A Convexity Theorem for Twisted Loop Groups

Ein Konvexitätssatz für getwistete Schleifengruppen

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Einleitung

Der abstrakteste Kontext, in dem der Hauptsatz dieser Dissertation formuliert werden kann, vereinigt zwei Konzepte aus der Lie Theorie: das einer *Wurzelraumzerlegung* einer (komplexen) *zerfällbaren Lie-Algebra*, und das einer adjungierten Wirkung einer reellen Lie-Gruppe.
Sei G zunächst eine Lie-Gruppe mit der Eigenschaft, dass ihre Lie-Algebra $\mathfrak{g} := \mathbf{L}(G)$ eine maximale abelsche Unteralgebra $\mathfrak{h} \subset \mathfrak{g}$ enthält, deren Komplexifizierung $\mathfrak{h}^{\mathbb{C}} := \mathfrak{h} \oplus i\mathfrak{h}$ in $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$ *zerfällend* ist. Im endlichdimensionalen Fall heißt das, wenn V' den topologischen Dualraum eines Vektorraums V bezeichnet, und

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})[h, x] = \alpha(h)x\} \ \text{ mit } \ \alpha \in (\mathfrak{h}^{\mathbb{C}})'$$

die *Wurzelräume* sind, und

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}' \backslash \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$$

das *Wurzelsystem* ist, dass es eine entsprechende *Wurzelraumzerlegung* gibt:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_\alpha.$$

In dem unendlichdimensionalen Szenario, an dem wir hauptsächlich interessiert sind, ist $\mathfrak{g}$ eine topologische Lie-Algebra, d.h. $\mathfrak{g}$ ist ein topologischer Vektorraum mit stetiger Lie Klammer. Wurzelräume sind hier genauso definiert, aber die Wurzelraumzerlegung ist topologisch zu verstehen in dem Sinn, dass

$$\mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}} := \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_\alpha$$

als Unteralgebra *dicht* in $\mathfrak{g}^{\mathbb{C}}$ liegt. Man beachte, dass diese Zerlegung eine natürliche Einbettung von $(\mathfrak{h}^{\mathbb{C}})'$ in $(\mathfrak{g}^{\mathbb{C}})'$ ermöglicht, als den Unterraum der Funktionale, die alle Wurzelräume auf 0 abbilden.
Zusätzlich zu den bisherigen Annahmen müssen wir fordern, dass $\mathfrak{g}_{\mathrm{alg}} := \mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}} \cap \mathfrak{g}$ eine *unitäre reelle Form* von $\mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$ ist; dieser Begriff ist eine direkte Verallgemeinerung des Begriffes einer *kompakten reellen Form* einer endlichdimensionalen komplexen halbeinfachen Lie-Algebra auf die Klasse der komplexen zerfällbaren quadratischen Lie-Algebren. Sie ist definiert als die reelle Unteralgebra von Fixpunkten einer antilinearen Involution $* : \mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}} \to \mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$, die mit der Wurzelraumzerlegung und einer invarianten symmetrischen Bilinearform im folgenden Sinn verträglich ist:

(i) $\alpha(x) \in \mathbb{R}$ für alle Wurzeln $\alpha \in \Delta$ und $x = x^* \in \mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$.

(ii) $(\mathfrak{g}^{\mathbb{C}}_\alpha)^* = \mathfrak{g}^{\mathbb{C}}_{-\alpha}$ für alle $\alpha \in \Delta$.

(iii) $\kappa(x^*, y^*) = \overline{\kappa(x, y)}$ für alle $x, y \in \mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$.

Dies impliziert insbesondere, dass alle Wurzeln in Δ auf $\mathfrak{h}$ rein imaginäre Werte annehmen.

Eine Wurzel wird als *integrierbar* bezeichnet, wenn Elemente $x_\alpha \in \mathfrak{g}_\alpha$ und $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ existieren, so dass $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$ und $\mathrm{ad}(x_{\pm\alpha})$ *lokal nilpotent* sind, d.h. für jedes $y \in \mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$ die Folge $(\mathrm{ad}(x_{\pm\alpha})^n y)_{n\in\mathbb{N}}$ nur endlich viele Glieder ungleich 0 hat.
Die integrierbaren Wurzeln $\Delta_i \subset \Delta \subset i\mathfrak{h}$ sind bijektiv mit *Kowurzeln* assoziiert. Dabei handelt es sich um bestimmte Elemente $\check{\alpha} \in i\mathfrak{h}$ mit $\alpha(\check{\alpha}) = 2$, so dass die linearen Abbildungen

$$\sigma_\alpha : i\mathfrak{h}' \to i\mathfrak{h}', \quad \sigma_\alpha(\lambda) := \lambda - \lambda(\check{\alpha})\alpha$$

Spiegelungen auf dem topologischen linearen Dualraum $i\mathfrak{h}'$ von $i\mathfrak{h}$ sind. Die Menge dieser Spiegelungen erzeugt die *Weyl-Gruppe* $\mathcal{W} := \mathcal{W}(\Delta)$, die mit der Wurzelraumzerlegung von $\mathfrak{g}^{\mathbb{C}}$ bezüglich $\mathfrak{h}^{\mathbb{C}}$ assoziiert ist.
Um auf die Lie-Gruppe G und ihrer adjungierten Wirkung $\mathrm{Ad} : G \rightharpoonup \mathfrak{g}$ zurückzukommen, betrachten wir die entsprechende *koadjungierte Wirkung*, welche für alle $x \in \mathfrak{g}$ durch

$$\mathrm{Ad}^* : G \rightharpoonup \mathfrak{g}', \quad \mathrm{Ad}^*(g)(\lambda)(x) := \lambda(\mathrm{Ad}(g^{-1})(x))$$

definiert ist, und komplex linear auf $(\mathfrak{g}^{\mathbb{C}})' \simeq \mathfrak{g}' \oplus i\mathfrak{g}'$ erweiterbar ist. Dieser Dualraum ist mit der schwach-$*$ Topologie ausgestattet, welche als die Initialtopologie bezüglich der Evaluationsmorphismen $\mathrm{ev}_x : (\mathfrak{g}^{\mathbb{C}})' \to \mathbb{C}, \mathrm{ev}_x(\lambda) := \lambda(x)$ für $x \in \mathfrak{g}^{\mathbb{C}}$ definiert ist.
Weiterhin schreiben wir konvexe Abschlüsse (d.h. Abschlüsse konvexer Hüllen) von Teilmengen $S \subseteq \mathfrak{g}'$ bezüglich der schwach-$*$ Topologie als $\overline{\mathrm{conv}}(S)$ und die koadjungierte Bahn von $\xi \in (\mathfrak{g}^{\mathbb{C}})'$ als $\mathcal{O}_\xi$ und setzen

$$\mathcal{O}_\xi|_{i\mathfrak{h}} := \{\gamma \in i\mathfrak{h}' : (\exists \chi \in \mathcal{O}_\lambda)\chi|_{i\mathfrak{h}} = \xi\}.$$

Doppelerweiterungen von *Hilbert-Schleifenalgebren* bilden eine Klasse von Lie-Algebren, die alle obengenannten Bedingungen erfüllen. Das **Hauptziel** dieser Dissertation ist, diese Lie-Algebren und damit assoziierten Lie-Gruppen ausreichend detailliert zu beschreiben, um zeigen zu können, dass

$$\mathcal{O}_\lambda|_{i\mathfrak{h}} \subseteq \overline{\mathrm{conv}}(\mathcal{W}.\lambda) \tag{0.1}$$

für „die meisten" Funktionale $\lambda \in i\mathfrak{h}' \subseteq i\mathfrak{g}'$.
Konkret ist eine Doppelerweiterung $(\mathbb{K}\mathbf{c} \oplus_\omega \mathfrak{a}) \rtimes \mathbb{K}\mathbf{d}$ einer Lie-Algebra $\mathfrak{a}$ über einem Körper $\mathbb{K}$ (normalerweise $\mathbb{R}$ or $\mathbb{C}$) eine direkte Summe von Vektorräumen $\mathbb{K}\mathbf{c} \oplus \mathfrak{a} \oplus \mathbb{K}\mathbf{d}$ mit der Klammer

$$[s_x\mathbf{c} + x_0 + t_x\mathbf{d}, s_y\mathbf{c} + y_0 + t_y\mathbf{d}] := \omega(x_0, y_0)\mathbf{c} + [x_0, y_0] + t_x\mathbf{d}(y_0) - t_y\mathbf{d}(x_0),$$

wobei $\mathbf{d} : \mathfrak{a} \to \mathfrak{a}$ eine derivation, $\mathbf{c}$ ein zentrales Element, und ω ein 2-Kozykel ist, d.h. eine alternierende bilineare Abbildung $\mathfrak{a} \times \mathfrak{a} \to \mathbb{K}$, welche die Bedingung

$$\omega(x, [y, z]) + \omega(z, [x, y]) + \omega(y, [z, x]) = 0 \ \text{ für alle } \ x, y, z \in \mathfrak{g} \tag{0.2}$$

erfüllt. Mit dieser Definition kann die Einschränkung der der Gewichte λ in (0.1) zu „$\lambda(i\mathbf{c}) \neq 0$" konkretisiert werden.

Im ersten Kapitel werden Doppelerweiterungen von Lie-Algebren und Lie-Gruppen eingeführt, zunächst für allgemeine Lie-Algebren, dann für die wichtige Unterklasse derjenigen topologischen Lie-Algebren, die ein *invariantes* inneres Produkt tragen, d.h. eine invariante, symmetrische, positiv definite Bilinearform. Im Kontext von Lie Algebren bedeutet „Invarianz" einer symmetrischen Bilinear- (oder Hermite-) form auf $\mathfrak{a}$, dass die Operatoren

$$\mathrm{ad}(x) : \mathfrak{a} \to \mathfrak{a}, \mathrm{ad}(x)(y) := [x, y]$$

für alle $x \in \mathfrak{a}$ bezüglich dieser Form schiefadjungiert sind.
Im Abschnitt 1.1 wird die Definition von Doppelerweiterungen vorbereitet, indem *zentrale Erweiterungen* und *semidirekte Produkte* von Lie-Algebren und Lie-Gruppen definiert werden. Grundlegende Eigenschaften von beiden werden hergeleitet, zusammen mit Beziehungen zwischen den jeweiligen Erweiterungen der Lie-Gruppen und ihrer Lie-Algebren. Doppelerweiterungen werden dann mittels einer Kompatibilitätsbedingung definiert, die notwendig und hinreichend dafür ist, dass eine zentrale Erweiterung und ein semidirektes Produkt eine Doppelerweiterung ergeben. Grundlegende Eigenschaften dieser Doppelerweiterungen werden dann auf die entsprechenden Eigenschaften der zugrundeliegenden Erweiterungen zurückgeführt. Ein besonders wichtiger Aspekt, der über einen großen Teil dieser Dissertation regelmäßig verwendet wird, ist die Definition einer bestimmten Wirkung: wenn K eine 1-zusammenhängende Lie-Gruppe ist, und $\mathfrak{g}$ eine Doppelerweiterung von $\mathfrak{k} := \mathbf{L}(K)$, dann gibt es eine adjungierte Wirkung von K auf $\mathfrak{g}$, unabhängig davon, ob eine Lie-Gruppe G mit $\mathfrak{g} = \mathbf{L}(G)$ existiert.
Abschnitt 1.2 untersucht Doppelerweiterungen, die durch eine Kombination aus einem invarianten inneren Produkt auf einer Lie-Algebra und einer schiefadjungierten äußeren Derivation entstehen. Diese Doppelerweiterungen sind mit einer invarianten Lorentz-Form

$$\begin{aligned}&\kappa : ((\mathbb{R}\mathbf{c} \oplus_\omega \mathfrak{a}) \rtimes \mathbb{R}\mathbf{d}) \times ((\mathbb{R}\mathbf{c} \oplus_\omega \mathfrak{a}) \rtimes \mathbb{R}\mathbf{d}) \to \mathbb{R},\\ &\kappa((c_1, x_1, t_1), (c_2, x_2, t_2)) := (x_1, x_2) - c_1 t_2 - c_2 t_1\end{aligned}$$

ausgestattet. Zusammen mit ihren assoziierten invarianten Lorentzschen Formen bilden diese doppelerweiterten Lie-Algebren die Klasse der *Lorentzschen Doppelerweiterungen*, und alle doppelerweiterten Lie-Algebren, die in dieser Dissertation behandelt werden, gehören zu dieser Klasse.
Auf jeder Lorentzschen Doppelerweiterung kann eine Familie von invarianten Lorentz-Formen konstruiert werden. Diese Familie wird von den reellen Zahlen parametrisiert und hat die Eigenschaft, dass jedes Element der doppelerweiterten Lie-Algebra, bis auf eine bestimmte Hyperebene, in einem offenen Lorentzkegel enthalten ist, der von einer invarianten Lorentz-Form definiert wird. Dies gibt einen ersten Einblick in die invariante konvexe Geometrie, welche die adjungierte Wirkung auf einer Lorentzschen Doppelerweiterung erzeugt.

Das zweite Kapitel beschreibt die adjungierte Wirkung von Hilbert-Schleifengruppen auf Doppelerweiterungen ihrer korrespondierenden Hilbert-Schleifenalgebren im Detail.

Abschnitt 2.1 gibt eine detaillierte Konstruktion dieser Objekte an. Zur Vorbereitung von Schleifengruppen und Schleifenalgebren, werden Hilbert–Lie-Algebren und Hilbert–Lie-Gruppen kurz vorgestellt, ebenso die *kompakt-offene C^k-Topologie* auf $C^k(M, N)$ für $k \in \mathbb{N}_0 \cup \{\infty\}$ und C^k-Mannigfaltigkeiten M und N. Informationen aus [GN20] werden verwendet, um zu zeigen, dass für jede Lie-Gruppe H das punktweise Produkt und die kompakt-offene C^k-Topologie $\mathcal{L}H := C^\infty(\mathbb{S}^1, H)$ zu einer Lie-Gruppe mit Lie-Algebra $\mathcal{L}\mathfrak{h} := C^\infty(\mathbb{S}^1, \mathfrak{h})$ machen, wobei $\mathfrak{h} := \mathbf{L}(H)$ und die Lie Klammer die punktweise Klammer und die Topologie die kompakt-offene C^∞-Topologie ist.
Es gibt auch eine *getwistete* Version von Schleifengruppen, die man mit einem Automorphismus von H von endlicher Ordnung N und $r := \frac{2\pi}{N}$ durch die Definition

$$C^\infty_{\Phi,r}(\mathbb{R}, H) := \{f \in C^\infty(\mathbb{R}, H) : (\forall t \in \mathbb{R})\ f(t + r) = \Phi(f(t))\}$$

erhält. Die natürliche Injektion von 2π-periodischen Funktionen $\mathbb{R} \to H$ in die Menge der Funktionen $\mathbb{S}^1 \to H$ macht $\mathcal{L}_\Phi H := C^\infty_{\Phi,r}(\mathbb{R}, H)$ zu einer Untergruppe von $\mathcal{L}H$. Eine Lie-Gruppen Topologie auf $\mathcal{L}H$ erhält man dann folgendermaßen:
Es gibt einige Methoden aus dem Kontext der endlichdimensionalen Lie-Theorie, die nicht allgemein auf das unendlichdimensionale Szenario übertragbar sind. Im Fall von *lokal-exponentiellen* Lie-Gruppen, siehe [GN20], kann ein großer Teil davon wiederhergestellt werden. Hilbert–Lie-Gruppen sind lokal-exponentiell, und $\mathcal{L}H$ ist lokal exponentiell, wenn H lokal exponentiell ist. Dieser Begriff wird verwendet, um eine Lie-Gruppen Topologie auf $\mathcal{L}_\Phi H$ einzuführen, indem man sie als eine Fixpunkt-Untergruppe von $\mathcal{L}H$ unter einem bestimmten Automorphismus begreift.
In Abschnitt 2.2 werden Doppelerweiterungen von Hilbert-Schleifenalgebren konstruiert. Diese sind Lorentzsche Doppelerweiterungen, wobei das invariante innere Produkt durch Integration über den Einheitskreis aus dem invarianten inneren Produkt der zugrundeliegenden Hilbert–Lie-Algebra konstruiert wird, und die schiefadjungierte Derivation ist die Ableitung glatter Kurven.
Die Familie invarianter Lorentz-Formen wird verwendet, um folgende Formel für die adjungierte Wirkung der Identitätskomponente einer Hilbert-Schleifengruppe auf einer Doppelerweiterung ihrer Lie-Algebra herzuleiten:

Proposition 0.1. *Wenn K eine Hilbert–Lie-Gruppe ist, $\mathfrak{k} = \mathbf{L}(K)$ und*

$$\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi \mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$$

eine doppelerweiterte Schleifenalgebra, dann ist die adjungierte Wirkung von $\mathcal{L}_\Phi K$ mit $\varphi = \mathbf{L}(\Phi)$ gegeben durch

$$\mathrm{Ad}(g)(a, x_0, t) = \Big(a - (\delta^r(g), x_0) - \frac{t}{2}(\delta^r(g), \delta^r(g)), \mathrm{Ad}_{\mathfrak{k}}(g)(x_0) - t\delta^r(g), t\Big),$$

wobei $\delta^r(g) = g'g^{-1}$ die rechte logarithmische Ableitung der Kurve g ist.

Abschnitt 2.3 beschließt das zweite Kapitel mit einem Thema, das in engem Zusammenhang mit Konvexitätssätzen steht. Die grundlegende Idee ist die folgende: angenommen, für eine Lie-Algebra $\mathfrak{g}$ mit einer maximalen abelschen Unteralgebra $\mathfrak{h}$ so, dass $\mathfrak{g}^{\mathbb{C}}$ über $\mathfrak{h}^{\mathbb{C}}$ zerfällt, gibt es einen Konvexitätssatz der Form $p_{\mathfrak{h}}(\mathcal{O}_x) \subseteq \overline{\mathrm{conv}}(\mathcal{W}.x)$ für alle $x \in \mathfrak{h}$, wobei $\mathcal{O}_x$ die Bahn von x unter der adjungierten Wirkung einer passenden Lie-Gruppe bezeichnet, und $p_{\mathfrak{h}}$ die lineare Projektion auf $\mathfrak{h}$ entlang der Wurzelraumzerlegung von $\mathfrak{g}^{\mathbb{C}}$ bezüglich $\mathfrak{h}^{\mathbb{C}}$. Dann kann dieser Satz auch auf alle Elemente $y \in \mathfrak{g}$, die $\mathcal{O}_y \cap \mathfrak{h} \neq \emptyset$ erfüllen, angewendet werden in dem Sinn, dass $p_{\mathfrak{h}}(\mathcal{O}_y) \subseteq \overline{\mathrm{conv}}(\mathcal{W}.z)$ für alle $z \in \mathcal{O}_y \cap \mathfrak{h}$. Um Elemente $y \in \mathfrak{g}$ mit dieser Eigenschaft zu finden, wenn $\mathfrak{g}$ eine doppelerweiterte Hilbert-Schleifenalgebra ist, wird eine Klassifikation von unendlichdimensionalen einfachen Hilbert–Lie-Algebren verwendet. Nach dem Satz von Schue, siehe [Sc60] und [Sc61], sind diese alle realisierbar als die Lie-Algebren der schiefadjungierten Hilbert–Schmidt Operatoren auf einem Hilbertraum über $\mathbb{R}, \mathbb{C}$ oder $\mathbb{H}$; entsprechend sind alle unendlichdimensionalen Hilbert–Lie-Gruppen realisiert als Schnitte der isometrischen Automorphismen von $\mathcal{H}$ und den Hilbert–Schmidt Störungen der Identität auf $\mathcal{H}$, d.h.

Proposition 0.2. *Wenn K eine einfache Hilbert–Lie-Gruppe ist, dann existiert ein Hilbertraum $\mathcal{H}$ über $\mathbb{R}, \mathbb{C}$ oder $\mathbb{H}$, so dass*

$$K \simeq U_2(\mathcal{H}) := U(\mathcal{H}) \cap (\mathrm{id}_{\mathcal{H}} + \mathfrak{gl}_2(\mathcal{H})),$$

wobei $\mathfrak{gl}_2(\mathcal{H})$ für die Menge der Hilbert–Schmidt-Operatoren auf $\mathcal{H}$ steht.

Damit werden wohlbekannte Spektralsätze anwendbar, wodurch folgender Satz abgeleitet werden kann:

Proposition 0.3. *Wenn $K \simeq U_2(\mathcal{H})$ eine einfache Hilbert–Lie-Gruppe ist, $\mathcal{H}$ ein komplexer oder quaternionischer Hilbertraum, $x \in \mathfrak{k} = \mathbf{L}(K)$, die Unteralgebra $\mathfrak{h} \subset \mathfrak{k}$ maximal abelsch, und $\mathcal{O}_x$ die Bahn von x unter $\mathrm{Aut}(\mathfrak{k})_0$ bezeichnet, dann ist $\mathcal{O}_x \cap \mathfrak{h} \neq \emptyset$.*
Wenn statt dessen $\mathcal{H}$ ein reeller Hilbertraum ist, bilden die maximal abelschen Unteralgebren von $\mathfrak{k}$ unter $\mathrm{Aut}(\mathfrak{k})_0$ zwei Konjugationsklassen.

Analoge Aussagen sind wahr wenn $\mathfrak{g}$ die Lorentzsche Doppelerweiterung der Schleifenalgebra $\mathcal{L}_\varphi\mathfrak{k}$ mit $\varphi = \mathbf{L}(\Phi) \in \mathrm{Aut}(\mathfrak{k})$ ist, und eine der beiden folgenden Bedingungen erfüllt ist:

- Entweder ist K endlichdimensional, und folglich kompakt, dann ist $\mathcal{L}_\Phi K$ die passende Gruppe, die durch Adjunktion auf $\mathfrak{g}$ wirkt, und das entscheidende Konjugationstheorem kann von einem Konjugationstheorem für nicht-zusammenhängende kompakte Lie-Gruppen aus [Se68] abgeleitet werden.
- Oder K ist eine einfache, unendlichdimensionale Hilbert–Lie-Gruppe, und $\Phi = \mathrm{id}_K$. In diesem Fall ist die passende Lie-Gruppe $\mathcal{L}K \rtimes \overline{K}$, wobei $\overline{K} = \mathrm{Aut}(K)_0$.

In beiden Fällen verwendet der Beweis die Äquivarianzeigenschaft einer *Holonomieabblidung* Hol : $\mathcal{L}_\varphi\mathfrak{k} \to K$, um ein Konjugationstheorem von K auf die Schleifenalgebra zurückzuziehen.

Kapitel 3 konzentriert sich auf die rechte Seite der Inklusion (0.1). Es beginnt mit der formalen Einführung der *Wurzelraumzerlegung* durch eine *zerfällende Cartan-Unteralgebra* und dem wichtigen Konzept von *integrierbaren Wurzeln.* In *zerfällbaren quadratischen* Lie-Algebren werden diese Begriffe mit invarianten, symmetrischen, nicht-ausgearteten Bilinearformen zusammengebracht; die wichtigsten Eigenschaften werden zur Referenz aufgeführt. *Lokal endliche Wurzelsysteme* werden definiert, und Ergebnisse von John R. Schue und Nina Stumme werden zitiert, welche diese als die Wurzelsysteme von Komplexifizierungen von Hilbert–Lie-Algebren identifizieren. Das heißt, die Komplexifizierung jeder einfachen Hilbert–Lie-Algebra hat eine Wurzelraumzerlegung bezüglich eines lokal endlichen Wurzelsystems, und jede komplexe zerfällbare Lie-Algebra mit einem lokal endlichen Wurzelsystem kann topologisch zu einer Hilbert–Lie-Algebra vervollständigt werden; dieser Kontext umfasst auch eine Klassifikation dieser Wurzelsysteme in Begriffen von realisierungen von Hilbert–Lie Algebren als Algebren von schiefadjungierten Hilbert–Schmidt-Operatoren.
Diese Identifikation liefert die Notation, die benötigt wird, um eine Bedingung anzugeben, unter der Doppelerweiterungen von (getwisteten) Schleifenalgebren zu glatten Doppelerweiterungen von korrespondierenden Schleifengruppen integriert werden können.
Abschnitt 3.2 beschreibt die Wurzelraumzerlegung einer gegeben doppelerweiterten Schleifenalgebra im Detail. Dies gipfelt in der Folgerung, dass das Wurzelsystem einer Komplexifizierung einer doppelerweiterten Schleifenalgebra ein *lokal affines Wurzelsystem* ist, was impliziert, dass $\mathfrak{g}^{\mathbb{C}}$ eine dichte *lokal affine Lie-Algebra* $\mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$ enthält. Dies ermöglicht es, eine dichte Unteralgebra $\mathfrak{g}_{\mathrm{fin}} \subset \mathfrak{g}$ mit einer Familie $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ von doppelerweiterten Schleifenalgebren über *kompakten* Lie-Algebren auszuschöpfen. Diese Familie hat die Eigenschaft, dass jedes Glied $\mathfrak{g}_n$ eine zerfällende Cartan-Unteralgebra $\mathfrak{h}_n$ hat, so dass die gerichtete Vereinigung dieser $\mathfrak{h}_n$ eine Cartan-Unteralgebra von $\mathfrak{g}_{\mathrm{fin}}$ ausschöpft.
Am Ende dieses Kapitels wird die Weyl-Gruppe zu einer Wurzelraumzerlegung einer lokal affinen Lie-Algebra und einer doppelerweiterten Schleifenalgebra definiert.

Im ersten Abschnitt von Kapitel 4 wird die Geometrie konvexer Abschlüsse von Bahnen von Weyl-Gruppen im Kontext einer doppelerweiterten Schleifenalgebra über einer einfachen kompakten, d.h. endlichdimensionalen Hilbert–Lie Algebra untersucht. Dies wird vorbereitet, indem gezeigt wird, dass die Cartan-Unteralgebra mit ihrer Weyl Gruppe als *lineares Coxeter System* aufgefasst werden kann. Diese linearen Coxeter Systeme werden axiomatisch eingeführt. *Coxeter-Gruppen*, die *Fundamentalkammer*, der *Tits-Kegel*, *Wurzeln* und *Kowurzeln* werden definiert, letztere a priori unabhängig vom homonymen Begriff aus dem Kontext von Wurzelraumzerlegungen. Im nächsten Schritt werden *einfache Systeme* in den Wurzelsystemen von Doppelerweiterungen von getwisteten Wurzelalgebren identifiziert. Diese ermöglichen es, den Begriff einer Coxeter-Gruppe auf diesen Kontext zu übertragen und zu zeigen, dass diese Wurzelraumzerlegungen zu linearen Coxeter Systemen führen, wobei die Coxeter-Gruppe mit der entsprechenden Weyl-Gruppe identisch ist und eine Fundamentalkammer durch den Dualkegel eines einfachen Systems von Wur-

zeln gegeben ist.
Das wichtigste Resultat, das auf diese Weise auf Wurzelraumzerlegungen und Weyl-Gruppen angewendet werden kann, ist [HN14, Theorem 2.7], welches impliziert, dass für jedes v aus der Fundamentalkammer eines linearen Coxeter Systems gilt:

$$\operatorname{conv}(\mathcal{W}.v) \subseteq \bigcap_{\sigma \in \mathcal{W}} \sigma(v - \check{C}_S), \tag{0.3}$$

wobei $\check{C}_S$ ein abgeschlossener, spitzer Kegel ist, der von v unabhängig ist. Dies kann auf alle inneren Elemente des Tits Kegels angewendet werden, denn die Coxeter-Gruppe konjugiert jedes solche Element in die Fundamentalkammer.
Der größere Teil dieses Abschnitts beschäftigt sich in der Folge damit, die Inklusion (0.3) zu einer Gleichheit zu verfeinern, was in einem Theorem erreicht wird, welches sicherstellt, dass der konvexe Abschluss von $\mathcal{W}.v$ mit $\bigcap_{\sigma \in \mathcal{W}} \sigma(v - \check{C}_S)$ identisch ist. Dies kann wieder auf alle inneren Punkte des Tits Kegels angewendet werden.

Im letzten Abschnitt werden die Themen aus den vorangegangen Abschnitten zusammengetragen, um folgenden Konvexitätssatz zu beweisen:

Theorem. *Wenn $\mathfrak{g}$ eine doppelerweiterte Schleifenalgebra über einer einfachen Hilbert–Lie-Algebra $\mathfrak{k}$ ist, und $\lambda \in i\mathfrak{t}_{\mathfrak{g}}'$ ein Gewicht mit $\lambda(\mathbf{c}) \neq 0$, dann ist*

$$\mathcal{O}_\lambda|_{i\mathfrak{t}_{\mathfrak{g}}} \subseteq \overline{\operatorname{co}}(\lambda).$$

Der grundlegende Ansatz zu diesem Beweis ist, einen Konvexitätssatz für eine Teilmenge der imaginären Gewichte auf einer Cartan-Unteralgebra $\mathfrak{t}_{\mathfrak{g}}$ einer doppelerweiterten Hilbert-Schleifenalgebra zu zeigen, und diesen dann auf immer größere Teilmengen auszuweiten, bis er auf alle imaginären Gewichte verallgemeinert ist, die auf dem Zentrum nicht verschwinden.
Die erste Version dieses Konvexitätssatzes gilt für das Gitter der *ganzzahligen Gewichte*, also den Gewichten, die ganzzahlige Werte auf den Kowurzeln annehmen (und auf dem Zentrum nicht verschwinden); es ist abgeleitet von der unitären Darstellungstheorie von lokal affinen Lie-Algebren, die in [Ne10] entwickelt worden ist. Um dies vorzubereiten, werden die relevanten Begriffe von *unitären* und *integrierbaren* Darstellungen von zerfällbaren Lie-Algebren mit unitären reellen Formen definiert. Eine unitäre, integrierbare Darstellung $\rho_\lambda : \mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}} \rightharpoonup L_{\mathrm{alg}}(\lambda)$ von *höchstem Gewicht* λ wird definiert, die insbesondere die Eigenschaft hat, dass die Wirkung von $\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}} \subset \mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$ auf dem komplexen prä-Hilbertraum $L_{\mathrm{alg}}(\lambda)$ durch das Gewichtssystem

$$\mathcal{P}_\lambda = \operatorname{conv}(\mathcal{W}.\lambda) \cap (\lambda + \operatorname{span}_{\mathbb{Z}}(\Delta)) \subset (\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})' \tag{0.4}$$

diagonalisiert wird.
Die Methode der *holomorphen Induktion*, nach der Formulierung in [JN18, Proposition 8.6], wird verwendet um ρ_λ zu einer unitären Darstellung einer Lie-Gruppe G mit $\mathfrak{g} = \mathbf{L}(G)$ auf der Hilbertraum-Vervollständigung $\mathcal{H}_\lambda$ von

$L_{\mathrm{alg}}(\lambda)$ zu integrieren. Anschließend, mit der Notation $\mathcal{H}_\lambda^\infty \subseteq \mathcal{H}_\lambda$ für den dichten Unterraum von Vektoren mit glatten Bahnen, wird die *Impulsabbildung*

$$\Phi_\pi : \mathbb{P}(\mathcal{H}_\lambda^\infty) \to \mathfrak{g}', \quad \Phi_\pi([v])(x) := -i\frac{\langle \rho(x)v, v\rangle}{\langle v, v\rangle} \text{ für } x \in \mathfrak{g}$$

verwendet, um die gewünschte Konvexitätseigenschaft von (0.4) herzuleiten.
Im nächsten Schritt wird dieser Konvexitätssatz für den Fall einer Schleifengruppe über einer kompakten Lie-Gruppe zu allen Gewichten $\lambda \in i\mathfrak{t}_\mathfrak{g}$ (die auf dem Zentrum nicht verschwinden) verallgemeinert. Der Beweis verallgemeinert die Konvexitätseigenschaft zunächst auf die dichte Untermenge der *rationalen* Gewichte und verwendet dann Ergebnisse aus dem 3. Kapitel über die konvexe Coxeter Geometrie auf $i\mathfrak{t}_\mathfrak{g}$, um ein Approximationsargument zu ermöglichen.
Die allgemeinste Version des Konvexitätstheorems setzt nur voraus, dass $\mathfrak{g}$ eine doppelerweiterte Schleifenalgebra über einer einfachen Hilbert–Lie-Algebra ist, und dass das Gewicht $\lambda \in i\mathfrak{t}_\mathfrak{g}'$ wiederum auf dem Zentrum nicht verschwindet. Der Beweis verwendet eine Ausschöpfung von $\mathfrak{g}$ durch doppelerweiterte Schleifenalgebren über kompakten Lie-Algebren, welche in Kapitel 3 vorbereitet worden ist. Dies erlaubt ein Approximationsargument aus dem kompakten Kontext.

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Introduction

The most abstract setting in which the main theorem of this thesis can be formulated brings together two concepts of Lie theory: that of a *root space decomposition* of a (complex) *split Lie algebra*, and that of the adjoint action of a real Lie group.
To start with, let G be a Lie group with the property that its Lie algebra $\mathfrak{g} := \mathbf{L}(G)$ contains a maximal abelian subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that its complexification $\mathfrak{h}^{\mathbb{C}} := \mathfrak{h} \oplus i\mathfrak{h}$ is *splitting* in $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \oplus i\mathfrak{g}$. In the finite-dimensional case this means that, if V' denotes the topological dual space of a topological vector space V, and

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})[h, x] = \alpha(h)x\} \quad \text{with} \quad \alpha \in (\mathfrak{h}^{\mathbb{C}})'$$

are *root spaces* of, and

$$\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}'\backslash\{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$$

is the *root system*, then we have a *root space decomposition*:

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_\alpha.$$

In the infinite-dimensional scenario we are mostly interested in, $\mathfrak{g}$ is a topological Lie algebra, i.e. carries a topology such that the Lie bracket is continuous. Here, root spaces and the root system are defined the same way, but the root space decomposition is to be understood in a topological way in the sense that

$$\mathfrak{g}^{\mathbb{C}}_{\text{alg}} := \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^{\mathbb{C}}_\alpha$$

is a *dense* subalgebra of $\mathfrak{g}^{\mathbb{C}}$. Note that this decomposition provides a natural way of injecting $(\mathfrak{h}^{\mathbb{C}})'$ into $(\mathfrak{g}^{\mathbb{C}})'$ as the subspace of functionals which send all root spaces to 0.
In addition to the preceding assumptions, one has to assume that $\mathfrak{g}_{\text{alg}} := \mathfrak{g}^{\mathbb{C}}_{\text{alg}} \cap \mathfrak{g}$ is a *unitary real form* of $\mathfrak{g}^{\mathbb{C}}_{\text{alg}}$; this notion is a straightforward generalisation of the notion of a *compact real form* of a finite-dimensional complex semisimple Lie algebra to the class of complex split quadratic Lie algebras. It is defined as the real fixed point subalgebra of an antilinear involution $* : \mathfrak{g}^{\mathbb{C}}_{\text{alg}} \to \mathfrak{g}^{\mathbb{C}}_{\text{alg}}$ which is compatible with the root space decomposition and an invariant symmetric bilinear form κ on $\mathfrak{g}^{\mathbb{C}}_{\text{alg}}$ in the following sense:

(i) $\alpha(x) \in \mathbb{R}$ for all roots $\alpha \in \Delta$ and $x = x^* \in \mathfrak{g}^{\mathbb{C}}_{\text{alg}}$.

(ii) $(\mathfrak{g}^{\mathbb{C}}_\alpha)^* = \mathfrak{g}^{\mathbb{C}}_{-\alpha}$ for all $\alpha \in \Delta$.

(iii) $\kappa(x^*, y^*) = \overline{\kappa(x, y)}$ for all $x, y \in \mathfrak{g}^{\mathbb{C}}_{\text{alg}}$.

It implies in particular that all roots in Δ take purely imaginary values on $\mathfrak{h}$.
A root α is called *integrable*, if there exist $x_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$, such that $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$ and $\text{ad}(x_{\pm\alpha})$ are *locally nilpotent*, i.e. for every $y \in \mathfrak{g}^{\mathbb{C}}_{\text{alg}}$ the

sequence $(\mathrm{ad}(x_{\pm\alpha})^n y)_{n\in\mathbb{N}}$ has only finitely many non-zero members.
The integrable roots $\Delta_i \subset \Delta \subset i\mathfrak{h}$ have *coroots* associated to them bijectively, which are particular elements $\check{\alpha} \in i\mathfrak{h}$ with $\alpha(\check{\alpha}) = 2$ so that the linear maps

$$\sigma_\alpha : i\mathfrak{h}' \to i\mathfrak{h}', \quad \sigma_\alpha(\lambda) := \lambda - \lambda(\check{\alpha})\alpha$$

are reflections on the topological linear dual space $i\mathfrak{h}'$ of $i\mathfrak{h}$. The set of these reflections generates the *Weyl group* $\mathcal{W} := \mathcal{W}(\Delta)$ associated to the root space decompositions of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$.
Returning to the Lie group G and its adjoint action $\mathrm{Ad} : G \rightharpoondown \mathfrak{g}$, we consider the corresponding *coadjoint action*, which is defined by

$$\mathrm{Ad}^* : G \rightharpoondown \mathfrak{g}', \quad \mathrm{Ad}^*(g)(\lambda)(x) := \lambda(\mathrm{Ad}(g^{-1})(x)) \text{ for all } x \in \mathfrak{g},$$

and extends complexly linear to $(\mathfrak{g}^{\mathbb{C}})' \simeq \mathfrak{g}' \oplus i\mathfrak{g}'$. This dual space carries the *weak-$*$ topology*, which is the initial topology with respect to the family of evaluation morphisms $\mathrm{ev}_x : (\mathfrak{g}^{\mathbb{C}})' \to \mathbb{C}, \mathrm{ev}_x(\lambda) := \lambda(x)$ for $x \in \mathfrak{g}^{\mathbb{C}}$.
We further write convex closures (i.e. closures of convex hulls) of subsets $S \subseteq \mathfrak{g}'$ with respect to the weak-$*$ topology as $\overline{\mathrm{conv}}(S)$ and the coadjoint orbit of any $\xi \in (\mathfrak{g}^{\mathbb{C}})'$ as $\mathcal{O}_\xi$, and set

$$\mathcal{O}_\xi|_{i\mathfrak{h}} := \{\gamma \in i\mathfrak{h}' : (\exists \chi \in \mathcal{O}_\lambda)\chi|_{i\mathfrak{h}} = \xi\}.$$

Double extensions of *Hilbert loop algebras* form a class of Lie algebras which satisfy all the conditions above. The *main goal* of this thesis is to describe these Lie algebras and their associated Lie groups in sufficient detail to show that

$$\mathcal{O}_\lambda|_{i\mathfrak{h}} \subseteq \overline{\mathrm{conv}}(\mathcal{W}.\lambda) \tag{0.1}$$

for "most" functionals $\lambda \in i\mathfrak{h}' \subseteq i\mathfrak{g}'$.
Concretely, a double extension $(\mathbb{K}\mathbf{c} \oplus_\omega \mathfrak{a}) \rtimes \mathbb{K}\mathbf{d}$ of a Lie algebra $\mathfrak{a}$ over a field $\mathbb{K}$ (usually $\mathbb{R}$ or $\mathbb{C}$) is the direct vector space sum $\mathbb{K}\mathbf{c} \oplus \mathfrak{a} \oplus \mathbb{K}\mathbf{d}$ endowed with the bracket

$$[s_x\mathbf{c} + x_0 + t_x\mathbf{d}, s_y\mathbf{c} + y_0 + t_y\mathbf{d}] := \omega(x_0, y_0)\mathbf{c} + [x_0, y_0] + t_x\mathbf{d}(y_0) - t_y\mathbf{d}(x_0),$$

where $\mathbf{d} : \mathfrak{a} \to \mathfrak{a}$ is a derivation, $\mathbf{c}$ is a central element, and ω is a 2-cocycle, i.e. an alternating bilinear map $\mathfrak{a} \times \mathfrak{a} \to \mathbb{K}$ satisfying

$$\omega(x, [y, z]) + \omega(z, [x, y]) + \omega(y, [z, x]) = 0 \text{ for all } x, y, z \in \mathfrak{g}. \tag{0.2}$$

With this definition, the constraint on the weight λ in (0.1) is expressed by the assumption that $\lambda(i\mathbf{c}) \neq 0$.

In the first chapter, double extensions of Lie algebras and Lie groups are introduced, first for Lie algebras in general, then for the important subclass of those topological real Lie algebras which carry an *invariant* inner product, i.e. an invariant, symmetric, positive definite bilinear form. In the context of Lie algebras, "invariance" of a symmetric bilinear (or hermitian) form means that, with respect to this form, a Lie algebra acts on itself via the adjoint action as

skew-adjoint operators.
In Subsection 1.1, the definition of double extensions is prepared by defining *central extensions* and *semidirect products* of both Lie algebras and Lie groups. Basic properties of both are derived, along with relations between the respective extensions of Lie groups and their Lie algebras. Double extensions are defined by means of a compatibility condition which is necessary and sufficient for a central extension and a semidirect product to give rise to a double extension. Basic properties of the resulting double extension are reduced to corresponding properties of the underlying extensions. A particular important aspect of this, which will frequently be applied throughout a major part of this thesis, is the definition of a certain adjoint action: if K is a 1-connected Lie group, and $\mathfrak{g}$ a double extension of $\mathfrak{k} := \mathbf{L}(K)$, then there exists an adjoint action of K on $\mathfrak{g}$, regardless of whether a Lie group G with $\mathfrak{g} = \mathbf{L}(G)$ exists.
Subsection 1.2 examines double extensions which arise from the combination of an invariant inner product on a Lie algebra and a skew-adjoint outer derivation. These doubly extended Lie algebras can be equipped with the invariant Lorentz form

$$\kappa : ((\mathbb{R}\mathbf{c} \oplus_\omega \mathfrak{a}) \rtimes \mathbb{R}\mathbf{d}) \times ((\mathbb{R}\mathbf{c} \oplus_\omega \mathfrak{a}) \rtimes \mathbb{R}\mathbf{d}) \to \mathbb{R},$$
$$\kappa((c_1, x_1, t_1), (c_2, x_2, t_2)) := (x_1, x_2) - c_1 t_2 - c_2 t_1. \tag{0.3}$$

These doubly extended Lie algebras, together with their associated invariant Lorentz forms, form the class of *Lorentzian double extensions*, and the double extensions of loop algebras which are studied in this thesis belong to this class. On every Lorentzian double extension, a family of invariant Lorentz forms can be constructed. This family is parametrised by the real numbers, and has the property that every point of the doubly extended Lie algebra, excluding one specific hyperplane, is contained in the open Lorentzian double-cone defined by an invariant Lorentz form. This gives a first insight into the invariant convex geometry which the adjoint action generates on a Lorentzian double extension.

The second chapter describes the adjoint action of Hilbert loop groups on double extensions of their corresponding Hilbert loop algebras in depth.
Subsection 2.1 provides a detailed construction of the objects in question. To prepare the definition of loop groups and loop algebras, Hilbert–Lie algebras and Hilbert–Lie groups are introduced shortly, as well as the *compact open C^k-topology* on $C^k(M, N)$ for $k \in \mathbb{N}_0 \cup \{\infty\}$ and C^k-manifolds M and N. Information from [GN20] is used to show that, for any Lie group H, the pointwise product and the compact open C^∞-topology turn $\mathcal{L}H := C^\infty(\mathbb{S}^1, H)$ into a Lie group with Lie algebra $\mathcal{L}\mathfrak{h} := C^\infty(\mathbb{S}^1, \mathfrak{h})$, where $\mathfrak{h} := \mathbf{L}(H)$, and the Lie bracket is the pointwise bracket and the topology is the compact open C^∞-topology.
There is also a *twisted* version of loop groups, which is obtained using an automorphism Φ of H of finite order N and defining

$$C^\infty_{\Phi,r}(\mathbb{R}, H) := \{f \in C^\infty(\mathbb{R}, H) : (\forall t \in \mathbb{R})\ f(t + r) = \Phi(f(t))\}$$

for $r := \frac{2\pi}{N}$. The natural injection of 2π-periodic maps $\mathbb{R} \to H$ into the set of maps $\mathbb{S}^1 \to H$ turns $\mathcal{L}_\Phi H := C^\infty_{\Phi,r}(\mathbb{R}, H)$ into a subgroup of $\mathcal{L}H$. A Lie group

topologiy on $\mathcal{L}_\Phi H$ is then obtained as follows:
There is a wide range of methods from finite-dimensional Lie theory, which are not generally available in the infinite-dimensional setting. A large part of these can be re-established in the case of *locally exponential* Lie groups, see [GN20]. Hilbert–Lie groups are locally exponential, and $\mathcal{L}H$ is locally exponential for locally exponential H. This notion is then applied to establish a Lie group topology on $\mathcal{L}_\Phi H$ by regarding it as a fixed point subgroup of $\mathcal{L}H$ under a certain automorphism.
In Subsection 2.2, double extensions of Hilbert loop algebras are constructed. These are Lorentzian double extensions, where the invariant inner product comes from the invariant inner product of the underlying Hilbert–Lie algebra by integration over the circle, and the skew-adjoint derivation is the derivative of smooth curves.
The family of invariant Lorentz forms is used to derive the following formula for the adjoint action of the identity component of a Hilbert loop group on a double extension of its Lie algebra:

Proposition 0.1. *If K is a Hilbert–Lie group, $\mathfrak{k} = \mathbf{L}(K)$, and*

$$\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi \mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$$

is a doubly extended loop algebra, then the adjoint action of $\mathcal{L}_\Phi K$ with $\varphi = \mathbf{L}(\Phi)$ is given by

$$\mathrm{Ad}(g)(a, x_0, t) = \Big(a - (\delta^r(g), x_0) - \frac{t}{2}(\delta^r(g), \delta^r(g)), \mathrm{Ad}_\mathfrak{k}(g)(x_0) - t\delta^r(g), t\Big),$$

where $\delta^r(g) = g'g^{-1}$ is the right-logarithmic derivative of the curve g.

Subsection 2.3 concludes the second chapter with a topic that is closely related to convexity theorems. The basic idea is the following: suppose that, for some Lie algebra $\mathfrak{g}$ with a maximal abelian subalgebra $\mathfrak{h}$ such that $\mathfrak{h}^\mathbb{C} \subset \mathfrak{g}^\mathbb{C}$ is splitting, there is a convexity theorem of the form $p_\mathfrak{h}(\mathcal{O}_x) \subseteq \overline{\mathrm{conv}}(\mathcal{W}.x)$ for all $x \in \mathfrak{h}$, where $\mathcal{O}_x$ denotes the orbit of x under the adjoint action of some appropriate Lie group, and $p_\mathfrak{h}$ denotes the linear projection onto $\mathfrak{h}$ along the root space decomposition of $\mathfrak{g}^\mathbb{C}$ with respect to $\mathfrak{h}^\mathbb{C}$. Then this theorem can also be applied to all elements $y \in \mathfrak{g}$ which satisfy $\mathcal{O}_y \cap \mathfrak{h} \neq \varnothing$ in the sense that $p_\mathfrak{h}(\mathcal{O}_y) \subseteq \overline{\mathrm{conv}}(\mathcal{W}.z)$ for any $z \in \mathcal{O}_y \cap \mathfrak{h}$. To find elements $y \in \mathfrak{g}$ with this property when $\mathfrak{g}$ is a double extension of a Hilbert loop algebra, a classification of the infinite-dimensional simple Hilbert–Lie algebras is employed. By Schue's Theorem, see [Sc60] and [Sc61], all of these appear as the Lie algebras of skew-adjoint *Hilbert–Schmidt* operators on some Hilbert space $\mathcal{H}$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$; accordingly, infinite-dimensional simple Hilbert–Lie groups are generally (isomorphic to) the intersections of the group of isometric automorphisms of $\mathcal{H}$ with the group of Hilbert–Schmidt perturbations of the identity on $\mathcal{H}$, i.e.:

Proposition 0.2. *If K is a simple Hilbert–Lie group, then there exists a Hilbert space $\mathcal{H}$ over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, so that*

$$K \simeq U_2(\mathcal{H}) := U(\mathcal{H}) \cap (\mathrm{id}_\mathcal{H} + \mathfrak{gl}_2(\mathcal{H})),$$

where $\mathfrak{gl}_2(\mathcal{H})$ *denotes the set of Hilbert–Schmidt operators on* $\mathcal{H}$*.*

This makes well-known spectral theorems applicable, from which the following proposition can be derived:

Proposition 0.3. *If* $K \simeq U_2(\mathcal{H})$ *is a simple Hilbert–Lie group,* $\mathcal{H}$ *is a complex or quaternionic Hilber space,* $x \in \mathfrak{k} = \mathbf{L}(K)$ *and* $\mathfrak{h} \subset \mathfrak{k}$ *is maximal abelian, and* $\mathcal{O}_x$ *denotes the orbit of* x *under* $\mathrm{Aut}(\mathfrak{k})_0$*, then* $\mathcal{O}_x \cap \mathfrak{h} \neq \emptyset$*.*
If, instead, $\mathcal{H}$ *is a real Hilbert space, the maximal abelian subalgebras in* $\mathfrak{k}$ *under* $\mathrm{Aut}(\mathfrak{k})_0$ *form two conjugacy classes.*

Analogous statements hold if $\mathfrak{g}$ is the Lorentzian double extension of the loop algebra $\mathcal{L}_\varphi\mathfrak{k}$ with $\varphi = \mathbf{L}(\Phi) \in \mathrm{Aut}(\mathfrak{k})$, and if one of the following two conditions is satisfied:

- Either K is finite-dimensional and therefore compact. Then the appropriate group acting on $\mathfrak{g}$ by the adjoint action is $\mathcal{L}_\Phi K$, and the relevant conjugacy theorem can be derived from a conjugacy theorem for non-connected compact Lie groups from [Se68].
- Or K is a simple, infinite-dimensional Hilbert–Lie group, and $\Phi = \mathrm{id}_K$. In this case, the appropriate Lie group is $\mathcal{L}K \rtimes \overline{K}$, where $\overline{K} = \mathrm{Aut}(K)_0$.

In both cases, the proof employs an equivariance property of a *holonomy map* $\mathrm{Hol} : \mathcal{L}_\varphi\mathfrak{k} \to K$ to pull back a conjugacy theorem from K to the loop algebra.

Chapter 3 focuses on the right hand side of the inclusion (0.1). It starts out by formally introducing a *root space decomposition* by a *splitting Cartan subalgebra* and the important concept of *integrable roots.* These notions are combined with invariant, symmetric, non-degenerate bilinear forms in the *split quadratic* Lie algebras, and important properties of these are listed for reference. *Locally finite root systems* are defined and results from John R. Schue and Nina Stumme are used to identify them as the root systems of (complexified) Hilbert–Lie algebras, which means that every complexification of a simple Hilbert–Lie algebra has a root space decomposition with respect to a locally finite root system, and every complex split Lie algebra with a locally finite root system can be topologically completed to a Hilbert–Lie algebra. This context also includes a classification of these root systems in terms of realisations of Hilbert–Lie algebras as algebras of skew-adjoint Hilbert–Schmidt operators.
This identification provides the notation needed to give a condition for a double extension of a (twisted) Hilbert loop algebra to integrate to a smooth double extension of a corresponding loop group.
Subsection 3.2 describes the root space decomposition of a given double extended loop algebra in detail. This culminates in the conclusion that the root system of a complexification of a double extended Hilbert loop algebra $\mathfrak{g}$ is a *locally affine root system*, which implies that $\mathfrak{g}^{\mathbb{C}}$ contains a dense *locally affine Lie algebra* $\mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$. This allows to exhaust a dense subalgebra $\mathfrak{g}_{\mathrm{fin}} \subset \mathfrak{g}$ with a family $(\mathfrak{g}_n)_{n\in\mathbb{N}}$ of double extended loop algebras over *compact* Lie algebras. This family has the property that every member $\mathfrak{g}_n$ has a splitting Cartan subalgebra

$\mathfrak{h}_n$ such that the directed union of these $\mathfrak{h}_n$ exhausts a given Cartan subalgebra of $\mathfrak{g}_{\text{fin}}$.
At the end of this chapter, the Weyl group corresponding to the root space decomposition of a locally affine Lie algebra and of a double extended loop algebra is defined.

In the first subsection of Chapter 4, the geometry of convex closures of the orbits of the Weyl group are studied in the context of a double extended loop algebra over a simple compact Lie algebra, i.e. the finite-dimensional case of a simple Hilbert–Lie algebra. This is prepared by showing that the Cartan subalgebra together with its Weyl group action can be seen as a *linear Coxeter system*. These linear Coxeter systems are defined by a list of axioms. *Coxeter groups*, the *fundamental chamber*, *Tits cone*, *roots* and *coroots* are defined, the latter a priory independently from the homonymous notion in the context of root space decompositions. *Simple systems* are identified in the root systems of double extensions of twisted loop algebras. These allow to transfer the notion of a Coxeter group to this context, and to show that these root space decompositions give rise to linear Coxeter systems, where the Coxeter group is identical with the respective Weyl group, and a fundamental chamber is given as the dual cone of any simple system of roots.
The most important result which can be applied to root space decompositions and Weyl groups in this way is [HN14, Theorem 2.7], which implies that, for any v in the fundamental chamber of a linear Coxeter system,

$$\operatorname{conv}(\mathcal{W}.v) \subseteq \bigcap_{\sigma\in\mathcal{W}} \sigma(v - \check{C}_S), \tag{0.4}$$

where $\check{C}_S$ is a closed pointed cone which is independent of v. This can be applied to all inner elements of a Tits cone, because the Coxeter group conjugates every such element to some element of the fundamental chamber.
The larger part of this subsection then deals with sharpening inclusion (0.4) to an equality, which is finally achieved in a Theorem which asserts that the convex closure of $\mathcal{W}.v$ equals $\bigcap_{\sigma\in\mathcal{W}} \sigma(v - \check{C}_S)$. Again, this can be applied to all inner points of the Tits cone.

In the final subsection, the topics prepared in the previous sections are brought together to derive the following convexity theorem:

Theorem. *If $\mathfrak{g}$ is a doubly extended loop algebra over some simple Hilbert–Lie algebra $\mathfrak{k}$, and $\lambda \in i\mathfrak{t}'_{\mathfrak{g}}$ a weight with $\lambda(\mathbf{c}) \neq 0$, then*

$$\mathcal{O}_\lambda|_{i\mathfrak{t}_{\mathfrak{g}}} \subseteq \overline{\operatorname{co}}(\lambda).$$

The basic approach is to show a convexity theorem for a subset of the imaginary weights on the Cartan subalgebra $\mathfrak{t}_{\mathfrak{g}}$ of a double extended Hilbert loop algebra $\mathfrak{g}$ and then extend it to increasingly larger subsets, until it is generalised to all imaginary weights on $\mathfrak{t}_{\mathfrak{g}}$ not vanishing on the centre.
The first version of the convexity theorem holds for the grid of *integral weights*, i.e. those weights that take integral values on the coroots (and do not vanish

on the centre); it is derived from the unitary representation theory of locally affine Lie algebras developed in [Ne10]. To prepare this, the relevant notions of *unitary* and *integrable* representations of split Lie algebras with a unitary real form are defined. A unitary, integrable representation $\rho_\lambda : \mathfrak{g}^{\mathbb{C}} \rightharpoonup L_{\mathrm{alg}}(\lambda)$ of *highest weight* λ is defined, which has the property that the action of $\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}$ on the complex Hilbert space $L_{\mathrm{alg}}(\lambda)$ is diagonalised by the weight system

$$\mathscr{P}_\lambda = \mathrm{conv}(\mathscr{W}.\lambda) \cap (\lambda + \mathrm{span}_{\mathbb{Z}}(\Delta)) \subset (\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})'. \tag{0.5}$$

The method of *holomorphic induction*, in the formulation from [JN18, Proposition 8.6], is used to integrate ρ_λ to a unitary representation of a Lie group G with $\mathfrak{g} = \mathbf{L}(G)$ on the Hilbert space completion $\mathscr{H}_\lambda$ of $L_{\mathrm{alg}}(\lambda)$. Then, with $\mathscr{H}_\lambda^\infty \subseteq \mathscr{H}_\lambda$ denoting the dense subspace of vectors with smooth orbit map, the *momentum map*

$$\Phi_\pi : \mathbb{P}(\mathscr{H}_\lambda^\infty) \to \mathfrak{g}', \quad \Phi_\pi([v])(x) := -i\frac{\langle \rho(x)v, v\rangle}{\langle v, v\rangle} \text{ for } x \in \mathfrak{g};$$

is used to derive the desired convexity property from (0.5).
In the next step, this convexity theorem is generalised to all weights $\lambda \in i\mathfrak{t}_{\mathfrak{g}}$ (not vanishing on the centre) in the case of a loop group over a compact Lie group. The proof first generalises the convexity property to the dense subset of *rational* weights, then employs the results from Chapter 3 about the convex Coxeter geometry on $i\mathfrak{t}_{\mathfrak{g}}$ to allow an approximation argument.
The most general version of the convexity theorem only requires $\mathfrak{g}$ to be a double extended loop algebra over a simple Hilbert–Lie algebra, and, again, the weight $\lambda \in i\mathfrak{t}_{\mathfrak{g}}'$ to not vanish on the centre. The proof employs an exhaustion of $\mathfrak{g}$ by double extended loop algebras over compact Lie algebras, which has been prepared in Chapter 3. This allows an approximation argument from the compact context.

Preface

The interest in invariant convex sets in the topological dual of Lie algebras stems from the unitary representation theory of Lie groups, and this correlation will be outlined in the following.
Right now, we cannot hope to develop a satisfying general theory of unitary representations of infinite-dimensional Lie groups, but the situation becomes more manageable if we focus on representations that satisfy appropriate regularity conditions. In this regard, *semiboundedness* is particularly promising. It is closely related to the physical concept of a "Hamiltonian", which is, abstractly spoken, an essentially self-adjoint operator whose spectrum is bounded from below. Every semibounded unitary representation of a Lie group G with Lie algebra $\mathfrak{g} := \mathbf{L}(G)$ contains such operators in $i\mathbf{d}\pi(\mathfrak{g})$, and in many cases the converse statement is true, i.e., under certain assumptions on the group in question, the existence of a "Hamiltonian" implies the semiboundedness of a given representation. This is in particular the case for representations of finite-dimensional Lie groups, groups with Kac–Moody algebras and Hilbert loop groups (see [Ne14]).
For a unitary Lie group action $\pi : G \rightharpoonup \mathcal{H}$ and $v \in \mathcal{H}$, let $\pi^v : G \to \mathcal{H}$ denote the orbit map; then π is said to be *smooth* if the subspace of *smooth vectors* $\mathcal{H}^\infty := \{v \in \mathcal{H} : \pi^v \in C^\infty(G, \mathcal{H})\}$ is dense in $\mathcal{H}$. A smooth unitary representation is called *semibounded* if there exists an open subset of the topological Lie algebra $\mathfrak{g} := \mathbf{L}(G)$ on which the convex functional

$$s_\pi : \mathfrak{g} \to (-\infty, +\infty], \quad x \to \sup(\mathrm{Spec}(-i\mathbf{d}\pi(x)))$$

is bounded. Then the interior of the domain of s_π can be written as

$$B_\pi := \{x \in \mathfrak{g} : (\exists \text{ neighbourhood } U \ni x, r \in \mathbb{R})\, s_\pi(U) \subset (-\infty, r)\}.$$

This is an open convex cone invariant under the adjoint action of G, and s_π is continuous on this cone [Ne08a].
The function s_π coincides with the *support functional*

$$\mathfrak{g} \to (-\infty, +\infty], \quad x \to \sup(I_\pi(x))$$

of the closed convex *momentum set* $I_\pi \subset \mathfrak{g}'$, which is defined as the weak-$*$ convex closure of the image of the *momentum mapping*

$$\Phi_\pi : \mathbb{P}(\mathcal{H}^\infty) \to \mathfrak{g}', \quad \Phi_\pi([v])(x) := -i\frac{\langle \mathbf{d}\pi(x)v, v\rangle}{\langle v, v\rangle} \text{ for } x \in \mathfrak{g};$$

see [Mi90] for general properties of this map. Subsets X of the topological dual V' of a real vector space V which exhibit the property that their support functional is bounded on an open subset of V are called *semi-equicontinuous.*
Considering the *coadjoint* action

$$\mathrm{Ad}^* : G \rightharpoonup \mathfrak{g}', \quad \mathrm{Ad}^*_g(\lambda) := \lambda \circ \mathrm{Ad}_{g^{-1}},$$

we find that Φ_π is equivariant with respect to the actions π and Ad^* of G on $\mathscr{H}$ and $\mathfrak{g}'$, so the momentum set is invariant under Ad^*.
This means that, in the case of a semibounded representation π, information about the spectral bounds of the operators $i\mathbf{d}\pi(x), x \in \mathfrak{g}$ is encoded in the invariant convex set $I_\pi \subset \mathfrak{g}'$ and the invariant open convex cone $B_\pi \subset \mathfrak{g}$. This can be employed to study various classes of unitary representations of Lie groups for which the invariant convex geometry of the dual of their Lie algebras is known; for example, one could easily determine whether all semibounded representations are bounded (which is the case for compact Lie groups, as follows from the linear version of Kostant's convexity theorem [Ko73]) or that non-trivial semibounded representations do not exist.
Early general results about invariant convex cones in semisimple (finite-dimensional) Lie algebras include works of Kostant and Segal [Se76], Vinberg [Vin80] and Paneitz [Pa81], [Pa83]. They will be summarised here very briefly:
If a simple Lie algebra $\mathfrak{g}$ contains non-trivial invariant cones, it contains a unique (up to sign) minimal one $C_{\min}$, which is itself contained in a maximal one $C_{\max}$. The minimal cone $C_{\min}$ is generated by a half-line which is invariant under the action of a maximal compactly embedded subgroup of inner automorphisms, and

$$C_{\max} = \{x \in \mathfrak{g} : \sup \kappa(C_{\min}, x) \leqslant 0\},$$

where κ denotes the Cartan-Killing form.
Further, every invariant convex cone in a semisimple Lie algebra $\mathfrak{g}$ is uniquely determined by its intersection with the Cartan subalgebra $\mathfrak{t}$ of a maximal compact subalgebra $\mathfrak{k} \subset \mathfrak{g}$ (the one corresponding to the subgroup of inner automorphisms just mentioned). These intersections are convex cones invariant under the natural action of the *Weyl group* $\mathscr{W}$ corresponding to the root space decomposition of $\mathfrak{k}$ with respect to $\mathfrak{t}$. If $\mathfrak{g}$ is compact, the relation between Ad_G-invariant subsets of $\mathfrak{g}$ and $\mathscr{W}$-invariant convex sets in $\mathfrak{t}$ can be described explicitly using the projection $p_\mathfrak{t} : \mathfrak{g} \to \mathfrak{t}$ with respect to the root space decomposition:

$$p_\mathfrak{t}(\mathrm{Ad}_G(x)) = \mathrm{conv}(\mathscr{W}.x) \text{ for all } x \in \mathfrak{t}. \tag{0.1}$$

This leads to a complete classification of invariant open convex cones in $\mathfrak{g}$ [HHL89].
In [KP84], Kac and Peterson achieved a very similar result for *Kac–Moody algebras*, which are a close infinite-dimensional analogon of finite-dimensional *split* Lie algebras, i.e. those Lie algebras which admit a root space decomposition with respect to a maximal abelian subalgebra. As has already been known from [Ka83, Theorem 8.5], these are (up to isomorphism) exactly the complexifications of Lie algebras of algebraic loops into compact Lie algebras.

In this Thesis, we derive a convexity theorem along the lines of 0.1 for a large class of loop algebras over Hilbert–Lie algebras, along with related conjugation theorems.

Contents

1 Double extensions and invariant Lorentz cones

Invariant Lorentz cones are maybe the most "obvious" class of cones that appear in Lie algebras and their dual spaces; they emerge whenever a real Lie algebra is equipped with an invariant *Lorentzian form*. These can always be constructed as *double extensions* of Lie algebras with an invariant inner product. Any such Lorentzian form leads to a convenient description of the adjoint and coadjoint orbits, as well as a classification of invariant convex semi-equicontinuous sets.

In this chapter, double extensions will be defined and constructed in a setting appropriate for later application to loop groups. First, some basic facts about central extensions and semidirect products will be gathered separately, then those will be put together to draw the first, rough picture of the invariant convex geometry of Lorentzian double extensions.

1.1 Construction of Lie algebra extensions

In this subsections, the "building blocks" of double extensions of Lie algebras are constructed, namely central extensions and extensions by derivation. In the last step, these are put together in the construction of double extensions, and the basic properties of these will be reduced to properties of the underlying extensions. This includes their relation to appropriate extensions of Lie groups, and the interplay of the extension structures with the smooth structure of any related Lie group.

Definition 1.1. ([HN12, p.201]) A *central extension* of a Lie algebra $\mathfrak{g}$ by a Lie algebra $\mathfrak{a}$ is a short exact sequence

$$\mathfrak{a} \hookrightarrow \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$$

such that the image of $\mathfrak{a}$ lies in the center of $\widetilde{\mathfrak{g}}$.

Note that this implies that $\mathfrak{a}$ is abelian.

Definition 1.2. ([HN12, pp.195 ff]) For Lie algebras $\mathfrak{g}$ and $\mathfrak{a}$, the latter assumed to be abelian, a *Lie algebra 2-cocycle* is a bilinear alternating map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ with the property:

$$\forall x, y, z \in \mathfrak{g} : \omega(x, [y, z]) + \omega(z, [x, y]) + \omega(y, [z, x]) = 0. \tag{1.1}$$

If $\mathfrak{g}$ and $\mathfrak{a}$ are topological Lie algebras, then we denote the set of all *continuous* 2-cocycles by $Z^2(\mathfrak{g}, \mathfrak{a})$.

This definition quite directly leads to an explicit construction of central extensions:

Proposition 1.3. *In the situation of Definition* 1.2*, the Lie algebra* $\widetilde{\mathfrak{g}}$*, defined as the vector space* $\mathfrak{a} \oplus \mathfrak{g}$ *with the Lie bracket*

$$[(a, x), (b, y)] := (\omega(x, y), [x, y]) \;\; \text{for} \;\; a, b \in \mathfrak{a}, x, y \in \mathfrak{g}, \tag{1.2}$$

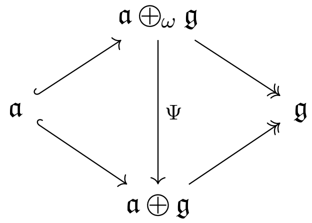
gives rise to a central extension $\mathfrak{a} \hookrightarrow \widetilde{\mathfrak{g}} \twoheadrightarrow \mathfrak{g}$ *by* $(a, x) \to x$ *for all* $x \in \mathfrak{g}$ *and* $a \in \mathfrak{a}$.

If $\mathfrak{g}$ *and* $\mathfrak{a}$ *are topological Lie algebras, and* ω *is continuous, then the product topology makes* $\widetilde{\mathfrak{g}}$ *into a topological Lie algebra. In this case, we use the designation "topological central extension".*

We denote the Lie algebra obtained this way by $\mathfrak{a} \oplus_\omega \mathfrak{g}$*, and also call it a "central extension of* $\mathfrak{g}$ *by* $\mathfrak{a}$*".*

The proof is just a quick calculation to verify axioms, with the obvious homomorphisms. More interesting is the fact that there is also a reverse construction, which enables a description of isomorphism classes of central extensions in terms of cohomology.

Definition 1.4. A continuous central extension $\mathfrak{a} \oplus_\omega \mathfrak{g}$ is called *trivial*, if there is a continuous isomorphism $\Psi : \mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{a} \oplus \mathfrak{g}$ such that the following diagram commutes:



Proposition 1.5. *For every continuous central extension*

$$\mathfrak{a} \overset{s}{\hookrightarrow} \widetilde{\mathfrak{g}} \overset{t}{\twoheadrightarrow} \mathfrak{g},$$

which admits a continuous linear section σ_1 *of* t*, there exists a continuous cocycle* $\omega_1 \in Z(\mathfrak{g}, \mathfrak{a})$ *such that* $\widetilde{\mathfrak{g}} \simeq \mathfrak{a} \oplus_{\omega_1} \mathfrak{g}$.
Furthermore, two continuous central extensions $\mathfrak{a} \oplus_{\omega_1} \mathfrak{g}$ *and* $\mathfrak{a} \oplus_{\omega_2} \mathfrak{g}$ *are equivalent if and only if* $\omega_1 = \omega_2 + l \circ [\cdot, \cdot]$ *for some linear map* $l : \mathfrak{g} \to \mathfrak{a}$*, which then is automatically continuous.*

Proof. We define

$$\varepsilon_1 : \mathfrak{g} \times \mathfrak{g} \to \widetilde{\mathfrak{g}},\ \varepsilon_1(x, y) := [\sigma_1(x), \sigma_1(y)] - \sigma_1([x, y]). \tag{1.3}$$

From t being a homomorphism and σ_1 a section of t follows that

$$\mathrm{Im}(\varepsilon_1) \subset \mathrm{Ker}(t) = \mathrm{Im}(s),$$

which allows to define $\omega_1 := s^{-1} \circ \varepsilon_1$. The following calculation shows that ε_1 and, therefore, ω_1 fulfil the Jacobi identity:

$$\begin{aligned}
&\varepsilon_1(x, [y, z]) + \varepsilon_1(z, [x, y]) + \varepsilon_1(y, [z, x]) \\
&\quad = [\sigma_1(x), \sigma_1([y, z])] + [\sigma_1(z), \sigma_1([x, y])] + [\sigma_1(y), \sigma_1([z, x])] \\
&\qquad - (\sigma_1([x, [y, z]]) + \sigma_1([z, [x, y]]) + \sigma_1([y, [z, x]]))
\end{aligned}$$

$$
\begin{aligned}
&=[\sigma_1(x), \sigma_1([y,z])] + [\sigma_1(z), \sigma_1([x,y])] + [\sigma_1(y), \sigma_1([z,x])] \\
&=[\sigma_1(x), [\sigma_1(y), \sigma_1(z)] - \varepsilon_1(y,z)] + [\sigma_1(z), [\sigma_1(x), \sigma_1(y)] - \varepsilon_1(x,y)] \\
&\quad + [\sigma_1(y), [\sigma_1(z), \sigma_1(x)] - \varepsilon_1(z,x)] \\
&= - [\sigma_1(x), \varepsilon_1(y,z)] - [\sigma_1(z), \varepsilon_1(x,y)] - [\sigma_1(y), \varepsilon_1(z,x)] = 0
\end{aligned}
$$

for $x, y, z \in \mathfrak{g}$, where in the last line we used that $\mathrm{Im}(\varepsilon_1) \subset \mathrm{Ker}(t)$, which lies in the center of $\widetilde{\mathfrak{g}}$.
An isomorphism $\varphi_1 : \mathfrak{a} \oplus_{\omega_1} \mathfrak{g} \to \widetilde{\mathfrak{g}}$ is then given by $\varphi_1((a,x)) := s(a) + \sigma_1(x)$.
Now, for $\omega_2 \in Z^2(\mathfrak{g}, \mathfrak{a})$ and an isomorphism $\varphi_2 : \mathfrak{a} \oplus_{\omega_2} \mathfrak{g} \to \widetilde{\mathfrak{g}}$ we get a section σ_2 of t by defining $l_0 : \mathfrak{g} \to \mathfrak{a} \oplus_{\omega_2} \mathfrak{g}, l_0(x) := (0, x)$ and $\sigma_2 := \varphi_2 \circ l_0$.
Thus we can rewrite ω_2 in analogy to ω_1 as $\omega_2 = s^{-1} \circ \varepsilon_2$ with

$$
\varepsilon_2(x,y) := [\sigma_2(x), \sigma_2(y)] - \sigma_2([x,y]).
$$

Because σ_1 and σ_2 are both sections of t, there exists a linear $l_0 : \mathfrak{g} \to \mathrm{Ker}(t)$ such that $\sigma_1 = \sigma_2 + l_0$, and we get

$$
\begin{aligned}
\varepsilon_1(x,y) =& \sigma_2([x,y]) + l_0([x,y]) - [\sigma_2(x) + l_0(x), \sigma_2(y) + l_0(y)] \\
=& \sigma_2([x,y]) + l_0([x,y]) - [\sigma_2(x), \sigma_2(y)] = \varepsilon_2(x,y) + l_0([x,y]).
\end{aligned}
$$

Applying s^{-1} on both sides yields $\omega_1 = \omega_2 + l \circ [\cdot,\cdot]$ with $l := s^{-1} \circ l_0$; the later is once again defined because $\mathrm{Im}(l_0) \subset \mathrm{Ker}(t) = \mathrm{Im}(s)$. This shows that $\omega_1 = \omega_2 + l \circ [\cdot,\cdot]$ if $\mathfrak{a} \oplus_{\omega_1} \mathfrak{g} \simeq \mathfrak{a} \oplus_{\omega_2} \mathfrak{g}$.

To prove the reverse implication, we start with $\omega_1 \in Z(\mathfrak{g}, \mathfrak{a})$, $l \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{a})$ and $\omega_2 := \omega_1 + l \circ [\cdot,\cdot]$. Then $\varphi(a,x) := (a + l(x), x)$ is an isomorphism $\mathfrak{a} \oplus_{\omega_1} \mathfrak{g} \to \mathfrak{a} \oplus_{\omega_2} \mathfrak{g}$. □

This proposition implies in particular that a continuous central extension $\mathfrak{a} \oplus_{\omega} \mathfrak{g}$ is trivial if and only if $\omega = l \circ [\cdot,\cdot]$ for some continuous linear $l : \mathfrak{g} \to \mathfrak{a}$.

Definition 1.6. A 2-cocycle of Lie algebras $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$ is called *trivial* or a *coboundary*, if $\omega = l \circ [\cdot,\cdot]$ for some $l \in \mathrm{Lin}(\mathfrak{g}, \mathfrak{a})$.
Two 2-cocycles $\omega_1, \omega_2 \in Z^2(\mathfrak{g}, \mathfrak{a})$ are called equivalent or *cohomologous* if there exists a linear $l : \mathfrak{g} \to \mathfrak{a}$ such that $\omega_1 = \omega_2 + l \circ [\cdot,\cdot]$.
The set of all trivial 2-cocycles is denoted by $B^2(\mathfrak{g}, \mathfrak{a})$, and the set of *cohomology classes* is denoted by

$$
H^2(\mathfrak{g}, \mathfrak{a}) := Z^2(\mathfrak{g}, \mathfrak{a}) / B^2(\mathfrak{g}, \mathfrak{a}).
$$

In the special case of finite-dimensional semisimple Lie algebras, this gives an exhaustive answer to the question for a classification of the central extensions:

Proposition 1.7 (Whitehead Lemma). [HN12, Lemma 7.5.27] *If $\mathfrak{g}$ is a finite dimensional, semisimple Lie algebra over $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$, then every 2-cocycle $\omega \in Z^2(\mathfrak{g}, \mathbb{F})$ is trivial.*

For the notion of central extensions to work well on the Lie group level, an additional smoothness assumption is convenient; it is closely related to the continuity assumption regarding Lie brackets.

Definition 1.8. Let $\alpha : H \curvearrowleft M$ be a smooth right action of the Lie group H on the smooth manifold M; then, a surjective morphism $p : M \to N$ of smooth manifolds is called an H-principal bundle, if for every $x \in N$, there exists a neighbourhood $U_x \ni x$ with a local trivialization $\varphi_U : p^{-1}(U) \to U \times H$ which is equivariant to α and the natural right action $H \curvearrowleft U \times H, (y, h).g := (y, hg)$.

Definition 1.9. A *central extension* of a group G by an abelian group A is a short exact sequence

$$A \overset{\sigma}{\hookrightarrow} \widetilde{G} \overset{\tau}{\twoheadrightarrow} G \tag{1.4}$$

such that the image of A lies in the center of $\widetilde{G}$.
If this short exact sequence is defined in the category of Lie groups, and (1.4) defines an A-principal bundle $\widetilde{G}$, it is called a *smooth central extension.*

Definition 1.10. For a group G and an abelian group A, a *group 2-cocycle* is a map $\varphi : G \times G \to A$ with the following properties for all $g, h, a \in G$:

$$\varphi(g, \mathbf{1}) = \varphi(\mathbf{1}, g) = \mathbf{1}, \tag{1.5}$$

$$\varphi(g, h)\varphi(gh, a) = \varphi(g, ha)\varphi(h, a). \tag{1.6}$$

If G and A are Lie groups, and there exists an open neighbourhood U of $\mathbf{1} \in G$ such that $\varphi|_{U\times U}$ is smooth, it is called a *locally smooth cocycle.*
φ is said to be *trivial* or a *coboundary*, if there exists a map $e : G \to A$ with $\varphi(g, h) = e(gh)e(g)^{-1}e(h)^{-1}$ for all $g, h \in G$.
Just as in the Lie algebra case, the set of all group 2-cocycles $G \times G \to A$ is called $Z^2(G, A)$, and the trivial cocycles are denoted by $B^2(G, A)$. These sets inherit an abelian group structure from A, so we can define the *cohomology classes* as $H^2(G, A) := Z^2(G, A)/B^2(G, A)$. If G and A are Lie groups, this notation always refers to the respective sets of locally smooth cocycles.

The following lemma goes without proof, because that is just verifying axioms through the obvious calculations:

Lemma 1.11. *Let G and A be groups, the latter abelian, and $\varphi \in Z^2(G, A)$. The product*

$$(a, g) \cdot (b, h) := (\varphi(g, h)ab, gh) \ \textit{ for all } \ g, h \in G, a, b \in A, \tag{1.7}$$

makes the set $A \times G$ into a group, which we denote by $A \times_\varphi G$.
Then, the short exact sequence

$$A \overset{\sigma_\varphi}{\hookrightarrow} A \times_\varphi G \overset{\tau_\varphi}{\twoheadrightarrow} G$$

with $\sigma_\varphi(a) := (a, \mathbf{1})$ and $\tau_\varphi((a, g)) := g$ is a central extension.

Remark 1.12. Let us consider any central extension as in (1.4), and a section $t : G \to \widetilde{G}$ of τ which maps $\mathbf{1} \in G$ to $\mathbf{1} \in \widetilde{G}$. In close analogy to the proof of Proposition 1.5, we define $\varepsilon : G \times G \to \widetilde{G}$, $\varepsilon(g,h) := t(g)t(h)t(gh)^{-1}$, and accordingly find that $\mathrm{Im}(\varepsilon) \subset \mathrm{Im}(\sigma)$.
This implies in particular that all elements of the image of ε commute with anything in $\widetilde{G}$, so that for all $g, h, a \in G$ we get:

$$\begin{aligned}\varepsilon(g,h)^{-1}\varepsilon(h,a) =& t(gh)t(h)^{-1}\varepsilon(h,a)t(g)^{-1}\\ =& t(gh)t(a)t(ha)^{-1}t(g)^{-1}\\ =& t(gh)t(a)t(gha)^{-1}t(gha)t(ha)^{-1}t(g)^{-1}\\ =& \varepsilon(gh,a)\varepsilon(g,ha)^{-1},\end{aligned}$$

which is (again by commutativity) equivalent to

$$\varepsilon(g,h)\varepsilon(gh,a) = \varepsilon(g,ha)\varepsilon(h,a).$$

The cocycle condition (1.6) now follows immediately for

$$\varphi := \sigma^{-1} \circ \varepsilon.$$

The condition (1.5) is verified by a quick calculation using the supposition that $t(\mathbf{1}) = \mathbf{1}$.

With this procedure, we can obviously for all central extensions $A \overset{\sigma}{\hookrightarrow} \widetilde{G} \overset{\tau}{\twoheadrightarrow} G$ and sections t of τ get a cocycle φ such that $\widetilde{G} \simeq A \times_\varphi G$. By the Axiom of Choice, there exists a "wild" section for every surjective map, so we can find such a cocycle for every central extension; however, "wild" means that it will not be generally compatible with any additional structure imposed on the groups in question and be rather useless without additional information.

The following proposition, which is essentially quoted from [Ne02, Proposition 4.2], sums up the relation between smooth central extensions of Lie groups and locally smooth cocycles.

Proposition 1.13. *Let G be a connected, and A an abelian Lie group; then*

i) for any locally smooth cocycle $\varphi_1 \in Z^2(G,A)$ as in Definition 1.10*, the group $A \times_\varphi G$ from Lemma* 1.11 *admits a Lie group structure such that the short exact sequence*

$$A \overset{\sigma}{\hookrightarrow} A \times_\varphi G \overset{\tau}{\twoheadrightarrow} G$$

with $\sigma(a) := (a, \mathbf{1})$ and $\tau((a,g)) := g$ is a smooth central extension of Lie groups.

ii) If a short exact sequence

$$A \overset{\sigma_2}{\hookrightarrow} \widetilde{G} \overset{\tau_2}{\twoheadrightarrow} G,$$

is a smooth central extension, then there exists a locally smooth 2-cocycle $\varphi_2 : G \times G \to A$ and an isomorphism of Lie groups $f : \widetilde{G} \to A \times_\varphi G$ satisfying $f(\sigma_2(A)) = \sigma(A)$.

Remark 1.14. There are examples of central extensions of Lie groups which are not principal bundles, but those won't be discussed here. Since "being smooth" is necessary and sufficient for a central extension to be expressed in terms of locally smooth cocycles, which is, in turn, necessary to relate those to continuous Lie algebra cocycles, all central extensions of Lie groups are assumed to be smooth from here on.

Definition 1.15. Let V and W be vector spaces, and let the Lie group H act linearly on V. A map $\alpha : H \times V \to W$ is called a *cocycle* if it is linear in the second argument and satisfies

$$\alpha_{gh}(x) = \alpha_h(x) + \alpha_g(h.x) \tag{1.8}$$

for all $g, h \in H$ and $x \in V$.

The next proposition helps us lifting smooth group actions on Lie algebras to given central extensions; see also [MN03, Corollary V.10] for more general results on this topic.

Proposition 1.16. *Let $\mathfrak{g}$ and $\mathfrak{a}$ be Lie algebras, $\mathfrak{a}$ abelian, $\omega \in Z^2(\mathfrak{g}, \mathfrak{a})$, and $R : H \rightharpoondown \mathfrak{g}$ an automorphic action of a Lie group H on $\mathfrak{g}$. The action R lifts to a smooth action $\widetilde{R}$ of H on the central extension $\mathfrak{a} \oplus_\omega \mathfrak{g}$ which fixes $\mathfrak{a}$ point-wise if and only if there exists a smooth cocycle $\alpha : H \times \mathfrak{g} \to \mathfrak{a}$ with*

$$\omega(g.x, g.y) = \omega(x, y) + \alpha_g([x, y]) \tag{1.9}$$

for all $x, y \in \mathfrak{g}$ and $g \in H$; then the lift is obtained by

$$g.(a, x) := \widetilde{R}_g(a, x) := (a + \alpha_g(x), g.x). \tag{1.10}$$

Proof. We denote the natural injection $\mathfrak{a} \hookrightarrow \mathfrak{a} \oplus_\omega \mathfrak{g}$ by s. If $\widetilde{R}$ is a lift of R to $\mathfrak{a} \oplus_\omega \mathfrak{g}$ which is trivial on $\mathfrak{a}$, we set

$$\alpha : H \times \mathfrak{g} \to \mathfrak{a}, \quad \alpha_g(x) := s^{-1}\big(\widetilde{R}_g(0, x) - (0, R_g(x))\big).$$

Smoothness of α follows directly from the smoothness of R and $\widetilde{R}$, and linearity implies

$$\widetilde{R}_g(a, x) = \widetilde{R}_g(a, 0) + \widetilde{R}_g(0, x) = (a + \alpha_g(x), g.x).$$

It is immediate that α is linear in the second argument, and the cocycle property follows by evaluating $\widetilde{R}_{gh}(a, x)$ two times, which yields:

$$(a + \alpha_h(x) + \alpha_g(h.x), (gh).x) = (a + \alpha_{gh}(x), (gh).x).$$

Formula (1.9) encodes the automorphism property of $\widetilde{R}_g$ for every $g \in H$, i.e.:

$$\widetilde{R}_g(\omega(x, y), [x, y]) = [(\alpha_g(x), g.x), (\alpha_g(y), g.y)]$$

for $x, y \in \mathfrak{g}$.
On the other hand, starting with a map α meeting the requirements (1.9) and (1.8), the same calculations show that

$$\widetilde{R}_g(a, x) := (a + \alpha_g(x), g.x)$$

defines an automorphic action on $\mathfrak{a} \oplus_\omega \mathfrak{g}$ fixing $\mathfrak{a}$ and interwined with R via the natural projection $\mathfrak{a} \oplus_\omega \mathfrak{g} \to \mathfrak{g}$. $\square$

Corollary 1.17. *If, in addition to the prerequisites of Proposition* 1.16*, the 2-cocycle ω is invariant under the action of H, we may take $\alpha_s = 0$ for all $s \in H$, and $\widetilde{R}_s(a, x) = (a, s.x)$ defines a lift of the action $H \rightharpoonup \mathfrak{g}$ to $\mathfrak{a} \oplus_\omega \mathfrak{g}$.*

Lemma 1.18. *Let $f \in Z^2(G, A)$ for a connected Lie group G and an abelian Lie group A be locally smooth, and $\mathfrak{g} := \mathbf{L}(G), \mathfrak{a} := \mathbf{L}(A)$; then:*

(i) *The formula for the conjugation action $\mathbf{c} : A \times_f G \rightharpoonup A \times_f G$ is:*

$$c_{(a,g)}((b, h)) = (bf(g, h)f(ghg^{-1}, g)^{-1}, ghg^{-1}). \tag{1.11}$$

(ii) *There is a cocycle $\theta : G \times \mathfrak{g} \to \mathfrak{a}$ such that:*

$$\mathrm{Ad}_{(a,g)}(r, x) = (r + \theta_g(x), \mathrm{Ad}_g x). \tag{1.12}$$

(iii) *The bilinear map $\omega : \mathbf{L}(G) \oplus \mathbf{L}(G) \to \mathbf{L}(A)$ defined by*

$$\omega(x, y) := \mathbf{d}^2 f(\mathbf{1}, \mathbf{1})(x, y) - \mathbf{d}^2 f(\mathbf{1}, \mathbf{1})(y, x) \tag{1.13}$$

is a continuous Lie algebra cocycle and $\mathbf{L}(A \times_f G) \simeq \mathfrak{a} \oplus_\omega \mathfrak{g}$.

Proof. The first two statements are taken from [Ne02, p.1390], and the proof can be found there.
For the last statement, we apply [Ne04, Theorem B.6] with $n = 2$. $\square$

Alas, the reverse statement does not generally hold, i.e. for Lie groups G and A and a Lie algebra cocycle $\omega \in Z^2(\mathbf{L}(G), \mathbf{L}(A))$ we cannot generally expect to find a Lie group cocycle $f \in Z^2(G, A)$ such that $\omega(x, y) = Df(\mathbf{1}, \mathbf{1})(y, x) - Df(\mathbf{1}, \mathbf{1})(x, y)$. This implies that there are central extensions of Lie algebras $\mathbf{L}(A) \oplus_\omega \mathbf{L}(G)$ which do *not* correspond to any central extension of Lie groups $A \times_f G$. The details can be found in [Ne02], but see also [EK64].
However, in the case such a central extension exists, A lies in its centre, which in turn is the kernel of $\mathrm{Ad} : A \times_f G \to \mathrm{Aut}(\mathfrak{a} \oplus \mathfrak{g})$. In this sense, the A-factor does not actually act on $\mathbf{L}(A) \oplus_\omega \mathbf{L}(G)$ at all, and the adjoint action is rather dependent on the crossed homomorphism θ alone, as in (1.12).

Proposition 1.19. *For a connected Lie group G, $\mathfrak{g} := \mathbf{L}(G)$ and an abelian Lie algebra $\mathfrak{a}$, let $\theta : G \times \mathfrak{g} \to \mathfrak{a}$ be a smooth cocycle, and let $\mathbf{d}\theta(x, y) := \mathbf{d}\theta(\mathbf{1}, y)(x, 0)$ denote the derivative in the first argument in the direction $x \in \mathfrak{g}$ for all $y \in \mathfrak{g}$. Then, $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{a}$, $\omega(x, y) := \mathbf{d}\theta(x)(y)$ is a Lie algebra cocycle.*

Proof. Because θ is required to be a smooth cocycle, the map

$$R : G \times (\mathfrak{g} \times \mathfrak{a}) \to \mathfrak{g} \times \mathfrak{a}, \quad R_g(y, a) := (\mathrm{Ad}_g(y), a + \theta_g(y))$$

is a smooth linear action of G. Thus it induces a linear action $\mathbf{L}(R)$ of $\mathfrak{g}$ on $\mathfrak{g} \times \mathfrak{a}$, which can be expressed as

$$\mathbf{L}(A)(x)(y, a) = (\mathrm{ad}(x)(y), \mathbf{d}\theta(x, y)) = ([x, y], \omega(x, y)). \qquad \square$$

These considerations, together with Proposition 1.16, lead to the conclusion, that an adjoint action of a Lie group G can be defined on a central extension of Lie algebras $\mathfrak{a} \oplus_\omega \mathbf{L}(G)$ if there exists a crossed homomorphism $\theta \in Z^1(G, \mathrm{Lin}(\mathfrak{g}, \mathfrak{a}))$ such that $\mathbf{d}\theta$ is antisymmetric, even if a corresponding central extension of Lie groups does not exist. To make this precise:

Definition 1.20. Let G be a Lie group, and $\mathfrak{a} \oplus_\omega \mathbf{L}(G)$ a central extension of Lie algebras, such that there exists a smooth cocycle $\theta : G \times \mathfrak{g} \to \mathfrak{a}$ with $\omega(x, y) = \mathbf{d}\theta(x)(y)$ for all $x, y \in \mathfrak{g}$. The adjoint action $\widetilde{\mathrm{Ad}} : G \rightharpoonup \mathfrak{a} \oplus_\omega \mathbf{L}(G)$ is defined by

$$\widetilde{\mathrm{Ad}}_g(r, x) := (r + \theta_g(x), \mathrm{Ad}_g\, x)$$

for all $g \in G, r \in \mathfrak{a}, x \in \mathbf{L}(G)$.

In the case where $\mathfrak{a} \oplus_\omega \mathfrak{g} = \mathbf{L}(A \times_f G)$ for some $f \in Z^2(G, A)$, the subgroup A lies in the kernel of the action

$$\mathrm{Ad} : A \times_f G \rightharpoonup \mathfrak{a} \oplus_\omega \mathfrak{g},$$

so Lemma 1.18 means that the above definition is equivalent to factoring out $A \subseteq \ker(\mathrm{Ad})$.
The following lemma means that we can quite generally use Definition 1.20 to construct an adjoint group action for a given central extension of Lie algebras; it is stated without proof, which can be found in [Ne02, Corollary 7.7].

Lemma 1.21. *Let G be a 1-connected Lie group with $\mathfrak{g} := \mathbf{L}(G)$, and $\mathfrak{a}$ be a sequentially complete locally convex vector space. Then, for every $\omega \in Z^2(\mathfrak{g}, \mathfrak{a})$, there exists a smooth cocycle $\theta : G \times \mathfrak{g} \to \mathfrak{a}$ such that $\omega(x, y) = \mathbf{d}\theta(y)(x)$ for all $x, y \in \mathfrak{g}$.*

When we proceed to studying the coadjoint action

$$\widetilde{\mathrm{Ad}}^* : G \rightharpoonup (\mathfrak{a} \oplus_\omega \mathfrak{g})', \widetilde{\mathrm{Ad}}^*_g(\lambda) := \lambda \circ \widetilde{\mathrm{Ad}}_{g^{-1}},$$

we quickly find the following formula for every $\lambda = \lambda_\mathfrak{a} + \lambda_\mathfrak{g} \in (\mathfrak{a} \oplus_\omega \mathfrak{g})'$ and $(a, x) \in \mathfrak{a} \oplus_\omega \mathfrak{g}$:

$$\widetilde{\mathrm{Ad}}^*_g(\lambda)(a, x) = \lambda((a + \theta_{g^{-1}}(x), \mathrm{Ad}_{g^{-1}}\, x)) = \lambda_\mathfrak{a}(a + \theta_{g^{-1}}(x)) + \mathrm{Ad}^*_g\, \lambda_\mathfrak{g}(x),$$

which leads to the conclusion:

Proposition 1.22. *For every $\lambda_{\mathfrak{a}} \in \mathfrak{a}'$, the coadjoint action $\widetilde{\mathrm{Ad}}^*$ induces an affine action of G on the invariant affine subspace $\{\lambda_{\mathfrak{a}}\} \oplus \mathfrak{g}' \subset (\mathfrak{a} \oplus_\omega \mathfrak{g})'$.*

Now we turn to the other side of the double extensions, the semidirect products, which respect a specific Lie algebra or Lie group action.
We say that a Lie algebra extension $\mathfrak{f} \xhookrightarrow{s} \mathfrak{g} \xrightarrow{t} \mathfrak{h}$ *splits* if there exists a subalgebra $\mathfrak{h}_0 < \mathfrak{g}$ and an isomorphism $u : \mathfrak{h} \to \mathfrak{h}_0$ which is a section of t. This situation is described by the following:

Definition 1.23. Let $\mathfrak{f}, \mathfrak{h}$ be Lie algebras and $\rho : \mathfrak{f} \times \mathfrak{h} \to \mathfrak{h}$ a continuous action by derivations. Then, the vector space $\mathfrak{h} \oplus \mathfrak{f}$, equipped with the Lie bracket

$$[(x, r), (y, s)] := ([x, y] + \rho(r)y - \rho(s)x, [r, d])$$

for all $x, y \in \mathfrak{h}$, $r, s \in \mathfrak{f}$ is called a semidirect product of Lie algebras. It is denoted by $\mathfrak{h} \rtimes_\rho \mathfrak{f}$; the "$\rho$" may be omitted if it is clear from the context. Obviously, $\mathfrak{h}$ is an ideal in $\mathfrak{h} \rtimes_\rho \mathfrak{f}$.

Proposition 1.24. [GN20, Proposition 4.2.8] *When considering semidirect products of Lie groups, $H \rtimes_R F$, we always assume the map $R : F \times H \to H$ to be smooth, so that $H \rtimes_R F$ is itself a Lie group.*
This definition corresponds to semidirect products of Lie algebras via the Lie functor, i.e.

$$\mathbf{L}(H \rtimes_R F) \simeq \mathbf{L}(H) \rtimes_{\mathbf{L}(R)} \mathbf{L}(F), \tag{1.14}$$

where

$$\mathbf{L}(R)(x, y) := \mathbf{d}R(\mathbf{1}_H, \mathbf{1}_F)(x, y) \;\; \text{for} \;\; x \in \mathbf{L}(H) \;\; \text{and} \;\; y \in \mathbf{L}(F).$$

Proposition 1.25. *Let $G \rtimes_R A$ be a semidirect product of Lie groups, and A be abelian. With $\mathfrak{g} := \mathbf{L}(G)$ and $\mathfrak{a} := \mathbf{L}(A)$, the adjoint action $\mathrm{Ad} : G \rtimes_R A \rightharpoonup \mathfrak{g} \rtimes_\rho \mathfrak{a}$ reads:*

$$\mathrm{Ad}_{(g,a)}(x, r) = (\mathrm{Ad}_g(\rho_a(x)) + \zeta_g(r), r), \tag{1.15}$$

where $\rho_a(x) := \mathbf{L}(R_a)(x)$ denotes the induced Lie algebra morphism for all $a \in A$, and $\zeta : G \to \mathrm{Lin}(\mathfrak{a}, \mathfrak{g})$ is a cocycle with respect to the action

$$G \times \mathrm{Lin}(\mathfrak{a}, \mathfrak{g}) \to \mathrm{Lin}(\mathfrak{a}, \mathfrak{g}), l \to \mathrm{Ad}_g \circ l.$$

Proof. The formula for the conjugation action of $G \rtimes_R A$ is:

$$c_{(g,a)}(h, b) = (g, a)(h, b)(R_{a^{-1}}(g^{-1}), a^{-1}) = (gR_a(h)R_b(g^{-1}), b).$$

Differentiating this at $(1, 1) \in G \rtimes_R A$ in the direction $(x, 0) \in \mathfrak{g} \oplus \mathfrak{a}$ yields

$$\mathrm{Ad}_{(g,a)}(x, 0) = (\mathrm{Ad}_g(\rho_a(x)), 0).$$

To calculate the differential in the direction $(0, r) \in \mathfrak{g} \oplus \mathfrak{a}$ we note

$$c_{(g,a)}(1, b) = c_{(g,1)}(1, b) = (gR_b(g^{-1}), b),$$

and thus

$$\mathrm{Ad}_{(g,a)}(0,r) = (\zeta_g(r), r).$$

The cocycle property of ζ follows by evaluating $\mathrm{Ad}_{(g,\mathbf{1})} \circ \mathrm{Ad}_{(h,\mathbf{1})}(0,r)$ two times for $a, b \in A, g, h \in G$:

$$\begin{aligned} \mathrm{Ad}_{(gh,\mathbf{1})}(0,r) =& (\zeta_{gh}(r), r), \\ \mathrm{Ad}_{(g,\mathbf{1})}(\zeta_h(r), r) =& (\mathrm{Ad}_g(\zeta_h(r)) + \zeta_g(r), r). \end{aligned}$$

□

Remark 1.26. A special form of a split extension is the "extension by a derivation", which, for a $\mathbb{K}$-Lie algebra $\mathfrak{g}$ and $\mathbf{d} \in \mathrm{Der}(\mathfrak{g})$ is denoted by $\mathfrak{g} \rtimes_{\mathbf{d}} \mathbb{K}$ and defined as the semidirect product of $\mathfrak{g}$ by $\mathbb{K}\mathbf{d} \subset \mathrm{Der}(\mathfrak{g})$.
Starting with a smooth automorphic action R of the circle group $\mathbb{T}$ on a Lie group G, by taking the Lie functor $\mathbf{L}(R_s)$ at each point $s \in \mathbb{T}$, we get a smooth action of $\mathbb{T}$ on $\mathbf{L}(G)$; we denote this action by ρ.
By taking derivatives of the orbit maps $\rho^x : \mathbb{T} \to \mathfrak{g}, t \to \rho_t(x)$ at $1 \in \mathbb{T}$, we get a derivation $\mathbf{d}_R$ on $\mathfrak{g}$; we call this derivation the *infinitesimal generatorinfinitesimal generator* of R.

Now that all building blocks are described in sufficient detail, we can define and describe double extensions of Lie algebras. We will find that the properties of both, central extensions and semidirect products, can be conferred to the complete scenario rather directly. This includes a definition of an adjoint action of an appropriate Lie group.

Definition 1.27. [Ne14, Definition 2.3] Let $\mathfrak{g}_0$ be a $\mathbb{K}$-Lie algebra, $\omega \in Z^2(\mathfrak{g}_0, \mathbb{K})$, and $\mathbf{d} \in \mathrm{Der}(\mathfrak{g}_0)$ such that there exist a linear map $\delta : \mathfrak{g}_0 \to \mathbb{R}$ with

$$\delta([x,y]) = \omega(\mathbf{d}(x), y) + \omega(x, \mathbf{d}(y)) \quad \text{for all} \quad x, y \in \mathfrak{g}_0; \tag{1.16}$$

then,

$$\widetilde{\mathbf{d}} : \mathbb{K} \oplus_\omega \mathfrak{g}_0 \to \mathbb{K} \oplus_\omega \mathfrak{g}_0, \quad \widetilde{\mathbf{d}}(r,x) := (\delta(x), \mathbf{d}(x))$$

is a derivation, and

$$[(r,x,t),(s,y,u)] := (\omega(x,y) + t\delta(y) - u\delta(x), [x,y] + t\mathbf{d}(y) - u\mathbf{d}(x), 0), \tag{1.17}$$

defines a Lie bracket on the vector space $\mathbb{K} \oplus \mathfrak{g}_0 \oplus \mathbb{K}$. The resulting Lie algebra is called a *double extension* and usually denoted by

$$\mathfrak{g} := (\mathbb{K} \oplus_\omega \mathfrak{g}_0) \rtimes_{\widetilde{\mathbf{d}}} \mathbb{K}.$$

In the case $\delta = 0$, the condition on ω simplifies to $\omega(\mathbf{d}x, y) + \omega(x, \mathbf{d}y) = 0$ for all $x, y \in \mathfrak{g}$. A 2-cocycle ω with this property is said to be $\mathbf{d}$-*invariant*. In this case we write

$$\mathfrak{g} := (\mathbb{K} \oplus_\omega \mathfrak{g}_0) \rtimes \mathbb{K}\mathbf{d}.$$

Remark 1.28. In the definition above, instead of using δ to extend $\mathbf{d} \in \mathrm{Der}(\mathfrak{g}_0)$ to $\widetilde{\mathbf{d}} \in \mathrm{Der}(\mathbb{K} \oplus_\omega \mathfrak{g}_0)$, one can also extend $\omega \in Z^2(\mathfrak{g}_0, \mathbb{K})$ to $\widetilde{\omega} \in Z^2(\mathfrak{g}_0 \rtimes_{\mathbf{d}} \mathbb{K}, \mathbb{K})$ by setting

$$\widetilde{\omega}((x,t),(y,u)) := \omega(x,y) + t\delta(y) - u\delta(x)$$

for $t, u \in \mathbb{K}$ and $x, y \in \mathfrak{g}_0$. This defines an isomorphism

$$(\mathbb{K} \oplus_\omega \mathfrak{g}_0) \rtimes_{\widetilde{\mathbf{d}}} \mathbb{K} \simeq \mathbb{K} \oplus_{\widetilde{\omega}} (\mathfrak{g}_0 \rtimes_{\mathbf{d}} \mathbb{K}).$$

This allows to apply results about both central and split extensions to double extensions.

Remark 1.29. When referring to double extensions, we will often use the notational convention

$$\mathbf{c} := (1,0,0) \in \mathfrak{g} = (\mathbb{K} \oplus_\omega \mathfrak{g}_0) \rtimes_{\widetilde{d}} \mathbb{K}.$$

We further write $\mathbf{d}^*$ for the linear functional on $\mathfrak{g}$ that sends $\mathbf{d} = (0,0,1)$ to 1 and all of $(\mathbb{K}\mathbf{c} \oplus_\omega \mathfrak{g})$ to zero; likewise, $\mathbf{c}^*(\mathbf{c}) := 1$ and $\mathbf{c}^*(\mathfrak{g}_0) = \mathbf{c}^*(\mathbb{K}\mathbf{d}) := \{0\}$. In this notation, the topological dual $\widehat{\mathfrak{g}}'$ decomposes as $\mathbb{K}\mathbf{c}^* \oplus \mathfrak{g}_0' \oplus \mathbb{K}\mathbf{d}^*$.

Remark 1.30. A double extension of Lie groups could be defined in the following way:
Let $R : \mathbb{T} \rightharpoonup G_0$ be a smooth automorphic action, and $f : G_0 \to \mathbb{T}$ be a locally smooth cocycle. If there exists a lift $\widetilde{R} : \mathbb{T} \rightharpoonup \mathbb{T} \times_f G_0$ of R, then the semidirect product

$$(\mathbb{T} \times_f G_0) \rtimes_{\widetilde{R}} \mathbb{T}$$

is a double extension of G_0; in this case, successively applying Proposition 1.24 and Lemma 1.18 shows that this definition is compatible with the definition of double extensions of Lie algebras in 1.27.
It is a nontrivial task to determine whether any such lift $\widetilde{R}$ exists, but to study the adjoint action of this group, it is sufficient to consider only the action of $G_0 \rtimes_R \mathbb{T}$, by applying Remark 1.28. Then, for

$$\mathfrak{g} := (\mathbb{R} \oplus_{\mathbf{L}(f)} \mathbf{L}(G_0)) \rtimes_{\mathbf{L}(\widetilde{R})} \mathbb{R}$$

the adjoint action

$$\mathrm{Ad} : (G_0 \rtimes_R \mathbb{T}) \rightharpoonup \mathfrak{g}$$

is already covered by the formulas (1.10), (1.12) and (1.15). For some applications, this allows us to avoid the question of whether a lift $\widetilde{R}$ exists.

1.2 Lorentzian double extensions

In this subsection, our setting will get narrowed down to a class of Lie algebras which already exhibit an important part of the geometric properties we later want to study in the double extensions of loop algebras.

Definition 1.31. A continuous bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ on a topological Lie algebra $\mathfrak{g}$ is called *invariant* if $\kappa([x, y], z) = -\kappa(y, [x, z])$ for all $x, y, z \in \mathfrak{g}$. If κ is, in addition, non-degenerate and symmetric, then $\mathfrak{g}$ is called a *symmetric Lie algebra*.
Because of the non-degeneracy assumption, the continuous linear map

$$\flat : \mathfrak{g} \to \mathfrak{g}', \quad x^\flat(y) := \kappa(y, x)$$

is injective, and therefore has an inverse $\mathfrak{g}^\flat \to \mathfrak{g}$, which we write as $\lambda \to \lambda^\sharp$. This defines a pull-back:

$$\kappa^* : \mathfrak{g}^\flat \times \mathfrak{g}^\flat \to \mathbb{K}, \quad \kappa^*(\lambda, \gamma) := \kappa(\lambda^\sharp, \gamma^\sharp).$$

Example 1.32. Let $\mathfrak{g}_0$ be a Lie algebra, and $\mathbf{d} \in \mathrm{Der}(\mathfrak{g}_0)$ be skew-adjoint with respect to an invariant symmetric bilinear form β. Then,

$$\omega : \mathfrak{g}_0 \times \mathfrak{g}_0 \to \mathbb{R}, \quad \omega(x, y) := \beta(x, \mathbf{d}y) \tag{1.18}$$

is a 2-cocycle. This follows from

$$\omega(x, y) = \beta(x, \mathbf{d}y) = \beta(\mathbf{d}y, x) = -\beta(y, \mathbf{d}x) = -\omega(y, x)$$

and

$$\begin{aligned}\omega([x, y], z) &= -\beta(\mathbf{d}[x, y], z) = -\beta([\mathbf{d}x, y], z) - \beta([x, \mathbf{d}y], z) \\ &= -\beta(\mathbf{d}x, [y, z]) + \beta(\mathbf{d}y, [x, z]) = -\omega([y, z], x) - \omega([z, x], y)\end{aligned}$$

for all $x, y, z \in \mathfrak{g}_0$. Also, $\mathbf{d}$-invariance of ω follows directly from $\mathbf{d}$ being skew-adjoint, and thus we get a double extension for every pair $(\beta, \mathbf{d})$ of an invariant symmetric bilinear form and a skew-adjoint derivation on $\mathfrak{g}_0$.

Definition 1.33. A *Lorentzian space* is a pair (V, κ) of a real topological vector space V and a *Lorentzian form* κ, which is defined as a continuous symmetric bilinear form on V for which there exists a vector $v \in V$ such that $\kappa(v, v) < 0$ and $\kappa|_{v^\perp \times v^\perp}$ is positive definite.

The set C of all vectors $v \in V$ such that $\kappa(v, v) \leqslant 0$ for some Lorentzian form κ on V is a double cone. For an arbitrary v_0 from the interior of C, i.e. $\kappa(v_0, v_0) < 0$, the cone $\{v \in C : \kappa(v, v_0) \leqslant 0\}$ is called a *Lorentz cone*.

One of our more important tools are invariant Lorentz forms on Lie algebras. They have first been intensively studied in [MR85]; however, our results here are derived independently from that article.

Proposition 1.34. *Let $\mathfrak{g}_0$ be a real Lie algebra endowed with an invariant inner product $(\cdot, \cdot)$, $d \in \mathrm{Der}(\mathfrak{g}_0)$ be skew-symmetric and $\omega \in Z^2(\mathfrak{g}_0, \mathbb{R})$ be defined as in* (1.18) *with respect to the invariant inner product on $\mathfrak{g}_0$. Then, the bilinear map $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$,*

$$\kappa((c_1, x_1, t_1), (c_2, x_2, t_2)) := (x_1, x_2) - c_1 t_2 - c_2 t_1 \tag{1.19}$$

on the associated double extension $\mathfrak{g} = (\mathbb{R} \oplus_\omega \mathfrak{g}_0) \rtimes_d \mathbb{R}$ is an invariant Lorentzian form.

Proof. We have to show that

$$\kappa(\mathrm{ad}(\hat{x})\hat{y}_1, \hat{y}_2) = -\kappa(\hat{y}_1, \mathrm{ad}(\hat{x})\hat{y}_2) \tag{1.20}$$

for $\hat{x} = (b, x, s), \hat{y}_1 = (c_1, y_1, t_1), \hat{y}_2 = (c_2, y_2, t_2) \in \mathfrak{g}$, and we do so by splitting the formula by linearity:
Central elements of $\mathfrak{g}$, specifically $\hat{x} = \mathbf{c}$, just send both sides to 0, and for elements of the form $\hat{x} = s\mathbf{d}$ we get:

$$\begin{aligned}\kappa([\hat{x}, \hat{y}_1], \hat{y}_2) =& \kappa((0, s\mathbf{d}(y_1), 0), (c_2, y_2, t_2)) \\ =& -(s\mathbf{d}(y_1), y_2) = (y_1, s\mathbf{d}(y_2)) = -\kappa(\hat{y}_1, [\hat{x}, \hat{y}_2])\end{aligned}$$

by applying (1.17). The calculation for the case $\hat{x} = (0, x, 0)$ is similar:

$$\begin{aligned}\kappa([\hat{x}, \hat{y}_1], \hat{y}_2) &= \kappa((\omega(x, y_1), [x, y_1] - t_1\mathbf{d}(x), 0), (c_2, y_2, t_2)) \\ &= ([x, y_1] - t_1\mathbf{d}(x), y_2) - t_2\omega(x, y_1) \\ &= ([x, y_1], y_2) - t_2(x, \mathbf{d}(y)_1) - t_1(\mathbf{d}(x), y_2) \\ &= -(y_1, [x, y_2]) + t_1(x, \mathbf{d}(y)_2) + t_2(\mathbf{d}(x), y_1) = -\kappa(\hat{y}_1, [\hat{x}, \hat{y}_2]).\end{aligned}$$

To see that κ is Lorentzian, we insert $v := (1, 0, 1) \in \hat{\mathfrak{g}}$ and get $\kappa(v, v) = -2$; further, $v^\perp = \mathfrak{g}_0 \oplus w\mathbb{R}$, where $w = (1, 0, -1) \in \hat{\mathfrak{g}}$, and κ is obviously positive definite on this subspace. □

Corollary 1.35. *With the prerequesites from Proposition* 1.34 *and* $a \in \mathbb{R}$, *the bilinear map* $\kappa_a : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$,

$$\kappa_a((s_x, x_0, t_x), (s_y, y_0, t_y)) := \kappa((s_x, x_0, t_x), (s_y, y_0, t_y)) - at_xt_y$$

is also an invariant Lorentzian form.

Proof. From formula (1.17) we conclude that, if $x \in [\mathfrak{g}, \mathfrak{g}]$, then $t_x = 0$, and if $y \in [\mathfrak{g}, \mathfrak{g}]$, then $t_y = 0$, so in both cases $at_xt_y = 0$, so the invariance follows directly from the invariance of κ.
To show that κ_a is Lorentzian, we set $v := (0, 0, 1)$ and $w := (a, 0, -1)$. Then $\kappa_a(v, v) = -a$, $\kappa_a(w, w) = a$, and $\mathfrak{g}_0$, v and w are orthogonal to each other. If $a > 0$, then κ_a is negative definite on $\mathbb{R}v$ and positive on $\mathfrak{g} \oplus \mathbb{R}w$, and for $a < 0$, the roles of v and w are switched. □

This justifies the following definition:

Definition 1.36. We call a Lie algebra $\mathfrak{g} := (\mathbb{R} \oplus_\omega \mathfrak{g}_0) \rtimes \mathbb{R}\mathbf{d}$ a *Lorentzian double extension* if $\mathfrak{g}_0$ is a real topological Lie algebra equipped with a continuous invariant inner product, $\mathbf{d} \in \mathrm{der}(\mathfrak{g}_0)$ is skew-symmetric and $\omega(x, y) := (x, \mathbf{d}y)$ for all $x, y \in \mathfrak{g}_0$.
Every Lorentzian double extension comes equipped with the invariant Lorentzian form κ from Proposition 1.34.

Proposition 1.37. *Let $x = (s, x_0, t) \in \mathfrak{g}$ and $a, b \in \mathbb{R}$; let further $(\kappa_a)_{a\in\mathbb{R}}$ be the family of invariant Lorentzian forms on a Lorentzian double extension $\mathfrak{g}$ of $\mathfrak{g}_0$. Then the corresponding open and closed Lorentzian double cones*

$$C_a := \{x \in \mathfrak{g} : \kappa_a(x, x) < 0\} \text{ and } \overline{C}_a = \{x \in \mathfrak{g} : \kappa_a(x, x) \leqslant 0\}$$

have the following properties:

(i) *If $t \neq 0$, then the map $\mathbb{R} \to \mathbb{R} : a \to \kappa_a(x, x)$ is surjective.*

(ii) *If $t \neq 0$, then $\kappa_a(x, x) = 0$ for $a = -\frac{2s}{t} + \frac{(x_0,x_0)}{t^2}$.*

(iii) $\bigcap_{a\in\mathbb{R}} \overline{C}_a = \mathbb{R}\mathbf{c}$

(iv) *If $x = (s, x_0, t) \in \mathfrak{g}$ with $t \neq 0$, then $x \in \bigcup_{a\in\mathbb{R}} C_a$.*

Proof. (i) is obvious.

(ii) is proven by a simple computation.

(iii) $\kappa_a(\mathbf{c}, \mathbf{c}) = 0$ for all $a \in \mathbb{R}$, and thus $\mathbf{c} \in \bigcap_{a\in\mathbb{R}} \overline{C}_a$. If $t = 0$, then either $x_0 = 0$ and therefore $x \in \mathbb{R}\mathbf{c}$, or, by (i), there exists an $a \in \mathbb{R}$ with $\kappa_a(x, x) > 0$, which means that $x \notin \overline{C}_a$ and thus $x \notin \bigcap_{a\in\mathbb{R}} \overline{C}_a$.

(iv) is another application of (i).

□

Remark 1.38. If $x = (s_x, x_0, t_x) \in \mathfrak{g}$ with $t_x \neq 0$ and $\kappa_a(x, x) = 0$, then the adjoint orbit $\mathcal{O}_x$ is contained in the parabolic conic section

$$\partial C_a \cap \{(s, y_0, t) \in \mathfrak{g} : t = t_x\},$$

and thus

$$s_y = \frac{\|y_0\|^2 - at_x^2}{2t_x} \text{ for every } y = (s_y, y_0, t_x) \in \mathcal{O}_x.$$

Likewise, with a cocycle $\zeta : G_0 \to \mathrm{Lin}(\mathbb{R}, \mathfrak{g}_0) \simeq \mathfrak{g}_0$ as in Proposition 1.25, we find that $\kappa_a((s_y, y_0, t_x), (s_y, y_0, t_x)) = 0$ for

$$y_0 = \mathrm{Ad}(g)(x_0) + t_x\zeta(g) \text{ and } a = \frac{\|x_0\|^2 - 2s_xt_x}{t_x^2}.$$

If G_0 is connected, then s_y can be directly computed by inserting a and y_0, taking into account that $\|x_0\|^2 = \|\mathrm{Ad}(g)(x_0)\|^2$. This allows us to spell out the formula for the adjoint action of G_0 on $\mathfrak{g}$:

$$\begin{aligned}&\mathrm{Ad}(g)(s_x, x_0, t_x)\\ &\quad= \Big(s_x - \big(\mathrm{Ad}(g)(x_0), \zeta(g)\big) - \frac{t_x}{2}\|\zeta(g)\|^2, \mathrm{Ad}(g)(x_0) + t_x\zeta(g), t_x\Big).\end{aligned}$$

We are going to extend this Lorentzian geometry to the topological dual $\mathfrak{g}'$; this will also lead to a sufficient criterion for an element $\lambda \in \mathfrak{g}'$ to have a semi-equicontinuous orbit.

Remark 1.39. We consider the injection $\flat : \mathfrak{g} \to \mathfrak{g}'$ of a Lorentzian double extension into its dual space via its invariant Lorentzian form. Explicitly, for $x = (s_x, x_0, t_x)$ and $y := (s_y, y_0, t_y) \in \mathfrak{g}$, we have

$$y^\flat(x) = \kappa((s_x, x_0, t_x), (s_y, y_0, t_y)) = (x_0, y_0) - s_x t_y - s_y t_x = y_0^\flat(x_0) - s_x t_y - s_y t_x,$$

so comparing coefficients shows

$$y^\flat = (s_y, y_0, t_y)^\flat = t_y \mathbf{c}^* + y_0^\flat + s_y \mathbf{d}^*$$

in the notation of Remark 1.29. In particular, $\flat$ sends $\mathbf{c}$ to $\mathbf{d}^*$ and $\mathbf{d}$ to $\mathbf{c}^*$.

Definition 1.40. If $\mathfrak{g}_0$ is a topological Lie algebra with an invariant inner product, we write $\overline{\mathfrak{g}}_0$ for the completion of $\mathfrak{g}_0$ with respect to the topology induced by its invariant inner product norm.
If $\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathfrak{g}_0) \rtimes \mathbb{R}\mathbf{d}$ is a Lorentzian double extension, then we set

$$\overline{\mathfrak{g}} := \mathbb{R}\mathbf{c} \oplus \overline{\mathfrak{g}}_0 \oplus \mathbb{R}\mathbf{d}.$$

This space inherits the Lorentzian forms κ_a for $a \in \mathbb{R}$. If we are talking about "the" Lorentzian space $\overline{\mathfrak{g}}$, then we always refer to the space equipped with $\kappa = \kappa_0$.

Note that $\overline{\mathfrak{g}}_0$ is a Hilbert space, but, in general, neither $\overline{\mathfrak{g}}_0$ nor $\overline{\mathfrak{g}}$ is a Lie algebra. However:

Proposition 1.41. *The adjoint action $G_0 \rightharpoonup \mathfrak{g}$ on a Lorentzian Lie algebra constructed from $\mathfrak{g}_0 = \mathbf{L}(G_0)$ extends to an action of G_0 on $\overline{\mathfrak{g}}$. This action is smooth in the sense that the set $\overline{\mathfrak{g}}^\infty$ of* smooth vectors*, i.e. elements with smooth orbit maps, is dense in $\overline{\mathfrak{g}}$.*

Proof. To see that there is a continuous extension of the adjoint action, we look at Remark 1.38 and find that there is only to show that $\mathrm{Ad} : G_0 \rightharpoonup \mathfrak{g}_0$ extends to $\overline{\mathfrak{g}}_0$. For every $g \in G_0$, the operator $\mathrm{Ad}(g)$ is unitary and thus automatically norm-continuous. So it maps Cauchy sequences to Cauchy sequences, and therefore $\overline{\mathfrak{g}}_0$ to itself.
The smoothness statement is then obvious, because $\mathfrak{g} \subset \overline{\mathfrak{g}}^\infty$ is dense in $\overline{\mathfrak{g}}$ by definition. □

Remark 1.42. $\overline{\mathfrak{g}}$ inherits the invariant Lorentzian form κ, and thus the injection $\flat : \overline{\mathfrak{g}} \to \mathfrak{g}'$. By applying the invariance of κ to the coadjoint action on elements $y^\flat \in \overline{\mathfrak{g}}^\flat$, we find

$$\mathrm{Ad}(g)^*(y^\flat)(x) = y^\flat(\mathrm{Ad}(g^{-1})x) = \kappa(x, \mathrm{Ad}(g)y) = (\mathrm{Ad}(g)y)^\flat(x),$$

for all $x \in \mathfrak{g}$, so from the preceding proposition follows that the coadjoint action $\mathrm{Ad}^* : G_0 \rightharpoonup \mathfrak{g}'$ normalises the subspace $\overline{\mathfrak{g}} \hookrightarrow \mathfrak{g}'$. Now Remark 1.39 implies that the coadjoint and the extended adjoint action on $\overline{\mathfrak{g}}$ are basically the same.

It is more natural to refer to it as the coadjoint action, as it arises directly from duality. In particular, for every $a \in \mathbb{R}$, the coadjoint action respects the Lorentzian form κ_a in the sense that $\kappa_a(\mathrm{Ad}(g)^*\lambda, \gamma) = \kappa_a(\lambda, \mathrm{Ad}(g^{-1})^*\gamma)$ for all $g \in G_0$ and $\lambda, \gamma \in \overline{\mathfrak{g}}$.

Proposition 1.43. *Let $\lambda = s\mathbf{d}^* + \lambda_0 + t\mathbf{c}^* \in \overline{\mathfrak{g}}$ with $\lambda_0(\mathbf{d}) = \lambda_0(\mathbf{c}) = 0$ and $a, b \in \mathbb{R}$; let further $C_a \subset \overline{\mathfrak{g}}$ be an open Lorentzian cone with respect to the Lorentzian form κ_a on $\overline{\mathfrak{g}}$ and $\overline{C}_a$ be its closure. Then:*

(i) *If $t \neq 0$, then the map $\mathbb{R} \to \mathbb{R} : a \to \kappa_a(\lambda, \lambda)$ is surjective.*

(ii) *If $t \neq 0$, then $\kappa_a(\lambda, \lambda) = 0$ for $a = -\frac{2s}{t} + \frac{(\lambda_0, \lambda_0)}{t^2}$.*

(iii) $\bigcap_{a \in \mathbb{R}} \overline{C}_a = \mathbb{R}\mathbf{d}^*$

(iv) *If $t \neq 0$, then $\lambda \in \bigcup_{a \in \mathbb{R}} C_a$*

Proof. Every single item here is a reformulation of the respective item from Proposition 1.37; with that in mind, (i) through (iii) follow because they are stable under applying closures in the topology of $\overline{\mathfrak{g}}$, and (iv) follows from (i). □

Proposition 1.44. *For every $\lambda \in \overline{\mathfrak{g}} \subset \mathfrak{g}'$ with $\lambda(\mathbf{c}) \neq 0$, the coadjoint orbit $\mathrm{Ad}(G)^*\lambda = \mathcal{O}_\lambda$ is semi-equicontinuous.*

Proof. We pick $a \in \mathbb{R}$ such that $\kappa_a(\lambda, \lambda) < 0$ and consider the cones

$$D_a := \{\gamma \in \overline{\mathfrak{g}} : \kappa_a(\gamma, \gamma) < 0, \kappa_a(\gamma, \lambda) < 0\} \text{ and}$$

$$E_a := D_a^\sharp \cap \mathfrak{g} = \{x \in \mathfrak{g} : \kappa_a(x, x) < 0, \lambda(x) < 0\},$$

see Remarks 1.39 and 1.42. For any $y \in E_a$, we have

$$\mathcal{O}_\lambda(y) = \kappa(\mathrm{Ad}_G(y), \lambda^\sharp) = \lambda(\mathrm{Ad}(G_0)(y)),$$

and, because E_a is invariant, $\lambda(\mathrm{Ad}(g)(y)) < 0$ for all $g \in G_0$, which is just what we had to show because E_a is open in $\mathfrak{g}$. □

2 Conjugacy theorems in Hilbert loop algebras

Kostant's convexity theorem and theorems generalising or resembling it (see [Ko73], [KP84], [Neu00]) are built around the concept of projecting a Lie algebra $\mathfrak{g}$ onto a maximal abelian subalgebra $\mathfrak{h}$ and describing the action of the normaliser of $\mathfrak{h}$ in the inner automorphisms of $\mathfrak{g}$, as in Equation (0.1).
In the original context of compact Lie groups K, the Kostant convexity theorem could easily be applied to arbitrary elements of the Lie algebra $\mathfrak{k} := \mathbf{L}(K)$. Because every $x \in \mathfrak{k}$ is conjugate to some element of any Cartan subalgebra $\mathfrak{t}_\mathfrak{k}$, the projection of $\mathcal{O}_x$ onto $\mathfrak{t}_\mathfrak{k}$ can always be determined by conjugating x into $\mathfrak{t}_\mathfrak{k}$ and then applying Kostant's convexity theorem.
We are interested in extending this concept to (twisted) Hilbert loop algebras; this means that we have to determine the elements of $\mathfrak{g}$ that are conjugate to some element $x \in \mathfrak{t}_\mathfrak{g}$ under the adjoint action. We refer to propositions describing such elements and their orbits as 'conjugacy theorems".

2.1 The construction of loop groups and loop algebras

The following subsection contains a collection of definitions and basic properties, establishing the setting in which the findings from Section 1 will be applied and refined.
Compact, simply connected Lie groups correspond bijectively with semisimple complex Lie algebras, which makes both very accessible and leads to a rich theory; similar results about Hilbert–Lie groups will be quoted shortly.
The set $C^\infty(M, K)$ of smooth maps from a compact manifold M into a Lie group K is itself a Lie group. In the case of the circle $M = \mathbb{S}^1$ and a Hilbert–Lie group K, this group inherits quite some properties from K.
For compact K, these "loop groups" are closely related to the class of affine Kac–Moody algebras. Extensive disquisitions about both the loop groups and Kac–Moody algebras can be found in [PS86] and [Ka83].
[AP83] and [KP84] both describe the projections of adjoint orbits of (doubly extended) loop groups to Cartan subalgebras in terms of convex geometry, though they do so from different points of view. The following definitions treat the compact Lie algebras as a subclass of the Hilbert–Lie algebras as often as possible and only deviate from this approach in non conferrable details.

Definition 2.1. A *Hilbert–Lie algebra* is a pair of a (real or complex) topological Lie algebra $\mathfrak{k}$ and an invariant inner product which induces the topology of $\mathfrak{k}$.

Accordingly, a Hilbert–Lie group is defined as a Lie group K whose Lie algebra $\mathfrak{k}$ is a Hilbert–Lie algebra. If K is connected, then the invariance of the inner product immediately implies that K acts on $\mathfrak{k}$ by isometries.

Definition 2.2. As we are going to talk about Hilbert spaces a lot, it is customary to fix the following notation: for any family $(\mathcal{H}_j)_{j\in J}$ of Hilbert spaces over some field in $\{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, we denote the *direct Hilbert space sum* by

$$\widehat{\bigoplus_{j\in J}}\mathcal{H}_j := \Big\{x = \sum_{j\in J} x_j : x_j \in \mathcal{H}_j, \sum_{j\in J} \|x_j\|^2_{\mathcal{H}_j} < \infty\Big\}.$$

Definition 2.3. We call a Hilbert–Lie algebra $\mathfrak{k}$ *semisimple* if it is a direct Hilbert space sum of simple ideals.

We call a Lie algebra *reductive* if it is the direct sum of a semisimple and an abelian Lie algebra.

Any compact Lie algebra is reductive, and by [Sc60], the same is true for general Hilbert–Lie algebras. As an abelian summand in a direct sum only contributes a central factor to any associated Lie group, the (co)adjoint action of this factor is trivial, as is the action of the whole group on any abelian ideal in the Lie algebra. Therefore, as we aim at a description of orbits and invariant sets of the (co)adjoint action, we can safely ignore the abelian summands and factors and restrict our attention to semisimple Lie algebras.

Definition 2.4. For a locally convex Lie group H, an automorphism $\Phi : H \to H$, and some constant $r > 0$, the set

$$C^\infty_{\Phi,r}(\mathbb{R}, H) := \{f \in C^\infty(\mathbb{R}, H) : (\forall t \in \mathbb{R})\ f(t + r) = \Phi(f(t))\}$$

becomes a group with the point-wise multiplication

$$(gh)(t) := g(t)h(t) \ \text{ for } \ g, h \in C^\infty_{\Phi,r}(\mathbb{R}, H), t \in \mathbb{R}.$$

Analogously, for a (real or complex) locally convex Lie algebra $\mathfrak{h}$ and $\varphi \in \mathrm{Aut}(\mathfrak{h})$, we define a Lie algebra as

$$C^\infty_{\varphi,r}(\mathbb{R}, \mathfrak{h}) := \{x \in C^\infty(\mathbb{R}, \mathfrak{h}) : (\forall t \in \mathbb{R})\ x(t + r) = \varphi(x(t))\},$$

endowed with the point-wise defined Lie bracket

$$[x, y](t) := [x(t), y(t)] \ \text{ for } \ x, y \in C^\infty_{\varphi,r}(\mathbb{R}, \mathfrak{h}).$$

In the following, we consider the case of an automorphism Φ of order $N \in \mathbb{N}_0$ and, for $r := {}^{2\pi}/_{N}$ endow the group $C^\infty_{\Phi,r}(\mathbb{R}, H)$ with a group topology which makes it into a Lie group with Lie algebra $C^\infty_{\varphi,r}(\mathbb{R}, \mathfrak{h})$, where $\varphi := \mathbf{L}(\Phi)$ and $\mathfrak{h} := \mathbf{L}(H)$.

Definition 2.5. [GN20, Definition 3.5.1] For two topological spaces X and Y, the *compact open topology* on $C(X, Y)$ is the topology generated by all sets of the form

$$P(L, O) := \{f \in C(X, Y) : f(L) \subset O\}$$

for $L \subset X$ compact and $O \subseteq Y$ open.
When M, N are C^k-manifolds, then on the set of k-times continuously differentiable maps $C^k(M,N)$ the *compact open C^k-topology* is defined as the initial topology with respect to the injection

$$C^k(M,N) \hookrightarrow \prod_{r=0}^{k} C(T^r M, T^r N), \quad f \to (T^r f)_{0 \leqslant r \leqslant k}.$$

Here, T^r denotes the tangent functor applied iteratively r times.
If M and N are actually smooth manifolds, one can consider the *common refinement* of the compact open C^k-topologies for all $k \in \mathbb{N}_0$ on C^∞, i.e. the topology generated by the union of all compact open C^k-topologies. This topology is called the *compact open C^∞-topology.*

Theorem 2.6. [GN20, Theorem 4.4.2] *Let H be a Lie group with Lie algebra $\mathfrak{h}$, M a compact smooth manifold, and $k \in \mathbb{N}_0 \cup \{\infty\}$. Then the compact open C^k-topology makes $C^k(M,H)$ into a Lie group with Lie algebra $C^k(M,\mathfrak{h})$, endowed with the pointwise bracket.*

Definition 2.7. [GN20, Definition 4.6.1] For a Lie group H, a smooth map $\exp_H : \mathbf{L}(H) \to H$ is called *exponential map* if $\mathbb{R} \to H, t \to \exp_H(tx)$ is a smooth one-parameter-group and

$$\frac{\mathbf{d}}{\mathbf{d}t}\Big|_{t=0} \exp_K(tx) = x \text{ for all } x \in \mathbf{L}(H).$$

Note that this implies $T_0(\exp_H) = \mathrm{id}_{\mathbf{L}(H)}$.

H is called *locally exponential* if an exponential map $\exp_H$ exists and restricts to a diffeomorphism on some neighbourhood of $0 \in \mathbf{L}(H)$.

Proposition 2.8. [GN20, Proposition 4.6.12] *If $\varphi : G \to H$ is a morphism of Lie groups with an exponential map, then the following diagram commutes:*

$$\begin{array}{ccc} G & \xrightarrow{\varphi} & H \\ {\scriptstyle \exp_G}\uparrow & & \uparrow{\scriptstyle \exp_H} \\ \mathbf{L}(G) & \xrightarrow{\mathbf{L}(\varphi)} & \mathbf{L}(H) \end{array}$$

In terms of category theory, this property is called naturality.

Proposition 2.9. *If Φ is a Lie group automorphism of a locally exponential Lie group H, then the group of fixed points, H^Φ, is a locally exponential Lie subgroup of H with Lie algebra $\mathbf{L}(H)^{\mathbf{L}(\Phi)}$, the subalgebra of fixed points of the induced automorphism of a topological Lie algebra.*

Proof. We consider the subalgebra

$$\mathbf{L}^e(H^\Phi) := \{x \in \mathbf{L}(H) : \exp_H(\mathbb{R}x) \subset H^\Phi\}.$$

Using the naturality of the exponential map, we find that $\mathbf{L}^e(H^\Phi) = \mathbf{L}(H)^{\mathbf{L}(\Phi)}$; this implies that, if $U \ni 0$ is an open neighbourhood in $\mathbf{L}(H)$, we have

$$\exp_H(\mathbf{L}^e(H^\Phi) \cap U) = H^\Phi \cap \exp_H(U).$$

Now [GN20, Proposition 8.3.11] applies, which tells us that H^Φ is a Lie subgroup of H, and [GN20, Lemma 8.3.8] allows us to canonically identify $\mathbf{L}^e(H^\Phi)$, and therefore $\mathbf{L}(H)^{\mathbf{L}(\Phi)}$, with $\mathbf{L}(H^\Phi)$. □

Theorem 2.10. [GN20, Theorem 4.4.2] *Let H be a Lie group and M a compact smooth manifold. Then, the compact open C^∞-topology makes $C^\infty(M, H)$ into a Lie group with Lie algebra $C^\infty(M, \mathbf{L}(H))$.*

Remark 2.11. By [GN20, Example 4.6.7], if H admits an exponential map $\exp_H$, then there exists an exponential map $\exp_{C^\infty(M,H)}$; it is given by

$$\exp_{C^\infty(M,H)} : C^\infty(M, \mathbf{L}(H)) \to C^\infty(M, H), \quad \exp_{C^\infty(M,H)}(x) := \exp_H \circ x.$$

By [GN20, Example 6.1.4 (b)], if H is locally exponential, then $C^\infty(M, H)$ is locally exponential.

Corollary 2.12. *The compact open C^∞-topology makes $C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H)$ into a Lie group with Lie algebra $C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, \mathbf{L}(H))$.*
If H is locally exponential, then so is $C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H)$.

Proof. $C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H)$ can be identified with $C^\infty(\mathbb{S}^1, H)$, so that Theorem 2.10 and Remark 2.11 apply. □

Lemma 2.13. *If X and Y are topological spaces, and $\rho : X \to X$ a homeomorphism, then the translation operation*

$$\tau_\rho : C(X, Y) \to C(X, Y), \quad \tau_\rho(f) := f \circ \rho$$

is continuous in the compact open topology.

Proof. By Definition 2.5, it suffices to show that the preimage under τ_ρ of every subset of the form $P(L, O) \subset C(X, Y)$ with compact $L \subset X$ and open $O \subset Y$ is open. Clearly, $\tau_\rho^{-1}(P(L, O)) = P(\rho^{-1}(L), O)$, which is open in the compact open topology. □

Lemma 2.14. *For smooth manifolds M and N, and a diffeomorphism ρ of M, the corresponding translation operation*

$$\tau_\rho : C^\infty(M, N) \to C^\infty(M, N), \quad \tau_\rho(f) := f \circ \rho$$

is continuous in the compact open C^∞-topology.

Proof. For every $k \in \mathbb{N}_0$, by functoriality of T, we have

$$T^k(\tau_\rho(f)) = T^k f \circ T^k \rho \text{ for all } f \in C^\infty(M,N).$$

By Definition 2.5, the compact open C^∞-topology is generated by sets of the form $(T^k)^{-1}(P_k)$, where $P_k := P(L_k, O_k)$ with compact $L_k \subset T^k M$ and open $O_k \subset N$. We have to show that the preimage $\tau_\rho^{-1}((T^k)^{-1}(P_k))$ is open for every such set. So:

$$\begin{aligned}\tau_\rho^{-1}((T^k)^{-1}(P_k)) &= \{f \in C^\infty(M,N) : T^k(\tau_\rho(f)) \in P(L_k, O_k)\}\\ &= \{f \in C^\infty(M,N) : T^k f \circ T^k \rho \in P(L_k, O_k)\}\\ &= \{f \in C^\infty(M,N) : T^k f \in P(T^k(\rho^{-1})(L_k), O_k)\}\\ &= (T^k)^{-1}(P(T^k(\rho^{-1})(L_k), O_k));\end{aligned}$$

because ρ, and thus $T^k\rho$, is a diffeomorphism, $T^k(\rho^{-1})(L_k) \subset M$ is compact, which means that $P(T^k(\rho^{-1})(L_k), O_k)$ is open in the compact open topology on $C^\infty(T^k M, T^k N)$, and finally the rightmost term references an open set in the compact open C^∞-topology on $C^\infty(M,N)$ by Definition 2.5. □

Lemma 2.15. *For topological spaces X, Y, and a continuous map $g : Y \to Y$, the map*

$$F_g : C(X,Y) \to C(X,Y), \quad F_g(f) := g \circ f$$

is continuous w.r.t. the compact open topology.

Proof. The preimage $F_g^{-1}(P(L,O))$ of the open set $P(L,O)$ with compact $L \subset X$ and open $O \subset Y$ is $P(L, g^{-1}(O))$, which is again open. Because sets of the form $P(L,O)$ generate the compact open topology, this shows that F_g is continuous. □

Lemma 2.16. *If M and N are smooth manifolds, and $g : N \to N$ a diffeomorphism, then*

$$F_g : C^\infty(M,N) \to C^\infty(M,N), \quad F_g(f) := g \circ f$$

is continuous in the compact open C^∞-topology.

Proof. For $k \in \mathbb{N}_0$, let $L_k \in T^k M$ be compact, $O_k \in T^k M$ be open, and $P_k := P(L_k, O_k)$ as in Definition 2.5. Considering the injection

$$T^k : C^\infty(M,N) \hookrightarrow C^\infty(T^k M, T^k N),$$

the compact open C^∞-topology is generated by sets of the form $(T^k)^{-1}(P_k)$. So, we have to show that the preimage $F_g^{-1}((T^k)^{-1}(P_k))$ is open. We do this by applying the functoriality of T repeatedly, i.e.

$$T^k(F_g(f)) = T^k g \circ T^k f \text{ for all } f \in C^\infty(M,N),$$

thereby obtaining

$$\begin{aligned}F_g^{-1}((T^k)^{-1}(P_k)) &= \{f \in C^\infty(M,N) : T^k g(T^k f(L_k)) \subseteq O_k\}\\ &= \{f \in C^\infty(M,N) : T^k f(L_k) \subseteq T^k(g^{-1})(O_k)\}\\ &= (T^k)^{-1}(P(L_k, T^k(g^{-1})(O_k))).\end{aligned}$$

Analysing the rightmost term, we find that $T^k g$ is continuous by assumption, so that $T^k(g^{-1})(O_k) \subset Y$ is open, and thus $P(L_k, T^k(g^{-1})(O_k))$ is open in the compact open topology on $C^\infty(T^k M, T^k N)$, so that its preimage in $C^\infty(M,N)$ is open by definition. □

Lemma 2.17. *For a locally exponential Lie group H and $\Phi \in \mathrm{Aut}(H)$ of finite order, the map*

$$\widehat{\Phi} : C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H) \to C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H), \quad \widehat{\Phi}(x)(t) := \Phi^{-1}(x(t+r)) \ \textit{for all}\ \ t \in \mathbb{R}$$

is a Lie group automorphism.

Proof. The purely algebraic automorphism property is easy to see. Because H is assumed to be locally exponential, [GN20, Theorem 6.2.4] applies to $\widehat{\Phi}$, asserting that it is smooth if it is continuous. Because $\widehat{\Phi}$ is the composition of the translation operation

$$\tau_r : C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H) \to C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H), \quad \tau_r(x)(t) := x(t+r) \ \text{ for all }\ t \in \mathbb{R}$$

and the pointwise application of Φ^{-1}, we can show continuity of $\widehat{\Phi}$ by showing continuity of both of these operations; this is done by directly applying Lemmas 2.14 and 2.16 above, which completes the proof. □

Proposition 2.18. *For a locally exponential Lie group H, an automorphism Φ of H of finite order $N \in \mathbb{N}$ and $r := 2\pi/N$, the group $C^\infty_{\Phi,r}(\mathbb{R}, H)$ from Definition 2.4 is a locally exponential Lie group with Lie algebra $C^\infty_{\varphi,r}(\mathbb{R}, \mathfrak{k})$, where $\varphi := \mathbf{L}(\Phi)$ and $\mathfrak{k} := \mathbf{L}(K)$.*

Proof. By Lemma 2.17, the map

$$\widehat{\Phi} : C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H) \to C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H), \quad \widehat{\Phi}(f)(t) := \Phi^{-1}(f(t+r)) \ \text{ for all }\ t \in \mathbb{R}$$

is a Lie group automorphism, so Proposition 2.9 applies, which asserts that $C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H)^{\widehat{\Phi}}$ is a Lie subgroup, with Lie algebra $C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, \mathbf{L}(H))^{\mathbf{L}(\widehat{\Phi})}$. Using the naturality of the exponential map, Proposition 2.8, it is easy to see that the tangent Lie algebra automorphism of $\widehat{\Phi}$ is given by

$$\begin{aligned}&\mathbf{L}(\widehat{\Phi}) : C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, \mathbf{L}(H)) \to C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, \mathbf{L}(H)),\\ &\qquad \mathbf{L}(\widehat{\Phi})(x)(t) := \mathbf{L}(\Phi^{-1})(x(t+r)) \ \text{ for all }\ t \in \mathbb{R}.\end{aligned}$$

With this, the identities

$$C^\infty_{\mathrm{id},2\pi}(\mathbb{R}, H)^{\widehat{\Phi}} = C^\infty_{\Phi,r}(\mathbb{R}, H)$$

and

$$C^\infty_{\mathrm{id},2\pi}(\mathbb{R},\mathbf{L}(H))^{\mathbf{L}(\widehat{\Phi})} = C^\infty_{\Phi,r}(\mathbb{R},\mathbf{L}(H))$$

are immediate, which gives us the asserted Lie group structure on $C^\infty_{\Phi,r}(\mathbb{R},H)$. □

Corollary 2.12 and Proposition 2.18 justify the following definition:

Definition 2.19. For a Lie group H, we call the group $C^\infty_{\mathrm{id},2\pi}(\mathbb{R},H)$, equipped with the Lie group structure associated to the compact open C^∞-topology, the (untwisted) *loop group over* H and denote it by $\mathcal{L}H$.
If H is locally exponential, Φ an automorphism of H of order $N \in \mathbb{N}$, and $r := 2\pi/N$, we call the group $C^\infty_{\Phi,r}(\mathbb{R},H)$, equipped with the Lie group structure induced by

$$\mathrm{id} : C^\infty_{\Phi,r}(\mathbb{R},H) \hookrightarrow C^\infty_{\mathrm{id},2\pi}(\mathbb{R},H)$$

the Φ-*twisted loop group over* H and denote it by $\mathcal{L}_\Phi K$.

Likewise, for a topological Lie algebra $\mathfrak{h}$ and $\varphi \in \mathrm{Aut}(\mathfrak{h})$ of finite order N, we define the untwisted *loop algebra over* $\mathfrak{h}$ as $\mathcal{L}\mathfrak{h} := C^\infty_{\mathrm{id},2\pi}(\mathbb{R},\mathfrak{h})$, equipped with the compact open C^∞-topology, and the φ-*twisted* loop algebra as

$$\mathcal{L}_\varphi\mathfrak{h} := C^\infty_{\varphi,r}(\mathbb{R},\mathfrak{h}),$$

equipped with the topology induced by the natural injection

$$\mathrm{id} : C^\infty_{\varphi,r}(\mathbb{R},\mathfrak{h}) \hookrightarrow C^\infty_{\mathrm{id},2\pi}(\mathbb{R},\mathfrak{h}).$$

2.2 Double extensions of loop algebras and the adjoint action

In the following, Definition 1.36 is applied to the case of a Hilbert loop algebra to construct a Lorentzian double extension on it.

Definition 2.20. Let $\mathfrak{k}$ be a Hilbert–Lie algebra with invariant inner product $(\cdot,\cdot)_\mathfrak{k}$. This can be extended to a continuous invariant inner product on $\mathcal{L}\mathfrak{k}$ by

$$(x,y) := \frac{1}{2\pi}\int_0^{2\pi} (x(s),y(s))_\mathfrak{k}\mathbf{d}s \quad \text{for} \quad x,y \in \mathcal{L}\mathfrak{k}; \tag{2.1}$$

as $(\cdot,\cdot)_\mathfrak{k}$ and $(\cdot,\cdot)$ coincide on the constant loops $\mathfrak{k} \hookrightarrow \mathbf{L}\,\mathfrak{k}$, only the notation $(\cdot,\cdot)$ will be used henceforth.

Note that any twisted loop algebra $\mathcal{L}_\varphi\mathfrak{k} \hookrightarrow \mathcal{L}\mathfrak{k}$ inherits this invariant inner product.

We are now ready to define the double extensions of loop groups and algebras, which are needed to obtain an interesting representation theory and convex geometry.

Definition 2.21. We identify the circle group $\mathbb{T}$ with $\mathbb{R}/2\pi\mathbb{Z}$ and, accordingly, write its group multiplication as "+".
Let K be a locally exponential Lie group, $\mathfrak{k} := \mathbf{L}(K)$, and Φ a finite-order-automorphism of K. Then, the *rotation* action $R : \mathbb{T} \times \mathcal{L}_\varphi\mathfrak{k} \to \mathcal{L}_\varphi\mathfrak{k}$ on the loop algebra is defined via

$$R_t(x)(s) := x(s + t)$$

for $x \in \mathcal{L}\mathfrak{k}$ and $s, t \in \mathbb{T}$. The rotation action $R : \mathbb{T} \times \mathcal{L}_\Phi K \to \mathcal{L}_\Phi K$ is defined the same way:

$$R_t(g)(s) := g(s + t)$$

for $g \in \mathcal{L}_\Phi K$ and $s, t \in \mathbb{T}$.

Proposition 2.22. *The rotation actions of $\mathbb{T}$ on $\mathcal{L}_\varphi\mathfrak{k}$ and $\mathcal{L}_\Phi K$ have the following properties:*

(i) *Both, $R : \mathbb{T} \times \mathcal{L}_\varphi\mathfrak{k} \to \mathcal{L}_\varphi\mathfrak{k}$ and $R : \mathbb{T} \times \mathcal{L}_\Phi K \to \mathcal{L}_\Phi K$, are smooth maps.*

(ii) *$R_r \circ \exp_{\mathcal{L}_\Phi K} = \exp_{\mathcal{L}_\Phi K} \circ R_r$ for all $r \in \mathbb{T}$.*

(iii) *The rotation on $\mathcal{L}_\varphi\mathfrak{k}$ is induced by the rotation on $\mathcal{L}_\Phi K$ via the Lie functor.*

(iv) *The infinitesimal generator $\mathbf{d} := \mathbf{L}(R) \in \operatorname{der}(\mathcal{L}\varphi\mathfrak{k})$ of the rotation action $R : \mathbb{T} \rightharpoonup \mathcal{L}_\varphi\mathfrak{k}$ is the differentiation $x \to x'$ of smooth curves.*

Proof. i) is from [MN03, Lemma VI.1], ii) follows directly from the description of the exponential function in Remark (2.11), and (iii) follows from (ii).
For the last point, we can calculate $\mathbf{L}(R)$ directly as differential:

$$\mathbf{L}(R)(x)(s) = \frac{d}{dt}\Big|_{t=0} R_t(x)(s) = \lim_{t\to 0} t^{-1}(x(s) - x(s + t)) = x'(s)$$

for all $x \in \mathcal{L}_\varphi\mathfrak{k}$ and $s \in \mathbb{T}$. ☐

Corollary 2.23. *Considering the semidirect product $\mathcal{L}_\Phi K \rtimes_R \mathbb{T}$ for a finite-order-automorphism Φ of K, the associated Lie algebra is*

$$\mathbf{L}(\mathcal{L}_\Phi K \rtimes_R \mathbb{T}) = \mathcal{L}_\varphi\mathfrak{k} \rtimes_{\mathbf{d}} \mathbb{R},$$

where $\mathbf{d}x := x'$, so that the bracket on $\mathcal{L}_\varphi\mathfrak{k} \rtimes_{\mathbf{d}} \mathbb{R}$, with $\varphi := \mathbf{L}(\Phi)$ reads

$$[(x, s), (y, t)] := ([x, y] + sy' - tx', 0) \tag{2.2}$$

for all $x, y \in \mathcal{L}_\varphi\mathfrak{k}$ and $s, t \in \mathbb{R}$.

With this, we have collected all necessary "building blocks" to finally define the most important objects of our studies:

Definition 2.24. For $\mathfrak{k}$ compact or Hilbert, let us recall the invariant symmetric bilinear form (2.1) on $\mathcal{L}_\varphi\mathfrak{k}$; integration by parts shows that the derivation $\mathbf{d}$ is skew-adjoint, so that

$$\omega(x, y) := (x, \mathbf{d}y) \text{ for all } x, y \in \mathcal{L}_\varphi\mathfrak{k}$$

defines a Lie algebra cocycle $\mathcal{L}_\varphi\mathfrak{k} \times \mathcal{L}_\varphi\mathfrak{k} \to \mathbb{R}$, and

$$\mathfrak{g} := \widehat{\mathcal{L}_\varphi}\mathfrak{k} := (\mathbb{R} \oplus_\omega L_\varphi\mathfrak{k}) \rtimes_{\tilde{\mathbf{d}}} \mathbb{R}$$

is a Lorentzian double extension as by Definition 1.36, equipped with the invariant Lorentzian form κ from Proposition 1.34. The Lie bracket of $\mathfrak{g}$ reads

$$[(a, x, s), (b, y, t)] = (\omega(x, y), [x, y] + sy' - tx', 0)$$

for all $a, b, s, t \in \mathbb{R}$ and $x, y \in \mathcal{L}_\varphi\mathfrak{k}$.

When dealing with central extensions of Lie algebras, the question whether a corresponding Lie group exists arises naturally. However, because the answer to that question requires knowledge about root space decompositions of Hilbert–Lie algebras, this subject will be postponed to the next chapter.
As long as we are only interested in the Lie group corresponding to $\mathfrak{g}$ as far as its adjoint action is concerned, we may ignore the central extension of $\mathcal{L}_\Phi K \rtimes_R \mathbb{T}$ even if it exists, because central group elements are invisible in the adjoint action. In this sense, Lemma 1.21 allows us to evade the existence question when dealing with the adjoint action on central extensions.

Definition 2.25. For a Lie group K, $\mathfrak{k} := \mathbf{L}(K)$ and a differentiable curve $f : \mathbb{R} \to K$, the curve

$$\delta^r(f) : \mathbb{R} \to \mathfrak{k}, \quad \delta^r(f)(t) := f'(t).f(t)^{-1} \tag{2.3}$$

is called the *right logarithmic derivative* of f. Analogously, the *left logarithmic derivative* $\delta^l(f)$ is defined as $f^{-1}.f'$.

With the use of this notation, the full formula for the adjoint action of the identity component $(\mathcal{L}_\Phi K)_0$ on $\mathfrak{g} = (\mathbb{R}\mathbf{c} \times_\omega \mathcal{L}_\varphi\mathfrak{k}) \rtimes_{\mathbf{d}} \mathbb{R}$ reads

$$\operatorname{Ad}(g)(a, x_0, t) = \Big(a - (\delta^r(g), x_0) - \frac{t}{2}(\delta^r(g), \delta^r(g)), \operatorname{Ad}_{\mathfrak{k}}(g)(x_0) - t\delta^r(g), t\Big). \tag{2.4}$$

Modulo the centre, this can be shown by directly computing the induced action $\mathcal{L}_\varphi\mathfrak{k} \rightharpoonup \mathcal{L}_\varphi\mathfrak{k} \rtimes \mathbb{R}\mathbf{d}$; the central component is then obtained by inserting into the formula given in Remark 1.38, with $\zeta(g) = -\delta^r(g)$.
In this context, also note that $\delta^l(g) = -\delta^r(g^{-1})$.

2.3 Twisted conjugation and adjoint orbits

Every compact Lie algebra $\mathfrak{k}$ contains a maximal abelian subalgebra $\mathfrak{t}$, and every corresponding compact Lie group K contains a maximal torus $T = \exp_K(\mathfrak{t})$; in fact, every $x \in \mathfrak{k}$ is contained in some maximal abelian subalgebra, and all these subalgebras are conjugate under the adjoint action of K, see [HN14, Theorem 12.2.2].
In the case of Hilbert–Lie algebras, the situation is very similar:

Definition 2.26. Let $\mathcal{H}$ be an infinite-dimensional real, complex or quaternionic Hilbert space. We denote the space of linear self-maps on $\mathcal{H}$ of Hilbert–Schmidt class, endowed with the commutator bracket, by $\mathfrak{gl}_2(\mathcal{H})$, and set

$$GL_2(\mathcal{H}) := \{g \in GL(\mathcal{H}) : g - \mathrm{id}_{\mathcal{H}} \in \mathfrak{gl}_2(\mathcal{H})\}.$$

For continuous operators g on a Hilbert space there is always a well-defined adjoint $g \to g^*$, and with this we further write

$$U(\mathcal{H}) := \{g \in GL(\mathcal{H}) : g^* = g^{-1}\} \text{ and } \mathfrak{u}(\mathcal{H}) := \{x \in \mathfrak{gl}(\mathcal{H}) : x^* = -x\}.$$

If $\mathcal{H}$ is real, we also use the notation

$$O(\mathcal{H}) := U(\mathcal{H}) \text{ and } \mathfrak{o}(\mathcal{H}) := \mathfrak{u}(\mathcal{H}),$$

and if $\mathcal{H}$ is quaternionic we put

$$Sp(\mathcal{H}) := U(\mathcal{H}) \text{ and } \mathfrak{sp}(\mathcal{H}) := \mathfrak{u}(\mathcal{H})$$

for the orthogonal and symplectic groups and Lie algebras. At last, we set

$$O_2(\mathcal{H}) := O(\mathcal{H}) \cap GL_2(\mathcal{H}), \ U_2(\mathcal{H}) := U(\mathcal{H}) \cap GL_2(\mathcal{H}) \text{ and}$$
$$Sp_2(\mathcal{H}) := Sp(\mathcal{H}) \cap GL_2(\mathcal{H}),$$

as well as

$$\mathfrak{o}_2(\mathcal{H}) := \mathfrak{o}(\mathcal{H}) \cap \mathfrak{gl}_2(\mathcal{H}), \ \mathfrak{u}_2(\mathcal{H}) := \mathfrak{u}(\mathcal{H}) \cap \mathfrak{gl}_2(\mathcal{H}) \text{ and}$$
$$\mathfrak{sp}_2(\mathcal{H}) := \mathfrak{sp}(\mathcal{H}) \cap \mathfrak{gl}_2(\mathcal{H}).$$

Theorem 2.27 (Schue's Theorem). [Ne14, Chapter 1.1] *Every infinite-dimensional simple real Hilbert–Lie algebra is isomorphic to $\mathfrak{u}_2(\mathcal{H})$ for a real, complex or quaternionic Hilbert space $\mathcal{H}$, with the natural inner product*

$$(\cdot,\cdot) : \mathfrak{u}_2(\mathcal{H}) \times \mathfrak{u}_2(\mathcal{H}) \to \mathbb{R}, \ (x, y) := \mathrm{Tr}(xy^*).$$

Remark 2.28. There are spectral theorems for all cases of $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$ which state that for any normal, compact operator x on $\mathcal{H}_{\mathbb{K}}$, which includes normal Hilbert–Schmidt operators, there exists an ONB $\mathcal{B}$ of $\mathcal{H}_{\mathbb{K}}$ which, in a sense, diagonalises x.
The case $\mathbb{K} = \mathbb{C}$ is the most well-known and in some form included in virtually all textbooks on functional analysis, e.g. [Ru73, Theorem 12.22]; in this case, we have the "usual" notion of diagonalisation, i.e. x acts on the Hilbert space

$$\mathcal{H}_{\mathbb{C}}(\mathcal{B}) := \left\{ f \in \mathrm{Map}(\mathcal{B}, \mathbb{C}) : \sum_{b \in \mathcal{B}} |f(b)|^2 < \infty \right\}$$

of square-summable maps as a multiplication operator. Operators which are diagonal with respect to the same ONB form a maximal abelian subalgebra of $\mathfrak{u}_2(\mathcal{H}_{\mathbb{C}})$, which means that:

(CD) For any complex Hilbert space $\mathcal{H}_{\mathbb{C}}$, the maximal abelian subalgebras of $\mathfrak{u}_2(\mathcal{H})$ correspond bijectively to the ONB's of $\mathcal{H}_{\mathbb{C}}$.

The quaternionic case, $\mathbb{K} = \mathbb{H}$, is a slight variation of the previous, details can be looked up in [Vis71, Theorem 3.4]. A "diagonal" operator x in this case is a multiplication operator on the space $\mathcal{H}_{\mathbb{H}}(\mathcal{B}) \subset \mathrm{Map}(\mathcal{B}, \mathbb{H})$ such that x commutes with all operators of the form $\mathcal{H}_{\mathbb{H}}(\mathcal{B}) \to \mathcal{H}_{\mathbb{H}}(\mathcal{B}), f \to lf$, where $l \in \{i, j, k\}$ is one of the fundamental quaternion units. This implies that,

(QD) for any quaternionic Hilbert space $\mathcal{H}_{\mathbb{H}}$, the maximal abelian subalgebras in $\mathfrak{sp}_2(\mathcal{H})$ correspond bijectively to pairs $(\mathcal{B}, l)$ of an ONB and a fundamental quaternion unit.

In [Ba69], $\mathfrak{u}_2(\mathcal{H})$ and $\mathfrak{sp}_2(\mathcal{H})$ are called *standard L^*-algebras of types A and C*; in that paper it is shown that

(QC) All maximal abelian subalgebras of $\mathfrak{u}_2(\mathcal{H}_{\mathbb{C}})$ and $\mathfrak{sp}_2(\mathcal{H}_{\mathbb{H}})$ are conjugate under the conjugation action of $U(\mathcal{H}_{\mathbb{C}})$ or $Sp(\mathcal{H}_{\mathbb{H}})$, respectively.

Remark 2.29. In the case $\mathbb{K} = \mathbb{R}$, the situation is more complicated; the closest we get to a spectral theorem is the following statement, which is transcribed from [AK94, Theorem 2.7]: for any $x \in \mathfrak{o}_2(\mathcal{H}_{\mathbb{R}})$, there exists an x-invariant decomposition of $\mathcal{H}_{\mathbb{R}}$ into either mutually orthogonal planes, or, if $\ker(x)$ is of finite and odd dimension, into mutually orthogonal planes and exactly one line, which then lies in the kernel. x then acts on every plane by an antisymmetric matrix.
To make this more formal, let $(P_j^0)_{j \in J_0}$ denote an orthogonal Hilbert space decomposition of $\mathcal{H}_{\mathbb{R}}$ into planes, i.e. $(P_j^0)_{j \in J_0}$ is a family of mutually orthogonal, 2-dimensional subspaces of $\mathcal{H}_{\mathbb{R}}$ such that

$$\mathcal{H}_{\mathbb{R}} = \widehat{\bigoplus_{j \in J_0}} P_j^0;$$

let further $v_1 \in \mathcal{H}_{\mathbb{R}}$ denote any one vector and $(P_j^1)_{j \in J_1}$ denote an orthogonal Hilbert space decomposition of $v_1^{\perp} \subset \mathcal{H}_{\mathbb{R}}$ into 2-dimensional planes, i.e.

$$\mathcal{H}_{\mathbb{R}} = \mathbb{R} v_1 \oplus \widehat{\bigoplus_{j \in J_1}} P_j^1.$$

Writing the set of real antisymmetric operators on a plane as $\mathfrak{o}(\mathbb{R}^2)$, equipped with the norm $\|A\|_2 := \mathrm{Tr}(AA^*)$ for all $A \in \mathfrak{o}(\mathbb{R}^2)$, an operator $x \in \mathfrak{o}_2(\mathcal{H}_{\mathbb{R}})$ is diagonal with respect to the decomposition $(P_j^0)_{j \in J_0}$ if it respects this decomposition, and hence can be represented by a map $\xi_0 : J_0 \to \mathfrak{o}(\mathbb{R}^2)$ satisfying:

$$x(w) = \sum_{j \in J_0} \xi_0(j)(w_j) \ \text{ for all } \ w = \sum_{j \in J_0} w_j \in \mathcal{H}_{\mathbb{R}},$$

$$\text{and } \sum_{j \in J_0} \|\xi_0(j)\|_2^2 < \infty.$$

Likewise, an operator $y \in \mathfrak{o}_2(\mathcal{H}_\mathbb{R})$ is diagonal with respect to the decomposition $(\mathbb{R}v_1, (P_j^1)_{j\in J_1})$ if it respects $(P_j^1)_{j\in J_1})$ and annihilates v_1, so that we have a map $\xi_1 : J_1 \to \mathfrak{o}(\mathbb{R}^2)$ with

$$y(w) = \sum_{j\in J_1} \xi_1(j)(w_j) \text{ for all } w = rv_1 + \sum_{j\in J_1} w_j \in \mathcal{H}_\mathbb{R},$$

$$\text{and } \sum_{j\in J_1} \|\xi_1(j)\|_2^2 < \infty.$$

Similarly, such a decomposition exists for every $g \in O_2(\mathcal{H}_\mathbb{R})$; for a decomposition of the form $(P_j^0)_{j\in J_0}$, the operator g is diagonal if there is a map $\gamma_0 : J_0 \to O(\mathbb{R}^2)$ such that

$$g(w) = \sum_{j\in J_0} \gamma_0(j)(w_j) \text{ for all } w = \sum_{j\in J_0} w_j \in \mathcal{H}_\mathbb{R},$$

$$\text{and } \sum_{j\in J_0} \|\gamma_0(j) - \mathrm{id}_{\mathbb{R}^2}\|_2^2 < \infty,$$

and for the case of a decomposition $(\mathbb{R}v_1, (P_j^1)_{j\in J_1})$, an operator $h \in O_2(\mathcal{H}_\mathbb{R})$ is diagonal if we have a map $\gamma_2 : J_1 \to O(\mathbb{R}^2)$ satisfying

$$h(w) = rv_1 + \sum_{j\in J_1} \gamma_1(j)(w_j) \text{ for all } w = rv_1 + \sum_{j\in J_1} w_j \in \mathcal{H}_\mathbb{R},$$

$$\text{and } \sum_{j\in J_1} \|\gamma_1(j) - \mathrm{id}_{\mathbb{R}^2}\|_2^2 < \infty.$$

In addition, we get a conjugacy theorem from [Ba69], where the Lie algebras $\mathfrak{o}_2(\mathcal{H}_\mathbb{R})$ are called *standard L^*-algebras of type B*. We summarise these findings so far:

(RD) For every $x \in \mathfrak{o}_2(\mathcal{H}_\mathbb{R})$, and every $g \in O_2(\mathcal{H}_\mathbb{R})$, there is a decomposition of $\mathcal{H}_\mathbb{R}$ of the form $(P_j^0)_{j\in J_0}$ or $(\mathbb{R}v_1, (P_j^1)_{j\in J_1})$ which diagonalises x or g, respectively.

(RC) There are two conjugation classes of maximal abelian subalgebras in $\mathfrak{o}_2(\mathcal{H}_\mathbb{R})$ under the conjugation action of $O(\mathcal{H}_\mathbb{R})$, corresponding to the two types of decompositions above.

Our goal in this chapter is to study the extent to which conjugation theorems like these can be transferred to loop algebras over Hilbert–Lie algebras.

Convention 2.30. From here on, in this chapter we will use the conventions that K denotes a semisimple Hilbert–Lie group and $\mathfrak{k} = \mathbf{L}(K)$ its Lie algebra, as well as $\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi\mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$ a double extended loop algebra with twist φ of order N.

Remark 2.31. The fixed point algebra $\mathfrak{k}^\varphi$ is Hilbert if $\mathfrak{k}$ is, and if $\mathfrak{k}$ is compact, then so is $\mathfrak{k}^\varphi$. As such, they contain maximal abelian subalgebras $\mathfrak{t}_0$, and it is evident that subalgebras of $\mathfrak{g}$ of the form

$$\mathfrak{t}_\mathfrak{g} := \mathbb{R}\mathbf{c} \oplus \mathfrak{t}_0 \oplus \mathbb{R}\mathbf{d}.$$

are maximal abelian.

To prove conjugation theorems in loop algebras over compact Lie algebras, a concept from gauge theory will be adopted, namely the holonomy. For a Hilbert–Lie group K and $\mathfrak{k} := \mathbf{L}(K)$, it will yield maps from loop spaces over $\mathfrak{k}$ to the group K which are equivariant with respect to certain actions of loop groups over K on the loop algebras and K itself. It is introduced here in a very specific form which, admittedly, is not very evocative of its gauge-theoretic origin.

Definition 2.32. By [GN20, Definition 5.1.1], a Lie group H is called *regular*, if for every smooth $f : [0, 2\pi] \to \mathbf{L}(H)$, the initial value problem

$$\delta^l(\gamma) = f, \quad \gamma(0) = \mathbf{1}$$

has a solution γ_f. By [GN20, Theorem 5.3.4], every Banach–Lie group is regular, which in particular includes all Hilbert–Lie groups. This justifies the following definition:

For every $x \in \mathcal{L}_\varphi\mathfrak{k}$, we call the value

$$\mathrm{Hol}(x) := \gamma_x({}^{2\pi}/_{\mathrm{ord}(\varphi)}) \in K$$

the *holonomy* of x.

As $\mathfrak{t}_\mathfrak{g}$ contains $\mathbb{R}\mathbf{c}$, the central component of $x \in \mathfrak{g}$ is irrelevant to the question whether the adjoint orbit $\mathcal{O}_x \subset \mathfrak{g}$ intersects $\mathfrak{t}_\mathfrak{g}$. Also taking into account the invariance of the coefficient of $\mathbf{d}$, see Formula (2.4) after Definition 2.25, we can focus on the setting described by the following definition:

Definition 2.33. By factoring out the central component and restricting Ad to $\mathcal{L}_\Phi K$ and the invariant affine hyperplane $\mathcal{L}_\varphi\mathfrak{k} + \mathbf{d}$, we get an affine action

$$* : \mathcal{L}_\Phi K \curvearrowright \mathcal{L}_\varphi\mathfrak{k}, \quad g * x := \mathrm{Ad}(g)(x + \mathbf{d}) - \mathbf{d};$$

From Formula (2.4) for the adjoint action of loop groups, see Definition 2.25, we obtain the more explicit formula $g * x = \mathrm{Ad}(g)x + \delta^l(g^{-1})$.

Remark 2.34. It can be concluded from its appearance in the formula for the affine action that δ^l has to satisfy a cocycle property; this could also be proven by direct calculation and explicitly reads

$$\delta^l(gh) = \mathrm{Ad}(h^{-1})(\delta^l(g)) + \delta^l(h)$$

for all $g, h \in \mathcal{L}_\Phi K$.

Definition 2.35. Let Φ be an automorphism of some group H; then we call the following (non-automorphic) action of H on itself the Φ-*twisted conjugation*:

$$c^\Phi : H \curvearrowright H, c^\Phi_g(h) := gh\Phi(g^{-1}).$$

Note that this action is smooth if H is a Lie group and Φ a Lie group automorphism.

Lemma 2.36. *The holonomy* Hol : $\mathcal{L}_\varphi\mathfrak{k} \to K$ *has the following equivariance property with respect to the affine action* $*$: $\mathcal{L}_\Phi K \rightharpoondown \mathcal{L}_\varphi\mathfrak{k}$ *and the twisted conjugation* $c^\Phi : K \rightharpoondown K$*:*

$$\mathrm{Hol}(\gamma * x) = c^\Phi_{\gamma(0)}(\mathrm{Hol}(x)). \tag{2.5}$$

Lemma 2.37. *The fibres of* Hol : $\mathcal{L}_\Phi\mathfrak{k} \to K$ *coincide with the affine orbits* $\Omega_\Phi K * x \subset \mathcal{L}_\varphi\mathfrak{k}$*, where*

$$\Omega_\Phi K := \{g \in \mathcal{L}_\Phi K : g(0) = \mathbf{1}\}$$

denotes the normal subgroup of loops based in $\mathbf{1}$.

Proof. Both Lemmas are implications of [Ne14, Proposition 2.14]. □

These Lemmas allow to classify the affine $\mathcal{L}_\Phi K$-orbits in $\mathcal{L}_\varphi\mathfrak{k}$ by means of an 1-to-1 correspondence with "twisted conjugation classes" in K, if we are able to provide a classification of these. This has been done in [Se68] for compact K, and we will cite and employ this classification in the following. However, the proof given by Segal uses fixed point index theory and is therefore not transferable to the infinite dimensional scenario.

Definition 2.38. A topological group S is called *topologically cyclic*, if there exists an $s \in S$ such that the subgroup

$$\mathbb{Z}s := \{s^n \in G : n \in \mathbb{Z}\} = \langle s \rangle$$

is dense in S. Such an element is called a *topological generator*.

Definition 2.39. A *Segal–Cartan subgroup* of a compact Lie group is a topologically cyclic subgroup that has finite index in its normaliser.

Remark 2.40. In [Se68, Definition 1.1], G. Segal called subgroups like this just *Cartan subgroups*, and also already noted that there exists a conflicting notion of Cartan subgroups; as there are currently at least three non-equivalent such notions in use, it makes sense to refer to Segal´s version as defined above.

Note also that, a priory, this definition is independent from the notion of a *Cartan subalgebra*, and is designed to work especially well in the context of non-connected compact Lie groups, which is the application we have in mind here.

Before getting to the twisted conjugation theorem, we have to show that this notion is actually applicable to our setting:

Lemma 2.41. *Let* K *be compact and connected,* $\Phi \in \mathrm{Aut}(K)$ *of finite order* N*, and* $P \subset \mathrm{Aut}(K)$*,* $P := \langle \Phi \rangle \simeq \mathbb{Z}_N$*. If* T_S *is a maximal torus subgroup of* K^Φ*, then* $S := T_S \times P$ *is a Segal–Cartan subgroup of* $K \rtimes P$*.*

Proof. By [Se68, Proposition 1.2], there is a Segal–Cartan subgroup $Q \subset K \rtimes P$ containing Φ and topologically generated by some q_1 in the coset $Q_0\Phi$. Its identity component Q_0 is a torus subgroup of K^Φ, which is also compact; if $T_Q \subset K^\Phi$ is a maximal torus containing Q_0, then every $t \in T_Q$ commutes with both Φ and Q_0, hence with the topological generator $q_1 \in Q_0\Phi$, and thus $T_Q \subset N_K(Q)$. Because $N_K(Q)/Q$ is finite by definition, this shows that $Q_0 = T_Q$ is already maximal in K^Φ.
Now, by [HN12, Theorem 12.2.2], there exists $k \in K^\Phi$ such that $T_S = c_k(Q_0)$, and, because c_k and Φ commute, $S = c_k(Q)$. Thus,

$$N_K(S)/S = c_k(N_K(Q))/c_k(Q) \simeq N_K(Q)/Q$$

is finite, and, for any topological generator $s \in Q$, the element $c_k(s)$ is a topological generator of S, so S is a Segal–Cartan subgroup. □

Proposition 2.42. *[Se68, Proposition 1.4] If C is a compact Lie group, and S a Segal–Cartan subgroup generated by s, then every $h \in C_0 s$ is conjugate to an element of S by some $g \in C_0$.*

Remark 2.43. Note that, in the last proposition, the compact group C is *not* assumed to be connected.

Lemma 2.44. *If K is compact and connected, and $T \subset K^\Phi$ a maximal torus, then every orbit of the twisted conjugation action intersects T.*

Proof. [JN18, Appendix 2] With $N := \mathrm{ord}(\Phi)$, we set $P := \langle \Phi \rangle \simeq \mathbb{Z}_N$ and consider the compact group

$$C := K \rtimes P;$$

for $g, h \in K$ we then have

$$(g, \mathrm{id}_K)(h, \Phi)(g, \mathrm{id}_K)^{-1} = (gh\Phi(g^{-1}), \Phi), \tag{2.6}$$

so that the K-conjugacy classes in the coset $K\Phi$ correspond 1-to-1 to the twisted conjugacy classes in K. By Lemma 2.41,

$$S := T \times P$$

is a Segal–Cartan subgroup. Thus, Proposition 2.42 applies to $S \subset C$, and (2.6) now shows that every $k \in K$ is Φ-twisted conjugate to some $t \in T$. □

Theorem 2.45. *If K is compact and connected, then for every $x \in \mathcal{L}_\varphi\mathfrak{k}$ the affine orbit $\mathcal{L}_\Phi K * x$ intersects any maximal abelian subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}^\varphi$.*

Proof. If $N := \mathrm{ord}(\Phi)$ and $y \in \mathcal{L}_\Phi K$ is a constant loop, then

$$\mathrm{Hol}(y) = \gamma_y\left(\frac{2\pi}{N}\right) = \exp_K\left(y\right);$$

this applies in particular to all elements of $\mathfrak{t}_0 \subset \mathfrak{k}^\varphi$, so that the maximal torus $T_0 := \exp_K(\mathfrak{t}_0) \subset K^\Phi$ is contained in the image of Hol.
By Lemma 2.44, the element $\mathrm{Hol}(x)$ is twisted-conjugate to some $g \in T_0$, so that, by Lemma 2.36, the affine orbit of x intersects the fibre $\mathrm{Hol}^{-1}(\{g\})$. This, in turn, intersects $\mathfrak{t}_0$, and from Lemma 2.37 it follows that $\mathcal{L}_\Phi K * x$ contains $\mathrm{Hol}^{-1}(\{g\})$, and therefore intersects $\mathfrak{t}_0$. □

It has already been mentioned that not all of these methods are available in the infinite-dimensional Hilbert space setting. However, they do apply to the untwisted case, where $\Phi = \mathrm{id}_K$.
In this context, recall Definition 2.26, Theorem 2.27, and the relevant spectral theorems from the beginning of this subsection. By [Ne14, Theorem 1.15], if $\mathfrak{k}$ is infinite-dimensional and simple, all automorphisms of K and $\mathfrak{k}$ can be written as conjugation with elements from $O(\mathcal{H}_\mathbb{R})$, $AU(\mathcal{H}_\mathbb{C})$ or $Sp(\mathcal{H}_\mathbb{H})$, where $AU(\mathcal{H}_\mathbb{C})$ denotes the group of linear and antilinear unitary operators.
Note also that, for $\overline{K}$ defined as $O(\mathcal{H}_\mathbb{R})$, $U(\mathcal{H}_\mathbb{C})$ or $Sp(\mathcal{H}_\mathbb{H})$ respectively, the affine action $* : \mathcal{L}K \rightharpoonup \mathcal{L}\mathfrak{k}$ extends to an affine action

$$* : \mathcal{L}K \rtimes \overline{K} \rightharpoonup \mathcal{L}\mathfrak{k}, \quad (g,k) * x := \mathrm{Ad}(gk)x + \delta^l(g^{-1}).$$

Lemma 2.46. *If K is simple and $\Phi = \mathrm{id}_K$, then $\mathrm{Hol} : \mathcal{L}\mathfrak{k} \to K$ is equivariant with the affine action of $\mathcal{L}K \rtimes \overline{K}$ on $\mathcal{L}\mathfrak{k}$ and its conjugation action on K.*

Proof. Considering any $k \in \overline{K}$ as a constant curve, we can calculate in the ambient Banach space of bounded operators on some appropriate Hilbert space as follows:

$$\delta^l(k\gamma_x k^{-1}) = k\gamma_x^{-1}\gamma_x' k^{-1} = k\delta^l(\gamma_x)k^{-1} = kxk^{-1}$$

for any $x \in \mathcal{L}\mathfrak{k}$, and with that the assertion follows from Definition 2.32. □

Lemma 2.47. *The exponential functions $\exp_{U_2(\mathcal{H}_\mathbb{C})} : \mathfrak{u}_2(\mathcal{H}_\mathbb{C}) \to U_2(\mathcal{H}_\mathbb{C})$ and $\exp_{Sp_2(\mathcal{H}_\mathbb{H})} : \mathfrak{sp}_2(\mathcal{H}_\mathbb{H}) \to Sp_2(\mathcal{H}_\mathbb{H})$ are surjective.*

Proof. We only prove the lemma for $\exp := \exp_{U_2(\mathcal{H}_\mathbb{C})}$, because the proof for the quaternionic case is literally the same.
We already know that every $g \in U_2(\mathcal{H}_\mathbb{C})$ is diagonalisable, i.e., for some ONB $(b_j)_{j\in J}$ of $\mathcal{H}_\mathbb{C}$ we have $g = \mathrm{diag}((g_n)_{n\in J})$ with $g_n \in \mathbb{C}$ and $|g_n| = 1$ for all $n \in J$.
From the surjectivity of $\exp : i\mathbb{R} \to \mathbb{S}^1$ follows that $g_n = \exp(y_n)$ for some family $(y_n)_{n\in J} \subset i\mathbb{R}$, so we have to show that we can choose $(y_n)_{n\in J}$ so that $y := \mathrm{diag}((y_n)_{n\in J}) \in \mathfrak{u}_2(\mathcal{H}_\mathbb{C})$, i.e. that $(y_n)_{n\in J}$ is square-summable.
For this, we note that the restriction $\exp : i(-\pi,\pi] \to \mathbb{S}^1$ is bijective, so that we can choose every y_n from $i(-\pi,\pi]$. Now we consider the function

$$f : [-\pi,\pi] \to \mathbb{R}, \quad f(t) := |\exp(it) - 1|^2 = 2(1-\cos(t))$$

and its derivative: for all $t \in [0,\pi]$ we have $2t \leqslant \pi 2\sin(t) = \pi f'(t)$ and, because $f(0) = 0$, this implies $t^2 \leqslant \pi f(t)$; both sides are symmetric around 0, so this inequality holds for all $t \in [-\pi, \pi]$, and from this we get

$$\sum_{n \in J} |y_n|^2 \leqslant \sum_{n \in J} \pi |\exp(y_n) - 1|^2 = \pi \sum_{n \in J} |g_n - 1|^2 < \infty. \qquad \square$$

Lemma 2.48. *The exponential function* $\exp_{O_2(\mathcal{H}_\mathbb{R})} : \mathfrak{o}_2(\mathcal{H}_\mathbb{R}) \to O_2(\mathcal{H}_\mathbb{R})_0$ *is surjective.*

Proof. W.l.o.g., we may assume that $g \in O_2(\mathcal{H}_\mathbb{R})$ does not have a finite-dimensional fixed point space in $\mathcal{H}_\mathbb{R}$, lest the restriction of g to that space is already trivially diagonal and we can proceed to study the restriction of g to $(\mathcal{H}_\mathbb{R}^g)^\perp \subset \mathcal{H}_\mathbb{R}$.
So, by (RD) in Remark 2.29, there is a g-invariant orthogonal decomposition of $\mathcal{H}_\mathbb{R}$ into planes, on which g acts by 2×2 rotation matrices:

$$SO(\mathbb{R}^2) = \left\{ \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix} \in O(\mathbb{R}^2) : t \in (-\pi, \pi] \right\}$$

and note that the map

$$\mathrm{rot} : (-\pi, \pi] \to SO(\mathbb{R}^2), \quad \mathrm{rot}(t) := \begin{pmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{pmatrix}$$

is a bijection. We also define

$$as : (-\pi, \pi] \to \mathfrak{o}(\mathbb{R}^2), \quad as(t) := \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix},$$

so that we have the following commuting, bijective correspondences:

$$\begin{array}{ccc} & & SO(\mathbb{R}^2) \\ & \overset{\mathrm{rot}}{\nearrow\!\!\swarrow} & \updownarrow \scriptstyle\exp \\ (-\pi, \pi] & \overset{as}{\longleftrightarrow} & \mathrm{im}(as) \subset \mathfrak{o}(\mathbb{R}^2). \end{array}$$

Thus, if we write g as a family $(g_n)_{n \in J}$ of rotation matrices $g_n = \mathrm{rot}(t_n)$ with $t_n \in (-\pi, \pi]$ for all $n \in J$, then we obtain a corresponding family $(as(t_n))_{n \in J}$ satisfying $g_n = \exp_{O(\mathbb{R}^2)}(as(t_n))$. We compare the functions $f : [-\pi, \pi] \to \mathbb{R}$,

$$f(t) := \|as(t)\|^2 = \mathrm{Tr}(as(t)^* as(t)) = 2t^2$$

and $h : [-\pi, \pi] \to \mathbb{R}$,

$$h(t) := \|\mathrm{rot}(t) - \mathrm{id}_{\mathbb{R}^2}\|^2 = \mathrm{Tr}((\mathrm{rot}(t) - \mathrm{id}_{\mathbb{R}^2})^*(\mathrm{rot}(t) - \mathrm{id}_{\mathbb{R}^2})) = 4(1 - \cos(t));$$

Their derivatives are $f'(t) = 4t$ and $h'(t) = 4\sin(t)$, so we have $f' \leqslant \pi h'$ on $[0, \pi]$, and further $f(0) = h(0) = 0$ and symmetry around 0, so we conclude that $f \leqslant \pi h$ on $[-\pi, \pi]$. With that,

$$\sum_{n \in J} \|as(t_n)\|^2 \leqslant \sum_{n \in J} \pi \|\mathrm{rot}(t_n) - \mathrm{id}_{\mathbb{R}^2}\|^2 = \pi \sum_{n \in J} \|g_n - \mathrm{id}_{\mathbb{R}^2}\|^2 < \infty$$

follows, so that $y := \mathrm{diag}_{n \in J}(as(t_n)) \in \mathfrak{o}_2(\mathcal{H}_\mathbb{R})$, the diagonal operator defined by $(as(t_n))_{n \in J}$, satisfies $\exp_{O_2(\mathcal{H}_\mathbb{R})}(y) = g$. $\square$

Theorem 2.49. *If K is a simple connected Hilbert–Lie group, then for every affine orbit $\mathcal{O}_x \subset \mathcal{L}\mathfrak{k}$ of $\mathcal{L}K \rtimes \overline{K}$, there exists a maximal abelian subalgebra of $\mathfrak{k} \hookrightarrow \mathcal{L}\mathfrak{k}$ intersecting $\mathcal{O}_x$.*

Remark 2.50. If $K \simeq U_2(\mathcal{H}_\mathbb{C})$ or $K \simeq Sp_2(\mathcal{H}_\mathbb{H})$ for a complex or quaternionic Hilbert space, then this implies that every affine orbit in $\mathcal{L}\mathfrak{k}$ intersects every maximal abelian subalgebra in $\mathfrak{k} \hookrightarrow \mathcal{L}\mathfrak{k}$; in the case $K \simeq O_2(\mathcal{H}_\mathbb{C})_0$, every affine orbit intersects all maximal abelian subalgebras of one of the two conjugacy classes, see (RC) from Remark 2.29.

Proof. It is immediate from Definition 2.32 that $\mathrm{Hol}(x)$ is contained in the identity component $K_0 \subset K$. For any given $x \in \mathcal{L}\mathfrak{k}$, we employ the appropriate spectral theorem, i.e. (RD), (CD) or (QD) from Remarks 2.28 and 2.29, to diagonalise $\mathrm{Hol}(x)$, which gives us a maximal torus subgroup $T_0 \subset K_0$ containing $\mathrm{Hol}(x)$. From the lemmas 2.48 and 2.47, we know that there exists an $y \in \mathfrak{t}_0 := \mathbf{L}(T_0) \subset \mathfrak{k}$ with $\exp_K(2\pi y) = \mathrm{Hol}(x)$, and from Lemma 2.46 it follows that $y \in \mathcal{O}_x$. □

3 The locally affine structure of Hilbert loop algebras

For finite-dimensional Lie algebras, there is already a well-developed structure theory of invariant convex cones ([HHL89]), complemented by a classification of Lie algebras containing pointed generating invariant convex cones ([Ne94]). This theory revolves mainly around maximal compactly embedded subgroups, the maximal abelian subalgebras in their Lie algebras, and the associated root space decompositions.
To ensure that these methods can be applied in an infinite-dimensional context, we focus on infinite-dimensional Lie algebras with particularly "well-behaved" root space decompositions. The starting point for appropriate generalisations is the compact Lie algebras and the (semi)simple complex Lie algebras. The close tie between these types of objects is standard: every compact Lie group K has a reductive Lie algebra, and a semisimple Lie algebras if the centre of K does not contain a torus, and the complexification of a real semisimple Lie algebra is again semisimple.
These concepts have been generalised in two directions: first, by "keeping" the invariant inner product as in compact Lie algebras, but forfeiting any restrictions to dimensionality, one obtains the category of Hilbert–Lie algebras ([Sc60]); second, the notion of a *root decomposition* can be systematically extended, which leads to the category of "extended affine Lie algebras" ([AA97]), of which the Kac–Moody algebras ([Ka83]) are the most well-studied subclass due to the completeness of their classification and their accessible representation theory. For us, their important feature is the existence of certain isomorphisms with an algebraic version of loop algebras, to be expounded later.
From the perspective of representation theory, these two concepts exhibit similarities which have been made rigorous in [Ne10]; the key notion for this reunification of concepts is that of a *locally affine Lie algebra*, which is in turn a special case of a *locally extended affine Lie algebra*, which has been introduced in [MY06].

3.1 Integrable roots and locally finite root systems

This subsection briefly introduces the elemental concepts of root space decompositions; integrable roots correlate to subalgebras isomorphic to $\mathfrak{sl}(\mathbb{C}^2)$ and to real subalgebras isomorphic to $\mathfrak{su}(\mathbb{C}^2)$, which makes the representation theory of these Lie algebras available as a tool to study Hilbert loop algebras. The description of locally finite root systems comes mainly as an example; they are of interest because they correspond to the Hilbert–Lie algebras which are the starting point of our construction, and also represent the subalgebras of constant loops.

Definition 3.1. [Ne14] A subalgebra $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ is called a *splitting Cartan subalgebra* if it is maximal abelian and the representation $\mathrm{ad}|_{\mathfrak{h}} : \mathfrak{h} \rightharpoonup \mathfrak{g}$ is diagonalisable.
A Lie algebra $\mathfrak{g}$ which contains a splitting Cartan subalgebra is called a *split*

Lie algebra, and it has a *root space decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha; \tag{3.1}$$

here, $\Delta := \Delta(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}' \backslash \{0\}$, the subspaces

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} : (\forall h \in \mathfrak{h})[h, x] = \alpha(h)x\}$$

are the *root spaces*, and

$$\Delta(\mathfrak{g}, \mathfrak{h}) := \{\alpha \in \mathfrak{h}' \backslash \{0\} : \mathfrak{g}_\alpha \neq \{0\}\}$$

is called the *root system.*

A root α is called *integrable*, if there exist $x_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$, such that $\alpha([x_\alpha, x_{-\alpha}]) \neq 0$ and $\mathrm{ad}(x_{\pm\alpha})$ are *locally nilpotent*, i.e. for every $y \in \mathfrak{g}$ the sequence $(\mathrm{ad}(x_{\pm\alpha})^n y)_{n \in \mathbb{N}}$ has only finitely many non-zero members.

The set of integrable roots is usually denoted by $\Delta_i \subset \Delta$.

Lemma 3.2. *For every integrable root α of the split Lie algebra $\mathfrak{g}$, the root spaces $\mathfrak{g}_{\pm\alpha}$ are 1-dimensional.*

The proof for this can be found in [Ne00b, Proposition I.6].

Definition 3.3. Thus we obtain a subalgebra $\mathfrak{g}(\alpha) := [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \oplus \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ which is isomorphic to $\mathfrak{sl}_2$. Further, for a set of integrable roots $\Pi \subset \Delta_i$, let $\mathfrak{g}(\Pi)$ denote the subalgebra generated by the union of $\mathfrak{g}(\alpha), \alpha \in \Pi$.

The unique element $\check{\alpha} \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ which satisfies $\alpha(\check{\alpha}) = 2$ is called the *coroot* of α.

Δ_i is called *connected* if, for every pair $\alpha, \beta \in \Delta_i$, there exists a subset $\{\alpha_k : k \in \{0, 1 \ldots, m\}\} \subset \Delta_i$ such that $\alpha_0 = \alpha, \alpha_m = \beta$ and $\alpha_{k-1}(\check{\alpha_k}) \neq 0$ for all $k \in \{1 \ldots, m\}$.

The subalgebra $\mathfrak{g}_c := \mathfrak{g}(\Delta_i) < \mathfrak{g}$ is called the *core* of $\mathfrak{g}$, and a split Lie algebra whose root spaces are all generated by the root spaces of integrable roots, i.e. $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_c$, is called *coral.*

Definition 3.4. A pair $(\mathfrak{g}, \kappa)$ of a Lie algebra $\mathfrak{g}$ and an invariant symmetric non-degenerate bilinear form κ on $\mathfrak{g}$ is called a *quadratic* Lie algebra.

Remark 3.5. [Ne14] We list some fundamental algebraic properties of the root systems of split quadratic Lie algebras: Let $\mathfrak{g}$ be a split Lie algebra and κ be an invariant symmetric non-degenerate bilinear form on $\mathfrak{g}$. Let further $\alpha, \beta \in \Delta$.

(1) For $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ and $h \in \mathfrak{h}$, we get

$$\alpha(h)\kappa(x, y) = \kappa([h, x], y) = -\kappa(x, [h, y]) = -\beta(h)\kappa(x, y),$$

so either $\alpha = -\beta$ or $\kappa(x, y) = 0$. If we consider another element $h_2 \in \mathfrak{h}$, then the analogous calculation yields

$$\alpha(h_2)\kappa(x, h) = -\kappa(x, [h_2, h]) = 0,$$

which implies $\kappa(x, h) = 0$. Thus, the root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_\beta$ are orthogonal if $\alpha + \beta \neq 0$, and all root spaces are orthogonal to $\mathfrak{h}$.
In particular, $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, and $\mathfrak{g}_\alpha$ is non-degenerately paired with $\mathfrak{g}_{-\alpha}$.

(2) For every $h \in \mathfrak{h}$ and non-zero $x_\alpha \in \mathfrak{g}_\alpha$ and $x_{-\alpha} \in \mathfrak{g}_{-\alpha}$ we get

$$\alpha(h)\kappa(x_\alpha, x_{-\alpha}) = \kappa([h, x_\alpha], x_{-\alpha}) = \kappa(h, [x_\alpha, x_{-\alpha}]),$$

and because of (1), we have $r := \kappa(x_\alpha, x_{-\alpha}) \neq 0$, so that $\alpha(h)$ can be expressed as $\kappa(h, x)$, where $x = r^{-1}[x_\alpha, x_{-\alpha}] \in \mathfrak{h}$, which, together with the non-degeneracy of κ, implies $\alpha \in \mathfrak{h}^\flat$ and

$$[x_\alpha, x_{-\alpha}] = \kappa(x_\alpha, x_{-\alpha})\alpha^\sharp.$$

(3) Note that the preceding paragraph implies that, for all $\beta \in \Delta_i$, the conditions $\alpha(\check{\beta}) \neq 0$ and $(\alpha, \beta) \neq 0$ are equivalent. So, if Δ_1 and Δ_2 are distinct connected components in Δ_i, then $(\alpha, \beta) = 0$ for $\alpha \in \Delta_1$, $\beta \in \Delta_2$. Conversely, if $\Delta_i = \Delta_1 \dot{\cup} \Delta_2$, and we assume that $\alpha_0 \in \Delta_1$ and $\alpha_n \in \Delta_2$ are connected, we conclude that the connecting chain $\alpha_0, \ldots, \alpha_n$ has an index $0 \leqslant k \leqslant n$ with $\alpha_k \in \Delta_1$ and $\alpha_{k+1} \in \Delta_2$, and therefore

$$0 \neq (\alpha_k, \alpha_{k+1}) \in (\Delta_1, \Delta_2).$$

In particular, Δ_i is connected if and only if it is not decomposable into non-empty mutually orthogonal subsets.

Definition 3.6. Let V be a real topological vector space, $\alpha \in V'$ a continuous linear functional and $x \in V$ such that $\alpha(x) = 2$. Then the *reflection* on the hyperplane $\ker(\alpha)$ in the direction x is defined as $\sigma(v) := v - \alpha(v)x$ for all $v \in V$.
If, in the above definition, V carries a symmetric bilinear form $(\cdot, \cdot)$ satisfying $(x, x) \neq 0$ and $\alpha(v) = 2\frac{(v,x)}{(x,x)}$ for all $v \in V$, then the reflection is denoted by σ_x.

Definition 3.7. [LN04, Definition 3.3] A pair (V, Δ) consisting of a real pre-Hilbert space V and a subset $\Delta \subset V$ satisfying the conditions

(i) $\sigma_\alpha(\Delta) = \Delta$,

(ii) $\Delta(\check{\alpha}) \subset \mathbb{Z}$, where $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}$,

(iii) $\mathrm{span}_{\mathbb{R}}(\Delta) = V$,

(iv) $\mathbb{R}\alpha \cap \Delta = \{\alpha, -\alpha\}$,

for all $\alpha \in \Delta$ is called a *locally finite root system*.
Note that in [LN04] the condition (iv) was not required for all locally finite root systems and those satisfying it were called *reduced* locally finite root systems.
If Δ is finite (equivalently, if V is finite-dimensional) then (V, Δ) is called a *finite root system*.

Example 3.8. Let $\mathfrak{k}$ be a compact semisimple Lie algebra, and $\mathfrak{k}^{\mathbb{C}} := \mathfrak{k} \oplus i\mathfrak{k}$ its complexification; then there exists a maximal abelian subalgebra $\mathfrak{t}^{\mathbb{C}} \subseteq \mathfrak{k}^{\mathbb{C}}$ such that $\mathfrak{t} := \mathfrak{k} \cap \mathfrak{t}^{\mathbb{C}}$ is maximal abelian in $\mathfrak{k}$ and a root space decomposition of $\mathfrak{k}^{\mathbb{C}}$ with respect to $\mathfrak{t}^{\mathbb{C}}$ and some root system $\Delta \subset (\mathfrak{t}^{\mathbb{C}})^*$.
In this situation, $i\mathfrak{t}' = \operatorname{span}_{\mathbb{R}}(\Delta)$ carries an inner product pushed forward from $\mathfrak{t}$, and the pair $(i\mathfrak{t}', \Delta)$ is a finite root system.
$\mathfrak{k}^{\mathbb{C}}$ is simple if and only if Δ does not decompose into two proper mutually orthogonal subsets.
Accessible proofs of these claims can be found in [HN12, Chapters 6.3, 6.4 and 12.2]. By Serre's Theorem, this example actually exhausts the class of finite root systems, see [HN12, Chapter 7.2.3].

In [Sc60] and [Sc61], John R. Schue has shown that all separable simple Hilbert–Lie algebras contain a dense simple split subalgebra, and has classified these subalgebras by their root decompositions. In [St99], this has been extended to all Hilbert–Lie algebras. To summarise these findings, we recall Definition 2.26 and Theorem 2.27 for the classification of Hilbert–Lie groups as groups of orthogonal, unitary or symplectic operators.

Remark 3.9. For a (skew-) field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$, let $\mathcal{B} := (e_j)_{j \in J}$ be an orthonormal Hilbert space basis of a $\mathbb{K}$-Hilbert space $\mathcal{H}_{\mathbb{K}}$ and consider the Lie algebra $\mathfrak{k}_{\text{fin}}$ of skew-adjoint $J \times J$ matrices with only finitely many nonzero entries; its complexification $\mathfrak{k}_{\text{fin}}^{\mathbb{C}}$ admits a root space decomposition

$$\mathfrak{k}_{\text{fin}}^{\mathbb{C}} = \mathfrak{t}_{\text{fin}}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{k}_\alpha,$$

where $\mathfrak{t}_{\text{fin}}$ are the diagonal matrices in $\mathfrak{k}_{\text{fin}}$. Note that $\mathfrak{k}_{\text{fin}}$ is a dense subalgebra of the real Hilbert–Lie algebra $\mathfrak{k} := \mathfrak{u}_2(\mathcal{H}_{\mathbb{K}})$ of skew-adjoint Hilbert–Schmidt operators. Thus $\mathfrak{k}_{\text{fin}}^{\mathbb{C}}$ and $\mathfrak{t}_{\text{fin}}^{\mathbb{C}}$ are also dense in $\mathfrak{k}^{\mathbb{C}}$ respectively $\mathfrak{t}^{\mathbb{C}}$ with respect to the complex Hilbert space structure obtained as the hermitian extension of the inner product from $\mathfrak{k}$.
In the case of $\mathbb{K} = \mathbb{C}$ or $\mathbb{K} = \mathbb{H}$, $\mathfrak{t} \subset \mathfrak{u}_2(\mathcal{H}_{\mathbb{K}})$ is the subalgebra of skew-adjoint diagonal operators on $\mathcal{H}_{\mathbb{K}}$ with respect to $\mathcal{B}$, see (CD) and (QD) in Remark 2.28, and for every $j \in J$ we can define $\varepsilon_j : \mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ as the unique linear functional satisfying

$$x(e_j) = \varepsilon_j(x) e_j \ \text{ for all } \ x \in \mathfrak{t}^{\mathbb{C}}.$$

On the other hand, in the case $\mathbb{K} = \mathbb{R}$, $\mathfrak{t}$ consists of the block-diagonal operators with skew-symmetric 2×2-blocks. There still exists a basis $\mathcal{B}' := (f_j)_{j \in J}$ of

$\mathcal{H}_{\mathbb{R}}^{\mathbb{C}}$ which simultaneously diagonalises $\mathfrak{t}^{\mathbb{C}}$. With this, we can define functionals ε_j as above on $\mathfrak{t} \subset \mathfrak{u}_2(\mathcal{H}_{\mathbb{R}})$ satisfying

$$x(f_j) = \varepsilon_j(x) f_j.$$

Then, the root system Δ equals one of the following (see [Ne14, Examples 1.10, 1.12 and 1.13]):

$$\begin{aligned} A_J &:= \{\varepsilon_j - \varepsilon_k : j,k \in J, j \neq k\}, \\ B_J &:= \{\pm\varepsilon_j, \pm\varepsilon_j \pm \varepsilon_k : j,k \in J, j \neq k\}, \\ C_J &:= \{\pm\varepsilon_j \pm \varepsilon_k : j,k \in J\}, \\ D_J &:= \{\pm\varepsilon_j \pm \varepsilon_k : j,k \in J, j \neq k\}. \end{aligned} \tag{3.2}$$

The root systems of the form A_J correspond to unitary Lie algebras, i.e. $\mathbb{K} = \mathbb{C}$, type C_J to symplectic Lie algebras (the case $\mathbb{K} = \mathbb{H}$), and B_J and D_J both to orthogonal Lie algebras of operators on real Hilbert spaces. The difference between D_J and B_J is that, in the latter case, $\mathfrak{t}$ has a nontrivial (1-dimensional) common kernel in $\mathcal{H}_{\mathbb{R}}$.

Remark 3.10. Together with Theorem 2.27, the previous remark implies that every semisimple Hilbert–Lie algebra has a dense subalgebra which admits a root space decomposition; the Cartan subalgebra is the direct sum of the Cartan subalgebras of its simple ideals, and its root system the disjoint union of their root systems.

This classification allows us to single out a specific inner product for any Hilbert–Lie algebra, which in the following will be employed to obtain Lie groups corresponding to double extensions of Hilbert loop algebras.

Definition 3.11. (see [Ne14, Definition 3.3 and Remark]) For a simple Hilbert–Lie algebra $\mathfrak{k}$, the *normalized inner product* is defined as the inner product with $(\check{\alpha}, \check{\alpha}) = 2$ for all *long* roots $\alpha \in \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, i.e. the roots with *minimal* $(\check{\alpha}, \check{\alpha})$. Specifically, if $\mathfrak{k} = \mathfrak{u}_2$ or $\mathfrak{k} = \mathfrak{sp}_2$, then $\operatorname{tr}_{\mathbb{C}}(xy^*)$ for all $x, y \in \mathfrak{k}^{\mathbb{C}}$ defines the normalized inner product, and $\frac{1}{2}\operatorname{tr}_{\mathbb{C}}(xy^*)$ defines the normalized inner product in the case $\mathfrak{k} = \mathfrak{o}_2$.

Theorem 3.12. *[Ne14, Theorem 3.4] Let K be a 1-connected simple Hilbert–Lie group and let $(\cdot,\cdot)$ be the normalized inner product on $\mathfrak{k} := \mathbf{L}(K)$. Then the cocycle*

$$\omega(x,y) := \frac{1}{2\pi}\int_0^{2\pi} (x'(t), y(t))\mathbf{d}t \;\; \textit{for all} \;\; x,y \in \mathcal{L}_\varphi\mathfrak{k}$$

integrates to a locally smooth Lie group cocycle $\Omega : \mathcal{L}_\Phi K \times \mathcal{L}_\Phi K \to \mathbb{T}$ such that the smooth central extension $\mathbb{T} \oplus_\Omega \mathcal{L}_\Phi K$ is locally exponential and satisfies $\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi\mathfrak{k} = \mathbf{L}(\mathbb{T} \oplus_\Omega \mathcal{L}_\Phi K)$.

Proposition 3.13. *For a 1-connected Hilbert-Lie group K and a normalized inner product on $\mathfrak{k} = \mathbf{L}(K)$, and with $R := \exp(i\mathbb{R}\mathbf{d}) \simeq \mathbb{T}$ and the cocycle*

$\Omega : \mathcal{L}_\Phi K \times \mathcal{L}_\Phi K \to \mathbb{T}$ *from Theorem 3.12, the smooth double extension of Lie groups*

$$G := (\mathbb{T} \oplus_\Omega \mathcal{L}K) \rtimes R$$

satisfies $\mathbf{L}(G) = \mathfrak{g} = (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi \mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$.

Proof. By [MN03, Thmeorem V.9], the rotation action of $\mathbb{R}$ on $\mathcal{L}_\Phi K$ lifts to a smooth action on the central extension $\mathbb{T} \oplus_\Omega \mathcal{L}K$. □

3.2 Root space decomposition of loop algebras

With the basic notions and properties of the locally finite root systems of Hilbert–Lie algebras established, we are now going to apply them to describe the structure and classification of twisted Hilbert loop algebras.
In the course of this section, the root systems of Hilbert loop algebras will be described and linked to their algebraic counterparts, the *affine* and *locally affine root systems*; these provide sufficient information for their application to convex and, in particular, *Coxeter* geometry.

Convention 3.14. Any automorphism of finite order $N \in \mathbb{N}$ of a complex vector space V is diagonalisable and there exists a primitive N-th root of unity $\sigma := e^{\frac{2\pi}{N}} \in \mathbb{T}$ such that every eigenvalue is of the form σ^k for some $k \in \mathbb{Z}$. With that in mind, all eigenvalues of finite-order automorphisms will be denoted by σ^k from here on, and the corresponding eigenspaces by $V_k := \{x \in V : \varphi(x) = \sigma^k x\}$. Note that $\sigma^k = \sigma^{k+mN}$ for all $m \in \mathbb{Z}$, and thus $V_k = V_{k+mN}$, so the indices denote congruence classes rather than integers in this context.

Definition 3.15. We will use the abbreviation

$$e_k := \mathbb{R} \to \mathbb{S}^1,\ e_k(t) := e^{ikt} \text{ for } k \in \mathbb{Z};$$

because these maps are $1/k$-periodic, we will mostly interpret them as maps $\mathbb{S}^1 \to \mathbb{S}^1$ and use them to write "trigonometric monomials".

A loop $x \in \mathcal{L}\mathfrak{k}$, where $\mathfrak{k}$ is a Hilbert–Lie algebra, is called a *trigonometric polynomial* if there exists a number $n \in \mathbb{N}$ and a tuple $(y_k)_{-n \leqslant k \leqslant n} \subset \mathfrak{k}$ such that

$$x(t) = \sum_{k=-n}^{n} e_k(t) y_k \text{ for all } t \in \mathbb{R}.$$

The set of all trigonometric polynomials with values in $\mathfrak{k}$ is denoted by $\mathcal{L}^{\mathrm{pol}}\mathfrak{k}$; we also write $\mathcal{L}^{\mathrm{pol}}_\varphi \mathfrak{k} := \mathcal{L}_\varphi \mathfrak{k} \cap \mathcal{L}^{\mathrm{pol}}\mathfrak{k}$.

Lemma 3.16. *For a complex Hilbert–Lie algebra* $\mathfrak{k}$ *and* $\varphi \in \mathrm{Aut}(\mathfrak{k})$ *of finite order,* $\mathcal{L}^{\mathrm{pol}}_\varphi \mathfrak{k}$ *is dense in* $\mathcal{L}_\varphi \mathfrak{k}$.

Proof. As smooth functions, $x \in \mathcal{L}\mathfrak{k}$ and all its derivatives are particularly of bounded variation, and $\mathfrak{k}$ is in particular a Banach space, so the corollary on

[Ka04, p.62] applies, which means that x can be approximated uniformly in all derivatives by trigonometric polynomials $x_n(t) = \sum_{k=-n}^{n} e_k(t)y_k$, where $y_k \in \mathfrak{k}^{\mathbb{C}}$. With $N := \mathrm{ord}(\varphi)$ and $r := 2\pi/N$, we specify an automorphism $\widehat{\varphi}$ of $\mathcal{L}\mathfrak{k}$ by

$$\widehat{\varphi}(x)(t) := \varphi^{-1}(x(t+r)) \text{ for all } t \in \mathbb{R}. \tag{3.3}$$

Obviously, $(\mathcal{L}\mathfrak{k})^{\widehat{\varphi}} = \mathcal{L}_\varphi\mathfrak{k}$; it is also clear that a monomial $e_k y_k$ is invariant under $\widehat{\varphi}$ if and only if $y_k \in \mathfrak{k}_k$ (see Convention 3.14), and that $\widehat{\varphi}$ does not change the degree of any monomial. Thus a polynomial $p \in \mathcal{L}\mathfrak{k}, p := \sum_{k=m}^{n} e_k p_k$ is invariant if and only if all its monomials are, i.e. $p \in \bigoplus_{k\in\mathbb{Z}}(e_k\mathfrak{k}_k)$.
According to the last theorem on [Ka04, p.103], the convergence of the Fourier series is absolute at every point $t \in \mathbb{R}$, so $\widehat{\varphi}(x) = \sum_{k=-\infty}^{\infty} \widehat{\varphi}(e_k c_k)$.
Therefore, every partial sum of the Fourier series of a $\widehat{\varphi}$-invariant $x \in \mathcal{L}\mathfrak{k}$ is itself $\widehat{\varphi}$-invariant, which proves the claim. □

Definition 3.17. A *real form* $\mathfrak{a}^{\mathbb{R}}$ a of a complex Lie algebra $\mathfrak{a}$ is a *real* subalgebra of $\mathfrak{a}$ such that $\mathfrak{a} = \mathfrak{a}^{\mathbb{R}} \oplus i\mathfrak{a}^{\mathbb{R}}$.
An *antilinear involution* is a map $* : \mathfrak{a} \to \mathfrak{a}$, $x \to x^*$ which, for all $a \in \mathbb{C}$ and $x, y \in \mathfrak{a}$, satisfies

(i) $(ax + y)^* = \overline{a}x^* + y^*$,

(ii) $[x, y]^* = -[x^*, y^*]$, and

(iii) $*^2 = \mathrm{id}_{\mathfrak{a}}$.

Remark 3.18. Every real form $\mathfrak{a}^{\mathbb{R}}$ in $\mathfrak{a}$ gives rise to an antilinear involution by writing elements $z \in \mathfrak{a}$ as $z \simeq (x, iy) \in \mathfrak{a}^{\mathbb{R}} \oplus i\mathfrak{a}^{\mathbb{R}}$ and defining

$$* : \mathfrak{a} \to \mathfrak{a}, \quad (x + iy)^* := -x + iy.$$

This satisfies $\mathfrak{a}^{\mathbb{R}} = \{z \in \mathfrak{a} : z^* = -z\}$. Vice versa, every antilinear involution defines a real form $\mathfrak{a}^{\mathbb{R}} \subset \mathfrak{a}$ as $\mathfrak{a}^{\mathbb{R}} := \{z \in \mathfrak{a} : z^* = -z\}$.
Now if we start with some real Lie algebra $\mathfrak{r}$, and $*$ is the antilinear involution associated to the real form $\mathfrak{r}$ of the complexification $\mathfrak{r}^{\mathbb{C}} := \mathfrak{r} \oplus i\mathfrak{r}$, then $*$ extends pointwise to an antilinear involution of $\mathcal{L}_\Phi(\mathfrak{r}^{\mathbb{C}})$, with fixed point set $\mathcal{L}_\Phi\mathfrak{r}$.
This means that the complexification of a loop algebra $\mathcal{L}_\Phi\mathfrak{r}$ over a real Lie algebra $\mathfrak{r}$ is the loop algebra $\mathcal{L}_\Phi(\mathfrak{r}^{\mathbb{C}})$.

Definition 3.19. ([Ne10, Definition 3.17]) Let $\mathfrak{s}$ be a complex split quadratic Lie algebra with root system Δ and bilinear form κ; then an antilinear involution on $\mathfrak{s}$ and its corresponding real form $\mathfrak{s}^{\mathbb{R}}$ are called a *unitary real form* if:

(i) $\alpha(x) \in \mathbb{R}$ for all roots $\alpha \in \Delta$ and $x = x^* \in \mathfrak{s}$.

(ii) $(\mathfrak{s}_\alpha)^* = \mathfrak{s}_{-\alpha}$ for all $\alpha \in \Delta$.

(iii) $\kappa(x^*, y^*) = \overline{\kappa(x, y)}$ for all $x, y \in \mathfrak{s}$.

Proposition 3.20. *If $\mathfrak{k}^{\mathbb{C}}$ is the complexification of a Hilbert–Lie algebra with a root space decomposition with respect to $\Delta := \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, then the antilinear involution $*$ from Remark 3.18 and the bilinear extension of the inner product from $\mathfrak{k}$ to $\mathfrak{k}^{\mathbb{C}}$ are a unitary real form of $\mathfrak{k}^{\mathbb{C}}$.*

Proof. (i) This is because the ad-invariance of κ implies that the elements of $\mathfrak{t}$ act as skew-symmetric operators, and thus have only imaginary eigenvalues. The elements satisfying $x = x^*$ are the ones in $i\mathfrak{t}$, therefore having real eigenvalues.

(ii) For all $h \in \mathfrak{t}$ and $x \in \mathfrak{k}_\alpha^\mathbb{C}$ we have

$$[h, x^*] = -[h^*, x]^* = [h, x]^* = \overline{\alpha(h)}x^*,$$

and if, instead, $h \in i\mathfrak{t}$ then $[h, x^*] = -\overline{\alpha(h)}x^*$, so that ii) follows from i).

(iii) For $x = x_0 + ix_1$ and $y = y_0 + iy_1 \in \mathfrak{k}^\mathbb{C}$ we get

$$\begin{aligned}\kappa(x^*, y^*) &= \kappa(-x_0 + ix_1, -y_0 + iy_1)\\ &= \kappa(x_0, y_0) - \kappa(x_1, y_1) - i(\kappa(x_0, y_1) + \kappa(x_1, y_0)) = \overline{\kappa(x, y)}.\end{aligned}$$ □

Remark 3.21. For any semisimple Hilbert–Lie algebra $\mathfrak{k}$ and $\Phi \in \mathrm{Aut}(\mathfrak{k})$ of finite order N, the subalgebra $\mathfrak{k}_0 = \mathfrak{k}^\varphi$ is non-trivial ([Ne14, Lemma D.1]). For every maximal abelian subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}_0$, its centraliser $\mathfrak{t} := \mathfrak{z}_\mathfrak{k}(\mathfrak{t}_0)$ is maximal abelian in $\mathfrak{k}$ ([Ne14, Lemma D.2]).
Because $\mathrm{ad}(\mathfrak{t}_0)$ and Φ commute, they can be diagonalised simultaneously over $\mathfrak{k}^\mathbb{C}$, which means that $\mathfrak{k}^\mathbb{C}$ has a simultaneous weight space decomposition with respect to $\mathfrak{t}_0^\mathbb{C}$ and Φ (Convention 3.14):

$$\mathfrak{k}^\mathbb{C} = \mathfrak{t}_0^\mathbb{C} \oplus \bigoplus_{0 \leqslant k \leqslant N-1} \Big(\widehat{\bigoplus_{\alpha \in \Delta_k}} \mathfrak{k}_{\alpha,k}^\mathbb{C} \Big).$$

Here, the sets of weights Δ_k are subsets of $\Delta := \Delta(\mathfrak{k}, \mathfrak{t})|_{\mathfrak{t}_0^\mathbb{C}} \cup \{0\}$, defined by

$$\begin{aligned}&\Delta_k := \{\alpha \in \Delta : \mathfrak{k}_\alpha^\mathbb{C} \cap \mathfrak{k}_k^\mathbb{C} \neq \{0\}\} \text{ for } 1 \leqslant k \leqslant N-1 \text{ and}\\ &\Delta_0 := \{\alpha \in \Delta : \alpha \neq 0, \mathfrak{k}_\alpha^\mathbb{C} \cap \mathfrak{k}_k^\mathbb{C} \neq \{0\}\}\end{aligned}$$

According to [Ne14, Appendix D], the weight spaces $\mathfrak{k}_\alpha^\mathbb{C} \cap \mathfrak{k}_k^\mathbb{C}$ are 1-dimensional.

Convention 3.22. From here on, we resume to use Convention 2.30, in particular $\mathfrak{k}$ denotes a semisimple real Hilbert–Lie algebra, and $\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi \mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$ a double extended loop algebra with twist Φ of order N over $\mathfrak{k}$.
We further denote any maximal abelian subalgebra of $\mathfrak{g}$ which contains $\mathbf{d}$ by $\mathfrak{t}_\mathfrak{g}$. It equals $\mathbb{R}\mathbf{c} \oplus \mathfrak{t}_0 \oplus \mathbb{R}\mathbf{d}$ for a maximal abelian subalgebra $\mathfrak{t}_0 \subset \mathfrak{k}^\varphi$.

Proposition 3.23. *The complexification $\mathfrak{g}^\mathbb{C}$ has a root space decomposition with respect to $\mathfrak{t}_\mathfrak{g}^\mathbb{C}$. With the notation of the preceding remark and $\delta \in \mathbb{C}\mathbf{d}^*$ defined by $\delta(d) := i$, the subset of integrable roots $\Delta_i \subset \Delta$ equals:*

$$\Delta_i = \dot{\bigcup_{k \in \mathbb{Z}}} (\Delta_k + k\delta),$$

and $\Delta = \Delta_i \dot\cup (\mathbb{Z}\backslash\{0\})\delta$. The root decomposition then reads:

$$\mathfrak{g}^\mathbb{C} = \mathfrak{t}_\mathfrak{g}^\mathbb{C} \oplus \widehat{\bigoplus_{\alpha + k\delta \in \Delta}} e_k \mathfrak{k}_{\alpha,k}^\mathbb{C}$$

and all root spaces are 1-dimensional.

Proof. For all $(\alpha, k) \in \Delta$ with $\alpha \neq 0$, the subspaces $e_k \mathfrak{k}^{\mathbb{C}}_{\alpha,k}$ are the weight spaces $\mathfrak{k}^{\mathbb{C}}_{\alpha} \cap \mathfrak{k}^{\mathbb{C}}_{k}$ from the preceding remark, multiplied with the periodic maps satisfying $e'_k = ike_k$, making them into root spaces with respect to $\mathbb{C}\mathbf{c} \oplus \mathfrak{t}^{\mathbb{C}}_0 \oplus \mathbb{C}\mathbf{d}$.
In [Ne14, Appendix D], shortly before Lemma D.3, is shown that

$$(\alpha + k\delta)([x, y]) \neq 0$$

for all roots $\alpha + k\delta \in \Delta_i$ and non-zero elements $x \in e_k \mathfrak{k}^{\mathbb{C}}_{\alpha,k}$ and $y \in e_{-k} \mathfrak{k}^{\mathbb{C}}_{-\alpha,-k}$, and that the root space $e_k \mathfrak{k}^{\mathbb{C}}_{\alpha,k}$ is 1-dimensional. Further, from [Ne14, Lemma D.2], we know that $\mathfrak{t}^{\mathbb{C}} := \mathfrak{z}_{\mathfrak{k}^{\mathbb{C}}}(\mathfrak{t}^{\mathbb{C}}_0)$ is maximal abelian in $\mathfrak{k}^{\mathbb{C}}$. It is obvious that the roots with $\alpha = 0$ are not integrable.
To show that $\mathrm{ad}(x)$ is locally nilpotent, we consider another root $\beta + l\delta \in \Delta$, and expressions of the form $(\beta + l\delta) + m(\alpha + k\delta)$ and $RS_m := \beta + m\alpha$ for $m \in \mathbb{N}$; clearly, if the latter is *not* a root in $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}_0)$, then neither is $(\beta + l\delta) + m(\alpha + k\delta)$ a root in $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}_{\mathfrak{g}})$. If $\beta = 0$, then $[x, b] \in e_{k+l} \mathfrak{k}^{\mathbb{C}}_{\alpha,(k+l)}$ for all $b \in \mathfrak{k}^{\mathbb{C}}_{\beta,l}$, so we can replace $\beta + l\delta = l\delta$ with $\alpha + (k + l)\delta$. Thus we can w.l.o.g. assume that $\beta \neq 0$. We further consider preimages of the restriction operation $(\mathfrak{t}^{\mathbb{C}})' \to (\mathfrak{t}^{\mathbb{C}}_0)'$; for $\gamma \in \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}_0) \subset (\mathfrak{t}^{\mathbb{C}}_0)'$, let

$$P_\gamma := \{\widehat{\gamma} \in \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}) \subset (\mathfrak{t}^{\mathbb{C}})' : \widehat{\gamma}|_{\mathfrak{t}^{\mathbb{C}}_0} = \gamma\}.$$

Because $e_k \mathfrak{k}^{\mathbb{C}}_{\alpha,k}$ is 1-dimensional, and $\mathfrak{k}^{\mathbb{C}}_{\alpha,(k+N)} = \mathfrak{k}^{\mathbb{C}}_{\alpha,k}$, the preimage P_α contains at most N elements, and the same is true for P_β. This leaves us with at most N^2 combinations of roots $\widehat{\alpha}, \widehat{\beta} \in \Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ such that $\widehat{\beta} + m\widehat{\alpha}$ restricts to RS_m. The root system $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$ is locally finite, so, for every one of these combinations, there exists a maximal $m_j \in \mathbb{N}$, $1 \leqslant j \leqslant N^2$ such that $\widehat{\beta} + m_j\widehat{\alpha}$ is a root in $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, and if we define $M := \max_{1 \leqslant j \leqslant N^2} m_j$, then $\beta + (M + 1)\alpha$ is not in $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}_0)$, and thus $(\beta + l\delta) + (M + 1)(\alpha + k\delta) \notin \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}_{\mathfrak{g}})$. From

$$\mathrm{ad}(\mathfrak{k}^{\mathbb{C}}_{\alpha,k})^{M+1} \mathfrak{k}^{\mathbb{C}}_{\beta,l} \subseteq \mathfrak{k}^{\mathbb{C}}_{(\beta+(M+1)\alpha),(l(M+1)+k)} = \{0\}$$

it now follows that $\mathfrak{k}^{\mathbb{C}}_{\alpha,k}$ acts on the dense subalgebra

$$\mathfrak{t}^{\mathbb{C}}_{\mathfrak{g}} \oplus \bigoplus_{\alpha+k\delta\in\Delta} e_k \mathfrak{k}^{\mathbb{C}}_{\alpha,k} \subset \mathfrak{g}^{\mathbb{C}}$$

by locally nilpotent operators. Thus, the roots of the form $\alpha + k\delta$ with $\alpha \neq 0$ are integrable. This completes the characterisation of the integrable roots in Δ, which was the last part we had left to show. □

Corollary 3.24. *The integrable roots $\beta \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}_{\mathfrak{g}})$ for a double extended loop algebra $\mathfrak{g}$ are exactly the ones satisfying $\beta|_{\mathfrak{t}_0} \neq 0$, which is also equivalent to $\kappa(\beta, \beta) \neq 0$.*

Lemma 3.25. $\mathfrak{g} \cap \mathfrak{g}(\alpha) \simeq \mathfrak{su}(\mathbb{C}^2)$ *for every integrable root $\alpha \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}}_{\mathfrak{g}})$.*

Proof. This follows from [Ne14, Appendix D] and [Ne14, Lemma 1.8]. □

Proposition 3.26. *For the double extended Hilbert loop algebra* $\mathfrak{g}$*, the involution* $* : \mathfrak{g}^{\mathbb{C}} \to \mathfrak{g}^{\mathbb{C}}$ *from Remark 3.18 and the bilinear extension* $\kappa_{\mathfrak{g}^{\mathbb{C}}}$ *of the invariant Lorentzian form to* $\mathfrak{g}^{\mathbb{C}}$ *are a unitary real form of* $\mathfrak{g}^{\mathbb{C}}$*.*

Proof. We want to show that $\mathfrak{g}^{\mathbb{C}}$ and $\kappa_{\mathfrak{g}^{\mathbb{C}}}$ inherit the list of properties from Definition 3.19 from the underlying Hilbert–Lie algebra and its complexification $\mathfrak{k}^{\mathbb{C}}$, so we are going through the list:

(i) We write any root as $\alpha + n\delta \in \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}}) \subset i\mathfrak{t}_{\mathfrak{g}}' \oplus \mathbb{R}\delta$ and see that α is a restriction of an element in $\Delta(\mathfrak{k}^{\mathbb{C}}, \mathfrak{t}^{\mathbb{C}})$, so it takes imaginary values on $\mathfrak{t}^{\Phi} \subset \mathfrak{t}$. The other summand, $n\delta$, is imaginary on $\mathbf{d}$ by definition (in Proposition 3.23).

(ii) This follows from (i) as in the proof of 3.20.

(iii) is immediate. $\square$

Now that the root systems of double extended Hilbert loop algebras are described in sufficient detail, they can be linked to the purely algebraic concept of *locally affine Lie algebras*, which will provide a detailed representation theory and exhaustion arguments.

Definition 3.27. [Ne10, Definition 2.4] Let V be a rational vector space with a symmetric positive semidefinite bilinear form $(\cdot,\cdot)$, and $\mathcal{R} \subset V$. Then $(V, \mathcal{R}, (\cdot,\cdot))$ is called a *locally affine root system*, in short LARS, if, for all $\alpha, \beta \in \mathcal{R}$, the following conditions are satisfied:

(i) $(\alpha, \alpha) \neq 0$.

(ii) $\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$.

(iii) If $\alpha, \beta \in \mathcal{R}$ and $\sigma_\alpha : V \to V, \sigma_\alpha(v) := v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha$, then $\sigma_\alpha(\beta) \in \mathcal{R}$.

(iv) $\mathcal{R}$ is connected in the sense that there exists a finite sequence of elements $\alpha_k \in \mathcal{R}$, $0 \leqslant k \leqslant n$ with $\alpha = \alpha_0$, $\beta = \alpha_n$ and $(\alpha_k, \alpha_{k+1}) \neq 0$ for $0 \leqslant k < n$.

(v) The subspace of degenerate elements $V^0 := \{v \in V : (v, V) = \{0\}\}$ intersects $\operatorname{span}_{\mathbb{Z}}(\mathcal{R})$ in a non-trivial cyclic group.

A LARS is called *reduced*, if it satisfies

(vi) $\mathbb{R}\alpha \cap \mathcal{R} = \{\pm\alpha\}$.

Definition 3.28. [Ne10, cf. Definitions 1.2 and 3.1] We consider a split quadratic Lie algebra $(\mathfrak{a}, \mathfrak{h}, \kappa)$ and its root system $\Delta := \Delta(\mathfrak{a}, \mathfrak{h})$; we further recall Remark 3.5(2), which implies that $\alpha^\sharp := \flat^{-1}(\alpha)$ exists for all roots $\alpha \in \Delta$, and thus every root can be assigned a length $\|\alpha\| := \sqrt{\kappa(\alpha^\sharp, \alpha^\sharp)}$.
The split quadratic Lie algebra $(\mathfrak{a}, \mathfrak{h}, \kappa)$ is called a *locally extended affine Lie algebra*, in short LEALA, if its set of integrable roots, Δ_i, is connected, and all roots $\alpha \in \Delta$ satisfying $\|\alpha\| \neq 0$ are integrable.

With $V := \operatorname{span}_{\mathbb{Q}}(\Delta_i)$, it is called a *locally affine Lie algebra* , in short LALA, if $(V, \Delta_i, \kappa|_V)$ is a locally affine root system.

In order to apply these notions to the root decomposition of twisted Hilbert–Lie algebras, we need to prepare an appropriate quadratic form on

$$i\mathfrak{t}_0' \oplus \mathbb{R}\delta = \operatorname{span}_{\mathbb{R}}(\Delta_i).$$

It is most convenient to define it on all of $i\mathfrak{t}_{\mathfrak{g}}'$ first, pointing out its relation to the split-quadratic structure of $\mathfrak{g}^{\mathbb{C}}$.

Definition 3.29. As in Definition 1.31, the Lorentz-form κ on $\mathfrak{g}$ can be regarded as an injective homomorphism $\mathfrak{g} \to \mathfrak{g}'$. Its image is dense, and its restriction to $\mathfrak{t}_{\mathfrak{g}}$ is an isomorphism to $\mathfrak{t}_{\mathfrak{g}}'$. We consider the push-forward of κ to $\mathfrak{t}_{\mathfrak{g}}'$ and denote its hermitian extension by $\eta : (\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})' \times (\mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})' \to \mathbb{C}$.
Note that the restriction of this hermitian form to both $\mathfrak{t}_{\mathfrak{g}}'$ and $i\mathfrak{t}_{\mathfrak{g}}' \simeq (i\mathfrak{t})_{\mathfrak{g}}'$ are Lorentz-forms. We will also denote them by η and it will be clear from the context which types of arguments are considered.

Proposition 3.30. *Let $\mathfrak{k}$ be a simple Hilbert–Lie algebra with complexification $\mathfrak{k}^{\mathbb{C}}$, and $V := \operatorname{span}_{\mathbb{Q}}(\Delta_i)$, which is a dense rational subspace of $i\mathfrak{t}_{\mathfrak{g}}' \oplus \mathbb{R}\delta$. Then $(V, \Delta_i, \eta|_V)$ with $\Delta_i \subset \Delta$ from Proposition 3.23 is a reduced LARS.*

Proof. We are verifying the list from Definition 3.27 point by point:

(i) is proven in [Ne14], between the Lemmas D.2 and D.3.

(ii) We consider $(\mathfrak{k}^{\mathbb{C}})^{\varphi}$, which is a Hilbert–Lie algebra, and write any two roots $\alpha, \beta \in \Delta_i$ as

$$\alpha = \alpha_0 + n_\alpha \delta \text{ and } \beta = \beta_0 + n_\beta \delta \in \mathbb{C}\mathbf{c}^* \oplus \mathfrak{t}_0^{\mathbb{C}} \oplus \mathbb{C}\mathbf{d}^*.$$

Then the corresponding coroots have the form

$$\check{\alpha} = (s_{\check{\alpha}}, \check{\alpha}_0, 0) \text{ and } \check{\beta} = (s_{\check{\beta}}, \check{\beta}_0, 0) \in \mathbb{C}\mathbf{c} \oplus \mathfrak{t}_0^{\mathbb{C}} \oplus \mathbb{C}\mathbf{d}.$$

Thus, $(\alpha, \beta) = (\alpha_0, \beta_0) = \beta_0^\sharp(\alpha_0)$, and the property $\frac{(\alpha,\beta)}{(\alpha,\alpha)} \in \mathbb{Z}$ follows from $\frac{(\alpha_0,\beta_0)}{(\alpha_0,\alpha_0)} \in \mathbb{Z}$, which can be derived from the explicit description of the root systems of Hilbert–Lie algebras in Equation (3.2) in Remark 3.23 using $\check{\beta}_0 = \frac{2}{(\beta_0^\sharp, \beta_0^\sharp)} \beta_0^\sharp$.

(iii) follows from the construction of the root system $\Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})$ from the subsystems $(\Delta_k)_{0 \leqslant k \leqslant \operatorname{ord}(\varphi)-1}$ in Remark 3.21 and Proposition 3.23.

(iv) is equivalent to the irreducibility proven in [Ne14, Lemma D.4].

(v) is an application of 3.23: The set of degenerate roots equals $(\mathbb{Z}\backslash\{0\})\delta$.

(vi) follows from [Ne00b, Proposition I.6]. □

Corollary 3.31. *We consider the dense subspace*

$$\mathfrak{t}_{\mathrm{alg}}^{\mathbb{C}} := \mathrm{span}_{\mathbb{C}}(\check{\Delta}, \mathbf{d}) \subset \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}}$$

and the dense subalgebra

$$\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}} := \mathfrak{t}_{\mathrm{alg}}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_{\alpha}^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}.$$

Then, $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$ is a locally affine Lie algebra.

Lemma 3.32. *If $\mathfrak{k}$ is separable and simple, then there exists a strictly ascending sequence $(\mathfrak{g}_n)_{n \in \mathbb{N}}$ of subalgebras of $\mathfrak{g}$ such that*

(i) $\mathfrak{g}_{\mathrm{fin}} := \bigcup_{n \in \mathbb{N}} \mathfrak{g}_n$ *is dense in $\mathfrak{g}$,*

(ii) *every $\mathfrak{g}_n$ is isomorphic to a double extended loop algebra over some compact Lie algebra $\mathfrak{k}_n$,*

(iii) *the abelian subalgebras $\mathfrak{t}_n := \mathfrak{g}_n \cap \mathfrak{t}_{\mathfrak{g}}$ are Cartan subalgebras of the $\mathfrak{g}_n$.*

Proof. This is a direct translation of [Ne10, Proposition 3.3(ii)] to the setting of loop algebras. That proposition asserts the existence of an exhaustion of Δ_i by an ascending sequence of *affine* root systems Δ_n for $n \in \mathbb{N}$, which correspond to *affine Kac–Moody algebras* $\mathfrak{a}_n := \mathfrak{g}(\Delta_n) \oplus \mathbb{C}\mathbf{d}$. The theory of affine Kac–Moody algebras is fully developed in [Ka83], the only part we need here is [Ka83, Theorem 8.5], according to which $\mathfrak{a}_n$ is isomorphic to a double extended Lie algebra of polynomial loops over some complex finite-dimensional simple Lie algebra $\mathfrak{s}_n$, i.e. $\mathfrak{a}_n = (\mathbb{C}\mathbf{c} \oplus_{\omega} \mathcal{L}_{\varphi_n}^{\mathrm{pol}} \mathfrak{s}_n) \rtimes \mathbb{C}\mathbf{d}$ for an appropriate automorphism φ_n. Now, to get back to real Lie algebras, we note that intersecting $\mathfrak{a}_n$ with $\mathfrak{g}$ is the same as taking the unitary real form $\mathfrak{a}_n^{\mathbb{R}}$ of $\mathfrak{a}_n$ which corresponds to $\mathfrak{g} \subset \mathfrak{g}^{\mathbb{C}}$. As in Remark 3.18, this equals $(\mathbb{R}\mathbf{c} \oplus_{\omega} \mathcal{L}_{\varphi_n}^{\mathrm{pol}} \mathfrak{k}_n) \rtimes \mathbb{R}\mathbf{d}$ for a suitable real form $\mathfrak{k}_n$ of $\mathfrak{s}_n$.
Recalling the latest corollary, we find that $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}} = \mathfrak{g}(\Delta) \oplus \mathbb{C}\mathbf{d}$, and from this it becomes clear that the $\mathfrak{a}_n$ exhaust $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$. Thus, the real forms $\mathfrak{a}_n^{\mathbb{R}}$ exhaust $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}} \cap \mathfrak{g}$, and all that is left to do is to define the $\mathfrak{g}_n$ as the Fréchet completions of the $\mathfrak{a}_n^{\mathbb{R}}$. Now (i) follows from $\bigcup_{n \in \mathbb{N}} \mathfrak{g}_n$ containing $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}} \cap \mathfrak{g}$, which is dense in $\mathfrak{g}$, (ii) is the definition of the $\mathfrak{g}_n$, and (iii) follows from the construction out of $\Delta_n \subset \Delta_i$, which determines the Cartan subalgebra of $\mathfrak{g}_n$. ☐

Locally finite and locally affine root systems exhibit their own convex geometries, which are invariant under the action of their respective *Weyl groups*:

Definition 3.33. The *Weyl group* of a LARS $(V, \mathcal{R}, (\cdot,\cdot))$ is defined as

$$\mathcal{W}(\mathcal{R}) := \langle \sigma_\alpha : \alpha \in \mathcal{R} \rangle.$$

Remark 3.34. Recall the LARS $(V, \Delta_i, \eta|_V)$ from Proposition 3.30. In this case, the reflections σ_α generating the associated Weyl group $\mathcal{W}(\Delta_i)$ are of the form

$$\sigma_\alpha(\lambda) := \lambda - 2\frac{\eta(\lambda,\alpha)}{\eta(\alpha,\alpha)}\alpha \quad \text{for all} \quad \lambda \in V \subset i\mathfrak{t}_{\mathfrak{g}}'.$$

Obviously, these generators, and thus the natural Weyl group action on V, immediately extend to $i\mathfrak{t}'_{\mathfrak{g}}$, and can be pulled back to $i\mathfrak{t}_{\mathfrak{g}}$ via the injection $i\mathfrak{t}_{\mathfrak{g}} \to i\mathfrak{t}'_{\mathfrak{g}}$ defined by $-\kappa$. This yields a faithful representation which we will identify with $\mathscr{W}(\Delta_i)$.

4 Lowest weight representations and a convexity theorem

The bridge we have established between Hilbert loop algebras and locally affine root systems, and in particular the connection between the adjoint action and the Weyl group, would already allow to prove a convexity statement

$$\mathcal{O}_\lambda|_{\mathfrak{t}_\mathfrak{g}} \subseteq \overline{\mathrm{conv}}(\mathcal{W}.\lambda)$$

for certain weights $\lambda \in \mathfrak{t}_\mathfrak{g}'$, by applying convexity theorems from the representation theory of locally affine Lie algebras, see [Ne10, Section 4]. We will come to this in the second subsection of this chapter, but in order to generalise this statement to as many weights as possible, which is all weights with $\lambda(\mathbf{c}) \neq 0$, we first need to understand the $\mathcal{W}$-invariant convex geometry on $\mathfrak{t}_\mathfrak{g}'$.

4.1 Convex geometry of Weyl group orbits

At some point in the development of a convexity-theorem for coadjoint orbits, we will need sharp information about the convex geometry which the Weyl group action induces on $\mathfrak{t}_\mathfrak{g}'$. The main source for this information will be [HN14].

For any subset $E \subset V$ of a locally convex real vector space, we write

$$E^\star := \{\lambda \in V' : (\forall x \in E)\lambda(x) \geqslant 0\}, \text{ and } B(E) := \{\lambda \in V' : \sup(\lambda(E)) < \infty\},$$

and for $F \subset V'$:

$$F^\star := \{x \in V : (\forall \lambda \in V')\lambda(x) \geqslant 0\}, \text{ and } B(F) := \{x \in V : \sup(F(x)) < \infty\};$$

The cones $E^\star$ and $F^\star$ are called the *dual cones* of E and F.

[HN14, Definition 1.1] With an arbitrary index set S, we consider a triple $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ of a finite-dimensional real vector space V, a family $(\alpha_s)_{s\in S}$ of linear functionals on V and a family $(\check{\alpha}_s)_{s\in S}$ of elements of V with $\alpha_s(\check{\alpha}_s) = 2$ for all $s \in S$. In this context, we denote the group generated by the reflections

$$\sigma_s : V \to V, \quad \sigma(x) := x - \alpha_s(x)\check{\alpha}$$

by $\mathcal{W}$; the connection of these groups with the Weyl groups of Lie algebras will become clear soon.
Such a triple is called a *linear Coxeter system* if it has the following properties:

(LCS1) The *fundamental chamber*, i.e. the convex cone

$$K := \{x \in V : (\forall s \in S)\alpha(s) \geqslant 0\}$$

has inner points.

(LCS2) For all $s \in S$, the functional α_s is not contained in $\mathrm{cone}\{\alpha_r : r \neq s\}$.

(LCS3) $\sigma K^0 \cap K^0 = \varnothing$ for every $\sigma \in \mathcal{W}\backslash\{\mathbf{1}\}$.

If S is finite, then $\mathcal{W}$ is called a *Coxeter group*.

Further important notions and notation from the context of convex and Coxeter geometry are:

(CG1) The convex cone $T := \mathcal{W}.K$ is referred to as the *Tits cone*.

(CG2) For the faithful *dual action* of $\mathcal{W}$ on V', in analogy to the coadjoint action, we use the notation $\sigma^*\lambda := \lambda \circ \sigma$ for all $\sigma \in \mathcal{W}, \lambda \in V'$. The elements of $\mathcal{R} := \bigcup_{s\in S} \mathcal{W}^*\alpha_s$ are called *roots*.

(CG3) If $\alpha = \sigma^*\alpha_s \in \mathcal{R}$ is a root, then $\check{\alpha} := \sigma\check{\alpha}_s$ is called the associated *coroot*; the set of coroots is denoted by $\check{\mathcal{R}}$, accordingly.

(CG4) $C_S := \operatorname{cone}\{\alpha_s \in V' : s \in S\}$ and $\check{C}_S := \operatorname{cone}\{\check{\alpha}_s \in V : s \in S\}$.

(CG5) $C_x := \operatorname{cone}\{\alpha \in \mathcal{R} : \alpha(x) > 0\}$ for $x \in V$ and

$$\check{C}_\lambda := \operatorname{cone}\{\check{\alpha} \in \check{\mathcal{R}} : \lambda(\check{\alpha}) > 0\} \text{ for } \lambda \in V'.$$

In the following we show how these notions apply to the root space decomposition with respect to $\Delta := \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})$ of a double extended twisted loop algebra $\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi\mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$ with compact simple $\mathfrak{k}$.

Definition 4.1. A *simple system*, also called *root basis*, is a linearly independent, and therefore finite, subset $\Pi \subset \Delta_i$ such that for every $\alpha \in \Delta$ there exists a family of roots $(\alpha_s)_{s\in I} \subset \Pi$ and a family of natural numbers $(z_s)_{s\in I} \subset \mathbb{N}$ with either

$$\alpha = \sum_{s\in I} z_s\alpha_s \quad \text{or} \quad \alpha = -\sum_{s\in I} z_s\alpha_s.$$

To a simple system $\Pi \subset \Delta_i$ we associate a fundamental chamber

$$K(\Pi) := \{x \in i\mathfrak{t}_{\mathfrak{g}} : (\forall\alpha \in \Pi) i\alpha(x) \geqslant 0\}$$

and the Coxeter group

$$\mathcal{W}(\Pi) := \langle\sigma_\alpha \in \mathrm{GL}(i\mathfrak{t}_{\mathfrak{g}}) : \alpha \in \Pi\rangle; \tag{4.1}$$

the definition of the Weyl group in [Ka83, §3.7] coincides with this definition, which means that the Weyl group in [Ka83] is automatically a Coxeter group. Furthermore:

Proposition 4.2. [Ka83, Proposition 3.12] *If $S \to \Pi, s \to \alpha_s$ is an indexing of a simple system $\Pi \subset \Delta$ and $V := i\mathfrak{t}_{\mathfrak{g}}$, then $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ is a linear Coxeter system.*

In Definition 3.33, we have defined the Weyl group $\mathcal{W}(\Delta_i)$ without referring to any simple system of Δ_i, because in the locally affine case, these are not well-behaved. As a consequence, we have to make sure that, in the affine case, Definition 3.33 is equivalent to (4.1). This has already been shown in [Ka80] and summarised in [Ne00b, Theorem II.7] for a larger class of Lie algebras, of which the following is a direct application:

Proposition 4.3. *If $\Delta_i \subset \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})$ for $\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi \mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$ with compact simple $\mathfrak{k}$, and $\Pi \subset \Delta_i$ a simple systems as defined in Definition 4.1, then*

(i) $\mathcal{W}(\Pi) = \mathcal{W}(\Delta_i)$ *and*

(ii) $\mathcal{W}(\Pi).\Pi = \Delta_i$.

Definition 4.4. For a linear Coxeter system $(V, (\check{\alpha}_s)_{s\in S}, (\alpha_s)_{s\in S})$ we set

$$\check{C}_S := \operatorname{cone}\{\check{\alpha}_s : s \in S\} \subseteq V \text{ and } C_S := \operatorname{cone}\{\check{\alpha}_s : s \in S\} \subseteq V'.$$

With $\mathcal{R}$ and $\check{\mathcal{R}}$ denoting the roots and coroots of the Coxeter system as in (CG2) and (CG3), for all $v \in V$, we further define

$$\check{C}_v := \operatorname{cone}\{\check{\alpha} \in \check{\mathcal{R}} : \alpha(v) > 0\} \subseteq V \text{ and } C_v := \operatorname{cone}\{\alpha \in \mathcal{R} : \alpha(v) > 0\} \subseteq V'.$$

Remark 4.5. If a linear Coxeter system $(V, (\check{\alpha}_s)_{s\in S}, (\alpha_s)_{s\in S})$ comes from the root space decomposition of an affine Lie algebra, then the members of $(\check{\alpha}_s)_{s\in S}$ and $(\alpha_s)_{s\in S}$ are related via the non-degenerate symmetric form η from Definition 3.29, specifically $\check{\alpha} = \frac{2}{\eta(\alpha,\alpha)}\alpha^\sharp$, which implies that $\check{C}_S$ is pointed, and thus $\check{C}_S^\star$ is a fundamental chamber of another linear Coxeter system, namely $(V', (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$, which shares its Weyl group with the first one. Depending on a choice of sign, its Tits cone equals either

$$(V')_0^+ := \{\lambda \in V' : \lambda(i\mathbf{c}) > 0\} \cup i\mathbb{R}\mathbf{d}^*$$

or $-(V')_0^+$.

With Proposition 4.3 it has become clear that Coxeter geometry is directly applicable to Cartan subalgebras and Weyl groups. We proceed to the inspection of the convex geometry of Weyl group orbits. By far the greater part of this has been accomplished in [HN14], and the part of that paper which deals with Coxeter systems culminates in the following Theorem:

Theorem 4.6. [HN14, Theorem 2.7] *If $(V, (\check{\alpha}_s)_{s\in S}, (\alpha_s)_{s\in S})$ is a linear Coxeter system, then*

$$\mathcal{W}.v \subset v - \check{C}_v \text{ for all } v \in T.$$

Corollary 4.7. *Because* $\operatorname{conv}(\mathcal{W}.v)$ *is $\mathcal{W}$-invariant and $\sigma\check{C}_v = \check{C}_{\sigma v}$ is convex for every $\sigma \in \mathcal{W}$, we immediately get*

$$\operatorname{conv}(\mathcal{W}.v) \subseteq \bigcap_{\sigma\in\mathcal{W}} \sigma(v - \check{C}_v).$$

In the following, this inclusion will be sharpened to the point where equality holds. We will use the abbreviations

$$\operatorname{co}(v) := \operatorname{conv}(\mathcal{W}.v) \text{ and } \overline{\operatorname{co}}(v) := \overline{\operatorname{conv}}(\mathcal{W}.v),$$

as well as

$$\sec(v) := \bigcap_{\sigma \in \mathcal{W}} \sigma(v - \check{C}_v).$$

Because we will not assume that the linear Coxeter system $(V, (\alpha_s)_{s\in S}, (\check{\alpha}_s)_{s\in S})$ comes from the root space decomposition of a Lie algebra, we use the definition $\Delta := \bigcup_{s\in S} \mathcal{W}^* \alpha_s$ for the set of roots, and $\Delta^+ := \Delta \cap C_S$.

Lemma 4.8. *Let $\mathcal{W}_v$ denote the stabilizer in $\mathcal{W}$ of some $v \in T$; we collect some important properties of linear Coxeter systems:*

(i) $C_S = K^\star = \operatorname{cone}\{\alpha \in \Delta : (\forall x \in K^0)\alpha(x) > 0\}$.

(ii) $\check{C}_v \subseteq \check{C}_S$ *for all* $v \in K$. *Equality holds if, and only if,* $v \in K^0$.

(iii) $\check{C}_v$ *is invariant under* $\mathcal{W}_v$.

In the following, we assume that $v \in T^0$.

(iv) $\check{C}_v = \bigcap_{\sigma\in\mathcal{W}_v} \sigma\check{C}_S$ *for all* $v \in K$.

(v) $\sec(v) = \bigcap_{\sigma\in\mathcal{W}} \sigma(v - \check{C}_S)$ *for all* $v \in K$.

(vi) $\check{C}_v$ *is closed.*

Proof. (i) This follows from every $\alpha \in \Delta$ being either positive or negative on K^0, [HN14, Remark 1.11].

(ii) For $v \in K$, it follows from (i) that $C_v \subseteq C_S$, with equality if and only if $v \in K^0$, and [HN14, Theorem 1.10] implies in particular that any root satisfies $\alpha \in C_S$ if and only if $\check{\alpha} \in \check{C}_S$.

(iii) If $\alpha(v) > 0$, then for $\sigma \in \mathcal{W}_v$ we get $(\sigma^{-1})^*\alpha(v) = \alpha(\sigma v) = \alpha(v) > 0$, so the invariance follows from

$$\sigma\{\check{\alpha} \in \check{\Delta} : \alpha(v) > 0\} = \{\check{\alpha} \in \check{\Delta} : (\sigma^{-1})^*\alpha(v) > 0\}.$$

(iv) For $v \in V$ we set $S_v := \{s \in S : \alpha_s(v) = 0\}$; if $\alpha \in \Delta^+$, but $\check{\alpha} \notin \check{C}_v$, i.e. $\alpha(v) = 0$, then $\check{\alpha} \in \check{C}_{S_v} := \operatorname{cone}\{\check{\alpha}_s : s \in S_v\}$, so that $\check{C}_S = \check{C}_v + \check{C}_{S_v}$. To this, [Ne00a, Corollary V.2.10] (with $x := 0$) applies, yielding

$$\bigcap_{\sigma\in\mathcal{W}_v} \sigma\check{C}_S = \bigcap_{\sigma\in\mathcal{W}_v} \sigma(\check{C}_{S_v} + \check{C}_v) = \check{C}_v.$$

(v) By applying (iv), we get

$$\sec(v) = \bigcap_{\sigma\in\mathcal{W}} \bigcap_{\sigma'\in\mathcal{W}_v} \sigma\sigma'(v - \check{C}_S) = \bigcap_{\sigma\in\mathcal{W}} \sigma(v - \check{C}_S). \qquad \square$$

(vi) Because $T = \mathcal{W}K$, we may assume that $v \in K$. Further, $\check{C}_S$ is a finitely generated convex cone in a finite-dimensional vector space, so, by (iv), $\check{C}_v$ is an intersection of closed sets.

That $\check{C}_v$ is closed implies $\overline{\mathrm{co}}(v) \subseteq \sec(v)$. To prove a first version of the reverse inclusion, we need two more technical lemmas:

Proposition 4.9. *Let $C \neq \varnothing$ be an open convex cone in a finite-dimensional real vector space V, $\Gamma \subset \mathrm{GL}(V)$ be a subgroup that stabilises C and satisfies $|\det(A)| = 1$ for all $A \in \Gamma$, and $v \in C$. Then $\overline{\mathrm{conv}}(\Gamma.v) \subset C$.*

Proof. By [HN93, Theorem 1.8] there exists a smooth, and in particular continuous, convex Γ-invariant function $f : C \to (0, \infty)$ such that for every $\mathbf{c} > 0$ the sublevel set $F_c := \{v \in C : f(v) \leqslant \mathbf{c}\}$ is a convex subset of C which is closed in V. Because f is Γ-invariant, $\mathrm{conv}(\Gamma.v) \subseteq F_c$ for all $v \in C$, and because F_c is closed, $\overline{\mathrm{conv}}(\Gamma.v) \subseteq F_c \subset C$ as well. □

Corollary 4.10. *If V is the vector space of a linear Coxeter system, then $\overline{\mathrm{co}}(v) \subset T^0$ for all $v \in T^0$.*

Lemma 4.11. *If $v \in T$ and $w \in \check{C}_v$, there exists $\varepsilon > 0$ such that $v - \varepsilon w \in \mathrm{co}(v)$.*

Proof. By definition, $w = \sum_{j=1}^{k} c_j \check{\alpha}_j$ for some $k \in \mathbb{N}$, $c_j > 0$ and roots satisfying $\alpha_j(v) > 0$. Obviously, $\sigma_{\alpha_j}(v) = v - \alpha_j(v)\check{\alpha}_j \in \mathrm{co}(v)$, so for all combinations $d_j > 0$ with $\sum_{j=1}^{k} d_j \leqslant 1$ and $d_0 := 1 - \sum_{j=1}^{k} d_j$ we find

$$v - \sum_{j=1}^{k} d_j \alpha_j(v)\check{\alpha}_j = d_0 v + \sum_{j=1}^{k} d_j (v - \alpha_j(v)\check{\alpha}_j) \in \mathrm{co}(v).$$

So, if we set $v_t := v - tw$, then

$$v_t = v - \sum_{j=1}^{k} t c_j \check{\alpha}_j = v - \sum_{j=1}^{k} t \frac{c_j}{\alpha_j(v)} \alpha_j(v)\check{\alpha}_j,$$

which, by the previous formula, is contained in $\mathrm{co}(v)$ if t is small enough so that $\sum_{j=1}^{k} t \frac{c_j}{\alpha_j(v)} \leqslant 1$. □

Theorem 4.12. *If $v \in T^0$, then $\overline{\mathrm{co}}(v) = \sec(v)$.*

Proof. The inclusion $\mathrm{co}(v) \subseteq \sec(v)$ is Corollary 4.7 above, and with Lemma 4.8(vi) this becomes $\overline{\mathrm{co}}(v) \subseteq \sec(v)$. Further, because $T = \mathcal{W}K$, we may assume that $v \in K$.
In an intermediate step, we show that $K \cap (v - \check{C}_v) \subseteq \overline{\mathrm{co}}(v)$.
So, let $u \in K \cap (v - \check{C}_v)$, and consider the line segment

$$\gamma : [0,1] \to K, \quad \gamma(t) := v + t(u - v);$$

then $\gamma^{-1}(\overline{\mathrm{co}}(v)) = [0, r]$ for some $r \in [0,1]$, and we have to show that $r = 1$.
If $\alpha(v) > 0$ for some $\alpha \in \Delta$, then $\alpha \in K^\star$, which implies $\alpha(u) \geqslant 0$. For $0 \leqslant t < 1$, this leads to $\alpha(\gamma(t)) > 0$ and thus $\check{C}_v \subseteq \check{C}_{\gamma(t)}$.

Suppose that $r < 1$ and set $u' := \gamma(r) \in \overline{\text{co}}(v) = (1-r)v + cu$; because $u \in v - \check{C}_v$ according to our assumption, $v - u \in \check{C}_v$, and thus

$$u' - u = (1-r)(v-u) \in \check{C}_v \subseteq \check{C}_{u'}.$$

By Lemma 4.11, there exists $\varepsilon > 0$ such that $u' - \varepsilon(u' - u) \in \text{co}(u') \subseteq \overline{\text{co}}(v)$, where the last inclusion comes from $u' \in \overline{\text{co}}(v)$. Now

$$u' - \varepsilon(u' - u) = v + r(u - v) + \varepsilon(1-r)(u-v) = \lambda(r + \varepsilon(1-r)),$$

which contradicts the maximality of r, so that we have shown $r = 1$ and thus $K \cap (v - \check{C}_v) \subseteq \overline{\text{co}}(v)$.
Again, because $T = \mathcal{W}K$, for every $w \in T \cap \text{sec}(v)$, there exists a $\sigma \in \mathcal{W}$ such that $\sigma(w) \in K$, and because $w \in \sigma^{-1}(v - \check{C}_v)$ in particular, $\sigma(w) \in K \cap (v - \check{C}_v)$, which shows that $T \cap \text{sec}(v) \subseteq \overline{\text{co}}(v)$.
For the last step we note that $\overline{\text{co}}(v) \subset T^0$, so $T \cap \text{sec}(v)$ cannot intersect ∂T, and because of connectedness, $\text{sec}(v) \subset T^0$, which finally shows $\text{sec}(v) \subseteq \overline{\text{co}}(v)$. □

4.2 The convexity theorem for weights

Definition 4.13. Let $\mathfrak{a}$ be a complex quadratic Lie algebra with a unitary real form $*$. A *unitary representation* of $\mathfrak{a}$ is a complex pre-Hilbert space $\mathcal{h}$ with an action $\rho : \mathfrak{a} \to B(\mathcal{h})$ such that, for every $x \in \mathfrak{a}$, the adjoint operator of $\rho(x)$ exists and $\rho(x^*) = \rho(x)^*$.
This implies that the real subalgebra $\mathfrak{a}^{\mathbb{R}} := \{x \in \mathfrak{a} : x^* = -x\}$ acts by skew-symmetric operators, i.e. $\rho(\mathfrak{a}^{\mathbb{R}}) \subseteq \mathfrak{u}(\mathcal{h})$, hence the name.

Definition 4.14. Let $\mathfrak{a}$ be a split Lie algebra. Then an $\mathfrak{a}$-module L is called *split* with respect to the splitting maximal abelian subalgebra $\mathfrak{h}$ if $\mathfrak{h}$ acts on L by diagonalisable operators. We will generally denote the weight spaces by

$$L_\lambda := \{v \in L : (\forall h \in \mathfrak{h}) h.v = \lambda(h)v\} \text{ for } \lambda \in \mathfrak{h}'$$

and the set of weights (also called *weight system*) by

$$\mathcal{P}_L := \{\lambda \in \mathfrak{h}' : L_\lambda \neq \{0\}\}.$$

The module L is called *integrable* if for every integrable root α every $x \in \mathfrak{a}_\alpha$ acts as a locally nilpotent operator.

Definition 4.15. [Ne10, Definition 4.4] If $\mathfrak{h}$ is a splitting Cartan subalgebra of some Lie algebra $\mathfrak{a}$, then a subset $\Delta^+ \subset \Delta := \Delta(\mathfrak{a}, \mathfrak{h})$ is called a *positive* system if

(i) $\Delta = \Delta^+ \dot{\cup} - \Delta^+$ and

(ii) $\sum_{\alpha \in F} \alpha \neq 0$ for every finite subset $F \subset \Delta^+$.

Remark 4.16. For every positive system Δ^+ the subspaces

$$\mathfrak{n}_+ := \bigoplus_{\alpha \in \Delta^+} \mathfrak{a}_\alpha \quad \text{and} \quad \mathfrak{n}_- := \bigoplus_{\alpha \in -\Delta^+} \mathfrak{a}_\alpha$$

are subalgebras. If $\mathfrak{a} = \mathfrak{g}^{\mathbb{C}}$ for a double extended Hilbert loop algebra $\mathfrak{g}$ and, accordingly, $\mathfrak{h} = \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}}$, then, for $\mathfrak{t}_{\mathrm{alg}}^{\mathbb{C}}$ as in Corollary 3.31, $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}} = \mathfrak{n}_- \oplus \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}} \oplus \mathfrak{n}_+$.

Definition 4.17. Let $\mathfrak{h}$ be a splitting Cartan subalgebra of some Lie algebra $\mathfrak{a}$, and $\Delta^+ \subset \Delta := \Delta(\mathfrak{a}, \mathfrak{h})$ be a positive system. An $\mathfrak{h}$-split $\mathfrak{a}$- module is called a module of *highest weight* λ, if there exists a weight $\lambda \in \mathcal{P}_L$ such that the weight space L_λ satisfies $\mathfrak{n}_+(L_\lambda) = \{0\}$ and L_λ generates L.

Definition 4.18. If $\mathfrak{h}$ is the splitting Cartan subalgebra of a Lie algebra $\mathfrak{a}$, and $\Delta := \Delta(\mathfrak{a}, \mathfrak{h})$ the corresponding root system, then a weight $\lambda \in \mathfrak{h}'$ is called *integral* if $\lambda(\check{\Delta}_i) \subseteq \mathbb{Z}$.

Proposition 4.19. *For a simple Hilbert–Lie algebra $\mathfrak{k}$, we consider the double extended loop algebra $\mathfrak{g} := (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi \mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$ with root system $\Delta := \Delta(\mathfrak{g}^{\mathbb{C}}, \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}})$, and recall from Corollary 3.31 the dense subspace $\mathfrak{t}_{\mathfrak{g}_{\mathrm{alg}}}^{\mathbb{C}} := \mathrm{span}_{\mathbb{C}}(\check{\Delta}, \mathbf{d}) \subset \mathfrak{t}_{\mathfrak{g}}^{\mathbb{C}}$ and the dense locally affine subalgebra*

$$\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}} := \mathfrak{t}_{\mathfrak{g}_{\mathrm{alg}}}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha^{\mathbb{C}} \subseteq \mathfrak{g}^{\mathbb{C}}.$$

Then, for every integral weight $\lambda \in i\mathfrak{t}_{\mathfrak{g}}'$ with $\lambda(\mathbf{c}) \neq 0$ and positive system $\Delta^+ \subset \Delta$ such that $\lambda(\check{\alpha}) \geqslant 0$ for all $\alpha \in \Delta^+$, there exists a simple (i.e. not containing any proper submodule) highest weight module $L_{\mathrm{alg}}(\lambda, \Delta^+)$ of $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$.

Proof. For a locally affine Lie algebra as in [Ne10, Definition 3.1], such a module is constructed in [Ne10, Definition 4.2].
From Proposition 3.30 and Corollary 3.31 we know that $\Delta_i \subset \Delta$ is a LARS and $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$ is a locally affine Lie algebra. □

Definition 4.20. According to [Ne10, Proposition 4.9], if Δ_1^+ and Δ_2^+ are positive systems satisfying the assumptions of Proposition 4.19, then

$$L_{\mathrm{alg}}(\lambda, \Delta_1^+) \simeq L_{\mathrm{alg}}(\lambda, \Delta_2^+).$$

Hence, we will write $L_{\mathrm{alg}}(\lambda)$ for the $\mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$-module of highest weight λ as in the prerequisites of Proposition 4.19.
We will denote the corresponding action by $\rho_\lambda : \mathfrak{g}^{\mathbb{C}} \rightharpoonup L_{\mathrm{alg}}(\lambda)$.

Proposition 4.21. *$L_{\mathrm{alg}}(\lambda)$ has the following properties:*

(i) $\mathcal{P}_\lambda := \mathcal{P}_{L_{\mathrm{alg}}(\lambda)} \subseteq \lambda - \mathrm{span}_{\mathbb{N}}(\Delta^+)$.

(ii) $\mathcal{P}_\lambda = \mathrm{conv}(\mathcal{W}.\lambda) \cap (\lambda + \mathrm{span}_{\mathbb{Z}}(\Delta))$.

(iii) *$L_{\mathrm{alg}}(\lambda)$ is a pre-Hilbert space, on which $\mathfrak{g}_{\mathrm{alg}}$ acts unitarily, i.e. by skew-adjoint operators.*

Proof. (i) For any root space $\mathfrak{g}_\alpha^{\mathbb{C}} \subset \mathfrak{g}_{\mathrm{alg}}^{\mathbb{C}}$, weight space $L_\gamma \subset L_{\mathrm{alg}}(\lambda)$ and non-zero $x \in \mathfrak{g}_\alpha^{\mathbb{C}}$ and $v \in L_\gamma$, we have $x.v \in L_{\gamma+\alpha}$, so (i) follows from Definition 4.17.

(ii) follows from [Ne10, Theorem 4.10(b)], and

(iii) from [Ne10, Theorem 4.11]. □

Proposition 4.22. *Let K be a 1-connected compact simple Lie group and $\mathfrak{k} = \mathbf{L}(K)$ endowed with the appropriate normalized inner product, and G a connected Lie group with $\mathbf{L}(G) = \mathfrak{g} = (\mathbb{R}\mathbf{c} \oplus_\omega \mathcal{L}_\varphi \mathfrak{k}) \rtimes \mathbb{R}\mathbf{d}$ as in Proposition 3.13. Let further $\lambda \in i\mathfrak{t}_{\mathfrak{g}}'$ be an integral weight with $\lambda(\mathbf{c}) \neq 0$ which is* $\mathbf{d}$-minimal *in the sense that $\lambda(-i\mathbf{d}) = \min(\mathcal{W}.\lambda(-i\mathbf{d}))$.*
Then, there exists a unitary Lie group action $\pi_\lambda : G \rightharpoonup \mathcal{H}_\lambda$ on the Hilbert space completion $\mathcal{H}_\lambda := \overline{L_{\mathrm{alg}}(\lambda)}$ such that the induced action

$$\mathbf{L}(\pi_\lambda) : \mathfrak{g} \rightharpoonup \mathcal{H}_\lambda^\infty$$

on the dense subspace of smooth vectors $\mathcal{H}_\lambda^\infty$, *i.e. those vectors whose orbit map is smooth, satisfies*

$$\mathbf{L}(\pi_\lambda)|_{\mathfrak{g}_{\mathrm{alg}}} = \rho_\lambda|_{\mathfrak{g}_{\mathrm{alg}}}.$$

Proof. Because λ is $\mathbf{d}$-minimal, by [HN14, Corollary 2.6], we have $\lambda \in -C_{\mathbf{d}}^\star$, where $C_{\mathbf{d}} = \{\check{\alpha} : \alpha(-i\mathbf{d}) > 0\}$, and

$$C_{\mathbf{d}}^\star = \{\gamma \in i\mathfrak{t}_{\mathfrak{g}}^* : (\forall \alpha \in \Delta)\alpha(-i\mathbf{d}) > 0 \Rightarrow \gamma(\check{\alpha}) \geqslant 0\},$$

i.e. $\lambda(\check{\alpha}) \leqslant 0$ for all roots α with $\alpha(-i\mathbf{d}) > 0$.
This implies that there is a positive system Δ^+ with $\lambda(\check{\Delta}^+) \subset \mathbb{N}_0$ which contains all roots with $\alpha(-i\mathbf{d}) > 0$ and with respect to which the underlying module $L_{\mathrm{alg}}(\lambda) \subset \mathcal{H}_\lambda$ equals the highest weight module $L_{\mathrm{alg}}(\lambda, \Delta^+)$.
By Proposition 4.21, $\mathcal{P}_\lambda \subseteq \lambda - \mathrm{span}_{\mathbb{N}}(\Delta^+)$, so $\lambda(-\check{\Delta}^+) \subseteq -\mathbb{N}_0$ implies that the spectrum of $\rho_\lambda(\check{\alpha})$ is negative for all $\alpha \in \Delta^+$.
Now, consider the Lie algebra $\widehat{\mathfrak{k}^\varphi} := \mathbb{R}\mathbf{c} \oplus \mathfrak{k}^\varphi \oplus \mathbb{R}\mathbf{d}$ with the natural injection $(\widehat{\mathfrak{k}^\varphi})^{\mathbb{C}} \hookrightarrow \mathfrak{g}^{\mathbb{C}}$, and let $S(\lambda) \subset L_{\mathrm{alg}}(\lambda)$ be the unitary $(\widehat{\mathfrak{k}^\varphi})^{\mathbb{C}}$-module generated by the weight space $L_\lambda \subset L_{\mathrm{alg}}(\lambda)$.
This is a $(\widehat{\mathfrak{k}^\varphi})^{\mathbb{C}}$-module of highest weight λ with respect to the positive system

$$\Delta_s^+ := \Delta^+ \cap \Delta((\mathfrak{k}^\varphi)^{\mathbb{C}}, \mathfrak{t}_0^{\mathbb{C}}).$$

From [HN12, Proposition 7.3.14] follows that $S(\lambda)$ is finite-dimensional, because $(\widehat{\mathfrak{k}^\varphi})^{\mathbb{C}}$ is a direct sum of a semisimple and an abelian finite-dimensional Lie algebra, and the abelian summand acts by scalar multiplications. This implies that the action $(\widehat{\mathfrak{k}^\varphi})^{\mathbb{C}} \rightharpoonup S(\lambda)$ integrates to a unitary action of the Lie group $\widehat{K}^\Phi = \mathbb{S}^1 \times K^\Phi \times \mathbb{S}^1 \subset G$, which is automatically bounded. We denote this action by $\chi : \widehat{K}^\Phi \rightharpoonup S(\lambda)$.
This and the aforementioned negativity of the spectrum make [JN18, Proposition 8.6] applicable, which implies that there exists a complex Hilbert space

$\mathcal{H}$ with an irreducible smooth unitary action $\vartheta : G \rightharpoonup \mathcal{H}$ and an injection $\iota : S(\lambda) \hookrightarrow \mathcal{H}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{K}^{\Phi} \times S(\lambda) & \xhookrightarrow{\mathrm{id}_G \times \iota} & G \times \mathcal{H} \\ \downarrow \chi & & \downarrow \vartheta \\ S(\lambda) & \xhookrightarrow{\quad \iota \quad} & \mathcal{H}. \end{array}$$

By construction, the complex linear extension of the induced Lie algebra action $\mathbf{L}(\chi) : \widehat{\mathfrak{k}}^{\varphi} \rightharpoonup S(\lambda)$ is identical with the restriction of ρ_λ to $(\widehat{\mathfrak{k}}^{\varphi})^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$. We also consider the induced action $\mathbf{L}(\vartheta) : \mathfrak{g} \rightharpoonup \mathcal{H}^{\infty}$, where

$$\mathcal{H}^{\infty} = \{v \in \mathcal{H} : G \to \mathcal{H}, g \to g.v \text{ is smooth }\},$$

and its complex linear extension to $\mathfrak{g}^{\mathbb{C}}$, which we also denote by $\mathbf{L}(\vartheta)$.
With these, we get a corresponding commuting diagram for the complex linearly extended induced actions on the smooth vectors:

$$\begin{array}{ccc} (\widehat{\mathfrak{k}}^{\varphi})^{\mathbb{C}} \times S(\lambda) & \xhookrightarrow{\mathrm{id}_{\mathfrak{g}} \times \iota} & \mathfrak{g}^{\mathbb{C}} \times \mathcal{H}^{\infty} \\ \downarrow \rho_\lambda & & \downarrow \mathbf{L}(\vartheta) \\ S(\lambda) & \xhookrightarrow{\quad \iota \quad} & \mathcal{H}. \end{array}$$

Looking at the image $\iota(v_\lambda) \in \mathcal{H}^{\infty}$ of a generating weight vector $v_\lambda \in L_\lambda \subset S(\lambda)$, we find that it generates a $\mathfrak{g}^{\mathbb{C}}$-module contained in $\mathcal{H}^{\infty}$, and thus, because $\mathcal{H}$ is irreducible, it generates $\mathcal{H}^{\infty}$. This allows us to identify $\mathcal{H}^{\infty}$ with $\mathcal{H}^{\infty}_{\lambda}$, and thus to pull back the action ϑ of G on $\mathcal{H}$ to $\mathcal{H}_\lambda$. This completes the proof. □

Theorem 4.23. *If G is a double extended loop group over a 1-connected, compact, simple Lie group K and $\mathfrak{g} = \mathbf{L}(G)$, then, for every integral weight $\lambda \in i\mathfrak{t}'_{\mathfrak{g}}$ with $\lambda(\mathbf{c}) \neq 0$, its coadjoint orbit $\mathcal{O}_\lambda$ satisfies*

$$\mathcal{O}_\lambda|_{i\mathfrak{t}_{\mathfrak{g}}} \subseteq \overline{\mathrm{co}}(\lambda). \tag{4.2}$$

Proof. Because λ is assumed to be integral, [Ne14, Remark 4.5] applies, which states that the orbit of λ under the Weyl group contains a $\mathbf{d}$-minimal element. Thus, because our claim only deals with the Weyl group orbit of λ instead of λ itself, we may w.l.o.g. assume that λ is $\mathbf{d}$-minimal, so that, by Proposition 4.22, we have a group action π_λ of G on the $\mathfrak{g}^{\mathbb{C}}_{\mathrm{alg}}$-module $\mathcal{H}^{\infty}_{\lambda} \subset \mathcal{H}_\lambda$ of smooth vectors.
On the projective space $\mathbb{P}(\mathcal{H}^{\infty}_{\lambda})$ we consider the *momentum map*

$$\Phi_\lambda : \mathbb{P}(\mathcal{H}^{\infty}_{\lambda}) \to \mathfrak{g}', \quad \Phi_\lambda([v])(x) := -i\frac{\langle \rho_\lambda(x)v, v\rangle}{\langle v, v\rangle} \text{ for all } x \in \mathfrak{g}. \tag{4.3}$$

For a generating weight vector v_λ and any $g \in G$, we may calculate

$$\begin{aligned} \Phi_\lambda([\pi_\lambda(g)(v_\lambda)])(x) &= -i\langle \rho_\lambda(x)\pi_\lambda(g)v_\lambda, \pi_\lambda(g)v_\lambda\rangle \\ &= -i\langle \rho_\lambda(\mathrm{Ad}(g^{-1})x)v_\lambda, v_\lambda\rangle = \Phi_\lambda([v_\lambda])(\mathrm{Ad}(g^{-1})x). \end{aligned} \tag{4.4}$$

By another direct calculation, for any weight $\gamma \in \mathcal{P}_\lambda$ (including λ in particular) and corresponding weight vector $v_\gamma \in L_\gamma \subset \mathcal{H}_\lambda^\infty$, we find

$$\Phi_\lambda([v_\gamma])(x) = -i\frac{\langle \rho_\lambda(x)v_\gamma, v_\gamma\rangle}{\langle v_\gamma, v_\gamma\rangle} = -i\frac{\langle \gamma(x)v_\gamma, v_\gamma\rangle}{\langle v_\gamma, v_\gamma\rangle} = -i\gamma(x) \text{ for all } x \in \mathfrak{t}_{\mathfrak{g}_{\mathrm{alg}}}^{\mathbb{C}},$$

which means $\Phi_\lambda([v_\gamma])|_{\mathfrak{t}_\mathfrak{g}} = -i\gamma$, so (4.4) implies that

$$\mathcal{O}_\lambda|_{\mathfrak{t}_\mathfrak{g}} \subseteq \mathrm{Im}(\Phi_{\pi_\Lambda})|_{\mathfrak{t}_\mathfrak{g}}.$$

Next, we relate the right hand side to convexity; for this, we note that $\mathcal{H}_\lambda$ has an ONB of weight vectors so that every $[v_0] \in \mathbb{P}(\mathcal{H}_\lambda)$ can be represented by a normed vector $v \in \mathcal{H}_\lambda$ satisfying $v = \sum_{\gamma\in\mathcal{P}_\lambda} q_\gamma b_\gamma$ with orthonormal basis vectors b_γ, coefficients $q_\gamma \in [0,1]$ and $\sum_{\gamma\in\mathcal{P}_{i\lambda}} q_\gamma^2 = 1$. Thus, for every $x \in \mathfrak{t}_\mathfrak{g}$:

$$\Phi_\lambda([v_0])(x) = -i\langle \rho(x)v, v\rangle = -i \sum_{\gamma\in\mathcal{P}_{i\lambda}} q_\gamma^2 \gamma(x),$$

and therefore $\mathrm{Im}(\Phi_\lambda)|_{\mathfrak{t}_\mathfrak{g}} \subseteq -i\overline{\mathrm{conv}}(\mathcal{P}_{i\lambda})$. Proposition 4.21(ii) now implies that

$$\overline{\mathrm{conv}}(\mathcal{P}_\lambda) = \overline{\mathrm{conv}}(\mathcal{W}.\lambda),$$

which completes the proof. □

Proposition 4.24. *If $\mathfrak{k}$ is finite-dimensional, then*

$$\mathcal{O}_\lambda|_{i\mathfrak{t}_\mathfrak{g}} \subseteq \overline{\mathrm{co}}(\lambda) \tag{4.5}$$

for all weights $\lambda \in i\mathfrak{t}_\mathfrak{g}'$ with $\lambda(\mathbf{c}) \neq 0$.

Proof. We choose a positive system $\Delta^+ \subset \Delta_i$ with $\lambda(\Delta^+) \subseteq \mathbb{N}_0$, and from that a family $(\alpha_s)_{s\in S}$ such that $\{\alpha_s : s \in S\}$ is a simple system of Δ_i. Theorem 4.6 applies to $i\mathfrak{t}_\mathfrak{g}'$ and $\mathcal{W}$, which means that $\mathcal{W}.\gamma \subseteq \gamma + C_\gamma$ for every $\gamma \in i\mathfrak{t}_\mathfrak{g}'$ that satisfies $\mathrm{sgn}(\gamma(\mathbf{c})) = \mathrm{sgn}(\lambda(\mathbf{c})) \neq 0$. If, in addition, γ lies in the Tits cone associated with $(\alpha_s)_{s\in S}$, then

$$\mathcal{W}.\gamma \subseteq \gamma + C_\gamma \subseteq \gamma + C_S. \tag{4.6}$$

We denote the set of integral weights by $\mathcal{Q}$ and note that they form a lattice generating $i(\mathfrak{t}_0' + \mathbb{R}\mathbf{d}^*) \hookrightarrow i\mathfrak{t}_\mathfrak{g}'$, which implies that

$$\mathcal{Q}_\mathbb{Q} := \mathrm{span}_\mathbb{Q}(\mathcal{Q}) = \frac{1}{\mathbb{N}}\mathcal{Q}$$

is dense in $i(\mathfrak{t}_0' + \mathbb{R}\mathbf{d}^*)$. Note that Proposition 4.2 and Remark 4.5 imply that $C_S \subset i(\mathfrak{t}_0' + \mathbb{R}\mathbf{d}^*)$ has interior points, and thus $\lambda - C_S$ has nonempty interior relative to the affine hyperplane $\lambda + i(\mathfrak{t}_0' + \mathbb{R}\mathbf{d}^*)$. Note further that both the prerequisites and conclusion of this proposition are stable under multiplication with positive constants, so we can w.l.o.g. assume that $\lambda(\mathbf{c})$ is rational; from this follows that there exists a sequence $(\nu_j)_{j\in\mathbb{N}}$ approximating λ, such that

$\nu_j \in \mathcal{Q}_{\mathbb{Q}} \cap (\lambda - C_S)$, and therefore $\nu_j - C_S \subseteq \lambda - C_S$ for every $j \in \mathbb{N}$.
Because the inclusion (4.2) from Theorem 4.23 is stable under multiplication with positive constants, it also applies to ν_j for every $j \in \mathbb{N}$. Combining it with (4.6) we get

$$\mathcal{O}_{\nu_j}|_{i\mathfrak{t}_\mathfrak{g}} \subseteq \nu_j - C_{\nu_j} \subseteq \lambda - C_S,$$

and thus

$$\mathcal{O}_{\nu_j}|_{i\mathfrak{t}_\mathfrak{g}} \subseteq \sec(\nu_j) \subseteq \sec(\lambda).$$

Now the right hand side is independent from j, and from the closedness of $\sec(\lambda)$ we conclude that, for every $g \in G$,

$$\lim_{j\to\infty} (\mathrm{Ad}^*(g)\nu_j)|_{i\mathfrak{t}_\mathfrak{g}} \in \sec(\lambda),$$

i.e. $\mathcal{O}_\lambda|_{i\mathfrak{t}_\mathfrak{g}} \subseteq \sec(\lambda)$, and by applying Theorem 4.12 and Lemma 4.8(v) this becomes

$$\mathcal{O}_\lambda|_{i\mathfrak{t}_\mathfrak{g}} \subseteq \overline{\mathrm{co}}(\lambda). \qquad \square$$

Theorem 4.25. *If $\mathfrak{g}$ is a double extended loop algebra over some simple Hilbert–Lie algebra $\mathfrak{k}$, then*

$$\mathcal{O}_\lambda|_{i\mathfrak{t}_\mathfrak{g}} \subseteq \overline{\mathrm{co}}(\lambda)$$

for all weights $\lambda \in i\mathfrak{t}'_\mathfrak{g}$ with $\lambda(\mathbf{c}) \neq 0$.

Proof. By 3.32, there exists an ascending sequence $(\mathfrak{g}_j)_{j\in\mathbb{N}}$ of double extensions of loop algebras such that $\mathfrak{g}_{\mathrm{fin}} = \bigcup_{j\in\mathbb{N}} \mathfrak{g}_j$ is dense in $\mathfrak{g}$. Then for every $j \in \mathbb{N}$ we set $\mathfrak{g}_{j,0} := \mathfrak{g}_j \cap \mathcal{L}_\varphi\mathfrak{k}$ and obtain a decomposition $\mathfrak{g} = \mathfrak{g}_j \oplus \mathfrak{r}_j$, where $\mathfrak{r}_j := \mathfrak{g}_{j,0}^\perp$ with respect to the invariant inner product on $\mathcal{L}_\varphi\mathfrak{k}$. The orthogonality implies that $\alpha(z) = (z, \alpha^\sharp) = 0$ for all $\alpha \in \Delta_j := \Delta(\mathfrak{g}_j^\mathbb{C}, \mathfrak{t}_\mathfrak{g}^\mathbb{C} \cap \mathfrak{g}_j^\mathbb{C})$ and $z \in \mathfrak{r}_j \cap \mathfrak{t}_0$, and thus $[z, \mathfrak{g}_j] = \{0\}$; we can use this to approximate adjoint and coadjoint orbits via an exhaustion of Lie groups:
Every $\mathfrak{g}_j$ contains $\mathbb{R}\mathbf{c}$, the center of $\mathfrak{g}$, so we can define Lie groups $G_j \subset G$ acting on $\mathfrak{g}$ via the adjoint action as $G_j := \langle \exp_{\mathcal{L}_\Phi K \rtimes_R \mathbb{T}}(\mathfrak{g}_j/\mathbb{R}\mathbf{c})\rangle$.
Thus, for every $x \in \mathfrak{g}$ we can write $x = y_j + z_j \in \mathfrak{g}$ with $y_j \in \mathfrak{t}_\mathfrak{g} \cap \mathfrak{g}_j$ and $z_j \in \mathfrak{t}_\mathfrak{g} \cap \mathfrak{r}_j$, and get $\mathrm{Ad}(G_j)(x) = \mathrm{Ad}(G_j)(y_j) + z_j$ for the adjoint orbit of x under G_j.
A corresponding decomposition of $\mathfrak{g}'$ is obtained by $\lambda = \mu_j + \nu_j$ with $\mu_j(\mathfrak{r}_j) = \{0\}$ and $\nu_j(\mathfrak{g}_j) = \{0\}$ for all $\lambda \in \mathfrak{g}'$, i.e. $\mathfrak{g}' = \mathfrak{g}'_j \oplus \mathfrak{r}'_j$.
These decompositions are compatible with the injection of $\mathfrak{t}'_\mathfrak{g}$ into $\mathfrak{g}'$ corresponding to κ, and they extend linearly to the complexifications of $\mathfrak{g}$ and $\mathfrak{g}'$. So for every $g \in G_j$ and $x = y_j + z_j \in i\mathfrak{t}_\mathfrak{g}$ we have

$$\mathrm{Ad}_g^*(\lambda)(x) = \lambda(\mathrm{Ad}_{g^{-1}}(x)) = \lambda(\mathrm{Ad}_{g^{-1}}(y_j) + z_j) = \mu(\mathrm{Ad}_{g^{-1}}(y_j)) + \nu_j(z_j),$$

which implies

$$\mathrm{Ad}^*(G_j)(\lambda)|_{i\mathfrak{t}_\mathfrak{g}} = \mathrm{Ad}^*(G_j)(\mu_j)|_{i\mathfrak{t}_\mathfrak{g}} + \nu_j \text{ for all } \lambda \in i\mathfrak{t}'_\mathfrak{g}.$$

Now for every $j \in \mathbb{N}$, Proposition 4.24 applies to G_j and $\mu_j \in i\mathfrak{g}'_j$, so

$$\mathrm{Ad}^*(G_j)(\lambda)|_{i\mathfrak{t}_\mathfrak{g}} \subseteq \overline{\mathrm{conv}}(\mathcal{W}_j.\mu_j) + \nu_j \subseteq \overline{\mathrm{co}}(\lambda),$$

where $\mathcal{W}_j := \mathcal{W}(\Delta_j)$ and the last inequality is due to $\mathcal{W}_j \subseteq \mathcal{W}$ and ν_j being invariant under $\mathcal{W}_j$. Now every $g \in G$ can be approximated by some sequence $(g_j)_{j\in\mathbb{N}}$ with $g_j \in G_j$, and we thus obtain our assertion. □

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