset \Omega_{\mathbb{Z}}^{2n+1}(M)/\Omega^{2n+1}(M)$. The functional measure associated with the generalised Chern-Simons action reads:

$$d\mu_k([\omega]) = \mathcal{D}[\omega] \exp\left\{ 2i\pi k \int_M [\omega] *_D [\omega] \right\} \, , \tag{3.179}$$

Here again a more precise meaning has to be given to this measure by relying on the affine bundle structure of $H_D^{2p+1}(M, \mathbb{Z})^*$. Since Poincaré duality still holds true, one will use it to fix as origins on fibers of this bundle a family of chosen $(2l+1)$-cycles (seen as elements of $H_D^{2p+1}(M, \mathbb{Z})^*$ by the now familiar inclusion (2.50). The zero cycle will be a special origin on the fiber over $0 \in \check{H}^{2n+2}(M, \mathbb{Z})$. We will obtain a decomposition totally similar to (3.121), except that all objects are of degree p, not 1.

As for Wilson loops, they are generated by fundamental loops (cycles) γ of dimension $2n+1$, according to:

$$W_n(\gamma) = \exp\left\{ 2i\pi \int_\gamma [\omega] \right\} = \exp\left\{ 2i\pi \int_M [\omega] *_D [\gamma] \right\} \, . \tag{3.180}$$

Let us notice that the degrees of the objects $[\omega]$ and $[\gamma]$ play an important role ensuring that $[\omega] *_D [\gamma] = [\gamma] *_D [\omega]$. The fundamental difference with the $m = 3$ case is that fundamental loops appearing in (3.180) are NOT necessarily diffeomorphic to spheres S^{2n+1}. For instance, one can have $\gamma \cong S^1 \times S^{2n}$, or $\gamma \cong S^1 \times (S^1)^{2n}$, or many different types of geometrical objects. Nevertheless, it is quite obvious that the methodology we have exposed in the previous three-dimensional examples will apply straightforwardly here. In particular, the zero regularisation and the finer framing regularisation will allow to deal with DB squares of fundamental loops. Also, we will define the charge of a loop as the number of times a fundamental loop is covered and a $(2l+1)$-link will denotes a formal combination of charged loops. At the end one will get for the expectation value of the Wilson line of a homologically trivial link:

$$\langle W_n(L) \rangle_k = \exp\left\{ -2i\pi \frac{1}{4k} \sum_{i,j=1}^{n} q_i L^{ij}_{(\gamma)} q_j \right\} , \tag{3.181}$$

whereas such a expectation value will prove to be zero for non-trivial links.

As for torsion one can expect to treat it the same way as in the 3-dimensional case.

Let us note that $m = 1$ is a trivial case since the Chern-Simons action is then an integer (zero in $\mathbb{R}/\mathbb{Z}$). The same holds for $m = 5$ and therefore the first non-trivial new case will be $m = 7$. But the topological sphere S^7 has many different and inequivalent differential structures. One can then wonder whether these structures plays a role in the computation of links invariants.

To end this chapter, let us note that a quantum field theory approach to these higher-dimension Chern-Simons theory can be used ([32]), but only in the case of $\mathbb{R}^{4l+3}$. To be able to go further, one would need Euclidean Quantum Field Theory to be defined on closed manifold which is far from being achieved. However, Chern-Simons theories provide a very interesting playground to try and test some possible extension of the usual QFT to theories on closed manifold. The reasons are that we know the fields exactly (I mean their representative for a given good cover are known), we know how to define Chern-Simons action in terms of these fields, and everything is local in this way. Furthermore, it is only topology, not physics. This either gives hope or despair.

4 Conclusion: what to do now?

In this last section I would like to present a list of possible developments in the framework of Topological models of cohomological type.

4.1 Equivariant Cohomology

- First, let us return to the problem of determining Mumford invariants using Equivariant Cohomology. We have seen that beside applying this cohomology theory on the space of Beltrami differential of a Riemann surface one can also obtain a collection of invariants by using metrics instead of Beltrami fields. Up to our knowledge, there is no clear demonstration showing that the two sets of invariants are identical. It should be so but it is not easy to show. This could appear as an exercise, but we have the feeling it might also provide new light in the field of topological models: how to decide if a set of equivariant observables are equivalent or not?

- What we have done only dealt with closed smooth manifolds: compact smooth manifolds without boundary. How could we extend this to smooth manifold with boundary? One could have in mind ADS/CFT which relates two models one on a manifold and the other on its boundary. This could be related to the supersymmetric version of Equivariant models like the one done in [37]. This is however more a guess (and so very speculative) than an evidence.

- One knows that Equivariant Cohomology admits a supersymmetric interpretation (see [38]). For instance one can easily show that a twisted supersymmetric conformal algebra can be seen, via its fundamental OPE, as the structure of an equivariant model. One can wonder what could be done in higher dimensions.

4.2 Deligne-Beilinson Cohomology

This is manifestly where most of our open questions can be asked.

- Chern-Simons theories are based on a quadratic functional measure, itself based on the DB square of (generalised) p-connections: $A *_D A$. But we have already noticed that the DB product is more general (see (2.38)) than the simple square. One could then imagine products of p-connections with q-connections: $A *_D B$, thus providing new quantum actions. With a closer look one can check that this would correspond to consider so-called abelian "BF" systems (or models). This is known in physics but as always from a purely QFT point of view. Using from the start DB Cohomology could be a new way to deal with these BF models. In particular, one sees that when $B = A$ one recovers the Chern-Simons lagrangian. So having a good mathematical "control" on the basic objects might also give a better understanding of the physical quantities these BF theories provide and also put some light on the link with Chern-Simons theories. Note that from DB point of view, the name BF is totally ambiguous. One should speak of an AB theory considering the lagrangian to be $A *_D B$ rather than $A \wedge dB$. We have learnt that a curvature is not enough to define a connection. Of course one can expect the Wilson lines of a BF models to be made of cycles (loops) of degree p and q and the invariants to just be intersections, as in the Chern-Simons case. Then a new challenge appears: what about trying to study non quadratic theories: $A *_D B *_D C$, and even more general ones. Of course all the tricks we have used in the quite simple quadratic case might failed to apply. However, they should give access, if not to all the observables, at least to some of them. Are there other observables providing computable "physical" quantities, and does DB Cohomology shed any light on this? These are interesting questions.

In the abelian Chern-Simons theory treated from the DB Cohomology point of view there are many open questions left (if not all) concerning the relation between this approach and surgery. Many theorems can be established using surgery and one can wonder how these theorems could be demonstrate within the DB framework. We are actually investigating these question with E. Guadagnini starting with the Reshetikhin-Turaev theorem relating expectation values of a Wilson line on a closed manifold to Wilson lines on the sphere. At first sight it should be possible to establish this via Chern-Simons and DB cohomology, but it has to be worked out.

- Of course, the Grail stands in the possible understanding of the non-abelian case. First, one knows that the non abelian Chern-Simons lagrangian, the one written $Tr(A \wedge dA + \frac{2}{3}A^3)$ by physicists, is actually a DB class (or a translation from a DB class to another depending on the point of view) of degree 3. This is actually why the level of these theories is also quantised, even if physicists like to say that it is due to gauge invariance. In fact gauge transformations provide terms that belongs to the (large) gauge group $\Omega^3_{\mathbb{Z}}(M)$ defining $H^3_D(M, \mathbb{Z})$ within the exact sequence (2.36). This means that unlike the abelian case, one could quantised the level k of the non-abelian theory by only considering $Tr(A \wedge dA + \frac{2}{3}A^3)$ as the lagrangian. Anyway, in QFT one uses this former lagrangian arguing about the quantisation of k from the point of view of $M = S^3$ which is not totally satisfactory. However, non abelian QFT gives a perturbative answer to the computation of the expectation value of non abelian Wilson lines (non abelian holonomies) (for given representations of the underlying lie group, usually $SU(N)$). Of course it is shown that these perturbative results coincide with the equivalent development of some link polynomials, at least up to some fixed order (maybe three at the moment). A large family of polynomials are then perturbatively "generated": HOMFLY polynomials. Then one can use surgery to get polynomials for any three-dimensional closed manifold. On the other hand, it seems hard to find a relation between Wilson lines (or their trace within a representation of the Lie group considered) and a DB class, unlike in the abelian case. Nevertheless there exists a non abelian Stokes theorem. It provides in a quite complicated way a relation between a non abelian connection and an abelian one. However, it is not totally clear yet if the abelian gauge field thus obtained is really a DB class. And if it is so, what is its relation with the Chern-Simons lagrangian itself? Mathematically there are higher-order invariants named Massey products which allow to distinguished the borromean link from a totally trivial one, whereas we know that the linking number fails to do so. Furthermore, there are extensions of Massey products to

DB classes. Therefore, one can wonder whether these Massey products are related to the non-abelian Chern-Simons theory and the invariants it (still perturbatively) generates. Also there is a classification (up to link homotopy) of three-component links in S^3 [48] using generalised Gauss maps and integrals. One can wonder wether this could be understand from the point of view of a Deligne-Beilinson Quantum Field Theory.

There has been attempts to define a non abelian version of Deligne-Beilinson cohomology (see for instance [39]): as in the abelian case it is supposed to classify non-abelian principal bundles with connections, as well as generalisations of such geometrical data. However, there doesn't seem to be more specific descriptions of these cohomology spaces, including for instance the knowledge of their embedding into some simple exact sequences, as the abelian DB spaces are. It has to be noticed that we were more interested in the structure of their Pontrjagin dual than on the DB spaces themselves. This was so because we wanted cycles to lie in the Quantum configuration space of the model. Accordingly, one could naively expect that the relevant structure is the one of a dual of the space of classes of non abelian principal bundles with connections, that is to say something like distributional connections. Ashtekar names such objects "generalized non abelian connections". It is from these spaces of generalized connections that Loop Gravity is built (see for instance ([40]). However we would like to point out a subtle difference between such an approach and what we have done in the abelian case: in Loop Gravity (and actually in all known case of QFT dealing with a non abelian group) one fixes a principal bundle on which gauge fields (*i.e.* connections) are supposed to live. In our abelian Chern-Simons models based on Deligne-Beilinson Cohomology we have considered the whole set of classes of bundle with connections. In other words, in the functional integration we also integrate over (classes of) principal bundles, not just on connections over a fixed bundle[6]. In the best case the non-abelian Deligne hypercohomology will allow to identify a canonical dual in which quantum fields stand. Then one would have to understand how these fields (or classes of fields) are related to the abelian DB theory and more precisely how they are related to the DB class represented by $Tr(A \wedge dA + \frac{2}{3}A^3)$. There should also be a relation with the non abelian Stokes theorem since this last one provides a link between non abelian and abelian holonomies (except for the fact it is a functional relation [41]). Actually, it is already possible to formally see the abelian DB theory on the self-linking part of the non abelian invariants, which is not surprising since it is mainly related to the quadratic part of the Lagrangian, formally of the same kind as the abelian lagrangian: $A \wedge dA$. Of course the possible light shaded by DB on non abelian Chern-Simons theory could have consequences in other theories like BF, Loop Gravity or even "more physical" Yang-Mills theories. As a final remark concerning these non abelian theories, let us point out that it is not possible to define straightforwardly a non abelian Chern-Simons lagrangian the way we did it in the abelian case (as a DB square). This is mainly due to the presence of the non quadratic part in the Lagrangian, which itself follows

[6]In fact, the space of generalized non-abelian connections which play the role of fundamental fields in Loop Gravity is obtained via some completion of the space of G-connections on a fixed principal G-bundle P over M. This completion naturally extends (generalized) G-connections on P to (generalized) connections on all G-bundles over M. Accordingly, our CS theory appears as an example of this completion in the abelian case.

from the quadratic part ($A \wedge A$) appearing in the non abelian curvature. This must not be confused with the fact that there are higher-dimensional DB classes built from a non-abelian connection and its curvature and which are associated with higher-dimensional Chern classes. However one could consider objects such as $F_A(B) = dB + A \wedge B$ and wonder what are the largest ambiguities on A and B for $F_A(B)$ to be still a p-form taking its values in some Lie algebra. Note that by construction A would have to be of degree 1, and B of degree $(p-1)$.The theory of non-abelian Gerbes could be involved.

- Recently with E. Guadagnini we wonder about doing a modes decomposition of the gauge fields in order to do a canonical quantisation. Of course, in order for these modes to be easily handled the manifold on which we have to consider our theory has to be simple, *e.g.* a product of two circles: $S^1 \times S^1$. It is possible to represent such a space by $\mathbb{R}^n$ divided by some discrete group: $S^1 \times S^1 \cong \mathbb{R}^2/\mathbb{Z}^2$. In this representation a connection can be describe as an object made of two parts: one is corresponding to a well-defined 1-form on $S^1 \times S^1$ and it is represented by a periodic 1-form on $\mathbb{R}^n$ compatible with the action of the discrete group, while the other part is made of a 1-form term not compatible with the action of the discrete group (we call it a periodicity breaking term for obvious reasons). It is quite remarkable that one can have a complete description of the DB Cohomology classes using these modes (*i.e.* we recover the standard exact sequences). Of course one think immediately about adapting this to the non-abelian case. If this approach were successful it would provide something equivalent to DB classes what would be of course very interesting in order to understand the non-abelian Chern-Simons theory. It would then be remarkable to know the answer on $S^1 \times S^1 \times S^1$ but not on S^3 (there the modes decomposition turns out to be much more difficult to handle because of the use of spherical harmonics). Nevertheless, it wouldn't be a total surprise since there are clues which lead to think that on a 3-manifold made of at least one non trivial circle, this circle can be used to gauge fixe non-abelian connections (see for instance [42]).

- With E. Pilon and L. Gallot we have studied recently the dimensional extension of abelian Chern-Simons theories. Our main results were produced as a last example of this kind in this review (see section 3.2.3). We would like to explain a bit more what one could have in mind concerning the use of this work in order to try to define a QFT (at least a topological one) on a closed manifold: first of all there will be no way to perturbatively compute expectation values since by essence a perturbative development is local. However, it turns out that the computation in abelian QFT can be done non perturbatively. So the whole point is about the use of DB classes instead of naive differential forms. We have noticed in this review that not only the DB classes admit nice representatives within the Čech-de Rham framework, but so do their DB products. In other words, we know the correct expression to use in action. The problem is that it is made of collections of fields defined in the open sets of a good cover of the manifold and in their intersections. This finally gives four terms to write the lagrangian in a correct way. Each of these terms has to be considered as a lagrangian on its own, each of which corresponds to some field theory in some open set (made of intersections of the elements of the good cover). It would be a first direction along which we may start the study. One could also use the expression of the DB product in another way: first make a polyhedral decomposition of the manifold according to the Weil techniques mentioned in section 2.2. This gives now a set of lagrangians one for each polyhedra of the decomposition, but also one for each faces of these polyhedra and another one for the edges of these faces and finally one set for the ends of the edges. This give four family like before, but now each family is define on a space of lower and lower dimension. the next step would be to define propagators and see how the gluing conditions for the various lagrangians (necessary to say that they actually build the Chern-Simons lagrangian) translate on these propagators. Then one could hope to make some computations using all these ingredients. If this can be achieved the computation will be non perturbative for the same reason as in the case of $\mathbb{R}^{4l+3}$. the simplest case to be treated should be of course S^3 and later S^{4l+3}.

- There is another domain where DB Cohomology could be very useful. It is well known that this cohomology is at the base of Geometric (pre)Quantisation. In this approach of Quantum Mechanics, one has to turn the symplectic form of the classical model into a closed 2-form (which it is already) with integral periods (which is a new requirement). This turns the symplectic form into a curvature suggesting the existence of some $U(1)$ bundle over configuration space. The lagrangian then turns into a DB class (a connection actually) the most famous example of which is provided by the Aharonov-Bohm effect. The interesting question is: if we start with a 3-curvature, or more precisely with an abelian 2-connection (on some abelian gerbe), what is the classical structure associated with the quantised curvature. It should be a generalisation of the Poisson brackets. Some authors, like J. Baez, have already investigated this, but directly from the point of view of generalized Poisson brackets, and not starting from a quantum version and trying to go back to its classical version. Note that if such a classical structure based on generalised Poisson brackets can be found (and it should be), it implies that it will be a trilinear object, suggesting that the new "configuration space" should be made of three independent coordinates. What would they be? And how could they be interpreted physically? Are there still positions and momenta? Is the new coordinate related to a derivative of the momentum, that is to say forces? Or on the contrary are positions related to this new coordinate? This game seems quite interesting and could have amusing consequences.

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6) Deligne-Beilinson cohomology and Abelian link invariants Torsion case.
7) Higher dimensional abelian Chern-Simons theories and their link invariants.

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ALGEBRAIC STRUCTURE OF COHOMOLOGICAL FIELD THEORY MODELS AND EQUIVARIANT COHOMOLOGY

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Abstract : The definition of observables within conventional gauge theories is settled by general consensus. Within cohomological theories considered as gauge theories of an exotic type, that question has a much less obvious answer. It is shown here that in most cases these theories are best defined in terms of equivariant cohomologies both at the field level and at the level of observables.

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I) Introduction.

In a classical article [W88], E. Witten proposed a Euclidean field theory scheme which should allow one to compute cohomology classes of orbit spaces using field theory methods. The example treated in that article is that of the Donaldson invariants [DK90]. Whereas the corresponding classical action was found thanks to $N = 2$ supersymmetry arguments, it was progressively realized that equivariant cohomology could be thought of as the proper mathematical background at the root of such constructions [W88, BS88, B92, OSB89, BS91, K93].

While equivariant cohomology is more than twenty years old [C50, GHV73, AB84, MQ86, BGV91, K93] relatively little is known about the corresponding field theory models in which both ultraviolet and infrared problems arise. Here, we shall have in mind a perturbative local field theory approach which is probably suitable since it is conjectured that the semiclassical approximation is exact. This sheds no light on the infrared problem, and in particular, the question of integration over moduli. These notes will focus on algebraic aspects needed to constrain the above mentioned field theories. Two models will be studied to some extent : "topological" Yang-Mills in four dimensions (YM_4^{top}), pure topological gravity in two dimensions (Gr_2^{top}). These are the examples for which equivariant cohomology is needed. Topological σ-models [WBS88] barely need such refinements unless they are coupled to Gr_2^{top}. In all models, on the other hand, field theory is the ideal set up to perform "fiber" integration.

These notes will be divided into three parts :

Section II will be devoted to a description of equivariant cohomology with emphasis on the points needed in the following sections.

Section III is devoted to YM_4^{top}.

Section IV is devoted to Gr_2^{top}.

The point of view taken here will be as algebraic as possible since it is the first step to control the perturbative renormalization problems to be solved next.

II) Equivariant cohomology [K93].

Let $\mathcal{M}$ be a smooth manifold and $\Omega^*(\mathcal{M})$ the exterior algebra of differential forms on $\mathcal{M}$ endowed with the differential $d_\mathcal{M}$. A Lie group $\mathcal{G}$ is assumed to be acting on $\mathcal{M}$ and its Lie algebra will be denoted Lie$\mathcal{G}$. For any $\lambda \in \mathrm{Lie}\mathcal{G}$ there is a vector field $\lambda_\mathcal{M}$ representing the infinitesimal action of λ on $\mathcal{M}$. This vector field $\lambda_\mathcal{M}$ is usually called the fundamental vector field associated with λ. We shall denote $i_\mathcal{M}(\lambda) = i_\mathcal{M}(\lambda_\mathcal{M})$ and $l_\mathcal{M}(\lambda) = l_\mathcal{M}(\lambda_\mathcal{M}) = [i_\mathcal{M}(\lambda), d_\mathcal{M}]$ the contraction (or inner derivative) and Lie derivative acting on $\Omega^*(\mathcal{M})$. Let us recall that $i_\mathcal{M}(\lambda)$ takes n-forms into (n−1)-forms while $l_\mathcal{M}(\lambda)$ acts on forms without changing their degrees. Elements of $\Omega^*(\mathcal{M})$ which are annihilated by both $i_\mathcal{M}(\lambda)$ and $l_\mathcal{M}(\lambda)$, for any $\lambda \in \mathrm{Lie}\mathcal{G}$, are the so-called ***basic*** elements of $\Omega^*(\mathcal{M})$ for the action of $\mathcal{G}$. The ***basic cohomology*** of $\mathcal{M}$, for the action of $\mathcal{G}$, is accordingly defined [C50].

We now consider the Weil algebra $\mathcal{W}$ of Lie$\mathcal{G}$. It is generated by the "connection" ω and its curvature Ω :

$$\Omega = d_{\mathcal{W}}\omega + \frac{1}{2}[\omega,\omega] \tag{2.1}$$

where $d_{\mathcal{W}}$ is the differential of $\mathcal{W}$. Of course, one has the Bianchi identity :

$$d_{\mathcal{W}}\Omega + [\omega,\Omega] = 0 \tag{2.2}$$

There is an action $i_{\mathcal{W}}(\lambda)$, $l_{\mathcal{W}}(\lambda)$ for $\lambda \in$ Lie$\mathcal{G}$:

$$i_{\mathcal{W}}(\lambda)\omega = \lambda \quad , \quad l_{\mathcal{W}}(\lambda)\omega = -[\lambda , \omega] \tag{2.3a}$$

$$i_{\mathcal{W}}(\lambda)\Omega = 0 \quad , \quad l_{\mathcal{W}}(\lambda)\Omega = -[\lambda , \Omega] \tag{2.3b}$$

For instance, ω may be a connection on a principal $\mathcal{G}$-bundle Π and Ω its curvature. In that case $i_{\mathcal{W}}(\lambda)$ and $l_{\mathcal{W}}(\lambda)$ are generated by the action of $\mathcal{G}$ on Π, and $\mathcal{W}$ will be referred to as $\mathcal{W}_{\Pi}$.

We now consider the graded algebra $\Omega^*(\mathcal{M})\otimes\mathcal{W}$ equipped with the differential $d_{\mathcal{M}} + d_{\mathcal{W}}$ so that $(\Omega^*(\mathcal{M})\otimes\mathcal{W},\ d_{\mathcal{M}} + d_{\mathcal{W}})$ turns into a graded differential algebra. Finally, the operations $i_{\mathcal{M}} + i_{\mathcal{W}}$ and $l_{\mathcal{M}} + l_{\mathcal{W}}$ are defined on $(\Omega^*(\mathcal{M})\otimes\mathcal{W},\ d_{\mathcal{M}} + d_{\mathcal{W}})$. The so-called ***equivariant cochains*** are the elements of $\Omega^*(\mathcal{M})\otimes\mathcal{W}$ that are annihilated by $(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda)$ and $(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda)$ for any $\lambda \in$ Lie$\mathcal{G}$, and the ***equivariant cohomology***, for the action of $\mathcal{G}$, is accordingly defined. This is what is called the ***Weil scheme*** for equivariant cohomology.

Equivariant cohomology can be alternatively described in the so-called ***intermediate scheme***, which was introduced in [K93] and which will be repeatedly used in the sequel. It is obtained from the Weil scheme via of the following algebra isomorphism :

$$x \to \exp\{-i_{\mathcal{M}}(\omega)\}x \tag{2.4}$$

for any $x \in \Omega^*(\mathcal{M})\otimes\mathcal{W}$. This isomorphism changes the original differential and operations on $\Omega^*(\mathcal{M})\otimes\mathcal{W}$ by conjugation[3] :

$$d_{\mathcal{M}} + d_{\mathcal{W}} \quad \to \quad D = d_{\mathcal{W}} + d_{\mathcal{M}} + l_{\mathcal{M}}(\omega) - i_{\mathcal{M}}(\Omega) \tag{2.5a}$$

$$(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda) \quad \to \quad i_{\mathcal{W}}(\lambda) = e^{-i_{\mathcal{M}}(\omega)}(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda)e^{i_{\mathcal{M}}(\omega)} \tag{2.5b}$$

$$(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) \quad \to \quad (l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) = e^{-i_{\mathcal{M}}(\omega)}(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda)e^{i_{\mathcal{M}}(\omega)} \tag{2.5c}$$

---

[3] These equations can be easily obtained by introducing the family of isomorphisms $x \to \exp\{-t \cdot i_{\mathcal{M}}(\omega)\}x$, $0 \le t \le 1$, and solving the differential equations for the transformed differential and operations, recalling that $i_{\mathcal{W}}(\lambda)\omega = \lambda$.

Finally, the so-called ***Cartan model*** is obtained from the intermediate scheme by putting $\omega = 0$ so that $D^2|_{\omega=0}$ vanishes when restricted to invariant cochains. This is the most popular model, although many calculations are better automatized in the intermediate scheme.

Another item which will be repeatedly used is "Cartan's theorem 3" [C50] : let us assume that $(\Omega^*(\mathcal{M}), d_{\mathcal{M}}, i_{\mathcal{M}}, l_{\mathcal{M}})$ admits a $\mathcal{G}$-connection, that is to say a Lie$\mathcal{G}$-valued 1-form θ on $\mathcal{M}$ such that $i_{\mathcal{M}}(\lambda)\theta = \lambda$ and $l_{\mathcal{M}}(\lambda)\theta = -[\lambda, \theta]$ for any $\lambda \in \mathrm{Lie}\mathcal{G}$, with curvature Θ. Then any equivariant cohomology class of $\Omega^*(\mathcal{M}) \otimes \mathcal{W}$ with representative $P(\omega,\Omega)$ gives rise canonically to a basic cohomology class of $\Omega^*(\mathcal{M})$ with representative $P(\theta,\Theta)$. This can be easily proven by using the homotopy which allows to prove the triviality of the cohomology of the Weil algebra [MSZ85]. It follows from the construction that the cohomology class of $P(\theta,\Theta)$ does not depend on θ (see Appendix B).

One convenient way to produce equivariant cohomology classes is as follows [BGV91] : we consider an H-bundle $\mathcal{P}(\mathcal{M},H)$ on which there exists an action of $\mathcal{G}$ which lifts the action of $\mathcal{G}$ on $\mathcal{M}$. In general, the Lie group H has nothing to do with the Lie group $\mathcal{G}$. As before, $\mathcal{P}(\mathcal{M},H)$ is endowed with a differential $d_{\mathcal{P}}$, a contraction $i_{\mathcal{P}}$ and a Lie derivative $l_{\mathcal{P}}$.

Next, let Γ be a $\mathcal{G}$-invariant H-connection on $\mathcal{P}(\mathcal{M},H)$:

$$l_{\mathcal{P}}(\lambda)\Gamma = 0 \qquad \text{for any } \lambda \in \mathrm{Lie}\mathcal{G} \tag{2.6}$$

The pull-back $\hat{\Gamma}$ of Γ on $\Omega^*(\mathcal{M}) \otimes \mathcal{W} \otimes \mathrm{LieH}$ is a 1-form on $\mathcal{P}(\mathcal{M},H)$ and a 0-form in $\mathcal{W}$. It follows that[4] :

$$i_{\mathcal{W}}(\lambda)\hat{\Gamma} = 0 \tag{2.7}$$

for any $\lambda \in \mathrm{Lie}\mathcal{G}$.

In $\Omega^*(\mathcal{P}(\mathcal{M},H)) \otimes \mathcal{W}$, the equivariant curvature of $\hat{\Gamma}$ is defined by :

$$R^{eq}_{int}(\hat{\Gamma},\omega,\Omega) = D\hat{\Gamma} + \frac{1}{2}[\hat{\Gamma},\hat{\Gamma}] \tag{2.8}$$

where $D = d_{\mathcal{W}} + d_{\mathcal{P}} + l_{\mathcal{P}}(\omega) - i_{\mathcal{P}}(\Omega)$. Then, if I_H is a symmetric invariant polynomial on LieH, we consider the H-characteristic class $I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = I_H(R^{eq}_{int}(\hat{\Gamma},\omega,\Omega))$. It is defined on $\mathcal{M}$ and fulfills :

$$(d_{\mathcal{W}} + d_{\mathcal{M}} + l_{\mathcal{M}}(\omega) - i_{\mathcal{M}}(\Omega)) I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = 0 \tag{2.9a}$$

$$i_{\mathcal{W}}(\lambda) I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = 0 \tag{2.9b}$$

$$(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) I^{eq}_{H,int}(\hat{\Gamma},\omega,\Omega) = 0 \tag{2.9c}$$

---

[4] This construction may be extended by choosing for $\mathcal{W}$ a $\mathcal{W}_\Pi$ for some Π as above, and have Γ depending parametrically on points of Π. Equation (2.6) has then to be replaced by : $(l_\Pi + l_{\mathcal{P}})(\lambda)\Gamma = 0$ whereas (2.7) still holds.

In the Weil scheme, the equivariant curvature is defined by :

$$\mathrm{R}_{\mathrm{W}}^{\mathrm{eq}}(\hat{\Gamma},\omega,\Omega)=\left(\mathrm{d}_{\mathcal{G}}+\mathrm{d}_{\mathcal{W}}\right)\left(\hat{\Gamma}+i_{\mathcal{G}}(\omega)\hat{\Gamma}\right)+\frac{1}{2}\left[\left(\hat{\Gamma}+i_{\mathcal{G}}(\omega)\hat{\Gamma}\right),\left(\hat{\Gamma}+i_{\mathcal{G}}(\omega)\hat{\Gamma}\right)\right]_{\mathrm{H}} \quad (2.10)$$

We may similarly consider $I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\omega,\Omega\right)=I_{\mathrm{H}}\left(\mathrm{R}_{\mathrm{W}}^{\mathrm{eq}}(\hat{\Gamma},\omega,\Omega)\right)=\exp\left\{-i_{\mathcal{G}}(\omega)\right\}I_{\mathrm{H}}\left(\mathrm{R}_{\mathrm{int}}^{\mathrm{eq}}(\hat{\Gamma},\omega,\Omega)\right)$ which fulfills :

$$(\mathrm{d}_{\mathcal{M}}+\mathrm{d}_{\mathcal{W}})I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\omega,\Omega\right)=0 \quad (2.11\mathrm{a})$$

$$(i_{\mathcal{M}}+i_{\mathcal{W}})(\lambda)I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\omega,\Omega\right)=0 \quad (2.11\mathrm{b})$$

$$(l_{\mathcal{M}}+l_{\mathcal{W}})(\lambda)I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\omega,\Omega\right)=0 \quad (2.11\mathrm{c})$$

Finally, if $\mathcal{M}$ admits a $\mathcal{G}$-connection θ with curvature Θ, we can apply "Cartan's theorem 3", and substitute θ and Θ instead of ω and Ω in $I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\omega,\Omega\right)$[5], so that :

$$\mathrm{d}_{\mathcal{M}}I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\theta,\Theta\right)=0 \quad (2.12\mathrm{a})$$

$$i_{\mathcal{M}}(\lambda)I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\theta,\Theta\right)=0 \quad (2.12\mathrm{b})$$

$$l_{\mathcal{M}}(\lambda)I_{\mathrm{H,W}}^{\mathrm{eq}}\left(\hat{\Gamma},\theta,\Theta\right)=0 \quad (2.12\mathrm{c})$$

By standard arguments, these cohomology classes do not depend either on $\hat{\Gamma}$ or on θ. In the following $\mathcal{P}(\mathcal{M},\mathrm{H})$ will be a family of H-bundles over a finite dimensional manifold Σ and $\mathcal{M}$ will be itself an infinite dimensional fibered manifold with fiber Σ and base, a space of fields defined on Σ. In this set up, the generators of the Weil algebra can be also realized as fields on Σ.

As we shall see in the sequel, this rather modest equipment proves quite useful to understand many features of the cohomological theories. The interesting aspects lie in the interconnection between various equivariant cohomologies, schematically, one attached to fields and one attached to observables as just described.

One final remark is in order : the above constructions only involve Lie algebras. In practice, this may not be enough and global group properties may have to be checked.

III) Topological Yang-Mills (YM_4^{top}) [W88, BS88].

At the geometric level as well as at the field theory level, one has to distinguish the fields and the observables.

[5] One may wonder why one does not use such a connection right from the beginning. The reader may convince himself that doing so would spoil the main algebraic properties of the whole construction, e.g. $\mathrm{D}^2=0$, with D the differential of equation (2.5a).

In YM_4^{top}, the idea is to produce cohomology classes of $\mathcal{A}/\mathcal{G}$ where $\mathcal{A}$ is a suitably defined space of connections a on some principal G-bundle $P(\Sigma,G)$ over a four-dimensional space-time manifold Σ and $\mathcal{G}$ is a suitably defined gauge group (group of vertical automorphisms of $P(\Sigma,G)$) [DK90]. The differential and operations are respectively denoted by d_Σ, i_Σ and l_Σ for Σ, d_P, i_P and l_P for $P(\Sigma,G)$ and δ, $\mathcal{I}$ and $\mathcal{L}$ for $\mathcal{A}$.

To produce the structure equations of the model, we follow section II. Here, $\mathcal{M}=\mathcal{A}$ and $\mathcal{W}$ is realized by a $\mathcal{G}$-connection $\tilde{\omega}$ and its curvature $\tilde{\Omega}$ on another copy $\tilde{\mathcal{A}}$ of $\mathcal{A}$. The differential and operations on $\tilde{\mathcal{A}}$ are denoted by $\tilde{\delta}$, $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{L}}$. The fields will be chosen as a, δa, $\tilde{\omega}$ and $\tilde{\Omega}$.

The structure equations then read :

$$s^{top}a = \Psi + \mathcal{L}(\tilde{\omega})a = \Psi + \mathcal{L}^{top}(\tilde{\omega})a \equiv \Psi - D_a\tilde{\omega} \tag{3.1a}$$

$$s^{top}\Psi = -\mathcal{L}^{top}(\tilde{\Omega})a + \mathcal{L}^{top}(\tilde{\omega})\Psi \equiv -D_a\tilde{\Omega} + [\Psi,\tilde{\omega}] \tag{3.1b}$$

$$s^{top}\tilde{\omega} = \tilde{\Omega} - \frac{1}{2}[\tilde{\omega},\tilde{\omega}] \tag{3.1c}$$

$$s^{top}\tilde{\Omega} = [\tilde{\Omega},\tilde{\omega}] \tag{3.1d}$$

where :

$$s^{top} = \tilde{\delta} + \delta + \mathcal{L}(\tilde{\omega}) - \mathcal{I}(\tilde{\Omega}) \quad , \quad \Psi = \delta a \equiv \Psi_{int} \tag{3.2}$$

in the intermediate scheme, whereas :

$$s^{top} = \tilde{\delta} + \delta \quad , \quad \Psi = \delta a - \mathcal{L}(\tilde{\omega})a = \delta a - \mathcal{L}^{top}(\tilde{\omega})a \equiv \Psi_W \tag{3.3}$$

in the Weil scheme, and $\mathcal{L}^{top} = \tilde{\mathcal{L}} + \mathcal{L}$ in both schemes[6]. One can check that :

$$\Psi_{int} = \exp\{-\mathcal{I}(\tilde{\omega})\}\Psi_W \tag{3.4}$$

$$\mathcal{I}^{top}(\lambda)\tilde{\omega} = \lambda \quad , \quad \mathcal{I}^{top}(\lambda)(\text{other}) = 0 \tag{3.5}$$

for any $\lambda \in \mathrm{Lie}\mathcal{G}$, with $\mathcal{I}^{top}(\lambda) = \tilde{\mathcal{I}}(\lambda)$ in the intermediate scheme and $\mathcal{I}^{top}(\lambda) = \tilde{\mathcal{I}}(\lambda) + \mathcal{I}(\lambda)$ in the Weil scheme.

Now choose $\mathcal{M} = \mathcal{A} \times \Sigma$, $\mathcal{P}(\mathcal{M},H) = \mathcal{P}(\mathcal{A} \times \Sigma, G) = \mathcal{A} \times P(\Sigma,G)$ and $\hat{\Gamma} = \hat{a}$: for any point a of $\mathcal{A}$ we consider the principal bundle $P(\Sigma,G)$ equipped with the connection a. This is a family $\hat{a}$ of G-connections such that $\hat{a}(a,p) = a(p)$ for any $(a,p) \in \mathcal{A} \times P(\Sigma,G)$, which defines a G-connection on $\mathcal{P}(\mathcal{A} \times \Sigma, G)$. We extend $\hat{a}$ to $\tilde{\mathcal{A}} \times \mathcal{P}(\mathcal{A} \times \Sigma, G)$. As a zero-form on $\mathcal{A}$ and a LieG-valued 1-form on $P(\Sigma,G)$, $\hat{a}$ is a LieG-valued 1-form on $\mathcal{P}(\mathcal{A} \times \Sigma, G)$.

---

[6] To get equation (3.1b) in the Weil scheme, one can either use (3.4) together with (2.5c) or directly compute it by using : $\tilde{\mathcal{L}}(\tilde{\omega})\mathcal{L}(\tilde{\omega})a = \mathcal{L}\big(\tilde{\mathcal{L}}(\tilde{\omega})\tilde{\omega}\big)a = -\mathcal{L}\big([\tilde{\omega},\tilde{\omega}]\big)a = 2\mathcal{L}^{top}(\tilde{\omega})\mathcal{L}(\tilde{\omega})a$.

From Appendix A, the fundamental vector field $\underline{\lambda}$ associated with the action of $\lambda \in \mathrm{Lie}\mathcal{G}$ on $\mathcal{P}(\mathcal{A} \times \Sigma, G)$ takes the following expression at $(a,p) \in \mathcal{P}(\mathcal{A} \times \Sigma, G)$:

$$\underline{\lambda} = l_P(\lambda_P) a_\mu \frac{\delta}{\delta a_\mu} - \lambda_P^\alpha e_\alpha \tag{3.6a}$$

where λ_P is the fundamental vector field on $P(\Sigma,G)$ associated with λ (for the natural left-action of $\mathcal{G}$ on $P(\Sigma,G)$) and e_α the fundamental vector field associated with a basis of LieG indexed by α. Noting that $\hat{a}$ does not really depend on $\widetilde{\mathcal{A}}$, the actions of Lie$\mathcal{G}$ on $\hat{a}$ reads :

$$\begin{aligned}
\left(\widetilde{\mathcal{I}} + \mathcal{I} + i_P\right)(\lambda)\hat{a} &\equiv \mathcal{I}(\lambda)\hat{a} - i_P(\lambda)\hat{a} = -\hat{\lambda} \\
\left(\widetilde{\mathcal{L}} + \mathcal{L} + l_P\right)(\lambda)\hat{a} &\equiv \mathcal{L}(l_P(\lambda_P) a_\mu \frac{\delta}{\delta a_\mu})\hat{a} - l_P(\lambda_P)\hat{a} \\
&= l_P(\lambda_P)\hat{a} - l_P(\lambda_P)\hat{a} = 0
\end{aligned} \tag{3.6b}$$

where $\hat{\lambda}$ is a LieG-valued function on $\mathcal{A} \times P(\Sigma,G)$ defined by : $\hat{\lambda}(a,p) = \lambda(p)$ for any element (a,p) of $\mathcal{A} \times P(\Sigma,G)$. From equations (3.6b), one sees that $\hat{a}$ is Lie$\mathcal{G}$-invariant.

In the intermediate scheme, the equivariant curvature of $\hat{a}$ is :

$$F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\widetilde{\omega},\widetilde{\Omega}\right) = D\hat{a} + \frac{1}{2}[\hat{a},\hat{a}] \tag{3.7}$$

with :

$$D = \widetilde{\delta} + \left(\delta + d_P\right) + \left(\mathcal{L} + l_P\right)(\widetilde{\omega}) - \left(\mathcal{I} + i_P\right)(\widetilde{\Omega}) \tag{3.8}$$

Taking into account the Lie$\mathcal{G}$-invariance of $\hat{a}$, we get :

$$F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\widetilde{\omega},\widetilde{\Omega}\right) = \hat{F}(\hat{a}) + \delta\hat{a} + i_P(\widetilde{\Omega})\hat{a} = \hat{F}(a) + \hat{\hat{\Psi}}_{\mathrm{int}} + \hat{\hat{\Omega}} \tag{3.9}$$

where $\hat{F}(\hat{a}) = d_P\hat{a} + \frac{1}{2}[\hat{a},\hat{a}]$. Notice the similarity of equation (3.9) with equation (3.2) up to the symbol ^. Moreover, using the $\mathcal{G}$-invariance of $\hat{a}$, one can verify the $\mathcal{G}$-basicity condition :

$$\widetilde{\mathcal{I}}(\lambda) F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\widetilde{\omega},\widetilde{\Omega}\right) = 0 \quad , \quad \left(\widetilde{\mathcal{L}} + \mathcal{L} + l_P\right)(\lambda) F_{\mathrm{int}}^{\mathrm{eq}}\left(\hat{a},\widetilde{\omega},\widetilde{\Omega}\right) = 0 \tag{3.10}$$

holding for any $\lambda \in \mathrm{Lie}\mathcal{G}$.

In order to go to the Weil scheme, we transform $\hat{a}$ as follows :

$$\hat{a} \rightarrow e^{(\mathcal{I} + i_P)(\widetilde{\omega})}\hat{a} = \hat{a} + \left(\mathcal{I} + i_P\right)(\widetilde{\omega})\hat{a} = \hat{a} - i_P(\widetilde{\omega})\hat{a} \equiv \hat{a} + \hat{\hat{\omega}} \tag{3.11}$$

The corresponding equivariant curvature is :

$$F_W^{\mathrm{eq}}\left(\hat{a},\widetilde{\omega},\widetilde{\Omega}\right) = \left(\widetilde{\delta} + \delta + d_P\right)\left(\hat{a} - i_P(\widetilde{\omega})\hat{a}\right) + \frac{1}{2}\left[\left(\hat{a} - i_P(\widetilde{\omega})\hat{a}\right),\left(\hat{a} - i_P(\widetilde{\omega})\hat{a}\right)\right] \tag{3.12}$$

or equivalently :

$$F^{eq}_{W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=\hat{F}(\hat{a})+\left(\delta\hat{a}+D_{a}\hat{\tilde{\omega}}\right)+\hat{\tilde{\Omega}}=\hat{F}(\hat{a})+\hat{\tilde{\Psi}}_{W}+\hat{\tilde{\Omega}} \tag{3.13}$$

By construction : $F^{eq}_{W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=\exp\{(\mathcal{I}+i_{P})(\tilde{\omega})\}F^{eq}_{int}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)$ and consequently :

$$\begin{aligned}\left(\tilde{\mathcal{I}}+\mathcal{I}+i_{P}\right)(\lambda)F^{eq}_{W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=0\\ \left(\tilde{\mathcal{L}}+\mathcal{L}+l_{P}\right)(\lambda)F^{eq}_{W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=0\end{aligned} \tag{3.14}$$

Now, for any symmetric invariant polynomial I_{G} on LieG, $I^{eq}_{G,W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=I_{G}\left(F^{eq}_{W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)\right)$ fulfills :

$$(\tilde{\delta}+\delta+d_{P})I^{eq}_{G,W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=0 \tag{3.15a}$$

$$\left(\tilde{\mathcal{I}}+\mathcal{I}+i_{P}\right)(\lambda)I^{eq}_{G,W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=0 \tag{3.15b}$$

$$\left(\tilde{\mathcal{L}}+\mathcal{L}+l_{P}\right)(\lambda)I^{eq}_{G,W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)=0 \tag{3.15c}$$

for any $\lambda\in\mathrm{Lie}\mathcal{G}$.

Last but not least, we apply "Cartan's theorem 3" to $I^{eq}_{G,W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)$. Let ω be a $\mathcal{G}$-connection on $\mathcal{A}$ and Ω its curvature. It does define a $\mathcal{G}$-connection on $\mathcal{M}=\mathcal{A}\times\Sigma$. Accordingly, we just replace $\tilde{\omega}$ and $\tilde{\Omega}$ respectively by ω and Ω in $I^{eq}_{G,W}\left(\hat{a},\tilde{\omega},\tilde{\Omega}\right)$. Then :

$$(\delta+d_{P})I^{eq}_{G,W}(\hat{a},\omega,\Omega)=(\delta+d_{\Sigma})I^{eq}_{G,W}(\hat{a},\omega,\Omega)=0 \tag{3.16a}$$

$$(\mathcal{I}+i_{P})(\lambda)I^{eq}_{G,W}(\hat{a},\omega,\Omega)=(\mathcal{I}+i_{\Sigma})(\lambda)I^{eq}_{G,W}(\hat{a},\omega,\Omega)=0 \tag{3.16b}$$

$$(\mathcal{L}+l_{P})(\lambda)I^{eq}_{G,W}(\hat{a},\omega,\Omega)=(\mathcal{L}+l_{\Sigma})(\lambda)I^{eq}_{G,W}(\hat{a},\omega,\Omega)=0 \tag{3.16c}$$

for any $\lambda\in\mathrm{Lie}\mathcal{G}$. Recall that :

$$F^{eq}_{W}(\hat{a},\omega,\Omega)=\hat{F}(a)+\left(\delta\hat{a}+D_{a}\hat{\omega}\right)+\hat{\Omega}=\hat{F}(a)+\hat{\Psi}_{W}+\hat{\Omega} \tag{3.17}$$

with : $\hat{\omega}\equiv-i_{P}(\omega)\hat{a}$ and $\hat{\varphi}\equiv i_{P}(\Omega)\hat{a}$.

In fact, $I^{eq}_{G,W}(\hat{a},\omega,\Omega)$ fulfills a horizontality property stronger than (3.16b), namely :

$$\mathcal{I}(\lambda)I^{eq}_{G,W}(\hat{a},\omega,\Omega)=0=i_{P}(\lambda)I^{eq}_{G,W}(\hat{a},\omega,\Omega)=i_{\Sigma}(\lambda)I^{eq}_{G,W}(\hat{a},\omega,\Omega) \tag{3.18}$$

and is defined on $\mathcal{A}\times\Sigma$. Now, let us decompose $I^{eq}_{G,W}(\hat{a},\omega,\Omega)$ according to :

$$I^{eq}_{G,W}(\hat{a},\omega,\Omega)=\sum_{k=0}^{2n}I^{k}_{2n-k} \tag{3.19}$$

where I^{k}_{2n-k} is a form of degree $2n-k$ on Σ and of degree k on $\mathcal{A}$ such that :

$$
\begin{aligned}
&d_\Sigma I^{k}_{2n-k} + \delta I^{k-1}_{2n-k+1} = 0 \\
&\mathcal{I}(\lambda) I^{k+1}_{2n-k-1} - i_\Sigma(\lambda) I^{k}_{2n-k} = 0 \\
&\mathcal{L}(\lambda) I^{k}_{2n-k} - l_\Sigma(\lambda) I^{k}_{2n-k} = 0
\end{aligned}
\tag{3.20}
$$

Then, the integration over a cycle γ_{2n-k} on Σ yields a k-form on $\mathcal{A}$:

$$
\mathcal{O}^k = \int_{\gamma_{2n-k}} I^{k}_{2n-k}
\tag{3.21}
$$

From the descent equations (3.20), we deduce :

$$
\delta \mathcal{O}^k = - \int_{\gamma_{2n-k}} d_\Sigma I^{k+1}_{2n-k-1} = 0
\tag{3.22}
$$

and because of the detailed horizontality condition expressed in equation (3.18) :

$$
\mathcal{I}(\lambda) \mathcal{O}^k = \int_{\gamma_{2n-k}} \mathcal{I}(\lambda) I^{k}_{2n-k} = 0
\tag{3.23}
$$

Finally :

$$
\begin{aligned}
\mathcal{L}(\lambda)\mathcal{O}^k &= \int_{\gamma_{2n-k}} l_\Sigma(\lambda) I^{k}_{2n-k} = \int_{\gamma_{2n-k}} i_\Sigma(\lambda) d_\Sigma I^{k}_{2n-k} = \\
&= \int_{\gamma_{2n-k}} i_\Sigma(\lambda) \delta I^{k-1}_{2n-k+1} = \delta \int_{\gamma_{2n-k}} i_\Sigma(\lambda) I^{k-1}_{2n-k+1} = 0
\end{aligned}
\tag{3.24}
$$

Hence, the k-form $\mathcal{O}^k$ defines a basic cohomology class[7]. This class does not depend on $\hat{a}$, $\hat{\omega}$, $\hat{\Omega}$ and $\hat{\Psi}_W$ provided that they are related by equation (3.17), so that one may average it out over these fields variables, which is the formal reason why the topological YM_4^{top} field theory should be a tool able to construct such cohomology classes. Of course, this is so provided the field theory treatment (e.g. renormalized perturbation theory which in the present case ought to be exact) retains enough properties of the averaging out process, which, in turn will be insured by the fulfillment of the proper Ward identities entailed by the requirement of s^{top}, $\mathcal{I}(\lambda)$ and $\mathcal{L}(\lambda)$ invariance. Note that the equivalence between the structure equations (3.1-3) and those leading to the construction of the observables (equations (3.17)) is insured by "Cartan's theorem 3", which, at the cohomology level allows one to replace $\tilde{\omega}$ and $\tilde{\Omega}$ by $\hat{\omega}$ and $\hat{\Omega}$.

For a review of the field theory context, we refer to [OSB89] supplemented with the proof, provided in [K93], that the basic cohomology proposed there is isomorphic with that proposed in [W88] in view of the equivalence between the intermediate model and the Cartan model. Of course, these (ultraviolet) considerations do not touch the problem of the integration of the relevant cohomology classes over orbit space.

---

[7] An alternative much faster construction is given in Appendix C. This one is identical to that used for 2d topological gravity. That of Appendix C takes advantage of the product structure $\mathcal{P}(\mathcal{M},G) = \mathcal{A} \times P(\Sigma,G)$.

IV) Topological 2d gravity (Gr_2^{top}).

Let Σ be a compact Riemann surface without boundary, of genus larger than one. We recall that the space of complex structures on Σ can be canonically identify with the space $\mathcal{B}(\Sigma)$ of Beltrami differentials on Σ. The origin in $\mathcal{B}(\Sigma)$ is nothing but the complex analytic structure defining Σ itself. Let us introduce more notations : $\mathcal{M}(\Sigma)$ is the space of metrics on Σ; $\mathcal{W}(\Sigma)$ is the group of Weyl transformations acting on $\mathcal{M}(\Sigma)$ by local scaling of the metrics; the space $\mathcal{M}(\Sigma)\ /\ \mathcal{W}(\Sigma)$ of conformal classes of metrics on Σ is denoted by $\mathcal{CM}(\Sigma)$ and is naturally isomorphic to $\mathcal{B}(\Sigma)$; finally, $\mathcal{D}_0(\Sigma)$ is the component of the group $\mathcal{D}(\Sigma)$ of diffeomorphisms of Σ connected to the identity. We recall that the Lie algebra of $\mathcal{D}_0(\Sigma)$ is the opposite of the Lie algebra $\mathcal{V}(\Sigma)$ of vector fields on Σ [Mi].

Let $\{(U_\alpha,(z_\alpha,\bar z_\alpha))\}$ be an atlas defining the complex analytic structure of Σ, and let g be a metric on Σ. With respect to this atlas, the metric element takes the form :

$$ds^2 = \rho_{z_\alpha \bar z_\alpha}\left|dz_\alpha + \mu^{z_\alpha}{}_{\bar z_\alpha} d\bar z_\alpha\right|^2 \tag{4.1}$$

where $\mu^{z_\alpha}{}_{\bar z_\alpha}$ is the component in $(z_\alpha,\bar z_\alpha)$ of the Beltrami differential $\mu = \mu^{z_\alpha}{}_{\bar z_\alpha}\partial_{z_\alpha}\otimes d\bar z_\alpha$ parametrizing the conformal class of the metric g. Note that equation (4.1) produces an isomorphism between $\mathcal{CM}(\Sigma)$ and $\mathcal{B}(\Sigma)$.

In topological (Euclidean) 2d gravity, one first wishes to study the Teichmüller space $\mathcal{T}(\Sigma)$ of Σ and later go over to the moduli space (as already explain in section II, we do not consider the global group properties and hence do not look at the whole group of diffeomorphisms). There are two ways to define $\mathcal{T}(\Sigma)$. In the first one, that we shall refer to as the "Riemannian route", one considers $\mathcal{M}(\Sigma)$ as the parameter space together with the action of $\mathcal{W}(\Sigma)\tilde{\times}\mathcal{D}_0(\Sigma)$[8] on it :

$$\mathcal{T}(\Sigma) = \frac{\mathcal{M}(\Sigma)}{\mathcal{W}(\Sigma)\tilde{\times}\mathcal{D}_0(\Sigma)} \tag{4.2a}$$

In the second approach, the space of parameters is $\mathcal{B}(\Sigma)$ and the "gauge group" $\mathcal{D}_0(\Sigma)$ so that :

$$\mathcal{T}(\Sigma) = \frac{\mathcal{B}(\Sigma)}{\mathcal{D}_0(\Sigma)} \tag{4.2b}$$

This will be referred to as the "Conformal route". The equivalence between the Conformal and Riemannian routes comes from the canonical identification of $\mathcal{CM}(\Sigma)$ with $\mathcal{B}(\Sigma)$. The former is natural from the mathematical point of view but presumably less amenable to a field theory treatment by virtue of the non-linearities involved. The latter, closer to field theory [BCI94], will be exhibited as an alternative.

[8] Where $\tilde{\times}$ denotes the semi-direct product.

In the Conformal route, the Weyl transformations are eliminated from the start by fixing the factor ρ of equation (4.1), as a function of μ and $\bar{\mu}$, through the $\mathcal{D}(\Sigma)$ invariant constraint :

$$R(\rho,\mu,\bar{\mu}) = -1 \tag{4.3}$$

where R is the scalar curvature of the metric (4.1). (Recall this is possible because the genus of Σ was assumed to be larger than 1).

We take a $\mathcal{D}_0(\Sigma)$-connection $\tilde{\omega}$ on another copy $\tilde{\mathcal{B}}(\Sigma)$ of $\mathcal{B}(\Sigma)$. So, $\tilde{\omega}$ and its curvature $\tilde{\Omega}$ are vector field on Σ :

$$\begin{aligned} \tilde{\omega} &= \tilde{\omega}^z \partial_z + \tilde{\omega}^{\bar{z}} \partial_{\bar{z}} \\ \tilde{\Omega} &= \tilde{\Omega}^z \partial_z + \tilde{\Omega}^{\bar{z}} \partial_{\bar{z}} \end{aligned} \tag{4.4}$$

If we denote δ, $\mathcal{I}$ and $\mathcal{L}$ the differential and operations on $\mathcal{B}(\Sigma)$ and $\tilde{\delta}$, $\tilde{\mathcal{I}}$ and $\tilde{\mathcal{L}}$ those on $\tilde{\mathcal{B}}(\Sigma)$, the action of $\lambda = \lambda^z\partial_z + \lambda^{\bar{z}}\partial_{\bar{z}} \in \mathcal{V}(\Sigma)$ on $\mathcal{B}(\Sigma)$ is :

$$\begin{aligned} \mathcal{L}(\lambda)\mu &= \left[\left(\partial_{\bar{z}} - \mu^z{}_{\bar{z}}\partial_z + \left(\partial_z\mu^z{}_{\bar{z}}\right)\right)\Lambda^z_\mu\right] d\bar{z}\otimes\partial_z = \left(\mathcal{L}(\lambda)\mu^z{}_{\bar{z}}\right) d\bar{z}\otimes\partial_z \\ &= \left(D_{\mu,\bar{z}}\Lambda^z_\mu\right) d\bar{z}\otimes\partial_z = \bar{D}_\mu\Lambda_\mu \end{aligned} \tag{4.5}$$

where we have introduced the type (1,0) vector field :

$$\Lambda_\mu = \Lambda^z_\mu\,\partial_z = \left(\lambda^z + \mu^z_{\bar{z}}\lambda^{\bar{z}}\right)\partial_z \tag{4.6}$$

and to emphasize the similarities with YM_4^{top} we have defined the operator $\bar{D}_\mu$:

$$\bar{D}_\mu = D_{\mu,\bar{z}}\, d\bar{z} = \left(\partial_{\bar{z}} - \mu^z{}_{\bar{z}}\partial_z + \left(\partial_z\mu^z{}_{\bar{z}}\right)\right) d\bar{z} = \bar{\partial} - \{\mu,\ \} \tag{4.7}$$

acting on type (1,0) vector fields. In equation (4.7), $\bar{\partial}$ is the usual Dolbeault operator, μ is considered as a $\mathcal{V}(\Sigma)$-valued one-form on Σ and $\{\ ,\ \}$ is the natural Lie bracket that turns $\mathcal{V}(\Sigma)$ into a Lie algebra[9]. Finally, noting that $\tilde{\omega}$,$\bar{D}_\mu$ and μ are odd while $\tilde{\Omega}$ is even, we get the structure equations :

$$s^{top}\mu = \nu + \mathcal{L}(\tilde{\omega})\mu = \nu + \mathcal{L}^{top}(\tilde{\omega})\mu = \nu + \bar{D}_\mu\tilde{\omega}_\mu \tag{4.8a}$$

$$s^{top}\nu = -\mathcal{L}^{top}(\tilde{\Omega})\mu + \mathcal{L}^{top}(\tilde{\omega})\nu = -\bar{D}_\mu\tilde{\Omega}_\mu - \left\{\nu,\tilde{\omega}_\mu\right\} \tag{4.8b}$$

$$s^{top}\tilde{\omega}_\mu = \tilde{\Omega}_\mu + \frac{1}{2}\left\{\tilde{\omega}_\mu,\tilde{\omega}_\mu\right\} \tag{4.8c}$$

$$s^{top}\tilde{\Omega}_\mu = -\left\{\tilde{\Omega}_\mu,\tilde{\omega}_\mu\right\} \tag{4.8d}$$

where we have introduced the μ-dependent basis :

---

[9] Since $\mathcal{V}(\Sigma)$ is the opposite of Lie$\mathcal{D}(\Sigma$: $\{\underline{\lambda}_1,\underline{\lambda}_2\} = -[\underline{\lambda}_1,\underline{\lambda}_2]$, where [,] is the Lie bracket of Lie$\mathcal{D}(\Sigma)$ and we have denoted by the same symbol an element of Lie$\mathcal{D}(\Sigma)$ and its image in $\mathcal{V}(\Sigma)$ by the canonical isomorphism between these two spaces [Mi]. Now, compare equation (4.9) with equation (3.1).

$$\widetilde{\omega}_\mu = \widetilde{\omega}^z_\mu \partial_z = \left(\widetilde{\omega}^z + \mu^z{}_{\bar z} \widetilde{\omega}^{\bar z}\right)\partial_z$$
$$\widetilde{\Omega}_\mu = \widetilde{\Omega}^z_\mu \partial_z = \left(\widetilde{\Omega}^z + \mu^z{}_{\bar z}\widetilde{\Omega}^{\bar z} + v^z{}_{\bar z}\widetilde{\omega}^{\bar z}\right)\partial_z \tag{4.9}$$

with :

$$s^{top} = \widetilde{\delta} + \delta + \mathcal{L}(\widetilde{\omega}) - \mathcal{I}(\widetilde{\Omega}) \quad , \quad v = \delta\mu = \left(\delta\mu^z{}_{\bar z}\right) d\bar z \otimes \partial_z \tag{4.10}$$

in the intermediate scheme, whereas :

$$s^{top} = \widetilde{\delta} + \delta \quad , \quad v = \delta\mu + \overline{D}_\mu \widetilde{\omega}_\mu = \left(\delta\mu^z{}_{\bar z} - D_{\mu,\bar z}\widetilde{\omega}^z_\mu\right) d\bar z \otimes \partial_z \tag{4.11}$$

in the Weil scheme, and $\mathcal{L}^{top}(\lambda) = \widetilde{\mathcal{L}}(\lambda) + \mathcal{L}(\lambda)$ in both schemes.

The action of $\mathcal{V}(\Sigma)$ is given by :

$$\mathcal{I}^{top}(\lambda)\widetilde{\omega}_\mu = \Lambda_\mu$$
$$\mathcal{I}^{top}(\lambda)\widetilde{\Omega}_\mu = -\left(v^z{}_{\bar z}\lambda^{\bar z}\right)\partial_z \tag{4.12}$$
$$\mathcal{I}^{top}(\lambda) other = 0$$

with $\mathcal{I}^{top}(\lambda) = \widetilde{\mathcal{I}}(\lambda)$ in the intermediate scheme and $\mathcal{I}^{top}(\lambda) = \widetilde{\mathcal{I}}(\lambda) + \mathcal{I}(\lambda)$ in the Weil scheme. The formulae for $\mathcal{L}(\lambda)$ follow from equations (4.8) and (4.12). Had we stuck to the initial basis, there would be complete similarity with the gauge case YM_4^{top}. The μ dependent basis, more appropriate to discuss holomorphic factorization [KLS91] introduces however an inconvenience : the second of equations (4.12).

Now, choose $\mathcal{M} = \mathcal{B}(\Sigma) \times \Sigma$ equipped with the complex structure defined by the complex variables μ, Z_μ, where Z_μ are complex coordinates on Σ which fulfill (locally) the Beltrami equation :

$$\left(\partial_{\bar z} - \mu^z{}_{\bar z}\partial_z\right) Z_\mu = 0 \tag{4.13}$$

allowing to construct from Σ and μ a Riemann surface denoted by Σ_μ.

For each $\mu \in \mathcal{B}(\Sigma)$ we consider the holomorphic tangent bundle of Σ_μ. This generate the family $T^{(1,0)}_{\{\mu\}}(\Sigma)$ of holomorphic tangent bundles of Σ, and the associated GL(1,ℂ) principal bundle is denoted $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ ($\mathcal{P}(\mathcal{M},H)$ of section II). A set of holomorphic coordinates on $T^{(1,0)}_{\{\mu\}}(\Sigma)$ is given locally by μ, Z_μ, V^{Z_μ}, or $E^{Z_\mu} \in GL(1,\mathbb{C})$ on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$[10]. $\mathcal{D}(\Sigma)$ acts holomorphically on these coordinates so that along an orbit of $\mathcal{D}(\Sigma)$ one may choose the action :

---

[10] The fiber of $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ over (μ,x) is the set of all frames (i.e. bases) of $T^{(1,0)}_x\Sigma_\mu$. Hence, with respect to the chart (U,Z_μ) at $x \in \Sigma_\mu$, the coordinates E^{Z_μ} of a frame E_x are the entries of the GL(1,ℂ) matrix transforming the natural frame ∂_{Z_μ} of $T^{(1,0)}_x\Sigma_\mu$ into E_x : $E_x = E^{Z_\mu}\,\partial_{Z_\mu}$.

$$\begin{aligned} \mu &\longrightarrow \mu^{\varphi} \\ Z_\mu(z,\bar z) &\longrightarrow Z_{\mu^\varphi}(\varphi^{-1}(z,\bar z),\bar\varphi^{-1}(z,\bar z)) = Z_\mu(z,\bar z) \\ E^{Z_\mu}(z,\bar z) &\longrightarrow E^{Z_{\mu^\varphi}}(\varphi^{-1}(z,\bar z),\bar\varphi^{-1}(z,\bar z)) = E^{Z_\mu}(z,\bar z) \end{aligned} \tag{4.14}$$

(cf. Appendix D). The fundamental vector field on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ representing $\lambda \in \mathcal{V}(\Sigma)$ thus reads :

$$\underline{\lambda} = \int_\Sigma \delta_\lambda \mu \frac{\delta}{\delta\mu}\bigg|_{Z_\mu, E^{Z_\mu}} d^2z \; + \; c.c = \underline{\lambda}^h + \underline{\lambda}^{\bar h} \tag{4.15}$$

where, as indicated, the derivative with respect to μ is evaluated at fixed Z_μ and E^{Z_μ}, and with : $\delta_\lambda \mu = \mathcal{L}(\lambda)\mu = \bar D_\mu \Lambda_\mu$ (cf. equation (4.5)). The Dolbeault operators on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ are accordingly given by :

$$\mathcal{D} = \int_\Sigma \delta_\lambda \mu \frac{\delta}{\delta\mu}\bigg|_{Z_\mu, A^{Z_\mu}} d^2z + dZ_\mu \frac{\partial}{\partial Z_\mu} + dE^{Z_\mu} \frac{\partial}{\partial E^{Z_\mu}} \tag{4.16}$$

and its complex conjugate, and the total differential is :

$$D = \mathcal{D} + \bar{\mathcal{D}} \tag{4.17}$$

By construction, the contraction I and the Lie derivative L on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ split up :

$$\begin{aligned} I(\lambda) &= I(\underline{\lambda}^h) + I(\underline{\lambda}^{\bar h}) \equiv I^h(\lambda) + I^{\bar h}(\lambda) \\ L(\lambda) &= L(\underline{\lambda}^h) + L(\underline{\lambda}^{\bar h}) \equiv L^h(\lambda) + L^{\bar h}(\lambda) \end{aligned} \tag{4.18}$$

with :

$$\begin{aligned} L^h(\lambda) &= \left[I^h(\lambda), \mathcal{D}\right]_+ \\ L^{\bar h}(\lambda) &= \left[I^{\bar h}(\lambda), \bar{\mathcal{D}}\right]_+ \end{aligned} \tag{4.19}$$

so that :

$$\left[L^h(\lambda), \mathcal{D}\right] = \left[L^{\bar h}(\lambda), \bar{\mathcal{D}}\right] = 0 \tag{4.20}$$

and the operators carrying the label h, together with $\mathcal{D}$ commute (in the graded sense) with those carrying the label $\bar h$ together with $\bar{\mathcal{D}}$.

Now, for each $\mu \in \mathcal{B}(\Sigma)$ choose the metric $ds^2 = \rho_{Z_\mu \bar Z_\mu} dZ_\mu d\bar Z_\mu$ solution of the constant curvature equation :

$$\partial_{\bar Z_\mu} \partial_{Z_\mu} \ln \rho_{Z_\mu \bar Z_\mu} = \rho_{Z_\mu \bar Z_\mu} \tag{4.21}$$

equivalent to equation (4.3). In local coordinates, the canonical $GL(1,\mathbb{C})$-connection on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ associated with $\rho_{Z_\mu \bar Z_\mu}$ is :

$$\Gamma = \mathcal{D}\ln\rho_{Z_\mu \bar{Z}_\mu} + \mathrm{D}\ln \mathrm{E}^{Z_\mu} \tag{4.22}$$

where $\mathrm{D}\ln \mathrm{E}^{Z_\mu}$ denotes the Maurer-Cartan form of GL(1,$\mathbb{C}$).

It is immediate that :

$$\mathrm{L}(\lambda)\Gamma = 0 \tag{4.23}$$

since the solution of equation (4.21) is independent of the point of any orbit in $\mathcal{B}(\Sigma)$ under $\mathcal{D}(\Sigma)$.

Now, given a $\mathcal{D}(\Sigma)$-connection $\tilde{\omega}$ and its curvature $\tilde{\Omega}$ on another copy of $\mathcal{B}(\Sigma)$, the connection becomes in the Weil scheme :

$$\hat{\Gamma} = \mathcal{D}\ln\rho_{Z_\mu \bar{Z}_\mu} + \mathrm{D}\ln \mathrm{E}^{Z_\mu} + \mathrm{I}(\tilde{\omega})\mathcal{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \tag{4.24}$$

and its equivariant curvature is given by :

$$\begin{aligned} \mathrm{R}^{\mathrm{eq}}_{\mathrm{W}}(\hat{\Gamma}) &= \left(\tilde{\delta} + \mathcal{D} + \bar{\mathcal{D}}\right)\hat{\Gamma} \\ &= \bar{\mathcal{D}}\mathcal{D}\ln\rho_{Z_\mu \bar{Z}_\mu} + \mathrm{I}(\tilde{\omega})\bar{\mathcal{D}}\mathcal{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \\ &\quad - \frac{\mathrm{I}^{h}(\tilde{\Omega})\mathcal{D} - \mathrm{I}^{\bar{h}}(\tilde{\Omega})\bar{\mathcal{D}}}{2}\ln\rho_{Z_\mu \bar{Z}_\mu} \\ &\quad + \frac{\mathrm{I}(\tilde{\omega})\mathrm{I}(\tilde{\omega})}{2}\bar{\mathcal{D}}\mathcal{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \end{aligned} \tag{4.25}$$

(details are given in Appendix E).

There remains to replace $\tilde{\omega}$ and $\tilde{\Omega}$ by a $\mathcal{D}(\Sigma)$-connection θ on $\mathcal{B}(\Sigma)\times\Sigma$ and its curvature Θ (Cartan's theorem 3). This is obtained by pulling back on $\mathcal{B}(\Sigma)\times\Sigma$ a $\mathcal{D}(\Sigma)$-connection on $\mathcal{B}(\Sigma)$. All in all, we may write :

$$\mathrm{R} \equiv \mathrm{R}^{\mathrm{eq}}_{\mathrm{W}}(\hat{\Gamma},\theta,\Theta) = \mathrm{R}^{0}_{2} + \mathrm{R}^{1}_{1} + \mathrm{R}^{2}_{0} \tag{4.26}$$

where the lower index labels the form degree on Σ and the upper index labels the form degree on $\mathcal{B}(\Sigma)$, and :

$$\left(\delta + d_{\Sigma}\right)\mathrm{R} = \left(\mathcal{I} + i_{\Sigma}\right)(\lambda)\mathrm{R} = \left(\mathcal{L} + l_{\Sigma}\right)(\lambda)\mathrm{R} = 0 \tag{4.27}$$

Observables are extracted from[11] :

$$\begin{aligned} \mathrm{R}^{n} &= \left(\mathrm{R}^{2}_{0}\right)^{n} + n\left(\mathrm{R}^{2}_{0}\right)^{n-1}\mathrm{R}^{1}_{1} + \left(n\left(\mathrm{R}^{2}_{0}\right)^{n-1}\mathrm{R}^{0}_{2} + \frac{n(n-1)}{2}\left(\mathrm{R}^{2}_{0}\right)^{n-1}\left(\mathrm{R}^{1}_{1}\right)^{2}\right) \\ &= \mathcal{O}^{2}_{0} + \mathcal{O}^{2n-1}_{1} + \mathcal{O}^{2n-2}_{2} \end{aligned} \tag{4.28}$$

with :

[11] There is no need of making Gl(1,$\mathbb{C}$) invariant polynomial, because R is already invariant.

$$\begin{cases} \delta\mathcal{O}_0^{2n} = 0 \\ d_\Sigma\mathcal{O}_0^{2n} + \delta\mathcal{O}_1^{2n-1} = 0 \\ d_\Sigma\mathcal{O}_1^{2n-1} + \delta\mathcal{O}_2^{2n-2} = 0 \end{cases} \tag{4.29a}$$

$$\begin{cases} \left(\mathcal{L} + l_\Sigma\right)(\lambda)\mathcal{O}_0^{2n} = \left(\mathcal{L}(\lambda) - l_\Sigma(\lambda)\right)\mathcal{O}_0^{2n} = 0 \\ \left(\mathcal{L} + l_\Sigma\right)(\lambda)\mathcal{O}_1^{2n-1} = \left(\mathcal{L}(\lambda) - l_\Sigma(\lambda)\right)\mathcal{O}_1^{2n-1} = 0 \\ \left(\mathcal{L} + l_\Sigma\right)(\lambda)\mathcal{O}_2^{2n-2} = \left(\mathcal{L}(\lambda) - l_\Sigma(\lambda)\right)\mathcal{O}_2^{2n-2} = 0 \end{cases} \tag{4.29b}$$

$$\begin{cases} \mathcal{I}(\lambda)\mathcal{O}_2^{2n-2} = 0 \\ \mathcal{I}(\lambda)\mathcal{O}_1^{2n-1} - i_\Sigma(\lambda)\mathcal{O}_2^{2n-2} = 0 \\ \mathcal{I}(\lambda)\mathcal{O}_0^{2n} - i_\Sigma(\lambda)\mathcal{O}_1^{2n-1} = 0 \end{cases} \tag{4.29c}$$

Let us introduce :

$$\begin{aligned} \mathcal{O}^{2n-2} &= \int_\Sigma \mathcal{O}_2^{2n-2} \\ \mathcal{O}_{(\gamma)}^{2n-1} &= \int_\gamma \mathcal{O}_1^{2n-1} \\ \mathcal{O}_{(x)}^{2n-1} &= \mathcal{O}_0^{2n}(x) \end{aligned} \tag{4.30}$$

where γ (resp. x) is a one cycle (resp. 0 cycle) in Σ. One verifies that $\mathcal{O}^{2n-2}$ represents a basic cohomology class on $\mathcal{B}(\Sigma)$ since :

$$\delta\mathcal{O}^{2n-2} = \mathcal{I}(\lambda)\mathcal{O}^{2n-2} = \mathcal{L}(\lambda)\mathcal{O}^{2n-2} = 0 \tag{4.31}$$

However :

$$\delta\mathcal{O}_{(\gamma)}^{2n-1} = 0 \tag{4.32a}$$

but :

$$\mathcal{I}(\underline{\lambda})\mathcal{O}_{(\gamma)}^{2n-1} = \int_\gamma i_\Sigma(\lambda)\mathcal{O}_2^{2n-2} \neq 0 \tag{4.32b}$$

$$\begin{aligned} \mathcal{L}(\lambda)\mathcal{O}_{(\gamma)}^{2n-1} &= \int_\gamma l_\Sigma(\lambda)\mathcal{O}_1^{2n-1} = \int_\gamma i_\Sigma(\lambda)d_\Sigma\mathcal{O}_1^{2n-1} \\ &= \delta\int_\gamma i_\Sigma(\lambda)\mathcal{O}_2^{2n-2} \neq 0 \end{aligned} \tag{4.32c}$$

Hence, $\mathcal{O}_{(\gamma)}^{2n-1}$ does not represent a basic cohomology class. Similarly :

$$\delta\mathcal{O}_{(x)}^{2n} = 0 \quad \text{while} \quad \mathcal{I}(\lambda)\mathcal{O}_{(x)}^{2n} = i_\Sigma(\lambda)\mathcal{O}_1^{2n-1} \neq 0 \tag{4.33}$$

This is different from what happened in the YM_4^{top} case and is essentially due to the fact that $\mathfrak{D}(\Sigma)$ moves points on Σ.

One should realize at this point that one should make sure that whatever cohomology classes have been constructed are non trivial. It is known that modular invariance plays a crucial role in that respect [Mu, BCI94].

On the other hand, the choice of the metric ρ fulfilling the constant negative curvature condition (4.21) is immaterial provided it behaves properly under diffeomorphisms, i.e. a change in ρ produces a coboundary.

Whereas the holomorphic fibration of $\mathfrak{B}(\Sigma) \times \Sigma$ over $\mathfrak{T}(\Sigma)$ is essential (the smooth fibration being trivial), a real approach is possible whereby the GL(1,$\mathbb{C}$) bundle is reduced to U(1) and the canonical connection Γ is replaced by the unitary connection :

$$\Gamma^{\text{unit}} = \frac{\mathcal{D} - \overline{\mathcal{D}}}{2i} \ln \rho_{Z\overline{Z}} \tag{4.34}$$

This make the bridge with the Riemannian route chosen in [BCI94], as follows.

The structure equations for the action of $\mathfrak{D}(\Sigma)$ on $\mathfrak{M}(\Sigma)$ read :

$$s^{top} g = \gamma + \mathfrak{L}(\tilde{\omega}) g = \gamma + \mathfrak{L}^{top}(\tilde{\omega}) g \tag{4.35a}$$

$$s^{top} \gamma = -\mathfrak{L}^{top}(\tilde{\Omega}) g + \mathfrak{L}^{top}(\tilde{\omega}) \gamma \tag{4.35b}$$

$$s^{top} \tilde{\omega} = \tilde{\Omega} - \frac{1}{2}[\tilde{\omega}, \tilde{\omega}] \tag{4.35c}$$

$$s^{top} \tilde{\Omega} = [\tilde{\Omega}, \tilde{\omega}] \tag{4.35d}$$

where $g \in \mathfrak{M}(\Sigma)$, $\tilde{\omega}$ a $\mathfrak{D}_0(\Sigma)$-connection on another copy $\tilde{\mathfrak{M}}(\Sigma)$ of $\mathfrak{M}(\Sigma)$ and $\tilde{\Omega}$ its curvature, $\gamma = \delta g$ in the intermediate scheme, $\gamma = \delta g - \mathfrak{L}^{top}(\tilde{\omega}) g$ in the Weil scheme, and $\mathfrak{L}^{top} = \tilde{\mathfrak{L}} + \mathfrak{L}$ in both schemes. Of course, δ, $\mathfrak{I}$ and $\mathfrak{L}$ are the differential and operations on $\mathfrak{M}(\Sigma)$ while $\tilde{\delta}$, $\tilde{\mathfrak{I}}$ and $\tilde{\mathfrak{L}}$ are those on $\tilde{\mathfrak{M}}(\Sigma)$.

One may wonder why one does not write down the structure equations for the action of $\mathfrak{W}(\Sigma) \tilde{\times} \mathfrak{D}(\Sigma)$. The main reason is that there is no known Weyl invariant connection on a bundle over $\mathfrak{M}(\Sigma) \times \Sigma$ to provide non trivial cohomology classes.

We now consider the family $\mathcal{P}F_{\{g\}}(\Sigma) = \mathfrak{M}(\Sigma) \times P(\Sigma, Gl(2,\mathbb{R}))$ of frame bundles over Σ indexed by $g \in \mathfrak{M}(\Sigma)$[12]. As usual, we wish to provide $\mathcal{P}F_{\{g\}}(\Sigma)$ with a $\mathfrak{D}(\Sigma)$-invariant Gl(2,$\mathbb{R}$)-connection. Accordingly, we look for a Gl(2,$\mathbb{R}$)-connection Γ that leaves g invariant. In terms of local coordinates :

[12] It is a Gl(2,$\mathbb{R}$) principal bundle over $\mathfrak{M}(\Sigma) \times \Sigma$ whose fiber $F_{(g,x)}\Sigma$ over (g,x) is made of all the frames of $T_x\Sigma$. With respect to coordinates (x^k) of x, the coordinates of a frame $E_{(g,x)} = (E_k)$ of $F_{(g,x)}\Sigma$ will be the entries $A^j{}_k$ of the Gl(2,$\mathbb{R}$) matrix A changing the natural frame (∂_k) associated with (x^k) into $E_{(g,x)}$: $E_j = A^k{}_j \partial_k$.

$$(\delta + d_\Sigma)g_{\mu\nu} - \Gamma^\lambda{}_\mu g_{\lambda\nu} - \Gamma^\lambda{}_\nu g_{\lambda\mu} = 0 \tag{4.36}$$

A solution of this compatibility equation is given by :

$$\Gamma^\lambda{}_\mu = {}^{LC}\Gamma^\lambda{}_\mu + \frac{1}{2} g^{\lambda\nu}\delta g_{\nu\mu} \tag{4.37}$$

where ${}^{LC}\Gamma$ is the Levi-Civita connection :

$${}^{LC}\Gamma^\lambda{}_\mu = \frac{1}{2} g^{\lambda\nu}\left(\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu}\right)dx^\rho \tag{4.38}$$

while the general solution is obtained according to :

$$\delta g_{\mu\nu} \longrightarrow \delta g_{\mu\nu} + a_{[\mu\nu]} \tag{4.39}$$

$a_{[\mu\nu]}$ being antisymmetric[13]. In equation (4.37) we have chosen $a_{[\mu\nu]} = 0$.

One lifts Γ to get a gl(2,$\mathbb{R}$)-valued one form $\hat{\Gamma}$ on $\mathcal{P}F_{\{g\}}(\Sigma)$:

$$\hat{\Gamma}^\sigma{}_\tau = (A^{-1})^\sigma{}_{\sigma'}\Gamma^{\sigma'}{}_{\tau'}A^{\tau'}{}_\tau + (A^{-1})^\sigma{}_{\sigma'} d_{\mathcal{P}} A^{\sigma'}{}_\tau \tag{4.40a}$$

or in a more compact notation :

$$\hat{\Gamma} = A^{-1}\Gamma A + A^{-1}d_{\mathcal{P}}A \tag{4.40b}$$

As explained in Appendix F, the action of $\mathcal{D}(\Sigma)$ extends to $\mathcal{P}F_{\{g\}}(\Sigma)$, and the fundamental vector field associated with $\lambda\in\mathcal{V}(\Sigma)$ reads :

$$\underline{\lambda} = \left((l_\Sigma(\lambda)g_{\mu\nu})\frac{\delta}{\delta g_{\mu\nu}} - \lambda^\alpha\partial_\alpha - (\partial_\rho\lambda^\sigma)A^\rho{}_\tau \frac{\delta}{\delta A^\sigma{}_\tau}\right) \tag{4.41}$$

It follows that :

$$\begin{aligned}(i_{\mathcal{P}}(\lambda)\hat{\Gamma})^\sigma{}_\tau \equiv (i_{\mathcal{P}}(\underline{\lambda})\hat{\Gamma})^\sigma{}_\tau &= (A^{-1})^\sigma{}_{\sigma'}\left(\frac{1}{2}g^{\sigma'\rho}\left({}^{LC}D_\rho\bar{\lambda}_{\tau'} - {}^{LC}D_{\tau'}\bar{\lambda}_\rho\right)\right)A^{\tau'}{}_\tau \\ &\underset{\text{Def}}{=} \left[A^{-1}\left(\frac{1}{2}{}^{LC}D\wedge\bar{\lambda}\right)A\right]^\sigma_\tau\end{aligned} \tag{4.42}$$

and :

$$l_{\mathcal{P}}(\lambda)\hat{\Gamma} \equiv l_{\mathcal{P}}(\underline{\lambda})\hat{\Gamma} = 0 \tag{4.43}$$

where $\bar{\lambda}_\rho = g_{\rho\sigma}\lambda^\sigma$ and ${}^{LC}D$ is the covariant derivative associated with ${}^{LC}\Gamma$. Hence, $\hat{\Gamma}$ is the $\mathcal{D}(\Sigma)$-invariant Gl(2,$\mathbb{R}$)-connection on $\mathcal{P}F_{\{g\}}(\Sigma)$ we are looking for, and its curvature reads :

$$\begin{aligned}\hat{R}(\hat{\Gamma}) = d_{\mathcal{P}}\hat{\Gamma} + \frac{1}{2}[\hat{\Gamma},\hat{\Gamma}] &= A^{-1}\left((\delta + d_\Sigma)\Gamma + \frac{1}{2}[\Gamma,\Gamma]\right)A \\ &= A^{-1}R(\Gamma)A\end{aligned} \tag{4.44}$$

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[13] We wish to thank M. Dubois Violette for communicating the above construction of Γ.

with :

$$R(\Gamma)=(\delta+d_\Sigma)\left({}^{LC}\Gamma+\frac{1}{2}g^{-1}\delta g\right)+\frac{1}{2}\left[{}^{LC}\Gamma+\frac{1}{2}g^{-1}\delta g,{}^{LC}\Gamma+\frac{1}{2}g^{-1}\delta g\right]$$
$$={}^{LC}R+{}^{LC}D\left(\frac{1}{2}g^{-1}\delta g\right)+\delta{}^{LC}\Gamma-\left(\frac{1}{2}g^{-1}\delta g\right)^2 \tag{4.45}$$

LC referring to the Levi-Civita part of the connection.

Let $\tilde{\omega}$ be a $\mathcal{V}(\Sigma)$-valued connection on another copy $\widetilde{\mathcal{M}}(\Sigma)$ of $\mathcal{M}(\Sigma)$, and $\tilde{\Omega}$ its curvature. In the intermediate scheme, the equivariant curvature of $\hat{\hat{\Gamma}}$ (the pull-back of $\hat{\Gamma}$ on $\widetilde{\mathcal{M}}(\Sigma)\times\mathcal{P}F_{\{g\}}(\Sigma)$) is given by :

$$\hat{\hat{R}}^{eq}_{int}(\hat{\Gamma},\tilde{\omega},\tilde{\Omega})=\left(\tilde{\delta}+d_{\mathcal{P}}+l_{\mathcal{P}}(\tilde{\omega})-i_{\mathcal{P}}(\tilde{\Omega})\right)\hat{\hat{\Gamma}}+\frac{1}{2}\left[\hat{\hat{\Gamma}},\hat{\hat{\Gamma}}\right]$$
$$=\hat{\hat{R}}(\hat{\Gamma})-i_{\mathcal{P}}(\tilde{\Omega})\hat{\hat{\Gamma}}=A^{-1}\left(\hat{R}(\hat{\Gamma})-\frac{1}{2}{}^{LC}D\wedge\overline{\tilde{\Omega}}\right)A \tag{4.46}$$
$$\equiv A^{-1}\hat{R}^{eq}_{int}A$$

with $\hat{R}(\hat{\Gamma})$ the pull-back of $R(\Gamma)$ on $\widetilde{\mathcal{M}}(\Sigma)\times\mathcal{P}F_{\{g\}}(\Sigma)$. In view of the particular form taken by $\hat{\hat{R}}^{eq}_{int}$, and since the invariants we are looking for are constructed in terms of curvatures, one can forget about the Gl(2,$\mathbb{R}$) fibration (represented by the A^{-1} and A terms) since one deals with forms globally defined on Σ, such as $\hat{R}(\hat{\Gamma})$ and $\hat{R}^{eq}_{int}$.

As before, $\mathcal{I}$, $\mathcal{L}$ and i_Σ, l_Σ refer to the action of $\mathcal{D}(\Sigma)$ on $\mathcal{M}(\Sigma)$ and Σ respectively. In the Weil scheme, the equivariant curvature is given by :

$$R^{eq}_W(\hat{\Gamma},\tilde{\omega},\tilde{\Omega})=\exp\{(\mathcal{I}+i_\Sigma)(\tilde{\omega})\}R^{eq}_{int}(\hat{\Gamma},\tilde{\omega},\tilde{\Omega})$$
$$=\hat{R}(\hat{\Gamma})+(\mathcal{I}+i_\Sigma)(\tilde{\omega})\hat{R}(\hat{\Gamma})+\frac{(\mathcal{I}+i_\Sigma)(\tilde{\omega})(\mathcal{I}+i_\Sigma)(\tilde{\omega})}{2}\hat{R}(\hat{\Gamma}) \tag{4.47}$$
$$-\frac{1}{2}{}^{LC}D\wedge\overline{\tilde{\Omega}}$$

The equivariant Euler class, which plays the role of the invariant polynomial $I^{eq}_{H,W}$ of section II, is defined by :

$$\mathcal{E}^{eq}_W=\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\nu}\left(R^{eq}_W(\hat{\Gamma})\right)^{\mu}_{\ \nu} \tag{4.48}$$

Once equation (4.47) has been made explicit (using equations (G.8) and (G.11) of Appendix G), $\mathcal{E}^{eq}_W$ can be written as :

$$\mathcal{E}_W^{eq} = \frac{\varepsilon^{\mu\rho}}{\sqrt{g}} g_{\rho\nu}\left({}^{LC}R - i_\Sigma(\tilde\omega){}^{LC}R + \frac{i_\Sigma(\tilde\omega) i_\Sigma(\tilde\omega)}{2}{}^{LC}R \right.$$

$$\left. + \frac{1}{2}{}^{LC}D \wedge \bar{\bar{\gamma}} - \frac{1}{2} i_\Sigma(\tilde\omega){}^{LC}D \wedge \bar{\bar{\gamma}} - \frac{1}{4}\tilde\psi\tilde\psi \right)_\mu^\nu \tag{4.49}$$

$$+ \frac{1}{2}\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}\left({}^{LC}D_\mu \bar{\bar{\Omega}}_\nu - {}^{LC}D_\nu \bar{\bar{\Omega}}_\mu \right)$$

with :

$$\begin{aligned} \bar{\bar{\gamma}}_\mu &= \tilde\gamma_{\lambda\mu} dx^\lambda = \left(\delta g_{\lambda\mu} - l_\Sigma(\tilde\omega) g_{\lambda\mu}\right) dx^\lambda = \left(\delta \bar{g}_\mu - l_\Sigma(\tilde\omega)\bar{g}_\mu\right) \\ \tilde\psi_\mu^\nu &= g^{\nu\lambda}\left(\delta g_{\lambda\mu} - l_\Sigma(\tilde\omega) g_{\lambda\mu}\right) = g^{\nu\lambda}\tilde\gamma_{\lambda\mu} = \left(g^{-1}\tilde\gamma\right)_\mu^\nu \end{aligned} \tag{4.50}$$

and ${}^{LC}D \wedge \bar{\bar{\gamma}}$ as defined in Appendix G.

Using the basicity property :

$$\left(\tilde{\mathcal{I}} + \mathcal{I} + i_\Sigma\right)(\lambda)\bar{\bar{\gamma}} = 0 \tag{4.51}$$

one easily checks :

$$\left(\tilde{\mathcal{I}} + \mathcal{I} + i_\Sigma\right)(\lambda)\mathcal{E}_W^{eq} = 0 \tag{4.52}$$

Now, because $\frac{\varepsilon^{\mu\rho}}{\sqrt{g}} g_{\rho\nu}$ is covariant constant (see Appendix G) and because of the Bianchi identity for $\hat{R}_W^{eq}$:

$$\left(\tilde\delta + \delta + d_\Sigma\right)\mathcal{E}_W^{eq} = 0 \tag{4.53}$$

And consequently :

$$\left(\tilde{\mathcal{L}} + \mathcal{L} + l_\Sigma\right)(\lambda)\mathcal{E}_W^{eq} = 0 \tag{4.54}$$

(Compare with [BCI94]).

The last step is to apply Cartan's theorem 3 : one has to replace $\tilde\omega$ by a $\mathcal{D}(\Sigma)$-connection on $\mathcal{M}(\Sigma) \times \Sigma$ [BGV91]. An obvious solution is given by a $\mathcal{D}(\Sigma)$-connection ω on $\mathcal{M}(\Sigma)$, so that the form of equation (4.47) is maintained with the replacement :

$$\begin{cases} \tilde\omega \longrightarrow \omega \\ \tilde\Omega \longrightarrow \Omega = \delta\omega + \frac{1}{2}[\omega, \omega] \\ \tilde\gamma \longrightarrow \gamma = \delta g - \mathcal{L}(\omega) g \\ \tilde\psi \longrightarrow \psi = g^{-1}\gamma = g^{-1}(\delta g - \mathcal{L}(\omega) g) \end{cases}$$

Furthermore, since $\mathcal{M}(\Sigma)$ is a principal bundle over $\mathcal{CM}(\Sigma)$ with structure group the Weyl group, one may choose ω a $\mathcal{D}(\Sigma)$-connection on $\mathcal{CM}(\Sigma)$. It is very likely that the conformal picture is recovered by choosing the section provided by the (negative) constant scalar curvature condition (4.21), but, at the time of writing this has not been explicitely checked.

V) Concluding Remarks.

Cohomological field theories are gauge theories of an exotic type. The question of the definition of the observables is crucial. The definition has to be such that "physics" - e.g. correlation functions of observables - be gauge independent, i.e. be independent of the parameters or external fields needed to define a perturbatively computable Lagrangian, namely, a Lagrangian whose quadratic part is non degenerate. The fact that the equivariant cohomology classes defined in the previous section do not depend on the various connections used to define them suggests that they be computed by "averaging out" over these connections. This is formally realizable in terms of functional integrals. The well known difficulties in defining those result in ambiguities which are well understood at the perturbative level provided that they are properly constrained algebraically. The equivariant cohomology framework exhibited here both at the level of fields and at the level of observables is a compelling ingredient whose necessity has often not been fully appreciated.

The construction reviewed here may not give all observables. Note that those which have been constructed here emerge as integrated local expressions in the fields. Whereas these are basic cohomology classes [OSB89], it is not clear a priori which cohomology classes the local densities represent. Another delicacy in the definition of observables has to do with global aspects which are known to be crucial [DK90, Mu83].

The corresponding mathematics should in each case guarantee that one is not describing a trivial cohomology class via complicated formulae. A final remark is in order since it provides a bridge with the origin of cohomological theories. The introduction of the connection ω -the Faddeev-Popov ghost introduced by L. Baulieu and I.M. Singer-, rather natural from the geometrical point of view may however look somewhat redundant, since only curvatures are involved in the final formulae. A similar impression may prevail from the field point of view. It has however several advantages, one being the necessity to introduce the operations $\widetilde{\mathcal{I}}$, $\mathcal{I}$ and i. The devoted reader will easily establish the bridge between equivariant cohomology and twisted N = 2 supersymmetry.

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APPENDIX A

Some basic facts and conventions about associated bundles.

Let $\Pi(B,\mathcal{G})$ be a smooth principal fiber bundle with a right action of the Lie group $\mathcal{G}$. The right-transformed by $\gamma\in\mathcal{G}$ of a point $\pi\in\Pi$ is written π^{γ}. An infinitesimal transformation, represented by $\lambda\in\mathrm{Lie}\mathcal{G}$, gives rise to a so-called fundamental vector field λ_{Π} on Π. The operations on Π are denoted by :

$$i_{\Pi}(\lambda) \underset{\mathrm{Def}}{=} i_{\Pi}(\lambda_{\Pi}) \quad \text{and} \quad l_{\Pi}(\lambda) \underset{\mathrm{Def}}{=} l_{\Pi}(\lambda_{\Pi}) \qquad \text{(A.1)}$$

while the differential is d_{Π}.

Now, let us consider a smooth manifold $\mathfrak{F}$ with a left-action of $\mathcal{G}$ on it. The left-transformed by $\gamma\in\mathcal{G}$ of a point $f\in\mathfrak{F}$ is written $\gamma(f)$. Here again, to any $\lambda\in\mathrm{Lie}\mathcal{G}$ there corresponds a fundamental vector field $\lambda_{\mathfrak{F}}$ on $\mathfrak{F}$, and the operations on $\mathfrak{F}$ are also written :

$$i_{\mathfrak{F}}(\lambda) \underset{\mathrm{Def}}{=} i_{\mathfrak{F}}(\lambda_{\mathfrak{F}}) \quad \text{and} \quad l_{\mathfrak{F}}(\lambda) \underset{\mathrm{Def}}{=} l_{\mathfrak{F}}(\lambda_{\mathfrak{F}}) \qquad \text{(A.2)}$$

Finally, we consider the right-action on the smooth product manifold $\Pi \times \mathfrak{F}$ defined by :

$$(\pi, f)^{\gamma} = (\pi^{\gamma}, \gamma^{-1}(f)) \qquad \text{(A.3)}$$

for $\gamma\in\mathcal{G}$. Hence, for an infinitesimal transformation $\lambda\in\mathrm{Lie}\mathcal{G}$, the corresponding fundamental vector field on $\Pi \times \mathfrak{F}$ will be :

$$\lambda_{\Pi\times\mathfrak{F}} = \lambda_{\Pi} - \lambda_{\mathfrak{F}} \qquad \text{(A.4)}$$

where λ_{Π} and $\lambda_{\mathfrak{F}}$ are the fundamental vector fields associated to the original right and left actions of λ on Π and $\mathfrak{F}$ respectively.

One can show that $\Pi \times \mathfrak{F}$ with this right-action of $\mathcal{G}$ can be made into a smooth principal bundle with structure group $\mathcal{G}$, whose base space, denoted by $\Pi \times_{\mathcal{G}} \mathfrak{F}$, is itself a smooth fiber bundle (not principal) over B, with typical fiber $\mathfrak{F}$ [GHV73].

Whereas the differentials on $\Pi \times \mathfrak{F}$ are $d_{\Pi\times\mathfrak{F}} = d_{\Pi} + d_{\mathfrak{F}}$, the operations become :

$$\begin{aligned} i_{\Pi\times\mathfrak{F}}(\lambda) &\underset{\mathrm{Def}}{=} i_{\Pi\times\mathfrak{F}}(\lambda_{\Pi\times\mathfrak{F}}) = (i_{\Pi} + i_{\mathfrak{F}})(\lambda_{\Pi\times\mathfrak{F}}) \\ &= i_{\Pi}(\lambda_{\Pi}) - i_{\mathfrak{F}}(\lambda_{\mathfrak{F}}) = i_{\Pi}(\lambda) - i_{\mathfrak{F}}(\lambda) \end{aligned} \qquad \text{(A.5)}$$

and the same for $l_{\Pi\times\mathfrak{F}}$. Note that $\lambda_{\mathfrak{F}}$ is the fundamental vector field associated with the original action of $\mathcal{G}$ on $\mathfrak{F}$ (here a left action) which explains the relative sign in the last term of (A.5).

APPENDIX B

Cartan's Theorem 3 [C50].

Let $P(\omega,\Omega)$ represent an equivariant cohomology class of $\Omega^*(\mathcal{M})\otimes \mathcal{W}_\Pi$. Let θ be a connection in $\Omega^*(\mathcal{M})$, i.e. a Lie$\mathcal{G}$ valued one form on $\mathcal{M}$ such that :

$$i_{\mathcal{M}}(\lambda)\omega = \lambda \quad \text{and} \quad l_{\mathcal{M}}(\lambda)\omega = -[\lambda , \omega] \tag{B.1}$$

for any $\lambda\in$Lie$\mathcal{G}$, and Θ its curvature.

Let :

$$\begin{aligned} \omega_t &= t\omega + (1-t)\theta \\ \Omega_t &= \left(d_\Pi + d_{\mathcal{M}}\right)\Omega + \frac{1}{2}\left[\omega_t,\omega_t\right] \end{aligned} \qquad 0 \le t \le 1 \tag{B.2}$$

It is easy to check that :

$$\begin{aligned} \left(i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)\right)\omega_t &= \lambda \\ \left(l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)\right)\omega_t &= -\left[\lambda,\omega_t\right] \end{aligned} \tag{B.3}$$

For all polynomials P in ω and Ω, define [MSZ85] the derivation k by :

$$(kP)(\omega,\Omega) = \int_{[0,1]} k_t P(\omega_t,\Omega_t) \tag{B.4}$$

with :

$$\begin{aligned} k_t\omega_t &= 0 \\ k_t\Omega_t &= d_t\omega_t \end{aligned} \tag{B.5}$$

such that :

$$k_t\left(d_\Pi + d_{\mathcal{M}}\right) - \left(d_\Pi + d_{\mathcal{M}}\right)k_t = d_t \tag{B.6}$$

and :

$$\left[i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda), k_t\right] = \left[l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda), k_t\right] = 0 \tag{B.7}$$

Thus :

$$\left[i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda), k\right] = \left[l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda), k\right] = 0 \tag{B.8}$$

and :

$$\begin{aligned} P(\omega,\Omega) - P(\theta,\Theta) &= \int_{[0,1]} d_t P(\omega_t,\Omega_t) \\ &= \int_{[0,1]} k_t\left[\left(d_\Pi + d_{\mathcal{M}}\right)P\right](\omega_t,\Omega_t) - \int_{[0,1]} \left(d_\Pi + d_{\mathcal{M}}\right)k_t P(\omega_t,\Omega_t) \\ &= k\left(d_\Pi + d_{\mathcal{M}}\right)P - \left(d_\Pi + d_{\mathcal{M}}\right)kP \end{aligned} \tag{B.9}$$

Now, since P represents an equivariant cohomology class :

$$\left(d_\Pi + d_{\mathcal{M}}\right)P = \left(i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)\right)P = \left(l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)\right)P = 0 \tag{B.10}$$

Then :

$$P(\omega,\Omega) = P(\theta,\Theta) - \left(d_\Pi + d_{\mathcal{M}}\right)kP \tag{B.11}$$

Because of the commutativity of k with $i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)$ and $l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)$,

$$\left(i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)\right)kP = \left(l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)\right)kP = 0 \tag{B.12}$$

It follows that :

$$\begin{aligned} \left(d_\Pi + d_{\mathcal{M}}\right)P(\theta,\Theta) &= d_{\mathcal{M}}P(\theta,\Theta) = 0 \\ \left(i_\Pi(\lambda) + i_{\mathcal{M}}(\lambda)\right)P(\theta,\Theta) &= i_{\mathcal{M}}(\lambda)P(\theta,\Theta) = 0 \\ \left(l_\Pi(\lambda) + l_{\mathcal{M}}(\lambda)\right)P(\theta,\Theta) &= l_{\mathcal{M}}(\lambda)P(\theta,\Theta) = 0 \end{aligned} \tag{B.13}$$

$P(\theta,\Theta)$ is an element of the basic cohomology of $\mathcal{M}$, cohomologous to $P(\omega,\Omega)$ within the equivariant cochain algebra and this correspondence is obviously defined at the level of cohomology. The same calculation shows that given two connections θ_1, θ_2 on $\Omega^*(\mathcal{M})$, $P(\theta_1,\Theta_1)$ and $P(\theta_2,\Theta_2)$ are cohomologous within the basic cohomology of $\Omega^*(\mathcal{M})$.

APPENDIX C

An Alternative construction of equivariant cohomology classes of $\mathfrak{A}$.

The construction given in the text may look unnecessarily complicated. In the present case where $\mathcal{P}(\mathcal{M},G) = \mathfrak{A} \times P(\Sigma,G)$ and $\mathcal{G}$ acts separately on $\mathfrak{A}$ and $P(\Sigma,G)$, the situation can be simplified as follows : construct equivariant cohomology classes of $P(\Sigma,G)$ using a connection ω on $\mathfrak{A}$, with curvature Ω. Now, choose $\hat{\Gamma} = \hat{a}$:

$$\mathcal{I}(\lambda)\hat{\Gamma} = 0 \tag{C.1}$$

$$\left(\mathcal{L} + l_P\right)(\lambda)\hat{\Gamma} = 0 \tag{C.2}$$

The equivariant curvature of $\hat{a}$ in the intermediate scheme is given by :

$$F^{eq}_{int}(\hat{a}) = \left(\delta + d_P + l_P(\tilde{\omega}) - i_P(\tilde{\Omega})\right)\hat{a} + \frac{1}{2}[\hat{a},\hat{a}] \tag{C.3}$$

One easily finds that this is the same as the equivariant curvature in the Weil scheme :

$$F^{eq}_{W}(\hat{a}) = \left(\delta + d_P\right)\left(\hat{a} + i_P(\omega)\hat{a}\right) + \frac{1}{2}\left[\left(\hat{a} + i_P(\omega)\hat{a}\right),\left(\hat{a} + i_P(\omega)\hat{a}\right)\right] \tag{C.4}$$

and they both coincide with $F^{eq}_{W}(\hat{a},\omega,\Omega)$ of equation (3.17).

APPENDIX D

An action of $\mathcal{D}(\Sigma)$ on $T^{(1,0)}_{\{\mu\}}(\Sigma)$ and $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$.

If (x,μ,V_x) is a point of $T^{(1,0)}_{\{\mu\}}(\Sigma)$, we choose the following right-action of $\mathcal{D}(\Sigma)$:

$$\forall\varphi\in\mathcal{D}(\Sigma)\ ,\ (\mu,x,V_x)^{\varphi} = (\mu^{\varphi},\ \varphi^{-1}(x),\ d_x\varphi^{-1}(V_x)) \tag{D.1}$$

where $d_x\varphi^{-1} : T_x\Sigma \to T_{\varphi^{-1}(x)}\Sigma$ is the differential of $\varphi\in\mathcal{D}(\Sigma)$ at $x\in\Sigma$:

$$\forall V_x\in T_x\Sigma\ ,\ \forall f\in\mathcal{C}^{\infty}(\Sigma)\ ,\ d_x\varphi^{-1}V_x(f) = V_x(f\circ\varphi^{-1}) \tag{D.2}$$

and μ^{φ} is the element of $\mathcal{B}(\Sigma)$ with components :

$$\left(\mu^{\varphi}\right)^{w}{}_{\overline{w}} = \frac{\left(\partial_{\overline{w}}\varphi^{w}\right)+\left(\partial_{\overline{w}}\varphi^{\overline{w}}\right)\left(\mu^{z}{}_{\overline{z}}\circ\varphi\right)}{\left(\partial_{w}\varphi^{w}\right)+\left(\partial_{w}\varphi^{\overline{w}}\right)\left(\mu^{z}{}_{\overline{z}}\circ\varphi\right)} \tag{D.3}$$

where $(z,\overline{z})$ and $(w,\overline{w})$ are coordinates at x and $\varphi^{-1}(x)$ respectively, and $(\varphi^{w},\varphi^{\overline{w}})$ the local representative of φ with respect to $(z,\overline{z})$ and $(w,\overline{w})$. Equation (D.3) defines the natural right-action of $\mathcal{D}(\Sigma)$ on $\mathcal{B}(\Sigma)$.

For an infinitesimal diffeomorphism represented by $\underline{\lambda} = \lambda^{z}\partial_{z} + \lambda^{\overline{z}}\partial_{\overline{z}}\in\mathcal{V}(\Sigma)$:

$$z(x) \to z(\varphi(x)) = z(x) + \lambda^{z}(x)\ ,\ \text{and c.c.} \tag{D.4}$$

we get :

$$\mu^{\varphi} = \mu + \delta_{\lambda}\mu = \mu + \overline{D}_{\mu}\Lambda_{\mu} \tag{D.5}$$

with the notations of equation (4.5).

Now, at $x\in\Sigma$ with coordinates $(z,\overline{z})$ we can solve the Beltrami equation (4.14) thus obtaining new complex coordinates $(Z_{\mu},\overline{Z}_{\mu})$ at x. The component $V^{Z_{\mu}}(x)$ of V_x with respect to the natural frame $\partial_{Z_{\mu}}$ associated to $(Z_{\mu},\overline{Z}_{\mu})$ are chosen to be coordinates of V_x. Similarly, at $\varphi^{-1}(x)\in\Sigma$ with coordinates $(w,\overline{w})$ we solve the Beltrami equation for $\mu+\delta_{\lambda}\mu$ and obtain complex coordinates $(Z_{\mu+\delta_{\lambda}\mu},\overline{Z}_{\mu+\delta_{\lambda}\mu})$ at $\varphi^{-1}(x)$, and the coordinates of $V_{\varphi^{-1}(x)}$ are the component $V^{Z_{\mu+\delta_{\lambda}\mu}}(\varphi^{-1}(x))$ of $V_{\varphi^{-1}(x)}$ with respect to the natural frame $\partial_{Z_{\mu+\delta_{\lambda}\mu}}$. This is how we define a complex analytic structure on $T^{(1,0)}_{\{\mu\}}(\Sigma)$. Hence, at the coordinates level, the infinitesimal action of $\mathcal{D}(\Sigma)$ is :

$$\left(\mu,Z_{\mu}(x),V^{Z_{\mu}}(x)\right) \longrightarrow \left(\mu+\delta_{\lambda}\mu,Z_{\mu+\delta_{\lambda}\mu}(\varphi^{-1}(x)),V^{Z_{\mu+\delta_{\lambda}\mu}}(\varphi^{-1}(x))\right) \tag{D.6}$$

Combining the Beltrami equations which define the coordinates $(Z_{\mu},\overline{Z}_{\mu})$ and $(Z_{\mu+\delta_{\lambda}\mu},\overline{Z}_{\mu+\delta_{\lambda}\mu})$ with equation (D.3), one can show that $Z_{\mu+\delta_{\lambda}\mu}(\varphi^{-1}(x))$ is an invertible holomorphic function of $Z_{\mu}(x)$. Hence, since the complex coordinates $Z_{\mu+\delta_{\lambda}\mu}$ are unique up to a biholomorphic mapping, we can choose [L] :

$$Z_{\mu+\delta_\lambda\mu}(\varphi^{-1}(x)) = Z_\mu(x) \tag{D.7}$$

Accordingly, for the coordinates $V^{Z_\mu}(x)$ of V(x) we get :

$$V^{Z_{\mu+\delta_\lambda\mu}}(\varphi^{-1}(x)) = \left(V^{Z_\mu} \circ \varphi\right)(\varphi^{-1}(x)) \frac{\partial Z_{\mu+\delta_\lambda\mu}}{\partial Z_\mu}(Z_\mu(x)) = V^{Z_\mu}(x) \tag{D.8}$$

Finally, it is straightforward to see that on $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$:

$$E^{Z_{\mu+\delta_\lambda\mu}}(\varphi^{-1}(x)) = \left(E^{Z_\mu} \circ \varphi\right)(\varphi^{-1}(x)) \frac{\partial Z_{\mu+\delta_\lambda\mu}}{\partial Z_\mu}(Z_\mu(x)) = E^{Z_\mu}(x) \tag{D.8}$$

with $E^{Z_\mu}(x)$ and $E^{Z_{\mu+\delta_\lambda\mu}}(\varphi^{-1}(x))$ coordinates for $\mathcal{P}T^{(1,0)}_{\{\mu\}}(\Sigma)$ (see main text).

APPENDIX E

Calculation of the equivariant curvature of $\hat{\Gamma}$.

$$
\begin{aligned}
R_W^{eq}(\hat{\Gamma}) &= \left(\tilde{\delta} + \mathfrak{D} + \overline{\mathfrak{D}}\right)\left(\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} + D\ln E^{Z_\mu} + I(\tilde{\omega})\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu}\right) \\
&= \left(\tilde{\delta} + \mathfrak{D} + \overline{\mathfrak{D}}\right)\left(\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} + I(\tilde{\omega})\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu}\right) \\
&= \overline{\mathfrak{D}}\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} - I\left(\tilde{\Omega} - \frac{1}{2}[\tilde{\omega},\tilde{\omega}]\right)\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} + I(\tilde{\omega})\overline{\mathfrak{D}}\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu}
\end{aligned}
\tag{E.1}
$$

where we have used the invariance of $\hat{\Gamma}$. The third term is :

$$
\begin{aligned}
\frac{1}{2}I\left([\tilde{\omega},\tilde{\omega}]\right)\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} &= -\frac{1}{2}\left[L(\tilde{\omega}), I(\tilde{\omega})\right]\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \\
&= -\frac{1}{2}L(\tilde{\omega})I(\tilde{\omega})\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \\
&= \frac{1}{2}\left(I^h(\tilde{\omega})\mathfrak{D} + I^{\bar{h}}(\tilde{\omega})\overline{\mathfrak{D}}\right)I^h(\tilde{\omega})\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \\
&= \frac{1}{2}\left(-I^h(\tilde{\omega})\mathfrak{D}I^{\bar{h}}(\tilde{\omega})\overline{\mathfrak{D}} + I^{\bar{h}}(\tilde{\omega})\overline{\mathfrak{D}}I^h(\tilde{\omega})\mathfrak{D}\right)\ln\rho_{Z_\mu \bar{Z}_\mu} \\
&= I^{\bar{h}}(\tilde{\omega})I^h(\tilde{\omega})\overline{\mathfrak{D}}\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \\
&= \frac{1}{2}I(\tilde{\omega})I(\tilde{\omega})\overline{\mathfrak{D}}\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu}
\end{aligned}
\tag{E.2}
$$

Finally :

$$
\begin{aligned}
R_W^{eq}(\hat{\Gamma}) = {} & \overline{\mathfrak{D}}\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} - I(\tilde{\omega})\overline{\mathfrak{D}}\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \\
& + I^{\bar{h}}(\tilde{\omega})I^h(\tilde{\omega})\overline{\mathfrak{D}}\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} - I(\tilde{\Omega})\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu}
\end{aligned}
\tag{E.3}
$$

The last term can be antisymmetrized since :

$$
\begin{aligned}
I(\tilde{\Omega})\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} &= I^h(\tilde{\Omega})\mathfrak{D}\ln\rho_{Z_\mu \bar{Z}_\mu} \\
&= -I^{\bar{h}}(\tilde{\Omega})\overline{\mathfrak{D}}\ln\rho_{Z_\mu \bar{Z}_\mu} = \frac{1}{2}\left(I^h(\tilde{\Omega})\mathfrak{D} - I^{\bar{h}}(\tilde{\Omega})\overline{\mathfrak{D}}\right)\ln\rho_{Z_\mu \bar{Z}_\mu}
\end{aligned}
\tag{E.4}
$$

due to the invariance of $\ln\rho_{Z_\mu \bar{Z}_\mu}$. Thus, as expected, $R_W^{eq}(\hat{\Gamma})$ is of type (1,1) for the natural complex structure of $\mathfrak{B}(\Sigma) \times \Sigma$.

APPENDIX F
The action of $\mathcal{D}(\Sigma)$ on $\mathcal{P}F_{\{g\}}(\Sigma)$.

Let (x, E_x) be a point of $F(\Sigma)$ the frame bundle of Σ, where, by definition, E_x is a frame (a basis) of $T_x\Sigma$: $E_x = (E_{x\,i})$. One defines coordinates for E_x as follows. One selects coordinates (x^k) for $x\in\Sigma$ and denoted by (∂_k) the natural basis of $T_x\Sigma$ defined by these coordinates : $\partial_k = \partial/\partial x^k$. Then, the coordinates of E_x are the components $A^j{}_k$ of the decomposition of E_x with respect to the natural basis (∂_k) :

$$E_{x\,k} = A^j{}_k\partial_j \tag{F.1}$$

Each vector $E_{x\,i}$ belongs to $T_x\Sigma$. As explained in Appendix C, there is a natural (left) action of $\varphi\in\mathcal{D}(\Sigma)$ on $T_x\Sigma$, given by $d_x\varphi : T_x\Sigma \to T_{\varphi(x)}\Sigma$ the differential of φ at x :

$$\forall V_x\in T_x\Sigma \;,\; \forall f\in\mathcal{C}^\infty(\Sigma) \;,\; d_x\varphi V_x(f) = V_x(f\circ\varphi) \tag{F.2}$$

In terms of coordinates, this gives :

$$\left(d_x\varphi(V_x)\right)^i = V_x^m\left(\partial_m\varphi^i\right) \tag{F.3}$$

where φ^i means the local representative of φ with respect to the coordinates (x^k) : $\varphi(x) = y = (y^i) = (\varphi^i(x^k))$. Applying equation (F.3) to the frame vectors $E_{x\,j}$, one gets :

$$A'^i{}_j = A^m{}_j\left(\partial_m\varphi^i\right) \tag{F.4}$$

and at the infinitesimal level, for $\lambda\in\mathcal{V}(\Sigma)$:

$$A'^i{}_j = A^m{}_j\left(\partial_m(x^i+\lambda^i)\right) = A^m{}_j\left(\delta^i_m + \partial_m\lambda^i\right) = A^i{}_j + (\partial_m\lambda^i)A^m{}_j \tag{F.5}$$

where $A'^i{}_j$ are the coordinates of the transformed frame at $\varphi(x)$.

Finally, at the coordinates level, the natural left-action of $\lambda\in\mathcal{V}(\Sigma)$ on $F(\Sigma)$ is :

$$\left((x^k),(A^i{}_j)\right) \longrightarrow \left((x^k+\lambda^k),(A^i{}_j+(\partial_m\lambda^i)A^m{}_j)\right) \tag{F.6}$$

Hence, the fundamental vector field on $F(\Sigma)$ defined by the action of $\lambda\in\mathcal{V}(\Sigma)$ reads :

$$\lambda^k\partial_k + A^m{}_j(\partial_m\lambda^i)\frac{\delta}{\delta A^i{}_j} \tag{F.7}$$

Now, if we consider $\mathcal{P}F_{\{g\}}(\Sigma)$ instead of $F(\Sigma)$, we need a right-action of $\mathcal{D}(\Sigma)$ on $\mathcal{P}F_{\{g\}}(\Sigma)$ and thus a right-action on $F(\Sigma)$:

$$\left((g_{\mu\nu}),(x^\alpha),(A^\sigma{}_\tau)\right)\longrightarrow\left((g_{\mu\nu}+l_\Sigma(\lambda)g_{\mu\nu}),(x^\alpha-\lambda^\alpha),(A^\sigma{}_\tau-(\partial_\rho\lambda^\sigma)A^\rho{}_\tau)\right) \tag{F.8}$$

at the coordinates level, for $\lambda \in \mathcal{V}(\Sigma)$, and the corresponding fundamental vector field is given by :

$$\underline{\lambda} = \left((l_\Sigma(\lambda) g_{\mu\nu}) \frac{\delta}{\delta g_{\mu\nu}} - \lambda^\alpha \partial_\alpha - (\partial_\rho \lambda^\sigma) A^\rho{}_\tau \frac{\delta}{\delta A^\sigma{}_\tau} \right) \tag{F.9}$$

In particular :

$$\begin{aligned} l_{\mathcal{G}}(\lambda) g_{\lambda\gamma} \equiv l_{\mathcal{G}}(\underline{\lambda}) g_{\lambda\gamma} &= \mathfrak{L}((l_\Sigma(\lambda) g_{\mu\nu}) \frac{\delta}{\delta g_{\mu\nu}}) g_{\lambda\gamma} + l_\Sigma(-\lambda) g_{\lambda\gamma} \\ &= l_\Sigma(\lambda) g_{\mu\nu} \delta^\mu_\lambda \delta^\nu_\gamma - l_\Sigma(\lambda) g_{\lambda\gamma} = 0 \end{aligned} \tag{F.10}$$

so that :

$$l_{\mathcal{G}}(\lambda) g = 0 \tag{F.10}$$

for any $g \in \mathfrak{M}(\Sigma)$.

APPENDIX G

Calculation of the equivariant curvature $\hat{R}_W^{eq}$
and of the corresponding Euler class $\mathcal{E}_W^{eq}$.

Recall :

$$\Gamma^{\lambda}{}_{\mu} = {}^{LC}\Gamma^{\lambda}{}_{\mu} + \frac{1}{2}(g^{-1}\delta g)^{\lambda}{}_{\mu} \equiv {}^{LC}\Gamma^{\lambda}{}_{\mu} + \frac{1}{2}g^{\lambda\nu}\delta g_{\nu\mu} \tag{G.1}$$

where ${}^{LC}\Gamma$ is the Levi-Civita connection :

$$^{LC}\Gamma^{\lambda}{}_{\mu} = \frac{1}{2}g^{\lambda\nu}\left(\partial_{\rho}g_{\mu\nu} + \partial_{\mu}g_{\rho\nu} - \partial_{\nu}g_{\rho\mu}\right)dx^{\rho} \tag{G.2}$$

Now :

$$\begin{aligned} R(\Gamma) &= (\delta + d_{\Sigma})\left({}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right) + \frac{1}{2}\left[{}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g, {}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right] \\ &= {}^{LC}R + {}^{LC}D\left(\frac{1}{2}g^{-1}\delta g\right) + \delta\,{}^{LC}\Gamma - \left(\frac{1}{2}g^{-1}\delta g\right)^{2} \end{aligned} \tag{G.3}$$

where ${}^{LC}R$ stands for the Levi-Civita curvature :

$$^{LC}R = d_{\Sigma}{}^{LC}\Gamma + \frac{1}{2}\left[{}^{LC}\Gamma, {}^{LC}\Gamma\right] \tag{G.4}$$

and ${}^{LC}D$ stands for the Levi-Civita covariant derivative. We also recall that, by definition :

$$^{LC}Dg = 0 \tag{G.5}$$

Differentiating equation (G.5), one gets :

$$^{LC}D_{\lambda}\left(\delta g_{\mu\nu}\right) - \left(\delta\,{}^{LC}\Gamma_{\lambda}{}^{\rho}{}_{\nu}\right)g_{\rho\mu} - \left(\delta\,{}^{LC}\Gamma_{\lambda}{}^{\rho}{}_{\mu}\right)g_{\rho\nu} = 0 \tag{G.6}$$

from which one deduces that :

$$\delta\,{}^{LC}\Gamma_{\lambda}{}^{\nu}{}_{\mu} = \frac{1}{2}g^{\rho\nu}\left({}^{LC}D_{\lambda}\delta g_{\rho\mu} + {}^{LC}D_{\mu}\delta g_{\lambda\rho} - {}^{LC}D_{\rho}\delta g_{\mu\lambda}\right) \tag{G.7}$$

Since δ and d_{Σ} anticommute, it follows that :

$$\begin{aligned} \left(\delta\,{}^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g\right)^{\nu}_{\mu} &= \frac{1}{2}g^{\rho\nu}\left({}^{LC}D_{\mu}\delta g_{\lambda\rho} - {}^{LC}D_{\rho}\delta g_{\mu\lambda}\right)dx^{\lambda} \\ &\underset{\text{Def}}{=} \frac{1}{2}\left({}^{LC}D \wedge \delta\bar{g}\right)^{\nu}_{\mu} \end{aligned} \tag{G.8}$$

where :

$$\bar{g}_{\mu} = g_{\mu\lambda}dx^{\lambda} \tag{G.9}$$

In the same way, using :

$$\mathcal{I}(\tilde{\omega})\left(g^{-1}\delta g\right) = g^{-1}\mathcal{I}(\tilde{\omega})\delta g = -g^{-1}\mathfrak{L}(\tilde{\omega})g = -g^{-1}l_{\Sigma}(\tilde{\omega})g \tag{G.10}$$

one obtains :

$$\begin{aligned}\left(\mathcal{I}(\tilde{\omega})(\delta^{LC}\Gamma + \frac{1}{2}g^{-1}\delta g)\right)^{\nu}_{\mu} &= -\frac{1}{2}g^{\rho\nu}\left({}^{LC}D_{\mu}l_{\Sigma}(\tilde{\omega})g_{\lambda\rho} - {}^{LC}D_{\rho}l_{\Sigma}(\tilde{\omega})g_{\mu\lambda}\right)dx^{\lambda} \\ &= -\frac{1}{2}\left({}^{LC}D \wedge l_{\Sigma}(\tilde{\omega})\bar{g}\right)^{\nu}_{\mu}\end{aligned} \tag{G.11}$$

Going over to the Weil scheme, the same construction occurs in the transformation of the quadratic term $\left(\frac{1}{2}g^{-1}\delta g\right)^2$, leading to the term : $\frac{1}{4}\tilde{\psi}\tilde{\psi}$, with $\tilde{\psi}$ defined in equation (4.50).

Finally, one needs the property that $\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\nu}$ is covariant constant for the connection Γ.

First, for the Levi-Civita part :

$$\begin{aligned}{}^{LC}D_{\lambda}\left(\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\nu}\right) &= g_{\rho\nu}\,{}^{LC}D_{\lambda}\frac{\varepsilon^{\mu\rho}}{\sqrt{g}} \\ &= \frac{1}{\sqrt{g}}\left(\Gamma^{\mu}_{\lambda\nu}\varepsilon^{\nu\rho} + \Gamma^{\rho}_{\lambda\nu}\varepsilon^{\mu\nu}\right) - \frac{1}{2}\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}\left(g^{\alpha\beta}\partial_{\lambda}g_{\alpha\beta}\right)\end{aligned} \tag{G.12}$$

Using the identity :

$$V^{\mu}\varepsilon^{\nu\rho} + V^{\rho}\varepsilon^{\mu\nu} + V^{\nu}\varepsilon^{\rho\mu} = 0 \tag{G.13}$$

together with :

$$\Gamma^{\nu}_{\lambda\nu} = \frac{1}{2}g^{\alpha\beta}\partial_{\lambda}g_{\alpha\beta} \tag{G.14}$$

the property follows.

Finally, for the second part of the connection, one needs :

$$\delta\left(\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\nu}\right) + \frac{1}{2}\left(g^{-1}\delta g\right)^{\mu}_{\lambda}\frac{\varepsilon^{\lambda\rho}}{\sqrt{g}}g_{\rho\nu} - \frac{1}{2}\left(g^{-1}\delta g\right)^{\lambda}_{\nu}\frac{\varepsilon^{\mu\rho}}{\sqrt{g}}g_{\rho\lambda} = 0$$

which again follows from the identity (G.13).

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Representatives of the Thom class of a vector bundle ★

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Abstract

After a review of several methods designed to produce equivariant cohomology classes, we apply one introduced by Berline et al. (1992) to get a family of representatives of the universal Thom class of a vector bundle. Surprisingly, this family does not contain the representative given by Mathaï and Quillen (1986). However, it contains the particularly simple and symmetric representative of Harvey and Lawson (1993). © 1998 Elsevier Science B.V.

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1. Introduction

In a recent paper [STW94] it has been shown how equivariant cohomology is related to the so-called (cohomological) topological models [BI92,BS88,BS91,OSB89,W88,WBS88]. In the same work, a way to compute some representatives of equivariant cohomology classes (i.e. observables of the corresponding topological model) was exhibited.

Here, we shall use this method in order to generate a family of representatives of the Thom class of a vector bundle depending on two arbitrary functions. As we shall see, these representatives are quite different from the Mathaï–Quillen representative. They offer a good deal of flexibility at the price of being slightly complicated. Special choices allow to

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find a very special representative with remarkable symmetry properties. However, its slow decrease at infinity makes it necessary to consider a cohomology theory with coefficients with sufficiently fast decrease (instead of compact). Some of these representatives (in particular the most symmetric one) already appeared in a quite different framework in the work of Harvey and Lawson [HL93] on singular connections, a fact we learned after this work was completed.

This work is divided into three parts. In Section 2 we recall basic facts about equivariant cohomology as well as the way to compute representatives of equivariant cohomology classes. This section parallels the explanations given in [STW94]. Section 3 is devoted to the Mathaï–Quillen representative of the Thom class. Finally, Section 4 exhibits a large family of representatives of the Thom class.

2. Equivariant cohomology

Let us consider the following setting: $\mathcal{M}$ is a smooth manifold and $\mathcal{G}$ a connected Lie group acting smoothly on $\mathcal{M}$. We would like to define a cohomology of the quotient space $\mathcal{M}/\mathcal{G}$ which coincides with the De Rham cohomology when this quotient is a smooth manifold but which also exists when it is not, i.e. when $\mathcal{G}$ acts with fixed points. Equivariant cohomology solves this problem.

Let $\mathcal{M}$ be a smooth manifold and $\Omega^*(\mathcal{M})$ the exterior algebra of differential forms on $\mathcal{M}$ endowed with the differential $d_{\mathcal{M}}$. A Lie group $\mathcal{G}$ is assumed to be acting on $\mathcal{M}$ as well as its Lie algebra, denoted $Lie\,\mathcal{G}$. For any $\lambda \in Lie\,\mathcal{G}$ there is a vector field $\lambda_{\mathcal{M}}$ representing the infinitesimal action of λ on $\mathcal{M}$. This vector field $\lambda_{\mathcal{M}}$ is usually called the fundamental vector field associated with λ. We shall denote by $i_{\mathcal{M}}(\lambda) = i_{\mathcal{M}}(\lambda_{\mathcal{M}})$ and $l_{\mathcal{M}}(\lambda) = l_{\mathcal{M}}(\lambda_{\mathcal{M}}) = [d_{\mathcal{M}}, i_{\mathcal{M}}(\lambda)]_+$ the contraction (or inner derivative) and Lie derivative acting on $\Omega^*(\mathcal{M})$. Let us recall that $i_{\mathcal{M}}(\lambda)$ takes n-forms into $(n-1)$-forms while $l_{\mathcal{M}}(\lambda)$ acts on forms without changing the degree. Elements of $\Omega^*(\mathcal{M})$ which are annihilated by both $i_{\mathcal{M}}(\lambda)$ and $l_{\mathcal{M}}(\lambda)$, for any $\lambda \in Lie\,\mathcal{G}$, are the so-called *basic* elements of $\Omega^*(\mathcal{M})$ for the action of $\mathcal{G}$. As $d_{\mathcal{M}}$ maps basic elements into basic elements, this leads to the definition of the *basic cohomology* of $\mathcal{M}$ for the action of $\mathcal{G}$ [C50].

We now consider the Weil algebra $\mathcal{W}(\mathcal{G})$ of $Lie\,\mathcal{G}$.[2] It is a graded differential algebra generated by two $Lie\,\mathcal{G}$-valued indeterminates, the "connection" ω, of degree 1, and its "curvature" Ω, of degree 2, such that

$$\Omega = d_{\mathcal{W}}\omega + \tfrac{1}{2}[\omega, \omega], \tag{1}$$

where $d_{\mathcal{W}}$ is the differential of $\mathcal{W}(\mathcal{G})$. Of course, one has the Bianchi identity

$$d_{\mathcal{W}}\Omega + [\omega, \Omega] = 0. \tag{2}$$

There is an action $i_{\mathcal{W}}(\lambda)$, $l_{\mathcal{W}}(\lambda)$ for $\lambda \in Lie\,\mathcal{G}$:

[2] This is a harmless abuse of notation, but it is to be remembered that equivariant cohomology deals only with the local structure of $\mathcal{G}$.

$$i_{\mathcal{W}}(\lambda)\omega = \lambda, \qquad l_{\mathcal{W}}(\lambda)\omega = -[\lambda, \omega], \tag{3}$$

$$i_{\mathcal{W}}(\lambda)\Omega = 0, \qquad l_{\mathcal{W}}(\lambda)\Omega = -[\lambda, \Omega]. \tag{4}$$

For instance, ω may be a connection on a principal $\mathcal{G}$-bundle Π and Ω its curvature. In that case $i_{\mathcal{W}}(\lambda)$ and $l_{\mathcal{W}}(\lambda)$ are generated by the action of $\mathcal{G}$ on Π, and in this case $\mathcal{W}(\mathcal{G})$ will be referred to as $\mathcal{W}_\Pi$.

We now consider the graded differential algebra $(\Omega^*(\mathcal{M}) \otimes \mathcal{W}(\mathcal{G}), d_{\mathcal{M}} + d_{\mathcal{W}})$, on which the operations $(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda)$ and $(\rho_{\mathcal{M}} + \rho_{\mathcal{W}})(\lambda)$ for any $\lambda \in Lie\,\mathcal{G}$ are well-defined. There common kernel is a graded differential subalgebra of $\Omega^*(\mathcal{M}) \otimes \mathcal{W}(\mathcal{G})$. By definition, the so-called *equivariant cochains* are the elements of this subalgebra annihilated by the differential $d_{\mathcal{M}} + d_{\mathcal{W}}$, leading to the *equivariant cohomology* of $\mathcal{M}$ for the action of $\mathcal{G}$: this is the so-called *Weil model* for equivariant cohomology.

Equivariant cohomology can be alternatively described in the so-called *intermediate model*, which was introduced in [K93] and which will be repeatedly used in the sequel. It is obtained from the Weil model via the following algebra isomorphism: [3]

$$x \longmapsto \exp\{-i_{\mathcal{M}}(\lambda)\}x \tag{5}$$

for any $x \in \Omega^*(\mathcal{M}) \otimes \mathcal{W}(\mathcal{G})$. This isomorphism changes the original differential and operations on $\Omega^*(\mathcal{M}) \otimes \mathcal{W}(\mathcal{G})$ by conjugation:

$$d_{\mathcal{M}} + d_{\mathcal{W}} \longrightarrow D_{\text{int}} = d_{\mathcal{M}} + d_{\mathcal{W}} + l_{\mathcal{M}}(\omega) - i_{\mathcal{M}}(\Omega), \tag{6}$$

$$(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda) \longrightarrow i_{\mathcal{W}}(\lambda) = e^{-i_{\mathcal{M}}(\lambda)}(i_{\mathcal{M}} + i_{\mathcal{W}})(\lambda)e^{i_{\mathcal{M}}(\lambda)}, \tag{7}$$

$$(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) \longrightarrow (l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda) = e^{-i_{\mathcal{M}}(\lambda)}(l_{\mathcal{M}} + l_{\mathcal{W}})(\lambda)e^{i_{\mathcal{M}}(\lambda)}. \tag{8}$$

Finally, the so-called *Cartan model* is obtained from the intermediate model by putting $\omega = 0$ so that $D^2_{\text{int}}\,|_{\omega=0}$ vanishes when restricted to invariant cochains. This is the most popular model, although many calculations are better automatized in the intermediate model.

Another item which will be repeatedly used is "Cartan's Theorem 3" [C50]: let us assume that $(\Omega^*(\mathcal{M}), d_{\mathcal{M}}, i_{\mathcal{M}}, l_{\mathcal{M}})$ admits a $\mathcal{G}$-connection θ [4], with curvature Θ. Then any equivariant cohomology class of $\Omega^*(\mathcal{M}) \otimes \mathcal{W}(\mathcal{G})$ with representative $P(\omega, \Omega)$ gives rise canonically to a basic cohomology class of $\Omega(\mathcal{M})$ with representative $P(\theta, \Theta)$. There is a simple proof using the homotopy that expresses the triviality of the cohomology of the Weil algebra [MSZ85]. It follows from the construction that the cohomology class of $P(\theta, \Theta)$ does not depend on θ.

One convenient way to produce equivariant cohomology classes is as follows [BGV91]: we consider a H-bundle $\mathcal{P}(\mathcal{M}, H)$ over $\mathcal{M}$ on which there exists an action of $\mathcal{G}$ which lifts the action of $\mathcal{G}$ on $\mathcal{M}$. In general, the Lie group H has nothing to do with the Lie group $\mathcal{G}$. As before, $\mathcal{P}(\mathcal{M}, H)$ is endowed with a differential $d_{\mathcal{P}}$, a contraction $i_{\mathcal{P}}$ and a Lie derivative $l_{\mathcal{P}}$.

[3] See [DV93] for a more general theorem.

[4] that is to say a $Lie\,\mathcal{G}$-valued 1-form on $\mathcal{M}$ such that $i_{\mathcal{M}}(\lambda)\theta = \lambda$ and $l_{\mathcal{M}}(\lambda)\theta = -[\lambda, \theta]$ for any $\lambda \in Lie\,\mathcal{G}$.

Next, let Γ be a $\mathcal{G}$-invariant H-connection on $\mathcal{P}(\mathcal{M}, H)$:

$$l_{\mathcal{P}}(\lambda)\Gamma = 0, \qquad \text{for any } \lambda \in Lie\,\mathcal{G}. \tag{9}$$

The pull-back $\hat{\Gamma}$ of Γ on $\Omega^*(\mathcal{M}) \otimes \mathcal{W}(\mathcal{G})$ is a 1-form on $\mathcal{P}(\mathcal{M}, H)$ and a 0-form in $\mathcal{W}(\mathcal{G})$. It follows that

$$i_{\mathcal{W}}(\lambda)\hat{\Gamma} = 0 \tag{10}$$

for any $\lambda \in Lie\,\mathcal{G}$.

In $\Omega^*(\mathcal{M}) \otimes \mathcal{W}(\mathcal{G})$, the equivariant curvature of $\hat{\Gamma}$ is defined by

$$R^{\text{eq}}_{\text{int}}(\hat{\Gamma}, \omega, \Omega) = D_{\text{int}}\hat{\Gamma} + \tfrac{1}{2}[\hat{\Gamma}, \hat{\Gamma}], \tag{11}$$

where $D_{\text{int}} = d_{\mathcal{W}} + d_{\mathcal{P}} + l_{\mathcal{P}}(\omega) - i_{\mathcal{P}}(\Omega)$. Then, if I_H is a symmetric invariant polynomial on $Lie\,H$, we consider the H-characteristic class $I^{\text{eq}}_{H,\text{int}}(\hat{\Gamma}, \omega, \Omega) = I_H(R^{\text{eq}}_{H,\text{int}}(\hat{\Gamma}, \omega, \Omega))$. It is defined on $\mathcal{M}$ and fulfills

$$(d_{\mathcal{W}} + d_{\mathcal{M}} + l_{\mathcal{M}}(\omega) - i_{\mathcal{M}}(\Omega))\, I^{\text{eq}}_{H,\text{int}}(\hat{\Gamma}, \omega, \Omega) = 0, \tag{12}$$

$$i_{\mathcal{W}}(\lambda)\, I^{\text{eq}}_{H,\text{int}}(\hat{\Gamma}, \omega, \Omega) = 0, \tag{13}$$

$$(l_{\mathcal{W}} + l_{\mathcal{M}})(\lambda)\, I^{\text{eq}}_{H,\text{int}}(\hat{\Gamma}, \omega, \Omega) = 0 \tag{14}$$

for any $\lambda \in Lie\,\mathcal{G}$.

In the Weil model, the equivariant curvature is defined by

$$R^{\text{eq}}_{\text{W}}(\hat{\Gamma}, \omega, \Omega) = (d_{\mathcal{W}} + d_{\mathcal{P}})\hat{\Gamma} + \tfrac{1}{2}[\hat{\Gamma} + i_{\mathcal{P}}(\Omega)\hat{\Gamma}, \hat{\Gamma} + i_{\mathcal{P}}(\Omega)\hat{\Gamma}]. \tag{15}$$

We may similarly consider

$$I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \omega, \Omega) = I_H(R^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \omega, \Omega)) = \mathrm{e}^{-i_{\mathcal{M}}(\lambda)} I^{\text{eq}}_{H,\text{int}}(\hat{\Gamma}, \omega, \Omega), \tag{16}$$

which fulfills

$$(d_{\mathcal{W}} + d_{\mathcal{M}})\, I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \omega, \Omega) = 0, \tag{17}$$

$$(i_{\mathcal{W}} + i_{\mathcal{M}})(\lambda)\, I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \omega, \Omega) = 0, \tag{18}$$

$$(l_{\mathcal{W}} + l_{\mathcal{M}})(\lambda)\, I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \omega, \Omega) = 0 \tag{19}$$

for any $\lambda \in Lie\,\mathcal{G}$.

Finally, if $\mathcal{M}$ admits a $\mathcal{G}$-connection θ with curvature Θ, we can apply "Cartan's Theorem 3", and substitute θ and Θ instead of ω and Ω in $I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \omega, \Omega)$, so that

$$d_{\mathcal{M}}\, I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \theta, \Theta) = 0, \tag{20}$$

$$i_{\mathcal{M}}(\lambda)\, I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \theta, \Theta) = 0, \tag{21}$$

$$l_{\mathcal{M}}(\lambda)\, I^{\text{eq}}_{H,\text{W}}(\hat{\Gamma}, \theta, \Theta) = 0 \tag{22}$$

for any $\lambda \in Lie\,\mathcal{G}$.

By standard arguments, these cohomology classes do not depend either on $\hat{\Gamma}$ or on θ.

3. Thom class of a vector bundles: The Mathaï–Quillen strategy [MQ86]

Let V be a real oriented Euclidean vector space of dimension $n = 2d$ with scalar product $(\ ,\ )_V$. On V, we choose a canonical basis $\{\mathbf{e}_k\}$ orthonormal with respect to $(\ ,\ )_V$:

$$(\mathbf{e}_i, \mathbf{e}_j)_V = \delta_{ij}. \tag{23}$$

Any vector on V can be decomposed as

$$\mathbf{v} = v^k \mathbf{e}_k. \tag{24}$$

Such a decomposition gives a coordinates system (v^k) on V, turning V into a manifold. Due to the linear space structure of V, only $GL(n, \mathbb{R})$ transformations define allowed coordinate changes. The group of isometries of V, with respect to $(\ ,\ )_V$, is $SO(n) \subset GL(n, \mathbb{R})$, with Lie algebra $so(n)$ and Weil algebra $\mathcal{W}(SO(n))$. Finally, we endow V and $\mathcal{W}(SO(n))$ with the standard differential operations d_V, i_V, l_V, $d_{\mathcal{W}}$, $i_{\mathcal{W}}$ and $l_{\mathcal{W}}$.

Now, let $E(\mathcal{M}, V)$ be a vector bundle over a smooth manifold $\mathcal{M}$ with typical fiber V, equipped with differential operations: d_E, i_E and l_E,. We denote $\Omega^n_{r\,\mathrm{d}v}(E)$ the space of n-forms on E whose restriction to each fiber of E is rapidly decreasing. The corresponding cohomology space is written $H^n_{r\,\mathrm{d}v}(E)$. The Thom Class of E is the element $T(E)$ of $H^n_{r\,\mathrm{d}v}(E)$ such that

$$\int_V T(E) = 1, \tag{25}$$

which means that integration of $T(E)$ along the fiber produces the constant function 1 on $\mathcal{M}$.

Actually, following Mathaï and Quillen [MQ86], we would like to exhibit a representative of $T(E)$ in the form of an integral representation. Then, we consider V^*, the dual space of V, equipped with the scalar product $(\ ,\ )_{V^*}$, dual to $(\ ,\ )_V$ on V. Moreover, we introduce coordinates (ϖ_k) for the Grassmann algebra ΛV^* of V^* together with the differential operations δ, I and L, dual to those on V.

We take as structure equations:

$$\begin{aligned}
s^{\mathrm{top}} v^k &= \Psi^k + L^{\mathrm{top}}(\omega) v^k, \\
s^{\mathrm{top}} \Psi^k &= -L^{\mathrm{top}}(\Omega) v^k + L^{\mathrm{top}}(\omega) \Psi^k, \\
s^{\mathrm{top}} \varpi_k &= b_k + L^{\mathrm{top}}(\omega) \varpi_k, \\
s^{\mathrm{top}} b_k &= -L^{\mathrm{top}}(\Omega) \varpi_k + L^{\mathrm{top}}(\omega) b_k, \\
s^{\mathrm{top}} \omega &= \Omega - \tfrac{1}{2}[\omega, \omega], \\
s^{\mathrm{top}} \Omega &= -[\omega, \Omega],
\end{aligned} \tag{26}$$

with

$$s^{\mathrm{top}} = d_{\mathcal{W}} + (d_V + \delta) + (l_V + L)(\omega) - (i_V + I)(\Omega), \tag{27}$$

$$\Psi^k = d_V v^k \equiv \Psi^k_{\mathrm{int}} \tag{28}$$

in the intermediate model, and

$$s^{\rm top} = d_{\mathcal{W}} + d_V + \delta, \tag{29}$$

$$\Psi^k = (d_V - L^{\rm top}(\omega))v^k \equiv \Psi^k_{\rm W} \tag{30}$$

in the Weil model, while

$$L^{\rm top} = l_V + L \tag{31}$$

in any model.

The null section s_0 of $E(\mathcal{M}, V)$ that sends any point of $\mathcal{M}$ into the null vector, diffeomorphically maps $\mathcal{M}$ into $s_0(\mathcal{M}) \subset E$. Then, the Thom Class $T(E)$ of E is nothing but the Poincaré dual of $s_0(\mathcal{M})$ in E [BT82], and the Dirac form on E:

$$\delta(\mathbf{v})\,\mathrm{d}v^1 \wedge \cdots \wedge \mathrm{d}v^n \tag{32}$$

represents the Poincaré dual of $s_0(\mathcal{M})$ in E. This form can be written as a Fourier transform:

$$\frac{1}{(2\pi)^n}\int \mathrm{d}b\,\mathrm{d}\varpi\,\exp\mathrm{i}\,\{b\cdot\mathbf{v} + \varpi\cdot\Psi\} = \frac{1}{(2\pi)^n}\int \mathrm{d}b\,\mathrm{d}\varpi\,\exp\mathrm{i}\{b_k v^k + \varpi_k\Psi^k\}. \tag{33}$$

From the structure equations (26), we deduce

$$b\cdot\mathbf{v} + \varpi\cdot\Psi = s^{\rm top}(\varpi\cdot\mathbf{v}). \tag{34}$$

However, we can consider a smoother representative, with a gaussian behavior for instance. That means that we must insert a term of the form:

$$i(b, b)_{V^*} \tag{35}$$

into (34). Now, we can try to write the new argument as an $s^{\rm top}$-exact term:

$$s^{\rm top}(\varpi\cdot\mathbf{v} + i(\varpi, b)_{V_*}) = b\cdot\mathbf{v} + \varpi\cdot\Psi + i(b,b)_{V^*} - i(L^{\rm top}(\Omega)\varpi, \varpi)_{V^*} \tag{36}$$

so that we are led to define

$$U = \frac{1}{(2\pi)^n}\int \mathrm{d}b\,\mathrm{d}\varpi\,\exp\mathrm{i}\{s^{\rm top}(\varpi\mathbf{v} + i(\varpi, b)_{V_*})\}. \tag{37}$$

Note that U is an element of $\mathcal{W}(SO(n)) \otimes \Omega^*(V)$.

In order to prove that U maps into a representative of $T(E)$, let us proceed in the intermediate model where we write $U_{\rm int}$ instead of U. Then, since in (37) $\omega \in \mathcal{W}(SO(n))$ does not appear, we immediately conclude that $U_{\rm int}$ does not explicitly depend on ω, that is to say:

$$\forall\lambda \in so(n), \quad i_{\mathcal{W}}(\lambda)U_{\rm int} = 0, \tag{38}$$

which express the basicity condition within the intermediate model. Now, there remains to show that $U_{\rm int}$ is closed with respect to $D_{\rm int} = d_{\mathcal{W}} + d_V + l_V(\omega) - i_V(\Omega)$. Indeed

$$D_{\rm int}U_{\rm int} = \frac{1}{(2\pi)^n} D_{\rm int} \int {\rm d}b\,{\rm d}\varpi \exp\{{\rm i}\cdot s^{\rm top}(\varpi.{\bf v} + i(\varpi, b)_{V_*})\} \tag{39}$$

$$= \frac{1}{(2\pi)^n} \int {\rm d}b\,{\rm d}\varpi (s^{\rm top} - D_{V^*}) \exp\{{\rm i}\cdot s^{\rm top}(\varpi\cdot{\bf v} + i(\varpi, b)_{V_*})\}, \tag{40}$$

where $D_{V^*} = \delta + L(\omega) - I(\Omega)$. Hence

$$D_{\rm int}U_{\rm int} = -\frac{1}{(2\pi)^n} \int {\rm d}b\,{\rm d}\varpi [D_{V^*} \exp\{{\rm i}\cdot s^{\rm top}(\varpi\cdot{\bf v} + {\rm i}(\varpi, b)_{V_*})\}]. \tag{41}$$

Now, from the structure equations (26), we get

$$D_{V^*} = (b_k + L^{\rm top}(\omega)\varpi_k)\frac{\partial}{\partial\varpi_k} + (-L^{\rm top}(\Omega)\varpi_k + L^{\rm top}(\omega)b_k)\frac{\partial}{\partial b_k} \tag{42}$$

$$= \left(L^{\rm top}(\omega)\varpi_k\frac{\partial}{\partial\varpi_k} + L^{\rm top}(\omega)b_k\frac{\partial}{\partial b_k}\right) + \left(b_k\frac{\partial}{\partial\varpi_k} - L^{\rm top}(\Omega)\varpi_k\frac{\partial}{\partial b_k}\right). \tag{43}$$

The first term in D_{V^*} corresponds to an $so(n)$-transformation. Due to the $so(n)$-invariance of the measure ${\rm d}b\,{\rm d}\varpi$, it does not contribute to (41). The last term in (43) vanishes upon integration by parts. Then

$$D_{\rm int}U_{\rm int} = 0. \tag{44}$$

Finally, combining Eqs. (38) and (44), we deduce that

$$\forall\lambda \in so(n), \quad (l_{\mathcal W} + l_V)(\lambda)U_{\rm int} = 0, \tag{45}$$

and conclude that $U_{\rm int}$ is a representative in $\mathcal{W}(SO(n)) \otimes \Omega^*(V)$ of the Thom Class of $E(\mathcal{M}, V)$. The corresponding representative in the Weil model is obtained by setting

$$s^{\rm top} = d_{\mathcal W} + d_V + \delta, \tag{46}$$

$$\Psi^k = (d_V - L^{\rm top}(\omega))v^k \equiv \Psi^k_W \tag{47}$$

within Eq. (37).

Actually, it can be easily shown that Fourier transform (denoted $\mathcal{F}$) commutes with equivariant differential operations. More precisely

$$\mathcal{F}[(d_{\mathcal W} + \delta + L(\omega) - I(\Omega))\Phi] = (d_{\mathcal W} + d_V + l_V(\omega) - i_V(\Omega))\mathcal{F}[\Phi], \tag{48}$$

$$\mathcal{F}[i_{\mathcal W}(\lambda)\Phi] = i_{\mathcal W}(\lambda)\mathcal{F}[\Phi], \tag{49}$$

$$\mathcal{F}[(l_{\mathcal W} + L)(\lambda)\Phi] = (l_{\mathcal W} + l_V)(\lambda)\mathcal{F}[\Phi] \tag{50}$$

in the intermediate model. The same holds in the Weil model with suitable differentials. Let us point out that this mainly relies on the identity $b\cdot{\bf v} + \varpi\cdot\Psi = s^{\rm top}(\varpi\cdot{\bf v})$.

Then, since $\phi = (b, b)_{V^*} + (L^{\rm top}(\Omega)\varpi, \varpi)_{V^*}$ is equivariant, it is straightforward to find that its Fourier transform is also equivariant. This simple remark allows to construct representatives of equivariant cohomology classes using Fourier transform of functions of ϕ.

Finally, we can consider a principal $SO(n)$-bundle P over $\mathcal{M}$. It is well known that $P \times_{SO(n)} V$ is a vector bundle isomorphic to E, and $P \times V$ is called the principal $SO(n)$-bundle associated with $E(\mathcal{M}, V)$. Hence, as an n-form on E, any representative of the Thom Class $T(E)$ of E comes from a closed $SO(n)$-basic n-form on the associated bundle $P \times V$ of E. In order to produce such a representative of $T(E)$, we use Cartan's Theorem 3, that is to say we replace (ω, Ω) (in the representative U) by (θ, Θ), a connection and its curvature on $P(\mathcal{M}, SO(n))$.

4. Construction of representatives of Thom class of vector bundles: The Berline–Getzler–Vergne strategy [BGV91]

In this section, we shall use the strategy explained in Section 2 in order to produce representatives of $T(E)$.

To begin with, we are going to turn V into a Riemannian manifold, i.e. a manifold V with a metric. The tangent bundle of V, denoted by TV, is obviously isomorphic to $V \times V$. The only $SO(n)$-invariants formed with $\mathbf{v}$ and $\mathrm{d}\mathbf{v}$ are the three scalar products, so that the general $SO(n)$-invariant metric on V is:

$$\mathrm{d}s^2(\mathbf{v}) = \mathrm{e}^{\varphi}((\mathrm{d}v^i)^2 + \sigma(v^i\,\mathrm{d}v^i)^2), \tag{51}$$

where φ and σ are smooth functions of $t = (\mathbf{v}, \mathbf{v})_V$ only. The above expression is positive definite if and only if $1 + \sigma(t)t > 0$ for $t \geq 0$. One can assume if convenient that the metric is asymptotically flat (i.e. that the curvature vanishes at infinity).

We can consider the principal $GL(n, \mathbb{R})$-bundle associated with TV, i.e. the frame bundle $R(V)$ of V. It is made of the points $(\mathbf{v}, b_{\mathbf{v}})$ where $b_{\mathbf{v}}$ is a frame (i.e. a basis) at $\mathbf{v}$. Coordinates for $b_{\mathbf{v}}$ are defined as follows. We denoted by (∂_k) the natural basis of $T_{\mathbf{v}}V$ defined by the canonical coordinates (v^k) of V: $\partial_k = \partial/\partial v^k$. Then, the coordinates of $b_{\mathbf{v}}$ are the components b_k^j of the decomposition of $b_{\mathbf{v}}$ with respect to the natural basis (∂_k):

$$b_{\mathbf{v}k} = b_k^j \partial_j \tag{52}$$

with $b_{\mathbf{v}k}$ the kth frame vector of the frame $b_{\mathbf{v}}$. The isometry group of $(V, (\ ,\ )_V)$, namely $SO(n)$, acts both on elements of V and on frames, that is to say on $R(V)$. This goes as follows. For any $\Phi \in SO(n)$,

$$\Phi^k(\mathbf{v}) = \Phi_m^k v^m. \tag{53}$$

At the infinitesimal level, if we write $\Phi_m^k = \delta_m^k + \varphi_m^k$, we get

$$\Phi^k(\mathbf{v}) = (\delta_m^k + \varphi_m^k)v^m = v^k + \varphi_m^k v^m = v^k + \xi^k, \tag{54}$$

where $\xi^k = \varphi_m^k v^m$ defines a vector field on V, the so-called fundamental vector field associated with the action of $\varphi \in so(n)$:

$$\xi = \xi^k \frac{\partial}{\partial v^k}. \tag{55}$$

The natural action of $\Phi \in SO(n)$ on $T_{\mathbf{v}}V$ is given by the so-called differential of Φ at $\mathbf{v}$, $\mathrm{d}_{\mathbf{v}}\Phi : T_{\mathbf{v}}V \to T_{\Phi(\mathbf{v})}V$:

$$\forall X_{\mathbf{v}} \in T_{\mathbf{v}}V,\ \forall f \in C^{\infty}(V) \quad \mathrm{d}_{\mathbf{v}}\Phi X_{\mathbf{v}}\,(f) = X_{\mathbf{v}}(f \circ \Phi). \tag{56}$$

Applying this definition to the frame vectors $b_{\mathbf{v}k}$, one gets

$$\tilde{b}^i_j = b^m_j(\partial_m \Phi^i(\mathbf{v})) = b^m_j \Phi^i_m, \tag{57}$$

where $\tilde{b}^i_j$ are the coordinates of the transformed frame at $\Phi(\mathbf{v})$. At the infinitesimal level, for $\varphi \in so(n)$,

$$\tilde{b}^i_j = b^m_j(\delta^i_m + \varphi^i_m) = b^i_j + b^m_j\varphi^i_m = b^i_j + b^m_j\varphi^i_m = b^i_j + \Xi^i_j. \tag{58}$$

Combining Eqs. (54) and (58), we deduce that the fundamental vector field associated with the action of $\varphi \in so(n)$ on $R(V)$ reads

$$\lambda_R = \xi^k \frac{\partial}{\partial v^k} + \Xi^p_q \frac{\partial}{\partial b^p_q} = \varphi^k_m v^m \frac{\partial}{\partial v^k} + b^m_q \varphi^p_m \frac{\partial}{\partial b^p_q}. \tag{59}$$

Now, let $P(\mathcal{M}, SO(n))$ be some principal $SO(n)$-bundle over a smooth manifold $\mathcal{M}$. It is well known that there is a vector bundle over $\mathcal{M}$ associated with P for the action of $SO(n)$ on V. The group $SO(n)$ acts on the right on P and on the left on V. We first define a right-action of $SO(n)$ on $P \times V$ by setting

$$(p, \mathbf{v})^{\Phi} = (p \cdot \Phi, \Phi^{-1}(\mathbf{v})) \tag{60}$$

so that, the fundamental vector field representing the action of $\varphi \in so(n)$ on $P \times V$ reads

$$\lambda_{P\times V} = \lambda_P - \xi^k \frac{\partial}{\partial v^k} = \lambda_P - \varphi^k_m v^m \frac{\partial}{\partial v^k}, \tag{61}$$

where λ_P is the fundamental vector field representing the action of φ on P.

Finally, the action of any $\varphi \in so(n)$ on the $GL(n, \mathbb{R})$-principal bundle $P \times R(V)$ is given by following fundamental vector field:

$$\lambda = \lambda_P - \lambda_R \tag{62}$$

with λ_R defined in Eq. (59).

In the following, V, $R(V)$ and P are equipped with the following differential operations: $d_V, d_R, d_P, i_V, i_R, i_P, l_V, l_R$ and l_P, respectively, exterior differentials, inner products and Lie derivatives.

Now, since we are looking for representatives of equivariant cohomology classes, we can mimic the construction made in [STW94] in the case of two-dimensional Gravity. We first look for a $GL(n, \mathbb{R})$-connection on $\mathcal{P}(P \times V, GL(n, \mathbb{R})) = P \times R(V)$ invariant under the action of $SO(n)$. If we notice that, by construction, the metric $\mathbf{g}$ on V is $SO(n)$-invariant, we can consider the Levi-Cevita connection ${}^{\mathrm{LC}}\Gamma$ associated with $\mathbf{g}$. Due to the $SO(n)$-invariance of $\mathbf{g}$, ${}^{\mathrm{LC}}\Gamma$ is an $SO(n)$-invariant connection. More precisely, the lift of ${}^{\mathrm{LC}}\Gamma$ into a connection 1-form Γ on $R(V)$ according to

$$\Gamma = b^{-1}({}^{\mathrm{LC}}\Gamma)b + b^{-1}d_R b \tag{63}$$

is invariant under the action of $SO(n)$. The fundamental vector field for the action of $so(n)$ was given before, so that

$$(i_{\mathcal{P}}(\lambda)\Gamma)^{\sigma}_{\tau} = (b^{-1})^{\sigma}_{\nu}(-\,^{\mathrm{LC}}D_{\mu}\xi^{\nu})b^{\mu}_{\tau}, \tag{64}$$

where $i_{\mathcal{P}}(\lambda) = (i_P + i_R)(\lambda)$, and

$$l_{\mathcal{P}}(\lambda)\Gamma = 0 \tag{65}$$

with $l_{\mathcal{P}}(\lambda) = (l_P + l_R)(\lambda)$.

The next step is to consider the Weil algebra $\mathcal{W}(SO(n))$ of $so(n)$. The relevant formulae were given in Section 2. We recall that the equivariant curvature of Γ in the intermediate model is

$$R^{\mathrm{eq}}_{\mathrm{int}}(\Gamma, \omega, \Omega) = (d_{\mathcal{W}} + d_{\mathcal{P}} + l_{\mathcal{P}}(\omega) - i_{\mathcal{P}}(\Omega))\,\Gamma + \tfrac{1}{2}\,[\Gamma, \Gamma] \tag{66}$$

while the corresponding curvature in the Weil model is obtained as

$$R^{\mathrm{eq}}_{\mathrm{W}}(\Gamma, \omega, \Omega) = \mathrm{e}^{i_{\mathcal{P}}(\omega)} R^{\mathrm{eq}}_{\mathrm{int}}(\Gamma, \omega, \Omega), \tag{67}$$

which gives

$$R^{\mathrm{eq}}_{\mathrm{W}}(\Gamma, \omega, \Omega) = R(\Gamma) + i_{\mathcal{P}}(\omega)R(\Gamma) + \tfrac{1}{2} i_{\mathcal{P}}(\omega) i_{\mathcal{P}}(\omega) R(\Gamma) - i_{\mathcal{P}}(\Omega)\Gamma. \tag{68}$$

The Weil equivariant Euler class is defined by

$$E^{\mathrm{eq}}_{\mathrm{W}} = \frac{\varepsilon^{\mu_1\rho_1\cdots\mu_d\rho_d}}{\sqrt{\mathbf{g}}} g_{\rho_1 v_1} \cdots g_{\rho_d v_d} (R^{\mathrm{eq}}_{\mathrm{W}})^{v_1}_{\mu_1} \wedge \cdots \wedge (R^{\mathrm{eq}}_{\mathrm{W}})^{v_d}_{\mu d}, \tag{69}$$

which after normalization gives rise to a representative of $T(E)$ in $P(\mathcal{M}, SO(n)) \times V$.

It is now time to use the explicit form of the metric to get a formula for the Thom class. Surprisingly, we shall see there is no choice of metric that allows to recover the Mathaï–Quillen representative of $T(E)$. From now on, the computations, if painful, are straightforward. We use the intermediate model so that $d_V v^i \equiv \Psi^i$. As (63) looks formally like a change of coordinates in the fiber, we know that its effect on curvature will be a simple conjugation which disappears completely on the Thom class. So we can forget it in the computation. From (51) we find that the metric is:

$$g_{ij} = \mathrm{e}^{\varphi}(\delta_{ij} + \sigma v_i v_j). \tag{70}$$

Our notations need some comment: we start with global coordinates v^i on V, so the exponent i is not a tensor component but just a label. The metric is expressed with respect to this particular coordinate system. However, it is convenient to deal consistently with formal lower and upper indices in the Einstein summation convention. So we define $v_i \equiv v^i$ and $\delta_{ij} \equiv \delta^{ij} \equiv \delta^i_j \equiv \delta^j_i = 1$ if $i = j$ and 0 else. For instance we use the notation v_i and δ_{ij} in g_{ij} and we write $t = v_i v^i$. This becomes slightly less formal if we restrict the diffeomorphism group of V to linear orthogonal transformations.

A simple computation shows that the inverse metric is

$$g^{ij} = \mathrm{e}^{-\varphi}(\delta^{ij} + \tilde{\sigma} v^i v^j), \tag{71}$$

where $\tilde{\sigma}$ is defined by $(1 + t\tilde{\sigma})(1 + t\sigma) = 1$.

First, we need a formula for the connection and curvature. The fact that φ and σ depend only on t leads to many simplifications in the computation. We use dots for derivatives with respect to t.

With the expression of $\mathbf{g}$, we get for the connection: [5]

$$\Gamma^k_{ij} = (1 + t\tilde{\sigma}) v^k [(\sigma - \dot{\varphi})\delta_{ij} + (\dot{\sigma} - \sigma\dot{\varphi}) v_i v_j] + \dot{\varphi}(v_i \delta^k_j + v_j \delta^k_i) \tag{72}$$

so that the connection matrix is

$$\Gamma^j_i \equiv \Psi^k \Gamma^j_{ik} = A v^j v_i v_k \Psi^k + B v^j \Psi_i + C(v_i \Psi^j + \delta^j_i v_k \Psi^k), \tag{73}$$

where we have set

$$A = (1 + t\tilde{\sigma})(\dot{\sigma} - C\sigma), \qquad B = (1 + t\tilde{\sigma})(\sigma - C), \qquad C = \dot{\varphi}. \tag{74}$$

The curvature matrix is given by

$$R^j_i \equiv d\Gamma^j_i + \Gamma^j_k \wedge \Gamma^k_i. \tag{75}$$

A tedious computation leads to

$$\begin{aligned} R^{ij} &\equiv g^{ik} R^j_k \\ &= \mathrm{e}^{-\varphi}(1 + t\tilde{\sigma})[M_{11} \Psi^i \Psi^j + M_{12}(v^i (v_k \Psi^k) \Psi^j - v^j (v_k \Psi^k) \Psi^i)], \end{aligned} \tag{76}$$

where

$$M_{11} = C^2 t + 2C - \sigma, \qquad M_{12} = 2\dot{C} - C^2 - \dot{\sigma} + (1 + t\tilde{\sigma})(\sigma - C)t\dot{\sigma}. \tag{77}$$

To get the full equivariant curvature, we need the part involving Ω. In accordance with our convention on indices, we define $\Omega^{ij} \equiv \Omega^j_i$. By definition Ω^{ij} is antisymmetric. According to formulae (64) and (66), the part of the equivariant curvature containing Ω is the covariant derivative of $\Omega^j_k v^k$, the $so(n)$ vector field associated to Ω. Consequently

$$(-i_{\mathcal{P}}(\Omega)\Gamma)^j_k = \Omega^j_k + \Omega^l_m v^m \Gamma^j_{l=k}. \tag{78}$$

The antisymmetry of Ω leads to further simplifications. The outcome is:

$$\begin{aligned} & g^{ik}(\Omega^j_k + \Omega^l_m v^m \Gamma^j_{lk}) \\ &\quad = \mathrm{e}^{-\varphi}(1 + t\tilde{\sigma})[M_{21} \Omega^{ij} + M_{22}(v^i v_k \Omega^{kj} - v^j v_k \Omega^{ki})], \end{aligned} \tag{79}$$

where

$$M_{21} = 1 + t\sigma, \qquad M_{22} = C - \sigma. \tag{80}$$

[5] Remember that $\Gamma^k_{ij} \equiv \frac{1}{2} g^{kl}(\partial_i g_{lj} + \partial_j g_{il} - \partial_l g_{ij})$.

We note the striking similarity between the two contributions. If we define a 2×2 matrix N^{ij} by

$$N^{ij}=\begin{pmatrix}\Psi^i\Psi^j & \Omega^{ij}\\ v^i(v_k\Psi^k)\Psi^j-v^j(v_k\Psi^k)\Psi^i & v^iv_k\Omega^{kj}-v^jv_k\Omega^{ki}\end{pmatrix},\tag{81}$$

the equivariant curvature can be written as a trace

$$(R^{\mathrm{eq}}_{\mathrm{int}})^{ij}=\mathrm{e}^{-\varphi}(1+t\tilde{\sigma})\mathrm{Tr}\ MN^{ij}.\tag{82}$$

The equivariant Euler class is

$$E^{\mathrm{eq}}_{\mathrm{int}}=2^{n/2}\sqrt{\mathbf{g}}\,\mathrm{Pfaff}\,(R^{\mathrm{eq}}_{\mathrm{int}})^{ij}\tag{83}$$

with the usual definition of the Pfaffian. Note that $\sqrt{\mathbf{g}}=\mathrm{e}^{n\varphi/2}(1+t\sigma)^{1/2}$.

This is the explicit formula for the universal Thom class that we were after. It involves two arbitrary functions of t, φ and σ (with the mild restriction $1+t\sigma>0$) which may be localized at will thus so leaving a fair amount of flexibility.

The first comment to make is that apparently the above representative, which is of course $so(n)$ invariant when $so(n)$ acts on V, Ω and Ψ at the same time, is not invariant when $so(n)$ acts only on V. To state it more simply, the V dependence of the Thom class is not only through t. This is to be contrasted with the Mathaï–Quillen representative.

Let us deal with a special case first. When $n=2$, it is easy to see that

$$\varepsilon_{ij}N^{ij}=2\begin{pmatrix}\Psi^1\Psi^2 & \Omega^{12}\\ t\Psi^1\Psi^2 & t\Omega^{12}\end{pmatrix},\tag{84}$$

so we have some hope to recover the Mathaï–Quillen formula as a special case. After some manipulations one finds

$$E^{\mathrm{eq}}_{\mathrm{int}}=4\dot{F}\Psi^1\Psi^2+2F\Omega^{12},\tag{85}$$

where

$$F\equiv\frac{1+tC}{(1+t\sigma)^{1/2}}.\tag{86}$$

So the Thom class depends only on one arbitrary function of t, namely F, which can easily be adjusted to recover the Mathaï–Quillen representative. The correct choice is $F=-(1/(4\pi))\exp(-t/4)$.

When $n>2$ the situation is more complicated. We shall use a trick to see how much the symmetry of the $so(n)$ action on V is broken.

The first observation is that under a similarity, the Pfaffian has a simple behavior: if A is a square antisymmetric matrix and S an arbitrary square matrix of the same size with transpose S^{t}, $S^{\mathrm{t}}AS$ is again antisymmetric and Pfaff $S^tAS=\mathrm{Det}\ S\,\mathrm{Pfaff}\ A$. The square of this equation just follows from the multiplicative property of the determinant, and the sign is fixed by the case when S is the identity matrix. So if we can find a matrix S^{ij} (independent of Ψ and Ω) such that $S^{\mathrm{t}}R^{\mathrm{eq}}_{\mathrm{int}}S$ simplifies, we shall end with a simpler formula for the Thom class.

Define a symmetric matrix $S(D)$ of parameter D by

$$S(D)^i_j = \delta^i_j + Dv^i v_j. \tag{87}$$

This matrix is easily diagonalized: the vectors orthogonal to v^i are left invariant and v^i is multiplied by $1 + tD$. So

$$\mathrm{Det}\ S(D) = 1 + tD \tag{88}$$

and

$$S(D)S(E) = S(D + E + tDE). \tag{89}$$

Moreover, if A^{ij} is any antisymmetric matrix,

$$(S(D)AS(D))^{ij} = A^{ij} + D(v^i v_k A^{kj} - v^j v_k A^{ki}). \tag{90}$$

We apply this identity to the four antisymmetric objects building the 2×2 matrix N^{ij} to get

$$S(D)NS(D) = \bar{S}(D)N, \tag{91}$$

where $\bar{S}(D)$ is the 2×2 matrix

$$\begin{pmatrix} 1 & D \\ 0 & 1 + tD \end{pmatrix}. \tag{92}$$

In Eq. (91), the left-hand side involves a product of $n \times n$ matrices, and the 2×2 indices are spectators whereas on the right-hand side the opposite occurs.

So we can write

$$S(D)R^{\mathrm{eq}}_{\mathrm{int}}S(D) = \mathrm{e}^{-\varphi}(1 + t\tilde{\sigma})\ \mathrm{Tr}\ M(D)N \tag{93}$$

with $M(D) \equiv M\bar{S}(D)$, and the Thom class is

$$E^{\mathrm{eq}}_{\mathrm{int}} = (1 + t\sigma)^{(1-n)/2}(1 + tD)^{-1}\mathrm{Pfaff}\,(\mathrm{Tr}\ M(D)N). \tag{94}$$

We can choose D to simplify the expression of $E^{\mathrm{eq}}_{\mathrm{int}}$.

First we take

$$D = D_1, \quad \text{where } M(D_1) = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix}.$$

This makes it easy to compute the term in $E^{\mathrm{eq}}_{\mathrm{int}}$ that does not involve Ω. The outcome is

$$E^{\mathrm{eq}}_{\mathrm{int}} = n!(1 + t\sigma)^{(1-n)/2}\left(1 + t\frac{M_{12}}{M_{11}}\right)M_{11}^{n/2}\Psi^1 \cdots \Psi^n + \text{terms involving } \Omega. \tag{95}$$

One can check that this is compatible with (85) for $n = 2$.

Second, we take

$$D = D_2, \quad \text{where} M(D_2) = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix}.$$

This makes it easy to compute the term in $E^{\text{eq}}_{\text{int}}$ that does not involve Ψ. The outcome is

$$E^{\text{eq}}_{\text{int}} = 2^{n/2}(1+t\sigma)^{(1-n)/2}\left(1+t\frac{M_{22}}{M_{21}}\right)M_{21}^{n/2}\,\text{Pfaff}\ \Omega + \text{terms involving}\ \Psi, \quad (96)$$

a result which is again compatible with (85) for $n = 2$.

Those two terms in $E^{\text{eq}}_{\text{int}}$ automatically depend only on t. On the other hand, the other terms are not scalars for the $so(n)$ action on V. To see this we keep $D = D_2$, set $A^{ij} = M_{11}\Psi^i\Psi^j + M_{21}\Omega^{ij}$ and $B^{ij} = v^i(v_k\Psi^k)\Psi^j - v^j(v_k\Psi^k)\Psi^i$. Using the fact that $x_k\Psi^k$ squares to 0 we get

$$E^{\text{eq}}_{\text{int}} = (1+t\sigma)^{(1-n)/2} \times \left(2^{n/2}\left(1+t\frac{M_{22}}{M_{21}}\right)\text{Pfaff}\ A - n/2\frac{\text{Det}\ M}{M_{21}}\varepsilon_{i_1 j_1\cdots i_n j_n}B^{i_1 j_1}A^{i_2 j_2}\ldots A^{i_n j_n}\right). \quad (97)$$

As Ω and Ψ are independent families of indeterminates, the matrix elements of A^{ij} are independent of each other (except for antisymmetry) and of the matrix elements of B^{ij}. So in the expansion of

$$\varepsilon_{i_1 j_1\cdots i_n j_n}B^{i_1 j_1}A^{i_2 j_2}\ldots A^{i_n j_n}, \quad (98)$$

no compensation can occur between A-factors and B-factors or between different B-factors. Moreover, B-factors contain the full non-$so(n)$ invariant part of the V dependence of the Thom class. So we have the following three possibilities. Either Det M is 0, or B^{ij} is invariant for the action of $so(n)$ on V, or the representative of the Thom class is not invariant for the action of $so(n)$ on V. The first term of the alternative depends on our choice of φ and σ. The second is easily checked to occur if and only if $n = 2$, a case we have already treated.

So finally, we have shown that if $n > 2$ the representative of the Thom class is invariant for the $so(n)$ action on V if and only if Det $M = 0$.

We shall now see that despite the fact that apparently our representative of the Thom class depends on two arbitrary functions, the single condition Det $M = 0$ fixes it completely. This can be seen as a manifestation of the topological character of the Thom class. We shall also see that the representative we end up with is not the Mathaï–Quillen representative.

From now on, we set Det $M = 0$. Explicit computation shows that this equation has a first integral. Namely Det $M = 0$ is equivalent to

$$(1+tC)^3\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\sigma}{(1+tC)^2} + \frac{1}{t(1+tC)^2} - \frac{1}{t}\right) = 0. \quad (99)$$

The term in parenthesis can be written as

$$-\frac{C^2t + 2C - \sigma}{(1+tC)^2} \quad \text{or} \quad \frac{(1+t\sigma)-(1+tC)^2}{t(1+tC)^2}. \quad (100)$$

Now, we distinguish two cases.

Suppose first that for some value of t the function $1 + tC$ vanishes together with its first derivative. Then

$$M = \frac{1 + t\sigma}{t} \begin{pmatrix} -1 & 1/t \\ t & -1 \end{pmatrix}. \tag{101}$$

As a byproduct, $M_{21} + tM_{22}$ vanishes, and the equivariant Euler class vanishes. So clearly, the function $1 + tC$ cannot vanish everywhere if we are to find a non-trivial class. Anyway, the vanishing of $1 + tC$ would mean that $e^{\varphi} = t_0/t$ for some constant t_0 leading to a metric singular at the origin. It is likely that in this case, a careful computation with distributions would give a curvature concentrated at the origin, but we are not interested in this anyway.

On the open intervals where $1 + tC \neq 0$ the second factor of (99) has to vanish. We get

$$\frac{\sigma}{(1 + tC)^2} + \frac{1}{t(1 + tC)^2} - \frac{1}{t} = \frac{1}{t_0} \tag{102}$$

for some constant t_0. Using (100), one obtains

$$M = \frac{1 + tC}{t_0} \begin{pmatrix} -(1 + tC) & (1 + (t_0 + t)C)(t_0 + t)^{-1} \\ (1 + tC)(t_0 + t) & -(1 + (t_0 + t)C) \end{pmatrix} \tag{103}$$

leading to a remarkable simplification of (97):

$$E^{\mathrm{eq}}_{\mathrm{int}} = 2^{n/2} \left(\frac{t_0}{t_0 + t} \right)^{1/2} \operatorname{Pfaff} \left(\Omega^{ij} - \frac{1}{t_0 + t} \Psi^i \Psi^j \right). \tag{104}$$

Now, as $M_{21} = 1 + t\sigma$, which has to remain strictly positive, $1 + tC$ cannot vanish at the boundary of an open interval where it is non-zero. This means that $1 + Ct$ vanishes nowhere, and that formula (104) is valid everywhere. This is our final formula for the equivariant Euler class if we decide to trade flexibility (arbitrary choice of φ and σ) for simplicity ($so(n)$ invariance on V, leading to a simple Pfaffian). The Mathaï–Quillen representative never shows up for $n > 2$.

Some comments are in order. Usually the Thom class is defined by using function with compact support (differential topology) or rapid decrease at infinity (quantum field theory) on V. The Mathaï–Quillen representative belongs to this second category. With the general formula, the freedom on φ and σ allows us to impose any behavior at infinity.[6] On the other hand our rigid proposal for the Thom class does not decrease fast at infinity. Despite the fact that this may be inconvenient in certain applications, we would like to point that it makes sense nevertheless. To define the Thom class, the crucial point is that the cohomology of V with coefficients having compact support or rapid decrease at infinity is concentrated in the dimension of V and one dimensional there. It seems clear that a cohomology of V can be build such as to retain this property and accept our rigid representative as a well-defined cohomology class. For instance k-forms on V such that for any non-negative integer l the partial derivatives of order l of the coefficients exist and are $O(t^{-(k+l+1)/2})$ at infinity, endowed with the usual exterior derivative, should work.

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[6] In fact to localize the Thom class on arbitrary spherical shells if this proves useful.

In particular, we can normalize things in such a way that the integral on V of the term independent of Ω is 1, as is usual for the Thom class. A simple calculation gives for the normalized Thom class

$$T_V = \frac{1}{(2\pi)^d}\left(\frac{t_0}{t_0+t}\right)^{1/2} \mathrm{Pfaff}\left(\frac{1}{t_0+t}\Psi^i\Psi^j - \Omega^{ij}\right). \tag{105}$$

Playing with the value of t_0 allows to localize around the zero section. This formula already appears in [HL93] as a specialization of another formula for the Thom class.

5. Conclusion

In these notes, we have obtained formulae for the universal Thom class of a vector bundle. A special choice leads to a rigid representative involving Cauchy-type kernels. It would be very interesting to know whether the Mathaï–Quillen representative, with its Gaussian-type kernel, is also a rigid member of some natural family of representatives.

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Some remarks on topological 4D-gravity

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Abstract

We show that the method of Wu [J. Geom. Phys. 12 (1993) 205] to study topological 4D-gravity can be understood within a standard method now designed to produce equivariant cohomology classes. Next, this general framework is applied to produce some observables of the topological 4D-gravity. © 1998 Elsevier Science B.V.

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1. Introduction

Since their appearance in 1988 in a famous article of Witten [13], topological field theories have played an important role in theoretical physics as well as in mathematics. Actually, the 1988 article gave a prototype of topological field theories of cohomological type. Witten has recognized that these cohomological field theories are related to equivariant cohomology and more precisely to the so-called Cartan model of equivariant cohomology.

Although cohomological field theories can be described independently of the models used for equivariant cohomology, the construction by Kalkman [9] of the so-called intermediate model [12] is of considerable technical help. In [12], topological Yang–Mills [1,3,13] and topological 2D gravity [4,5] were studied from this point of view. In [2], new representatives of the Thom class of a vector bundle were produced using this general framework.

Wu [14] explained the role of the universal bundle in 4D gravity,[2] and exhibited some observables of the corresponding topological model. We shall explain here how his method can be deduced from the general approach of [12] and which observables are obtained.

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2. From the intermediate to the Weil model of equivariant cohomology

In [12] it was explained how one can generate representatives of equivariant cohomology classes using an idea of [6] which benefits from Kalkman's construction [9] as follows: let us assume that $\mathcal{M}$ is a smooth manifold with a smooth $\mathcal{G}$-action for some connected Lie group $\mathcal{G}$ (with Lie algebra Lie $\mathcal{G}$). Let $d_{\mathcal{M}}, i_{\mathcal{M}}, l_{\mathcal{M}}$ be the standard exterior derivative, inner product and Lie derivative on $\mathcal{M}$. The action of $\mathcal{G}$ induces an action of Lie $\mathcal{G}$, and to any $\lambda \in$ Lie $\mathcal{G}$, there corresponds a so-called fundamental vector field $\lambda_{\mathcal{M}}$ on $\mathcal{M}$. The space of forms on $\mathcal{M}$ is denoted by $\Omega(\mathcal{M})$, and its basic elements are those annihilated both by $i_{\mathcal{M}}(\lambda)$ and $l_{\mathcal{M}}(\lambda)$, for any $\lambda \in$ Lie $\mathcal{G}$. We recall that $l_{\mathcal{M}} = [d_{\mathcal{M}}, i_{\mathcal{M}}]_+$.

The Weil algebra $(\mathcal{W}(\mathcal{G}), d_{\mathcal{W}}, i_{\mathcal{W}}, l_{\mathcal{W}})$ of $\mathcal{G}$ is the graded differential algebra generated by the "connection ω" and its "curvature Ω"

$$d_{\mathcal{W}}\omega = \Omega - \tfrac{1}{2}[\omega, \omega], \tag{1}$$

$$d_{\mathcal{W}}\Omega = -[\omega, \Omega], \tag{2}$$

$$i_{\mathcal{W}}(\lambda)\omega = \lambda, \tag{3}$$

$$i_{\mathcal{W}}(\lambda)\Omega = 0, \tag{4}$$

$$l_{\mathcal{W}}(\lambda)\omega = -[\lambda, \omega], \tag{5}$$

$$l_{\mathcal{W}}(\lambda)\Omega = -[\lambda, \Omega], \tag{6}$$

for any $\lambda \in$ Lie $\mathcal{G}$.

Then the equivariant cohomology for the action of $\mathcal{G}$ on $\mathcal{M}$ is the basic cohomology of the graded differential algebra $(\mathcal{W}(\mathcal{G}) \otimes \Omega(\mathcal{M}), d_{\mathcal{W}} + d_{\mathcal{M}}, i_{\mathcal{W}} + i_{\mathcal{M}}, l_{\mathcal{W}} + l_{\mathcal{M}})$. It generates the so-called Weil model of equivariant cohomology.

Now let us consider another Lie group H such that $\mathcal{M}$ is the base space of some principal H-bundle $\mathcal{P}(\mathcal{M}, H)$ on which the action of $\mathcal{G}$ can be lifted. This bundle is also equipped with standard differential operations: $d_{\mathcal{P}}, i_{\mathcal{P}}, l_{\mathcal{P}}$. Then some equivariant cohomology classes can be represented as follows: consider a $\mathcal{G}$-invariant H-connection Γ on $\mathcal{P}$. Extend Γ to $\mathcal{W}(\mathcal{G}) \otimes \Omega(\mathcal{M})$, still denoting it Γ. Since Γ does not depend on ω, it fulfills

$$i_{\mathcal{W}}(\lambda)\Gamma = 0, \tag{7}$$

$$(l_{\mathcal{W}} + l_{\mathcal{P}})(\lambda)\Gamma = 0, \tag{8}$$

for any $\lambda \in$ Lie $\mathcal{G}$. This expresses the basicity of Γ in the so-called intermediate model of equivariant cohomology. In this model, the exterior derivative reads

$$D_{\text{int}} = d_{\mathcal{W}} + d_{\mathcal{P}} + l_{\mathcal{P}}(\omega) - i_{\mathcal{P}}(\Omega) \tag{9}$$

so that

$$D_{\text{int}}\Gamma = d_{\mathcal{P}}\Gamma - i_{\mathcal{P}}(\Omega)\Gamma \tag{10}$$

and the equivariant curvature of Γ in the intermediate model reads

$$R^{\text{eq}}_{\text{int}}(\Gamma, \omega, \Omega) = D_{\text{int}}\Gamma + \tfrac{1}{2}[\Gamma, \Gamma]. \tag{11}$$

It satisfies

$$D_{\rm int} R^{\rm eq}_{\rm int} = [R^{\rm eq}_{\rm int}, \Gamma], \tag{12}$$

$$i_{\mathcal{W}}(\lambda) R^{\rm eq}_{\rm int} = 0, \tag{13}$$

$$(l_{\mathcal{W}} + l_{\mathcal{P}})\lambda R^{\rm eq}_{\rm int} = 0. \tag{14}$$

The H-fibration is eliminated by considering symmetric H-invariant polynomials $I^{\rm eq}_{\rm int} = I(R^{\rm eq}_{\rm int})$.

To go to the more usual Weil model, we use the Kalkman differential algebra isomorphism $\exp\{i_{\mathcal{P}}(\omega)\}$, thus obtaining

$$(d_{\mathcal{W}} + d_{\mathcal{P}}) I^{\rm eq}_{\rm W} = 0, \tag{15}$$

$$(i_{\mathcal{W}} + i_{\mathcal{P}})(\lambda) I^{\rm eq}_{\rm W} = 0, \tag{16}$$

$$(l_{\mathcal{W}} + l_{\mathcal{P}})(\lambda) I^{\rm eq}_{\rm W} = 0, \tag{17}$$

where $I^{\rm eq}_{\rm W} = \exp\{i_{\mathcal{P}}(\omega)\} I^{\rm eq}_{\rm int}$. Now since the H-fibration has disappeared, $I^{\rm eq}_{\rm W}$ lies in $\mathcal{W}(\mathcal{G}) \otimes \Omega(\mathcal{M})$. Under the assumption that $\mathcal{M}$ is a principal $\mathcal{G}$-bundle over $\mathcal{M}/\mathcal{G}$, we can replace ω and Ω by a $\mathcal{G}$-connection θ and its curvature Θ on $\mathcal{M}$. Cartan's Theorem 3 guarantees that our new representative gives a representative of the same equivariant cohomology class [7,12]. Still denoting this representative by $I^{\rm eq}_{\rm W}$, we verify that

$$d_{\mathcal{M}} I^{\rm eq}_{\rm W} = 0, \tag{18}$$

$$i_{\mathcal{M}}(\lambda) I^{\rm eq}_{\rm W} = 0, \tag{19}$$

$$l_{\mathcal{M}}(\lambda) I^{\rm eq}_{\rm W} = 0. \tag{20}$$

Now, we are ready to use this method in topological 4D-gravity.

3. Wu's construction [14] in topological 4D-gravity

Let Σ be a 4D smooth manifold. The fundamental objects in ${\rm Gr}^{\rm top}_4$ are the metrics of Σ and the generators of the Weil algebra of ${\rm Diff}_0(\Sigma)$, the connected component of the diffeomorphism group of Σ. The structure equations then read

$$s^{\rm top} g = \Psi + L^{\rm top}(\omega) g, \tag{21}$$

$$s^{\rm top} \Psi = -L^{\rm top}(\Omega) g + L^{\rm top}(\omega)\Psi, \tag{22}$$

$$s^{\rm top} \omega = \Omega - \tfrac{1}{2}[\omega, \omega], \tag{23}$$

$$s^{\rm top} \Omega = -[\omega, \Omega]. \tag{24}$$

Let us note that the form of these structure equations is universal (i.e. independent of the model we choose). Now, let us apply the precepts of the previous section. The group of diffeomorphisms of Σ plays the role of the gauge group $\mathcal{G}$ over ${\rm Met}(\Sigma)$. The H-fibration is obtained by considering the frame bundle over Σ, $F(\Sigma)$,[3] and our final principal

[3] Note that $F(\Sigma)$ is the principal bundle associated to the tangent vector bundle $T\Sigma$ of Σ.

$GL(4,\mathbb{R})$-bundle $\mathcal{P}$ is just $\mathrm{Met}(\Sigma) \times F(\Sigma)$. The $\mathrm{Diff}(\Sigma)$-invariant $GL(4,\mathbb{R})$-connection Γ on $\mathrm{Met}(\Sigma) \times F(\Sigma)$ is given by

$$\Gamma_\mu{}^\lambda = \Gamma^{\mathrm{LC}}(g)_\mu{}^\lambda + \tfrac{1}{2} g^{\lambda\nu} \delta g_{\nu\mu}, \tag{25}$$

where $\Gamma^{\mathrm{LC}}(g)$ is the Levi-Civita connection of $g \in \mathrm{Met}(\Sigma)$, and δ is the exterior derivative on $\mathrm{Met}(\Sigma)$ [4,8].

This $GL(4,\mathbb{R})$-connection is used in the intermediate model. Before going any further, let us notice that in the Weil model, this connection reads

$$\tilde{\Gamma}_\mu{}^\lambda = \Gamma_\mu{}^\lambda - (i_{\mathcal{P}}(\Omega)\Gamma)_\mu{}^\lambda, \tag{26}$$

which is comparable with (2.5) in [14]. Now, the intermediate curvature

$$R^{\mathrm{eq}}_{\mathrm{int}}(\Gamma, \omega, \Omega) = D_{\mathrm{int}}\Gamma - \tfrac{1}{2}[\Gamma, \Gamma] \tag{27}$$

gives the corresponding Weil curvature

$$\begin{aligned} R^{\mathrm{eq}}_{\mathrm{W}}(\Gamma, \omega, \Omega) &= \exp\{i_{\mathcal{P}}(\omega)\} R^{\mathrm{eq}}_{\mathrm{int}}(\Gamma, \omega, \Omega) \\ &= (d_{\mathcal{W}} + d_{\mathcal{P}})\tilde{\Gamma} + \tfrac{1}{2}[\tilde{\Gamma}, \tilde{\Gamma}], \end{aligned} \tag{28}$$

which is of the form (2.6) of Wu [14].

Now, let us construct some observables.

4. Some observables for topological 4D-gravity

In order to generate observables of the theory, we first eliminate the GL(4, $\mathbb{R}$)-fibration. As explained in Section 2 this is achieved by considering symmetric GL(4, $\mathbb{R}$)-invariant polynomials. The Euler class and the Pontrjagin classes generated by $R^{\mathrm{eq}}_{\mathrm{W}}$ are such polynomials [10]. Actually, only the first Pontrjagin class is relevant. [4] Up to normalization factors, those two cohomology classes are given by

$$E^{\mathrm{eq}}_{\mathrm{W}} = \frac{\epsilon^{\mu\nu\rho\sigma}}{\sqrt{g}} g_{\nu\lambda} g_{\sigma\chi} (R^{\mathrm{eq}}_{\mathrm{W}})^\lambda_\mu \wedge (R^{\mathrm{eq}}_{\mathrm{W}})^\chi_\rho, \tag{29}$$

$$P^{\mathrm{eq}}_{\mathrm{W}} = (\delta^\mu_\lambda \delta^\rho_\chi - \delta^\mu_\chi \delta^\rho_\lambda)(R^{\mathrm{eq}}_{\mathrm{W}})^\lambda_\mu \wedge (R^{\mathrm{eq}}_{\mathrm{W}})^\chi_\rho, \tag{30}$$

and decompose into five terms

$$E^{\mathrm{eq}}_{\mathrm{W}} = Q^4_0 + Q^3_1 + Q^2_2 + Q^1_3 + Q^0_4, \tag{31}$$

$$P^{\mathrm{eq}}_{\mathrm{W}} = G^4_0 + G^3_1 + G^2_2 + G^1_3 + G^0_4, \tag{32}$$

[4] The zeroth class is trivially 1 while the second (and the highest) class is the square of the Euler class.

where the upper index refers to the form degree on Met(Σ) while the lower one refers to the form degree on Σ. These expressions are to be compared with (2.9) of [14].[5] Observables extracted from monomials $(E_{\mathrm{W}}^{\mathrm{eq}})^m (P_{\mathrm{W}}^{\mathrm{eq}})^n$.

$$(E_{\mathrm{W}}^{\mathrm{eq}})^m (P_{\mathrm{W}}^{\mathrm{eq}})^n = V_0^{4(m+n)} + V_1^{4(m+n)-1} - V_2^{4(m+n)-2} + V_3^{4(m+n)-3} + V_4^{4(m+n)-4}, \tag{33}$$

with

$$V_0^{4(m+n)} = (Q_0^4)^m (G_0^4)^n, \tag{34}$$

$$V_1^{4(m+n)-1} = n(Q_0^4)^m (G_0^4)^{n-1} G_1^3 + m(Q_0^4)^{m-1} Q_1^3 (G_0^4)^n, \tag{35}$$

$$\begin{aligned} V_2^{4(m+n)-2} = {} & n(Q_0^4)^m (G_0^4)^{n-1} G_2^2 + \frac{n(n-1)}{2} (Q_0^4)^m (G_0^4)^{n-2} (G_1^3)^2 \\ & + mn(Q_0^4)^{m-1} Q_1^3 (G_0^4)^{n-1} G_1^3 + m(Q_0^4)^{m-1} Q_2^2 (G_0^4)^n \\ & + \frac{m(m-1)}{2} (Q_0^4)^{m-2} Q_2^2 Q_1^3 (G_0^4)^n, \end{aligned} \tag{36}$$

$$\begin{aligned} V_3^{4(m+n)-3} = {} & n(Q_0^4)^m (G_0^4)^{n-1} G_3^1 + \frac{n(n-1)}{2} (Q_0^4)^m (G_0^4)^{n-2} G_2^2 G_1^3 \\ & + \frac{n(n-1)(n-2)}{6} (Q_0^4)^m (G_0^4)^{n-3} (G_1^3)^3 \\ & + mn(Q_0^4)^{m-1} Q_1^3 (Q_0^4)^{n-1} G_2^2 \\ & + m\frac{n(n-1)}{2} (Q_0^4)^{m-1} Q_1^3 (G_0^4)^{n-2} (G_1^3)^2 \\ & + mn(Q_0^4)^{m-1} Q_2^2 (G_0^4)^{n-1} G_1^3 \\ & + n\frac{m(m-1)}{2} (Q_0^4)^{m-2} (Q_1^3)^2 (G_0^4)^{n-1} G_1^3 \\ & + m(Q_0^4)^{m-1} Q_3^1 (G_0^4)^n + \frac{m(m-1)}{2} (Q_0^4)^{m-2} Q_2^2 Q_1^3 (G_0^4)^n \\ & + \frac{m(m-1)(m-2)}{6} (Q_0^4)^{m-3} (Q_1^3)^3 (G_0^4)^n, \end{aligned} \tag{37}$$

$$\begin{aligned} V_4^{4(m+n)-4} = {} & n(Q_0^4)^m (G_0^4)^{n-1} G_0^4 \\ & + \frac{n(n-1)}{2} (Q_0^4)^m (G_0^4)^{n-2} ((G_2^2)^2 + G_1^3 G_3^1) \\ & + \frac{n(n-1)(n-2)}{6} (Q_0^4)^m (G_0^4)^{n-3} (G_1^3)^2 G_2^2 \\ & + \frac{n(n-1)(n-2)(n-3)}{24} (Q_0^4)^m (G_0^4)^{n-4} (G_1^3)^4 \\ & + mn(Q_0^4)^{m-1} Q_1^3 (G_0^4)^{n-1} G_3^1 \end{aligned}$$

[5] In earlier references [11] devoted to algebraic studies of topological gravity, one can find similar formulae whose geometrical meaning is given here.

$$
\begin{aligned}
&+m\frac{n(n-1)}{2}(Q_0^4)^{m-1}Q_1^3(G_0^4)^{n-2}G_2^2G_1^3\\
&+m\frac{n(n-1)(n-2)}{6}(Q_0^4)^{m-1}Q_1^3(G_0^4)^{n-3}(G_1^3)^3\\
&+mn(Q_0^4)^{m-1}Q_2^2(G_0^4)^{n-1}G_2^2\\
&+m\frac{n(n-1)}{2}(Q_0^4)^{m-1}Q_2^2(G_0^4)^{n-2}(G_1^3)^2\\
&+n\frac{m(m-1)}{2}(Q_0^4)^{m-2}(Q_1^3)^2(G_0^4)^{n-1}G_2^2\\
&+\frac{mn(m-1)(n-1)}{4}(Q_0^4)^{m-2}(Q_1^3)^2(G_0^4)^{n-2}(G_1^3)^2\\
&+mn(Q_0^4)^{m-1}Q_3^1(G_0^4)^{n-1}G_1^3\\
&+n\frac{m(m-1)}{2}(Q_0^4)^{m-2}Q_2^2Q_1^3(G_0^4)^{n-1}G_1^3\\
&+n\frac{m(m-1)(m-2)}{6}(Q_0^4)^{m-3}Q_1^3(G_0^4)^{n-1}G_1^3\\
&+m(Q_0^4)^{m-1}Q_0^4(G_0^4)^{n}\\
&+\frac{m(m-1)}{2}(Q_0^4)^{m-2}((Q_2^2)^2+Q_1^3Q_3^1)(G_0^4)^{n}\\
&+\frac{m(m-1)(m-2)}{6}(Q_0^4)^{m-3}(Q_1^3)^2Q_2^2(G_0^4)^{n}\\
&+\frac{m(m-1)(m-2)(m-3)}{24}(Q_0^4)^{m-4}(Q_1^3)^4(G_0^4)^{n}.
\end{aligned}
\tag{38}
$$

Next, we replace ω and Ω by a Diff(Σ)-connection θ and its curvature Θ on Met(Σ). The corresponding forms fulfill the "descent" equations

$$\delta V_p^{4n-p} + d_\Sigma V_{p-1}^{4n-p+1} = 0, \tag{39}$$

$$\mathcal{I}(\lambda) V_p^{4n-p} + i_\Sigma(\lambda) V_{p+1}^{4n-p-1} = 0, \tag{40}$$

$$\mathcal{L}(\lambda) V_p^{4n-p} + l_\Sigma(\lambda) V_p^{4n-p} = 0, \tag{41}$$

where $\mathcal{I}$ and $\mathcal{L}$ are the inner product and Lie derivative on Met(Σ). Finally, we integrate over cycles on Σ to obtain forms on Met(Σ) only

$$V^{4n-p} = \oint_{\gamma_p} V_p^{4n-p}. \tag{42}$$

Exactly as in the 2D-gravity, only

$$V^{4n-4} = \oint_{\Sigma} V_4^{4n-4} \tag{43}$$

defines an equivariant form on Met(Σ). This gives observables of $\mathrm{Gr}_4^{\mathrm{top}}$, which are the analogues of the Mumford invariants appearing in $\mathrm{Gr}_2^{\mathrm{top}}$.

An explicit expression of the Q's and the G's is given in Appendix A.

5. Conclusion

All the work done above can be applied to higher-dimensional gravity theory. Of course this also applies to Yang–Mills topological theory. Nevertheless, in this last case things are much simpler since the gauge group does not act on the space–time manifold Σ, while in gravity theory the diffeomorphism group does.

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I would like to thank Raymond Stora for drawing to my attention the work of S. Wu and for many helpful discussions.

Appendix A

It was already shown in [12] that the Weil curvature takes the form

$$(R_{\mathrm{W}}^{\mathrm{eq}})_\mu^\nu = (R^{\mathrm{LC}} - i_\Sigma(\omega)R^{\mathrm{LC}} + \tfrac{1}{2}i_\Sigma(\omega)i_\Sigma(\omega)R^{\mathrm{LC}} + \tfrac{1}{2}D^{\mathrm{LC}} \wedge \tilde{\bar{\gamma}} - \tfrac{1}{2}i_\Sigma(\omega)D^{\mathrm{LC}} \wedge \tilde{\bar{\gamma}} - \tfrac{1}{4}\tilde{\psi}\tilde{\psi} + \tfrac{1}{2}D^{\mathrm{LC}} \wedge \tilde{\bar{\Omega}})_\mu^\nu, \tag{A.1}$$

where

$$\tilde{\bar{\gamma}}_\mu = (\delta g_{\rho\mu} - l\Sigma(\omega)g_{\rho\mu})\,\mathrm{d}x^\rho = \tilde{\gamma}_{\rho\mu}\,\mathrm{d}x^\rho, \tag{A.2}$$

$$\tilde{\psi}_\mu^\nu = g^{\rho\nu}(\delta g_{\rho\mu} - l_\Sigma(\omega)g_{\rho\mu}) = g^{\rho\nu}(\tilde{\gamma}_{\rho\mu}) = (g^{-1}\tilde{\gamma})_\mu^\nu, \tag{A.3}$$

$$(D^{\mathrm{LC}} \wedge \tilde{\bar{\gamma}})_\mu^\nu = g^{\rho\nu}(D\rho^{\mathrm{LC}}\tilde{\bar{\gamma}}_\mu - D\mu^{\mathrm{LC}}\tilde{\bar{\gamma}}^\rho). \tag{A.4}$$

Then, after a "straightforward" algebraic juggle, one finally obtains

$$Q_4^0 = \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{\mathbf{g}}} g_{\nu\lambda}g_{\sigma\chi}(R^{\mathrm{LC}})_\mu^\lambda \wedge (R^{\mathrm{LC}})_\rho^\chi = E_\Sigma, \tag{A.5}$$

$$Q_3^1 = 2\frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{\mathbf{g}}} g_{\nu\lambda}g_{\sigma\chi}(R^{\mathrm{LC}})_\mu^\lambda \wedge \left(-i_\Sigma(\omega)R^{\mathrm{LC}} + \frac{1}{2}D^{\mathrm{LC}} \wedge \tilde{\bar{\gamma}}\right)_\rho^\chi, \tag{A.6}$$

$$Q_2^2 = \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{\mathbf{g}}} g_{\nu\lambda}g_{\sigma\chi}[(i_\Sigma(\omega)(R^{\mathrm{LC}})_\mu^\lambda \wedge (i_\Sigma(\omega)R^{\mathrm{LC}})_\rho^\chi - 2(i_\Sigma(\omega)R^{\mathrm{LC}})_\mu^\lambda \wedge (D^{\mathrm{LC}} \wedge \tilde{\bar{\gamma}})_\rho^\chi + (D^{\mathrm{LC}} \wedge \tilde{\bar{\gamma}})_\mu^\lambda \wedge (D^{\mathrm{LC}} \wedge \tilde{\bar{\gamma}})_\rho^\chi$$

$$+ (R^{\rm LC})^{\lambda}_{\mu} \wedge (i_{\Sigma}(\omega) i_{\Sigma}(\omega) R^{\rm LC} - i_{\Sigma}(\omega)(D^{\rm LC} \wedge \bar{\tilde{\gamma}}) - \tfrac{1}{2}\tilde{\psi}\tilde{\psi} - D^{\rm LC} \wedge \bar{\tilde{\Omega}})^{\chi}_{\rho}], \tag{A.7}$$

$$Q^3_1 = \frac{\varepsilon^{\mu\nu\rho\sigma}}{\sqrt{\mathbf{g}}} g_{\nu\lambda} g_{\sigma\chi} (i_{\Sigma}(\omega) i_{\Sigma}(\omega) R^{\rm LC} - i_{\Sigma}(\omega)(D^{\rm LC} \wedge \bar{\tilde{\gamma}}) - \tfrac{1}{2}\tilde{\psi}\tilde{\psi} - D^{\rm LC} \wedge \bar{\tilde{\Omega}})^{\lambda}_{\mu} \wedge (-i_{\Sigma}(\omega) R^{\rm LC} + \tfrac{1}{2} D^{\rm LC} \wedge \bar{\tilde{\gamma}})^{\chi}_{\rho}, \tag{A.8}$$

$$Q^4_0 = \frac{\varepsilon^{\mu\nu\rho\sigma}}{4\sqrt{\mathbf{g}}} g_{\nu\lambda} g_{\sigma\chi} (i_{\Sigma}(\omega) i_{\Sigma}(\omega) R^{\rm LC} - i_{\Sigma}(\omega)(D^{\rm LC} \wedge \bar{\tilde{\gamma}}) - \tfrac{1}{2}\tilde{\psi}\tilde{\psi} - D^{\rm LC} \wedge \bar{\tilde{\Omega}})^{\lambda}_{\mu} \wedge (i_{\Sigma}(\omega)(i_{\Sigma}(\omega) R^{\rm LC} - i_{\Sigma}(\omega)(D^{\rm LC} \wedge \bar{\tilde{\gamma}}) - \tfrac{1}{2}\tilde{\psi}\tilde{\psi} - D^{\rm LC} \wedge \bar{\tilde{\Omega}})^{\chi}_{\rho}. \tag{A.9}$$

Finally, the G's are obtained by replacing $(\varepsilon^{\mu\nu\rho\sigma}/\sqrt{\mathbf{g}}) g_{\nu\lambda} g_{\sigma\chi}$ in the Q's by $(\delta^{\mu}_{\lambda}\delta^{\rho}_{\chi} - \delta^{\mu}_{\chi}\delta^{\rho}_{\lambda})$.

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A class of topological actions

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Abstract: We review definitions of generalized parallel transports in terms of Cheeger-Simons differential characters. Integration formulae are given in terms of Deligne-Beilinson cohomology classes. These representations of parallel transport can be extended to situations involving distributions as is appropriate in the context of quantized fields.

Keywords: Topological Field Theories.

Contents

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1. Introduction

Parallel transports and generalizations thereof have been repeatedly met both in mathematics [1 − 7] and in global aspects of gauge theories [8 − 10], which played a major role in elementary particle physics.

It has taken some time for the existing mathematics [11 − 15, 7] to become known to physicists [9, 10, 16 − 19].

At the semi-classical level one is lead to integrate objects more general than differential forms over cycles with a result defined modulo integers; Cheeger-Simons differential characters [2] are privileged candidates. Their integral representations in terms of Deligne-Beilinson smooth cohomology classes are particularly well adapted to field theory for two reasons : first of all, they involve locally defined fields subject to some gluing properties. Besides, they allow for natural generalizations well adapted to, at least, semi-classical quantization. Indeed the latter already requires regularizing (thickening) the integration cycles, an operation which can be performed easily within the Deligne-Beilinson cohomology framework. This operation is less naive than one might think; indeed the corresponding currents are not any longer differential forms (de Rahm currents) but Deligne-Beilinson classes. In view of this phenomenon we shall proceed in detail from the semi-classical situation for which the use of Cheeger-Simons characters is well adapted. In this case there exist canonical integral representations in terms of differential forms with discontinuous coefficients and therefore inappropriate for applications to quantum fields, even in the semi-classical approximation. Fortunately, these integral representations can be replaced by others with smooth coefficients. The latter are easily generalizable to situations involving distributions and therefore well adapted to quantum fields. There is however a price to pay: the differential forms involved in the classical formulae have to be replaced (non canonically) by Deligne-Beilinson smooth classes.

We start in section 2 with the prototype example of Maxwell's electromagnetism in which a functional integral is defined under "reasonable" hypotheses concerning the interaction with an external current. The rest of the paper is devoted to a sequence of constructions which give a mathematical foundation of the above hypotheses.

Section 3 proposes three equivalent ways to describe Cheeger-Simons differential characters in terms of the integration of Deligne-Beilinson cohomology classes.

Section 4 presents the natural generalizations required upon quantization: the integration of Deligne-Beilinson classes with distributional coefficients.

Section 5 contains our concluding remarks.

A number of technical details are collected in three appendices.

2. Maxwell Semi-Classical theory (à la Feynman)

While in a classical theory the action (when it exists) is optional (in principle, the equations of motion are sufficient), it becomes the keystone of the Feynman semi-classical point of view. Hence, such an action must be carefully defined. In the context of Maxwell's electromagnetism, we consider the euclidean action defined on a 4-dimensional, riemmannian,[1] compact manifold M_4

$$S_{EM} = \frac{1}{2}\int_{M_4} F\wedge *F \,+\, i\cdot \text{“}\!\int_{M_4} j\wedge A\text{”}\,. \tag{2.1}$$

Quotes emphasize that we have to make precise the meaning of the second integral since A is **not** a 1-form on M_4, but rather a connection on a U(1)-bundle over M_4, with curvature F. We defer until the next section a mathematically sound definition [2] of " $\int_{M_4} j\wedge A$" for j a 3-form with integral periods.

At this point, we only need to know that " $\int_{M_4} j\wedge A$" will be defined modulo $2\pi\mathbb{Z}$ and will fulfill the following natural property : if $A = A_0 + \alpha$ (with A_0 a fixed U(1)-connection and α a generic 1-form), then

$$\text{“}\!\int_{M_4} j\wedge A\text{”} = \text{“}\!\int_{M_4} j\wedge A_0\text{”} + \int_{M_4} j\wedge\alpha\,. \tag{2.2}$$

Gauge invariance requires $\int_{M_4} j\wedge(g^{-1}dg)\in 2\pi\mathbb{Z}$ which is less restrictive than the "classical" requirement $\int_{M_4} j\wedge(g^{-1}dg) = 0$, commonly assumed [20, 21] to hold at the quantum level.

Once the choice of definition of the action integral with the above property has been made, we can try to evaluate the state [3] ($\hbar = 1$)

$$\left\langle e^{-\,i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\right\rangle = \int \mathcal{D}A\, e^{-\frac{1}{2}\int_{M_4} F\wedge *F \,-\, i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\,, \tag{2.3}$$

where A is a U(1)-connection. First let

$$A = A_0 + \alpha\,, \tag{2.4}$$

with A_0 a background connection and α a globally defined 1-form. Then, denoting by $F_0 = dA_0$ the background curvature, we obtain

$$\begin{aligned}\left\langle e^{-\,i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\right\rangle &= e^{-\frac{1}{2}\int_{M_4} F_0\wedge *F_0 \,-\, i\cdot\text{“}\int_{M_4} j\wedge A_0\text{”}}\times\\ &\times\int \mathcal{D}\alpha\, e^{-\frac{1}{2}\int_{M_4} d\alpha\wedge *d\alpha \,-\, \int_{M_4} F_0\wedge *d\alpha \,-\, i\cdot\int_{M_4} j\wedge\alpha}\,.\end{aligned} \tag{2.5}$$

The 1-form α is linearly coupled to $(j + i\,d*F_0)$ and we need to gauge fix the α integration. Note that $\int j\wedge\alpha$ is an ordinary integral. Gauge transformations connected with the identity are eliminated by choosing a Green function (ξ, the gauge parameter)

$$G_\xi \;=\; [\delta d + \xi d\delta]^{-1}\;, \qquad \xi > 0\,, \tag{2.6}$$

in the subspace orthogonal to harmonic forms (the elimination of large gauge transformations will come later). So, we are led to

$$\begin{aligned}\left\langle e^{-\,i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\right\rangle &= e^{-\frac{1}{2}\int_{M_4} F_0\wedge *F_0 \,-\, i\cdot\text{“}\int_{M_4} j\wedge A_0\text{”}}\times\\ &\times e^{-\frac{1}{2}\int_{M_4}(j+i\,d*F_0)_\perp G_\xi\, *(j+i\,d*F_0)_\perp}\times Z\left(j_\parallel\right).\end{aligned} \tag{2.7}$$

The subscript $\perp$ (resp. $\parallel$) refers to the decomposition of forms into components orthogonal to (resp. along) harmonic forms. We shall come to the definition of $Z(j_\parallel)$ later.

[1] $*$ is the usual Hodge operator.

[2] It will turn out that more data than just the 3-form j will be needed.

[3] A linear functional on observables.

The A_0 dependence can be reduced to:

$$\left\langle e^{-\,i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\right\rangle = e^{-\frac{1}{2}\int_{M_4} F_{0\|}\wedge * F_0 \;-\; i\cdot\text{“}\int_{M_4} j_{\|}\wedge A_0\text{”}} \times$$
$$\times\, e^{-\frac{1}{2}\int_{M_4} j_\perp G_\xi * j_\perp}\cdot Z\left(j_{\|}\right). \tag{2.8}$$

The first term yields an overall normalization factor to be divided out. The third term is ξ independent by $dj = 0$. The forms $\alpha_{\|}$ and $j_{\|}$ being harmonic are necessarily closed (also co-closed). Using Poincaré duality and assuming no torsion, we can decompose them along a dual basis of integral 3-cycles and 1-cycles respectively

$$\alpha_{\|} = \sum_k \alpha_k\, {\zeta_k}^{(3)} + d(\cdots)\,, \qquad j_{\|} = \sum_k n_k {\zeta_k}^{(1)} + d(\cdots)$$

with

$$\left\langle {\zeta_k}^{(3)} {\zeta_l}^{(1)} \right\rangle = \delta_{kl}\,, \tag{2.9}$$

where the α_k's are real numbers since α is real, while the n_k's are integers since $j_{\|}$ has integral periods. With this decomposition of $\alpha_{\|}$ and $j_{\|}$, we can formally write

$$Z(j_{\|}) = \int \mathcal{D}\alpha_{\|}\, e^{\,i\cdot\int_{M_4} \alpha_{\|}\wedge j_{\|}} = \int\; d\vec{\alpha}\, e^{\,i\vec{n}\cdot\vec{\alpha}}\,, \tag{2.10}$$

where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_m)$ and $\vec{n} = (n_1, \ldots, n_m)$.

Now, large gauge transformations are :

$$\alpha_k \mapsto \alpha_k + p_k\,, \qquad p_k \in 2\pi\mathbb{Z} \tag{2.11}$$

and can be factored out by transforming α_k integration into ϑ_k integration $0 \le \vartheta_k < 2\pi$:

$$Z(j_{\|}) = \int d\vec{\vartheta}\, e^{\,i\vec{n}\cdot\vec{\vartheta}}\,. \tag{2.12}$$

These angles ϑ_k parametrize $H^1(M_4, \mathbb{R})/H^1(M_4, \mathbb{Z})$, still assuming no torsion (torsion yields an extra factor).

Similarly

$$e^{\,i\cdot\text{“}\int_{M_4} j_{\|}\wedge A_0\text{”}} = e^{\,i\vec{n}\cdot\vec{\vartheta}_0}\,, \tag{2.13}$$

where ϑ_{0k} are fixed angles which may be incorporated into ϑ_k.

To conclude, after normalization, the state $\langle\;\rangle$ can be decomposed into gauge invariant states labelled by the angles $\vec{\vartheta}$

$$\left\langle e^{\,i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\right\rangle = \int d\vec{\vartheta} \left\langle e^{\,i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\right\rangle_{\vec{\vartheta}}\,, \tag{2.14}$$

with

$$\left\langle e^{\,i\cdot\text{“}\int_{M_4} j\wedge A\text{”}}\right\rangle_{\vec{\vartheta}} = e^{\,i\vec{n}\cdot\vec{\vartheta}}\cdot e^{-\frac{1}{2}\int_{M_4} j_\perp G_\xi * j_\perp}\,, \tag{2.15}$$

a familiar situation which provides an alternative to the commonly accepted choice [20, 21] which amounts to integrate over $\vec{\vartheta}$'s with the result $\propto \delta(j_{\|})$; in the latter case $j = dm$ are the only possible integration currents for A, while for the states defined in (2.15) the currents j are only required to be closed forms with integral periods. In other words, homological triviality of Wilson loops or appropriately smeared version thereof are not consequences of gauge invariance, but rather, of some form of locality.

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3. Integral representations of differential characters

In section 2 we have described the physical consequences of " $\int_M j \wedge A$ " being defined modulo $2\pi\mathbb{Z}$ (with j a form with integral periods). We shall now proceed to give some substance to this assumption and write down explicit formulae.

To start with, let us recall that one can associate to any closed curve[4] Γ in a manifold M a closed current δ_Γ (i.e. a closed form whose local representatives have distributional coefficients) such that integration of a form ω along Γ formally reduces to the integration of $\delta_\Gamma \wedge \omega$ over the whole of M [22].

We shall first try to find a satisfactory definition of the *circulation integral* of A along a closed curve Γ by considering various situations. This study will naturally lead us to the mathematical notion of *differential character* introduced by Cheeger and Simons [2].

Then, while seeking for a representation of a differential character supported by Čech-de Rham cohomology theory, there will emerge a defining formula for " $\int_M j \wedge A$ " in terms of Deligne-Beilinson cohomology [13]. We will see that for $j = \delta_\Gamma$, there is a canonical definition of this integral, whereas for general j there is a whole class of adequate definitions.

From now on, M will be a **torsion-free** smooth n-dimensional oriented compact manifold without boundary.

3.1 Circulation of U(1) gauge fields as differential characters

Within Maxwell's theory of electromagnetism on M_4, due to the triviality of the homology and cohomology groups of $\mathbb{R}^4$ (i.e. any closed curve is a boundary, and any closed 3-form is exact), the circulation of a U(1)-gauge field A along a closed curve Γ is a perfectly well-defined and gauge invariant integral which measures the magnetic flux through any surface Σ with boundary $\Gamma = \partial\Sigma$, namely

$$\text{“}\oint_\Gamma A\text{”} \equiv \oint_{\Gamma=\partial\Sigma} A = \int_\Sigma F \,. \tag{3.1}$$

Of course, such a property fails for a general manifold M with non-trivial (co-)homology groups. Nevertheless, it may be asked whether (3.1) can be maintained for boundaries $\Gamma = \partial\Sigma$, assuming that "$\oint_\Gamma A$" has a mathematical meaning for any closed curve Γ in M. Let us then consider a closed curve Γ that splits a **closed** surface Σ into two components Σ_+ and Σ_- : $\Sigma = \Sigma_+ \cup \Sigma_-$ and $\Gamma = \partial\Sigma_+ = -\partial\Sigma_-$, where the minus sign takes care of orientations. Then, we would have

$$\text{“}\oint_\Gamma A\text{”} = \int_{\Sigma_+} F \tag{3.2}$$

since $\Gamma = \partial\Sigma_+$, and

$$\text{“}\oint_\Gamma A\text{”} = -\int_{\Sigma_-} F \tag{3.3}$$

since $\Gamma = \partial\Sigma_-$. Since F is a U(1) curvature, we know that

$$\oint_{\Sigma_-} F + \oint_{\Sigma_+} F = \oint_\Sigma F \in \mathbb{Z}(1) := 2i\pi\mathbb{Z} \tag{3.4}$$

[4] By *curve* we mean a 1-dimensional embedded smooth submanifold of M.

on any closed surface Σ. This suggests that, if it exists, "$\oint_\Gamma A$" is only defined modulo $\mathbb{Z}(1) := 2i\pi\mathbb{Z}$. Otherwise stated, we can expect, for fixed A, "$\frac{1}{2i\pi}\oint_\Gamma A$" to be some $\mathbb{R}/\mathbb{Z}$-valued linear functional on the space of closed curves (cycles). Let us have a closer look at such an assumption.

To begin with, a U(1)-gauge transformation, g, changes the connection A into the connection $A^g = A + g^{-1}dg$ with the same curvature F; therefore, **if (3.1) holds**

$$\text{“}\oint_{\partial\Sigma} A^g\text{”} = \text{“}\oint_{\partial\Sigma} A + g^{-1}dg\text{”} = \int_\Sigma F = \text{“}\oint_{\partial\Sigma} A\text{”}, \tag{3.5}$$

i.e. "$\oint_{\partial\Sigma} A$" is gauge invariant.

In fact, for any closed 1-form α on M, $A + \alpha$ is also a connection with curvature $F = dA$, so that we obtain a relation similar to (3.5) with α in place of $g^{-1}dg$. Consequently, we can infer that connections with the same curvature may define (*a priori* different) $\mathbb{R}/\mathbb{Z}$-valued linear functionals on cycles which *coincide on boundaries.* In this sense, the "integral" of A on boundaries is completely defined by F.

For a general closed curve Γ and any gauge transformation g, we would like to maintain gauge invariance of " $\oint_\Gamma A$ ", which is not immediate since the term $\oint_\Gamma g^{-1}dg$ may not vanish (Γ not being necessarily a boundary). However, since $g^{-1}dg$ is the pullback by g of the standard U(1) ($\simeq S^1$) volume 1-form, $z^{-1}dz$, we have

$$\oint_\Gamma g^{-1}dg \in \mathbb{Z}(1)\,.$$

Accordingly, still assuming that " $\oint_\Gamma A$ " is defined modulo $\mathbb{Z}(1)$, we obtain the sought after gauge invariance

$$\text{“}\oint_\Gamma A^g\text{”} = \text{“}\oint_\Gamma A\text{”}\,, \tag{3.6}$$

though Γ is **not** a boundary.

All these requirements can be satisfied if we ask for (3.1) and define "$\frac{1}{2i\pi}\oint_\Gamma A$" to be an $\mathbb{R}/\mathbb{Z}$-valued functional, linear in Γ and affine in A,[5], a property which is satisfied if we set

$$\text{“}\oint_\Gamma (A+\gamma)\text{”} = \text{“}\oint_\Gamma A\text{”} + \oint_\Gamma \gamma\,, \tag{3.7}$$

where the last integral is the ordinary integral of the 1-form γ -in the same line of thought recall (2.2)-. Then, for any closed 1-form α we have

$$\text{“}\oint_\Gamma (A+\alpha)\text{”} = \text{“}\oint_\Gamma A\text{”} \tag{3.8}$$

if and only if all periods of α take values in $\mathbb{Z}(1)$. In fact the 1-forms $g^{-1}dg$, with g running through the U(1)-gauge group, generate the space of closed 1-forms with $\mathbb{Z}(1)$-valued periods. That is, if $per(\alpha) \in \mathbb{Z}(1)$, we can write

$$\alpha = g^{-1}dg + d\lambda$$

[5]This is a natural demand since the space of connections is an affine space.

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for some U(1)-gauge transformation g and some function λ on M. Then, as far as "integration" of A on closed curves is concerned, gauge invariance is equivalent to invariance under $A \mapsto A + \alpha$, with α a form with $\mathbb{Z}(1)$-valued periods. Therefore, it is expected that two connections that differ by a form with $\mathbb{Z}(1)$-valued periods define the same $\mathbb{R}/\mathbb{Z}$-valued linear functional on the space of closed curves.

At this point, let us make some remarks. First, if the connection A is a 1-form on M (for instance when the corresponding U(1)-bundle is flat), we must require that the general definition of "$\oint_\Gamma A$" reduces to the usual definition of the integral of a form. Second, up to now, we have only considered U(1)-connections on M. In a more general situation we will consider objects $A^{(p)}$, representing antisymmetric tensor "gauge potentials" which appear in supergravities and string theories [23]. However, the geometric situation turns out to be more involved than in the case of connections. Indeed, a U(1)-connection, although it is not a 1-form on M, is lifted as a 1-form on some principal U(1)-bundle over M. Such $A^{(p)}$'s will in general not be p-forms on M. It turns out that they can be considered as connections on new mathematical objects called gerbes [13, 24]. Here we will not go into such an interpretation: we will consider locally defined differential forms "$A^{(p)}$" on M whose differentials, $F^{(p+1)}$, are globally defined $(p+1)$-forms with $\mathbb{Z}$-valued periods on M.[6] We will define an $\mathbb{R}/\mathbb{Z}$-valued linear functional,"$\oint_{S_p} A^{(p)}$" on the space of closed p-submanifolds, S_p, of M. Such linear functionals turn out to be *differential characters* in the sense of J. Cheeger and J. Simons. Differential characters have been constructed within the framework of Chern-Simons' theory of secondary characteristic classes, an extension of the Chern-Weil theory. They were introduced to describe, on the base space, secondary characteristic classes of principal bundles initially defined as differential forms on the whole bundle space (see [3] for a review, and [2] for the original reference).

Our integrals, "$\oint_{S_p} A^{(p)}$", are related to *Deligne-Beilinson cohomology classes* as presented in [13] and therefore (cf. section A.7) offer a parametrization of differential characters.

In appendix A the reader will find notations, basic definitions and results concerning smooth Deligne-Beilinson cohomology groups $H^q(\mathcal{C}_p, D)$.

Our basic example deals with a U(1)-connection on the n- dimensional manifold M. In this case there is a one to one correspondence between the second smooth Deligne cohomology group of M, $H^2(\mathcal{C}_2, D)$,[7] and the set of equivalence classes of U(1) principal bundles with connection, $(P[\mathrm{U}(1)], A)$ (cf. appendix C). We will show how to integrate an element of $H^2(\mathcal{C}_2, D)$ over a 1-cycle, z_1, and take this "integral" as a definition for "$\frac{1}{2i\pi} \oint_\Gamma A$". This generalizes to integrating elements of $H^{p+1}(\mathcal{C}_{p+1}, D)$ over p-cycles, z_p which provides a definition for "$\oint_{z_p} A^{(p)}$". As we shall see (section 3.2) the classical Weil construction, pertaining to singular homology, both suggests a natural definition of elements of $H^{p+1}(\mathcal{C}_{p+1}, D)$ and of their integration over a p-cycle. In [18] R. Zucchini gives integral representations of "relative" differential characters, essentially identical with ours, independently of the ex-

[6] In this framework, $A = (2i\pi)A^{(1)}$, and its curvature $F = (2i\pi)F^{(2)}$.

[7] cf. appendix B.

pression of the integrand in terms of Deligne-Beilinson classes. Later (section 3.3) we will give another definition of the integral which avoids Weil's analysis of the cycle and allows for generalization.

3.2 Integration over a cycle: the appearance of Deligne-Beilinson classes

There is a natural procedure to define integration over integral cycles, based on the classic work of André Weil [25]. In this paper, for any simple[8] covering $\mathcal{U}$ of M, "$\mathcal{U}$-p-chains" are defined as singular p-chains, C_p, such that

$$C_p = \partial C_{(0,p)} := \sum_\alpha C_{(0,p),\,\alpha}\,, \tag{3.9}$$

where every $C_{(0,p),\,\alpha}$ is a singular p-chain with carrier $\mathcal{U}_\alpha$ (here, ∂ is the boundary operator on Čech chains). A $\mathcal{U}$-p-cycle z_p is a closed $\mathcal{U}$-p-chain ($bz_p = 0$, with b the boundary operator on singular chains). Then, it is shown that for any $\mathcal{U}$-p-cycle z_p of M there exists a sequence of Čech (smooth) singular $\mathcal{U}$-chains, $z_{(k,p-k)}$

$$z^{\mathcal{W}}_{(p)} := \left(z_{(0,p)}, \ldots, z_{(k,p-k)}, \ldots, z_{(p,0)}\right), \tag{3.10}$$

where each $z_{(k,p-k)}$ has support in some open $(k+1)$-fold intersection of $\mathcal{U}$, such that

$$\begin{aligned} \partial z_{(0,p)} &= z_{(-1,p)} := z_p \\ b z_{(k,p-k)} &= \partial z_{(k+1,p-k-1)}\,, \qquad k \in \{1, \ldots, p-1\} \\ b_0 z_{(p,0)} &:= z_{(p,-1)}\,, \end{aligned} \tag{3.11}$$

where b_0 is just the "degree" operator on singular chains [25], $z_{(p,-1)}$ is an integral Čech p-cycle of $\mathcal{U}$ and $(\partial z_{(k,p-k)})_{\alpha_0\cdots\alpha_{k-1}} = \sum_\beta z_{(k,p-k),\beta\alpha_0\cdots\alpha_{k-1}}$.

The collection $z^{\mathcal{W}}_{(p)}$ is called a Weil descent of z_p, and the corresponding equations (3.11) a Weil descent equation of $z^{\mathcal{W}}_{(p)}$.

Now, if $Z^{\mathcal{W}}_{(p)}$ is another Weil descent of the same $\mathcal{U}$-p-cycle z_p, it differs from $z^{\mathcal{W}}_{(p)}$ according to

$$\begin{aligned} Z_{(0,p)} &= z_{(0,p)} + \partial t_{(1,p)} + b t_{(0,p+1)}, \\ Z_{(k,p-k)} &= z_{(k,p-k)} + b t_{(k,p-k+1)} + \partial t_{(k+1,p-k)}\,, \qquad k = 1, \ldots, p-1 \\ Z_{(p,0)} &= z_{(p,0)} + b t_{(p,1)} + \partial t_{(p+1,0)}\,, \end{aligned} \tag{3.12}$$

where the $t_{(k,p-k+1)}$ are some Čech $\mathcal{U}$-chains. Since z_p is fixed, we must have

$$\partial b\, t_{(0,p+1)} = 0 = b\partial\, t_{(0,p+1)}\,, \tag{3.13}$$

which means that $\partial t_{(0,p+1)}$ is a $\mathcal{U}$-$(p+1)$-cycle, $\tilde{z}_{p+1}$ which in turn gives rise to a Weil descent

$$\tilde{z}^{\mathcal{W}}_{(p+1)} := \left(\tilde{z}_{(0,p+1)} := t_{(0,p+1)}, \tilde{z}_{(1,p)}, \ldots, \tilde{z}_{(k,p-k+1)}, \ldots, \tilde{z}_{(p+1,0)}\right),$$

[8] Definitions and notations are given in appendix A.

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so that

$$\begin{aligned} Z_{(0,p)} &= z_{(0,p)} + \partial(t_{(1,p)} + \tilde{z}_{(1,p)})\,, \\ Z_{(k,p-k)} &= z_{(k,p-k)} + b(t_{(k,p-k+1)} + \tilde{z}_{(k,p-k+1)}) + \partial(t_{(k+1,p-k)} + \tilde{z}_{(k+1,p-k)})\,, \\ Z_{(p,0)} &= z_{(p,0)} + b(t_{(p,1)} + \tilde{z}_{(p,1)}) + \partial(t_{(p+1,0)} + \tilde{z}_{(p+1,0)})\,, \end{aligned} \tag{3.14}$$

with $k = 1, \ldots, p-1$. Accordingly, the general ambiguities on a Weil descent of a given cycle z_p of M take the form

$$\begin{aligned} Z_{(0,p)} &= z_{(0,p)} + \partial h_{(1,p)}\,, \\ Z_{(k,p-k)} &= z_{(k,p-k)} + b h_{(k,p-k+1)} + \partial h_{(k+1,p-k)}\,, \\ Z_{(p,0)} &= z_{(p,0)} + b h_{(p,1)} + \partial h_{(p+1,0)}\,, \end{aligned} \tag{3.15}$$

By identifying Weil descents that differ by ambiguities (3.15), one defines an equivalence relation between Weil descents whose corresponding equivalence classes *canonically* represent $\mathcal{U}$-p-cycles of M. Actually, one could introduce a boundary operator made of the operators b and ∂, turning what we have just done into a homological game in which Weil descent classes are homology classes.

Similarly — cf. appendix A —, a sequence

$$\omega_{\mathcal{D}}^{(p)} := \left(\omega^{(0,p)}, \omega^{(1,p-1)}, \ldots, \omega^{(p,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}\right), \tag{3.16}$$

where $\omega^{(k,p-k)} \in \check{\mathrm{C}}^{k}(\mathcal{U}, \Omega^{(p-k)}(M))$ and $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \in \check{\mathrm{C}}^{(p+1)}(\mathcal{U})$ defines a Deligne-Beilinson cocycle if

$$\left(\tilde{d} + \delta\right)\omega_{\mathcal{D}}^{(p)} = D\omega_{\mathcal{D}}^{(p)} = 0\,,$$

i.e.

$$d_{p-k}\omega^{(k,p-k)} = \delta\omega^{(k-1,p-k+1)}\,, \qquad k = 1, \cdots, p+1\,. \tag{3.17}$$

In the above equation δ is the Čech coboundary operator, $d_{-1}\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$ is the injection of numbers into $\Omega^{(0)}(M)$ and $\tilde{d}$ the differential of the Deligne complex (it coincides with the de Rham differential d, up to degree $p-1$ and is the **zero** map at degree p). By convention, cohomology (resp. homology) indices are upper (resp. lower) indices, those referring to Čech complex coming first.

Note that $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$ is necessarily a cocycle, and, although $\tilde{d}\omega^{(0,p)} \equiv 0$, $d\omega^{(0,p)}$ is the restriction of a globally defined closed form $\omega^{(-1,p+1)}$ with integral periods [25]. This $\omega^{(-1,p+1)}$ will be called the top form of the cocycle $\omega_{\mathcal{D}}^{(p)}$.

We can now proceed and build Deligne-Beilinson cohomology classes as equivalence classes of Deligne-Beilinson cocycles related as follows:

$$\varpi_{\mathcal{D}}^{(p)} = \omega_{\mathcal{D}}^{(p)} + D\,Q_{\mathcal{D}}\,,$$

i.e.

$$\begin{aligned} \varpi^{(0,p)} &= \omega^{(0,p)} + dq^{(0,p-1)}\,, \\ \varpi^{(k,p-k)} &= \omega^{(k,p-k)} + dq^{(k,p-k-1)} + \delta q^{(k-1,p-k)}\,, \qquad k = 1, \ldots, p\,, \\ \overset{\mathbb{Z}}{\varpi}{}^{(p+1,-1)} &= \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} + \delta\overset{\mathbb{Z}}{q}{}^{(p,-1)}\,, \end{aligned} \tag{3.18}$$

where $q^{(k,p-k-1)} \in \check{C}^k(\mathcal{U}, \Omega^{(p-k-1)}(M))$ and $\overset{\mathbb{Z}}{q}{}^{(p,-1)} \in \check{C}^p(\mathcal{U})$. As an immediate consequence, all cocycles belonging to the same Deligne-Beilinson cohomology class have the same top form.

The integral of a Deligne-Beilinson cocycle $\omega_{\mathcal{D}}^{(p)}$ over a p-cycle $z_{(p)}^{\mathcal{W}}$ is naturally defined as the pairing

$$\int_{z_{(p)}^{\mathcal{W}}} \omega_{\mathcal{D}}^{(p)} := \left\langle \omega_{\mathcal{D}}^{(p)}, z_{(p)}^{\mathcal{W}} \right\rangle := \sum_{k=0}^{p} \int_{z_{(k,p-k)}} \omega^{(k,p-k)}$$
$$:= \sum_{k=0}^{p} \frac{1}{(k+1)!} \sum_{\alpha_0,\cdots,\alpha_k} \int_{z_{(k,p-k),\alpha_0\cdots\alpha_k}} \omega_{\alpha_0\cdots\alpha_k}^{(k,p-k)} . \tag{3.19}$$

In (3.19) the ambiguities on the representatives $z_{(p)}^{\mathcal{W}}$ (resp. $\omega_{\mathcal{D}}^{(p)}$) of $[z_{(p)}^{\mathcal{W}}]$ (resp. $[\omega_{\mathcal{D}}^{(p)}]$) generate terms of the form

$$\int_{h_{(p+1,0)}} d_{-1} \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} + \delta \overset{\mathbb{Z}}{q}{}^{(p,-1)} \right) + \int_{z_{(p,0)}} d_{-1} \overset{\mathbb{Z}}{q}{}^{(p,-1)} . \tag{3.20}$$

These terms are necessarily integers since the chains and the cochains appearing there are integers. In other words, (3.19) extends to classes as long as we work modulo "integers". This also means that the duality so realized is over $\mathbb{R}/\mathbb{Z}$, not $\mathbb{R}$, i.e. of Pontrjagin type. Actually, this is not totally surprising since a Deligne-Beilinson cohomology class defines a form up to a form with integral periods (cf. appendix A).

Many of the equalities we will encounter only hold true *mod* $\mathbb{Z}$, accordingly we shall use the notation " $\overset{\mathbb{Z}}{=}$ " to mean " $= \ldots$ mod $\mathbb{Z}$".

With all this information, we finally set

$$\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)} \right] := \int_{\left[z_{(p)}^{\mathcal{W}} \right]} \left[\omega_{\mathcal{D}}^{(p)} \right] \overset{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_{z_{(k,p-k)}} \omega^{(k,p-k)} , \tag{3.21}$$

for any representative of $[\omega_{\mathcal{D}}^{(p)}]$ and $[z_{(p)}^{\mathcal{W}}]$ to which we shall refer to (3.21) as the *"Defining Formula"*.

Let us note that the linearity of (3.21) with respect to z_p is clear since all descents are linear.

3.2.1 Examples

Let us apply (3.21) to two simple cases. First, consider the situation where the cycle z_p is a boundary: $z_p = bc_{p+1}$. Due to the equivalence of singular and Čech homologies, any Čech p-cycle, $z_{(p,-1)}$, arising from the descent of z_p, is a Čech boundary, i.e.

$$z_{(p,-1)} = \partial c_{(p+1,-1)} , \tag{3.22}$$

for some integral Čech chain $c_{(p+1,-1)}$. Then, the corresponding descent has a representative of the form

$$z_{(p)}^{\mathcal{W}} := (z_{(0,p)} = bc_{(0,p+1)}, 0 \ldots, 0, \ldots, 0) , \tag{3.23}$$

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with $\partial c_{(0,p+1)} = c_{p+1}$. Accordingly, the integral of $[\omega_{\mathcal{D}}^{(p)}]$ over this trivial cycle z_p reads

$$\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] \stackrel{\mathbb{Z}}{=} \int_{bc_{(0,p+1)}} \omega^{(0,p)} \stackrel{\mathbb{Z}}{=} \int_{c_{(0,p+1)}} d\omega^{(0,p)} \stackrel{\mathbb{Z}}{=} \int_{c_{(0,p+1)}} \delta_{-1}\omega^{(-1,p+1)}$$
$$\stackrel{\mathbb{Z}}{=} \int_{\partial c_{(0,p+1)}} \omega^{(-1,p+1)} \stackrel{\mathbb{Z}}{=} \int_{c_{p+1}} \omega^{(-1,p+1)} . \tag{3.24}$$

This property is exactly what we were expecting when we considered the integration of a U(1)-connection (cf the introduction to this section).

Second, let us assume that the $(p+1)$-form associated to $[\omega_{\mathcal{D}}^{(p)}]$ is exact. Then, it can be shown that there is a Deligne-Beilinson representative

$$\omega_{\mathcal{D}}^{(p)} := \left(\omega^{(0,p)} = \delta_{-1}q^{(-1,p)}, 0 \ldots, 0, \ldots, 0\right) \tag{3.25}$$

of $[\omega_{\mathcal{D}}^{(p)}]$, where $\omega^{(-1,p+1)} = dq^{(-1,p)}$. The integration formula now reads

$$\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] \stackrel{\mathbb{Z}}{=} \int_{z_{(0,p)}} \omega^{(0,p)} \stackrel{\mathbb{Z}}{=} \int_{z_{(0,p)}} \delta_{-1}q^{(-1,p)}$$
$$\stackrel{\mathbb{Z}}{=} \int_{\partial z_{(0,p)}} q^{(-1,p)} \stackrel{\mathbb{Z}}{=} \int_{z_p} q^{(-1,p)} , \tag{3.26}$$

as expected. Indeed, on the one hand, as we write $\omega^{(-1,p+1)} = dq^{(-1,p)}$, we canonically associate to $\omega^{(-1,p+1)}$ a definite form, on the other hand, we have emphasized the fact that a Deligne-Beilinson cohomology class $[\omega_{\mathcal{D}}^{(p)}]$ defines a p-form on M, up to p-forms with integral periods, $q^{(-1,p)}$. It is then natural to find that the integral of $[\omega_{\mathcal{D}}^{(p)}]$ over a cycle coincides -up to integers- with the integral of $q^{(-1,p)}$ over this cycle.

3.3 An equivalent integration over the whole manifold

In the previous approach that led to the *Defining Formula*, we have only dealt with integrals defined over cycles. In view of further generalization we shall first express those as integrals over the whole manifold M. A way to do so is to construct a version of Pontrjagin duality in the Deligne-Beilinson framework. In other words, we construct a (non smooth) canonical Deligne-Beilinson cohomology class $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ associated to any singular p-cycle z on M and a cup product $(\cup_{\mathcal{D}})$ [6, 11, 13] such that

$$\int_{z} \left[\omega_{\mathcal{D}}^{(p)}\right] \stackrel{\mathbb{Z}}{=} \int_{M} \left[\omega_{\mathcal{D}}^{(p)}\right] \cup_{\mathcal{D}} \left[\eta_{\mathcal{D}}^{(n-p-1)}(z)\right] , \tag{3.27}$$

for any Deligne-Beilinson cohomology class $[\omega_{\mathcal{D}}^{(p)}]$. We refer the reader to appendix B for a construction of (a representative of) $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$. Now, let

$$\eta_{\mathcal{D}}^{(n-p-1)}(z) := \left(\eta^{(0,n-p-1)}, \ldots, \eta^{(n-p-1,0)}, \stackrel{\mathbb{Z}}{\eta}{}^{(n-p,-1)}\right) ,$$

be a representative of $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ and

$$\omega_{\mathcal{D}}^{(p)} := \left(\omega^{(0,p)}, \ldots, \omega^{(p,0)}, \stackrel{\mathbb{Z}}{\omega}{}^{(p+1,-1)}\right) ,$$

a representative of a Deligne-Beilinson cohomology class $[\omega_{\mathcal{D}}^{(p)}]$. Then a representative of the cup product $[\omega_{\mathcal{D}}^{(p)}] \cup_{\mathcal{D}} [\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ is given by

$$\Big(\omega^{(0,p)} \cup d\eta^{(0,n-p-1)}, \ldots, \omega^{(p,0)} \cup d\eta^{(0,n-p-1)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(0,n-p-1)}, \ldots,$$

$$\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \overset{\mathbb{Z}}{\eta}{}^{(n-p,-1)}\Big) . \tag{3.28}$$

The cup product $\cup$ within the Čech-de Rahm complex is defined in [27]. In this Deligne-Beilinson cohomology class, $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \overset{\mathbb{Z}}{\eta}{}^{(n-p,-1)}$ is an integral Čech $(n+1)$-cocycle which is necessarily trivial since the covering of M is simple. Hence

$$\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \overset{\mathbb{Z}}{\eta}{}^{(n-p,-1)} = \delta \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \tag{3.29}$$

for some integral Čech n-cochain $\overset{\mathbb{Z}}{\chi}{}^{(n,-1)}$. Accordingly, considering M itself as a cycle we can associate to it a Weil decomposition [9]

$$M^{\mathcal{W}} = (m_{(0,n)}, \ldots, m_{(k,n-k)}, \ldots, m_{(n,0)}) , \tag{3.30}$$

so that we obtain

$$\begin{aligned} \int_M \left[\omega_{\mathcal{D}}^{(p)}\right] \cup_{\mathcal{D}} \left[\eta_{\mathcal{D}}^{(n-p-1)}(z)\right] &\overset{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_{m_{(k,n-k)}} \omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)} + \\ &\quad + \sum_{k=p+1}^{n} \int_{m_{(k,n-k)}} \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)} \\ &\overset{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_{m_{(k,n-k)}} \omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)} . \end{aligned} \tag{3.31}$$

It has to be noted that not all representatives of $[M^{\mathcal{W}}]$ and of $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ are suitable. Indeed, representatives of $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ are de Rham currents and so cannot always be integrated on a singular chain. Strictly speaking, the integration is possible only when the current and the chain are transversal; this is the same problem as encountered in trying to define the product of distributions. Intersection theory of chains in $\mathbb{R}^n$ assures that there exist representatives of $[M^{\mathcal{W}}]$ and $[z_{(p)}^{\mathcal{W}}]$ for which (3.31) is well defined. More precisely the allowed ambiguities on the representatives of the m's and the η's are just those required to set the chains they represent in *"general position"*, so that their intersection can be defined (see for instance [26]). Then we can show that (3.31) gives, up to integers, the same result as (3.21).

We shall refer to formula (3.31) as the "*Long Formula*" which obviously allows to generalize the integration of $[\omega_{\mathcal{D}}^{(p)}]$ over cycles in the sense that we can now define the Deligne-Beilinson product of $[\omega_{\mathcal{D}}^{(p)}]$ with any Deligne-Beilinson cohomology class $[\eta_{\mathcal{D}}^{(n-p-1)}]$ (not necessarily representing a singular cycle) and integrate over M. As an exercise, one can check that the two simple cases presented in subsection (3.2.1) lead to the same results when using the *Long Formula*, instead of the *Defining Formula*.

[9]Which is nothing but a polyhedral decomposition of M, as defined in [22].

3.4 Smoothing

Instead of using singular chains as in the previous construction we use here de Rham chains which are equivalence classes of singular chains — *for which the integrals of any smooth form on M are the same ([22, p. 28])* —. Accordingly we introduce de Rham integration currents

$$T(z)^{n-p+k}_{k}$$

associated with $z_{(k,p-k)}$, elements of which can be seen as $(n-p)$-forms with compact supports (and distributional coefficients). In analogy with (3.10) we obtain a sequence of currents

$$T^{(p)}_{\mathcal{W}}(z) = (T(z)^{n-p}_{0}, \ldots, T(z)^{n-p+k}_{k}, \ldots, T(z)^{n}_{p}\ ), \tag{3.32}$$

and the descent equations

$$dT(z)^{n-p+k}_{k} = \partial T(z)^{n-p+k+1}_{k+1}\,, \tag{3.33}$$

for $k = 1, \ldots, p-1$ and

$$\partial T(z)^{n-p}_{0} = T(z)^{n-p}_{-1} := T(z) \qquad \int_M T(z)^{n}_{p} \in [z_{(p,-1)}]\,, \tag{3.34}$$

where $T(z)$ is the integration current of z and $[z_{(p,-1)}]$ is the Čech homology class of z in M. In terms of these de Rham currents, the *Defining Formula* reads

$$\int_z \left[\omega^{(p)}_{\mathcal{D}}\right] \overset{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_M T(z)^{n-p+k}_{k} \odot \omega^{(k,p-k)}\,, \tag{3.35}$$

where we define:

$$T(z)^{n-p+k}_{k} \odot \omega^{(k,p-k)} = \frac{1}{(k+1)!} \sum_{\alpha_0,\cdots,\alpha_k} T(z)^{n-p+k}_{k,\,\alpha_0\cdots\alpha_k} \wedge \omega^{(k,p-k)}_{\alpha_0\cdots\alpha_k}\,. \tag{3.36}$$

As a special case, the whole cycle M gives rise to a sequence

$$T_{\mathcal{W}}(M) = \Big(T(M)^0_0, \ldots, T(M)^k_k, \ldots, T(M)^n_n\Big)\,, \tag{3.37}$$

with

$$dT(M)^k_k = \partial T(M)^{k+1}_{k+1} \tag{3.38}$$

for $k = 1, \ldots, n-1$ and

$$\partial T(M)^0_0 = T(M)^{\,0}_{-1} := T(M) = 1; \qquad \int_M T(M)^n_n \in [m_{(n,-1)}]\,, \tag{3.39}$$

$[m_{(n,-1)}]$ being the Čech homology class of M. Accordingly, the *Long Formula* now reads

$$\begin{aligned}\int_M \left[\omega^{(p)}_{\mathcal{D}}\right] \cup_{\mathcal{D}} \left[\eta^{(n-p-1)}_{\mathcal{D}}(z)\right] \overset{\mathbb{Z}}{=} & \sum_{k=0}^{p} \int_M T(M)^k_k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)}\right) + \\ & + \sum_{k=p+1}^{n} \int_M T(M)^k_k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)}\right)\,. \end{aligned} \tag{3.40}$$

The allowed ambiguities of de Rham currents representing $[T_{\mathcal{W}}(M)]$ are bigger than those implied by the Weil descent in the decomposition of M, except at the first and the last steps — cf. (3.39) —. Indeed, in (3.40) an ambiguity may be any de Rham current and not necessarily the integration current of an integral chain as in the case of (3.30), in particular it can be any **smooth** form (but still with compact support). This freedom on the ambiguities allows us to smooth the $T(M)^k_k$ currents occurring in the *Long Formula*, replacing them by differential forms induced by a partition of unity on M, as shown below.

Let us seek for sequences of (smooth) forms that satisfy the same descent equations as $T_{\mathcal{W}}(M)$ and such that when substituted into (3.40) they define the same integrals. Concerning the descent equations, it is well-known (see for instance [25]) that a partition of unity on M, subordinate to the simple covering $\mathcal{U}$ of M, gives rise to a sequence of forms

$$\Theta_{\mathcal{W}}(M) := (\vartheta^0_0, \ldots, \vartheta^k_k, \ldots, \vartheta^n_n\ ), \tag{3.41}$$

which satisfy homological descent equations

$$d\vartheta^k_k = \partial\vartheta^{k+1}_{k+1} \tag{3.42}$$

$k = 1, \ldots, n-1$, as well as

$$\partial\vartheta^0_0 = \vartheta^{-1}_{-1} = 1\,. \tag{3.43}$$

Furthermore, since M is supposed to be compact, the forms ϑ^k_k can all be chosen with compact supports in their defining open sets. Due to the smoothness of all the components of $\Theta_{\mathcal{W}}(M)$, the second constraint of (3.39) reads

$$\int_M \vartheta^n_n := t_{(n,-1)} + \partial r_{(n+1,-1)}\,, \tag{3.44}$$

where $t_{(n,-1)}$ is an integral Čech cycle while $r_{(n+1,-1)}$ is a **real** Čech chain. That is to say, ϑ^n_n defines an integral cycle up to a real boundary. Using homological and cohomological descents, one can show that $t_{(n,-1)} \in [m_{(n,-1)}]$. This is mainly due to the fact that the integration of any closed n-form on M can be performed by means of a partition of unity on M.

Let us compare $T_{\mathcal{W}}(M)$ with $\Theta_{\mathcal{W}}(M)$ in order to replace $T_{\mathcal{W}}(M)$ by $\Theta_{\mathcal{W}}(M)$ in (3.40). To begin with,

$$\partial\vartheta^0_0 - \partial T^0_0 = 0 \quad \Rightarrow \quad \vartheta^0_0 - T^0_0 = \partial R^0_1\ + d_{-1}R^{-1}_0\,, \tag{3.45}$$

with $\partial d_{-1}R^{-1}_0 = 0$, hence $\partial R^{-1}_0 = 0$. As M is connected $H_0(M,\mathbb{R}) = 0$, $R^{-1}_0 = \partial R^{-1}_1$. T^0_0 can be replaced by ϑ^0_0 in (3.40) since

$$\int_M d_{-1}R^{-1}_0 \odot \left(\omega^{(0,p-0)} \cup d\eta^{(0,n-p-1)}\right) = \int_M d\left[R^{-1}_1 \odot \left(\omega^{(0,p-0)} \cup d\eta^{(0,n-p-1)}\right)\right] = 0\,. \tag{3.46}$$

Thus R^{-1}_0 can be ignored in (3.45) and the first step of the descent reads

$$\partial(\vartheta^1_1 - T^1_1) = d(\vartheta^0_0 - T^0_0) = d\partial R^0_1 = \partial d R^0_1\,, \tag{3.47}$$

so that

$$\vartheta^1_1 - T^1_1 = dR^0_1 + \partial R^1_2\,. \tag{3.48}$$

Similarly

$$\vartheta_k^k - T_k^k = dR_k^{k-1} + \partial R_{k+1}^k\,, \qquad k = 1, \ldots, n. \tag{3.49}$$

Finally, the constraints (3.34) and (3.44) give

$$\int_M (\vartheta_n^n - T_n^n) = \partial\lambda_{n+1}^{-1} = \partial \int_M R_{n+1}^n\,. \tag{3.50}$$

Now, if we replace T_k^k by ϑ_k^k and its ambiguities, the *Long Formula* reads:

$$\begin{aligned}\int_M (\ldots) \;\overset{\mathbb{Z}}{=}\; & \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)}\right) + \\ & + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)}\right) + \\ & + \int_M \partial R_{n+1}^n \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)}\right)\,.\end{aligned} \tag{3.51}$$

The last term in this equation gives

$$\begin{aligned}\int_M R_{n+1}^n \odot \delta\,(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)}) &= \int_M R_{n+1}^n \odot \delta \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \\ &= \int_M \partial R_{n+1}^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} = \int_M (\vartheta_n^n - T_n^n) \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)}\end{aligned} \tag{3.52}$$

Since all integrals of T_n^n's are integers, we obtain

$$\int_M R_{n+1}^n \odot \delta \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(n-p-1,0)}\right) \;\overset{\mathbb{Z}}{=}\; \int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)}\,,$$

so that the *(smoothed) Long Formula* reads

$$\begin{aligned}\int_z \left[\omega_{\mathcal{D}}^{(p)}\right] \;\overset{\mathbb{Z}}{=}\; & \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)}\right) \\ & + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)}\right) + \int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)}\,.\end{aligned} \tag{3.53}$$

Let us make some final remarks. First, if the simple covering $\mathcal{U}$ of M is such that all intersections of order larger than $n+1$ are empty — we shall say that $\mathcal{U}$ is *"excellent"* — we deduce that

$$\int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} \in \mathbb{Z}\,, \tag{3.54}$$

which leads to

$$\begin{aligned}\int_z [\omega_{\mathcal{D}}^{(p)}] \;\overset{\mathbb{Z}}{=}\; & \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \left(\omega^{(k,p-k)} \cup d\eta^{(0,n-p-1)}\right) + \\ & + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \left(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup \eta^{(k-p-1,n-k)}\right)\,.\end{aligned} \tag{3.55}$$

In other words, with respect to an excellent covering of M, the ϑ_k^k's play the role of the integration currents $T(M)_k^k$ of the Weil descent of M.

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Second, the previous construction, i.e. the smoothing, cannot be applied to the *Defining Formula* without care. Indeed, a simple covering of M does not always induce a simple covering on the p-cycle z_p, so that, although a $(p+1)$-cocycle on M reduces to a $(p+1)$-cocycle on z_p, this cocycle is not necessarily trivial. Therefore we cannot establish a smoothed *Defining Formula* in full generality. However, let us assume that z_p admits a tubular neighborhood, $\mathcal{V}_z$, such that $\mathcal{U}_{|\mathcal{V}_z}$ - the restriction to $\mathcal{V}_z$ of the simple covering $\mathcal{U}$ of M - is also simple. Then, as a tubular neighborhood, $\mathcal{V}_z$ has necessarily the same cohomology as z_p, and since $\mathcal{U}_{|\mathcal{V}_z}$ is simple, this cohomology is also the Čech cohomology of $\mathcal{U}_{|\mathcal{V}_z}$. In particular the Čech $(p+1)$-cocycle, $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$, on M is also a $(p+1)$-cocycle on $\mathcal{V}_z$ and is necessarily trivial on it, that is: $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} = \delta \overset{\mathbb{Z}}{\varpi}{}^{(p,-1)}$ for some integral Čech p-cochain $\overset{\mathbb{Z}}{\varpi}{}^{(p,-1)}$, just as in the case of the *Long Formula*. With all this, a natural candidate for a smoothed *Defining Formula* would be

$$\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] \overset{\mathbb{Z}}{=} \sum_{k=0}^{p-1} \int_{z_p} \vartheta_k^k \odot \omega^{(k,p-k)} + \int_{z_p} \vartheta_p^p \odot \left(\omega^{(p,0)} - \overset{\mathbb{Z}}{\varpi}{}^{(p,-1)}\right),$$

which compares to the smoothed *Long Formula* (3.53).

As a third remark, one can wonder what is the relation between the *Defining Formulas* and the decomposition $A = A_0 + \alpha$ used in section 2 in the case of U(1)-connections. Let us consider two Deligne-Beilinson classes, $[\omega_{\mathcal{D}}]$ and $[\chi_{\mathcal{D}}]$, representing the same Čech cohomology class, $[\check{\xi}]$, as detailed in appendix A.6. We know that $[\omega_{\mathcal{D}}]$ and $[\chi_{\mathcal{D}}]$ differ by a Deligne-Beilinson class, $[(\delta\alpha)_{\mathcal{D}}]$ coming from a global form α on M. This exactly corresponds to the standard decomposition $A = A_0 + \alpha$ for U(1)-connections met in section 2. This can also be seen at the level of the integrals : choose representatives $(\omega^{(0,p)}, \cdots, \omega^{(p,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)})$ and $(\chi^{(0,p)}, \cdots, \chi^{(p,0)}, \overset{\mathbb{Z}}{\chi}{}^{(p+1,-1)})$ of $[\omega_{\mathcal{D}}]$ and $[\chi_{\mathcal{D}}]$ respectively, and write the previous decomposition

$$\begin{aligned}\left(\chi^{(0,p)}, \ldots, \chi^{(p,0)}, \overset{\mathbb{Z}}{\chi}{}^{(p+1,-1)}\right) &= \left(\omega^{(0,p)}, \ldots, \omega^{(p,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}\right) + (0, \ldots, 0, \delta\alpha) + \\ &\quad + D(q^{(0,p-1)}, \cdots, q^{(p-1,0)}, \overset{\mathbb{Z}}{q}{}^{(p,-1)}), \qquad (3.56)\end{aligned}$$

for some $[q_{\mathcal{D}}]$. By assumption $\overset{\mathbb{Z}}{\chi}{}^{(p+1,-1)}$ and $\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}$ are cohomologous, so

$$\int_{z_p} [\chi_{\mathcal{D}}] \overset{\mathbb{Z}}{=} \int_{z_p} [\omega_{\mathcal{D}}] + \int_{z_p} \alpha\,.$$

This result also means that the standard decomposition $A = A_0 + \alpha$ of U(1)-connections, extends to any generalized p-connection.

A final remark on notations, we could have denoted the integral over z_p of the class $[\omega_{\mathcal{D}}^{(p)}]$ simply as:

$$\begin{aligned}\int_{z_p} \left[\omega_{\mathcal{D}}^{(p)}\right] &\overset{\mathbb{Z}}{=} \left\langle \left[\omega_{\mathcal{D}}^{(p)}\right], \left[z_{(p)}^{\mathcal{W}}\right] \right\rangle \\ &\overset{\mathbb{Z}}{=} \left\langle \omega^{(0,p)} + \cdots \omega^{(p,0)} + \overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)}, z_{(0,p)} + \cdots + z_{(p,0)} \right\rangle \\ &\overset{\mathbb{Z}}{=} \left\langle [\omega_{\mathcal{D}}^{(p)}] \cup_{\mathcal{D}} [\eta_{\mathcal{D}}^{(n-p-1)}(z)],\ M \right\rangle \qquad (3.57)\end{aligned}$$

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which has the advantage to make easier the proof of independence with respect to the various representatives.

4. Integration of Deligne-Beilinson classes with distributional coefficients

In any quantization procedure, ω will be by nature distributional and integration over a cycle will, in general, be ill defined so that the integration current of the cycle will have to be replaced by some regularized form. This is the situation which has been exhibited in the example of section 2. A canonical way to perform such an operation for $[\omega_{\mathcal{D}}^{(p)}]$ of distributional character is to use formula (3.53), (3.57) with $z_{(p)}^{\mathcal{W}}$ replaced by a smooth Deligne-Beilinson class $[j_{\mathcal{D}}^{(n-p-1)}]$, the integration formula being

$$\Big\langle [\omega_{\mathcal{D}}^{(p)}] \cup_{\mathcal{D}} [j_{\mathcal{D}}^{(n-p-1)}],\ M \Big\rangle \overset{\mathbb{Z}}{=} \sum_{k=0}^{p} \int_M \vartheta_k^k \odot \Big(\omega^{(k,p-k)} \cup dj^{(0,n-p-1)} \Big) + \\ + \sum_{k=p+1}^{n} \int_M \vartheta_k^k \odot \Big(\overset{\mathbb{Z}}{\omega}{}^{(p+1,-1)} \cup j^{(k-p-1,n-k)} \Big) \\ + \int_M \vartheta_n^n \odot \overset{\mathbb{Z}}{\chi}{}^{(n,-1)} . \tag{4.1}$$

Note that (4.1) is — **mod** $\mathbb{Z}$! — symmetric in $[\omega_{\mathcal{D}}^{(p)}]$ and $[j_{\mathcal{D}}^{(n-p-1)}]$, as can be easily verified. Whereas we have shown that to the current of a cycle z_p is associated a special Deligne-Beilinson class $[\eta_{\mathcal{D}}^{(n-p-1)}(z)]$, the map $z_p \to [\eta_{\mathcal{D}}^{(n-p-1)}(z)]$ being analogous to the cycle map in [6], we do not know of such an assignment in the case of a smoothed version. It is expected that after renormalization some of the characteristics of the regularized class $[j_{\mathcal{D}}^{(n-p-1)}]$ will survive.

5. Conclusions

We have described in some details a class of topological actions which are "topological" in the sense that they are defined modulo "integers", a situation repeatedly met in semi classical treatments of various field theories involving particular geometries (mostly gauge theories, including gravity). They are described by integral formulae which involve refinements of closed differential forms with integral periods named Deligne-Beilinson cohomology classes. The integrals are written as pairings of two such classes in such a way that one of them may have a distributional character as demanded in most field theory contexts.

A. Deligne-Beilinson cohomology

We have not been able to find an elementary discussion of Deligne-Beilinson cohomology in the mathematical literature. The purpose of this appendix is to fill in this gap, concentrating on the computation of Deligne-Beilinson cohomology and on the proof of its independence upon the covering. For more algebraic exposés we refer to [11, 13].

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A.1 Definitions and notations

As in the main text, M denotes a compact differentiable manifold of dimension n, and $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ a simple covering[10] of M, $M = \cup_{\alpha\in I}\mathcal{U}_\alpha$. A Čech cochain of degree k with values in an abelian group G is a collection of elements $c_{\alpha_0\cdots\alpha_k}$ of G, one for each intersection $\mathcal{U}_{\alpha_0\cdots\alpha_k}$, which is totally antisymmetric in all its indices and vanishes on empty intersections. A Čech cochain of degree -1 is a constant map from M to G.

The Čech differential, δ, maps $(k-1)$-cochains to k-cochains and squares to 0. Acting on (-1)-cochains, δ is the restriction : $(\delta c)_{\alpha_0} = c$ on any non empty $\mathcal{U}_{\alpha_0}$. For $k \geq 1$, if $c_{\alpha_0\cdots\alpha_{k-1}}$ is a $(k-1)$-cochain and $\mathcal{U}_{\alpha_0\cdots\alpha_k} \neq \emptyset$,

$$(\delta c)_{\alpha_0\cdots\alpha_k} = \sum_{i=0}^{k} (-)^i c_{\alpha_0\cdots\widehat{\alpha}_i\cdots\alpha_k} \tag{A.1}$$

were the $\widehat{}$ means omission.

The elements in the kernel of δ are Čech cocycles, those in the image of δ are Čech coboundaries.

In the sequel, we shall have no use of general abelian groups G, but $\mathbb{R}$ (for real Čech cochains), $\mathbb{Z}$ (for integral Čech cochains) and $\mathbb{R}/\mathbb{Z}$ will play preferred roles.

One can also consider Čech cochains where each $c_{\alpha_0\cdots\alpha_k}$ is a differential l-form defined on $\mathcal{U}_{\alpha_0\cdots\alpha_k}$; such cochains are often referred to as Čech-de Rham cochains of bidegree (k,l). In Čech degree -1, we retrieve global differential l-forms defined on M and δ is still defined by restriction. On these "extended" Čech $(k-1)$-cochains, $k \geq 1$, the action of δ is still given by (A.1) except for an overall multiplicative factor $(-)^{l+1}$ on the right hand side : each term makes sense with the proviso that it is restricted to the corresponding $(k+1)$-fold intersection . This leads to the space[11] denoted by $\check{C}^{(k)}(\mathcal{U},\Omega^l(M))$ in the main text. To save space in this appendix, we shall denote it simply by $\Omega^{(k,l)}(\mathbb{R})$, because most of the time M and $\mathcal{U}$ will be fixed.

By convention, a "purely Čech " cochain with constant coefficients (in a subgroup G of $\mathbb{R}$) receives form degree -1, so it belongs to $\Omega^{(k,-1)}(G)$. The de Rham differential d maps $\Omega^{(k,l)}(G)$ into $\Omega^{(k,l+1)}(G)$ for $k \geq 0$.[12] We extend d to (-1)-forms as the injection which maps an element of $G \subset \mathbb{R}$ to the corresponding constant function. This is sometimes denoted by the symbol d_{-1}. This extension still satisfies $d^2 = 0$.

Later in the appendix, we shall need to compare several simple coverings. Suppose that the simple covering $\mathcal{V} = \{\mathcal{V}_\sigma\}_{\sigma\in J}$ of M is a refinement of the simple covering $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in I}$: this means that there is the restriction map $r : J \longrightarrow I$ such that $\mathcal{V}_\sigma \subset \mathcal{U}_{r(\sigma)}$ for all indices $\sigma \in J$. A Čech k-cochain, c, for $\mathcal{U}$ can be restricted to $\mathcal{V}$: if the intersection $\mathcal{V}_{\sigma_0\cdots\sigma_k}$

[10] Such an open covering is alternatively called a *good* covering in [27]. This means that any finite intersection of $\mathcal{U}_\alpha$'s, $\mathcal{U}_{\alpha_0\cdots\alpha_q} = \mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_q}$, $(\alpha_0, \cdots, \alpha_q) \in I^{q+1}$, is either empty or diffeomorphic to $\mathbb{R}^n$.

[11] A more appropriate language for this setting involves sheaves, but we shall not use the corresponding terminology.

[12] The sign factor $(-1)^{l+1}$ insures that $d\,\delta + \delta\, d = 0$.

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is nonempty, then so is $\mathcal{U}_{r(\sigma_0)\cdots r(\sigma_k)}$, and

$$r(c)_{\sigma_0\cdots\sigma_k} \equiv \left(c_{r(\sigma_0)\cdots r(\sigma_k)}\right)_{|\mathcal{V}_{\sigma_0\cdots\sigma_k}} .$$

The Čech and de Rham differentials commute with restriction, i.e. $\delta \circ r = r \circ \delta$ (it being understood that the Čech differential on the left-hand side is for the covering $\mathcal{V}$ and on the right-hand side for the covering $\mathcal{U}$) and $d \circ r = r \circ d$.

A.2 Deligne-Beilinson cochains

Take an integer $0 \leq p \leq n+1$ (n the dimension of the manifold) and consider the double complex

$$\begin{array}{ccccccccc}
\Omega^{(0,-1)}(\mathbb{Z}) & \xrightarrow{d_{-1}} & \Omega^{(0,0)}(\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{(0,p-1)}(\mathbb{R}) & \xrightarrow{0} & 0 \\
\downarrow \delta & & \downarrow \delta & & & & \downarrow \delta & & \\
\Omega^{(1,-1)}(\mathbb{Z}) & \xrightarrow{d_{-1}} & \Omega^{(1,0)}(\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{(1,p-1)}(\mathbb{R}) & \xrightarrow{0} & 0 \\
\downarrow \delta & & \downarrow \delta & & & & \downarrow \delta & & \\
\Omega^{(2,-1)}(\mathbb{Z}) & \xrightarrow{d_{-1}} & \Omega^{(2,0)}(\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{(2,p-1)}(\mathbb{R}) & \xrightarrow{0} & 0 \\
\downarrow \delta & & \downarrow \delta & & & & \downarrow \delta & & \\
\vdots & & \vdots & & & & \vdots & &
\end{array}$$

The columns of this diagram form standard Čech complexes. The rows are Deligne complexes of index p, that is de Rham complexes extended to the left by d_{-1} (the injection of integral constants into real functions) and truncated on the right at $(p-1)$-forms by the 0 map. We denote by $\tilde{d}$ this modified differential, to avoid confusion with the de Rham differential, d.

We build a new "diagonal complex" from this double complex. The space C_p^q of Deligne-Beilinson cochains of degree $q \geq 0$ (with fixed index p) is defined by

$$C_p^q = \begin{cases} \Omega^{(q,-1)}(\mathbb{Z}) + \sum_{k=1}^{q} \Omega^{(q-k,k-1)}(\mathbb{R}) & \text{for } 0 \leq q < p \\ \Omega^{(p,-1)}(\mathbb{Z}) + \sum_{k=1}^{p} \Omega^{(p-k,k-1)}(\mathbb{R}) & \text{for } \ q = p \\ \Omega^{(q,-1)}(\mathbb{Z}) + \sum_{k=1}^{p} \Omega^{(q-k,k-1)}(\mathbb{R}) & \text{for } q > p \end{cases} .$$

Elements of these spaces are respectively represented by the following sequences :

$$c = \left(c^{(0,q-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(q,-1)}\right), \qquad c = \left(c^{(0,p-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(p,-1)}\right),$$

$$c = \left(c^{(q-p,p-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(q,-1)}\right),$$

with the last element $\mathbb{Z}$-valued.[13]

[13] Our complex contains $\Omega^{(q,-1)}(\mathbb{Z})$ while in the literature one usually finds $\Omega^{(q,-1)}(\mathbb{Z}(p))$, where $\mathbb{Z}(p) = (2i\pi)\mathbb{Z}$. This difference is irrelevant for our purpose.

We set $\mathcal{C}_p = C_p^0 \oplus C_p^1 \oplus \cdots$.

The operator $D = \tilde{d} + \delta$ maps C_p^q to C_p^{q+1} and, due to the sign convention in the definition of δ on l-forms, $D^2 = 0$. The complex $(\mathcal{C}_p, D)$ is called the Deligne-Beilinson complex,[14] and the elements of $\mathcal{C}_p$ Deligne-Beilinson cochains. We write $Z_p^q = \left\{ \operatorname{Ker} D : C_p^q \to C_p^{q+1} \right\}$ (resp. $B_p^q = \left\{ \operatorname{Im} D : C_p^{q-1} \to C_p^q \right\}$) for the space of Deligne-Beilinson cocycles (resp. coboundaries).

We are interested in the cohomology of $(\mathcal{C}_p, D)$. A priori, it depends on the covering, but we shall see later that the cohomologies for simple coverings are canonically isomorphic.

The projection $\pi : C_p^q \to \Omega^{(q,-1)}(\mathbb{Z})$ gives a chain map

$$\begin{array}{ccccccc}
\cdots & \xrightarrow{D} & C_p^q & \xrightarrow{D} & C_p^{q+1} & \xrightarrow{D} & \cdots \\
 & & \downarrow \pi & & \downarrow \pi & & \\
\cdots & \xrightarrow{\delta} & \Omega^{(q,-1)}(\mathbb{Z}) & \xrightarrow{\delta} & \Omega^{(q+1,-1)}(\mathbb{Z}) & \xrightarrow{\delta} & \cdots
\end{array}$$

so that in all cases, there is a canonical map $H^q(\mathcal{C}_p, D) \to H^q_{\text{Čech}}(M, \mathbb{Z})$. The computation of $H^q(\mathcal{C}_p, D)$ goes along different lines whether $q \leq p-1$ or $q > p-1$.

A.3 Computation of $H^q(\mathcal{C}_p, D)$, $q < p$

In this case, we use the Poincaré lemma for differential forms (ensuring that for forms of nonnegative degree, the de Rham cohomology is locally trivial) to show that

$$H^q(\mathcal{C}_p, D) \simeq H^{q-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) \qquad (I)$$

(the isomorphism is canonical). In particular, the canonical map

$$H^q(\mathcal{C}_p, D) \to H^q_{\text{Čech}}(M, \mathbb{Z})$$

maps $H^q(\mathcal{C}_p, D)$ onto the subgroup $H^q_{\text{Čech}}(M, \mathbb{Z})_{\text{torsion}}$ of torsion classes.

Proof. Suppose $c = (c^{(0,q-1)}, c^{(1,q-2)}, \cdots, c^{(q-1,0)}, \overset{\mathbb{Z}}{c}{}^{(q,-1)})$ is a Deligne-Beilinson cocycle. This implies that $\tilde{d}c^{(0,q-1)} = 0$, and since $q \leq p-1$, the operator $\tilde{d}$ in this equation is the standard de Rham differential. So, by the Poincaré lemma, there is an element $\rho^{(0,q-2)} \in \Omega^{(0,q-2)}(\mathbb{R})$ such that $c^{(0,q-1)} + \tilde{d}\rho^{(0,q-2)} = 0$. Accordingly the cocycle c is cohomologous to the cocycle

$$c + D\rho^{(0,q-2)} = \left(0, \underline{c}^{(1,q-2)}, \cdots, c^{(q-1,0)}, \overset{\mathbb{Z}}{c}{}^{(q,-1)} \right),$$

where $\underline{c}^{(1,q-2)} \equiv c^{(1,q-2)} + \delta\rho^{(0,q-2)}$.

The cocycle condition for $c + D\rho^{(0,q-2)}$ yields $d\underline{c}^{(1,q-2)} = 0$ were d is the standard exterior derivative. The procedure can be iterated to show that the cohomology class of c contains a representative of the form

$$\left(0, \cdots, 0, \underline{c}^{(q-1,0)}, \overset{\mathbb{Z}}{c}{}^{(q,-1)} \right)$$

[14] A better notation would be $(\mathcal{C}_p(M), \mathcal{U}, D)$.

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with the standard descent equations fulfilled :

$$d\underline{c}^{(q-1,0)} = 0\,, \qquad \delta\underline{c}^{(q-1,0)} = d_{-1}\overset{\mathbb{Z}}{c}{}^{(q,-1)}\,, \qquad \delta\overset{\mathbb{Z}}{c}{}^{(q,-1)} = 0\,.$$

The first equation just tells that $\underline{c}^{(q-1,0)} = d_{-1}\rho^{(q-1,-1)}$, where the components $\rho^{(q-1,-1)}$ are real constants. This, combined with the second equation, implies that the integral Čech cocycle $\overset{\mathbb{Z}}{c}{}^{(q,-1)}$ is exact as a real cocycle, so that it represents a torsion class.

Reduction modulo 1 turns $\rho^{(q-1,-1)}$ into an $\mathbb{R}/\mathbb{Z}$ Čech cocycle and the ambiguity on $\underline{c}^{(q-1,0)}(\text{mod } 1)$ is a Čech coboundary. So we have proved the announced result, (I), which is also the content of the following exact sequence [13]

$$0 \longrightarrow H^{q-1}(M,\mathbb{Z}(p)) \longrightarrow H^{q-1}(M,\mathbb{R}) \longrightarrow H^q(\mathcal{C}_p, D) \longrightarrow H^q(M,\mathbb{Z}(p))_{\text{torsion}} \longrightarrow 0.$$

A.4 The Čech homotopy operator

Here we introduce the Čech homotopy operator that we shall need to compute $H^q(\mathcal{C}_p, D)$ in the special cases $q \geq p$. This homotopy[15] operator, which depends on a partition of unity defined on M, is instrumental to establish the generalized Mayer-Vietoris exact sequence, the Čech-de Rham isomorphism and the *Collating Formula* [27], a construction we illustrate below.

A.4.1 The K operator on the enlarged double complex

Consider the following double complex :

$$\begin{array}{ccccccccc}
 & & \Omega^{(-1,0)}(\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{(-1,p-1)}(\mathbb{R}) & \xrightarrow{0} & 0 \\
 & & \downarrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} & & \\
\Omega^{(0,-1)}(\mathbb{Z}) & \xrightarrow{d_{-1}} & \Omega^{(0,0)}(\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{(0,p-1)}(\mathbb{R}) & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} & & \\
\Omega^{(1,-1)}(\mathbb{Z}) & \xrightarrow{d_{-1}} & \Omega^{(1,0)}(\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{(1,p-1)}(\mathbb{R}) & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} & & \\
\Omega^{(2,-1)}(\mathbb{Z}) & \xrightarrow{d_{-1}} & \Omega^{(2,0)}(\mathbb{R}) & \xrightarrow{d} & \cdots & \xrightarrow{d} & \Omega^{(2,p-1)}(\mathbb{R}) & \xrightarrow{0} & 0 \\
\downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} & & & & \downarrow{\scriptstyle\delta} & & \\
\vdots & & \vdots & & & & \vdots & &
\end{array}$$

where the de Rham complex of global differential forms truncated at degree $(p-1)$ has been added at the top. We extend the definition of D to this enlarged complex.

Let us choose a partition of unity ϑ_α subordinate to the simple covering $\{\mathcal{U}_\alpha\}_{\alpha\in I}$ of M: each ϑ_α is a (smooth) non-negative function on M with compact support in $\mathcal{U}_\alpha$, and $\sum_\alpha \vartheta_\alpha$ is the constant function 1 on M. On the enlarged complex, define an operator K (depending on the chosen partition of unity) as follows.

[15] Ensuring that the Čech cohomology for forms of nonnegative degree is trivial.

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Take $c = \{c_{\alpha_0\cdots\alpha_k}\} \in \Omega^{(k,l)}(\mathbb{R})$, $k, l \geq 0$. Due to the support properties of the ϑ_α's, $c_{\alpha_0\cdots\alpha_k} \cdot \vartheta_{\alpha_k}$ (extended by 0 outside $\mathcal{U}_{\alpha_k}$) is a smooth differential form in each nonempty $\mathcal{U}_{\alpha_0\cdots\alpha_{k-1}}$. Let $Kc \equiv \{(-)^{l+1} \sum_{\alpha_k} c_{\alpha_0\cdots\alpha_k} \cdot \vartheta_{\alpha_k}\} \in \Omega^{(k-1,l)}(\mathbb{R})$.

For $c \in \Omega^{(-1,l)}(\mathbb{R})$, $l \geq 0$, set $Kc \equiv 0$, and for $c \in \Omega^{(k,-1)}(\mathbb{Z})$, $k \geq 0$, set $Kc \equiv Kd_{-1}c \in \Omega^{(k-1,0)}(\mathbb{R})$.

Though we shall not try to compute its homology, note that $K^2 = 0$ so K is a boundary operator (or equivalently a co-differential).

A.4.2 The homotopy property and the fundamental identity

Algebraic manipulations show that $K\delta + \delta K$ is the identity operator on $\Omega^{(k,l)}(\mathbb{R})$, $k \geq -1, l \geq 0$ and d_{-1} on $\Omega^{(k,-1)}(\mathbb{Z})$, $k \geq 0$. In particular, in the enlarged double complex, the vertical Čech complexes in nonnegative de Rham degree have vanishing Čech cohomology, since K is a homotopy operator.

Acting on the enlarged double complex, $K\tilde{d}$ lowers the Čech degree by one unit, so $K\tilde{d}$ is locally nilpotent and $1 + K\tilde{d}$ is invertible : locally the geometric series for $(1 + K\tilde{d})^{-1}$ stops after a finite number of terms. Moreover, as a consequence of

$$\left(1 + K\tilde{d}\right)\left(\tilde{d} + \delta\right) - \delta(1 + K\tilde{d}) = \tilde{d} + K\tilde{d}\delta - \delta K\tilde{d} = (1 - K\delta - \delta K)\,\tilde{d} = 0\,,$$

(the first equality uses $\tilde{d}^2 = 0$, the second $\tilde{d}\delta = -\delta\tilde{d}$ and the last one that the image of $\tilde{d}$ lives in de Rham degree ≥ 0 where $K\delta + \delta K = 1$) one derives that on the enlarged double complex, D and δ are conjugate, that is

$$(1 + K\tilde{d})D = \delta(1 + K\tilde{d})\,. \qquad (\heartsuit)$$

This fundamental identity ($\heartsuit$) is at the heart of the computation of the Deligne-Beilinson cohomology when $q \geq p$ as shown later in A.5 and A.6. It can also be useful in other contexts as illustrated below.

A.4.3 Relation with the Čech-de Rham isomorphism

Suppose that in the first column of the enlarged complex we replace the coefficient group $\mathbb{Z}$ by $\mathbb{R}$, and that we take $p = n + 1$, n the dimension of the manifold, so that the lines are usual de Rham complexes, hence $\tilde{d} = d$ in this enlarged context and the ($\heartsuit$) identity can be written $(1 + Kd)D = \delta(1 + Kd)$. This double complex is a Čech-de Rham complex with differential $D = d + \delta$ and of course $q < p = n + 1$. In the sequel this is the complex we have in mind when we refer to Čech-de Rham cochains, cocycles or coboundaries.

On the one hand if $c^{(q,-1)} \in \Omega^{(q,-1)}(\mathbb{R})$ is a Čech cocycle, it is a D-cocycle, hence its top component $(-Kd)^{q+1}c^{(q,-1)}$ is a global closed q-form, i.e. a de Rham q-cocycle.

On the other hand if $c^{(q,-1)}$ is a Čech coboundary, $c^{(q,-1)} = \delta\gamma^{(q-1,-1)}$ for some $\gamma^{(q-1,-1)} \in \Omega^{(q-1,-1)}(\mathbb{R})$, then using ($\heartsuit$) $(1+Kd)^{-1}c^{(q,-1)} = D(1+Kd)^{-1}\gamma^{(q-1,-1)}$ is a D-coboundary. Identifying top form components, $(-Kd)^{q+1}c^{(q,-1)}$ is a de Rham coboundary $d(-Kd)^q\,\gamma^{(q-1,-1)}$.

Finally, if $c = (c^{(-1,q)}, \cdots, c^{(q-1,0)}, c^{(q,-1)})$ is a D-cocycle, $c^{(q,-1)}$ is a Čech cocycle, $c^{(-1,q)}$ is a closed global de Rham q-form, and c is D-cohomologous to $(1 + Kd)^{-1}c^{(q,-1)}$. Indeed, start from $D(1+Kd)^{-1}K(c-c^{(q,-1)}) = (1+Kd)^{-1}\delta K(c-c^{(q,-1)})$, a consequence of the (♡) identity. As $c - c^{(q,-1)}$ has no component in de Rham degree -1, $\delta K(c - c^{(q,-1)}) = (1 - K\delta)(c - c^{(q,-1)})$ by the homotopy property. By $Dc = 0 = \delta c^{(q,-1)}$, we obtain finally that $\delta K(c - c^{(q,-1)}) = (1 + Kd)c - c^{(q,-1)}$. Multiplication by $(1 + Kd)^{-1}$ leads to

$$c = (1 + Kd)^{-1} c^{(q,-1)} + D(1 + Kd)^{-1} K(c - c^{(q,-1)}), \quad (*)$$

proving that c is D-cohomologous to $(1 + Kd)^{-1} c^{(q,-1)}$. This implies that $c^{(-1,q)}$ is d-cohomologous to $(-Kd)^{q+1} c^{(q,-1)}$, explicitly,

$$c^{(-1,q)} = (-Kd)^{q+1} c^{(q,-1)} + d\left(K \sum_{r=0}^{q-1} (-dK)^r c^{(r,q-1-r)}\right),$$

which is the famous *Collating Formula*; see e.g. [27], where it is used to prove that $c^{(q,-1)} \to (-Kd)^{q+1} c^{(q,-1)}$ which maps (real) Čech cocycles to de Rham cocycles and (real) Čech coboundaries to de Rham coboundaries induces an isomorphism in cohomology. With notations closer to the ones used in the main text, the *Collating Formula* can be rewritten[16]

$$c^{(-1,q)} = d\left(\vartheta_0^0 \cdot c^{(0,q-1)} + \vartheta_1^1 \cdot c^{(1,q-2)} + \cdots + \vartheta_{q-1}^{q-1} \cdot c^{(q-1,0)}\right) + \vartheta_q^q \cdot c^{(q,-1)}.$$

The *Collating Formula* is related to the Weil theorem which can be rewritten neatly using the Deligne-Beilinson machinery.

First, observe that $C_{p+1}^p = C_p^p$ but $Z_{p+1}^p \subset Z_p^p$. Indeed on $\Omega^{(0,p-1)} \subset C_{p+1}^p$ the operator $\tilde{d}$ is the genuine de Rham differential, while on $\Omega^{(0,p-1)} \subset C_p^p$ it is the 0 map, so the condition to be D-closed is more stringent in the first case. If $c = (c^{(0,p-1)}, c^{(1,p-2)}, \cdots, c^{(0,p-1)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)})$ belongs to Z_p^p, the standard de Rham differential applied to $c^{(0,p-1)}$ leads to a global closed p-form. Indeed, $\delta d c^{(0,p-1)} = d\delta c^{(0,p-1)} = \pm d^2\, c^{(1,p-2)} = 0$, so $dc^{(0,p-1)}$ is the restriction of a global p-form, which is obviously closed. So there is a canonical map $\left\{\operatorname{Ker} D : C_p^p \to C_p^{p+1}\right\} \xrightarrow{m} \left\{\operatorname{Ker} d : \Omega^{(-1,p)} \to \Omega^{(-1,p+1)}\right\}$. The image of this map is not totally obvious, but this is precisely the content of Weil's theorem [25]: the sequence of abelian groups

$$0 \longrightarrow Z_{p+1}^p \xrightarrow{i} Z_p^p \xrightarrow{m} \left\{\begin{matrix}\text{Closed global } p\text{-forms} \\ \text{with integral periods}\end{matrix}\right\} \longrightarrow 0$$

is exact.

A.4.4 Refinements

If the simple covering $\mathcal{V} = \{\mathcal{V}_\sigma\}_{\sigma\in J}$ of M is a refinement of the simple covering $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in I}$ and $\{\varphi_\sigma\}$ is a partition of unity for $\mathcal{V}$, we define a (compatible) partition of unity for $\mathcal{U}$ $\{\vartheta_\alpha\} = \{\sum_{\substack{\sigma\in J \\ r(\sigma)=\alpha}} \varphi_\sigma\}$. For compatible partitions of unity, the homotopy operator commutes with restriction, i.e. $K \circ r = r \circ K$ (it being understood that the homotopy operator on the left-hand side is for the covering $\mathcal{V}$ and on the right-hand side for the covering $\mathcal{U}$). To

[16] Cf. (3.41)–(3.43) in the main text for properties of the θ_k^k's.

summarize, restriction commutes with δ, $\tilde{d}$, D and K :

$$\delta \circ r = r \circ \delta\,, \quad \tilde{d} \circ r = r \circ \tilde{d}\,, \quad K \circ r = r \circ K\,, \quad D \circ r = r \circ D\,. \tag{A.2}$$

We could say this more pedantically by drawing the Deligne-Beilinson complexes (or their enlarged versions) for $\mathcal{U}$ and $\mathcal{V}$ on top of each other (in three dimensions) and stating that restriction is a (co-)chain map for all differentials or co-differentials defined up to now.

A.5 Computation of $H^q(\mathcal{C}_p, D)$, $q > p$

We show that for $q > p$,

$$H^q(\mathcal{C}_p, D) \simeq H^q_{\check{\mathrm{C}}\mathrm{ech}}(M, \mathbb{Z}) \qquad (II)$$

(the isomorphism is canonical).

Proof. Start from the simple observation that for $q > p$ one has the inclusion $K\tilde{d}(C^q_p) \subset C^q_p$, so that one can freely use $(1 + K\tilde{d})D = \delta(1 + K\tilde{d})$ to compute $H^q(\mathcal{C}_p, D)$.

The middle cohomology in the complex $0 \longrightarrow C^{q-1}_p \xrightarrow{\delta} C^q_p \xrightarrow{\delta} C^{q+1}_p \longrightarrow 0$ is concentrated in de Rham degree -1 because δ does not change the de Rham degree and has no cohomology in nonnegative de Rham degree due to the existence of the homotopy operator. So this cohomology is simply $H^q_{\check{\mathrm{C}}\mathrm{ech}}(M, \mathbb{Z})$. If $\overset{\mathbb{Z}}{c}{}^{(q,-1)} \in \Omega^{(q,-1)}(\mathbb{Z}) \subset C^q_p$ is a Čech cocycle, $(1 + K\tilde{d})^{-1} \overset{\mathbb{Z}}{c}{}^{(q,-1)}$ is a D-cocycle. Conversely if the cochain $c = (c^{(q-p,p-1)}, \cdots, \overset{\mathbb{Z}}{c}{}^{(q,-1)}) \in C^q_p$ is a D-cocycle, $\overset{\mathbb{Z}}{c}{}^{(q,-1)}$ is a Čech cocycle and the relation $(*)$ is satisfied i.e.

$$c = (1 + K\tilde{d})^{-1} \overset{\mathbb{Z}}{c}{}^{(q,-1)} + D(1 + K\tilde{d})^{-1} K (c - \overset{\mathbb{Z}}{c}{}^{(q,-1)})\,.$$

Hence the projection map $\pi : C^q_p \to \Omega^{(q,-1)}(\mathbb{Z})$ descends to an isomorphism in cohomology which proves the announced result (II).

A.6 The case $q = p$

A full description of $H^p(\mathcal{C}_p, D)$ is complicated in general, but it fits in all cases into an exact sequence of abelian groups[17]

$$0 \longrightarrow \left\{ \begin{array}{c} \text{Closed global } (p-1)\text{-forms} \\ \text{with integral periods} \end{array} \right\} \longrightarrow \Omega^{p-1}(M, \mathbb{R}) \longrightarrow H^p(\mathcal{C}_p, D) \longrightarrow H^p_{\check{\mathrm{C}}\mathrm{ech}}(M, \mathbb{Z}) \longrightarrow 0 \qquad (III)$$

Proof. We shall treat separately the cases $p = q = 0$ and $p = q \neq 0$, starting with the latter.

Let $c = (c^{(0,p-1)}, \cdots, c^{(p-1,0)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)}) \in C^p_p$ be a D-cocycle, then $\overset{\mathbb{Z}}{c}{}^{(p,-1)}$ is a Čech cocycle and $(*)$ tells us that

$$c - (1 + K\tilde{d})^{-1} \overset{\mathbb{Z}}{c}{}^{(p,-1)} = D(1 + K\tilde{d})^{-1} K \left(c - \overset{\mathbb{Z}}{c}{}^{(p,-1)} \right).$$

[17] For instance, when $p = 2$, we recover the classification of line bundles with connection modulo gauge equivalence, as expected. This case is treated in detail in appendix C.

However, we now have $K\tilde{d}(C_p^q) \subset C_p^q + \Omega^{(-1,q)}(\mathbb{R})$, in contrast with the previous case for which we had the inclusion $K\tilde{d}(C_p^q) \subset C_p^q$. Accordingly, as an element of $C_p^{p-1} + \Omega^{(-1,p-1)}(\mathbb{R})$, $(1+K\tilde{d})^{-1}K(c - \overset{\mathbb{Z}}{c}{}^{(p,-1)})$ has a component, say $\gamma^{(-1,p-1)}$, in $\Omega^{(-1,p-1)}(\mathbb{R})$, so we cannot conclude that c and $(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(p,-1)}$ are D-cohomologous. Nevertheless $\tilde{d}\gamma^{(-1,p-1)} = 0$ (not d !), hence c is D-cohomologous to $(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}$.

Conversely, the cochain $(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}$ is a Deligne-Beilinson cocycle whenever $\gamma^{(-1,p-1)}$ is a global de Rham $(p-1)$-form and $\overset{\mathbb{Z}}{c}{}^{(p,-1)} \in \Omega^{(p,-1)}(\mathbb{Z})$ is a Čech cocycle.

So we have exhibited a family of "reduced" representatives

$$(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}\,, \qquad (**)$$

of Deligne-Beilinson cohomology classes.

Decomposition $(**)$ leads us to consider the following maps

$$\pi : c = \left(c^{(0,p-1)}, \cdots, c^{(p-1,0)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)}\right) \in C_p^p \;\mapsto\; \overset{\mathbb{Z}}{c}{}^{(p,-1)} \in \Omega^{(p,-1)}(M,\mathbb{Z}),$$

(already met in subsection A.2), and

$$\phi : \gamma^{(-1,p-1)} \in \Omega^{(-1,p-1)}(\mathbb{R}) \;\mapsto\; (\delta\gamma^{(-1,p-1)}, 0, \ldots, 0) \in C_p^p\,.$$

We provide $\Omega^{(-1,p-1)}$ with the trivial differential $= 0$, so that π and ϕ are maps between complexes. It is quite easy to check that these two maps are chain maps, i.e. $\phi \cdot 0 = D \cdot \phi$ and $\pi \cdot D = \delta \cdot \pi$, hence, passing to cohomology,

$$\Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\text{Čech}}(M,\mathbb{Z})\,.$$

Let us show that $\hat{\pi}$ is surjective. First, by definition and with obvious notations,

$$\hat{\pi}([c]) := \left[\overset{\mathbb{Z}}{c}{}^{(p,-1)}\right].$$

For any class $\xi \in H^p_{\text{Čech}}(M,\mathbb{Z})$, let us pick a representative $\overset{\mathbb{Z}}{c}{}^{(p,-1)}$ of ξ. From $(**)$, we deduce that

$$c = (1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(p,-1)}$$

is a Deligne-Beilinson cocycle which trivially fulfills $\pi(c) = \overset{\mathbb{Z}}{c}{}^{(p,-1)}$, so that

$$\hat{\pi}([c]) = [\overset{\mathbb{Z}}{c}{}^{(p,-1)}] = \xi\,.$$

This means that any integral Čech cohomology class is the image under $\hat{\pi}$ of a Deligne-Beilinson cohomology class, thus establishing the surjectivity of $\hat{\pi}$.

According to this, we can extend further the previous exact sequence to the right

$$\Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\text{Čech}}(M,\mathbb{Z}) \longrightarrow 0.$$

Now, let us show that this sequence is actually exact on the left, that is to say $\mathrm{Ker}(\hat{\pi}) = \mathrm{Im}(\hat{\phi})$.

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If $[c] \in \mathrm{Ker}(\hat{\pi})$ then $\hat{\pi}([c]) = [\overset{\mathbb{Z}}{c}{}^{(p,-1)}] = 0$, meaning that any representative of $[\overset{\mathbb{Z}}{c}{}^{(p,-1)}]$ is a Čech coboundary, $\delta\overset{\mathbb{Z}}{\lambda}{}^{(p-1,-1)}$. Thus, if $(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(p,-1)} + \delta\gamma^{(-1,p-1)}$ is a "reduced" representative of $[c] \in \mathrm{Ker}(\hat{\pi})$, we have

$$c = (1+K\tilde{d})^{-1}\delta\overset{\mathbb{Z}}{\lambda}{}^{(p-1,-1)} + \delta\gamma^{(-1,p-1)} = Dq + \delta\rho^{(-1,p-1)}\,,$$

with $\rho^{(-1,p-1)} = \gamma^{(-1,p-1)} + (-K\tilde{d})^p\overset{\mathbb{Z}}{\lambda}{}^{(p-1,-1)}$. In other words, $\mathrm{Ker}(\hat{\pi})$ is made of Deligne-Beilinson classes $[c]$ that admit a representative of the form $\delta\rho^{(-1,p-1)}$ for some global form $\rho^{(-1,p-1)} \in \Omega^{(-1,p-1)}(\mathbb{R})$. Conversely, for any global form $\rho^{(-1,p-1)} \in \Omega^{(-1,p-1)}(\mathbb{R})$ the Deligne-Beilinson class $[\delta\rho^{(-1,p-1)}]$ trivially belongs to $\mathrm{Ker}(\hat{\pi})$. This implies $\mathrm{Ker}(\hat{\pi}) = \mathrm{Im}(\hat{\phi})$.

So, the sequence

$$\Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\check{\mathrm{C}}\mathrm{ech}}(M,\mathbb{Z}) \longrightarrow 0\,,$$

is exact, and to extend it to the left, we have to compute $\mathrm{Ker}(\hat{\phi})$.

If $\gamma^{(-1,p-1)} \in \mathrm{Ker}(\hat{\phi})$, then $\hat{\phi}(\gamma^{(-1,p-1)}) = [\delta\gamma^{(-1,p-1)}] = 0$, which means that any representative of $[\delta\gamma^{(-1,p-1)}]$ is a Deligne-Beilinson coboundary. In particular

$$\delta\gamma^{(-1,p-1)} = \left(\delta\gamma^{(-1,p-1)}, 0, \ldots, 0\right) = D\tau\,,$$

for some $\tau = (\tau^{(0,p-2)}, \cdots, \tau^{(p-2,0)}, \overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)}) \in C^{p-1}_p$. This gives rise to the following Čech-de Rham cochain

$$\left(-\gamma^{(-1,p-1)}, \tau^{(0,p-2)}, \cdots, \tau^{(p-2,0)}, \overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)}\right),$$

which turns out to be a Čech-de Rham cocycle since $\delta\gamma^{(-1,p-1)} = D\tau$. Now, from Weil's theorem (see subsection A.4.3) we conclude that since $\overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)}$ is integral the global form $\gamma^{(-1,p-1)}$ has integral periods. Conversely, if $\gamma^{(-1,p-1)}$ has integral periods then, still from Weil's theorem, it gives rise to an integral Čech-de Rham cocycle $(\tau^{(-1,p-1)} = -\gamma^{(-1,p-1)}, \tau^{(0,p-2)}, \cdots, \tau^{(p-2,0)}, \overset{\mathbb{Z}}{\tau}{}^{(p-1,-1)})$ such that $\delta\gamma^{(-1,p-1)} = D\tau$. This shows that $\mathrm{Ker}(\hat{\phi})$ is nothing else but the space of $(p-1)$-forms with integral periods. So we can extend our exact sequence to the left using the canonical injection of $(p-1)$-forms with integral periods into $(p-1)$-forms

$$\left\{\begin{array}{c}\text{Closed global } (p-1)\text{-forms}\\ \text{with integral periods}\end{array}\right\} \xrightarrow{i} \Omega^{(-1,p-1)}(\mathbb{R}) \xrightarrow{\hat{\phi}} H^p(\mathcal{C}_p, D) \xrightarrow{\hat{\pi}} H^p_{\check{\mathrm{C}}\mathrm{ech}}(M,\mathbb{Z}) \longrightarrow 0\,.$$

Finally, it is obvious that $\mathrm{Ker}(i) = 0$. This last point definitively establishes the exactness of (III) for $p = q \neq 0$.

In the special case $p = q = 0$, identity $(**)$ reads

$$(1+K\tilde{d})^{-1}\,\overset{\mathbb{Z}}{c}{}^{(0,-1)} + \delta\gamma^{(-1,-1)} = \overset{\mathbb{Z}}{c}{}^{(0,-1)} + \delta\gamma^{(-1,-1)},$$

where $\gamma^{(-1,-1)}$ is just a real number. This means that reduced representatives of $[c] \in H^0(\mathcal{C}_0, D)$ are integral Čech cohomology classes (canonically imbedded in the real Čech cohomology). Conversely, any Čech cohomology class $\xi \in H^0_{\text{Čech}}(M, \mathbb{Z})$ defines a Deligne-Beilinson class $[(1+K\tilde{d})^{-1} \overset{\mathbb{Z}}{c}{}^{(0,-1)}]$, i.e. $H^0(\mathcal{C}_0, D) \simeq H^0_{\text{Čech}}(M, \mathbb{Z})$. As a side note, this can be combined with (II) to yield the more general result

$$H^q(\mathcal{C}_0, D) \simeq H^q_{\text{Čech}}(M, \mathbb{Z})\,.$$

If $H^p_{\text{Čech}}(M, \mathbb{Z})$ has no torsion, the sequence (III) is split : choose a basis of $H^p_{\text{Čech}}(M, \mathbb{Z})$, take a representative Čech cocycle in $\Omega^p(M, \mathbb{Z})$ for each basis element, and multiply it by $(1+K\tilde{d})^{-1}$ to get a Deligne-Beilinson cocycle, then extend by linearity. This gives an injection of $H^p_{\text{Čech}}(M, \mathbb{Z})$ into $H^p(\mathcal{C}_p, D)$ which is isomorphic (as an abelian group, but in a non canonical way) to

$$H^p_{\text{Čech}}(M, \mathbb{Z}) \oplus \Omega^{p-1}(M, \mathbb{R}) / \left\{ \begin{array}{c} \text{Closed global } (p-1) \text{ forms} \\ \text{with integral periods} \end{array} \right\}.$$

If $H^p_{\text{Čech}}(M, \mathbb{Z})$ has torsion there is no splitting and the above description is not correct. Finally note the special case $p = q = 1$: $H^1(\mathcal{C}_1, D)$ is canonically isomorphic to $C^\infty(M, \mathbb{R}/\mathbb{Z})$, the multiplicative group of smooth functions from M to the circle group, a more compact description than the one given by the exact sequence (III).

A.7 The isomorphism between Cheeger-Simons differential characters and Deligne-Beilinson classes for $q = p$

The Deligne-Beilinson cohomology group can be imbedded into another exact sequence

$$0 \longrightarrow H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow H^p(\mathcal{C}_p, D) \longrightarrow \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) \longrightarrow 0,$$

which fits better with the representation we have chosen for the classes, namely:

$$\omega = (\omega^{(0,p-1)}, \cdots, \omega^{(p-1,0)}, \overset{\mathbb{Z}}{\omega}{}^{(p,-1)})\,.$$

On the other hand, the Cheeger-Simons differential character group $\hat{H}^p(M, \mathbb{R}/\mathbb{Z})$ can also be imbedded into the same exact sequence [2, 14]

$$0 \longrightarrow H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \hat{H}^p(M, \mathbb{R}/\mathbb{Z}) \longrightarrow \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) \longrightarrow 0.$$

These two sequences can be combined into the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) & \longrightarrow & H^p(\mathcal{C}_p, D) & \longrightarrow & \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) & \longrightarrow & 0 \\ & & \text{id} \updownarrow & & \downarrow \int & & \text{id} \updownarrow & & \\ 0 & \longrightarrow & H^{p-1}_{\text{Čech}}(M, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \hat{H}^p(M, \mathbb{R}/\mathbb{Z}) & \longrightarrow & \Omega^p_{\mathbb{Z}}(M, \mathbb{R}) & \longrightarrow & 0 \end{array}$$

in which the descending map in the middle -$\int$- is given by (3.21). Then by the *5-Lemma* this map is an isomorphism.

A.8 The Deligne-Beilinson cohomology is the same for all good coverings

A proof is needed only when $q = p$, because in the other cases, we have given canonical isomorphisms with standard Čech cohomology spaces.

If the simple covering $\mathcal{V}$ of M is a refinement of the simple covering $\mathcal{U}$, it is a classical theorem that for Čech cohomology the restriction chain map induces an isomorphism in cohomology. This isomorphism is canonical because restriction is canonical.

We use this as a starting point to prove the corresponding result for Deligne-Beilinson cohomology. To avoid notational ambiguities, we write $\mathcal{C}_p(\mathcal{U})$ (resp. $\mathcal{C}_p(\mathcal{V})$) for the Deligne-Beilinson complex for the covering $\mathcal{U}$ (resp. $\mathcal{V}$).

Restriction gives a chain map from the complex $(\mathcal{C}_p(\mathcal{U}), D)$ to the complex $(\mathcal{C}_p(\mathcal{V}), D)$. So there is a canonical homomorphism

$$H^p(\mathcal{C}(\mathcal{U})_p, D) \overset{\text{restriction}}{\longrightarrow} H^p(\mathcal{C}(\mathcal{V})_p, D)\,.$$

We want to show that this homomorphism is one-to-one onto.[18] We start by showing that the homomorphism is one to one. Suppose that an element of $H^p(\mathcal{C}(\mathcal{U})_p, D)$, represented by a certain $c = (c^{(0,p-1)}, \cdots, c^{(p-1,0)}, \overset{\mathbb{Z}}{c}{}^{(p,-1)}) \in \mathcal{C}(\mathcal{U})^p_p$, maps to the trivial element in $H^p(\mathcal{C}(\mathcal{V})_p, D)$. This implies that the restriction of $c^{(p,-1)}$ to the covering $\{\mathcal{V}_\sigma\}_{\sigma\in J}$ is a trivial Čech cocycle, and by the isomorphism theorem for Čech cohomology, $c^{(p,-1)}$ itself is trivial. From the previous section, we know then that c is Deligne-Beilinson cohomologous to some $\delta\gamma^{(p-1)}$ where $\gamma^{(p)}$ is a global de Rham $(p-1)$ form, so we can assume that $c = \delta\gamma^{(p-1)}$ to start with. The condition of triviality is then the same for both coverings, i.e. $\gamma^{(p-1)}$ has to be closed with integral periods. We have proved that in the diagram

$$\begin{array}{ccccc}
0 & & & & \\
\downarrow & & & & \\
H^p(\mathcal{C}(\mathcal{U})_p, D) & \longrightarrow & H^p_{\text{Čech}}(M,\mathbb{Z}) & \longrightarrow & 0 \\
\downarrow r & & \downarrow \text{Id} & & \\
H^p(\mathcal{C}(\mathcal{V})_p, D) & \longrightarrow & H^p_{\text{Čech}}(M,\mathbb{Z}) & \longrightarrow & 0
\end{array}$$

the first column is exact (i.e. restriction is one to one) and the kernels of the top and bottom rows are canonically isomorphic via restriction.

To prove that the restriction map is onto, we take compatible partitions of unity $\{\vartheta_\alpha\}$ and $\{\varphi_\sigma\}$ for $\mathcal{U}$ and its refinement $\mathcal{V}$. Take a class in $H^p(\mathcal{C}(\mathcal{V})_p, D)$, represented by a cocycle $s = (s^{(0,p-1)}, \cdots, s^{(p-1,0)}, s^{(p,-1)})$ in $\mathcal{C}(\mathcal{V})^p_p$. Then $s^{(p,-1)}$ is a Čech cocycle for $\mathcal{V}$. If $s^{(p-1,-1)}$ is an integral Čech cochain of degree $(p-1)$ for $\mathcal{V}$, $s + Ds^{(p-1,-1)} = (\cdots, s^{(p,-1)} + \delta s^{(p-1,-1)})$ represents the same Deligne-Beilinson class, so by the isomorphism theorem for Čech cohomology, we can assume without loss of generality that $s^{(p,-1)}$ is the restriction of a Čech cocycle $c^{(p,-1)}$ for $\{\mathcal{U}_\alpha\}_{\alpha\in I}$. We have proved in the previous section

[18]The general canonical isomorphism theorem for two (arbitrary) simple coverings is an automatic consequence of the fact that on a compact manifold two simple coverings have a common simple refinement.

that s is Deligne-Beilinson cohomologous to $(1+K\tilde{d})^{-1}s^{(p,-1)} + \delta\gamma^{(p-1)} = (\ldots, s^{(p,-1)})$ for some global de Rham $(p-1)$-form $\gamma^{(p-1)}$ (in this formula, K, $\tilde{d}$ and δ are with respect to the covering $\mathcal{V}$) so we can assume without loss of generality that s is of that form to start with. Then $(1+K\tilde{d})^{-1}c^{(p,-1)} + \delta\gamma^{(p-1)}$ (where now K, $\tilde{d}$ and δ are with respect to the covering $\mathcal{U}$) is a Deligne-Beilinson cocycle, and (as restriction commutes with K, $\tilde{d}$ and δ), $s = r(c)$. So each element of $H^p(\mathcal{C}(\mathcal{V})_p, D)$ has a representative which is the restriction of a Deligne-Beilinson p-cocycle for $\mathcal{U}$: the restriction chain map leads to a surjective map in Deligne-Beilinson cohomology. Putting things together, the proof that restriction induces a canonical bijection from $H^p(\mathcal{C}(\mathcal{U})_p, D)$ to $H^p(\mathcal{C}(\mathcal{V})_p, D)$ is complete.

B. Deligne-Beilinson dual of a cycle

In this section we present a construction of a "cycle map" which associates a Deligne-Beilinson cohomology class to a given cycle. The kind of duality that is implied is not of the "Poincaré" type, but is rather an analog of Pontrjagin duality for Deligne-Beilinson cohomology.

Let z_p be a *singular* or rather a *de Rham* (cf. section 3.4) integral p-cycle of M and $\mathcal{U}$ a simple cover. We perform the following descent[19] using the singular boundary operator, b, and the Čech coboundary operator, δ:

$$(\delta z_p)_{\alpha_0} = z_{p\,|\alpha_0} = b\; c^0_{p+1,\,\alpha_0} \quad \text{in} \quad \mathcal{U}_{\alpha_0}\,. \tag{B.1}$$

Then

$$b\,(c^0_{p+1,\,\alpha_1} - c^0_{p+1,\,\alpha_0}) = z_{p\,|\alpha_1} - z_{p\,|\alpha_0} = 0 \quad \text{in} \quad \mathcal{U}_{\alpha_0\alpha_1}, \tag{B.2}$$

so that

$$\left(\delta\, c^0_{p+1}\right)_{\alpha_0\alpha_1} := c^0_{p+1,\,\alpha_1} - c^0_{p+1,\,\alpha_0} = b\, c^1_{p+2,\,\alpha_0\alpha_1} \quad \text{in} \quad \mathcal{U}_{\alpha_0\alpha_1}\,. \tag{B.3}$$

This descent goes on at level k (the fact that the covering is simple is crucial):

$$\delta\, c^k_{p+k+1} = b\, c^{k+1}_{p+k+2} \tag{B.4}$$

and stops for $k = n-p-2$

$$\delta\, c^{n-p-2}_{n-1} = b\; c^{n-p-1}_n\,. \tag{B.5}$$

As usual c^{k+1}_{p+k+2} is defined in $\mathcal{U}_{\alpha_0\cdots\alpha_{k+1}}$.

Finally,

$$\delta\, c^{n-p-1}_n = c^{n-p}_n \quad \text{with} \quad b\, c^{n-p}_n = 0\,, \tag{B.6}$$

in each $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$. Hence every $c^{n-p}_n{}_{,\,\alpha_0\cdots\alpha_{n-p}}$ is a integral n-cycle in $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$, so that we can write

$$c^{n-p}_n{}_{,\,\alpha_0\cdots\alpha_{n-p}} = \overset{\mathbb{Z}}{\eta}_{z,\,\alpha_0\cdots\alpha_{n-p}} \cdot \mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}\,, \tag{B.7}$$

once $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$ has been identified with a singular n-cycle in a natural way. Furthermore, the $\overset{\mathbb{Z}}{\eta}_{z,\,\alpha_0\cdots\alpha_{n-p}}$'s define a Čech cocycle in an obvious way. In terms of de Rham currents

$$c^k_{p+k+1} \longrightarrow \eta_z{}^{(k,n-p-k-1)}, \tag{B.8}$$

[19] All chains involved below are integral chains.

the above descent equations read

$$\delta\,\eta_z{}^{(k,n-p-k-1)} = d\eta_z{}^{(k+1,n-p-k-2)} \cdots \delta\,\eta_z{}^{(n-p-2,1)} = d\eta_z{}^{(n-p-1,0)} . \tag{B.9}$$

Now

$$\delta\,\eta_z{}^{(n-p-1,0)} = d_{-1}\eta_z{}^{(n-p,-1)} ,$$

where one can show, using integration of n-forms with compact supports in $\mathcal{U}_{\alpha_0\cdots\alpha_{n-p}}$, that

$$\eta^{(n-p,-1)}_{z,\alpha_0\cdots\alpha_{n-p}} = \overset{\mathbb{Z}}{\eta}_{z,\,\alpha_0\cdots\alpha_{n-p}} .$$

Therefore the sequence

$$\eta^{(n-p-1)}_{\mathcal{D}}(z) = (\eta_z{}^{(0,n-p-1)}, \ldots, \eta_z{}^{(n-p-1,0)}, \overset{\mathbb{Z}}{\eta}_z)$$

fulfilling the descent (B.9) is nothing but a Deligne-Beilinson cocycle with *distribution* coefficients.

The singular homology that was used here (in the intersections of the simple covering) is not the usual one (i.e. with compact support), but rather the "infinite" one where chains may have non-compact supports. Accordingly, the corresponding currents do not necessarily have compact support in the intersections either. Moreover, the Čech cocycle $\overset{\mathbb{Z}}{\eta}_z$ is *a priori* non trivial since it is obtained from a Čech-de Rham descent of the *a priori* non trivial integration current of z. In fact, $\overset{\mathbb{Z}}{\eta}_z$ is a Čech representative of the Poincaré dual of z on M.

Let us have a look at the ambiguities of the descent of the p-cycle z which led to $\eta^{(n-p-1)}_{\mathcal{D}}(z)$. At the level of the currents $\eta_z{}^{(n-p-k,k-1)}$, one can check that ambiguities of Deligne-Beilinson type (3.18) are obviously present. However, since our starting point is the integral current of z, we could also have ambiguities on $\eta_z{}^{(0,n-p-1)}$ corresponding to the restriction of a globally defined closed $(n-p-1)$-current, $\delta\eta_z{}^{(-1,n-p-1)}$. But, since all the currents of our descent **must** be integration currents of integral chains, $\delta\eta_z{}^{(-1,n-p-1)}$ must necessarily be the integration current of a $(p+1)$-cycle. Hence, it produces a Deligne-Beilinson ambiguity. The same argument holds at the bottom of the descent, where our integral chains will only produce integral Čech cochain ambiguities, which are also of Deligne-Beilinson type. In other words, the fact we use integral chains to produce a Deligne-Beilinson cocycle provides us with a canonical Deligne-Beilinson class $[\eta^{(n-p-1)}_{\mathcal{D}}(z)]$ associated with z.[20]

C. U(1) connections as Deligne-Beilinson cohomology classes

Let us briefly recall how connections over U(1)-bundles are related to Deligne-Beilinson cohomology classes [13]. Let $P := P(M, \mathrm{U}(1), E, \pi)$ be a principal U(1)-bundle with total space E over M and projection π. For a given **simple** open covering of $M, \mathcal{U}$, P is described by transition functions $g_{\alpha\beta} : \mathcal{U}_{\alpha\beta} \mapsto \mathrm{U}(1)$ which satisfy the cocycle condition

$$g_{\alpha_0\alpha_1} g_{\alpha_1\alpha_2} g_{\alpha_2\alpha_0} = 1 , \tag{C.1}$$

[20]This result can be obtained using *integrally flat* currents defined in [28], see also [14].

in any intersection $\mathcal{U}_{\alpha_0\alpha_1\alpha_2}$, or equivalently

$$\Lambda_{\alpha_0\alpha_1} + \Lambda_{\alpha_1\alpha_2} + \Lambda_{\alpha_2\alpha_0} := n_{\alpha_0\alpha_1\alpha_2} \in \mathbb{Z}\,, \tag{C.2}$$

with

$$g_{\alpha_0\alpha_1} = \exp(2i\pi\Lambda_{\alpha_0\alpha_1})\,. \tag{C.3}$$

Trivially

$$n_{\alpha_0\alpha_1\alpha_2} - n_{\alpha_0\alpha_1\alpha_3} + n_{\alpha_0\alpha_2\alpha_3} - n_{\alpha_1\alpha_2\alpha_3} = 0\,, \tag{C.4}$$

in $\mathcal{U}_{\alpha_0\alpha_1\alpha_2\alpha_3}$, which means that the collection $n^{(2,-1)}$ defined by these integers is an **integral Čech 2-cocycle** on M.

Given a collection of local sections, a connection $\tilde{A}$ on P induces a collection $(A)_\alpha$ of locally defined 1-forms on M which glue together on every $\mathcal{U}_{\alpha_0\alpha_1}$ according to

$$A_{\alpha_1} - A_{\alpha_0} = g^{-1}_{\alpha_0\alpha_1} dg_{\alpha_0\alpha_1} \;=\; (2i\pi)\, d\Lambda_{\alpha_0\alpha_1}\,. \tag{C.5}$$

We then obtain a family

$$(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)}) \in \check{C}^{(0)}(\mathcal{U},\Omega^1(M)) \times \check{C}^{(1)}(\mathcal{U},\Omega^0(M)) \times \check{C}^{(2)}(\mathcal{U},\mathbb{Z})\,, \tag{C.6}$$

such that

$$\begin{aligned}
(\delta A^{(0,1)})_{\alpha_0\alpha_1} &:= A_{\alpha_1} - A_{\alpha_0} = (2i\pi)\, d\Lambda_{\alpha_0\alpha_1}\,, \\
(\delta \Lambda^{(1,0)})_{\alpha_0\alpha_1\alpha_2} &:= \Lambda_{\alpha_0\alpha_1} + \Lambda_{\alpha_1\alpha_2} + \Lambda_{\alpha_2\alpha_0} = d_{-1} n_{\alpha_0\alpha_1\alpha_2} := n_{\alpha_0\alpha_1\alpha_2}\,, \\
(\delta n^{(2,-1)})_{\alpha_0\alpha_1\alpha_2\alpha_3} &:= n_{\alpha_0\alpha_1\alpha_2} - n_{\alpha_0\alpha_1\alpha_3} + n_{\alpha_0\alpha_2\alpha_3} - n_{\alpha_1\alpha_2\alpha_3} = 0\,,
\end{aligned} \tag{C.7}$$

in the appropriate intersections. As described in detail above such a sequence makes up a **Deligne-Beilinson cocycle**.

The curvature of $\tilde{A}$ also admits canonical local representatives on M, $F_\alpha := dA_\alpha$, which are globally defined since

$$F_{\alpha_1} - F_{\alpha_0} = d(A_{\alpha_1} - A_{\alpha_0}) = 2i\pi d(d\Lambda_{\alpha_0\alpha_1}) = 0\,, \tag{C.8}$$

Obviously, the existence of F on M is a direct consequence of the existence of $A^{(0,1)}$, and we can formally write "$F = dA^{(0,1)}$".

For a given triple $(\mathcal{U}, P, \tilde{A})$ the Deligne-Beilinson cocycle $(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)})$ is not unique. More precisely, ambiguities on the local representatives of P and $\tilde{A}$ (that is allowed changes of transition functions and local sections) induce ambiguities on the Deligne-Beilinson cocycle (C.6) of the following form

$$\Big(dq^{(0,0)}, \delta q^{(0,0)} + d_{-1} m^{(1,-1)}, \delta m^{(1-,1)}\Big)\,, \tag{C.9}$$

with $(m^{(1,-1)}, q^{(0,0)}) \in \check{C}^{(1)}(\mathcal{U},\mathbb{Z}) \times \check{C}^{(0)}(\mathcal{U},\Omega^0(M))$. Such ambiguities correspond precisely to Deligne-Beilinson coboundaries and thus represent the ambiguities among the representatives of the relevant Deligne-Beilinson cohomology classes.[21]

[21] See appendix A

Two triples $(\mathcal{U}, P, \tilde{A})$ and $(\mathcal{U}, P', \tilde{A}')$ are said to be U(1)-*equivalent* if there is a U(1) isomorphism $\Phi : P \mapsto P'$, such that $\tilde{A}' = \Phi_* \tilde{A}$. Locally, this means that the transition functions of P and P' are related according to

$$g'_{\alpha\beta} = h_\alpha{}^{-1} \cdot g_{\alpha\beta} \cdot h_\beta \,, \tag{C.10}$$

or equivalently

$$\Lambda'_{\alpha\beta} = \Lambda_{\alpha\beta} + q_\beta - q_\alpha \,, \tag{C.11}$$

where the $h_\alpha = \exp(2i\pi q_\alpha)$. In the same way the local representatives of the connections fulfill

$$A'{}_\alpha = A_\alpha + 2i\pi dq_\alpha \,. \tag{C.12}$$

Then we clearly see that these relations assume the same form as the ambiguities in (C.9), showing that two equivalent triples are associated to the same Deligne-Beilinson cohomology class in $H^2_{\mathcal{D}}(M, \mathbb{Z}(2))$.

This correspondence can be established in the reverse way. Indeed, consider a representative $(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)})$ of a Deligne-Beilinson cohomology class, the U(1)-valued mappings $g_{\alpha\beta} := \exp 2i\pi\Lambda_{\alpha\beta}$ are U(1) transition functions over $\mathcal{U}$ since they satisfy the cocycle condition (C.1). With these functions, one can canonically build a principal U(1)-bundle over M, $P(M, \mathrm{U}(1), E, \pi)$ [29, 30]. Furthermore, there is only one connection $\tilde{A}$ on P whose local representatives on M coincide with those of $A^{(0,1)}$. Hence our Deligne-Beilinson cocycle defines a couple $(P, \tilde{A})$ in a canonical way.

Now, with another representative, $(A^{(0,1)} + dq^{(0,0)}, \Lambda^{(1,0)} + \delta q^{(0,0)}, n^{(2,-1)})$, we obtain another set of transition functions which defines an equivalent principal bundle — cf. (C.10). In the same way, $A^{(0,1)} + dq^{(0,0)}$ is related to $\tilde{A}$ through a U(1)-bundle isomorphism.

Finally, a representative $(A^{(0,1)}, \Lambda^{(1,0)} + m^{(1,-1)}, n^{(2,-1)} + \delta m^{(1,-1)})$ gives the same transition functions and the same connection. This establishes that the Deligne-Beilinson cohomology class of $(A^{(0,1)}, \Lambda^{(1,0)}, n^{(2,-1)})$ can be canonically associated to a whole class of U(1)-equivalent triples $(\mathcal{U}, P, \tilde{A})$.

The independence of this isomorphism upon the chosen covering $\mathcal{U}$ of M is a direct consequence of the results proven in (A.8).

Note added. While completing this paper, we became aware of the recent mathematical work of R. Harvey, B. Lawson and J. Zweck [14], who discuss in detail the Pontrjagin duality we use in section 3.3. The authors emphasize the differential character point of view rather than the Deligne-Beilinson one we have adopted here.

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Deligne–Beilinson Cohomology and Abelian Link Invariants

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Abstract. For the Abelian Chern–Simons field theory, we consider the quantum functional integration over the Deligne–Beilinson cohomology classes and we derive the main properties of the observables in a generic closed orientable 3-manifold. We present an explicit path-integral non-perturbative computation of the Chern–Simons link invariants in the case of the torsion-free 3-manifolds S^3, $S^1 \times S^2$ and $S^1 \times \Sigma_g$.

Key words: Deligne–Beilinson cohomology; Abelian Chern–Simons; Abelian link invariants

2000 Mathematics Subject Classification: 81T70; 14F43; 57M27

1 Introduction

The topological quantum field theory which is defined by the Chern–Simons action can be used to compute invariants of links in 3-manifolds [1, 2, 3, 4]. The algebraic structure of these invariants, which is based on the properties of the characters of simple Lie groups, is rather general. In fact, these invariants can also be defined by means of skein relations or of quantum group Hopf algebra methods [5, 6].

In the standard quantum field theory approach, the gauge invariance group of the Abelian Chern–Simons theory is given by the set of local $U(1)$ gauge transformations and the observables can directly be computed by means of perturbation theory when the ambient space is $\mathbb{R}^3$ (the result also provides the values of the link invariants in S^3). For a nontrivial 3-manifold M_3, the standard gauge theory approach presents some technical difficulties, and one open problem of the quantum Chern–Simons theory is to produce directly the functional integration in the case of a generic 3-manifold M_3. In this article we will show how this can be done, at least for a certain class of nontrivial 3-manifolds, by using the Deligne–Beilinson cohomology. We shall concentrate on the Abelian Chern–Simons invariants; hopefully, the method that we present will possibly admit an extension to the non-Abelian case.

The Deligne–Beilinson approach presents some remarkable aspects. The space of classical field configurations which are factorized out by gauge invariance is enlarged with respect to the standard field theory formalism. Indeed, assuming that the quantum amplitudes given by the exponential of the holonomies – which are associated with oriented loops — represent a complete set of observables, the functional integration must locally correspond to a sum over 1-forms modulo forms with integer periods, i.e. it must correspond to a sum over Deligne–Beilinson classes. In this new approach, the structure of the functional space admits a natural description in terms of the homology groups of the 3-manifold M_3. This structure will be used to compute the Chern–Simons observables, without the use of perturbation theory, on a class of torsion-free manifolds.

The article is organized as follows. Section 2 contains a description of the basic properties of the Deligne–Beilinson cohomology and of the distributional extension of the space of the equivalence classes. The framing procedure is introduced in Section 3. The general properties of the Abelian Chern–Simons theory are discussed in Section 4; in particular, non-perturbative proofs of the colour periodicity, of the ambient isotopy invariance and of the satellite relations are given. The solution of the Chern–Simons theory on S^3 is presented in Section 5. The computations of the observables for the manifolds $S^1 \times S^2$ and $S^1 \times \Sigma_g$ are produced in Sections 6 and 7. Section 8 contains a brief description of the surgery rules that can be used to derive the link invariants in a generic 3-manifold, and it is checked that the results obtained by means of the Deligne–Beilinson cohomology and by means of the surgery method coincide. Finally, Section 9 contains the conclusions.

2 Deligne–Beilinson cohomology

The applications of the Deligne–Beilinson (DB) cohomolgy [7, 8, 9, 10, 11] – and of its various equivalent versions such as the Cheeger–Simons Differential Characters [12, 13] or Sparks [14] – in quantum physics has been discussed by various authors [15, 16, 17, 18, 19, 21, 20, 22, 23]. For instance, geometric quantization is based on classes of $U(1)$-bundles with connections, which are exactly DB classes of degree one (see Section 8.3 of [24]); and the Aharanov–Bohm effect also admits a natural description in terms of DB cohomology.

In this article, we shall consider the computation of the Abelian link invariants of the Chern–Simons theory by means of the DB cohomology. Let L be an oriented (framed and coloured) link in the 3-manifold M_3; one is interested in the ambient isotopy invariant which is defined by the path-integral expectation value

$$\left\langle \exp\left\{2i\pi \int_L A\right\}\right\rangle_k \equiv \frac{\int DA \exp\left\{2i\pi k \int_{M_3} A\wedge dA\right\} \exp\left\{2i\pi \int_L A\right\}}{\int DA \exp\left\{2i\pi k \int_{M_3} A\wedge dA\right\}}, \tag{2.1}$$

where the parameter k represents the dimensionless coupling constant of the field theory. In equation (2.1), the holonomy associated with the link L is defined in terms of a $U(1)$-connection A on M_3; this holonomy is closely related to the classes of $U(1)$-bundles with connections that represent DB cohomology classes. The Chern–Simons lagrangian $A\wedge dA$ can be understood as a DB cohomology class from the Cheeger–Simons Differential Characters point of view, and it can also be interpreted as a DB "square" of A which is defined, as we shall see, by means of the DB $*$-product.

To sum up, the DB cohomology appears to be the natural framework which should be used in order to compute the Chern–Simons expectation values (2.1). As we shall see, this will imply the quantization of the coupling constant k and it will actually provide the integration measure DA with a nontrivial structure which is related to the homology of the manifold M_3. It should be noted that the gauge invariance of the Chern–Simons action and of the observables is totally included into the DB setting: working with DB classes means that we have already taken the quotient by gauge transformations.

Although we won't describe DB cohomology in full details, we shall now present a few properties of the DB cohomology that will be useful for the non-perturbative computation of the observables (2.1).

2.1 General properties

Let M be a smooth oriented compact manifold without boundary of finite dimension n. The Deligne cohomology group of M of degree q, $H_D^q(M,\mathbb{Z})$, can be described as the central term of the following exact sequence

$$0 \longrightarrow \Omega^q(M)/\Omega^q_{\mathbb{Z}}(M) \longrightarrow H_D^q(M,\mathbb{Z}) \longrightarrow H^{q+1}(M,\mathbb{Z}) \longrightarrow 0, \tag{2.2}$$

where $\Omega^q(M)$ is the space of smooth q-forms on M, $\Omega^q_{\mathbb{Z}}(M)$ the space of smooth closed q-forms with integral periods on M and $H^{q+1}(M,\mathbb{Z})$ is the $(q+1)^{th}$ integral cohomology group of M. This last space can be taken as simplicial, singular or Cech. There is another exact sequence into which $H_D^q(M,\mathbb{Z})$ can be embedded, namely

$$0 \longrightarrow H^q(M,\mathbb{R}/\mathbb{Z}) \longrightarrow H_D^q(M,\mathbb{Z}) \longrightarrow \Omega^{q+1}_{\mathbb{Z}}(M) \longrightarrow 0, \tag{2.3}$$

where $H^q(M,\mathbb{R}/\mathbb{Z})$ is the $\mathbb{R}/\mathbb{Z}$-cohomology group of M [11, 14, 25].

One can compute $H_D^q(M,\mathbb{Z})$ by using a (hyper) cohomological resolution of a double complex of Cech–de Rham type, as explained for instance in [9, 25]. In this approach, $H_D^q(M,\mathbb{Z})$ appears as the set of equivalence classes of DB cocycles which are defined by sequences $(\omega^{(0,q)}, \omega^{(1,q-1)}, \ldots, \omega^{(q,0)}, \omega^{(q+1,-1)})$, where $\omega^{(p,q-p)}$ denotes a collection of smooth $(q-p)$-forms in the intersections of degree p of some open sets of a good open covering of M, and $\omega^{(q+1,-1)}$ is an *integer* Cech $(p+1)$-cocycle for this open good covering of M. These forms satisfy cohomological descent equations of the type $\delta\omega^{(p-1,q-p+1)} + d\omega^{(p,q-p)} = 0$, and the equivalence relation is defined via the δ and d operations, which are respectively the Cech and de Rham differentials. The Cech–de Rham point of view has the advantage of producing "explicit" expressions for representatives of DB classes in some good open covering of M.

Definition 2.1. Let ω be a q-form which is globally defined on the manifold M. We shall denote by $[\omega] \in H_D^q(M,\mathbb{Z})$ the DB class which, in the Cech–de Rham double complex approach, is represented by the sequence $(\omega^{(0,q)} = \omega, \omega^{(1,q-1)} = 0, \ldots, \omega^{(q,0)} = 0, \omega^{(q+1,-1)} = 0)$.

From sequence (2.2) it follows that $H_D^q(M,\mathbb{Z})$ can be understood as an affine bundle over $H^{q+1}(M,\mathbb{Z})$, whose fibres have a typical underlying (infinite dimensional) vector space structure given by $\Omega^q(M)/\Omega^q_{\mathbb{Z}}(M)$. Equivalently, $\Omega^q(M)/\Omega^q_{\mathbb{Z}}(M)$ canonically acts on the fibres of the bundle $H_D^q(M,\mathbb{Z})$ by translation. From a geometrical point of view, $H_D^1(M,\mathbb{Z})$ is canonically isomorphic to the space of equivalence classes of $U(1)$-principal bundles with connections over M (see for instance [14, 25]). A generalisation of this idea has been proposed by means of Abelian Gerbes (see for instance [11, 26]) and Abelian Gerbes with connections over M. In this framework, $H^{q+1}(M,\mathbb{Z})$ classifies equivalence classes of some Abelian Gerbes over M, in the same way as $H^2(M,\mathbb{Z})$ is the space which classifies the $U(1)$-principal bundles over M, and $H_D^q(M,\mathbb{Z})$ appears as the set of equivalence classes of some Abelian Gerbes with connections. Finally, the space $\Omega^q_{\mathbb{Z}}(M)$ can be interpreted as the group of generalised Abelian gauge transformations.

We shall mostly be concerned with the cases $q = 1$ and $q = 3$. As for M, we will consider the three dimensional cases $M_3 = S^3$, $M_3 = S^1 \times S^2$ and $M_3 = S^1 \times \Sigma_g$, where Σ_g is a Riemann surface of genus $g \geq 1$. In particular, M_3 is oriented and torsion free. In all these cases, the exact sequence (2.2) for $q = 3$ reads

$$0 \longrightarrow \Omega^3(M_3)/\Omega^3_{\mathbb{Z}}(M_3) \longrightarrow H_D^3(M_3,\mathbb{Z}) \longrightarrow H^4(M_3,\mathbb{Z}) = 0 \longrightarrow 0,$$

where the first non trivial term reduces to

$$\frac{\Omega^3(M_3)}{\Omega^3_{\mathbb{Z}}(M_3)} \cong \frac{\mathbb{R}}{\mathbb{Z}}. \tag{2.4}$$

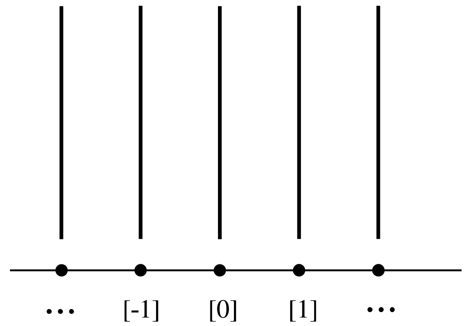

Figure 1. Presentation of the Deligne–Beilinson affine bundle $H_D^1\left(S^1\times S^2,\mathbb{Z}\right)$.

The validity of equation (2.4) can easily be checked by using a volume form on M_3. By definition, for any $(\rho,\tau_{\mathbb{Z}})\in\Omega^3(M_3)\times\Omega^3_{\mathbb{Z}}(M_3)$ one has

$$[\rho+\tau_{\mathbb{Z}}]=[\rho]\in H_D^3(M_3,\mathbb{Z})\,;$$

consequently

$$H_D^3(M_3,\mathbb{Z})\simeq\frac{\Omega^3(M_3)}{\Omega^3_{\mathbb{Z}}(M_3)}\cong\frac{\mathbb{R}}{\mathbb{Z}}.$$

These results imply that any Abelian 2-Gerbes on M_3 is trivial ($H^4(M_3,\mathbb{Z})=0$), and the set of classes of Abelian 2-Gerbes with connections on M_3 is isomorphic to $\mathbb{R}/\mathbb{Z}$. In the less trivial case $q=1$, sequence (2.2) reads

$$0\longrightarrow\Omega^1(M_3)/\Omega^1_{\mathbb{Z}}(M_3)\longrightarrow H_D^1(M_3,\mathbb{Z})\longrightarrow H^2(M_3,\mathbb{Z})\longrightarrow 0.\tag{2.5}$$

Still by definition, for any $(\eta,\omega_{\mathbb{Z}})\in\Omega^1(M_3)\times\Omega^1_{\mathbb{Z}}(M_3)$ one has

$$[\eta+\omega_{\mathbb{Z}}]=[\eta]\in H_D^1(M_3,\mathbb{Z})\,.$$

When $H^2(M_3,\mathbb{Z})=0$, sequence (2.5) turns into a short exact sequence; this also implies $H^1(M_3,\mathbb{Z})=0$ due to Poincaré duality. For the 3-sphere S^3, the base space of $H_D^1\left(S^3,\mathbb{Z}\right)$ is trivial. Whereas, the bundle $H_D^1\left(S^1\times S^2,\mathbb{Z}\right)$ has base space $H^2\left(S^1\times S^2,\mathbb{Z}\right)\cong\mathbb{Z}$ and, as depicted in Fig. 1, its fibres are (infinite dimensional) affine spaces whose underlying linear space identifies with the quotient space $\Omega^1\left(S^1\times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1\times S^2\right)$. In the general case $M_3=S^1\times\Sigma_g$ with $g\geq 1$, the base space $H^2\left(S^1\times\Sigma_g,\mathbb{Z}\right)$ is isomorphic to $\mathbb{Z}^{2g+1}$.

Finally, one should note that sequence (2.5) also gives information on $\Omega^1_{\mathbb{Z}}(M_3)$ since its structure is mainly given by the $H_D^1(M_3,\mathbb{Z})$. For instance, $\Omega^1_{\mathbb{Z}}\left(S^3\right)=d\Omega^0\left(S^3\right)$, all other cases being not so trivial.

2.2 Holonomy and pairing

As we have already mentioned, DB cohomology is the natural framework in which integration (or holonomy) of a $U(1)$-connection over 1-cycles of M_3 can be defined and generalised to objects of higher dimension (n-connections and n-cycles). In fact integration of a DB cohomology class $[\chi]\in H_D^q(M,\mathbb{Z})$ over a q-cycle of M, denoted by $C\in Z_q(M)$, appears as a $\mathbb{R}/\mathbb{Z}$-valued linear pairing

$$\begin{aligned}\langle\,,\,\rangle_q:\ & H_D^q(M,\mathbb{Z})\times Z_q(M)\longrightarrow\mathbb{R}/\mathbb{Z}=S^1,\\ & ([\chi],C)\longrightarrow\langle[\chi],C\rangle_q\equiv\int_C[\chi],\end{aligned}\tag{2.6}$$

which establishes the equivalence between DB cohomology and Cheeger–Simons characters [12, 13, 11, 14, 25]. Accordingly, a quantity such as

$$\exp\left\{2i\pi\int_C[\chi]\right\}$$

is well defined and corresponds to the fundamental representation of $\mathbb{R}/\mathbb{Z}=S^1\simeq U(1)$. Using the Chech–de Rham description of DB cocycles, one can then produce explicit formulae [25] for the pairing (2.6).

Alternatively, (2.6) can be seen as a dualising equation. More precisely, any $C\in Z_q(M)$ belongs to the Pontriagin dual of $H_D^q(M,\mathbb{Z})$, usually denoted by $\mathrm{Hom}\left(H_D^q(M,\mathbb{Z}),S^1\right)$, the pairing (2.6) providing a canonical injection

$$Z_q(M)\vec{\subset}\,\mathrm{Hom}\left(H_D^q(M,\mathbb{Z}),S^1\right). \tag{2.7}$$

A universal result [27] about the Hom functor implies the validity of the exact sequences, dualising (2.2) (via (2.3)),

$$0\longrightarrow\mathrm{Hom}\big(\Omega_{\mathbb{Z}}^{q+1}(M),S^1\big)\longrightarrow\mathrm{Hom}\left(H_D^q(M,\mathbb{Z}),S^1\right)\longrightarrow H^{n-q}(M,\mathbb{Z})\longrightarrow 0, \tag{2.8}$$

where $H^{n-q}(M,\mathbb{Z})\cong\mathrm{Hom}\left(H^q(M,\mathbb{R}/\mathbb{Z}),S^1\right)$.

The space $\mathrm{Hom}\left(H_D^q(M,\mathbb{Z}),S^1\right)$ also contains $H_D^{n-q-1}(M,\mathbb{Z})$, so that $Z_q(M)$ (or rather its canonical injection (2.7)) can be seen as lying on the boundary of $H_D^{n-q-1}(M,\mathbb{Z})$ (see details in [14]). Accordingly

$$Z_q(M)\oplus H_D^{n-q-1}(M,\mathbb{Z})\subset\mathrm{Hom}\left(H_D^q(M,\mathbb{Z}),S^1\right), \tag{2.9}$$

with the obvious abuse in the notation. Let us point out that, as suggested by equation (2.9), one could represent integral cycles by currents which are singular (i.e. distributional) forms. This issue will be discussed in detail in next subsection.

Now, sequence (2.8) shows that $\mathrm{Hom}\left(H_D^q(M,\mathbb{Z}),S^1\right)$ is also an affine bundle with base space $H^{n-q}(M,\mathbb{Z})$. In particular, let us consider the case in which $n=3$ and $q=1$; on the one hand, Poincaré duality implies

$$H^{n-q}(M,\mathbb{Z})=H^2(M_3,\mathbb{Z})\cong H^1(M_3,\mathbb{Z}).$$

On the other hand, one has

$$H_D^1(M,\mathbb{Z})\subset\mathrm{Hom}\left(H_D^1(M,\mathbb{Z}),S^1\right),$$

and, because of the Pontriagin duality,

$$Z_1(M)\oplus H_D^1(M,\mathbb{Z})\subset\mathrm{Hom}\left(H_D^1(M,\mathbb{Z}),S^1\right).$$

This is somehow reminiscent of the self-dual situation in the case of four dimensional manifolds and curvature.

2.3 The product

The pairing (2.6) is actually related to another pairing of DB cohomology groups

$$H_D^p(M,\mathbb{Z})\times H_D^q(M,\mathbb{Z})\longrightarrow H_D^{p+q+1}(M,\mathbb{Z}), \tag{2.10}$$

whose explicit description can be found for instance in [12, 14, 25]. This pairing is known as the DB product (or DB $*$-product). It will be denoted by $*$. In the Cech–de Rham approach, the DB product of the DB cocyle $(\omega^{(0,p)}, \omega^{(1,p-1)}, \dots, \omega^{(p,0)}, \omega^{(p+1,-1)})$ with the DB cocycle $(\eta^{(0,q)}, \eta^{(1,q-1)}, \dots, \eta^{(q,0)}, \eta^{(q+1,-1)})$ reads

$$\left(\omega^{(0,p)} \cup d\eta^{(0,q)}, \dots, \omega^{(p,0)} \cup d\eta^{(0,q)}, b\omega^{(p+1,-1)} \cup \eta^{(0,q)}, \dots, \omega^{(p+1,-1)} \cup \eta^{(n-p,-1)}\right), \tag{2.11}$$

where the product $\cup$ is precisely defined in [28, 9, 25], for instance.

Definition 2.2. Let us consider the sequence $(\eta^{(0,q)}, \eta^{(1,q-1)}, \dots, \eta^{(q,0)}, \eta^{(q+1,-1)})$, in which the components $\eta^{(k-q,k)}$ satisfy the same descent equations as the components of a DB cocycle but, instead of smooth forms, these components are currents (i.e. distributional forms). This allows to extend the (smooth) cohomology group $H_D^q(M, \mathbb{Z})$ to a larger cohomology group that we will denote $\widetilde{H}_D^q(M, \mathbb{Z})$.

Obviously, the DB product (2.11) of a smooth DB cocycle with a distributional one is still well-defined, and thus the pairing (2.10) extends to

$$H_D^p(M, \mathbb{Z}) \times \widetilde{H}_D^q(M, \mathbb{Z}) \longrightarrow \widetilde{H}_D^{p+q+1}(M, \mathbb{Z}).$$

Then, it can be checked [25] that any class $[\eta] \in \widetilde{H}_D^{n-q-1}(M, \mathbb{Z})$ canonically defines a $\mathbb{R}/\mathbb{Z}$-valued linear pairing as in (2.6) so that

$$\widetilde{H}_D^{n-q-1}(M, \mathbb{Z}) \subset \mathrm{Hom}\left(H_D^q(M, \mathbb{Z}), S^1\right).$$

It is important to note that, as it was shown in [25], to any $C \in Z_q(M)$ there corresponds a canonical DB class $[\eta_C] \in \widetilde{H}_D^{n-q-1}(M, \mathbb{Z})$ such that

$$\exp\left\{2i\pi \int_C [\chi]\right\} = \exp\left\{2i\pi \int_M [\chi] * [\eta_C]\right\},$$

for any $[\chi] \in H_D^q(M, \mathbb{Z})$. This means that we have the following sequence of canonical inclusions

$$Z_q(M) \subset \widetilde{H}_D^{n-q-1}(M, \mathbb{Z}) \subset \mathrm{Hom}\left(H_D^q(M, \mathbb{Z}), S^1\right).$$

Let us point out the trivial inclusion

$$H_D^{n-q-1}(M, \mathbb{Z}) \subset \widetilde{H}_D^{n-q-1}(M, \mathbb{Z}).$$

In the 3 dimensional case, let us consider the DB product

$$H_D^1(M_3, \mathbb{Z}) \times H_D^1(M_3, \mathbb{Z}) \longrightarrow H_D^3(M_3, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z}. \tag{2.12}$$

Starting from equation (2.12) and extending it to

$$H_D^1(M_3, \mathbb{Z}) \times \widetilde{H}_D^1(M_3, \mathbb{Z}) \longrightarrow \widetilde{H}_D^3(M_3, \mathbb{Z}) \cong \mathbb{R}/\mathbb{Z},$$

one finds that it is possible to associate with any 1-cycle $C \in Z_1(M_3)$ a canonical DB class $[\eta_C] \in \widetilde{H}_D^1(M_3, \mathbb{Z})$ such that

$$\exp\left\{2i\pi \int_C [\omega]\right\} = \exp\left\{2i\pi \int_{M_3} [\omega] * [\eta_C]\right\}, \tag{2.13}$$

for any $[\omega] \in H_D^1(M_3, \mathbb{Z})$. As an an alternative point of view, consider a smoothing homotopy of C within $H_D^1(M_3, \mathbb{Z})$, that is, a sequence of smooth DB classes $[\eta_\varepsilon] \in H_D^1(M, \mathbb{Z})$ such that (see [14] for details)

$$\lim_{\varepsilon \to 0} \exp\left\{2i\pi \int_M [A] * [\eta_\varepsilon]\right\} = \exp\left\{2i\pi \int_C [A]\right\}. \tag{2.14}$$

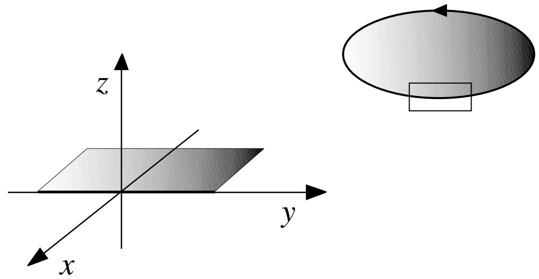

Figure 2. In a open domain with local coordinates (x, y, z), a piece of a homologically trivial loop C can be identified with the y axis, and the disc that it bounds (Seifert surface) can be identified with a portion of the half plane $(x < 0,\ y, z = 0)$.

This implies

$$\lim_{\varepsilon \to 0} [\eta_\varepsilon] = [\eta_C] \tag{2.15}$$

within the completion $\widetilde{H}^1_D(M_3, \mathbb{Z})$ of $H^1_D(M_3, \mathbb{Z})$; this is why in [14] $[\eta_C]$ is said to belong to the boundary of $H^1_D(M_3, \mathbb{Z})$. It should be noted that, by definition, the limit (2.14) and the corresponding limit (2.15) are always well defined. For this reason, in what follows we shall concentrate directly to the distributional space $\widetilde{H}^1_D(M_3, \mathbb{Z})$ and, in the presentation of the various arguments, the possibility of adopting a limiting procedure of the type shown in equation (2.14) will be simply understood.

Finally, let us point out that with the aforementioned geometrical interpretation of DB cohomology classes, the DB product of smooth classes canonically defines a product within the space of Abelian Gerbes with connections. For instance, the DB product of two classes of $U(1)$-bundles with connections over M turns out to be a class of $U(1)$-gerbe with connection over M.

2.4 Distributional forms and Seifert surfaces

How to construct the class $[\eta_C]$, which enters equation (2.13), is explained in detail for instance in [25]. Here we outline the main steps of the construction and we consider, for illustrative purposes, the case $M_3 \sim S^3$. The integral of a one-form ω along an oriented knot $C \subset S^3$ can be written as the integral on the whole S^3 of the external product $\omega \wedge J_C$, where the current J_C is a distributional 2-form with support on the knot C; that is, $\int_C \omega = \int_{S^3} \omega \wedge J_C$. Since J_C can be understood as the boundary of an oriented surface Σ_C in S^3 (called a Siefert surface), one has $J_C = d\eta_C$ for some 1-form η_C with support on Σ_C. One then finds, $\int_C \omega = \int_{S^3} \omega \wedge d\eta_C$, which corresponds precisely to equation (2.13) with $[\eta_C] \in \widetilde{H}^1_D(S^3, \mathbb{Z})$ denoting the Deligne cohomology class which is associated to η_C and with $[\omega] \in H^1_D(S^3, \mathbb{Z})$ denoting the class which can be represented by ω.

Let us consider, for instance, the unknot C in S^3, shown in Fig. 2, with a simple disc as Seifert surface. Inside the open domain depicted in Fig. 2, the oriented knot is described – in local coordinates (x, y, z) – by a piece of the y-axis and the corresponding distributional forms J_C and η_C are given by

$$J_C = \delta(z)\delta(x) dz \wedge dx, \qquad \eta_C = \delta(z)\theta(-x) dz. \tag{2.16}$$

For a generic 3-manifold M_3 and for each oriented knot $C \subset M_3$, the distributional 2-form J_C always exists, whereas a corresponding Seifert surface and the associated 1-form η_C can in general be (globally) defined only when the second cohomology group of M_3 is vanishing. Nevertheless, the class $[\eta_C] \in \widetilde{H}^1_D(M, \mathbb{Z})$ is always well defined for arbitrary 3-manifold M_3. In fact, when a Seifert surface associated with $C \subset M_3$ does not exist, the Chech–de Rham cocycle sequence representing $[\eta_C] \in \widetilde{H}^1_D(M,\mathbb{Z})$ is locally of the form $(\eta_C^{(0,1)}, \Lambda_C^{(1,0)}, N_C^{(2,-1)})$ where, inside sufficiently small open domains, the expression of $\eta_C^{(0,1)}$ is trivial or may coincide with the expression (2.16) for η_C, and $\Lambda_C^{(1,0)}$ and $N_C^{(2,-1)}$ are nontrivial components (when a Seifert surface exists, the components $\Lambda_C^{(1,0)}$ and $N_C^{(2,-1)}$ are trivial).

3 Linking and self-linking

As we have already mentioned, in the context of equation (2.13) the pairing $H^1_D(M_3,\mathbb{Z}) \times \widetilde{H}^1_D(M_3,\mathbb{Z}) \to \widetilde{H}^3_D(M_3,\mathbb{Z})$ is well defined. However, in what follows we shall also need to consider a pairing induced by the DB product of the type $\widetilde{H}^1_D(M_3,\mathbb{Z}) \times \widetilde{H}^1_D(M_3,\mathbb{Z}) \to \widetilde{H}^3_D(M_3,\mathbb{Z})$ and this presents in general ambiguities that we need to fix by means of some conventional procedure.

3.1 Linking number

Let us consider first the case $M_3 \sim S^3$. Let C_1 and C_2 be two non-intersecting oriented knots in S^3 and let η_1 and η_2 the corresponding distributional 1-forms described in Section 2.4, one has

$$\int_{S^3} \eta_1 \wedge d\eta_2 = \int_{S^3} \eta_2 \wedge d\eta_1 = \ell k(C_1, C_2), \tag{3.1}$$

where $\ell k(C_1, C_2)$ denotes the linking number of C_1 and C_2, which is an integer valued ambient isotopy invariant. In fact, $\eta_1 \wedge d\eta_2$ represents an intersection form counting how many times C_2 intersects the Seifert surface associated with C_1 (see also, for instance, [28, 29]). Let $[\eta_1]$ and $[\eta_2]$ denote the DB classes which are associated with η_1 and η_2; since the linking number is an integer, one finds

$$\exp\left\{2i\pi \int_{S^3} [\eta_1] * [\eta_2]\right\} = \exp\left\{2i\pi \int_{S^3} [\eta_2] * [\eta_1]\right\} = \exp\left\{2i\pi \int_{S^3} \eta_1 \wedge d\eta_2\right\} = 1. \tag{3.2}$$

Equations (3.1) and (3.2) show that the product $[\eta_1] * [\eta_2]$ is well defined and just corresponds to the trivial class

$$[\eta_1] * [\eta_2] = [0] \in \widetilde{H}^3_D\left(S^3,\mathbb{Z}\right). \tag{3.3}$$

In the next sections, we shall encounter the linking number in the DB cohomology context in the following form. Let x be a real number, since η_2 is globally defined in S^3, the 1-form $x\eta_2$ is also globally defined. Let us denote by $[x\eta_2]$ the DB class which is represented by the form $x\eta_2$. One has

$$\exp\left\{2i\pi \int_{S^3} [\eta_1] * [x\eta_2]\right\} = \exp\left\{2i\pi \int_{S^3} \eta_1 \wedge d(x\eta_2)\right\} = \exp\left\{2i\pi x \ell k(C_1, C_2)\right\}. \tag{3.4}$$

3.2 Framing

Let η_C be the distributional 1-form which is associated with the oriented knot $C \subset S^3$; for a single knot, the expression of the self-linking number

$$\int_{S^3} \eta_C \wedge d\eta_C \tag{3.5}$$

is not well defined because the self-intersection form $\eta_C \wedge d\eta_C$ has ambiguities. This means that, similarly to what happens with the product of distributions, at the level of the class $[\eta_C] \in \widetilde{H}^1_D\left(S^3, \mathbb{Z}\right)$, the product $[\eta_C] * [\eta_C]$ is not well defined a priori.

As shown in equations (2.14) and (2.15), $[\eta_C]$ can be determined by means of the $\varepsilon \to 0$ limit of $[\eta_\varepsilon] \in H^1_D\left(M_3, \mathbb{Z}\right)$. One could then try to define the product $[\eta_C] * [\eta_C]$ by means of the same limit

$$\lim_{\varepsilon \to 0} \int_{S^3} [\eta_\varepsilon] * [\eta_\varepsilon] = \int_{S^3} [\eta_C] * [\eta_C]. \tag{3.6}$$

Unfortunately, the limit (3.6) does not exist, because the value that one obtains for the integral (3.6) in the $\varepsilon \to 0$ limit nontrivially depends on the way in which $[\eta_\varepsilon]$ approaches $[\eta_C]$. This problem will be solved by the introduction of the framing procedure, which ultimately specifies how $[\eta_\varepsilon]$ approaches $[\eta_C]$. One should note that the ambiguities entering the integral (3.5) and the limit (3.6) also appear in the Gauss integral

$$\frac{1}{4\pi} \oint_C dx^\mu \oint_C dy^\nu \epsilon_{\mu\nu\rho} \frac{(x-y)^\rho}{|x-y|^3}, \tag{3.7}$$

which corresponds to the self-linking number. A direct computation [30] shows that the value of the integral (3.7) is a real number which is not invariant under ambient isotopy transformations; in fact, it can be smoothly modified by means of smooth deformations of the knot C in S^3. In order to remove all ambiguities and define the product $[\eta_C] * [\eta_C]$, we shall adopt the framing procedure [29, 31], which is also used for giving a topological meaning to the self-linking number.

Definition 3.1. A solid torus is a space homeomorphic to $S^1 \times D^2$, where D^2 is a two dimensional disc; in the complex plane, D^2 can be represented by the set $\{z, \text{ with } |z| \leq 1\}$. Consider now an oriented knot $C \subset S^3$; a tubular neighbourhood V_C of C is a solid torus embedded in S^3 whose core is C. A given homeomorphism $h : S^1 \times D^2 \to V_C$ is called a framing for C. The image of the standard longitude $h(S^1 \times 1)$ represents a knot $C_f \subset S^3$, also called the framing of C, which lies in a neighbourhood of C and whose orientation is fixed to agree with the orientation of C. Up to isotopy transformations, the homeonorphism h is specified by C_f.

Clearly, the thickness of the tubular neighbourhood V_C of C is chosen to be sufficiently small so that, in the presence of several link components for instance, any knot different from C belongs to the complement of $V_C \subset S^3$.

For each framed knot C, with framing C_f, the self-linking number of C is defined to be $\ell k(C, C_f)$,

$$\int_{S^3} \eta_C \wedge d\eta_C \equiv \int_{S^3} \eta_C \wedge d\eta_{C_f} = \ell k(C, C_f). \tag{3.8}$$

Definition 3.2. In agreement with equation (3.8), one can consistently define the product $[\eta_C] * [\eta_C]$ as

$$[\eta_C] * [\eta_C] \equiv [\eta_C] * [\eta_{C_f}]. \tag{3.9}$$

Definition (3.9) together with equations (3.8) and (3.3) imply that, for each framed knot C (in S^3), the product $[\eta_C] * [\eta_C]$ is well defined and corresponds to the trivial class

$$[\eta_C] * [\eta_C] = [0] \in \widetilde{H}^3_D\left(S^3, \mathbb{Z}\right).$$

Remark 3.1. The product $[\eta_C] * [\eta_C]$ also admits a definition which differs from equation (3.9) but, as far as the computation of the Chern–Simons observables is concerned, is equivalent to equation (3.9). Instead of dealing with a tubular neighbourhood V_C with sufficiently small but finite thickness, one can define a limit in which the transverse size of the neighbourhood V_C vanishes. Let $\rho > 0$ be the size of the diameter of the tubular neighbourhood $V_C(\rho)$ of the knot C; ρ is defined with respect to some (topology compatible) metric g. The homeomorphism $h(\rho) : S^1 \times D^2 \to V_C(\rho)$ is assumed to depend smoothly on ρ. Then, the corresponding framing knot $C_f(\rho)$ also smoothly depends on ρ. Consequently, the linking number $\ell k(C, C_f(\rho))$ does not depend on the value of ρ and it will be denoted by $\ell k(C, C_f)$. It should be noted that $\ell k(C, C_f)$ does not depend on the choice of the metric g. In the $\rho \to 0$ limit, the solid torus $V_C(\rho)$ shrinks to its core C and the framing $C_f(\rho)$ goes to C. One can then define $\eta_C \wedge d\eta_C$ according to

$$\int_{S^3} \eta_C \wedge d\eta_C \equiv \lim_{\rho \to 0} \int_{S^3} \eta_C \wedge d\eta_{C_f(\rho)} = \lim_{\rho \to 0} \ell k(C, C_f(\rho)) = \ell k(C, C_f). \tag{3.10}$$

In agreement with equation (3.10), one can put

$$[\eta_C] * [\eta_C] \equiv \lim_{\rho \to 0} [\eta_C] * [\eta_{C_f(\rho)}]. \tag{3.11}$$

Remark 3.2. The definition (3.9) of the DB product $[\eta_C] * [\eta_C]$ is consistent with equations (3.2)–(3.4) and is topologically well defined. In fact, in the case of an oriented framed link L with N components $\{C_1, C_2, \ldots, C_N\}$ the corresponding canonical class $[\eta_L] \in \widetilde{H}^1_D\left(S^3, \mathbb{Z}\right)$ is equivalent to the sum of the classes which are associated with the single components, i.e. $[\eta_L] = \sum_j [\eta_j]$. Thus one finds

$$[\eta_L] * [\eta_L] = \sum_j [\eta_j] * [\eta_j] + 2 \sum_{i<j} [\eta_i] * [\eta_j]. \tag{3.12}$$

The framing procedure which is used to define the DB product $[\eta_L] * [\eta_L]$ guarantees that, if one integrates the 3-forms entering expression (3.12), the result does not depend on the particular choice of the Seifert surface which is used to (locally) define the distributional forms associated with L. This means that the framing procedure preserves both gauge invariance and ambient isotopy invariance.

Remark 3.3. In order to define the extension of the DB product to distributional DB classes, one could try to start from equation (2.11). In this case, the product of the DB representatives of two cycles (2.11) would only contain local integral chains and the cup product $\cup$ would just reduce to the intersection number of such integral chains (once these chains have been placed into transverse position, which is always possible because of the freedom in the choice of the DB cocycles representing a given DB class). Accordingly, the extension of the product to the distributional case would only produce integral chains and eventually integers in the integrals. Finally, by using smooth approximations of the cycles within (2.11) and then performing the limits, as described above in equation (3.11), one would obtain the same result. Note that, in this last approach, the limit would be performed with the linking number $\ell k(C, C_f)$ fixed. This is similar to the definition of the charge density of a charged point particle by taking the limit $r \to 0$ of a uniformly charged sphere of radius r while keeping the total charge of the sphere fixed, which leads to the well-known Dirac delta-distribution.

Knots or links can be framed in any oriented 3-manifold M_3. In order to preserve the topological properties of the pairing $\widetilde{H}^1_D\left(S^3, \mathbb{Z}\right) \times \widetilde{H}^1_D\left(S^3, \mathbb{Z}\right) \to \widetilde{H}^3_D\left(S^3, \mathbb{Z}\right)$ which is defined by means of framing in S^3, we shall extend the framing procedure to the case of a generic 3-manifold M_3 by extending the validity of properties (3.3) and (3.9).

Definition 3.3. If $[\eta_1]$ and $[\eta_2]$ are the classes in $\widetilde{H}^1_D(M_3, \mathbb{Z})$ which are canonically associated with the oriented nonintersecting knots C_1 and C_2 in M_3, in agreement with equation (3.3) we shall eliminate the (possible) ambiguities of the product $[\eta_1] * [\eta_2]$ in such a way that

$$[\eta_1] * [\eta_2] = [0] \in \widetilde{H}^3_D(M_3, \mathbb{Z}). \tag{3.13}$$

Consequently, for each oriented framed knot $C \subset M_3$ with framing C_f, we shall use the definition

$$[\eta_C] * [\eta_C] \equiv [\eta_C] * [\eta_{C_f}] = [0] \in \widetilde{H}^3_D(M_3, \mathbb{Z}). \tag{3.14}$$

Remark 3.4. Definition (3.14) can also be understood by starting from equation (2.11) and by using the same arguments that have been presented in the case $M_3 \sim S^3$. Let us point out that, unlike the S^3 case, for generic M_3 one finds directly equation (3.14) without the validity of some intermediate relations like equation (3.8), which may not be well defined for $M_3 \nsim S^3$.

4 Abelian Chern–Simons field theory

4.1 Action functional

If one uses the Cech–de Rham double complex to describe DB classes, it can easily be shown that the first component of a DB product of a $U(1)$-connection A with itself is given by $A \wedge dA$ or, more precisely, it is given by the collection of all these products taken in the open sets of a good cover of M_3. This means that the expression of the Chern–Simons lagrangian of a $U(1)$-connection A can be understood as a DB class which coincides with the "DB square" of the class of A. Let $[A]$ denote the DB class associated to the $U(1)$-connection A, the Chern–Simons functional S_{CS} is given by

$$S_{CS} = \int_{M_3} [A] * [A].$$

By definition of the DB cohomology, the Chern–Simons action S_{CS} is an element of $\mathbb{R}/\mathbb{Z}$ and then it is defined modulo integers. Consequently, in the functional measure of the path-integral, the phase factor which is induced by the action has to be of the type

$$\exp\{2i\pi k S_{CS}\} = \exp\left\{2i\pi k \int_{M_3} [A] * [A]\right\},$$

where the coupling constant k must be a nonvanishing integer

$$k \in \mathbb{Z}, \qquad k \neq 0.$$

A modification of the orientation of M_3 is equivalent to the replacement $k \to -k$.

4.2 Observables

The observables that we shall consider are given by the expectation values of the Wilson line operators $W(L)$ associated with links L in M_3. An oriented coloured and framed link $L \subset M_3$ with N components is the union of non-intersecting knots $\{C_1, C_2, \ldots, C_N\}$ in M_3, where each knot C_j (with $j = 1, 2, \ldots, N$) is oriented and framed, and its colour is represented by an integer charge $q_j \in \mathbb{Z}$. For any given DB class $[A]$, the classical expression of $W(L)$ is given by

$$W(L) = \prod_{j=1}^{N} \exp\left\{2i\pi q_j \int_{C_j} [A]\right\} = \exp\left\{2i\pi \sum_j q_j \int_{C_j} [A]\right\}, \tag{4.1}$$

which actually corresponds to the pairing (2.6)

$$W(L) = \exp\left\{2i\pi \int_L [A]\right\} \equiv \exp\left\{2i\pi \left\langle [A]\,, L\right\rangle_1\right\}.$$

Once more, each factor

$$\exp\left\{2i\pi q_j \int_{C_j} [A]\right\}, \tag{4.2}$$

which appears in expression (4.1), is well defined if and only if $q_j \in \mathbb{Z}$; that is why the charges associated with knots must take integer values. A modification of the orientation of the knot C_j is equivalent to the replacement $q_j \to -q_j$. Obviously, any link component with colour $q = 0$ can be eliminated.

Remark 4.1. The classical expression (4.1) does not depend on the framing of the knots $\{C_j\}$; however, only for framed links are the Wilson line operators well defined. The point is that, in the quantum Chern–Simons field theory, the field components correspond to distributional valued operators, and the Wilson line operators are formally defined by expression (4.1) together with a set of specified rules which must be used to remove possible ambiguities in the computation of the expectation values. In the operator formalism, these ambiguities are related to the product of field operators in the same point [32, 33] and they are eliminated by means of a framing procedure. In the path-integral approach, we shall see that all the ambiguities are related to the definition of the pairing $\widetilde{H}^1_D(M_3,\mathbb{Z}) \times \widetilde{H}^1_D(M_3,\mathbb{Z}) \to \widetilde{H}^3_D(M_3,\mathbb{Z})$; as it has been discussed in Section 3, we shall use the framing of the link components to eliminate all ambiguities by means of the definition (3.14).

Remark 4.2. In equations (4.1) and (4.2), we have used the same symbol to denote knots and their homological representatives because a canonical correspondence [28] between them always exists. At the classical level, for any integer q one can identify the 1-cycle $qC \in Z_1(M)$ with the q-fold covering of the cycle C or the q-times product of C with itself. At the quantum level, this equivalence may not be valid when it is applied to the Wilson line operators because of ambiguities in the definition of composite operators; so, in order to avoid inaccuracies, we will always refer to Wilson line operators defined for oriented coloured and framed knots or links.

Definition 4.1. For each link component C_j, let $[\eta_j] \in \widetilde{H}^1_D(M_3,\mathbb{Z})$ be the DB class such that

$$\exp\left\{2i\pi q_j \int_{C_j} [A]\right\} = \exp\left\{2i\pi q_j \int_{M_3} [A] * [\eta_j]\right\}.$$

With the definition

$$[\eta_L] = \sum_j q_j [\eta_j], \tag{4.3}$$

one has

$$\exp\left\{2i\pi \int_{M_3} [A] * [\eta_L]\right\} = \exp\left\{2i\pi \sum_j q_j \int_{M_3} [A] * [\eta_j]\right\}.$$

The expectation values of the Wilson line operators can be written in the form

$$\langle W(L)\rangle_k \equiv \frac{\int D[A] \exp\left\{2i\pi k \int_{M_3} [A] * [A]\right\} W(L)}{\int D[A] \exp\left\{2i\pi k \int_{M_3} [A] * [A]\right\}}$$

$$= \frac{\int D\left[A\right] \exp\left\{2i\pi k \int_{M_3} \left[A\right] * \left[A\right]\right\} \exp\left\{2i\pi \int_{M_3} \left[A\right] * \left[\eta_L\right]\right\}}{\int D\left[A\right] \exp\left\{2i\pi k \int_{M_3} \left[A\right] * \left[A\right]\right\}}, \tag{4.4}$$

and our main purpose is to show how to compute them for arbitrary link L.

Remark 4.3. In the DB cohomology approach, the functional integration (4.4) locally corresponds to a sum over 1-form modulo forms with integer periods. So, the space of classical field configurations which are factorized out by gauge invariance is in general larger than the standard group of local gauge transformations. It should be noted that this enlarged gauge symmetry perfectly fits the assumption that the expectation values (4.4) form a complete set of observables. In the DB cohomology interpretation of the functional integral for the quantum Chern–Simons field theory, this enlargement of the "symmetry group" represents one of the main conceptual improvements with respect to the standard formulation of gauge theories and, as we shall show, allows for a description of the functional space structure in terms of the homology groups of the manifold M_3.

4.3 Properties of the functional measure

The sum over the DB classes $\int D[A]$ corresponds to a gauge-fixed functional integral in ordinary quantum field theory, where one has to sum over the gauge orbits in the space of connections. In the path-integral, smooth fields configurations or finite-action configurations have zero measure [34, 35]; so, the functional integral (4.4) has to be understood as the functional integral in the appropriate extension or closure $\mathcal{H}^1_D\left(M_3, \mathbb{Z}\right)$ of the space $H^1_D\left(M_3, \mathbb{Z}\right)$, with $\widetilde{H}^1_D\left(M_3, \mathbb{Z}\right) \subset \mathcal{H}^1_D\left(M_3, \mathbb{Z}\right)$ and, more generaly, with $\mathrm{Hom}\left(H^1_D\left(M, \mathbb{Z}\right), S^1\right) \subset \mathcal{H}^1_D\left(M_3, \mathbb{Z}\right)$. In order to guarantee the consistency of the functional integral and its correspondence with ordinary gauge theories, we assume that the quantum measure has the following two properties.

(**M1**) *The space $\mathcal{H}^1_D\left(M_3, \mathbb{Z}\right)$ inherits its structure from $H^1_D\left(M_3, \mathbb{Z}\right)$ in agreement with sequence* (2.5).

Equation (2.5) then implies that the sum over DB classes is locally equivalent to a sum over $\Omega^1\left(M_3\right)/\Omega^1_{\mathbb{Z}}\left(M_3\right)$, i.e. a sum over 1-forms modulo generalized gauge transformations.

(**M2**) *The functional measure is translational invariant.*

This implies in particular that, for any $[\omega] \in \widetilde{H}^1_D\left(M_3, \mathbb{Z}\right)$, the quadratic measure

$$d\mu_k\left([A]\right) \equiv D\left[A\right] \exp\left\{2i\pi k \int_{M_3} \left[A\right] * \left[A\right]\right\} \tag{4.5}$$

satisfies the equation

$$d\mu_k\left([A] + [\omega]\right) = d\mu_k\left([A]\right) \exp\left\{4i\pi k \int_{M_3} [A] * [\omega] + 2i\pi k \int_{M_3} [\omega] * [\omega]\right\}, \tag{4.6}$$

which looks like a Cameron–Martin formula (see for instance [36] and references therein).

Equation (4.6) will be used extensively in our computations. The importance of generalized Wiener measures in the functional integral – which necessarily imply the validity of the Cameron–Martin property – and of the singular connections was also stressed in the articles [37] and [38] in which, however, the space of the functional integral is supposed to coincide with the space of the classes of smooth connections on a fixed $U(1)$-bundle over M_3.

In the computation of the observables (4.4), we shall not use perturbation theory; only properties (**M1**) and (**M2**) of the functional measure will be utilized. We shall now derive the main properties of the observables of the Abelian Chern–Simons theory which are valid for any 3-manifold M_3.

4.4 Colour periodicity

The colour of each oriented knot or link component $C \subset M_3$ is specified by the value of its associated charge $q \in \mathbb{Z}$. For fixed nonvanishing value of the coupling constant k, the expectation values (4.4) are invariant under the substitution $q \to q + 2k$, where q is the charge of a generic link component. Consequently, one has

Proposition 4.1. *For fixed integer k, the colour space is given by $\mathbb{Z}_{2k}$ which coincides with the space of residue classes of integers* mod $2k$.

Proof. Let $\{q_j\}$ be the charges which are associated with the components $\{C_j\}$ $(j = 1, 2, \dots, N)$ of the link L. With the notation (4.5), the expectation value $\langle W(L) \rangle_k$ shown in equation (4.4) can be written as

$$\langle W(L) \rangle_k = \frac{\int d\mu_k([A]) \exp\left\{2i\pi \sum_j q_j \int_{M_3} [A] * [\eta_j]\right\}}{\int d\mu_k([A])}. \tag{4.7}$$

Property (**M2**) implies that, with the substitution $[A] \to [A] + [\eta_1]$, the numerator of expression (4.7) becomes

$$\int d\mu_k([A]) \exp\left\{2i\pi \sum_j q_j \int_{M_3} [A] * [\eta_j]\right\} = \int d\mu_k([A]) \exp\left\{2i\pi \sum_j q'_j \int_{M_3} [A] * [\eta_j]\right\}$$
$$\times \exp\left\{2i\pi k \int_{M_3} [\eta_1] * [\eta_1]\right\} \exp\left\{2i\pi \sum_j q_j \int_{M_3} [\eta_1] * [\eta_j]\right\},$$

where $q'_j = q_j + 2k\delta_{j1}$. In agreement with equation (3.13), for $j \neq 1$ one has $[\eta_1] * [\eta_j] \simeq [0] \in \widetilde{H}^3_D(M_3, \mathbb{Z})$, and then

$$\exp\left\{2i\pi q_j \int_{M_3} [\eta_1] * [\eta_j]\right\} = 1.$$

Similarly, in agreement with equation (3.14), by means of the framing procedure one obtains $[\eta_1] * [\eta_1] \simeq [0] \in \widetilde{H}^3_D(M_3, \mathbb{Z})$, and then

$$\exp\left\{2i\pi (q_1 + k) \int_{M_3} [\eta_1] * [\eta_1]\right\} = 1.$$

Consequently, the numerator of expression (4.7) can be written as

$$\int d\mu_k([A]) \exp\left\{2i\pi \sum_j q_j \int_{M_3} [A] * [\eta_j]\right\}$$
$$= \int d\mu_k([A]) \exp\left\{2i\pi \sum_j q'_j \int_{M_3} [A] * [\eta_j]\right\},$$

which proves that, for fixed k, the expectation values (4.4) are invariant under the substitution $q_1 \to q_1 + 2k$, where q_1 is the charge of the link component C_1. ∎

4.5 Ambient isotopy invariance

Two oriented framed coloured links L and L' in M_3 are ambient isotopic if L can be smoothly connected with L' in M_3.

Proposition 4.2. *The Chern–Simons expectation values (4.4) are invariants of ambient isotopy for framed links.*

Proof. Suppose that two oriented coloured framed links L and L' are ambient isotopic in M_3. The link L has components $\{C_1, C_2, \dots, C_N\}$ with colours $\{q_1, q_2, \dots, q_N\}$; whereas the link L' has components $\{C'_1, C_2, \dots, C_N\}$ with colours $\{q_1, q_2, \dots, q_N\}$, so that

$$[\eta_L] = q_1[\eta_1] + \sum_{j\neq 1} q_j[\eta_j], \qquad [\eta_{L'}] = q_1[\eta'_1] + \sum_{j\neq 1} q_j[\eta_j], \tag{4.8}$$

where the class $[\eta_1]$ refers to the knot $C_1 \subset M_3$ and $[\eta'_1]$ is associated to the knot $C'_1 \subset M_3$.

Let $\tau : [0, 1] \to C_1(\tau) \subset M_3$ be the isotopy which connects C_1 with C'_1 in M_3, with $C_1(0) = C_1$ and $C_1(1) = C'_1$. We shall denote by $\Sigma \subset M_3$ the 2-dimensional surface which has support on $\{C_1(\tau) \subset M_3; 0 \leq \tau \leq 1\}$; because of the freedom in the choice of τ within the same ambient isotopy class, it is assumed that Σ is well defined and presents no singularities. Σ belongs to the complement of the link components $\{C_2, C_3, \dots, C_N\}$ in M_3 and one can introduce an orientation on Σ in such a way that its oriented boundary is given by $\partial\Sigma = C'_1 \cup C_1^{-1}$, where C_1^{-1} denotes the knot C_1 with reversed orientation.

The distributional 1-form η_Σ, which is associated with Σ, is globally defined in M_3 and satisfies

$$d\eta_\Sigma = d\eta'_1 - d\eta_1. \tag{4.9}$$

where, with a small abuse of notation, $d\eta_1$ and $d\eta'_1$ denote the integration currents of C_1 and C'_1 respectively. For $j \neq 1$ one finds

$$\int_{M_3} \eta_\Sigma \wedge d\eta_j = 0, \tag{4.10}$$

because the link components $\{C_2, C_3, \dots, C_N\}$ do not intersect the surface Σ. Moreover, according to the framing procedure, the orientation of Σ implies

$$\int_{M_3} \eta_\Sigma \wedge (d\eta'_1 + d\eta_1) = \int_{C'_{1f}} \eta_\Sigma + \int_{C_{1f}} \eta_\Sigma = 0, \tag{4.11}$$

where C'_{1f} denotes the framing of C'_1 and C_{1f} represents the framing of C_1. Since η_Σ is globally defined in M_3, the 1-form $x\eta_\Sigma$ (with $x = (q_1/2k) \in \mathbb{R}$) is also globally defined. Let $[x\eta_\Sigma] \in \widetilde{H}^1_D(M_3, \mathbb{Z})$ be the DB class which can be represented by the 1-form $x\eta_\Sigma$; by construction, one has

$$\begin{aligned}&\exp\left\{4i\pi k \int_{M_3} [A] * [(q_1/2k)\eta_\Sigma]\right\} \\ &\qquad = \exp\left\{2i\pi q_1 \int_{M_3} [A] * [\eta'_1]\right\} \exp\left\{-2i\pi q_1 \int_{M_3} [A] * [\eta_1]\right\}.\end{aligned} \tag{4.12}$$

The expectation value $\langle W(L)\rangle_k$ is given by

$$\langle W(L)\rangle_k = \frac{\int d\mu_k([A]) \exp\left\{2i\pi \int_{M_3} [A] * [\eta_L]\right\}}{\int d\mu_k([A])}. \tag{4.13}$$

Equation (4.12) and property (**M2**) imply that, with the substitution $[A] \rightarrow [A] + [x\eta_\Sigma]$, the numerator of expression (4.13) can be written as

$$\int d\mu_k([A]) \exp\left\{2i\pi \int_{M_3} [A] * [\eta_{L'}]\right\}$$
$$\times \exp\left\{2i\pi k \int_{M_3} [x\eta_\Sigma] * [x\eta_\Sigma]\right\} \exp\left\{2i\pi \int_{M_3} [x\eta_\Sigma] * [\eta_L]\right\}.$$

By using the relations

$$\exp\left\{2i\pi k \int_{M_3} [x\eta_\Sigma] * [x\eta_\Sigma]\right\} = \exp\left\{(i\pi q_1^2/2k) \int_{M_3} \eta_\Sigma \wedge (d\eta_1' - d\eta_1)\right\},$$
$$\exp\left\{2i\pi \int_{M_3} [x\eta_\Sigma] * [\eta_L]\right\} = \exp\left\{(i\pi q_1^2/k) \int_{M_3} \eta_\Sigma \wedge d\eta_1\right\}$$
$$\times \exp\left\{(i\pi q_1/k) \sum_{j\neq 1} q_j \int_{M_3} \eta_\Sigma \wedge d\eta_j\right\},$$

and equations (4.9)–(4.11), one finds that the numerator of expression (4.13) assumes the form

$$\int d\mu_k([A]) \exp\left\{2i\pi \int_{M_3} [A] * [\eta_{L'}]\right\}.$$

Consequently, the expectation values of the Wilson line operators associated with the links L and L', entering equation (4.8), are equal. The same argument, applied to all the link components, implies that, for any two ambient isotopic links L and L', one has

$$\langle W(L)\rangle_k = \langle W(L')\rangle_k .$$

This concludes the proof. ■

4.6 Satellite relations

For the oriented framed knot $C \subset M_3$, let the homeomorphism $h : S^1 \times D^2 \rightarrow V_C$ be the framing of C, where V_C is a a tubular neighbourhood of C. Let us represent the disc D^2 by the set $\{z, \text{ with } |z| \leq 1\}$ of the complex plane. The framing C_f of C is given by $h(S^1 \times 1)$, whereas one can always imagine that the knot C just corresponds to $h(S^1 \times 0)$. Let P be a link in the solid torus $S^1 \times D^2$; if one replaces the knot $C \subset M_3$ by $h(P) \subset M_3$ one obtains the satellite of C which is defined by the pattern link P.

Definition 4.2. Let $B \subset S^1 \times D^2$ be the oriented link with two components $\{B_1, B_2\}$ given by $B_1 = (S^1 \times 0) \subset S^1 \times D^2$ and $B_2 = (S^1 \times 1/2) \subset S^1 \times D^2$. For any oriented framed knot $C \subset M_3$, let us denote by $C^{(2)} \in M_3$ the satellite of C with is obtained by means of the pattern link B. The two oriented components $\{K_1, K_2\}$ of $C^{(2)}$ are given by $K_1 = h(B_1)$ and $K_2 = h(B_2)$. Let us introduce a framing for the components of the link $C^{(2)}$; the knot K_1 has framing $K_{1f} = h(S^1 \times 1/4)$ and the knot K_2 has framing $K_{2f} = h(S^1 \times 3/4)$.

By construction, the satellite $C^{(2)}$ of C is an oriented framed link.

Proposition 4.3. *Let L and $\widetilde{L}$ be two oriented coloured framed links in M_3 in which $\widetilde{L}$ is obtained from $L = \{C_1, \ldots, C_N\}$ by substituting the component C_1, which has colour $q_1 \in \mathbb{Z}$, with its satellite $C_1^{(2)}$ whose components K_1 and K_2 have colours $\widetilde{q}_1 = q_1 \pm 1$ and $\widetilde{q}_2 = \mp 1$ respectively. Then, the corresponding Chern–Simons expectation values satisfy*

$$\langle W(L)\rangle_k = \langle W(\widetilde{L})\rangle_k. \tag{4.14}$$

Proof. Because of the ambient isotopy invariance of $\langle W(\widetilde{L})\rangle_k$, one can consider the limit in which the component K_1 approaches to K_2 and coincides with K_2. In this limit, for each field configuration (i.e. for each DB class) the associated holonomies $W(C_1)$ and $W(C_1^{(2)})$ coincides. This means that, at the classical level, equality (4.14) is satisfied. Thus, we only need to consider possible ambiguities in the expectation value of the composite Wilson line operator $W(C_1^{(2)}) = W(K_1)W(K_2)$ in the $K_1 \to K_2$ limit. In agreement with what we shall show in the following sections, we now assume that all the ambiguities which refer to composite Wilson line operators are eliminated by means of the framing procedure which is used to define the product $[\eta_{\widetilde{L}}] * [\eta_{\widetilde{L}}]$. According to the definition (4.3), one has

$$[\eta_L] = q_1[\eta_1] + \sum_{j=2}^{N} q_j[\eta_j] = q_1[\eta_1] + [\overline{\eta}_L],$$

$$[\eta_{\widetilde{L}}] = \widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}] + \sum_{j=2}^{N} q_j[\eta_j] = \widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}] + [\overline{\eta}_L],$$

and then

$$\begin{aligned}
[\eta_L] * [\eta_L] &= q_1^2[\eta_{C_1}] * [\eta_{C_1}] + 2q_1[\eta_{C_1}] * [\overline{\eta}_L] + [\overline{\eta}_L] * [\overline{\eta}_L],\\
[\eta_{\widetilde{L}}] * [\eta_{\widetilde{L}}] &= (\widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}]) * (\widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}])\\
&\quad + 2\,(\widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}]) * [\overline{\eta}_L] + [\overline{\eta}_L] * [\overline{\eta}_L].
\end{aligned}$$

As far as the computation of the Chern–Simons observables is concerned, ambient isotopy invariance and equality $q_1 = \widetilde{q}_1 + \widetilde{q}_2$ imply

$$2q_1[\eta_{C_1}] * [\overline{\eta}_L] = 2\,(\widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}]) * [\overline{\eta}_L],$$

moreover, by construction of the satellite $C_1^{(2)}$ and the definition (3.14), one also finds

$$q_1^2[\eta_{C_1}] * [\eta_{C_1}] = (\widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}]) * (\widetilde{q}_1[\eta_{K_1}] + \widetilde{q}_2[\eta_{K_2}])\,.$$

Therefore, as far as the computation of the Chern–Simons observables is concerned, one can replace $[\eta_L] * [\eta_L]$ by $[\eta_{\widetilde{L}}] * [\eta_{\widetilde{L}}]$, and then $\langle W(L)\rangle_k = \langle W(\widetilde{L})\rangle_k$. ■

Definition 4.3. In agreement with Proposition 4.3, for any oriented coloured framed link $L \subset M_3$, one can replace recursively all the link components which have colour given by $q \neq \pm 1$ by their satellites constructed with the pattern link B, in such a way that the resulting link $\overline{L} \subset M_3$ has the following property: each oriented framed component of $\overline{L}$ has colour which is specified by a charge $q = +1$ or $q = -1$. Remember that, for each link component C, the sign of the associated charge q is defined with respect to the orientation of C. So, with a suitable choice of the orientation of the link components, all the link components of $\overline{L}$ have charges $+1$. For each link $L \subset M_3$, the corresponding link $\overline{L} \subset M_3$ will be called the *simplicial satellite* of L and, as a consequence of Proposition 4.3, one has

$$\langle W(L)\rangle_k = \langle W(\overline{L})\rangle_k. \tag{4.15}$$

5 Abelian Chern–Simons theory on S^3

When $M_3 = S^3$, the DB cohomology group satisfies $H_D^1\left(S^3,\mathbb{Z}\right) \simeq \Omega^1\left(S^3\right)/\Omega^1_{\mathbb{Z}}\left(S^3\right)$ and one has $\Omega^1\left(S^3\right)/\Omega^1_{\mathbb{Z}}\left(S^3\right) = \Omega^1\left(S^3\right)/d\Omega^0\left(S^3\right)$. Since in general the path-integral of the Chern–Simons theory on M_3 locally corresponds to a sum over the space of 1-forms modulo forms with integer periods, it is convenient to introduce a new notation; with respect to the origin of $\Omega^1(S^3)/\Omega^1_{\mathbb{Z}}(S^3)$ that one can choose to correspond to the vanishing connection, an element of $\Omega^1(S^3)/\Omega^1_{\mathbb{Z}}(S^3)$ will be denoted by $[\alpha]$. So that, in agreement with property (**M1**), for any oriented coloured and framed link $L \subset S^3$ the expectation value (4.4) can be written as

$$\begin{aligned}\langle W(L)\rangle_k &= \frac{\int D[\alpha] \exp\left\{2i\pi k \int_{S^3} [\alpha] * [\alpha]\right\} \exp\left\{2i\pi \int_{S^3} [\alpha] * [\eta_L]\right\}}{\int D[\alpha] \exp\left\{2i\pi k \int_{S^3} [\alpha] * [\alpha]\right\}} \\ &= \frac{\int d\mu_k([\alpha]) \exp\left\{2i\pi \int_{S^3} [\alpha] * [\eta_L]\right\}}{\int d\mu_k([\alpha])}, \end{aligned} \tag{5.1}$$

where $[\alpha] \in \Omega^1(S^3)/\Omega^1_{\mathbb{Z}}(S^3)$ and $[\eta_L] \in \widetilde{H}^1_D(M_3, \mathbb{Z})$ denotes the class which is canonically associated with L. The integral (5.1) actually extends to $\mathcal{H}^1_D(S^3, \mathbb{Z})$ which has to be understood as a suitable extension of $\Omega^1(S^3)/\Omega^1_{\mathbb{Z}}(S^3)$. We shall now compute the observable $\langle W(L)\rangle_k$ for arbitrary link L.

Theorem 5.1. *Let the oriented coloured and framed link components $\{C_j\}$ of the link L, with $j = 1, 2, \ldots, N$, have charges $\{q_j\}$ and framings $\{C_{jf}\}$. Then*

$$\langle W(L)\rangle_k = \exp\left\{-(2i\pi/4k) \sum_{ij} q_i \mathbb{L}_{ij} q_j\right\}, \tag{5.2}$$

where the linking matrix $\mathbb{L}_{ij}$ is defined by

$$\begin{aligned}\mathbb{L}_{ij} &= \int_{S^3} \eta_i \wedge d\eta_j = \ell k(C_i, C_j), \qquad \text{for} \quad i \neq j, \\ \mathbb{L}_{jj} &= \int_{S^3} \eta_j \wedge d\eta_j = \ell k(C_j, C_{jf}).\end{aligned}$$

Proof. Since $H^2(S^3, \mathbb{Z}) = 0$, Poincaré duality implies that any 1-cycle on S^3 is homologically trivial. Equivalently, for each knot C_j one can find an oriented Seifert surface $\Sigma_j \subset S^3$ such that $\partial\Sigma_j = C_j$ (in fact, there is an infinite number of topologically inequivalent Seifert surfaces) and one can then define a distributional 1-form η_j (with support on Σ_j) which is globally defined in S^3. The distributional 1-form η_L associated with the link L,

$$\eta_L = \sum_j q_j \eta_j,$$

is globally defined in S^3 and, in the Chech–de Rham description of DB cocycles, the class $[\eta_L]$ can be represented by the sequence $(\eta_L, 0, 0)$. The distributional 1-form

$$\eta_L/2k = \sum_j (q_j/2k)\eta_j$$

is also globally defined in S^3 and we shall denote by $[\eta_L/2k] \in \widetilde{H}^1_D(M_3, \mathbb{Z})$ the DB class which, in the Chech–de Rham description of DB cocycles, is represented by the sequence $(\eta_L/2k, 0, 0)$. It should be noted that the class $[\eta_L/2k]$ does not depend on the particular choice of the 1-form η_L which represents $[\eta_L]$. (In turn, this implies that $[\eta_L/2k]$ does not depend on the particular choice of the Seifert surfaces.) In fact, any representative 1-form of $[\eta_L]$ can be written as $\eta_L + d\chi$ for some $\chi \in \Omega^0(S^3)$; therefore, for the corresponding class $[(\eta_L + d\chi)/2k]$ one finds

$$[(\eta_L + d\chi)/2k] = [\eta_L/2k + d\chi/2k] = [\eta_L/2k] + [d(\chi/2k)] = [\eta_L/2k].$$

By construction, the class $[\eta_L/2k]$ satisfies the relation

$$2k[\eta_L/2k] = [\eta_L],$$

therefore

$$\exp\left\{4i\pi k \int_{S^3} [\alpha] * [\eta_L/2k]\right\} = \exp\left\{2i\pi \int_{S^3} [\alpha] * [\eta_L]\right\}. \tag{5.3}$$

In agreement with property (**M2**), by means of the substitution $[\alpha] \to [\alpha] - [\eta_L/2k]$ the numerator of expression (5.1) assumes the form

$$\int d\mu_k([\alpha]) \exp\left\{-4i\pi k \int_{S^3} [\alpha] * [\eta_L/2k]\right\} \exp\left\{2i\pi k \int_{S^3} [\eta_L/2k] * [\eta_L/2k]\right\}$$
$$\times \exp\left\{2i\pi k \int_{S^3} [\alpha] * [\eta_L]\right\} \exp\left\{-2i\pi \int_{S^3} [\eta_L/2k] * [\eta_L]\right\}. \tag{5.4}$$

With the help of equation (5.3), expression (5.4) becomes

$$\exp\left\{-(2i\pi/4k) \int_{S^3} \eta_L \wedge d\eta_L\right\} \int d\mu_k([\alpha]),$$

and then

$$\langle W(L)\rangle_k = \exp\left\{-(2i\pi/4k) \int_{S^3} \eta_L \wedge d\eta_L\right\} \frac{\int d\mu_k([\alpha])}{\int d\mu_k([\alpha])}.$$

Assuming that, for the manifold S^3, one has

$$\int d\mu_k([\alpha]) \neq 0,$$

one finally obtains

$$\langle W(L)\rangle_k = \exp\left\{-(2i\pi/4k) \int_{S^3} \eta_L \wedge d\eta_L\right\}$$
$$= \exp\left\{-(2i\pi/4k) \sum_{ij} q_i q_j \int_{S^3} \eta_i \wedge d\eta_j\right\}, \tag{5.5}$$

which coincides with expression(5.2); and this concludes the proof. ■

Remark 5.1. Expression (5.2) describes an invariant of ambient isotopy (Proposition 4.2) for oriented coloured framed links. Since the matrix elements $\mathbb{L}_{ij}$ are integers, in agreement with Proposition 4.1 the observable (5.2) is invariant under the substitution $q_i \to q_i + 2k$ (for fixed i). Moreover, one can verify that Proposition 4.3 is indeed satisfied by expression (5.2).

Remark 5.2. The topological properties of knots and links in S^3 and in $\mathbb{R}^3$ are equal. Therefore, expression (5.2) also describes the Wilson line expectation values for the quantum Chern–Simons theory in $\mathbb{R}^3$ and, in fact, equation (5.2) is in agreement with the results which can be obtained by means of standard perturbation theory [33].

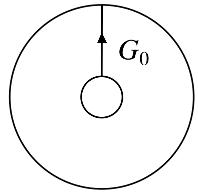

Figure 3. The region of $\mathbb{R}^3$ which is delimited by two spheres S^2, one into the other, with their face-to-face points identified, provides a description of $S^1 \times S^2$. The oriented fundamental loop $G_0 \subset S^1 \times S^2$ is also represented.

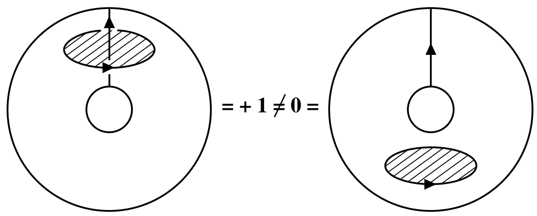

Figure 4. The trivial knot surrounding the non trivial knot G_0 is moved down (via an ambient isotopy). The intersection number of its associated surface – given by a disc – with G_0 goes from unity to 0.

6 Abelian Chern–Simons theory on $S^1 \times S^2$

One can represent $S^1 \times S^2$ by the region of $\mathbb{R}^3$ which is delimited by two concentric 2-spheres (of different radii), with the convention that the points on the two surfaces with the same angular coordinates are identified. The nontrivial knot G_0, which can be taken as generator of $H_1(S^1 \times S^2, \mathbb{Z}) \simeq \mathbb{Z}$, is shown in Fig. 3.

Let us recall that, since $H_2(S^1 \times S^2, \mathbb{Z})$ is not trivial, the linking number of two knots may not be well defined in $S^1 \times S^2$; one example is shown in Fig. 4.

Differently from S^3, the manifold $S^1 \times S^2$ has nontrivial cohomology and homology groups. While $H^3_D\left(S^1 \times S^2, \mathbb{Z}\right)$ is still canonically isomorphic to $\Omega^3\left(S^1 \times S^2\right)/\Omega^3_{\mathbb{Z}}\left(S^1 \times S^2\right)$, the group $H^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$ has the structure of a non trivial affine bundle over the second integral cohomology group $H^2\left(S^1 \times S^2, \mathbb{Z}\right) \simeq \mathbb{Z}$. As shown in Fig. 1, one can then represent $H^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$ by means of a collection of fibres over the base space $\mathbb{Z}$, each fibre has a linear space structure and is isomorphic to $\Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$. For the fiber over $0 \in \mathbb{Z}$ one can choose the trivial vanishing connection as canonical origin, so that this fibre can actually be identified with $\Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$. The fiber over $n \in \mathbb{Z}$, with $n \neq 0$, has not a canonical origin, but one can fix an origin and each element of this fibre will be written as a sum of this origin with an element of $\Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$.

6.1 Structure of the functional measure

The choice of an origin on each fibre of the affine bundle $H^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$ defines of a section s of $H^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$ over the discrete base space $\mathbb{Z} \cong H^2\left(S^1 \times S^2, \mathbb{Z}\right)$, with the convention that $s\left(0\right) = [0] \in H^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$. In agreement with property (**M1**), the quantum measure space $\mathcal{H}^1_D(S^1 \times S^2, \mathbb{Z})$ can also be understood as an affine bundle over $\mathbb{Z}$, and the section s will be used to make the structure of the functional integral explicit. Therefore, one can actually admit distributional values for s and, in fact, it is convenient to define the section s with values in $\widetilde{H}^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$.

Definition 6.1. The simplest choice for s is suggested by the additive structure of the base space. More precisely, let us pick up a nontrivial 1-cycle (or oriented knot) G_0 which is directed along the S^1 component of $S^1 \times S^2$ and is a generator of $H_1(S^1 \times S^2, \mathbb{Z}) \simeq \mathbb{Z}$. If $[\gamma_0] \in \widetilde{H}^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$ denotes the DB class which is canonically associated with G_0, we shall consider the section

$$\begin{aligned} s:\ & \mathbb{Z} \to \widetilde{H}^1_D\left(S^1 \times S^2, \mathbb{Z}\right), \\ & n \mapsto s\left(n\right) \equiv n\left[\gamma_0\right]. \end{aligned} \tag{6.1}$$

Each element $[A]$ of $\widetilde{H}^1_D\left(S^1 \times S^2, \mathbb{Z}\right)$ (and of $\mathcal{H}^1_D(S^1 \times S^2, \mathbb{Z})$) can then be written as

$$[A] = n\left[\gamma_0\right] + \left[\alpha\right],$$

for some integer n and $[\alpha] \in \Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$; and the functional measure takes the form

$$d\mu_k([A]) = \sum_{n=-\infty}^{+\infty} D[\alpha] \exp\left\{2i\pi k \int_{S^1 \times S^2} (n[\gamma_0] + [\alpha]) * (n[\gamma_0] + [\alpha])\right\}. \tag{6.2}$$

Remark 6.1. Because of the translational invariance of the quantum measure, the particular choice (6.1) of the section s will play no role in the computation of the observables. In fact, a modification of the origin of each fiber of $\mathcal{H}^1_D(S^1 \times S^2, \mathbb{Z})$ can be achieved by means of an element of $\Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$.

Expression (6.2) can be written as

$$\begin{aligned} d\mu_k([A]) = \sum_{n=-\infty}^{+\infty} & D[\alpha] \exp\left\{2i\pi k \int_{S^1 \times S^2} [\alpha] * [\alpha]\right\} \exp\left\{4i\pi k n \int_{S^1 \times S^2} [\alpha] * [\gamma_0]\right\} \\ & \times \exp\left\{2i\pi k n^2 \int_{S^1 \times S^2} [\gamma_0] * [\gamma_0]\right\}. \end{aligned} \tag{6.3}$$

As usual, in order to define $[\gamma_0] * [\gamma_0] \in \widetilde{H}^3_D\left(S^1 \times S^2, \mathbb{Z}\right)$ we shall introduce a framing G_{0f} for the knot G_0 and, in agreement with equations (3.13) and (3.14), we define $[\gamma_0] * [\gamma_0] \equiv [\gamma_0] * [\gamma_{0f}] = [0] \in \widetilde{H}^3_D(S^1 \times S^2, \mathbb{Z})$. Therefore, with integers k and n, the last factor entering expression (6.3) is well defined and it is equal to the identity. So, one obtains

$$d\mu_k([A]) = \sum_{n=-\infty}^{+\infty} D[\alpha] \exp\left\{2i\pi k \int_{S^1 \times S^2} [\alpha] * [\alpha]\right\} \exp\left\{4i\pi k n \int_{S^1 \times S^2} [\alpha] * [\gamma_0]\right\}, \tag{6.4}$$

with $[\alpha] \in \Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$.

6.2 Zero mode

Definition 6.2. Let S_0 be a oriented 2-dimensional sphere which is embedded in $S^1 \times S^2$ in such a way that it can represent a generator of $H_2(S^1 \times S^2, \mathbb{Z})$.

S_0 is isotopic with the component S^2 of $S^1 \times S^2$ and, if one represents $S^1 \times S^2$ by the region of $\mathbb{R}^3$ which is delimited by two concentric spheres, S_0 can just be represented by a third concentric sphere. We shall denote by β_0 the distributional 1-form which is globally defined in $S^1 \times S^2$ and has support on S_0; the overall sign of β_0 is fixed by the orientation of S_0 so that

$$\int_{G_0} \beta_0 = 1. \tag{6.5}$$

Since the boundary of the closed surface S_0 is trivial, one has $d\beta_0 = 0$. For any given real parameter x, the 1-form $x\beta_0$ is also globally defined in $S^1 \times S^2$; let us denote by $[x\beta_0] \in \Omega^1\left(S^1 \times S^2\right)/\Omega^1_{\mathbb{Z}}\left(S^1 \times S^2\right)$ the class which is represented by the form $x\beta_0$.

Proposition 6.1. *For each value m of the integer residues* mod $2k$, *the Chern–Simons measure* (6.4) *on $S^1 \times S^2$, with nontrivial coupling constant k, satisfies the relation*

$$d\mu_k([A]) = d\mu_k([A] + [(m/2k)\beta_0]). \tag{6.6}$$

Proof. From expression (6.4) one finds

$$\begin{aligned} & d\mu_k([A] + [(m/2k)\beta_0]) \\ & \quad = \sum_{n=-\infty}^{+\infty} D[\alpha] \exp\left\{2i\pi k \int_{S^1\times S^2} [\alpha] * [\alpha]\right\} \exp\left\{4i\pi k n \int_{S^1\times S^2} [\alpha] * [\gamma_0]\right\} \\ & \quad \times \exp\left\{4i\pi k \int_{S^1\times S^2} [\alpha] * [(m/2k)\beta_0]\right\} \exp\left\{2i\pi k \int_{S^1\times S^2} [(m/2k)\beta_0] * [(m/2k)\beta_0]\right\} \\ & \quad \times \exp\left\{4i\pi k n \int_{S^1\times S^2} [(m/2k)\gamma_0] * [\eta_0]\right\}, \end{aligned} \tag{6.7}$$

where the integer m takes the values $m = 0, 1, 2, \ldots, 2k-1$. From the equality $d\beta_0 = 0$ it follows that

$$4i\pi k \int_{S^1\times S^2} [\alpha] * [(m/2k)\beta_0] = 2i\pi m \int_{S^1\times S^2} \alpha \wedge d\beta_0 = 0,$$

where $\alpha \in \Omega^1\left(S^1 \times S^2\right)$ represents the class $[\alpha]$,

$$2i\pi k \int_{S^1\times S^2} [(m/2k)\beta_0] * [(m/2k)\beta_0] = i\pi(m^2/2k) \int_{S^1\times S^2} \beta_0 \wedge d\beta_0 = 0.$$

Finally, relation (6.5) implies

$$\exp\left\{4i\pi k n \int_{S^1\times S^2} [(m/2k)\beta_0] * [\gamma_0]\right\} = \exp\left\{2i\pi n m \int_{G_0} \beta_0\right\} = 1.$$

Therefore expressions (6.7) and (6.4) are equal. ■

6.3 Values of the observables

Let us consider an oriented coloured and framed link L in $S^1 \times S^2$; without loss of generality, one can always assume that L does not intersect the knot G_0. In agreement with equation (6.5), the integral

$$N_0(L) = \int_L \beta_0$$

takes integer values; more precisely, $N_0(L)$ is equal to the sum of the intersection numbers (weighted with the charges of the link components) of the link L with the surface S_0.

Theorem 6.1. *Given a link $L \subset S^1 \times S^2$,*

- *when $N_0(L) \not\equiv 0 \mod 2k$, one finds $\langle W(L)\rangle_k = 0$;*
- *whereas for $N_0(L) \equiv 0 \mod 2k$, one has*

$$\langle W(L)\rangle_k = \exp\left\{-(2i\pi/4k) \int_{S^1\times S^2} \eta_L \wedge d\eta_L\right\}, \tag{6.8}$$

where $\eta_L \wedge d\eta_L$ is defined by means of the framing procedure.

Proof. The expectation value of the Wilson line operator is given by

$$\langle W(L)\rangle_k = Z_k^{-1}\int d\mu_k([A])\exp\left\{2i\pi\int_{S^1\times S^2}[A]*[\eta_L]\right\}, \tag{6.9}$$

where $d\mu_k([A])$ is shown in equation (6.4) and

$$Z_k = \int d\mu_k([A]).$$

Equation (6.6) implies that $W(L)$ satisfies the following relation

$$\begin{aligned}\langle W(L)\rangle_k &= Z_k^{-1}\frac{1}{2k}\sum_{m=0}^{2k-1}\int d\mu_k([A]+[(m/2k)\beta_0])e^{2i\pi\int_{S^1\times S^2}([A]+[(m/2k)\beta_0])*[\eta_L]}\\ &= Z_k^{-1}\int d\mu_k([A])e^{2i\pi\int_{S^1\times S^2}[A]*[\eta_L]}\frac{1}{2k}\sum_{m=0}^{2k-1}e^{2i\pi\int_{S^1\times S^2}[(m/2k)\beta_0]*[\eta_L]}\\ &= \langle W(L)\rangle_k\frac{1}{2k}\sum_{m=0}^{2k-1}\exp\left\{2i\pi(m/2k)\int_L\beta_0\right\}. \end{aligned} \tag{6.10}$$

One has

$$\frac{1}{2k}\sum_{m=1}^{2k-1}\exp\left\{2i\pi N_0\left(L\right)m/2k\right\} = \begin{cases} 1 & \text{if} \quad N_0\left(L\right)\equiv 0 \mod 2k,\\ 0 & \text{otherwise.}\end{cases}$$

Therefore equation (6.10) shows that, when $N_0(L)\not\equiv 0 \mod 2k$, the expectation value $\langle W(L)\rangle_k$ is vanishing.

Let us now consider the case in which $N_0(L)\equiv 0$ mod $2k$. Because of Proposition 4.1, we only need to discuss the case $N_0(L)=0$. In fact, if $N_0(L)=2kp$ for some integer $p\neq 0$, at least one of the link components $C\subset L$ intersects S_0; one can then modify the value q_C of its charge according to $q_C\to q_C-2kp$ so that $N_0(L)$ vanishes. According to the decomposition $[A]=n[\gamma_0]+[\alpha]$, one finds

$$\begin{aligned}\exp\left\{2i\pi\int_{S^1\times S^2}[A]*[\eta_L]\right\} &= \exp\left\{2i\pi n\int_{S^1\times S^2}[\gamma_0]*[\eta_L]\right\}\exp\left\{2i\pi\int_{S^1\times S^2}[\alpha]*[\eta_L]\right\}\\ &= \exp\left\{2i\pi\int_{S^1\times S^2}[\alpha]*[\eta_L]\right\},\end{aligned}$$

where the last equality is a consequence of the identity $[\gamma_0]*[\eta_L]=[0]\in\widetilde{H}^3_D\left(S^1\times S^2,\mathbb{Z}\right)$, which follows from the framing procedure. Then, from equation (6.9) one gets

$$\langle W(L)\rangle_k = Z_k^{-1}\int\sum_{n=-\infty}^{+\infty}D[\alpha]e^{2i\pi k\int_{S^1\times S^2}[\alpha]*[\alpha]}e^{4i\pi kn\int_{S^1\times S^2}[\alpha]*[\gamma_0]}e^{2i\pi\int_{S^1\times S^2}[\alpha]*[\eta_L]}. \tag{6.11}$$

When $N_0(L)=0$, the link L is homological trivial and one can find a Seifert surface for L. More precisely, in agreement with Proposition 4.3 and equation (4.15), one can substitute L with its simplicial satellite $\overline{L}$, defined in Section 4, whose components have unitary charges. The oriented framed link $\overline{L}\subset S^1\times S^2$ also is homologically trivial and it is the boundary of an oriented surface that we shall denote by $\Sigma_{\overline{L}}\subset S^1\times S^2$. Let η_L be the distributional 1-form with support on $\Sigma_{\overline{L}}$ which is globally defined in $S^1\times S^2$; because of Proposition 4.3, in the Chech–de Rham description of the DB classes, $[\eta_L]$ can then be represented by the sequence $(\eta_L,0,0)$.

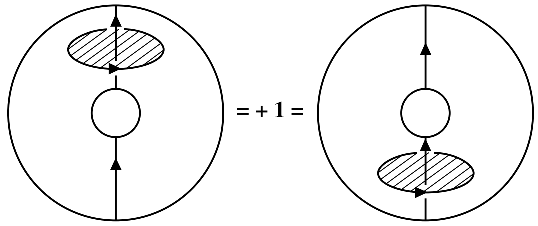

Figure 5. An example of conservation of the intersection number under ambient isotopy for a globally trivial 1-cycle.

The 1-form $(1/2k)\eta_L$ also is globally defined in $S^1 \times S^2$ and we shall denote by $[(1/2k)\eta_L]$ the DB class which is represented by the form $(1/2k)\eta_L$. By construction,

$$\exp\left\{-4i\pi k \int_{S^1\times S^2} [\alpha] * [(1/2k)\eta_L]\right\} = \exp\left\{-2i\pi \int_{S^1\times S^2} [\alpha] * [\eta_L]\right\}, \tag{6.12}$$

and the condition $N_0(L) = 0$ (or $N_0(L) \equiv 0 \bmod 2k$) implies that, for integer n,

$$\exp\left\{-4i\pi kn \int_{S^1\times S^2} [(1/2k)\eta_L] * [\gamma_0]\right\} = 1. \tag{6.13}$$

By means of the substitution $[\alpha] \to [\alpha] - [(1/2k)\eta_L]$ and with the help of equations (6.12) and (6.13), expression (6.11) assumes the form

$$\langle W(L)\rangle_k = \exp\left\{-(2i\pi/4k)\int_{S^1\times S^2} \eta_L \wedge d\eta_L\right\} Z_k^{-1} Z_k .$$

Therefore, assuming $Z_k \neq 0$, when $N_0(L) \equiv 0 \bmod 2k$ one gets

$$\langle W(L)\rangle_k = \exp\left\{-(2i\pi/4k)\int_{S^1\times S^2} \eta_L \wedge d\eta_L\right\},$$

and this concludes the proof. ■

Remark 6.2. Expression (6.8) formally coincides with the result (5.5) which has been obtained in the case $M_3 \sim S^3$. It should be noted that the integral (which appears in equation (6.8))

$$\int_{S^1\times S^2} \eta_L \wedge d\eta_L \equiv \int_{S^1\times S^2} \eta_L \wedge d\eta_{\overline{L}_f} = \int_{\overline{L}_f} \beta_L, \tag{6.14}$$

where $\overline{L}_f$ denotes the framing of $\overline{L}$, is well defined because it does not depend on the choice of the Seifert surface of $\overline{L}$. Indeed suppose that, instead of $\Sigma_{\overline{L}}$, we take $\Sigma'_{\overline{L}}$ as Seifert surface for the link $\overline{L}$. The difference between the intersection number (6.14) of $\overline{L}_f$ with $\Sigma'_{\overline{L}}$ and $\Sigma_{\overline{L}}$ is given by the intersection number of $\overline{L}_f$ with the closed surface $\Sigma'_{\overline{L}} \cup \Sigma^{-1}_{\overline{L}}$. This surface could be nontrivial in $S^1 \times S^2$ but, since $\overline{L}$ is homologically trivial, $\overline{L}_f$ also is homologically trivial and then its intersection number with a closed surface vanishes. The example of Fig. 5 illustrates the ambient isotopy invariance of the intersection number of a homologically trivial link with the Seifert surface of a trivial knot in $S^1 \times S^2$.

7 Abelian Chern–Simons theory on $S^1 \times \Sigma_g$

Let us now consider the manifold $M_3 \sim S^1 \times \Sigma_g$ where Σ_g is a closed Riemann surface of genus $g \geq 1$. In this case, the computation of the Chern–Simons observables is rather similar to the computation when $M_3 \sim S^1 \times S^2$. So, we shall briefly illustrate the main steps of the construction.

As it has been mentioned in Section 1, $H^1_D(S^1 \times \Sigma_g, \mathbb{Z})$ has the structure of a affine bundle over $H^2(S^1 \times \Sigma_g, \mathbb{Z}) \sim \mathbb{Z}^{2g+1}$ with $\Omega^1(S^1 \times \Sigma_g)/\Omega^1_{\mathbb{Z}}(S^1 \times \Sigma_g)$ acting canonically on each fibre by translation. In agreement with property (**M1**), the functional space $\mathcal{H}^1_D(S^1 \times \Sigma_g, \mathbb{Z})$ is assumed to have the same structure of $H^1_D(S^1 \times \Sigma_g, \mathbb{Z})$ and, in order to fix a origin in each fibre, we need to introduce a section $s : \mathbb{Z}^{2g+1} \to \mathcal{H}^1_D(S^1 \times \Sigma_g, \mathbb{Z})$.

Definition 7.1. Let the nonintersecting oriented framed knots $\{G_0, G_1, \dots, G_{2g}\}$ in $S^1 \times \Sigma_g$ represent the generators of $H_1\left(S^1 \times \Sigma_g, \mathbb{Z}\right)$. For each $j = 0, 1, \dots, 2g$, we shall denote by $[\gamma_j] \in \widetilde{H}^1_D(S^1 \times \Sigma_g, \mathbb{Z})$ the DB class which is canonically associated with the knot G_j.

Definition 7.2. If the elements of $\mathbb{Z}^{2g+1}$ are represented by vectors

$$\vec{n} \equiv (n_0, n_1, n_2, \dots, n_{2g}) \in \mathbb{Z}^{2g+1},$$

a possible choice for the section s is given by

$$s : \ \mathbb{Z}^{2g+1} \to \widetilde{H}^1_D\left(S^1 \times \Sigma_g, \mathbb{Z}\right),$$
$$\vec{n} \mapsto s\left(\vec{n}\right) = [n\gamma] \equiv \vec{n} \cdot [\vec{\gamma}] = \sum_{j=0}^{2g} n_j [\gamma_j].$$

Each class $[A] \in \widetilde{H}^1_D(S^1 \times \Sigma_g, \mathbb{Z})$ can then be written as

$$[A] = [n\gamma] + [\alpha],$$

for certain $\vec{n}$ and $[\alpha] \in \Omega^1(S^1 \times \Sigma_g)/\Omega^1_{\mathbb{Z}}(S^1 \times \Sigma_g)$. Consequently, the Chern–Simons functional measure takes the form

$$d\mu_k([A]) = \sum_{\vec{n}} D[\alpha] \exp\left\{2i\pi k \int_{S^1 \times S^2} [\alpha] * [\alpha]\right\} \exp\left\{4i\pi k \int_{S^1 \times S^2} [\alpha] * [n\gamma]\right\}, \tag{7.1}$$

which is the analogue of equation (6.4). The condition $[n\gamma] * [n\gamma] = 0 \in \widetilde{H}^3_D(S^1 \times \Sigma_g, \mathbb{Z})$, which results from the framing procedure, has already been used to simplify the expression of $d\mu_k([A])$.

Definition 7.3. Let the oriented closed surfaces $S_j \subset S^1 \times \Sigma_g$, with $j = 0, 1, \dots, 2g$, represent the generators of $H_2(S^1 \times \Sigma_g, \mathbb{Z}) \sim \mathbb{Z}^{2g+1}$. We shall denote by $\beta_j \in \widetilde{H}^1_D\left(S^1 \times \Sigma_g, \mathbb{Z}\right)$ the distributional 1-form which is globally defined in $S^1 \times \Sigma_g$ and has support on S_j. One can choose the generators of $H_2(S^1 \times \Sigma_g, \mathbb{Z})$ in such a way that the following orthogonality relations are satisfied

$$\int_{G_i} \beta_j = \delta_{ij}, \qquad i, j = 0, 1, \dots, 2g.$$

Since S_j are closed surfaces, one has $d\beta_j = 0$. For any real parameter x, the 1-form $x\beta_j$ also is globally defined in $S^1 \times \Sigma_g$ and the corresponding class, which can be represented by $x\beta_j$, will be denoted by $[x\beta_j] \in \Omega^1(S^1 \times \Sigma_g)/\Omega^1_{\mathbb{Z}}(S^1 \times \Sigma_g)$. The arguments that have been presented to prove Proposition 6.1 can also be used to prove the following

Proposition 7.1. *The quantum measure* (7.1) *of the Chern–Simons theory on* $S^1 \times \Sigma_g$*, with nontrivial coupling constant* k*, satisfies the relation*

$$d\mu_k([A]) = d\mu_k([A] + [(m/2k)\beta_j]).$$

for $m = 0, 1, 2, \ldots, 2k-1$ *and for each value of* $j = 0, 1, \ldots, 2g$.

Finally, the expectation values of the Wilson line operators are determined by the following

Theorem 7.1. *Let* L *be a oriented coloured framed link in* $S^1 \times \Sigma_g$*. For each* $j = 0, 1, \ldots, 2g$*, let us introduce the integer*

$$N_j(L) = \int_L \beta_j .$$

Then

- *when* $N_j(L) \not\equiv 0 \mod 2k$ *for at least one value of* $j = 0, 1, \ldots, 2g$*, one has* $\langle W(L) \rangle_k = 0$ *;*
- *whereas when* $N_j(L) \equiv 0 \mod 2k$ *for all values of* $j = 0, 1, \ldots, 2g$*, one finds*

$$\langle W(L) \rangle_k = \exp \left\{ -(2i\pi/4k) \int_{S^1 \times \Sigma_g} \eta_L \wedge d\eta_L \right\}, \tag{7.2}$$

where $\eta_L \wedge d\eta_L$ *is defined by means of the framing procedure.*

Proof. The proof is similar to the proof of Theorem 6.1. In fact, when $N_j(L) \not\equiv 0 \mod 2k$ for at least one value of $j = 0, 1, \ldots, 2g$, Proposition 7.1 implies that the Chern–Simons expectation value $\langle W(L) \rangle_k$ vanishes. On the other hand, when $N_j(L) \equiv 0 \mod 2k$ for all values of $j = 0, 1, \ldots, 2g$, the substitution $[\alpha] \to [\alpha] - [(1/2k)\eta_L]$ in the functional measure (7.1) leads to the equation (7.2). It should be noted that expression (7.2) is well defined because the link L and then its framing L_f are homologically trivial. ■

8 Surgery rules

For the quantum Abelian Chern–Simons theory on the manifolds $S^1 \times S^2$ and $S^1 \times \Sigma_g$ (and, in general, in any nontrivial 3-manifold), the standard gauge theory approach which is based on the gauge group $U(1)$ is in principle well defined but presents some technical difficulties, which are related, for instance, to the implementation of the gauge fixing procedure and the determination of the Feynman propagator. As a matter of facts, by means of the usual methods of quantum gauge theories, the computation of the Chern–Simons observables in a nontrivial 3-manifold has never been explicitly produced.

In order to determine the Wilson line expectation values in $M_3 \not\sim S^3$, one can use for instance the surgery rules of the Reshetikhin–Turaev type [6] as developed by Lickorish [39] and by Morton and Strickland [40]. In this section, we outline the surgery method which turns out to produce the Chern–Simons observables for the manifolds $S^1 \times S^2$ and $S^1 \times \Sigma_g$ in complete agreement with the results obtained in the DB approach of the path-integral.

Every closed orientable connected 3-manifold M_3 can be obtained by Dehn surgery on S^3 and admits a surgery presentation [29] which is described by a framed surgery link $\mathcal{L} \subset S^3$ with integer surgery coefficients. Each surgery coefficient specifies the framing of the corresponding component of $\mathcal{L}$ because it coincides with the linking number of this component with its framing. The manifold $S^1 \times S^2$ admits a presentation with surgery link given by the unknot with vanishing surgery coefficient, whereas $S^1 \times S^1 \times S^1$ for example corresponds to the Borromean rings with vanishing surgery coefficients. Any oriented coloured framed link $L \subset M_3$ can be described by a link $L' = L \cup \mathcal{L}$ in S^3 in which:

- the surgery link $\mathcal{L}$ describes the surgery instructions corresponding to a presentation of M_3 in terms of Dehn surgery on S^3;
- the remaining components of L' describe how L is placed in M_3.

Assuming that the expectation values of the Wilson line operators form a complete set of observables, one can find [33] consistent surgery rules, according to which the expectation value of the Wilson line operator $W(L)$ in M_3 can be written as a ratio

$$\langle W(L)\rangle_k|_{M_3} = \langle W(L)W(\mathcal{L})\rangle_k|_{S^3} \;/\; \langle W(\mathcal{L})\rangle_k|_{S^3}, \tag{8.1}$$

where to each component of the surgery link is associated a particular colour state ψ_0. Remember that, for fixed integer k, the colour space coincides with space of residue classes of integers mod $2k$, which has a canonical ring structure; let χ_j denote the residue class associated with the integer j. Then, the colour state ψ_0 is given by

$$\psi_0 = \sum_{j=0}^{2k-1} \chi_j.$$

One can verify that the surgery rule (8.1) is well defined and consistent; in fact, expression (8.1) is invariant under Kirby moves [41]. Finally, one can check that, according to the surgery formula (8.1), the expectation values of the Wilson line operators in $S^1 \times S^2$ and in $S^1 \times \Sigma_g$ are given precisely by the expressions of Theorems 6.1 and 7.1, which have been obtained by means of the DB cohomology.

9 Conclusions

In the standard field theory formulation of Abelian gauge theories, the (classical fields) configuration space is taken to be the set of 1-forms modulo closed forms. But when the observables of the theory are given by the exponential of the holonomies which are associated with oriented loops, the classical configuration space is actually given by the set of 1-forms modulo forms of integer periods; that is, the classical configuration space indeed coincides with space of the Deligne–Beilinson cohomology classes. So, in this article we have considered the Abelian Chern–Simons gauge theory, in which a complete set of observables is given by the set of exponentials of the holonomies which are associated with oriented knots or links in a 3-manifold M_3. We have explored the main properties of the quantum theory and of the corresponding quantum functional integral, which enters the computation of the observables, when the path-integral is really defined over the Deligne–Beilinson classes. Within this new approach, we have produced an explicit path-integral computation of the Chern–Simons link invariants in a class of torsion-free 3-manifolds. In facts, we have not used any standard gauge-fixing and perturbative method, as it has been done so far in literature. Our results are based on an explicit non-perturbative path-integral computation and are exact results.

Let us briefly summarize the main issues of our article. In Sections 2 and 3 we have discussed a few technical points which are important for the computation of the observables. The basic definitions and properties of the DB cohomology together with a distributional extension of the space of the equivalence classes have been illustrated. Then we have shown how the framing procedure, which is used to give a topological meaning to the self-linking number, can be naturally defined also in the DB context. The general features of the Abelian Chern–Simons theory in a generic 3-manifold M_3 have been derived in Section 4. The main achievements concerning the observables are the “colour periodicity” property (Proposition 4.1), the “ambient isotopy invariance” (Proposition 4.2) and the validity of appropriate “satellite relations” (Proposition 4.3).

With respect to the standard field theory approach, our proofs extend the validity of these properties from $\mathbb{R}^3$ to a generic (closed and oriented) manifold M_3.

The Abelian Chern–Simons theory formulated in S^3 is discussed in Section 5 and its solution is given by Theorem 5.1; in this case, the outcome is in agreement with the results obtained by means of standard perturbation theory in $\mathbb{R}^3$. The expressions of the observables for the Chern–Simons theory formulated in $S^1 \times S^2$ and in a generic 3-manifold of the type $S^1 \times \Sigma_g$ are contained in Theorems 6.1 and 7.1; in the standard field theory approach, no proof of these theorems actually exists.

Finally, we have checked the validity our path-integral results by means of an alternative "combinatorial method". Indeed, the link invariants defined in the Chern–Simons theory are related to the link invariants defined by means of the quantum group methods of Reshetikhin and Turaev. Given a surgery presentation in S^3 of a generic 3-manifold M_3 and knowing the values of the link invariants in S^3, one can use the surgery method of Lickorish and Morton–Strickland to determine the values of the link invariants in M_3. As far as the Abelian Chern–Simons is concerned, we have presented the basic aspects of this surgery method in Section 8. We have verified that the expression of the link invariants for the manifolds $S^1 \times S^2$ and $S^1 \times \Sigma_g$, which are described by Theorems 6.1 and 7.1, precisely coincide with the results obtained by means of the surgery method.

Clearly, in the case of a generic 3-manifold, the general features of the Deligne–Beilinson approach to the Abelian Chern–Simons functional integral remain to be fully explored. Possible applications of this formalism to the non-Abelian Chern–Simons theory would also give new hints on the topological meaning of the polynomial link invariants. Finally, we mention that extensions of Deligne–Beilinson cohomology approach to the topological field theories in lower dimensions can easily be produced, but the resulting structure of the observables appears to be quite elementary. Presumably, applications in higher dimensions will produce more interesting invariants.

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Deligne–Beilinson cohomology and Abelian link invariants: Torsion case

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For the Abelian Chern–Simons field theory, we consider the quantum functional integration over the Deligne–Beilinson cohomology classes and present an explicit path-integral nonperturbative computation of the Chern–Simons link invariants in $SO(3) \simeq \mathbb{R}P^3$, a toy example of a 3-manifold with torsion. © *2009 American Institute of Physics.* [doi:10.1063/1.3266178]

I. INTRODUCTION

In a quite recent paper,[9] we have shown how Deligne–Beilinson (DB) cohomology[5,2,6,12,3,1] within Chern–Simons (CS) quantum field theory (QFT) framework[21,10,18,22,13,19,8,7] can be used to provide a nonperturbative way to compute Abelian link invariants on some three dimensional manifolds, such as S^3, $S^2 \times S^1$, etc. In particular, quantization of the CS parameter k, as well as the charges q of the links, was a straightforward consequence of the use of DB cohomology, and the standard regularization via framing was directly interpreted as the problem of regularizing the product of two distributional DB cohomology classes.

Actually, this former article only deals with torsion free (oriented) 3-manifold. We are going to mend this lake of generality by explaining how to extend our approach to (oriented) 3-manifold with torsion. As a school case, we will consider the oriented 3-manifold $SO(3) \simeq \mathbb{R}P^3$.

In Sec. II, we will recall some basic facts concerning DB cohomology and how it relates to the functional measure based on the Abelian CS action. In Sec. III, we will deal with Wilson lines themselves.

Here are the following three results we will obtain:

(1) the CS level parameter k has to be even;
(2) trivial cycles give the same result than in S^3; and
(3) torsion cycles must hold an even charge,

in perfect agreement with surgery methods.

Throughout this paper we will use the notation $\underset{\mathbb{Z}}{=}$, which stands for equality modulo $\mathbb{Z}$.

II. DB COHOMOLOGY: CONSTRAINTS ON THE LEVEL *k* OF THE ABELIAN CS THEORY

Let us recall that DB cochains can be seen as generalizations of $U(1)$-connections on $U(1)$-principal bundles over smooth manifolds, their classes classifying the corresponding objects, i.e., $U(1)$-gerbes with connections.[3,15] Concentrating on the case of an oriented 3-manifold M, its DB cohomology space $H^1_D(M,\mathbb{Z})$ is canonically embedded into the following exact sequence:[3,11]

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$$0 \to \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \to H^1_D(M,\mathbb{Z}) \to \check{H}^2(M,\mathbb{Z}) \to 0, \tag{2.1}$$

where $\Omega^1(M)$ is the space of smooth 1-form on M, $\Omega^1_{\mathbb{Z}}(M)$ is the space of smooth closed 1-form with integral periods on M, and $\check{H}^2(M,\mathbb{Z})$ is the second integral Čech cohomology group of M. Actually, $H^1_D(M,\mathbb{Z})$ can also be embedded into[11]

$$0 \to \check{H}^1(M,\mathbb{R}/\mathbb{Z}) \to H^1_D(M,\mathbb{Z}) \to \Omega^2_{\mathbb{Z}}(M) \to 0, \tag{2.2}$$

where $\check{H}^1(M,\mathbb{R}/\mathbb{Z})$ is the first $\mathbb{R}/\mathbb{Z}$-valued Čech cohomology group of M and $\Omega^2_{\mathbb{Z}}(M)$ is the space of smooth closed 2-form with integral periods on M. Each one of these two exact sequences has its own interest to describe $H^1_D(M,\mathbb{Z})$, but both give this space the structure of an affine bundle, with (discrete) base $\check{H}^2(M,\mathbb{Z})$ and translation group $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$ from the former sequence, and with base $\Omega^2_{\mathbb{Z}}(M)$ and translation group $\check{H}^1(M,\mathbb{R}/\mathbb{Z})$ from the latter one.

The other important DB space we will need is $H^3_D(M,\mathbb{Z})$. However, the exact sequences of the previous type into which this space is embedded both lead to $H^3_D(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$.

A (graded) pairing between DB cohomology spaces can be introduced. In our particular case of interest, it reduces to a commutative product,

$$*_D : H^1_D(M,\mathbb{Z}) \times H^1_D(M,\mathbb{Z}) \to H^3_D(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}. \tag{2.3}$$

The "DB square" of a class $[\omega] \in H^1_D(M,\mathbb{Z})$,

$$cs_1([\omega]) \equiv [\omega] *_D [\omega] \tag{2.4}$$

canonically identifies with the Abelian CS Lagrangian, while the level k CS Lagrangian simply reads as

$$cs_k([\omega]) \equiv k \cdot cs_1([\omega]) = k \cdot [\omega] *_D [\omega]. \tag{2.5}$$

Of course, due to the $\mathbb{Z}$-module structure of DB spaces, $cs_k([\omega])$ belongs to $H^3_D(M,\mathbb{Z})$ if and only if $k \in \mathbb{Z}$.

In fact, DB classes are another point of view for what is called Cheeger–Simons differential characters (see, for instance, Refs. 3, 4, 14, 11, and 1). This implies that any DB cohomology class can be integrated over any (integral) cycle of M of the corresponding dimension. However, the result takes values in $\mathbb{R}/\mathbb{Z}$ and not $\mathbb{R}$ like in standard integration. Integral 3-cycles on an oriented 3-manifold are just integer multiples of M. Hence, the Lagrangian $cs_k([\omega])$ defines the well known level k CS action as

$$CS_k([\omega]) \equiv k \int_M cs_1([\omega]) = k \int_M [\omega] *_D [\omega], \tag{2.6}$$

which takes its values in $\mathbb{R}/\mathbb{Z}$ if and only if $k \in \mathbb{Z}$. We now have all the necessary ingredients to try to define the functional "CS measure" on $H^1_D(M,\mathbb{Z})$, denoted by

$$\mu_k([\omega]) \equiv D[\omega] \exp\left\{ 2i\pi k \int_M [\omega] *_D [\omega] \right\}. \tag{2.7}$$

Let us point out that (2.7) imposes quantization of the level k, that is to say,

$$k \in \mathbb{Z} \tag{2.8}$$

for the exponential to be well defined. The procedure giving a meaning to (2.7) was detailed in Ref. 9. To make it, short let us say that if we choose the exact sequence (2.1) as defining $H^1_D(M,\mathbb{Z})$, the measure will be made of a discrete sum indexed by elements of $\check{H}^2(M,\mathbb{Z})$; then, we pick up an origin on every (affine) fiber, and for each of these fibers, we consider a (formal) measure over the translation group $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. As already noted and extensively used in Ref. 9, the CS measure satisfies

$$\mu_k([\omega]+\bar{\alpha}) = \mu_k([\omega])\exp\left\{2i\pi k\int_M (2[\omega]*_D\bar{\alpha} + \bar{\alpha}*_D\bar{\alpha})\right\} \tag{2.9}$$

for all $\bar{\alpha}\in\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$, which is similar to the Cameron–Martin property cylindrical functional measures verified.

In addition to the product $*_D$, integration of elements of $H^1_D(M,\mathbb{Z})$ over 1-cycle on M also provides a pairing,

$$\oint : H^1_D(M,\mathbb{Z}) \times Z_1(M) \to \mathbb{R}/\mathbb{Z}, \tag{2.10}$$

where $Z_1(M)$ denotes the Abelian group of (integral) 1-cycle on M. This pairing allows us to see 1-cycle on M as elements of $H^1_D(M,\mathbb{Z})^* \equiv Hom(H^1_D(M,\mathbb{Z}),\mathbb{R}/\mathbb{Z})$, the Pontrjagin dual of $H^1_D(M,\mathbb{Z})$. This dual space is itself embedded into dual sequences,[11]

$$0 \to \check{H}^1(M,\mathbb{R}/\mathbb{Z}) \to H^1_D(M,\mathbb{Z})^* \to Hom(\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M),\mathbb{R}/\mathbb{Z}) \to 0 \tag{2.11}$$

and

$$0 \to Hom(\Omega^2_{\mathbb{Z}}(M),\mathbb{R}/\mathbb{Z}) \to H^1_D(M,\mathbb{Z})^* \to \check{H}^2(M,\mathbb{Z}) \to 0, \tag{2.12}$$

both being very similar to the original sequences (2.1) and (2.2). On the other hand, the DB product (2.3) also allows us to canonically identify $H^1_D(M,\mathbb{Z})$ as a subspace of $H^1_D(M,\mathbb{Z})^*$ via integration over M, which is also legitimated by the sequences above. However, since $Z_1(M)\subset H^1_D(M,\mathbb{Z})^*$, one is naturally led to consider the possibility to associate with each 1-cycle, z, on M a (distributional) DB class, $[\eta_z]$. Details of this association can be found in Ref. 1. These arguments look totally similar to how smooth functions can be considered as distributions via standard integration and how chains can be seen as de Rham currents, except that everything is done with respect to $\mathbb{R}/\mathbb{Z}$ and not $\mathbb{R}$.

The usefulness of the Pontrjagin dual in our problem is deeply related to the fact that in QFT, the quantum configuration space is made of distributional objects, and not just smooth ones. The first consequence will be an attempt to extend the CS measure to $H^1_D(M,\mathbb{Z})^*$. However, while the DB product (2.3) obviously extends to

$$*_D : H^1_D(M,\mathbb{Z}) \times H^1_D(M,\mathbb{Z})^* \to \mathbb{R}/\mathbb{Z}, \tag{2.13}$$

it is hopeless to try to extend it straightforwardly to

$$*_D : H^1_D(M,\mathbb{Z})^* \times H^1_D(M,\mathbb{Z})^* \to \mathbb{R}/\mathbb{Z} \tag{2.14}$$

since we will face the problem of defining product of distributions (or currents). Actually, we will not really need to give a meaning to the products of any two elements of $H^1_D(M,\mathbb{Z})^*$. We will only need to define products such as $[\eta_z]*_D[\eta_z]$, where $[\eta_z]$ is the DB representative of a 1-cycle, z, on M. For the rest, we just need to assume that there is a functional measure on the quantum configuration space ($\subseteq H^1_D(M,\mathbb{Z})^*$), which satisfies the Cameron–Martin-like property (2.9) (see Ref. 9 and references therein concerning this point).

Let us now deal with Wilson lines. We will explicitly consider $M=\mathbb{R}P^3$, although our treatment is quite obviously general.

III. EXPECTATION VALUE OF WILSON LINES WITH TORSION IN THE ABELIAN CS THEORY: $M=\mathbb{R}P^3$ CASE

The 3-manifold $M=\mathbb{R}P^3$ is among the simplest ones involving torsion. Indeed, and due to Poincaré duality, we have

$$\check{H}^2(M,\mathbb{Z}) \simeq \check{H}_1(M,\mathbb{Z}) = \mathbb{Z}_2,$$

$$\check{H}^1(M,\mathbb{Z}) \simeq \check{H}_2(M,\mathbb{Z}) = 0. \quad (3.1)$$

The first equation, together with (2.1), implies that $H_D^1(M,\mathbb{Z})$ is an affine fiber bundle with base space $\mathbb{Z}_2 \equiv \{\check{0},\check{1}\}$, with $2\cdot\check{1}=\check{0}$. The fiber over $\check{0}$ clearly contains the zero $U(1)$-connection, $[0]$, which plays the role of a canonical origin in this fiber so that a DB class $[\omega_0]$ over $\check{0}$ satisfies

$$[\omega_0] = [0] + \bar{\alpha} \quad (3.2)$$

for some $\bar{\alpha} \in \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$. Over $\check{1}$ there is unfortunately no such canonical choice. Nevertheless, from the exact sequence (2.12), we see that $H_D^1(M,\mathbb{Z})^*$ is also an affine bundle with base space $\mathbb{Z}_2 \equiv \{\check{0},\check{1}\}$. Thus, the choice of $[0]$ for origin on the fiber over $\check{0}$ still holds. Now, as explained in [GT], and because of the inclusion $Z_1(M) \subset H_D^1(M,\mathbb{Z})^*$, there is a family of "natural" choices of origin for the fiber over $\check{1}$ provided by 1-cycle, z, on M, or rather by their DB representatives $[\eta_z]$. All we have to assume is that such an origin also belongs to the quantum configuration space of the theory. We can then formally write the functional CS measure on $H_D^1(M,\mathbb{Z})^*$,

$$\mu_k([\omega]) \equiv D\bar{\alpha}\, \exp\left\{2i\pi k \int_M \bar{\alpha} *_D \bar{\alpha}\right\} + D\bar{\alpha}\, \exp\left\{2i\pi k \int_M ([\eta_1]+\bar{\alpha}) *_D ([\eta_1]+\bar{\alpha})\right\}, \quad (3.3)$$

where $[\eta_1]$ is the origin on the fiber over $\check{1}$ associated with some given (and so fixed) torsion cycle τ_1 on M. In the second term of (3.3), there appears the quantity $[\eta_1]*_D[\eta_1]$, which is ill defined as being a product of distributions (or rather de Rham currents). This is where regularization is required. Actually, and as mentioned earlier, regularization is only required later on when computing expectation values of Wilson lines. However, as we will see (check Ref. 9), the quantities to regularize are of the type $[\eta_1]*_D[\eta_1]$. This is why we are going to deal with regularization right now.

A. Regularization of $[\eta_1]*_D[\eta_1]$ via framing: Linking numbers of torsion cycles

When a cycle z is trivial, i.e., $z=bc$, with b as the usual boundary operator, one can define the self-linking number of z as the linking number of z with z^f, where z^f is a framing of z. This reads as

$$L(z,z) \equiv L(z,z^f) \equiv c \cap z^f, \quad (3.4)$$

with $\cap$ denoting the transverse intersection. Of course, the result fully depends on the chosen framing of z. This also provides a regularization procedure for $[\eta_1]*_D[\eta_1]$. Indeed, if z and z' are two trivial cycles in M without any common points, their DB representatives, $[\eta_z]$ and $[\eta_{z'}]$, satisfy

$$[\eta_z]*_D[\eta_{z'}] = [0] + \overline{\eta_z \wedge d\eta_{z'}} \in H_D^3(M,\mathbb{Z})^* \equiv \mathbb{R}/\mathbb{Z}, \quad (3.5)$$

where η_z ($\eta_{z'}$) is the de Rham current of the cycle z (z') such that $z=bc$ ($z'=bc'$). However, $\eta_z \wedge d\eta_{z'}$ is the de Rham current representing the intersection $c\cap z' = c'\cap z$. Accordingly, $\int_M \eta_z \wedge d\eta_{z'} \in \mathbb{Z}$ so that $[\eta_z]*_D[\eta_{z'}]=[0]$. Note that we did not use any regularizing at this stage. We can now apply this to z and z^f, leading to $[\eta_z]*_D[\eta_{z^f}]=[0]$. Thus, the framing procedure can be used to regularize $[\eta_z]*_D[\eta_z]$ into $[0]$. It can even be applied for a non trivial (but torsionless) cycle (see Refs. 1 and 9 for details).

For two torsion cycles τ and τ' on M, we have $2\tau = b\zeta$ and $2\tau' = b\zeta'$. Hence, $\zeta \cap \tau'$ and $\zeta' \cap \tau$ are still well defined integers. The linking number this torsion cycle is then

$$L(\tau,\tau') = \tfrac{1}{2}\zeta \cap \tau' \in \tfrac{1}{2}\mathbb{Z}. \tag{3.6}$$

Due to the occurrence of the $\frac{1}{2}$ factor in (3.6), we immediately conclude that there is no chance for the framing procedure to regularize $[\eta_\tau]*_D[\eta_\tau]$ into $[0]$. Accordingly, the term $[\eta_1]*_D[\eta_1]$ appearing within (3.3) will plague the CS measure since, by construction, it is built from a torsion cycle. Fortunately, there is the level parameter k also occurring in (3.3). Now, if $k=2l$, then $k[\eta_1]*_D[\eta_1] = l\cdot 2[\eta_1]*_D[\eta_1]$, and hence the framing procedure consistently applies to $2[\eta_1]*_D[\eta_1]$ because the factor $\frac{1}{2}$ into (3.6) is now vanishing. Thus, here comes a new constraint on the CS level parameter for $M=\mathbb{R}P^3$,

$$k = 2l, \quad l \in \mathbb{Z}. \tag{3.7}$$

Note that one could decide to regularize by using only an "even" framing, keeping $k\in\mathbb{Z}$. However, obviously, this would be totally equivalent to consider any framing and $k=2l$. This is this last point of view we will chose and from now on k will be even.

We are now ready to look at Wilson lines.

B. Expectation value of a Wilson line on $M=\mathbb{R}P^3$: Trivial cycles and torsion cycles with charge q

Let z be a 1-cycle on $M=\mathbb{R}P^3$. As previously explained, for any $[\omega]\in H_D^1(M,\mathbb{Z})$

$$\int_z [\omega] \in \mathbb{R}/\mathbb{Z}. \tag{3.8}$$

This integral defines parallel transport of the connection $[\omega]$ along the cycle z, and

$$\exp\left\{2i\pi\int_z [\omega]\right\} \tag{3.9}$$

is called the $U(1)$-holonomy of z with respect to the connection (or to the DB class) $[\omega]$. We also noticed that it is possible to write

$$\int_z [\omega] \underset{\mathbb{Z}}{=} \int_M [\omega]*_D[\eta_z] \tag{3.10}$$

for $[\eta_z]\in H_D^1(M,\mathbb{Z})^*$, canonically representing z. As long as $[\omega]$ is smooth, formula (3.10) is well defined, but since we need to go to $H_D^1(M,\mathbb{Z})^*$, once more, some regularization will be required. On the other hand, a fundamental loop is a continuous mapping, $f: S^1 \to M$, such that $f(S^1)\simeq S^1$. A singular decomposition of S^1 provides a singular decomposition of $f(S^1)$ so that this last quantity can be considered as a (singular) 1-cycle on M. Then, we can consider linear combinations,

$$z = \sum_i^N q_i Z_i, \tag{3.11}$$

where the Z_i are fundamental loops without any common points.

From now on, we will assume that the functional CS measure is (existing and) normalized so that

$$\int \mu_k([\omega]) = 1. \tag{3.12}$$

The expectation values of the Wilson line for a fundamental loop Z with respect to the level k CS measure formally read as

$$\langle W(Z)\rangle_k \equiv \left\langle \exp\left\{2i\pi\int_Z [\omega]\right\}\right\rangle \equiv \int \mu_k([\omega])\exp\left\{2i\pi\int_Z [\omega]\right\}, \tag{3.13}$$

and for a cycle $z=qZ$

$$\langle W(z=qZ)\rangle_k = \int \mu_k([\omega])\exp\left\{2iq\pi\int_Z [\omega]\right\}. \tag{3.14}$$

From (3.10), we can equivalently write

$$\langle W(z=qZ)\rangle_k = \int \mu_k([\omega])\exp\left\{2i\pi q\int_M [\omega]*_D[\eta_Z]\right\}. \tag{3.15}$$

Finally, substituting (3.3) into (3.15), we obtain

$$\langle W(z=qZ)\rangle_k = \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M \bar{\alpha}*_D(k\bar{\alpha}+q[\eta_Z])\right\} + \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M ([\eta_1]+\bar{\alpha})*_D(k[\eta_1]+k\bar{\alpha}+q[\eta_Z])\right\}. \tag{3.16}$$

There are two different cases to consider: either $Z=bC$ (trivial cycle) or $2Z=bC'$ but $Z\neq bC$ (torsion cycle).

When $Z=bC$ and with our choice of origin on the trivial fiber of $H_D^1(M,\mathbb{Z})^*$, we can write $[\eta_Z]=\overline{\beta_C}$ for some $\overline{\beta_C}\in Hom(\Omega^2_{\mathbb{Z}}(M),\mathbb{R}/\mathbb{Z})$. As explained in Ref. 1, $\overline{\beta_C}$ is built from the de Rham current β_C of the chain C. Unlike DB classes, β_C can be divided by $2k$, giving rise to $\overline{\beta_C/2k}\in Hom(\Omega^2_{\mathbb{Z}}(M),\mathbb{R}/\mathbb{Z})$. Now, as intensively done in Ref. 9, we perform the shift

$$\bar{\alpha}\rightarrow\bar{\chi}=\bar{\alpha}+q\frac{\overline{\beta_C}}{2k} \tag{3.17}$$

in both terms of (3.16), thus obtaining

$$\langle W(z=qZ)\rangle_k = \int D\bar{\chi}\,\exp\left\{2i\pi k\int_M \bar{\chi}*_D\bar{\chi}\right\}\exp\left\{-2i\pi kq^2\int_M \frac{\overline{\beta_C}}{2k}*_D\frac{\overline{\beta_C}}{2k}\right\} + \int D\bar{\chi}\,\exp\left\{2i\pi k\int_M ([\eta_1]+\bar{\chi})*_D([\eta_1]+\bar{\chi})\right\}\exp\left\{-2i\pi kq^2\int_M \frac{\overline{\beta_C}}{2k}*_D\frac{\overline{\beta_C}}{2k}\right\}, \tag{3.18}$$

where we used $2k\overline{\beta_C/2k}\underset{\mathbb{Z}}{=}\overline{\beta_C}$. Note that the result mainly derives from the Cameron–Martin property of the CS measure. Finally, since

$$\frac{\overline{\beta_C}}{2k}*_D\frac{\overline{\beta_C}}{2k}\underset{\mathbb{Z}}{=}\overline{\frac{\beta_C}{2k}\wedge d\frac{\beta_C}{2k}}\underset{\mathbb{Z}}{=}\overline{\frac{\beta_C\wedge d\beta_C}{4k^2}}, \tag{3.19}$$

we derive

$$\exp\left\{-2i\pi kq^2\int_M \frac{\overline{\beta_C}}{2k}*_D\frac{\overline{\beta_C}}{2k}\right\}=\exp\left\{-\frac{2i\pi q^2}{4k}\int_M \beta_C\wedge d\beta_C\right\}. \tag{3.20}$$

The product $\beta_C\wedge d\beta_C$ has to be regularized for its integral over M to have a meaning. Applying the framing procedure to Z leads to

$$\int_M \beta_C\wedge d\beta_C \equiv L(Z,Z^f)\equiv C\cap Z^f\in\mathbb{Z}. \tag{3.21}$$

We then conclude that

$$\langle W(z=qZ)\rangle_k=\exp\left\{-2i\pi\frac{q^2}{4k}L(Z,Z^f)\right\}=\exp\left\{-2i\pi\frac{q^2}{4k}C\cap Z^f\right\}, \tag{3.22}$$

which is, as expected, the same result as for $M=S^3$. Let us prove that the above procedure does not depend on our choice of β_C. Let $\tilde{C}$ be another chain bounding Z. Then, $b(\tilde{C}-C)=0$ which means that $\tilde{C}-C$ is a 2-cycle on M. Since here $M=\mathbb{R}P^3$, from (3.1), we deduce that $\tilde{C}-C=b\vartheta$. Then, $b\vartheta\cap Z^f=\vartheta\cap bZ^f=0$, and (3.22) will still hold. If M has free homology of degree 2, there will also be free cohomology of degree 2 (see universal coefficient theorem), and then the base space of $H_D^1(M,\mathbb{Z})$ [and $H_D^1(M,\mathbb{Z})^*$] will also have a free part so that we have to adapt our measure. However, it is almost obvious that (3.20) would then produce a term $(\tilde{C}-C)\cap Z^f=(\tilde{C}-C)\cap bC^f$ $=b(\tilde{C}-C)\cap C^f=0$, since by hypothesis Z, and so Z^f, is a trivial cycle.

In the torsion case, since $2Z=bC'$, we can obviously write $[\eta_{2Z}]=2[\eta_Z]=\overline{\beta_{C'}}$, with $\overline{\beta_{C'}}$ built from the de Rham current $\beta_{C'}$ of the chain C'. However, since $Z\neq bC$, we cannot find any de Rham current β_C of an integral chain such that $[\eta_Z]=\overline{\beta_C}$. This is because DB cohomology is defined over $\mathbb{Z}$ and not $\mathbb{Q}$. On the other hand, $[\eta_1]$, the DB representative of the fixed torsion cycle τ_1, has been chosen as origin of the fiber over $\check{1}$, so we can also write $[\eta_Z]=[\eta_1]+\overline{\beta_y}$, where $\overline{\beta_y}$ is made from the de Rham current $\overline{\beta_y}$ of the chain y relating Z and τ_1: $Z=\tau_1+by$. Substituting that into (3.16) gives

$$\langle W(z=qZ)\rangle_k=\int D\bar{\alpha}\,\exp\left\{2i\pi\int_M \bar{\alpha}*_D(k\bar{\alpha}+q[\eta_1]+q\overline{\beta_y})\right\}$$
$$+\int D\bar{\alpha}\,\exp\left\{2i\pi\int_M ([\eta_1]+\bar{\alpha})*_D(k[\eta_1]+k\bar{\alpha}+q[\eta_1]+q\overline{\beta_y})\right\}. \tag{3.23}$$

Since k is even, the quantity $k[\eta_1]*_D[\eta_1]$ occurring in the second term of this expression is consistently regularized into $[0]$ using the framing procedure. Unfortunately, in the same term we also see the quantity $q[\eta_1]*_D[\eta_1]$. It combines with the previous one to give $(k+q)[\eta_1]*_D[\eta_1]$. From the same regularization argument, which led us to impose k to be even, we deduce that $(k+q)$ has to be even too, and thus

$$q=2m,\quad m\in\mathbb{Z}. \tag{3.24}$$

In other words, charges inherit the same constraint than the level parameter and for exactly the same reasons. Note that when q is odd then the framing procedure might produce variations in the relative sign between the two terms of (3.23), depending on whether the framing is odd or even, hence implying that the expectation value would not be properly defined. Let us assume for the rest of this section that $q=2m$, and let us rewrite (3.23), accordingly,

$$\langle W(z=qZ)\rangle_k = \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M \bar{\alpha}*_D(k\bar{\alpha}+2m[\eta_1]+2m\overline{\beta_y})\right\} + \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M ([\eta_1]+\bar{\alpha})*_D(k[\eta_1]+k\bar{\alpha}+2m[\eta_1]+2m\overline{\beta_y})\right\}. \tag{3.25}$$

Since $[\eta_1]$ is the DB representative of the torsion cycle τ_1, there exists a chain C with de Rham current γ_C, such that $2\tau = bC$, that is to say, $2[\eta_1]=\overline{\gamma_C}$. Hence,

$$\begin{aligned}\langle W(z=qZ)\rangle_k &= \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M \bar{\alpha}*_D(k\bar{\alpha}+m\overline{\gamma_C}+2m\overline{\beta_y})\right\} \\ &\quad + \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M ([\eta_1]+\bar{\alpha})*_D(k[\eta_1]+k\bar{\alpha}+m\overline{\gamma_C}+2m\overline{\beta_y})\right\} \\ &= \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M \bar{\alpha}*_D(k\bar{\alpha}+m\overline{\rho_{C+2y}})\right\} \\ &\quad + \int D\bar{\alpha}\,\exp\left\{2i\pi\int_M ([\eta_1]+\bar{\alpha})*_D(k[\eta_1]+k\bar{\alpha}+m\overline{\rho_{C+2y}})\right\}.\end{aligned} \tag{3.26}$$

where we have introduce $\overline{\rho_{C+2y}}=\overline{\gamma_C}+2\overline{\beta_y}=\overline{\gamma_C+2\beta_y}$, with ρ_{C+2y} being the de Rham current of $C+2y$. Now, let us perform the usual shift

$$\bar{\alpha} \to \bar{\chi} = \bar{\alpha} + q\frac{\overline{\rho_{C+2y}}}{2k} \tag{3.27}$$

to obtain

$$\begin{aligned}\langle W(z=qZ)\rangle_k &= \int D\bar{\chi}\,\exp\left\{2i\pi k\int_M \bar{\chi}*_D\bar{\chi}\right\}\exp\left\{-2i\pi km^2\int_M \frac{\overline{\rho_{C+2y}}}{2k}*_D\frac{\overline{\rho_{C+2y}}}{2k}\right\} \\ &\quad + \int D\bar{\chi}\,\exp\left\{2i\pi k\int_M ([\eta_1]+\bar{\chi})*_D([\eta_1]+\bar{\chi})\right\}\exp\left\{-2i\pi km^2\int_M \frac{\overline{\rho_{C+2y}}}{2k}*_D\frac{\overline{\rho_{C+2y}}}{2k}\right\}.\end{aligned} \tag{3.28}$$

We are left with proving that the framing procedure provides a consistent regularization of $\overline{\rho_{C+2y}/2k}*_D\overline{\rho_{C+2y}/2k}$, giving a meaning to (3.28). Actually, if Z^f denotes a framing of Z,

$$km^2\int_M \frac{\overline{\rho_{C+2y}}}{2k}*_D\frac{\overline{\rho_{C+2y}}}{2k} \underset{\mathbb{Z}}{=} \frac{m^2}{4k}\int_M \rho_{C+2y}\wedge d\rho_{C+2y} = \frac{m^2}{4k}(C+2y)\cap 2Z^f, \tag{3.29}$$

which implies that

$$\langle W(z=qZ)\rangle_k = \exp\left\{-2i\pi\frac{q^2}{4k}\frac{(C+2y)\cap Z^f}{2}\right\}. \tag{3.30}$$

We also introduce the 2-chain C' such that $2Z=bC'$. Hence, $b(C'-C-2y)=0$, which means that $C'-C-2y$ is a 2-cycle on M. Since the second homology group of $\mathbb{R}P^3$ is trivial, in this case $C'-C-2y=b\vartheta$, which implies that $(C+2y)\cap Z^f = C'\cap Z^f$. Once more, if M had a nontrivial second homology group, then we would have $(C+2y)\cap Z^f = C'\cap Z^f + \Sigma\cap Z^f$ for some (possibly nontrivial) 2-cycle Σ. Yet, since $2Z^f = bC^f$ we would still obtain that $(C+2y)\cap Z^f = C'\cap Z^f$. Finally

$$\langle W(z=qZ)\rangle_k = \exp\left\{-2i\pi\frac{q^2}{4k}\cdot\frac{C'\cap Z^f}{2}\right\}, \tag{3.31}$$

with $2Z=bC'$, which is exactly the result coming from surgery.[20,7,16] This last series of results also prove that nothing depends on the choice we made for ρ_{C+2y}.

Finally, note that (3.31) is actually containing (3.22) since if $2Z=bC'$ and $Z=bC$ then $C'=2C$ is a possible choice and then $C'\cap Z^f/2=C\cap Z^f$ has expected. And consistently, we do not need q to be even within (3.22). One can convince himself that the factor 1/2 appearing in (3.30) is nothing but the torsion degree of Z, and thus in the case of a 3-manifold with torsion cycle of degree p, we would see a term such as $C'\cap Z^f/p=C\cap Z^f$. This is also in agreement with the case of trivial cycles, which can be seen as torsion cycles of degree 1.

IV. CONCLUSIONS

The treatment of Abelian CS to generate link invariants introduced in Ref. 9 straightforwardly extends to the case of oriented 3-manifold with torsion. Although we only considered $\mathbb{R}P^3$, it is clear that our results apply to any oriented 3-manifold with torsion. In Ref. 17, we will show how DB cohomology can also be applied to higher dimensional Abelian CS theories and link invariants, thus fulfilling some of the questions left opened in Ref. 9.

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Higher dimensional abelian Chern-Simons theories and their link invariants

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Abstract

The role played by Deligne-Beilinson cohomology in establishing the relation between Chern-Simons theory and link invariants in dimensions higher than three is investigated. Deligne-Beilinson cohomology classes provide a natural abelian Chern-Simons action, non trivial only in dimensions $4l+3$, whose parameter k is quantized. The generalized Wilson $(2l+1)$-loops are observables of the theory and their charges are quantized. The Chern-Simons action is then used to compute invariants for links of $(2l+1)$-loops, first on closed $(4l+3)$-manifolds through a novel geometric computation, then on $\mathbb{R}^{4l+3}$ through an unconventional field theoretic computation.

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1 Introduction

The role that Deligne-Beilinson cohomology [1, 2, 3, 4, 5, 6, 7] plays in establishing the relation between Chern-Simons Quantum Field Theory and link invariants [8, 9, 10, 11, 12, 13, 14, 15, 16], in the abelian case, has been stressed out in a series of papers [17, 18]. We will here complete these works by showing how higher dimensional Deligne-Beilinson (DB) cohomology classes, and their DB-products, provide a natural generalisation of the Chern-Simons action, and how they can be used to compute invariants for higher dimensional links [13, 19]. We will produce a novel, geometric computation for closed $(4l+3)$-manifolds. We will then compare it to a field theoretic computation made on $\mathbb{R}^{4l+3}$.

In section 2, we recall some basic facts concerning Deligne-Beilinson cohomology and how it relates to the functional measure based on the abelian Chern-Simons action. In section 3, we present a natural candidate for the generalized CS action. In section 4, we deal with generalized abelian loops and their expectation values for closed $(4l+3)$-manifolds within the DB approach. We further illustrate it with two specific examples. Section 5 is devoted to a quite unusual field theoretic computation of these expectation values in the $\mathbb{R}^{4l+3}$ case, and the extension of this type of computation to S^{4l+3} is sketched. In Appendix, a geometrical interpretation of the higher dimensional linking number relating it to the notions of solid angle and zodiacus is presented following the original ideas of Gauss [20].

Here are the main results elaborated in this article:

1. The abelian Chern-Simons generalised action is non trivial only in dimension $4l+3$, and its level parameter k has to be quantized;

2. The generalised Wilson $(2l+1)$-loops are observables of the theory and their charges are quantized.

3. In the geometric DB approach provided by functional integration over the space $[H_D^{2l+1}(M,\mathbb{Z})]^* \supset H_D^{2l+1}(M,\mathbb{Z})$, the $2k$-nilpotency property holds and the observables are given by (self-)linking numbers under the so-called zero-regularization choice (*i.e.* framing). Furthermore only homology is involved in abelian Chern-Simons theories and only homologically trivial links (modulo 2k) give non vanishing expectation values.

4. A field theoretic computation in $\mathbb{R}^{4l+3}$ can be handled in a non perturbative way, yet it still misses quantization of the level and charges. Once the latter are imposed by hand the result reproduces the one from the DB approach.

2 Basic facts about Deligne-Beilinson cohomology

Without recalling the whole theory let us remind the basic facts about DB-cohomology useful in this paper.

2.1 Definition via exact sequences

If M is a closed (*i.e.* compact and without boundary) n-dimensional smooth manifold, the p-th DB cohomology group of M, denoted $H^p_D(M,\mathbb{Z})$ ($p \leq dim M = n$), is canonically embedded into the following equivalent exact sequences [5, 21]:

$$0 \longrightarrow \Omega^p(M)/\Omega^p_{\mathbb{Z}}(M) \longrightarrow \quad H^p_D(M,\mathbb{Z}) \quad \longrightarrow \check{H}^{p+1}(M,\mathbb{Z}) \longrightarrow 0\,, \tag{2.1}$$

$$0 \longrightarrow \check{H}^p(M,\mathbb{R}/\mathbb{Z}) \longrightarrow \quad H^p_D(M,\mathbb{Z}) \quad \longrightarrow \Omega^{p+1}_{\mathbb{Z}}(M) \longrightarrow 0\,, \tag{2.2}$$

where $\Omega^p(M)$ is the space of smooth p-forms on M, $\Omega^p_{\mathbb{Z}}(M)$ the space of smooth closed p-forms with integral periods on M, $\check{H}^{p+1}(M,\mathbb{Z})$ is the $(p+1)$-th integral Čech cohomology group of M, and $\check{H}^1(M,\mathbb{R}/\mathbb{Z})$ is the p-th $\mathbb{R}/\mathbb{Z}$-valued Čech cohomology group of M. These exact sequences also occur in the context of Cheeger-Simons differential characters [22, 23] or Harvey-Lawson sparks [21].

Thanks to exact sequences (2.1) one can interpret $H^p_D(M,\mathbb{Z})$ as an affine bundle over $\check{H}^{p+1}(M,\mathbb{Z})$ (resp. $\Omega^{p+1}_{\mathbb{Z}}(M)$) with structure group $\Omega^p(M)/\Omega^p_{\mathbb{Z}}(M)$ (resp. $\check{H}^p(M,\mathbb{R}/\mathbb{Z})$). Note that in the former case $\Omega^p_{\mathbb{Z}}(M)$ plays the role of a gauge group, which is much bigger (in general) than the usual group of exact forms. An element of $H^p_D(M,\mathbb{Z})$ will be generically written $\omega^{[p]}$.

Let us pick up a normalized volume form on M, *i.e.* a n-form μ such that $\int_M \mu = 1$. For dimensional reasons any n-form on M is closed, hence for any n-form ω on M there exists a $(n-1)$-form ν such that $\omega = \tau\mu + d\nu$, with $\tau = \int_M \omega \in \mathbb{R}$. Furthermore, if ω has integral periods, then $\tau \in \mathbb{Z}$, since $d\nu$ is a closed n-form with zero periods ($\int_M d\nu = 0$ since M has no boundary). This proves that any element of $\Omega^n(M)/\Omega^n_{\mathbb{Z}}(M)$ can be written as $\theta\mu$, with $\theta \in \mathbb{R}/\mathbb{Z}$. Finally, integrating $\theta\mu$ over M makes the construction independent of μ and proves that $\Omega^n(M)/\Omega^n_{\mathbb{Z}}(M) \simeq \mathbb{R}/\mathbb{Z}$ (equivalently one can pick up another normalized volume form and see that it will give the same θ, and finally pick any volume form and prove the same). Still for dimensional reasons, $\check{H}^{n+1}(M,\mathbb{Z}) = 0$, so we conclude that $H^n_D(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$.

For later convenience, let us consider two special cases. First, when $M = S^{4l+3}$ and $p = 2l+1$, we have $\check{H}^{2l+1}(M,\mathbb{R}/\mathbb{Z}) = 0 = \check{H}^{2l+2}(M,\mathbb{R}/\mathbb{Z})$, then sequence (2.1) reduces to:

$$0 \longrightarrow \Omega^{2l+1}(M)\big/ d\Omega^{2l}(M) \longrightarrow \quad H^{2l+1}_D(M,\mathbb{Z}) \quad \longrightarrow 0\,. \tag{2.3}$$

Hence $H_D^{2l+1}(M,\mathbb{Z})$ is isomorphic to the quotient space $\Omega^{2l+1}(M)/d\Omega^{2l}(M)$, the gauge group reducing to the trivial group $d\Omega^{2l}(M)$. Although this is a quite trivial case, it is very close to the one of the field theoretic approach.

The second example is provided by $M = S^{2l+1} \times S^{2l+2}$, still with $p = 2l+1$. Since $\check{H}^{2l+1}(M,\mathbb{R}/\mathbb{Z}) = \mathbb{Z} = \check{H}^{2l+2}(M,\mathbb{R}/\mathbb{Z})$, sequence (2.1) reads:

$$0 \longrightarrow \Omega^{2l+1}(M)\big/\Omega_{\mathbb{Z}}^{2l+1}(M) \longrightarrow H_D^{2l+1}(M,\mathbb{Z}) \longrightarrow \check{H}^{2l+2}(M,\mathbb{Z}) = \mathbb{Z} \longrightarrow 0\,. \tag{2.4}$$

The DB $\mathbb{Z}$-module $H_D^{2l+1}(M,\mathbb{Z})$ is then a non trivial affine bundle over $\mathbb{Z}$, the gauge group $\Omega_{\mathbb{Z}}^{2l+1}(M)$ being also now non trivial.

2.2 Pontrjagin dual of DB-spaces

Due to the form of the exact sequences (2.1), one can consider dual sequences not with respect to $\mathbb{R}$ but to $\mathbb{R}/\mathbb{Z}$. This gives rise to the Pontrjagin dual space of $H_D^p(M,\mathbb{Z})$: $H_D^p(M,\mathbb{Z})^* \equiv Hom(H_D^p(M,\mathbb{Z}), S^1)$. In particular, $H_D^p(M,\mathbb{Z})^*$ belongs itself to an exact sequence (dualizing (2.2) in $\mathbb{R}/\mathbb{Z}$):

$$0 \longrightarrow Hom\big(\Omega_{\mathbb{Z}}^{p+1}(M), \mathbb{R}/\mathbb{Z}\big) \longrightarrow H_D^p(M,\mathbb{Z})^* \longrightarrow \check{H}^{n-p-1}(M,\mathbb{Z}) \longrightarrow 0\,, \tag{2.5}$$

This identifies $H_D^p(M,\mathbb{Z})^*$ as an affine bundle over the same base, $\check{H}^{n-p-1}(M,\mathbb{Z})$, than $H_D^{n-p-1}(M,\mathbb{Z})$. Of course there is a second exact sequence we could obtain from dualizing (2.1).

Thanks to integration over integral cycles on M, the quotient $\Omega^{n-p-1}(M)\big/\Omega_{\mathbb{Z}}^{n-p-1}(M)$ can be canonically embedded into $Hom\big(\Omega_{\mathbb{Z}}^{p+1}(M), \mathbb{R}/\mathbb{Z}\big)$. We have also noticed that $H_D^n(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$. This suggests that $H_D^{n-p-1}(M,\mathbb{Z})$ might be canonically identified as a subset of $H_D^p(M,\mathbb{Z})^*$, just as continuous functions can be seen as (regular) distributions. The notion of integration of DB-classes over cycles is needed to confirm this.

2.3 Integration of DB-classes over integral cycles

There is a canonical pairing between DB-class and cycles on M provided by integration of the later over the former:

$$\oint : H_D^p(M,\mathbb{Z}) \times Z_p(M) \longrightarrow \mathbb{R}/\mathbb{Z}\,, \tag{2.6}$$

where $Z_p(M)$ denotes the space of integral p-cycles on M. Let us stress that these integrals take their values in $\mathbb{R}/\mathbb{Z} \simeq S^1$, not $\mathbb{R}$.

Since M itself is a cycle, one can integrate any DB-class $\omega^{[n]} \in H_D^n(M,\mathbb{Z})$ over M. This confirms that $H_D^n(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z}$ and proves that $H_D^{n-p-1}(M,\mathbb{Z})$ can be canonically identified as a subset of $H_D^p(M,\mathbb{Z})^*$.

Incidentally, integration also shows that $Z_p(M)$ is canonically embedded into $H_D^p(M,\mathbb{Z})^*$ - which can be expressed [21] by saying that p-cycles live in the topological boundary of $H_D^p(M,\mathbb{Z})^*$. Hence:

$$H_D^{n-p-1}(M,\mathbb{Z}) \times Z_p(M) \subset H_D^p(M,\mathbb{Z})^* \;, \tag{2.7}$$

where $\subset$ has to be understood as the above canonical embeddings.

Property 1 *As in the three dimensional case, abelian holonomies defined by:*

$$\exp\left\{2i\pi \oint_z \omega^{[p]}\right\}, \tag{2.8}$$

are observables of the generalized abelian Chern-Simons theories.

2.4 DB-product and cycle map

There is a natural bilinear product, referred here as the DB-product:

$$*_D : H_D^p(M,\mathbb{Z}) \times H_D^q(M,\mathbb{Z}) \longrightarrow H_D^{p+q+1}(M,\mathbb{Z}) \;, \tag{2.9}$$

which is graded according to:

$$\omega_1^{[p]} *_D \omega_2^{[q]} = (-1)^{(p+1)(q+1)} \omega_2^{[p]} *_D \omega_1^{[q]} \,. \tag{2.10}$$

From our previous remarks, one straightforwardly verifies:

$$*_D : H_D^p(M,\mathbb{Z}) \times H_D^{n-p-1}(M,\mathbb{Z}) \longrightarrow H_D^n(M,\mathbb{Z}) \simeq \mathbb{R}/\mathbb{Z} \tag{2.11}$$

The "DB-square" operation satisfies the graded commutation property:

$$\omega^{[p]} *_D \omega^{[p]} = (-1)^{(p+1)(p+1)} \omega^{[p]} *_D \omega^{[p]} \,. \tag{2.12}$$

which implies in particular:

$$\omega^{[2l]} *_D \omega^{[2l]} = 0 \,, \tag{2.13}$$

for any $\omega^{[2l]} \in H_D^{2l}(M,\mathbb{Z})$.

The DB-classes introduced above are smooth ones. They can be extended to distributional DB-classes. relying on Pontrjagin duality. Setting $H_D^{-1}(M,\mathbb{Z}) \equiv \mathbb{Z}$, one extends the previous DB-product to a pairing of $H_D^p(M,\mathbb{Z})$ and $H_D^q(M,\mathbb{Z})^*$ into $H_D^{(q-p-1)}(M,\mathbb{Z})^* \supset H_D^{(n-q+p+1)}(M,\mathbb{Z})$ $(q \geq p)$. Note that $H_D^{-1}(M,\mathbb{Z})^* = \mathbb{R}/\mathbb{Z} = H_D^n(M,\mathbb{Z})$ hence $*_D : H_D^p(M,\mathbb{Z}) \times H_D^p(M,\mathbb{Z})^* \to H_D^{-1}(M,\mathbb{Z})^* = \mathbb{R}/\mathbb{Z}$ as expected. This is similar to the usual theory of de Rham currents.

We end this subsection with the following important result shown in [7]: to any p-cycle z on M one can associate a canonical distributional DB-class $\eta_z \in H^p_D(M,\mathbb{Z})^*$ such that:

$$\oint_z \omega^{[p]} = \int_M \omega^{[p]} \star_D \eta_z \,, \tag{2.14}$$

for any $\omega^{[p]} \in H^p_D(M,\mathbb{Z})$. Such distributional DB-classes thus appear as elements of $H^p_D(M,\mathbb{Z})^*$. This is just another way to see the inclusion $Z_p(M) \subset H^p_D(M,\mathbb{Z})^*$. In the particular case where the p-cycle is a boundary, $z = bc$, the associated DB-class $\eta_z^{[n-p-1]}$ reduces to the de Rham current of the integral $(p+1)$-chain c. See [7] for details.

3 Generalized Chern-Simons action, Chern-Simons functional measure, observables and framing

3.1 Generalized Chern-Simons action

It is standard from a physicist point of view to present the abelian Chern-Simons (CS) lagrangian on $\mathbb{R}^3$ as :

$$cs_1(A) \equiv A \wedge dA \,, \tag{3.15}$$

or, using the CS action:

$$CS_1(A) = 2i\pi \int_{\mathbb{R}^3} A \wedge dA \,, \tag{3.16}$$

where A is a $U(1)$-connection on some principal $U(1)$-bundle P over $\mathbb{R}^3$. A natural generalization for $\mathbb{R}^{4l+3}$ would be to replace A in eqn. 3.15 by a $(2l+1)$-form. This is what will be done in section 5 when dealing with the field theoretic formulation.

However $U(1)$-connections on M are actually not 1-forms for compactclosed 3-manifolds M. Hence, as explained in [17, 18], we rather have to use DB-classes to write the lagrangian (3.15), and hence the action (3.16). Let us recall that $H^1_D(M,\mathbb{Z})$ canonically identifies with the set of classes of $U(1)$-isomorphic principal $U(1)$-bundles with connection over M. Hence we must replace eqn. (3.16) by

$$CS_1(A) = 2i\pi \int_M A \star_D A \,, \tag{3.17}$$

where A has now to be understood as a DB class.

For a level k CS theory we set:

$$CS_k(A) = 2i\pi k \int_M A \star_D A \,. \tag{3.18}$$

We can extend the definition of the action (3.18) to any closed smooth n-dimensional manifold M as:

$$CS_k(\omega^{[p]}) = 2i\pi k \int_M \omega^{[p]} *_D \omega^{[p]} \,. \tag{3.19}$$

This will be our definition of the n-dimensional Chern-Simons theory of level k on M. Since integrals take values in $\mathbb{R}/\mathbb{Z}$ this quantity is well defined provided

$$k \in \mathbb{Z} \,, \tag{3.20}$$

which is the announced **quantization of the level parameter**.

We now consider the "quantum weight":

$$\exp\left\{CS_k(\omega^{[p]})\right\} = \exp\left\{2i\pi k \int_M \omega^{[p]} *_D \omega^{[p]}\right\} . \tag{3.21}$$

When $p = 2l$ the graded commutation property (2.12) leads to:

$$\exp\left\{CS_k(\omega^{[2l]})\right\} = \exp\left\{2i\pi k \int_M \omega^{[2l]} *_D \omega^{[2l]}\right\} = 1 \,. \tag{3.22}$$

thereby providing a trivial functional measure. Consequently, the non-trivial cases only occur when $p = 2l+1$ which implies that $n = 2p+1 = \mathbf{4l+3}$. In particular, if M is a sphere, the only non trivial abelian Chern-Simons theories will occur for

$$S^3 \,, S^7 \,, S^{11} \ldots . \tag{3.23}$$

Note that this is namely the set of spheres for which Hopf invariants are non-trivial, hence linking numbers are non trivial as well [24]. Furthermore, this expression for the CS action holds true for closed manifolds with torsion.

In summary:

Property 2 *The non trivial generalized abelian Chern-Simon lagrangian of level k is defined by the DB square product of (2l+1) dimensional DB classes on a (4l+3)-dimensional closed manifold, with k an integer.*

For a $(4l+3)$-dimensional manifold and its $(2l+1)$-loops, the inclusions stressed out after (2.5) and in (2.7) give:

$$\begin{aligned} H_D^{2l+1}(M,\mathbb{Z}) &\subset H_D^{2l+1}(M,\mathbb{Z})^* \,, \\ \Omega^{2l+1}(M)\big/\Omega_{\mathbb{Z}}^{2l+1}(M) &\subset Hom\left(\Omega_{\mathbb{Z}}^{2l+2}(M), \mathbb{R}/\mathbb{Z}\right) . \end{aligned} \tag{3.24}$$

We will assume that the space of quantum fields of a generalized abelian Chern-Simons theory in $(4l+3)$ dimensions is a subset of $H_D^{2l+1}(M,\mathbb{Z})^*$ which contains $H_D^{2l+1}(M,\mathbb{Z}) \times Z_{2l+1}(M)$.

3.2 Chern-Simons functional measure and zero mode property

The generalized Chern-Simons "gaussian" functional measure for a $(4l+3)$-manifold takes the form:

$$d\mu_k(\omega) \equiv D\omega \exp\{CS_k(\omega)\} \,. \tag{3.25}$$

Since we wish to use this measure to compute observables and identify them with $(2l+1)$-links invariants, let us have a closer look at it. First, $d\mu_k(\omega)$ is supposed to be a measure on $H_D^{2l+1}(M,\mathbb{Z})$ or rather on (some subset of) $H_D^{2l+1}(M,\mathbb{Z})^*$, its "quantum" version. Of course, and as usual for infinite dimensional spaces, the measure (3.25) is totally formal on both spaces: as a Lebesgue measure over $H_D^{2l+1}(M,\mathbb{Z})$, $D\omega$ is zero, and so is (3.25); considering globally on $d\mu_k(\omega)$ $H_D^{2l+1}(M,\mathbb{Z})^*$, we should need to regularize products of distributional DB classes appearing in the gaussian part of the measure - something common in Quantum Field Theory. In fact, we will only need the fundamental Cameron-Martin like property for the measure (3.25), that is to say:

$$d\mu_k(\omega+\zeta) = d\mu_k(\omega)\exp\left\{4i\pi k\int_M \omega *_D \zeta\right\}\exp\left\{2i\pi k\int_M \zeta *_D \zeta\right\} , \tag{3.26}$$

for any given $\zeta \in H_D^{2l+1}(M,\mathbb{Z})$. Note that this property is similar to the one of a finite-dimensional gaussian measure which relies on the translational invariance of the Lebesgue measure. In other words, we have to assume that the "existing measure" on the functional space has property (3.26) which holds true for (3.25) seen has a measure on any finite dimensional subset of $H_D^{2l+1}(M,\mathbb{Z})$.

Let us consider a $(2l+2)$-cycle Σ, whose integration $(2l+1)$-current in M is denoted β_Σ. While this current canonically represents the zero class in $H_D^{2l+1}(M,\mathbb{Z})$, in general the current $\frac{\beta_\Sigma}{2k}$ does not. From property (3.26), and identically denoting currents and the DB classes which they represent, we deduce:

$$d\mu_k(\omega+\frac{\beta_\Sigma}{2k}) = d\mu_k(\omega)\exp\left\{4i\pi k\int_M \omega *_D \frac{\beta_\Sigma}{2k}\right\}\exp\left\{2i\pi k\int_M \frac{\beta_\Sigma}{2k} *_D \frac{\beta_\Sigma}{2k}\right\} . \tag{3.27}$$

In contrast with the identity

$$\exp\left\{2i\pi k\int_M \frac{\beta_\Sigma}{2k} *_D \frac{\beta_\Sigma}{2k}\right\} = \exp\left\{\frac{2i\pi}{4k}\int_M \beta_\Sigma \wedge d\beta_\Sigma\right\} = 1 \,, \tag{3.28}$$

trivial since $d\beta_\Sigma = 0$, the following one:

$$\exp\left\{4i\pi k\int_M \omega *_D \frac{\beta_\Sigma}{2k}\right\} = \exp\left\{2i\pi\int_M \omega *_D \beta_\Sigma\right\} = 1 \,, \tag{3.29}$$

deserves some justification. The factor $4i\pi k = 2k\cdot(2i\pi)$ in eqn. (3.29) is of pivotal importance. Indeed, $\omega *_D \beta_\Sigma/2k$ is not the zero class, whereas $2k(\omega *_D \beta_\Sigma/2k) = \omega *_D \beta_\Sigma$

is, as β_Σ is trivial. Note that $\beta_\Sigma/2k$ is not an integer current, and that a DB class ω is not the restriction of a current in general (see for instance [7]). Of course, one should be careful when dealing with the product of currents $\beta_\Sigma \wedge d\beta_\Sigma$. However one can always smooth β_Σ around Σ (*i.e.* use a Poincaré representative with support as close to Σ as necessary) in order to consistently regularize $\beta_\Sigma \wedge d\beta_\Sigma$ to the zero DB class. More generally, for any integer m,

$$d\mu_k(\omega + m\,\frac{\beta_\Sigma}{2k}) = d\mu_k(\omega) \tag{3.30}$$

which provides the generalization of Property 4 of [17]:

Property 3 *The functional measure $d\mu_k(\omega)$ is invariant under translations by $m\frac{\beta_\Sigma}{2k}$, where β_Σ is the integration current of a $(2l+2)$-cycle Σ and m an integer.*

When Σ is homologically trivial ($\Sigma = b\mathcal{V}$) then $\beta_\Sigma = d\chi_\mathcal{V}$, and therefore $\frac{\beta_\Sigma}{2k} = d(\frac{\chi_\mathcal{V}}{2k})$. In this case the DB-class of $\frac{\beta_\Sigma}{2k}$ is also zero. This happens for any Σ when the $(2l+2)$th homology group of M is trivial. Conversely, as we shall see in the next section, when M has a non trivial $(2l+2)$-th homology group, Property 3 will provide a treatment of the so-called "zero modes", thus leading to the important result of this paper concerning the vanishing of links invariants.

3.3 Observables and Framing

Following Property 1, let us consider an observable of our level k generalized CS theory:

$$\exp\left\{2i\pi \oint_z \omega\right\} = \exp\left\{2i\pi \int_M \omega \star_D \eta_z\right\} . \tag{3.31}$$

Let us remind that a $(2l+1)$-loop is meant to be a continuous mapping $\gamma : \Sigma_{2l+1} \to M$, where Σ_{2l+1} is a closed $(2l+1)$-dimensional manifold. It is always possible to identify such a loop with a $(2l+1)$-cycle in M. Furthermore, if the mapping is an embedding (*i.e.* the image $\gamma(\Sigma_{2l+1})$ is isomorphic to Σ_{2l+1}) γ is said to be a **fundamental loop**. Then, seen as a cycle, any $(2l+1)$-loop in M can be written as: $\gamma = q\gamma_0$, for some fundamental loop γ_0 and $q \in \mathbb{Z}$. Hence, the abelian Wilson line of the gauge field ω of degree $(2l+1)$ along a $(2l+1)$-loop $\gamma = q\gamma_0$ in M reads:

$$W(\omega,\gamma) \equiv \exp\left\{2i\pi \oint_\gamma \omega\right\} = \exp\left\{2i\pi q \int_{\gamma_0} \omega\right\} , \tag{3.32}$$

Conversely, the righthand side of this expression has a meaning if and only if q is an integer. This leads to:

Property 4 *In the generalized CS theories, loops must have integer charges.*

The charge (or colour) of a loop γ can be geometrically interpreted as the number of times the fundamental loop associated with γ has been covered. When γ is not homologically trivial, its charge canonically identifies with its homology class. The charge can also be seen has defining a representation for the $U(1)$ holonomy of a fundamental loop. This is also true for the level k parameter which can be seen as a charge of M, or as a representation of the $U(1)$ 3-holonomy given by the Chern-Simons action.

If η_γ and η_0 are the DB classes ($\in H_D^{2l+1}(M,\mathbb{Z})^*$) associated with γ and γ_0 respectively, then $\eta_\gamma = q\eta_0$. Hence we can alternatively write:

$$W(\omega,\gamma) = \exp\left\{2i\pi q \int_M \omega *_D \eta_0\right\} . \tag{3.33}$$

The expectation values of the Wilson lines are given by:

$$< W(\omega,\gamma) >_{CS_k} = Z_k^{-1} \int d\mu_k(\omega) \exp\left\{2i\pi q \int_M \omega *_D \eta_0\right\} , \tag{3.34}$$

where Z_k is the normalization factor such that $< W(\omega,\gamma \equiv 0) >_{CS_k} = 1$.

For a generic homological combination $\gamma = \sum_{i=1}^n q_i \gamma_i^0$ with $q_i \in \mathbb{Z}$ and γ_i^0 fundamental, we get:

$$W(\omega,\gamma) = \exp\left\{2i\pi \sum_{i=1}^n q_i \int_{\gamma_i^0} \omega\right\} , \tag{3.35}$$

or in term of the DB representatives η_i^0 of these γ_i^0:

$$W(\omega,\gamma) = \exp\left\{2i\pi \sum_{i=1}^n q_i \int_M \omega *_D \eta_i^0\right\} . \tag{3.36}$$

Let us first exhibit the nilpotency property of the expectation values

$$< W(\omega,\gamma) >_{CS_k} = Z_k^{-1} \int d\mu_k(\omega) \exp\left\{2i\pi \sum_{i=1}^n q_i \int_M \omega *_D \eta_i^0\right\} , \tag{3.37}$$

For the loop $2k\gamma_0$, where γ_0 is fundamental with DB representative η_0:

$$< W(\omega,2k\gamma_0) >_{CS_k} = Z_k^{-1} \int d\mu_k(\omega) \exp\left\{2i\pi (2k) \int_M \omega *_D \eta_0\right\} . \tag{3.38}$$

Performing the shift

$$\omega \mapsto \omega + \eta_0 \,, \tag{3.39}$$

thanks to property (3.26), we obtain:

$$< W(\omega, 2k\gamma_0) >_{CS_k} = Z_k^{-1} \int d\mu_k(\omega) \exp\left\{ -2i\pi \int_M \eta_0 *_D \eta_0 \right\} . \tag{3.40}$$

Such an expression is ill-defined since η_0 is distributional. If we decide to regularize the quantities $\eta_0 *_D \eta_0$ into the zero DB class, which we refer to as the **zero-regularization**, then:

$$< W(\omega, 2k\gamma_0) >_{CS_k} = 1 =< W(\omega, \gamma \equiv 0) >_{CS_k} \,. \tag{3.41}$$

This gives:

Property 5 *The generalized CS theories satisfy the $2k$-nilpotency property.*

Zero-regularization calls for a comparison with framing. If γ_0 is a boundary (*i.e.* is homologically trivial), then

$$\int_M \eta_0 *_D \eta_0 \underset{\mathbb{Z}}{=} \int_M \chi_0 \wedge d\chi_0 \,, \tag{3.42}$$

where χ_0 is the current of a chain whose γ_0 is the boundary, while $d\chi_0$ is the de Rham current of γ_0. The symbol $\underset{\mathbb{Z}}{=}$ in eqn. (3.42) means "equals modulo integers". The framing procedure gives a meaning to the right hand side of eqn. (3.42): each framing choice assigns a well defined *i.e.* homotopically invariant integer value to the self-linking of γ_0. The difference between two choices of framing is an integer, which coincides with taking $\eta_0 *_D \eta_0 = 0$. However, when γ_0 is not a boundary the framing procedure is not a well-defined regularization as it does not provide a definite homotopically invariant integer for the self-linking number $\int_M \chi_0 \wedge d\chi_0$. Notwithstanding property (3.41) still holds, the zero-regularization is thus coarser than framing yet more "general". Let us point out that $2k$-nilpotency[1] is totally equivalent to zero-regularization.

4 Abelian $(2l+1)$-links invariants: a geometric computation

In this section we will show:

[1] This was called colour periodicity in [17]. Yet the name "nilpotency" accounts more accurately of property (3.41).

Property 6 *In generalized CS theories, the only Wilson loops having non vanishing expectation values are those of the homologically trivial links (modulo 2k). The expectation values of these Wilson loops are given by the self-linking of the corresponding link and the only required regularization is the one provided by framing (i.e. self-linking of the fundamental loops forming the link).*

We will first present the general ideas used to compute expectation values (3.37). Then we will consider the particular case $M = S^{4l+3}$, the closest to the field theoretical computation of section 5. We will next treat the less trivial case $M = S^{2l+1} \times S^{2l+2}$. In these two examples, we will present an alternative and more computational way to get Property 6. Since M is assumed without torsion, all its homology and cohomology groups are free and of finite type, *i.e* of the form $\mathbb{Z}^N$, for some integer N. If $(\vec{e})_{I=1,\dots,N}$ denotes the canonical basis of $\mathbb{Z}^N$, then any $\vec{u} \in \mathbb{Z}^N$ is written as

$$\vec{u} = \sum_{I=1}^{N} u^I \vec{e}_I \quad , \quad u^I \in \mathbb{Z}.$$

4.1 Abelian $(2l+1)$-links invariants on $(4l+3)$-dimensional manifolds

As already mentioned, $H_D^{2l+1}(M,\mathbb{Z})^*$, as well as its smooth version $H_D^{2l+1}(M,\mathbb{Z})$, are affine bundles over the discrete space $\check{H}^{2l+2}(M,\mathbb{Z})$. Although the Chern-Simons functional measure on this space is written as in eqn. (3.25), we need to give a more precise meaning to this expression before we perform any computation. First, since the base space is of the form $\mathbb{Z}^N$, the measure $d\mu_k(\omega)$ has to be decomposed into a sum of measures over each (affine) fiber of $H_D^{2l+1}(M,\mathbb{Z})^*$. On each of these fibers we choose an origin, say $\omega^0_{\vec{u}}$, where $\vec{u} \in \mathbb{Z}^N$ denotes the corresponding base point in $\check{H}^{2l+2}(M,\mathbb{Z})$. Thus, $d\mu_k(\omega)$ reduces to a "vectorial" measure on $Hom(\Omega^{2l+2}_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$. This amounts to pick up a global section for the affine bundle $H_D^{2l+1}(M,\mathbb{Z})^*$. The CS measure hence reads:

$$d\mu_k(\omega) = \sum_{\vec{u}\in\mathbb{Z}^N} D\alpha \exp\left\{CS_k(\omega^0_{\vec{u}} + \alpha)\right\} = \sum_{\vec{u}\in\mathbb{Z}^N} d\mu_k(\omega^0_{\vec{u}};\alpha)\,, \tag{4.43}$$

where $\alpha \in Hom(\Omega^{2l+2}_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$, $D\alpha$ is a measure on $Hom(\Omega^{2l+2}_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$, and each measure $d\mu_k(\omega^0_{\vec{u}};\alpha)$ satisfies the Cameron-Martin property (3.25).

On the other hand, inclusion (2.7) together with Poincaré duality imply that on each fiber of $H_D^{2l+1}(M,\mathbb{Z})^*$ we can use, as an origin on this fiber, a $(2l+1)$-cycle or equivalently its DB representative. In particular, a fundamental loop γ^0_I can be associated with each basis vector $\vec{e}_I$ of $\mathbb{Z}^N$. Its DB representative η^0_I then plays the role of origin on the fiber over $\vec{e}_I$. If $\vec{u} = \sum u^I \vec{e}_I$, then $\eta_{\vec{u}} \equiv \sum u^I \eta^0_I$ will be a possible origin for the fiber over $\vec{u}$.

Note that the de Rham current of γ_I^0 would play the role of the "curvature" of η_I^0, as an element of $Hom\left(\Omega^{2l+1}(M)/\Omega_{\mathbb{Z}}^{2l+1}(M),\mathbb{R}/\mathbb{Z}\right)$.

Once such an origin for each fiber of $H_D^{2l+1}(M,\mathbb{Z})^*$ has been chosen, any DB class ω can be decomposed as

$$\omega = \sum_{I=1}^{N} u_\omega^I \eta_I^0 + \alpha \equiv \vec{u}_\omega \cdot \vec{\eta}^{\,0} + \alpha\,, \tag{4.44}$$

with $\alpha \in Hom\left(\Omega_{\mathbb{Z}}^{2l+2}(M),\mathbb{R}/\mathbb{Z}\right)$, and $\vec{u}_\omega$ being the base point over which ω stands. In particular, the DB representative η of a cycle γ will decompose as

$$\eta = \sum_{I=1}^{N} u_\gamma^I \eta_I^0 + \alpha \equiv \vec{u}_\gamma \cdot \vec{\eta}^{\,0} + \alpha\,. \tag{4.45}$$

For a link L, we can express the expectation value of the corresponding Wilson line according to our choice of basis $(\eta_I^0)_{I=1,\dots,N}$:

$$< W(L) >_{CS_k} = Z_k^{-1} \sum_{\vec{u}} \int d\mu_k(\vec{u}\cdot\vec{\eta}^{\,0};\alpha)\, W(\vec{u},\alpha,\vec{v}_L,\beta)\,, \tag{4.46}$$

where

$$Z_k = \sum_{\vec{u}} \int d\mu_k(\vec{u}\cdot\vec{\eta}^{\,0};\alpha)\,, \tag{4.47}$$

and

$$W(\vec{u},\alpha,\vec{v}_L,\beta) = \exp\left\{2i\pi \int_M (\vec{u}\cdot\vec{\eta}^{\,0}+\alpha) \star_D (\vec{v}_L\cdot\vec{\eta}^{\,0}+\beta)\right\} \tag{4.48}$$

is a rewriting of the Wilson line of L with respect to the basis $(\eta_I^0)_{I=1,\dots,N}$, and with the decomposition $\eta_L = \vec{v}_L\cdot\vec{\eta}^{\,0}+\beta$ for the DB representative of L. We recall that L is a link (a formal combination of charged fundamental loops) hence a cycle.

Instead of evaluating the Wilson line (4.46), we rather use the zero mode property. Let $(\Sigma_0^I)_{I=1,\dots,N}$ be a collection of $(2l+2)$-cycles on M which generates $H_{2l+2}(M,\mathbb{Z})$ and are orthogonal to the fundamental loops γ_I^0:

$$\int_{\gamma_I^0} \beta_0^J = \delta_{IJ} = \Sigma_0^J \pitchfork \gamma_I^0\,, \tag{4.49}$$

β_0^J being the currents of the Σ_0^J, and $\pitchfork$ denoting transversal intersection. Due to Poincaré and Hom dualities there are as many β_0^J as γ_I^0.

Let us consider again:

$$< W(L) >_{CS_k} = Z_k^{-1} \int d\mu_k(\omega) \exp\left\{2i\pi \int_L \omega\right\}\,, \tag{4.50}$$

into which we perform the shift

$$\omega \to \omega + \sum_{I=1}^{N} m_I \frac{\beta_0^I}{2k} , \tag{4.51}$$

for a collection of integers m_I. This gives:

$$< W(L) >_{CS_k} = Z_k^{-1} \int d\mu_k(\omega + \sum_{I=1}^{N} m_I \frac{\beta_0^I}{2k}) \exp\left\{ 2i\pi \int_L \left(\omega + \sum_{I=1}^{N} m_I \frac{\beta_0^I}{2k} \right) \right\} . \tag{4.52}$$

Using Property 3, we obtain:

$$< W(L) >_{CS_k} = Z_k^{-1} \int d\mu_k(\omega) \exp\left\{ 2i\pi \int_L \omega \right\} \exp\left\{ 2i\pi \sum_{I=1}^{N} \frac{m_I}{2k} \int_L \beta_0^I \right\} . \tag{4.53}$$

That is to say:

$$< W(L) >_{CS_k} = < W(L) >_{CS_k} \exp\left\{ 2i\pi \sum_{I=1}^{N} \frac{m_I}{2k} \int_L \beta_0^I \right\} . \tag{4.54}$$

Since this has to hold for any collection of integers $(m_I)_{I=1,\ldots,N}$, we conclude that, for a non vanishing mean value:

$$\int_L \beta_0^I = 0 \ [2k], \tag{4.55}$$

$\forall I \in \{1, \ldots, N\}$. Thus, if we forget about $[2k]$, the link L has to be "orthogonal" to the generators of $H_{2l+2}(M, \mathbb{Z})$, which means that L must be homologically trivial, for the mean value of the corresponding Wilson loop to be non vanishing. When L is not trivial, the mean value of the Wilson loop it defines has to be zero. The modulo $2k$ appearing in eqn. (4.55) simply reminds us of the $2k$-nilpotency property (3.41).

Finally, let L be an homologically trivial link in M. This amounts to set $\vec{v}_L = \vec{0}$ in eqn. (4.46), thus reducing it to:

$$\sum_{\vec{u}} \int D\alpha \exp\left\{ CS_k(\vec{u} \cdot \vec{\eta}^{\,0} + \alpha) \right\} \exp\left\{ 2i\pi \int_M (\vec{u} \cdot \vec{\eta}^{\,0} + \alpha) \star_D \beta_L \right\} , \tag{4.56}$$

where β_L is the DB class of a current of a $(2l+2)$-chain with boundary L. Now let us perform into eqn. (4.56) the shift:

$$\alpha \to \alpha + \frac{\beta_L}{2k} , \tag{4.57}$$

what leads to:

$$\sum_{\vec{u}} \int D\alpha \exp\left\{ CS_k(\vec{u} \cdot \vec{\eta}^{\,0} + \alpha) \right\} \exp\left\{ -2i\pi k \int_M \frac{\beta_L}{2k} \star_D \frac{\beta_L}{2k} \right\} . \tag{4.58}$$

Hence, we obtain:

$$< W(L) >_{CS_k} = \exp\left\{ -\frac{2i\pi}{4k} \int_M \beta_L \wedge d\beta_L \right\} . \tag{4.59}$$

The integral in this expression is, modulo zero-regularization via framing, exactly the self-linking number of the link L [25, 26, 27], itself made of self-linking (defined via framing) and linking of the fundamental loops composing L. We stress out that while the link has to be homologically trivial, its components do not have to. This completes the proof of Property 6.

Of course we could have directly used property (3.26) together with the shift (4.57) to obtain eqn. (4.59). However we have preferred to use the explicit definition (4.43) of the functional integral rather than the formal one.

Let us have a closer look at a first example where zero modes are not required to be treated: the spheres. This will provide us with a general property concerning $(4l+3)$-manifolds whose $(2l+1)$-th homology group vanishes.

4.2 Abelian links invariants on S^{4l+3}

Since $\check{H}^{2l+2}(S^{4l+3}, \mathbb{Z}) = 0 = \check{H}^{2l+1}(S^{4l+3}, \mathbb{Z})$, the first of the exact sequences (2.1) reduces to:

$$\begin{aligned} H_D^{2l+1}\left(S^{4l+3}, \mathbb{Z}\right) &\simeq \Omega^{2l+1}\left(S^{4l+3}\right) / \Omega_{\mathbb{Z}}^{2l+1}\left(S^{4l+3}\right) \\ &= \Omega^{2l+1}\left(S^{4l+3}\right) / d\Omega^{2l}\left(S^{4l+3}\right) , \end{aligned} \tag{4.60}$$

and the dual sequence (2.5) to:

$$\begin{aligned} H_D^{2l+1}\left(S^{4l+3}, \mathbb{Z}\right)^* &\simeq Hom\left(\Omega_{\mathbb{Z}}^{2l+2}\left(S^{4l+3}\right), \mathbb{R}/\mathbb{Z}\right) \\ &= Hom\left(d\Omega^{2l+1}\left(S^{4l+3}\right), \mathbb{R}/\mathbb{Z}\right) . \end{aligned} \tag{4.61}$$

These isomorphisms are somehow canonical if we consider that the choice of the zero class, $\mathbf{0}$, as origin of these spaces is canonical. More explicitly, for any $\omega \in H_D^{2l+1}(S^{4l+3}, \mathbb{Z})^*$ there is a $\alpha \in Hom(\Omega_{\mathbb{Z}}^{2l+2}(S^{4l+3}), \mathbb{R}/\mathbb{Z})$ such that:

$$\omega = \mathbf{0} + \alpha \equiv \alpha , \tag{4.62}$$

This corresponds to choose the zero cycle $z \equiv 0$ as origin, the DB representative of this cycle being $\mathbf{0}$. Since $\check{H}_{2l+1}(S^{4l+3}, \mathbb{Z}) = 0$, any $(2l+1)$-cycle in S^{4l+3} is trivial, *i.e.* a boundary. Hence, if L denotes a $(2l+1)$-link which is the sum of charged fundamental $(2l+1)$-loops γ_i^0 on S^{4l+3}:

$$L = \sum_{i=1}^{N} q_i \gamma_i^0 , \tag{4.63}$$

then there exists some $(2l+2)$-chain, Σ_L, such that $L = b\Sigma_L$. Geometrically, Σ_L can be seen as a $(2l+2)$-surface in S^{4l+3}. This surface is of course not unique, but two of them only differ by a closed $(2l+2)$-surface. As explained in [7], the de Rham current of such a Σ_L, β_Σ, completely determines the DB representative, η_L, of L, according to:

$$\eta_L = \mathbf{0} + \beta_\Sigma \,, \tag{4.64}$$

with $\beta_\Sigma \in Hom\left(\Omega^{2l+2}_{\mathbb{Z}}\left(S^{4l+3}\right), \mathbb{R}/\mathbb{Z}\right)$. The Wilson line of L is then written:

$$W(\alpha, L) = \exp\left\{2i\pi \int_{S^{4l+3}} \alpha *_D \beta_\Sigma\right\} , \tag{4.65}$$

and its expectation value reads:

$$< W(L) >_{CS_k} = \frac{\int D\alpha \exp\left\{2i\pi k \int_{S^{4l+3}} \alpha *_D \alpha + 2i\pi \int_{S^{4l+3}} \alpha *_D \beta_\Sigma\right\}}{\int D\alpha \exp\left\{2i\pi k \int_{S^{4l+3}} \alpha *_D \alpha\right\}} \,. \tag{4.66}$$

Seen as an element of $Hom\left(\Omega^{2l+2}_{\mathbb{Z}}\left(S^{4l+3}\right), \mathbb{R}/\mathbb{Z}\right)$, $\beta_\Sigma/2k$ fulfills:

$$2k\left(\frac{\beta_\Sigma}{2k}\right) = \beta_\Sigma \,. \tag{4.67}$$

However, the corresponding DB class, $\mathbf{0} + (\beta_\Sigma/2k)$, is not the representative of any fundamental loop in S^{4l+3}.

Next, we perform the change of variable:

$$\alpha \mapsto \widetilde{\alpha} = \alpha + \frac{\beta_\Sigma}{2k} \,, \tag{4.68}$$

into eqn. (4.66). This turns the expectation value into:

$$< W(L) >_{CS_k} = \exp\left\{-2i\pi k \int_{S^{4l+3}} \frac{\beta_\Sigma}{2k} *_D \frac{\beta_\Sigma}{2k}\right\} . \tag{4.69}$$

Making explicit the DB product within this expression, we obtain:

$$< W(L) >_{CS_k} = \exp\left\{-\frac{2i\pi}{4k} \int_{S^{4l+3}} \beta_\Sigma \wedge d\beta_\Sigma\right\} , \tag{4.70}$$

what is exactly eqn. (4.59).

Finally in terms of the charged fundamental loops, γ_i^0, building L, we have

$$< W(L) >_{CS_k} = \exp\left\{-\frac{2i\pi}{4k} \sum_{i,j} q_i L(\gamma_i^0, \gamma_j^0) q_j\right\} , \tag{4.71}$$

where $L(\gamma_i^0, \gamma_j^0)$ is the linking number of γ_i^0 with γ_j^0, that is to say:

$$L(\gamma_i^0, \gamma_j^0) = \int_{S^{4l+3}} \alpha_i^0 \wedge d\alpha_j^0 \,, \tag{4.72}$$

with α_i^0 the de Rham current for which $\mathbf{0} + \alpha_i^0$ is the DB representative of the fundamental loop γ_i^0. As for "diagonal" terms $L(\gamma_i^0, \gamma_i^0)$ we regularize them using the usual framing procedure (what we have called zero-regularization):

$$L(\gamma_i^0, \gamma_i^0) \equiv L(\gamma_i^0, \gamma_i^{0f}) \,. \tag{4.73}$$

As in the three dimensional case extensively detailed in [17], the abelian invariants thus obtained are nothing but those coming from linking and self-linking numbers, that is to say intersection theory in S^{4l+3}. Let's note that this result is what we are supposed to recover via a quantum field theory approach. There, the gauge fixing procedure is supposed to provide a choice of representatives for DB classes, and the propagator thus obtained appears like an inverse of the de Rham differential d, deeply related to the Poincaré chain homotopy operator. The consistency of the procedure is ensured by the fact that if γ is a loop (a $(2l+1)$-cycle), and if Σ is a $(2l+2)$-chain such that $b\Sigma = \gamma$, which corresponds to $d\beta_\Sigma = \eta_\gamma$ in term of currents, then β_Σ (as the current of an integral chain) is unique up to closed $(2l+1)$-currents (of integral $(2l+2)$-cycles). However, on S^{4l+3} any $(2l+2)$-cycle is trivial so β_Σ is unique up to $d\chi$, where χ is the $2l$-current of an arbitrary $(2l)$-chain. This means $d\beta_\Sigma = \eta_\gamma$ has to be inverted on classes $\beta_\Sigma \sim \beta_\Sigma + d\chi$. This is exactly gauge invariance from the point of view of integral chains (and currents). This will be detailed in section 5.

What we have done here for S^{4l+3} can be straightforwardly applied to any $(4l+3)$-manifold M for which $\check{H}^{2l+1}(M, \mathbb{Z}) = 0 = \check{H}^{2l+2}(M, \mathbb{Z})$, leading to exactly the same final result.

Property 7 *Over a* $(4l+3)$*-dimensional closed manifold, without torsion, whose* $(2l+1)$*th homology groups vanishes, the generalized abelian Wilson loop of a link* L *defines a link invariant made of the self-linkings, the linkings and the charges of the fundamental loops composing* L*.*

The second example will present a homologically non trivial case which is the equivalent of the three dimensional pedagogical case $S^1 \times S^2$ widely discussed in [17].

4.3 Abelian links invariants on $S^{2l+1} \times S^{2l+2}$

Let us now consider the less trivial case $M \equiv S^{2l+1} \times S^{2l+2}$ for which $\check{H}^{2l+2}(M,\mathbb{Z}) = \mathbb{Z} = \check{H}^{2l+1}(M,\mathbb{Z})$, so that:

$$H_D^{2l+1}(M,\mathbb{Z}) \quad \simeq \quad \mathbb{Z} \times \frac{\Omega^{2l+1}(M)}{\Omega_{\mathbb{Z}}^{2l+1}(M)}, \tag{4.74}$$

and:

$$H_D^{2l+1}(M,\mathbb{Z})^* \quad \simeq \quad \mathbb{Z} \times Hom\left(\Omega_{\mathbb{Z}}^{2l+2}(M), \mathbb{R}/\mathbb{Z}\right), \tag{4.75}$$

none of these isomorphisms being canonical. However, over the base point $0 \in \mathbb{Z}$ we still have the zero DB class (which is again the representative of the zero cycle in M), so that this particular fiber of $H_D^{2l+1}(M,\mathbb{Z})^*$ can be (almost canonically) identified with the translation group $Hom(\Omega_{\mathbb{Z}}^{2l+2}(M), \mathbb{R}/\mathbb{Z})$. This is similar to what previously happened in the case of the sphere $S^{(4l+3)}$. However, we now have $\check{H}_{2l+1}(M,\mathbb{Z}) = \mathbb{Z}$, which means that there are non trivial $(2l+1)$-loops in M. Accordingly, we pick up a fundamental $(2l+1)$-loop γ^0 which generates $\check{H}_{2l+1}(M,\mathbb{Z})$. Formally γ^0 is given by a S^{2l+1} in M. Its DB representative, η^0 will play the role of the origin on the fiber over $1 \in \mathbb{Z}$ in $H_D^{2l+1}(M,\mathbb{Z})^*$. If L is a link in M, then its DB representative, η_L, satisfies

$$\eta_L = n_L \eta^0 + \beta_\Sigma, \tag{4.76}$$

with $n_L \in \mathbb{Z}$ the base point over which η_L stands in $H_D^{2l+1}(M,\mathbb{Z})^*$, and the translation term β_Σ belongs to $Hom(\Omega_{\mathbb{Z}}^{2l+2}(M), \mathbb{R}/\mathbb{Z})$. Once more, β_Σ alternatively denotes the de Rham current of a $(2l+2)$-chain Σ_L for which $L = n_L\gamma^0 + b\Sigma_L$ as well as the DB class this current defines via sequence (2.5). Such a chain is not unique, but two of them differ by a $(2l+2)$-cycle whose de Rham current belongs to the zero class in $Hom(\Omega_{\mathbb{Z}}^{2l+2}(M), \mathbb{R}/\mathbb{Z})$, making β_Σ unique from the DB class point of view.

So, up to the normalization factor Z_k^{-1}, the expectation value (4.46) reduces to:

$$\sum_{m\in\mathbb{Z}} \int D\alpha \exp\left\{2i\pi \int_M (m\eta^0 + \alpha) *_D (km\eta^0 + k\alpha + n_L\eta^0 + \beta_\Sigma)\right\}. \tag{4.77}$$

Instead of using the elegant zero-mode property, as was done to establish Property 6, we shall present a somehow more computational approach. Although this will be a bit "heavier", we make this choice in order to show more explicitly the usefulness of zero modes as well as of zero-regularization.

Since it provides the final answer, let us first consider the case where $n_L = 0$ (*i.e.* when L is homologically trivial). Then expression (4.77) takes the form:

$$\sum_{m\in\mathbb{Z}} \int D\alpha \exp\left\{2i\pi \int_M (m\eta^0 + \alpha) *_D (km\eta^0 + k\alpha + \beta_\Sigma)\right\}. \tag{4.78}$$

For the same reasons than in the previous example, $\beta_\Sigma/2k \in Hom\left(\Omega^{2l+2}_{\mathbb{Z}}\left(M\right),\mathbb{R}/\mathbb{Z}\right)$. So, we perform the shift:

$$\alpha \mapsto \widetilde{\alpha} = \alpha + \frac{\beta_\Sigma}{2k} \,. \tag{4.79}$$

The expectation value of the Wilson line of L then simplifies into:

$$\sum_{m\in\mathbb{Z}} \int D\alpha \exp\left\{2i\pi \int_M (m\eta^0 + \alpha) *_D (km\eta^0 + k\alpha)\right\} \times \exp\left\{-2i\pi \int_M \frac{\beta_\Sigma}{2k} *_D \frac{\beta_\Sigma}{2k}\right\} , \tag{4.80}$$

that is to say:

$$< W(L) >_{CS_k} = \exp\left\{-2i\pi k \int_M \frac{\beta_\Sigma}{2k} *_D \frac{\beta_\Sigma}{2k}\right\} , \tag{4.81}$$

or equivalently:

$$< W(L) >_{CS_k} = \exp\left\{-\frac{2i\pi}{4k} \int_M \beta_\Sigma \wedge d\beta_\Sigma\right\} , \tag{4.82}$$

just as in the S^{4l+3} case. Once more, this is totally similar to what happens in the three dimensional case $S^1 \times S^2$ detailed in [17]. This turns out to be the same expression as eqn. (4.70), and of course as eqn. (4.59): the link invariant is made of linking and self-linking numbers of the fundamental loops forming the link. However let us stress again that whereas the link L has to be homologically trivial, this is not the case of its components.

Let us now assume that n_L is not zero (nor an integral multiple of $2k$, although this can be dealt with straightforwardly). If we expand all the expressions within the exponentials appearing in eqn. (4.77), and then apply the zero-regularization to $\eta^0 *_D \eta^0$, we obtain the expression:

$$k\alpha *_D \alpha + \alpha *_D \beta_\Sigma + (2km + n_L)\eta^0 *_D \alpha + m\eta^0 *_D \beta_\Sigma \,. \tag{4.83}$$

Once more, we perform the shift (4.79), and get, after some simplifications:

$$k\alpha *_D \alpha + (2km + n_L)\eta^0 *_D \alpha - k\frac{\beta_\Sigma}{2k} *_D \frac{\beta_\Sigma}{2k} - n_L\eta^0 *_D \frac{\beta_\Sigma}{2k} \,. \tag{4.84}$$

The last two terms are independent of m and α, and then give rise to:

$$\exp\left\{-2i\pi \int_M \frac{\beta_\Sigma}{2k} *_D \left(k\frac{\beta_\Sigma}{2k} + n_L\eta^0\right)\right\} , \tag{4.85}$$

out of the integration and sum in eqn. (4.78). In the remaining factor, we can invert the sum over m with the integration over α, thus obtaining:

$$\int D\alpha e^{2i\pi k\int_M \alpha *_D \alpha} \sum_{m\in\mathbb{Z}} \exp\left\{2i\pi \int_M \left((2km+n_L)\eta^0 *_D \alpha\right)\right\}. \tag{4.86}$$

But:

$$\begin{aligned} \sum_{m\in\mathbb{Z}} \exp\left\{2i\pi \int_M \left((2k)m\eta^0 *_D \alpha\right)\right\} &= \sum_{m\in\mathbb{Z}} \exp\left\{2i\pi(2km) \int_{\gamma^0} \alpha\right\} \\ &= \sum_{K\in\mathbb{Z}} \delta\left(\int_{\gamma^0} \alpha - K/2k\right). \end{aligned} \tag{4.87}$$

Putting this back into eqn. (4.86), and performing some algebraic juggling, we obtain:

$$\sum_{K\in\mathbb{Z}} e^{2i\pi n_L K/2k} \int D\alpha\, \delta\left(\int_{\gamma^0} \alpha - K/2k\right) e^{2i\pi k\int_M \alpha *_D \alpha}. \tag{4.88}$$

Let us introduce a closed $(2l+2)$-surface Σ_0, with de Rham $(2l+1)$-current ρ^0, which satisfies:

$$\int_{\gamma^0} \rho^0 = 1 = \Sigma_0 \pitchfork \gamma^0. \tag{4.89}$$

This surface is a generator of $\check{H}^{2l+1}(M,\mathbb{Z}) \simeq \check{H}_{2l+2}(M,\mathbb{Z}) = \mathbb{Z}$ and is formally a sphere $S^{(2l+2)}$ in $M = S^{(2l+1)} \times S^{(2l+2)}$. The (trivial) DB class associated with ρ^0 (also denoted ρ^0) give rises to the DB class $\rho^0/2k$, which is non trivial since:

$$\int_{\gamma^0} \frac{\rho^0}{2k} \underset{\mathbb{Z}}{=} \frac{1}{2k}. \tag{4.90}$$

Actually, $\rho^0/2k \in Hom(\Omega^{2l+2}_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$ and the DB class it determines is $\mathbf{0} + \rho^0/2k$. Moreover, as seen when establishing the zero-mode property:

$$\int_M \frac{\rho^0}{2k} *_D \frac{\rho^0}{2k} \underset{\mathbb{Z}}{=} 0 \underset{\mathbb{Z}}{=} 2k \int_M \frac{\rho^0}{2k} *_D \alpha, \tag{4.91}$$

for any $\alpha \in Hom(\Omega^{2l+2}_{\mathbb{Z}}(M), \mathbb{R}/\mathbb{Z})$. Consequently, eqn. (4.88) reads:

$$\sum_{K\in\mathbb{Z}} e^{2i\pi n_L K/2k} \int D\alpha\, \delta\left(\int_{\gamma^0} (\alpha - K\frac{\rho^0}{2k})\right) e^{2i\pi k\int_M \alpha *_D \alpha}, \tag{4.92}$$

and for each value of K, if we perform the shift:

$$\alpha \mapsto \widetilde{\alpha} = \alpha - K\frac{\rho^0}{2k}, \tag{4.93}$$

and use eqn. (4.91), the expression under the integral in eqn. (4.88) turns out to be independent of K. Thus:

$$\sum_{K\in\mathbb{Z}} e^{2i\pi n_L K/2k}\,, \tag{4.94}$$

factorizes out of eqn. (4.92). The same procedure has to be applied to the denominator of expression (4.46) (which is the normalization factor needed to compute expectation values), producing a term:

$$\sum_{K\in\mathbb{Z}} 1\,. \tag{4.95}$$

None of the expressions (4.94) and (4.95) is well-defined. However, using $2k$-nilpotency, we can reduce each of these infinite sums to a sum over a period, thus obtaining:

$$\sum_{K=0}^{2k-1} e^{2i\pi n_L K/2k} = \left| \begin{array}{cl} 2k & \text{if } n_L = 0 \\ 0 & \text{otherwise} \end{array} \right. \tag{4.96}$$

for the former one and

$$\sum_{K=0}^{2k-1} 1 = 2k\,. \tag{4.97}$$

for the latter one. The “regularized” quotient defining the expectation value will then be taken as:

$$\lim_{N\mapsto\infty} \frac{N\sum_{K=0}^{2k-1} e^{2i\pi n_L K/2k}}{N\sum_{K=0}^{2k-1} 1} = \frac{\sum_{K=0}^{2k-1} e^{2i\pi n_L K/2k}}{\sum_{K=0}^{2k-1} 1} = \left| \begin{array}{cl} 1 & \text{if } n_L = 0\ [2k] \\ 0 & \text{otherwise}\,. \end{array} \right. \tag{4.98}$$

Hence, when $n_L \neq 0\ [2k]$, the expectation value of the corresponding Wilson line is zero, while when $n_L = 0$ the expectation value is given by eqn. (4.81). Due to $2k$-nilpotency, when $n_L = 2kN$, with $N \in \mathbb{Z}^*$, then the corresponding link invariant is trivial. These results are a clear generalization of those investigated in [17] for the three dimensional case. Also, it is quite obvious how to deal with a more general case than the quite simple product $S^{2l+1} \times S^{2l+2}$, as long as M is torsionless. The case of $(4l+3)$-manifolds with torsion might be treated extending [18].

5 Naive abelian gauge field theory and $(2l+1)$-links invariants

This section provides a formulation of the abelian $(4l+3)$-dimensional Chern Simons theory on $\mathbb{R}^{4l+3}$ with Euclidean metric in terms of a lagrangian density involving a $U(1)$

connection *i.e.* gauge field A, plus gauge fixing. This formulation, coined "naive gauge field theory" extends eqns. (3.15), (3.16) to the $(4l+3)$-dimensional case, and is the one familiar to field theorists. The presentation is formulated in a somewhat hybrid way conveniently using notations which keep track of the geometric nature of the fields and operations, combined with algebraic manipulations familiar in field theory. We aim here at emphasizing the ambiguities or weaknesses arising in this framework, in order to stress where the above non perturbative formulation in terms of DB cohomology classes brings clarification. In particular, the normalization of both the level k and loop charges e are a priori unspecified in the naive field theory approach: the prescription that they have to be integers is *ad hoc*, whereas they are bound to be integers *ab-initio* in the DB approach. Furthermore, the naive approach leads to ill-defined self-linking integrals which require to be given meaning and integer values by some extrinsic regularization procedure, such as framing, whereas the DB approach was shown above provides a natural *regularization independent normalization* prescription for the latter. Last, this study on $\mathbb{R}^{4l+3}$ also suggests which complications may arise when trying to extend the naive field theoretical framework to manifolds with non trivial cohomology.

5.1 Formulation and computation on $\mathbb{R}^{4l+3}$

The lagrangian density[2] $\mathcal{L}_{CS}\left(A^{(2l+1)}\right)$ of the abelian $(4l+3)$-dimensional Chern-Simons theory reads:

$$\mathcal{L}_{CS}\left(A^{(2l+1)}\right) = \frac{1}{2}A^{(2l+1)} \wedge d\,A^{(2l+1)} \,. \tag{5.99}$$

An extra factor 1/2 is introduced in the normalization of $\mathcal{L}_{CS}$ with respect to the normalization of $cs_1(A)$ in eq. (3.15). This normalization choice is convenient to calculate the propagator of the $A^{(2l+1)}$ field. This extra factor is subsequently compensated by defining the Chern Simons action as $4i\pi$ times the integral of $\mathcal{L}_{CS}$ indeed matching the normalization of $CS_1(A)$ in eq. (3.16).

The degeneracy coming from the gauge invariance $A^{(2l+1)} \to A^{(2l+1)} + d\,\Lambda^{(2l)}$ of this lagrangian density shall be fixed, in order that the functional integral giving the generating functional, and, in particular, the propagator of the $A^{(2l+1)}$ field be defined.

[2]Properly speaking the Chern-Simons lagrangian *density* familiar to field theorists is the Hodge * dual (on $\mathbb{R}^{4l+3}$ with Euclidean metric) of the lagrangian $(4l+3)$-*form* familiar to geometers introduced by eq. (3.15). The left hand side of eq. (5.99) should thus be $^*\mathcal{L}_{CS}\left(A^{(2l+1)}\right)$, and likewise for the gauge fixing lagrangian density $\mathcal{L}_{GF}$ in the forthcoming subsection 5.1.1. This sloppiness will hopefully not be confusing.

5.1.1 Covariant gauge fixing and corresponding propagator

In the three dimensional case, a common procedure consists in imposing the "covariant gauge fixing" $d\,{}^*A^{(3)} = 0$ by adding the following Lagrange constraint:

$$\mathcal{L}^{(3d)}_{GF} = B^{(0)} \wedge d\,{}^*A^{(3)} \tag{5.100}$$

where * here denotes the Hodge dual operation with respect to the Euclidean metric on $\mathbb{R}^3$ and the Lagrange multiplier $B^{(0)}$ is a scalar field *i.e.* a zero-form. Let from now on * denote the Hodge dual operation on flat Euclidean $\mathbb{R}^{4l+3}$, such that for any q-form $B^{(q)}$, ${}^{**}B^{(q)} = (-1)^{q(4l+3-q)}B^{(q)} = B^{(q)}$. The naive straightforward generalization of eqn. (5.100) by means of a single auxiliary $2l$-form $B^{(2l)}$ according to

$$\mathcal{L}^{naive}_{GF} = B^{(2l)} \wedge d\,{}^*A^{(2l+1)}$$

is not effective as $\mathcal{L}^{naive}_{GF}$ still has the residual gauge invariance $B^{(2l)} \to B^{(2l)} + d\,\Lambda^{(2l-1)}$. An appropriate formulation requires a collection of $2l+1$ auxiliary forms of decreasing degrees $(B^{(2l)}, B^{(2l-1)}, \cdots, B^{(0)})$, according to:

$$\mathcal{L}_{GF} = B^{(2l)} \wedge d\,{}^*A^{(2l+1)} \;+\; B^{(2l-1)} \wedge d\,{}^*B^{(2l)} \;+\; \cdots \;+\; B^{(0)} \wedge d\,{}^*B^{(1)}\,. \tag{5.101}$$

Regrouping all the fields into

$$\vec{\mathcal{A}} = (\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \cdots, \mathcal{A}_{2l+2}) \equiv (A^{(2l+1)}, B^{(2l)}, B^{(2l-1)}, \cdots, B^{(0)})$$

we can compactly write the full action given by $\mathcal{L}_{tot} = \mathcal{L}_{CS}(A^{(2l+1)}) + \mathcal{L}_{GF}$ as a scalar product:

$$\int \mathcal{L}_{tot} = \int_{\mathbb{R}^{4l+3}} \vec{\mathcal{A}} \wedge {}^*D\vec{\mathcal{A}} \equiv \frac{1}{2}\left(\vec{\mathcal{A}}, D\vec{\mathcal{A}}\right) \tag{5.102}$$

with:

$$D = \begin{bmatrix} {}^*d & -d & 0 & 0 & & & & & & \\ \delta & 0 & d & 0 & & & & & & \\ 0 & \delta & 0 & -d & & & & & & \\ 0 & 0 & \delta & 0 & & & & & & \\ & & & & \cdots & & & & & \\ & & & & & 0 & d & 0 & & \\ & & & & & \delta & 0 & -d & & \\ & & & & & 0 & \delta & 0 & & \\ & & & & & & & & \cdots & \\ & & & & & & & & 0 & -d \\ & & & & & & & & \delta & 0 \end{bmatrix} \tag{5.103}$$

where $\delta \equiv {}^*d^*$ is the co-differential associated with the Hodge dual. The Euler-Lagrange equations of motion of the $\vec{\mathcal{A}}$ field read:

$$D\vec{\mathcal{A}} = 0 \tag{5.104}$$

The propagator $< \vec{\mathcal{A}}(x) \otimes \vec{\mathcal{A}}(y) >$ of the field $\vec{\mathcal{A}}$ is the inverse of the operator D conveniently determined solving

$$D < \vec{\mathcal{A}}(x) \otimes \vec{\mathcal{A}}(y) >= \delta^{(4l+3)}(x-y)\, 1\!\mathrm{I}_{2l+2} \tag{5.105}$$

by means of Fourier transformation, taking advantage of translation invariance on Euclidean space $\mathbb{R}^{4l+3}$. It is especially convenient to use a Fourier transformation, defined by means of Berezin integration, which preserves the degrees of forms, as detailed in Appendix A. The Fourier transform of $D\,\delta^{(4l+3)}(x-y)$ reads:

$$\overrightarrow{D} = -i \begin{bmatrix} {}^*P & -P & 0 & 0 & & & & & & & \\ \Xi & 0 & P & 0 & & & & & & & \\ 0 & \Xi & 0 & -P & & & & & & & \\ 0 & 0 & \Xi & 0 & & & & & & & \\ & & & & \cdots & & & & & & \\ & & & & & 0 & P & 0 & & & \\ & & & & & \Xi & 0 & -P & & & \\ & & & & & 0 & \Xi & 0 & & & \\ & & & & & & & & \cdots & & \\ & & & & & & & & & 0 & -P \\ & & & & & & & & & \Xi & 0 \end{bmatrix} . \tag{5.106}$$

The expression for P and Ξ are given in eqns. (6.139) of Appendix A.

The Fourier transforms $\overrightarrow{N}_{jk}$ of the $< \mathcal{A}_{2l+2-j} \otimes \mathcal{A}_{2l+2-k} >$ satisfy:

$$-\,i\left({}^*P \wedge \overrightarrow{N}_{1,j} - P \wedge \overrightarrow{N}_{2,j}\right) = \delta_{1,j}\,\mathrm{Id}_{(2l+1)}\,, \;\; j \in [1,\ldots,2l+2] \tag{5.107}$$

$$-i\left(\Xi \wedge \overrightarrow{N}_{k-1,j} + (-)^k P \wedge \overrightarrow{N}_{k+1,j}\right) = \delta_{k,j}\,\mathrm{Id}_{(2l+2-j)}\,, \quad j \in [1,\ldots,2l+2]\,,\; k \in [2,\ldots,2l+1] \tag{5.108}$$

$$-i\left(\Xi \wedge \overrightarrow{N}_{2l+1,j}\right) = \delta_{2l+2,j}.\mathrm{Id}_{(0)}\,, \;\; j \in [1,\ldots,2l+2]\,. \tag{5.109}$$

A particular solution to the inhomogeneous eqns. (5.107)-(5.109) on the diagonal $j = k$ is suggested by the Hodge decomposition of the Laplacian operator whose Fourier transform reads: $\Xi \wedge P + P \wedge \Xi = p^2\,\mathrm{Id}$, and by the identities $P \wedge P = 0$, $\Xi \wedge \Xi = 0$:

$$\overrightarrow{N}_{1,1} = \frac{i}{p^2}\,{}^*P_{(2l+1)} \tag{5.110}$$

$$\overrightarrow{N}_{j-1,j} = \frac{i}{p^2}\,P_{(2l+1-j)}, \quad 2 \le j \le 2l+2 \tag{5.111}$$

$$\overrightarrow{N}_{j+1,j} = -\frac{i}{p^2}\,\Xi_{(2l+1+j)}, \quad 1 \le j \le 2l+1 \tag{5.112}$$

and all the other $\overrightarrow{N}_{i,j}$ vanishing. The particular solution thus found for the Fourier transform $\overrightarrow{N}_{1,1}$ of the propagator $< A^{(2l+1)} \otimes A^{(2l+1)} >$ involved in the computation of Wilson $(2l+1)$-loops correlators turns out to be the so-called Moore-Penrose pseudo-inverse[3] of the operator $i\,{}^*P$ which satisfies:

$$-\,i\,{}^*P\,\overrightarrow{N}_{1,1} = \Pi \tag{5.113}$$

where Π is the projector onto the subspace selected by the covariant gauge fixing condition.

The propagators $< \mathcal{A}_{2l+2-j} \otimes \mathcal{A}_{2l+2-k} >$ might differ from the particular solution above by terms corresponding to general solutions of the homogeneous equations associated with eqns. (5.107) - (5.109) *i.e.* with all right hand sides vanishing. The general solutions of these homogeneous equations on the space of tempered currents can be proven to be forms with harmonic coefficients. Hence in the present case on $\mathbb{R}^{4l+3}$ with Euclidean metrics the coefficient functions of these harmonic forms are harmonic polynomials of $(x-y)$. In a first step we shall ignore such potential terms and consider the $\overrightarrow{N}_{jk}$ entirely given by eqns.(5.110) - (5.112). We will comment on them in paragraph 5.1.2 and prove that they do not contribute insofar as we are only concerned with the computation of correlators of $(2l+1)$-loops.

Performing the inverse Fourier transforms of eqns.(5.110) - (5.112) yields the explicit expressions of the $< \mathcal{A}_j(x)\mathcal{A}_k(y) >$. The only one explicitly needed in the following is:

$$\begin{aligned}&\left\langle \mathcal{A}^{(2l+1)}_{\mu_1,\cdots,\mu_{2l+1}}(x)\,\mathcal{A}^{(2l+1)}_{\nu_1,\cdots,\nu_{2l+1}}(y)\right\rangle \\ &= \frac{\Gamma\left(\frac{4l+3}{2}\right)}{2\pi^{\frac{4l+3}{2}}}\epsilon_{\mu_1,\cdots,\mu_{2l+1},\nu_1,\cdots,\nu_{2l+1},\rho}\,\frac{(x-y)^\rho}{|x-y|^{4l+3}}\,,\end{aligned} \tag{5.114}$$

[3]This can be most simply and explicitly checked in the three dimensional case. The projector Π is then the projector transverse to p, which indeed corresponds to the subspace of Fourier modes $\widehat{A}(p)$ such that $p^\mu \widehat{A}_\mu(p) = 0$ *i.e.* the Fourier dual of the covariant gauge fixing condition $d^*A = 0$ imposed in x-space.

$\Gamma(w)$ being the Euler Gamma function and ϵ the $(4l+3)$-dimensional Levi-Civita symbol. The derivation of identity (5.114) relies on eqn. (6.137) of Appendix A.

The gauge field theory is provided by the generating functional in presence of arbitrary source currents $\vec{\mathcal{J}}$, which may be formally expressed by the following functional integral:

$$\mathcal{Z}(\vec{\mathcal{J}}) \;=\; \mathcal{N} \int \mathcal{D}\vec{A}\, e^{2i\pi k\,(\vec{\mathcal{A}},D\vec{\mathcal{A}})+i(\vec{\mathcal{A}},\vec{\mathcal{J}})} \tag{5.115}$$

in which $\mathcal{D}\vec{A}\,\exp\{2i\pi k\,(\vec{\mathcal{A}},D\vec{\mathcal{A}})\}$ is a functional integration measure on some (unspecified) appropriate functional space. This measure is assumed to have all nice properties of usual gaussian integrals, and $\mathcal{N}$ is a normalization constant such that $\mathcal{Z}(\vec{\mathcal{J}}=0)=1$. The correlator of two $(2l+1)$-loops γ_1 and γ_2 is provided by the quantity

$$\mathcal{N} \int \mathcal{D}\vec{A}\; e^{2i\pi k\,(\vec{\mathcal{A}},D\vec{\mathcal{A}})}\; e^{2i\pi\, e_1 \int_{\gamma_1} A^{(2l+1)}}\; e^{2i\pi\, e_2 \int_{\gamma_2} A^{(2l+1)}}\,. \tag{5.116}$$

Let us represent the $(2l+1)$-loop γ_s by the $(2l+2)$-current $j_s^{(2l+2)}$ so that

$$\int_{\gamma_s} A^{(2l+1)} = \int_{\mathbb{R}^{4l+3}} A^{(2l+1)} \wedge j_s^{(2l+2)} \tag{5.117}$$

hence

$$2\pi\, e_1 \int_{\gamma_1} A^{(2l+1)} \;+\; 2\pi\, e_2 \int_{\gamma_2} A^{(2l+1)} = \left(\vec{\mathcal{A}},\vec{\mathcal{J}}\right) \tag{5.118}$$

so that the loop correlator (5.116) is given by eqn. (5.115) identifying

$$\vec{\mathcal{J}} = 2\pi \left(e_1\, {}^*j_1^{(2l+2)} + e_2\, {}^*j_2^{(2l+2)}, 0, 0, \cdots, 0\right)\,. \tag{5.119}$$

The phase in the integrand of eqn. (5.116) involves:

$$k\left(\vec{\mathcal{A}},D\vec{\mathcal{A}}\right) + e_1 \int_{\gamma_1} A^{(2l+1)} + e_2 \int_{\gamma_2} A^{(2l+1)} = k\left(\vec{\mathcal{A}}',D\vec{\mathcal{A}}'\right) - \frac{1}{16\pi^2 k}\left(\vec{\mathcal{J}},D^{-1}\vec{\mathcal{J}}\right) \tag{5.120}$$

where

$$\vec{\mathcal{A}}' \;=\; \vec{\mathcal{A}} + \frac{1}{4\pi k}\, D^{-1}\vec{\mathcal{J}}\,. \tag{5.121}$$

The functional space $\{\vec{\mathcal{A}}\}$ is assumed to be stable[4] under the shift (5.121). This shift is namely the counterpart of the one performed in eqn. (4.68), and the gaussian properties of the functional measure $\mathcal{D}\vec{A}\,\exp\{2i\pi k\,(\vec{\mathcal{A}},D\vec{\mathcal{A}})\}$ are the mere counterparts of the Cameron-Martin property (3.26). We thus proceed as in the geometric approach.

[4] By passing let us notice that any current $j^{(2l+2)}$ representing a $(2l+1)$-loop is such that $j^{(2l+2)} = d\eta^{(2l+1)}$, the corresponding ${}^*j^{(2l+2)}$ thus belongs to the functional subspace of $\{A^{(2l+1)}\}$ obeying the covariant gauge fixing condition $d^*A^{(2l+1)}=0$. Furthermore this subspace is stable under the action of the operator $\left[D^{-1}\right]$, cf. eqn. (5.113), so that this subspace is itself stable under the shift (5.121).

The functional integration leads to:

$$\mathcal{N} \int \mathcal{D}\vec{A}\, e^{2i\pi k\,(\vec{\mathcal{A}},D\vec{\mathcal{A}})}\, e^{2i\pi\, e_1 \int_{\gamma_1} A^{(2l+1)}}\, e^{2i\pi\, e_2 \int_{\gamma_2} A^{(2l+1)}} \quad = \quad e^{-\frac{i}{8\pi k}(\vec{\mathcal{J}},D^{-1}\vec{\mathcal{J}})}\,. \qquad (5.122)$$

In the integral in the exponential in the r.h.s. of eqn. (5.122), the term of degree $(2l+1)$ is made of:

$$(D^{-1}\vec{\mathcal{J}}_{2l+1})_{\mu_1,\cdots,\mu_{2l+1}}(x) = \int_{\mathbb{R}^{4l+3}_y} \left\langle A^{(2l+1)}_{\mu_1,\cdots,\mu_{2l+1}}(x) A^{(2l+1)}_{\nu_1,\cdots,\nu_{2l+1}}(y)\right\rangle (\vec{\mathcal{J}}_{2l+1})^{\nu_1,\cdots,\nu_{2l+1}}_{2l+1} d^{4l+3}y \qquad (5.123)$$

and:

$$(\vec{\mathcal{J}}_{2l+1}, D^{-1}\vec{\mathcal{J}}_{2l+1}) = \int_{\mathbb{R}^{4l+3}_x} d^{4l+3}x(\vec{\mathcal{J}}_{2l+1})^{\mu_1,\cdots,\mu_{2l+1}}(x)(D^{-1}\vec{\mathcal{J}}_{2l+1})_{\mu_1,\cdots,\mu_{2l+1}}(x)\,. \qquad (5.124)$$

This yields two sorts of terms.

1. Those of the form:

$$\begin{aligned} L(\gamma_1,\gamma_2) &\equiv \int_{\mathbb{R}^{4l+3}_x} d^{4l+3}x({}^*j_1^{(2l+2)})^{\mu_1,\cdots,\mu_{2l+1}}(x)(D^{-1*}j_2^{(2l+2)})_{\mu_1,\cdots,\mu_{2l+1}}(x) \\ &= \int_{\mathbb{R}^{4l+3}_x \mathrm{x} \mathbb{R}^{4l+3}_y} j_1^{(2l+2)}(x) \wedge \left\langle A^{(2l+1)}(x) \otimes A^{(2l+1)}(y)\right\rangle \wedge j_2^{(2l+2)}(y) \\ &= \frac{1}{(2l+1)!^2} \oint_{\gamma_1} (dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{2l+1}}) \times \\ &\qquad \oint_{\gamma_2} (dy^{\nu_1} \wedge \cdots \wedge dy^{\nu_{2l+1}}) \left\langle A^{(2l+1)}_{\mu_1,\cdots,\mu_{2l+1}}(x) A^{(2l+1)}_{\nu_1,\cdots,\nu_{2l+1}}(y)\right\rangle\,. \end{aligned} \qquad (5.125)$$

   They turn out to be the linking of γ_1 and γ_2 since after injecting expression (5.114) in the last line of eqn. (5.125) one recognizes the generalized Gauss formula [19]. The latter is recalled in Appendix B providing a consistency check of all normalizations between the geometric and the "naive" approaches. However, at variance with the virtue of the geometric approach, it is important to notice in this respect that the values of the level k and of the loop charges e_j are *not* quantized in the naive approach: their prescribed integer natures here are *ad hoc* and imposed "by hand".

   This derivation sheds some light on the relation between the generalized Gauss formula (5.125) and the geometric approach developed in section 4. With respect to the variable $\vec{\mathcal{J}}$ the propagator identifies with $[{}^*d]^{-1}_{MP}$, the (Moore-Penrose pseudo-) inverse of ${}^*d$, whereas it identifies with $[d]^{-1}_{MP}$ the inverse of d with respect to the loops currents $j_1^{(2l+2)}$ and $j_2^{(2l+2)}$ in the following way. All loops are contractible in $\mathbb{R}^{4l+3}$, therefore there exists a de Rham current $\eta_2^{(2l+1)}$ such that:

$$j_2^{(2l+2)} = d\eta_2^{(2l+1)} , \tag{5.126}$$

whose general solution is

$$\eta_2^{(2l+1)} = [d]^{-1}_{MP} j_2^{(2l+2)} + \zeta_2^{(2l+1)} , \tag{5.127}$$

where $\zeta_2^{(2l+1)}$ is an arbitrary closed current. Indeed the current $\eta_2^{(2l+1)}$ is not unique since:

$$d(\eta_2^{(2l+1)} + \zeta_2^{(2l+1)}) = j_2^{(2l+2)} . \tag{5.128}$$

This reminds us of the definition of the Poincaré Homotopy:

$$\kappa \wedge d + d \wedge \kappa = \mathrm{Id}_{(2l+1)} \tag{5.129}$$

that encodes Poincaré Lemma (for $\mathbb{R}^{4l+3}$). The degeneracy associated with the inversion of d is exactly the one due to gauge invariance since on $\mathbb{R}^{4l+3}$, and still by virtue of Poincaré's lemma, one has:

$$\zeta_2^{(2l+1)} \in Ker[d] \Leftrightarrow \exists \xi^{(2l+1)},\ \zeta_2^{(2l+1)} = d\,\xi^{(2l+1)} .$$

We shall come back to this comment below when addressing the corresponding issue on topologically non trivial $(4l+3)$-dimensional manifolds instead of $\mathbb{R}^{4l+3}$.

2. It also involves the self-linkings of $(2l+1)$-loop γ_1 and of $(2l+1)$-loop γ_2 by means of formulas very similar to eqn. (5.125), yet the integrals involved here are ill-defined [25, 26, 27]. An extrinsic procedure is required to have them make sense as quantities defined modulo integers. Framing provides *one* such procedure in the present case, a given integer for each self-linking corresponding to a given framing choice. By contrast the zero regularization implemented in the geometric approach is less detailed as it does not prescribe any definite integer value to any given self-linking.

5.1.2 Harmonic terms do not contribute

So far we have ignored the presence of a harmonic contribution $H(x-y)$ to the propagator $< A^{(2l+3)}(x) \otimes A^{(2l+3)}(y) >$. At first sight one might be tempted to argue that the absence of such terms is implied by the cluster property meaning that $< A^{(2l+3)}(x) \otimes A^{(2l+3)}(y) > \to 0$ when $\|x-y\| \to +\infty$. However this is i) beside the point ii) not necessarily true.

i) It is beside the point insofar as we are interested in correlators of $(2l+1)$-loops *i.e. closed* curves. Assuming that the propagator involves such a harmonic term $H(x-y)$, let us generalize eqn. (5.125) by

$$\begin{aligned}
&\widetilde{L}(\gamma_1, \gamma_2) \\
&= \int_{\mathbb{R}_x^{4l+3} \mathrm{x} \mathbb{R}_y^{4l+3}} j_1^{(2l+2)}(x) \wedge \left\{ \left\langle A^{(2l+1)}(x) \otimes A^{(2l+1)}(y) \right\rangle + H(x-y) \right\} \wedge j_2^{(2l+2)}(y) \\
&\equiv L(\gamma_1, \gamma_2) + L'_H(\gamma_1, \gamma_2)
\end{aligned} \tag{5.130}$$

The currents $j_{1,2}^{(2l+2)}$ dualize $(2l+1)$-loops so that *e.g.* $j_1^{(2l+2)} = d\eta_1^{(2l+1)}$ so that through integration by part,

$$\begin{aligned}
L'_H(\gamma_1, \gamma_2) &= \int_{\mathbb{R}_x^{4l+3} \mathrm{x} \mathbb{R}_y^{4l+3}} \eta_1^{(2l+1)}(x) \wedge (d_y H(x-y)) \wedge j_2^{(2l+2)}(y) \\
&= 0
\end{aligned} \tag{5.131}$$

This suggests that the appropriate functional space on which the propagator has to be defined is a quotient modulo harmonic parts. Such a functional space has been studied in ref. [32].

By passing, eqn. (5.131) proves that harmonic contributions vanish even when $j_2^{(2l+2)}$ dualizes a non compactly supported loop, such as a $(2l+1)$-hyperplane. This property is expected to be particularly relevant in order to extend the present result to the sphere S^{4l+3}.

ii) The cluster property may not hold with another gauge fixing choice. See for instance the 3-dimensional case with axial gauge fixing.

5.1.3 Impact of the gauge fixing choice

Equation (5.125) was noticed to reproduce the generalized Gauss formula when the propagator $< A^{(2l+3)} \otimes A^{(2l+3)} >$ is given by eqn. (5.114). Another condition than the gauge fixing (5.100) would lead to a different propagator. Equation (5.125) would then provide an expression of the linking number different from the one obtained using the generalized Gauss invariant. For example in the three dimensional case, the "axial gauge" choice leads to a braiding interpretation of the linking number [29], rather than the solid angle interpretation reminded in Appendix B. Let us stress that all gauge fixing choices are equivalent ways of computing the generalized linking number. Indeed, the propagator in the covariant gauge and one with an alternative gauge choice differ by terms involving the derivative d whose actions on the closed currents dualizing $(2l+1)$-loops vanish.

In a Quantum Electro-Dynamical language, the latter are "conserved currents" which guarantees the gauge fixing independence of observables associated with these currents.

5.2 Further issues arising on the $\mathcal{S}^{4l+3}$ then on further non trivial manifolds

As we already mentioned it, Chern-Simons field theory cannot provide a quantization of the level k nor of the charge q. This is due to the fact such a theory is developed over the non compact space $\mathbb{R}^{4l+3}$. It's only when going on a closed manifold such as a sphere that the quantization naturally appeared in the geometric approach. This suggest that to get such a quantization of k and q within the field theoretic framework, one should have to first define a field theory over a closed manifold M, starting with S^{4l+3}. Since the CS lagrangian is not a globally defined 3-form, we anticipate two possible paths: one based on a partition of unity subordinated to a good covering of M and a second based on a polyhedral decomposition of M.

1. We could consider a polyhedral decomposition Δ of M and start with field theories on each of the fundamental *i.e.* $(4l+3)$-dimensional polyhedra Δ_α of the decomposition. Once this done on fundamental polyhedra we would have to see how things match on the $(4l+2)$-dimensional boundaries $\Delta_{\alpha\beta}$ of these polyhedra leading to $(4l+2)$-dimensional field theories on those boundaries. We would have to keep proceeding along this line till we reach the polyhedral elements of dimension 0 of the decomposition. This would be related to the short formula defining the integral of a DB class, as explained in [7].

2. We could provide M with a partition of unity subordinated to a good covering $\mathcal{U}$ in such a way that each open set $\mathcal{U}_\alpha$ supports a field theory in $\mathbb{R}^{4l+3}$. Matching these theories in the $(4l+3)$-dimensional intersections $\mathcal{U}_{\alpha\beta}$ would lead to considering extra field theories in these intersections then in the triple intersections $\mathcal{U}_{\alpha\beta\gamma}$ etc. The present point of view in which all supplemented field theories would be on $\mathbb{R}^{4l+3}$ is a smoothing of the former polyhedral approach. This would be related to the long formula appearing in [7].

We would like to stress out that our procedure to compute the propagator of the abelian CS field theory on $\mathbb{R}^{4l+3}$ exhibits a set of descent equations whose resolution is made simple because $\mathbb{R}^{4l+3}$ has no cohomology (except in dimension 0). Our results might be extended to S^{4l+3} since it shares the same cohomology properties for the concerned degrees. In the case of a general closed manifold, such has $S^{2l+1} \times S^{2l+2}$, this would not be true. However, locally that is to say with respect to a good covering and with an Euclidean metric on each open set, such a descent might still hold. Yet the gluing constraints on the whole manifold (*e.g.* via a partition of unity) would prevent the descent from being globally trivial. The simplest case to investigate would be S^3 and the first non trivial one $S^1 \times S^2$.

Concerning the propagator itself, the fact it coincides with the Gauss integral is once more only due to the fact we are working on $\mathbb{R}^{4l+3}$. One would expect a different expression for the propagator on a closed manifold. However there exist expressions of the Gauss integral on spheres [31]. One could also try to mimic Gauss zodiacus idea, at least in the case of S^3 identified with $SU(2)$, replacing the notion of translations acting on $\mathbb{R}^3$ by actions on $SU(2)$. From the point of view of the two possible approaches previously mentioned, we can expect a collection of propagators, associated with the different field theory arising from the construction (for instance one for each polyhedra type of the decomposition of the closed manifold), but also a gluing rule explaining how these propagators "communicate".

It appears as a very interesting problem how this could be properly handled because it would provide an example of a field theory over a closed manifold. We can have some hope about how this can be done, because the theory which we are dealing with is a topological one, and also because the geometric approach provides us with the final answer concerning Wilson observables.

6 Conclusions and outlook

The treatment of abelian Chern-Simons to generate link invariants introduced in [17] straightforwardly extends to the case of oriented closed $(4l + 3)$-dimensional manifolds without torsion. Actually, we didn't show that the expectation values of our generalised Wilson lines are ambient isotopy invariants. This can be easily checked extending what has been done in [17]. In the same way, it is possible to establish satellite relations for our generalised invariants. As for torsion, one could follow the approach developed for $\mathbb{R}P^3$ in [18]. One can wonder whether the DB strategy applies more generally to abelian BF systems. Using Deligne-Beilinson Cohomology technics might also provide a way to study higher order systems, that is to say systems whose classical lagrangian involves DB products of more than two DB classes. In any of these cases one should expect homology and intersection to play the fundamental role.

Appendix A: Forms and Fourier Transform

This appendix is devoted to the conventions and properties of Fourier transform applied to forms and linear operators acting on them. These properties are used in Section 5 in order to evaluate precisely the propagator of the vector potential in the covariant gauge.

Berezin-Fourier transform preserving forms degrees

The components of a q-form are defined through

$$B^{(q)} = B(x)_{\nu_1\ldots\nu_q}\,\psi^{\nu_1}\wedge\ldots\wedge\psi^{\nu_q} \tag{6.132}$$

where $\psi^\mu = dx^\mu$. This convention partially avoids clutter with factorial numbers.
The Fourier transform of a q-form is then defined as

$$\begin{aligned}
\overrightarrow{B}^{(q)} &\equiv \left[\int d^n x\,\mathrm{e}^{ip_\mu x^\mu} B(x)_{\nu_1\ldots\nu_q}\right]\left[\frac{1}{l_{(n-q)}}\int d^n\psi\,\mathrm{e}^{i\bar\omega_\mu\psi^\mu}\psi^{\nu_1}\wedge\ldots\wedge\psi^{\nu_q}\right]\\
&= \left[\int d^n x\,\mathrm{e}^{ip_\mu x^\mu} B(x)_{\nu_1\ldots\nu_q}\right]\\
&\quad \frac{\epsilon^{\nu_{q+1}\ldots\nu_n\ldots\nu_1\ldots\nu_q}}{(n-q)!}\frac{\epsilon^{\tau_{q+1}\ldots\tau_n\ldots\mu_1\ldots\mu_q}}{q!}\delta_{\nu_{q+1}\tau_{q+1}}\ldots\delta_{\nu_n\tau_n}\,\bar\omega_{\mu_1}\wedge\ldots\wedge\bar\omega_{\mu_q}\\
&= \frac{\widehat{B}(p)_{\nu_1\ldots\nu_q}}{q!(n-q)!}\,\epsilon^{\nu_{q+1}\ldots\nu_n\ldots\nu_1\ldots\nu_q}\,\delta_{\nu_{q+1}\tau_{q+1}}\ldots\delta_{\nu_n\tau_n}\,\epsilon^{\tau_{q+1}\ldots\tau_n\ldots\mu_1\ldots\mu_q}\,\bar\omega_{\mu_1}\wedge\ldots\wedge\bar\omega_{\mu_q}\\
&= \widehat{B}(p)_{\nu_1\ldots\nu_q}\,\delta^{\nu_1\mu_1}\ldots\delta^{\nu_q\mu_q}\,\bar\omega_{\mu_1}\wedge\ldots\wedge\bar\omega_{\mu_q}\\
&= \widehat{B}(p)^{\mu_1\ldots\mu_q}\,\bar\omega_{\mu_1}\wedge\ldots\wedge\bar\omega_{\mu_q}
\end{aligned} \tag{6.133}$$

where $l_{(a)} = 1$ if a is even and $l_{(a)} = i$ if a is odd, $\bar\omega_\mu \equiv dp_\mu$, and $\widehat{\ }$ denotes the usual Fourier transform on functions. With this definition, the Fourier transform of a q-form is itself a q-form, that is to say the Fourier transform respects the form degrees.
Inverse Fourier transform is accordingly defined as

$$\begin{aligned}
\overleftarrow{B}^{(q)} &\equiv \frac{1}{(2\pi)^n}\left[\int d^n p\,\mathrm{e}^{-ip_\mu x^\mu} B(p)^{\nu_1\ldots\nu_q}\right]\left[\frac{1}{l_{(n-q)}}\int d^n\bar\omega\,\mathrm{e}^{-i\bar\omega_\mu\psi^\mu}\bar\omega_{\nu_1}\wedge\ldots\wedge\bar\omega_{\nu_q}\right]\\
&= \frac{\check{B}(x)^{\nu_1\ldots\nu_q}}{q!(n-q)!}\epsilon_{\nu_{q+1}\ldots\nu_n\ldots\nu_1\ldots\nu_q}\,\delta^{\nu_{q+1}\tau_{q+1}}\ldots\delta^{\nu_n\tau_n}\,\epsilon_{\tau_{q+1}\ldots\tau_n\ldots\mu_1\ldots\mu_q}\,\psi^{\mu_1}\wedge\ldots\wedge\psi^{\mu_q}\\
&= \check{B}(x)^{\nu_1\ldots\nu_q}\,\delta_{\nu_1\mu_1}\ldots\delta_{\nu_q\mu_q}\,\psi^{\mu_1}\wedge\ldots\wedge\psi^{\mu_q}\\
&= \check{B}(x)_{\mu_1\ldots\mu_q}\,\psi^{\mu_1}\wedge\ldots\wedge\psi^{\mu_q}
\end{aligned} \tag{6.134}$$

where $\check{\ }$ is the inverse Fourier transform on functions. An explicit evaluation indeed confirms that

$$\overleftarrow{\overrightarrow{B}} = B\,. \tag{6.135}$$

An important property is that the Hodge operation and Berezin-Fourier transform do commute:

$$\begin{aligned} &{}^*\Bigg[\frac{1}{l_{(n-q)}}\int d^n\psi\, \mathrm{e}^{i\bar{\omega}_\mu\psi^\mu}\,\psi^{\nu_1}\wedge\,...\,\wedge\psi^{\nu_q}\Bigg] \\ &= \frac{1}{q!}\,\delta^{\nu_1...\nu_q}_{\sigma_1...\sigma_q}\,\delta^{\sigma_1\mu_1}\,...\,\delta^{\sigma_q\mu_q}\,\bar{\omega}_{\mu_1}\wedge\,...\,\wedge\bar{\omega}_{\mu_q} \\ &= \Bigg[\frac{1}{l_{(n-q)}}\int d^n\psi\, \mathrm{e}^{i\bar{\omega}_\mu\psi^\mu}\,{}^*(\psi^{\nu_1}\wedge\,...\,\wedge\psi^{\nu_q})\Bigg]\,. \end{aligned} \tag{6.136}$$

An useful Fourier transform

The explicit computation of the fundamental propagator (5.114) relies on the following Fourier transform

$$\overleftarrow{\left(\frac{p^\tau}{p^2}\right)} = \frac{1}{(2\pi)^{4l+3}}\int d^{4l+3}p\,\mathrm{e}^{-ip_\mu x^\mu}\,\frac{p^\tau}{p^2} = -i\frac{\Gamma\left(\frac{4l+3}{2}\right)}{2\pi^{\frac{4l+3}{2}}}\frac{x^\tau}{x^{4l+3}}\,. \tag{6.137}$$

Berezin-Fourier transform for linear operators

The Berezin-Fourier transform of a linear operator $\mathcal{O}$ acting on forms is defined by

$$\overrightarrow{\mathcal{O}}B \equiv \overrightarrow{\mathcal{O}\overleftarrow{B}}\,. \tag{6.138}$$

Accordingly, the (useful) Fourier transform of the differential, its Hodge dual and the co-differential read:

$$\overrightarrow{d} = -ip^\mu\bar{\omega}_\mu \equiv -iP \tag{6.139}$$

$$\overrightarrow{({}^*d)} = {}^*\left(\overrightarrow{d}\right) = -i\,{}^*P \tag{6.140}$$

$$\overrightarrow{\delta} = {}^*\left(\overrightarrow{d}\right)^* = -i\,{}^*P^* \equiv -i\,\Xi\,. \tag{6.141}$$

Appendix B: Generalized Gauss linking number

Definition of the linking number

We consider two $(2l+1)$-dimensional closed surfaces γ_{2l+1} and γ'_{2l+1} embedded in the space $\mathbb{R}^{4l+3}$. They are defined as a map from the $(2l+1)$-dimensional closed manifold T, respectively T', to $\mathbb{R}^{4l+3}$. Their linking number is given by [19]

$$L(\gamma_{2l+1},\gamma'_{2l+1}) = \mathcal{N}_l \oint\limits_{\gamma_{2l+1}} dx^{\mu} \oint\limits_{\gamma'_{2l+1}} dy^{\nu} \epsilon_{\mu,\nu,\sigma}\, \delta^{\sigma\tau}\, \partial_{\tau} |x-y|^{-4l-1} \tag{6.142}$$

where the xs (resp. ys) are the coordinates of points of γ_{2l+1} (resp. γ'_{2l+1}) and ϵ is the $(4l+3)$-dimensional Levi-Civita symbol. We have used the following shorthand notations

$$dx^{\mu} = dx^{\mu_1} \cdots dx^{\mu_{2l+1}} \ , \ dy^{\nu} = dy^{\nu_1} \cdots dy^{\nu_{2l+1}} \ , \ \epsilon_{\mu,\nu,\sigma} = \epsilon_{\mu_1 \cdots \mu_{2l+1} \nu_1 \cdots \nu_{2l+1} \sigma} \tag{6.143}$$

and set $\partial_{\tau} = \partial_{y^{\tau}}$. The other choice of the derivative, $\partial = \partial_x$, reverses the sign of the linking number, e.g. it corresponds to an orientation choice. The normalisation of the linking number is

$$\mathcal{N}_l = \frac{\Gamma\left(\frac{4l+3}{2}\right)}{(8l+2)\sqrt{\pi^{4l+3}}(2l+1)!^2} \,. \tag{6.144}$$

with Γ the Euler Gamma function, satisfying $\Gamma(n+1) = n!$ for an integer n.

The linking number can be given a more enlightening form as follows. For two points x (resp y) on γ_{2l+1} (resp. γ'_{2l+1}), we consider the unitary vector

$$e_{xy} = \frac{x-y}{|x-y|}. \tag{6.145}$$

The unitary vector e_{xy} thus defines a map from $T \times T'$ to the sphere S^{4l+2} whose degree is the linking number [33]. We now consider the quantity

$$[e_{xy}; dx; dy] = \frac{1}{(2l+1)!^2} \epsilon_{\mu,\nu,\sigma}\, dx^{\mu} dy^{\nu} e^{\sigma}_{xy} \tag{6.146}$$

which has a simple physical interpretation:

$$\frac{[e_{xy}; dx; dy]}{|x-y|^{4l+2}} \tag{6.147}$$

is the oriented solid angle formed by a simultaneous displacement dx on γ_{2l+1} and dy on γ'_{2l+1}.

The linking number can thus be given the following equivalent form

$$L(\gamma_{2l+1}, \gamma'_{2l+1}) = \frac{1}{S_{4l+2}} \oint\limits_{\gamma_{2l+1}} \oint\limits_{\gamma'_{2l+1}} \frac{[e_{xy}; dx; dy]}{|x-y|^{4l+2}} \tag{6.148}$$

and interpretation of a global solid angle. We have used the value of the surface of a unit sphere S^n is given by

$$S_n = \frac{2\pi^{\frac{n+1}{2}}}{\Gamma(\frac{n+1}{2})} \,. \tag{6.149}$$

This is also the total solid angle in dimension $n+1$.

The three dimensional case

In the three dimensional case ($l = 0$), the linking number (6.148) is the famous Gauss invariant [20]

$$L(\gamma, \gamma') = \frac{1}{4\pi} \oint\limits_{\gamma} \oint\limits_{\gamma'} d\vec{x} \times d\vec{y}. \frac{\vec{x} - \vec{y}}{|\vec{x} - \vec{y}|^3} \,. \tag{6.150}$$

The unitary vector

$$\vec{e}_{xy} = \frac{\vec{x} - \vec{y}}{|x - y|} \,. \tag{6.151}$$

defines a map e from $S^1 \times S^1$ to the sphere S^2 whose degree is the linking number [33]. The image of the map e is generically a surface called the zodiacus by Gauss who also obtained a necessary condition for a point to be on its boundary: the tangent vectors to the two curves at points x and y respectively and the vector $\vec{e}_{xy}$ are linearly dependent. In other words, these are points such that

$$[\vec{e}_{xy}; d\vec{x}; d\vec{y}] = 0 \tag{6.152}$$

and do not contribute to the Gauss integral. This condition is only necessary and not all solutions do represent actual boundaries of the zodiacus. Two cases have to be distinguished: (1) the two curves are not linked and the zodiacus has at least one boundary, (2) the two curves are linked and the curve defined by the previous condition cannot be a boundary of the zodiacus which is in fact the whole sphere.

Some intuition on these matters can be given by the following particular case. We consider a basic configuration of two circles γ, having radius one and centered at the origin, and γ', having radius R greater than one. This configuration has linking number one when the circle γ' intersects the disc defined by γ. In the extreme case where the radius $R \to \infty$, the γ' circle may be deformed to a straight line perpendicular to the plane containing the circle γ completed with an half circle at infinity whose contribution to the Gauss integral vanishes.

The circle γ can be parameterized as

$$x_1 = \cos(s)\,,\; x_2 = \sin(s)\,,\; x_3 = 0 \tag{6.153}$$

and the straight line γ' as

$$y_1 = 0\,,\; y_2 = y \tag{6.154}$$

and intersection with the disc bounded by γ occurs when $|y| < 1$.

We obtain the linking number by integrating over the straight line

$$L(\gamma,\gamma') = \frac{1}{4\pi}\int_0^{2\pi} ds \int_{-\infty}^{+\infty} dy_3\, \frac{1 - y\sin(s)}{(1 - 2y\sin(s) + y^2 + y_3^2)^{\frac{3}{2}}} \tag{6.155}$$

The integral over y_3 is classical and, for $|y| \neq 1$, one has

$$L(\gamma,\gamma') = \frac{1}{2\pi}\int_0^{2\pi} ds\, \frac{1 - y\sin(s)}{(1 - 2y\sin(s) + y^2)} \tag{6.156}$$

The evaluation of this integral can be done by expanding the integrand in powers of the sine, using then the classical values of integral of even powers of the sine function. The result is then

$$L(\gamma,\gamma') = 1 \text{ for } |y| < 1\,,\; L(\gamma,\gamma') = 0 \text{ for } |y| > 1\,. \tag{6.157}$$

The unitary vector $\vec{e}$ reads

$$\vec{e} = \frac{\cos(s)\,\vec{i} + (\sin(s) - y)\vec{j} - y_3\vec{k}}{(1 - 2y\sin(s) + y^2 + y_3^2)^{\frac{1}{2}}} \tag{6.158}$$

and the necessary condition for a point to be on the boundary of the zodiacus is

$$1 - y\sin(s) = 0. \tag{6.159}$$

A moment thought shows that for $|y| < 1$, there is no boundary and the vector $\vec{e}$ sweeps the whole sphere once. On the contrary, for $|y| > 1$, the zodiacus has two boundaries at the values $s = \arcsin(y^{-1})$ and $s = \pi - \arcsin(y^{-1})$ that join at antipodal points for $y_3 = \pm\infty$.

Higher dimensional cases

As in the three dimensional case, the unitary vector e_{xy} spans on the sphere S^{4l+2} the zodiacus associated with the two surfaces γ_{2l+1} and γ'_{2l+1}. The eventual boundaries of the zodiacus necessarily correspond to stationary points of e_{xy} upon infinitesimal displacements δx (resp. δy) on the surface γ_{2l+1} (resp. γ'_{2l+1}), that is to say $\delta e_{xy} = 0$ where

$$\delta e_{xy} = \frac{\delta(x-y) - e_{xy}(e_{xy}.\delta(x-y))}{|x-y|} \tag{6.160}$$

If the surfaces γ_{2l+1} and γ'_{2l+1} are parameterized by (even local) coordinates s_i, t_j respectively $(i,j = 1 \dots 2l+1)$, then

$$\delta(x-y) = a_i \frac{\partial x}{\partial s_i} - b_j \frac{\partial y}{\partial t_j} \tag{6.161}$$

where a_i and b_j are two families of infinitesimal coefficients. As a consequence of the stationarity conditions, the vector e_{xy} is thus a linear combination of the $4l+2$ tangent vectors $\partial_{s_i} x$ and $\partial_{t_j} y$. Hence the oriented solid angle formed by two simultaneous displacements on both curves vanishes at the boundary of the zodiacus:

$$[e_{xy}; \partial_i x; \partial_j y] = 0. \tag{6.162}$$

We shall now check the normalisation of the linking number considering a simple choice of linked surfaces. We choose a $(2l+1)$-sphere centered at the origin and an orthogonal $(2l+1)$-hyperplane containing the origin. They are given respectively by

$$\gamma_{2l+1} : x_1^2 + \dots + x_{2l+2}^2 = 1, \;\; x_{2l+3} = \dots = x_{4l+3} = 0 \tag{6.163}$$

and a $(2l+1)$-hyperplane

$$\gamma'_{2l+1} : y_1 = \dots = y_{2l+2} = 0 \tag{6.164}$$

with its completion (an half-sphere) at infinity whose contribution to the Gauss integral vanishes. The ball defined by the sphere γ_{2l+1} and the hyperplane γ'_{2l+1} intersect at the origin so we have a configuration with linking number equal to one and a moment thought shows that the zodiacus is the whole $(4l+2)$-sphere.

The linking number (6.148) here reads

$$L(\gamma_{2l+1}, \gamma'_{2l+1}) = \frac{1}{S_{4l+2}} \oint_{\gamma_{2l+1}} d^{2l+1}x \oint_{\gamma'_{2l+1}} d^{2l+1}y \frac{1}{(1+|\vec{y}|^2)^{\frac{4l+3}{2}}}. \tag{6.165}$$

The first integral yields the surface of the $(2l+1)$-sphere

$$\oint_{\gamma_{2l+1}} d^{2l+1}x = S_{2l+1}\,, \tag{6.166}$$

while the second integral can be decomposed in a surfacic and a radial ones as

$$\oint_{\gamma'_{2l+1}} d^{2l+1}y \, \frac{1}{(1+\vec{y}^2)^{\frac{4l+3}{2}}} = S_{2l} \int_0^\infty dy \, \frac{y^{2l}}{(1+y^2)^{\frac{4l+3}{2}}} \tag{6.167}$$

The radial integral is a classic one and may be computed after the change of variable $y = \tan(\theta)$

$$\int_0^\infty dy \, \frac{y^{2l}}{(1+y^2)^{\frac{4l+3}{2}}} = \int_0^{\frac{\pi}{2}} d\theta \sin^{2l}(\theta) \cos^{2l+1}(\theta) = \frac{\Gamma(l+\frac{1}{2})\Gamma(l+1)}{2\Gamma(2l+\frac{3}{2})} \ . \tag{6.168}$$

We thus obtain

$$L(\gamma_{2l+1}, \gamma'_{2l+1}) = \frac{S_{2l} S_{2l+1}}{S_{4l+2}} \frac{\Gamma(l+\frac{1}{2})\Gamma(l+1)}{2\Gamma(2l+\frac{3}{2})} \tag{6.169}$$

what drastically simplifies into the expected result

$$L(\gamma_{2l+1}, \gamma'_{2l+1}) = +1 \ . \tag{6.170}$$

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