Generalised Geometries for Type II and M Theory

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by

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I certify that this thesis, and the research to which it refers, are the product of my own work, and that any ideas or quotations from the work of other people, published or otherwise, are fully acknowledged in accordance with the standard referencing practices of the discipline.

André Janeiro Coimbra

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Para o meu pai, a minha mãe e o meu irmão.

Abstract

In this thesis a new formulation is presented of the low energy, supergravity limit of type II string theory and M theory, including fermions to leading order. This is performed by utilising the language of generalised geometry, which is shown to be the natural setting for these theories.

The core idea behind generalised geometry – an extension of ordinary differential geometry – and what makes it such a powerful tool for analysing supergravity, is that it recasts all the bosonic fields of the manifold as the natural geometric symmetries of an enlarged tangent space. There are two versions of generalised geometry which are of particular interest, namely $O(d, d)$ generalised geometry which will be used to formulate the NSNS sector of type II theories, and $E_{d(d)}$ generalised geometry (also known as exceptional generalised geometry) which enables the description of eleven-dimensional supergravity. For both cases, this work will show how one can introduce generalised connections to study the differential structure of the extended tangent spaces and define novel notions of generalised curvature. Specifying extra local structure defines a generalised notion of the Riemannian metric tensor, which contains all the relevant bosonic fields in a single, unified object.

With these tools one can then reformulate the supergravity equations very naturally, as they become simply the generalised geometry analogue of Einstein gravity. One thus obtains a formalism which is automatically fully covariant under all the bosonic symmetries of supergravity. Furthermore, generalised connections are shown to be intimately related to supersymmetry, with important consequences for future applications. As an example, in the concluding chapter it will be shown how the classic problem of solving the Killing spinor equations of supersymmetric compactifications can be equivalently recast as the statement that the background possesses the generalised analogue of special holonomy.

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List of Publications

This work was based on the following publications:

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Chapter 1

Introduction

The foundations of modern physics rest on two major pillars, two theoretical constructs which have been validated empirically to an extraordinary degree, general relativity and the quantum field theory of the standard model. Yet these two are mutually inconsistent – direct quantisation of Einstein-Hilbert gravity fails miserably due to its non-renormalisability. And while it could have been the case that there were no situations in the Universe where both theories need be applied, so that they could coexist peacefully, we are aware of phenomena in nature where they must, such as astronomical black holes and the early universe. Thus if physics is to provide a complete description of reality then one needs to find a new theory which replaces the two current standards while still being able to recover them at the appropriate limits, a theory-of-everything(-that-we-are-currently-aware-of). At the time of writing, the leading candidate for such a theory is string theory.

String theory is a quantum field theory (of one-dimensional objects, as opposed of point-like objects) which contains Einstein-Hilbert gravity at its lowest level in perturbation theory (and with several corrections at higher level as should be expected). Since it was originally formulated in the late 1960s, it has been established that string theory has to satisfy certain internal consistency conditions in order to obtain a proper quantum theory, which result in some very non-trivial consequences. These requirements include the absence of negative norm states, which can be attained by formulating the theory at the appropriate critical spacetime dimension, or a lack of tachyons in its physical spectrum, which results from positing worldsheet supersymmetry (or equivalently, supersymmetry in the target spacetime) and forces the critical dimension to be ten[1]. There are in fact five different string theories which satisfy these properties, type IIA, IIB, type I, $SO(32)$-heterotic and $E_8 \times E_8$-heterotic. The low energy limit of these theories is a particular supergravity theory in ten-dimensions, respectively, type IIA, type IIB and type I coupled to different super Yang-Mills theories, and it is by considering the supergravity limits that we can study several of the properties of the string theories, especially if we are to have any hope of relating them to any phenomenology. The fact that there exist five different string theories might seem problematic but in 1995 [1, 2] it was realised that this is not really an issue – all five theories are in fact describing the same reality, just "interpreting" it differently, in the sense that, for example, a solution to $SO(32)$-heterotic in a small coupling constant limit is the same as a solution to type I in the large coupling constant limit (and vice-versa). This is known as a duality transformation and when they are all put together they form a web of relations connecting all the string theories, with a surprising corollary of their existence being the discovery of a sixth theory, namely M theory, which does not contain strings but is nonetheless dual to the string theories. M theory is formulated on an eleven-dimensional spacetime and its low energy limit is eleven-dimensional supergravity.

Supergravity in eleven-dimensions, first constructed in [3], is special – it is uniquely determined by the requirement that the graviton multiplet does not contain fields of spin higher than two (a traditional requirement as there is no known way of formulating a consistent interacting theory with a finite number of fields with spin higher than two[2]), or equivalently, a maximum of 32 supercharges. Since generic spinors of $\mathrm{Cliff}(10,1)$ contain 32 components (and are non-chiral), the minimum amount of supercharges allowed is precisely 32 and $N = 1$ supergravity is the only possibility left for constructing a consistent theory. For dimensions higher than eleven spinors always have more than 32 components, making eleven dimensional supergravity the highest possible dimension on which one can formulate a consistent theory.

One can then descend from eleven dimensions to ten. If one does this while preserv-

[1] Note that in this regard string theory is one of very few examples of physical theories which actually are able to predict a dimension for spacetime, as opposed to taking it as an initial input or axiom.

[2] To clarify, consistent theories with an infinite number of fields with arbitrarily high spin have been constructed [4, 5] and are currently subject of intensive research.

ing the 32 supersymmetries one then obtains one of the two maximal supergravities in $D = 10$, type IIA, and then by performing a duality transformation one gets the other maximal supergravity, type IIB. Further, if one wishes to study the properties of other maximal supergravities, then one should be able to describe reductions down to $D = 11 - d$ dimensions and, indeed, if we wish to relate string/M theory to our observed four-dimensional reality, we better be able to account for the cases at least up to $d \leq 7$. In fact one of the main uses for eleven-dimensional supergravity was to study the properties of maximal $N = 8$ supergravity in four dimensions. In the late 1970s Cremmer and Julia [6, 7, 8] toroidally reduced eleven-dimensional supergravity down to $D = 4$ and discovered that it contains a large group of hidden symmetries – a global, non-compact $E_{7(7)}$ and a local $SU(8)$ group. Subsequently it was shown that reducing to $11 - d$ dimensions leads to a a global, non-compact $E_{d(d)}$ and a local symmetry group which is the maximal compact subgroup $H_d \subset E_{d(d)}$. As we will see, generalised geometry will allow us to describe a much broader class of backgrounds than rectangular tori.

First some historical background. Generalised geometry (also known as $O(d, d)$ generalised geometry, to distinguish from other versions that were introduced later) was introduced by Hitchin to provide a unified description of complex manifolds and symplectic manifolds, with an ultimate goal of generalising the notion of Calabi-Yau manifolds [9]. Instead of looking at structures defined on the tangent space TM of a manifold M, Hitchin proposed considering "generalised structures" on the bundle $TM \oplus T^*M$. For a d-dimensional manifold, this bundle comes automatically equipped with a global $O(d, d)$ metric η and admits a generalisation of the Lie bracket, the Courant bracket [10]. The Courant bracket possesses extra symmetries, it is invariant not just under diffeomorphisms but also under B-shifts, i.e. transformations by a closed two-form field B. Gualtieri expanded the formalism [11, 12], introducing the generalised metric – the natural object that arises when one introduces the local maximal compact subgroup $O(d) \times O(d) \subset O(d, d)$ and which unifies the ordinary Riemannian metric with the B gauge field – and generalised connections, which allow one to define covariant derivatives on the generalised tangent space. These are the basic tools that we will be making use of in this thesis.

All together, generalised geometry can be seen as a mechanism for covariantising the two-form Kalb-Rammond field B of the NSNS sector of string theory. For physicists, the formalism thus provided a systematic way of studying supersymmetric compactifications that went beyond Calabi-Yau by including the NSNS fluxes $H = \mathrm{d}B$. This has permitted the classification of entire new classes of solutions (for example in [13, 14, 15, 16, 17, 18, 19, 20]), with applications to, for instance, the AdS/CFT correspondence (see for example [21, 22, 23, 24, 25, 26]). It has in addition, been able to shed some light in the problem of non-geometric backgrounds.

Non-geometry is an intrinsic part of string theory, a result of its non-local version of quantum gravity. It can show up when one attempts to make a generic duality transformation away from a known background, resulting in a theory which no longer possesses a spacetime which can be described by the usual tools of differential geometry. U-duality, in its more narrow sense, is a non-pertubative symmetry that arises when one toroidally reduces a string theory down to $11-d$ dimensions, with $d \leq 8$. The resulting theory is then invariant under $E_{d(d)}(\mathbb{Z})$ transformations [1]. The moduli of solutions of the low energy limit of this theory, that is, maximal $11-d$-dimensional supergravity, is thus described by the coset $E_{d(d)}(\mathbb{Z})\backslash\, E_{d(d)}(\mathbb{R})/\, H_d$. A subset of the U-duality transformations that has been extensively studied is T-duality, which is given by the $O(d,d;\mathbb{Z}) \subset E_{d(d)}(\mathbb{Z})$ subgroup. Unlike generic U-dualities which are non-perturbative, T-duality invariance hold at each level of perturbation theory and in certain cases it can be interpreted in a very intuitive geometric picture from the point of view of the target space of the string worldsheet – for instance, if the background is a fluxless compactification on a circle of radius R, T-duality exchanges that with a compactification on a circle of radius $1/R$, and, for closed strings, the winding modes around the circle get exchanged with momentum modes. Non-geometry can be observed in the case where one starts with a d-dimensional torus bundle background with NSNS flux along the torus fibres. Performing a first T-duality along one of those directions results in a new background with a different topology due to presence of so-called geometric fluxes (see for instance [27, 28]). The map between the two spaces is by now well understood, especially so in the context of generalised geometry [29]. If one T-dualises again, now along a different direction of flux, then one finds that, while it is still possible to describe the resulting space locally in terms of generalised geometry [30], there is no longer a well defined global picture in that language. In [31] Hull proposed that for those cases one instead consider what he called T-folds – briefly, these are spaces with local patches which look like ordinary manifolds, but are glued together by $O(d, d; \mathbb{Z})$ T-duality transformations instead of just diffeomorphisms. T-folds can be constructed by defining a new "doubled torus" bundle, of dimension $2d$, and then specifying a d-dimensional (no longer geometric) subspace with those transition functions. The different T-dual configurations are then particular examples of allowed "slicings" of the doubled torus.

There have been other attempts at attaining a deeper understanding of T-duality, and early attempts at making those symmetries more manifest include [32, 33, 34]. A somewhat similar approach that has recently received quite a bit of attention is Double Field Theory (DFT), introduced in 2008 by Hull and Zweibach [35]. Inspired by the doubled torus example, in DFT fields live in a doubled manifold with "winding coordinates" (i.e. canonically conjugate to winding modes on a torus) which are dual to the usual coordinates (i.e. conjugate to momentum modes), with this duality being formalised by postulating the existence of the flat $O(d, d)$ metric η globally defined on the doubled manifold. This results in a theory that closely mirrors the earlier work of Siegel [36, 37] on "two-vierbein formalism", where he builds a gravitational theory based on local $GL(d; \mathbb{R}) \times GL(d; \mathbb{R})$ doubled frames. The reason the two formulations match can be traced to the fact that Siegel also demands compatibility of his two-vierbein-connections with the $O(d, d)$ metric, thus reducing the structure group to the common subgroup of $O(d, d)$ and $GL(d; \mathbb{R}) \times GL(d; \mathbb{R})$, that is, $O(d) \times O(d)$ which is precisely the same local structure in DFT. A more extensive comparison of Siegel's work and that of Hull and Zweibach can be found in [38].

DFT has more peculiarities. There exists a constraint, imposed by considering the closed string worldsheet perspective, that the fields are not allowed a completely general dependence on the coordinates of the doubled manifold, rather they must satisfy the so-called weak constraint: any field A must satisfy $\partial^2 A = 0$. In subsequent work [39, 40] it was realised that in order to formulate a completely background independent theory one must borrow several of the concepts of generalised geometry, which is possible since the tangent space of the doubled manifold coincides in many respects the $O(d, d)$ generalised tangent space. Indeed, it turns out that as a requirement to make the theory consistent – in particular for it to be diffeomorphism and gauge invariant – one must impose an even stronger constraint, also known as a section condition, that effectively forces the fields to depend only on half the coordinates of the doubled manifold. One can then always use the $O(d,d)$ symmetry to rotate locally this surviving set of coordinates so that they match the usual momentum coordinates – thus “undoubling” the theory. This means that under the strong constraint, DFT produces no new solutions as compared to the generalised geometrical description, at least locally. Nonetheless the formalism has been extensively developed [41, 42, 38, 43, 44, 45, 46, 47, 48, 49, 50, 51], and while originally the DFT version of the NSNS action was formulated in terms of first-order derivatives of doubled objects, new constructions [49] based on the DFT analogues of generalised connections have appeared which have enabled the construction of full DFT Riemann-like tensors. In particular, the work of Jeon, Lee and Park [52, 53, 54, 55, 56] in terms of “semi-covariant” derivatives on the doubled space has culminated in a full rewriting of the type II actions, matching the results obtained in [57].

Differences between generalised geometry and DFT can therefore only potentially arise in respect to global considerations or by relaxing the strong constraint. Recent work has moved in this direction by shifting the focus from the worldsheet perspective to looking directly at the closure of the symmetry algebras in DFT. In particular backgrounds one is able to leverage additional symmetries in order to formulate a consistent DFT which does not respect the strong constraint, and these constructions have been used to describe non-geometric backgrounds associated to gauged supergravities [46, 47, 45, 50, 58].

In this thesis we will actually consider a broader class of “generalised geometries”, as we are interested not just in the NSNS sector of the type II theories, but also in the gauge fields of eleven-dimensional supergravity. In 1986 de Wit and Nicolai [59] showed that the hidden symmetries of $N = 8$, $D = 4$ supergravity discovered by Cremmer and Julia – the global $E_{7(7)}$ and local $SU(8)$ – can in fact be realised at the $D = 11$ level. By partially gauge fixing the tangent space structure to $Spin(3,1) \times Spin(7) \subset Spin(10,1)$, they demonstrated that one can enhance this local group to obtain eleven-dimensional supergravity, to first order in fermions, with a manifest $Spin(3,1) \times SU(8)$ symmetry. The other groups that appear in toroidal reductions (all the way down to two dimensions) have also been shown to exist in eleven-dimensions, see for instance [60, 61, 62, 63]. The $E_{d(d)} \times \mathbb{R}^+$ formalism we will present in thesis should be seen as the natural geometrisation of these results.

$E_{d(d)} \times \mathbb{R}^+$ generalised geometry was initially developed independently by Hull [64] and Waldram and Pacheco [65], who named it exceptional generalised geometry. Like Hitchin's original $O(d,d)$ version, the key concept here is the introduction of an enlarged tangent space whereupon one covariantises the gauge transformations of eleven-dimensional supergravity as an intrinsic part of the geometry. And just like in the original version, one is able to introduce a generalised version of the Lie bracket, which was called the exceptional Courant bracket in [65]. It turns out that Gualtieri's formalism for generalised connections and generalised metric (now obtained by specifying the local $H_d \subset E_{d(d)}$ subgroup) also translates over almost verbatim.

There are other constructs inspired by the appearance of the $E_{d(d)}$ hidden symmetry groups in eleven-dimensional supergravity reductions. In particular, West in 2001 proposed that the full underlying symmetry of M theory should be described by E_{11} [66]. Recall that E_{11}, sometimes also denoted E_8^{+++}, is the infinite-dimensional Kac-Moody algebra obtained from triple extending the E_8 Dynkin diagram. Similarly to the DFT idea, in West's model spacetime is enlarged [67] (now by an infinite number of extra dual coordinates), such that its tangent space transforms as an E_{11} representation. A field theory was then constructed based on non-linear realisations of E_{11} over its local (also infinite dimensional) maximally compact subgroup [68, 69, 70]. Evidence for this conjecture had been found by decomposing E_{11} into one of its E_d subalgebras, and then truncating all the extra coordinates while keeping the part of the enlarged tangent space which transforms under E_d [71, 72, 73]. A line of research that was originally developed independently of the E_{11} programme is the one of Berman and Perry [74], which essentially reinterpreted the much earlier work of [75] from the perspective of DFT. In [75] Duff and Lu built upon [32] to provide an explanation for the existence of the $E_{d(d)}$ and H_d hidden symmetries of toroidal reductions (for $d < 5$) by showing that they can be realised directly in the worldvolume theory of (what is now known as) the M2 brane of M theory. Based on that result, Berman and Perry proposed that, just like in DFT, one should introduce a larger manifold with extra coordinates which here correspond to M2 wrapping modes instead of the string's winding modes. However, in order to produce a consistent theory (with diffeomorphism and gauge invariance) they again needed to reduce down to momentum coordinates. The mechanism for performing this reduction was called the "section condition", and it is essentially the $E_{d(d)}$ version of the strong constraint of DFT. Just like in that case, the resulting setup ends up matching very closely that of exceptional generalised geometry [76, 77]. This model has been expanded in [78, 79, 80] and the whole construction was incorporated into the E_{11} formalism in [73]. In that paper the authors succeed in providing expressions for Lagrangians based on this method for all $d < 8$ (in this thesis we also consider just those dimensions), though the "curvature" scalars they built were only first-order derivatives of fields, and thus the resulting actions are equivalent to the usual supergravity ones only up to total derivatives. Additionally, most of these constructions stop short of including fermions. One notable exception is the work of Hillmann [81, 82], who, picking up West's E_{11} non-realisations, focused on the $d = 7$ case and introduced a sixty dimensional spacetime (4+56, 56 being the dimension of the smallest nontrivial representation of E_7). By using "generalised $E_{7(7)}$ coset dynamics" and demanding that, upon truncating to 4+7 dimensions, the theory possess $\mathrm{Diff}(7)$ invariance, the author managed to show that the construction reproduces the results of [59]. This dimensional truncation, which mirrors the effects of the section condition, implies that again the enlarged tangent space of Hillmann is precisely the exceptional generalised tangent space of [65]. The geometrical objects he constructs, together with the fermion fields, can then be directly mapped to the ones we obtain in the $d = 7$ section of [83].

One can thus see that a lot of the success of these different approaches is a direct result of their several points of contact with generalised geometry. This is no coincidence, as we hope the reader will come to agree by the end of this thesis. Generalised geometry is the natural language to formulate supergravities. Its relation to their bosonic symmetries is precisely the same as that between Riemannian geometry and Einstein gravity. We will demonstrate this by proceeding as follows.

In the next chapter we will review type II and eleven-dimensional supergravities, establishing conventions and the expressions we intend to recover from generalised geometry. A word of caution to the reader – we try to provide formulations which strike a balance between the typical conventions of these supergravities and the ones which arise more naturally in generalised geometry. It is therefore possible that some steps we take in that chapter might seem unusual, as their reason will only become clear later in the thesis. This treatment of the supergravities was described in [57, 83].

In chapter three we review $O(d, d) \times \mathbb{R}^+$ generalised geometry. This is a very similar construction to that originally provided by Gualtieri. We examine the properties of the generalised tangent space, its differential structure and introduce generalised connections. The existence of a local $O(p,q)\times O(q,p)$ structure allows us to construct all the analogues of familiar objects of (pseudo-)Riemannian geometry such as metric and curvature. These tools allow us to geometrise the NSNS sector of ten-dimensional type II supergravity theories in chapter four and obtain a reformulation with the $O(9,1) \times O(1,9)$ symmetry manifest. These chapters follow from our work in [57].

To geometrise the RR fields we have to move to $E_{d(d)} \times\mathbb{R}^+$ generalised geometry with a local H_d symmetry, which we introduce in chapter six. We proceed in similar fashion to the $O(d,d) \times \mathbb{R}^+$ chapter, though the analysis here is more involved as, on the face of it, each dimension d would have to be considered separately. However, by constructing the relevant groups in terms of their $GL(d)$ and $SO(d)$ subgroups and introducing some new notation, we manage to maintain the discussion completely generic. The generalised metric, generalised connections and their curvatures are all then constructed in completely analogous ways to those of chapter three. Indeed, the method we use in this chapter is kept so general that it should be possible to adapt it directly to other kinds of generalised geometries, not just those based on $E_{d(d)} \times\mathbb{R}^+$. Most of the content in this chapter was first given in [76].

We are then able to rewrite dimensional reductions of eleven-dimensional supergravity in chapter seven. Here, in addition to providing the generic description for all d, we also study in explicit detail two cases: $d = 4$, corresponding to compactifications to $D = 11 - d = 7$ dimensions, which is a relatively simple case with a local group $H_4 = SO(5)$; and the more complex $d = 7$, local group $H_7 = SU(8)/\mathbb{Z}_2$, which, since it describes compactifications to four dimensions, is arguably the most important formulation of $E_{d(d)} \times\mathbb{R}^+$ generalised geometry. These results were first presented in [83].

In the conclusion we review the results and discuss their potential applications, with a particular emphasis on supersymmetric backgrounds, and possible extensions.

Chapter 2

Type II and eleven-dimensional supergravity theories

In this chapter we briefly review the usual formulations of ten-dimensional type II supergravity and eleven-dimensional supergravity. These will provide us with the expressions that we will try to reproduce in the latter chapters based on generalised geometry.

2.1 Type II supergravity

Our basic conventions are given in appendix A and for the $\mathrm{Cliff}(9,1;\mathbb{R})$ Clifford algebra conventions see appendix C.4. We essentially follow those of the democratic formalism [84], with the only difference which is not purely notational being that we take the opposite sign for the Riemann tensor, as discussed in appendix B. We consider only the leading-order fermionic terms. We introduce a slightly unconventional notation in a few places in order to match more naturally with the underlying generalised geometry. It is also helpful to considerably rewrite the fermionic sector, introducing a particular linear combination of dilatini and gravitini, to match more closely what follows.

The type II fields are denoted

$$\{g_{\mu\nu}, B_{\mu\nu}, \phi, A^{(n)}_{\mu_1 \ldots \mu_n}, \psi^{\pm}_{\mu}, \lambda^{\pm}\}, \tag{2.1}$$

where $g_{\mu\nu}$ is the metric, $B_{\mu\nu}$ the two-form potential, ϕ is the dilaton and $A^{(n)}_{\mu_1\dots\mu_n}$ are the RR potentials in the democratic formalism, with n odd for type IIA and n even for type IIB. In each theory there is also a pair of chiral gravitini $\psi^\pm_\mu$ and a pair of chiral dilatini $\lambda^\pm$. Here our notation is that $\pm$ does *not* refer to the chirality of the spinor but, as we will see, denote generalised geometrical subspaces. Specifically, in the notation of [84], for type IIA they are the chiral components of the gravitino and dilatino

$$\begin{aligned} \psi_\mu &= \psi^+_\mu + \psi^-_\mu \qquad \text{where} \quad \gamma^{(10)}\psi^\pm_\mu = \mp\psi^\pm_\mu \\ \lambda &= \lambda^+ + \lambda^- \qquad \text{where} \quad \gamma^{(10)}\lambda^\pm = \pm\lambda^\pm . \end{aligned} \tag{2.2}$$

(Note that ψ^+_μ and λ^+, and similarly ψ^-_μ and λ^-, have *opposite* chiralities.) For type IIB, in the notation of [84] one has two component objects

$$\begin{aligned} \psi_\mu &= \begin{pmatrix} \psi^+_\mu \\ \psi^-_\mu \end{pmatrix} \qquad \text{where} \quad \gamma^{(10)}\psi^\pm_\mu = \psi^\pm_\mu \\ \lambda &= \begin{pmatrix} \lambda^+ \\ \lambda^- \end{pmatrix} \qquad \text{where} \quad \gamma^{(10)}\lambda^\pm = -\lambda^\pm . \end{aligned} \tag{2.3}$$

and again the gravitini and dilatini have opposite chiralities.

In what follows, it will be very useful to consider the quantities

$$\rho^\pm := \gamma^\mu\psi^\pm_\mu - \lambda^\pm , \tag{2.4}$$

instead of $\lambda^\pm$. These are the natural combinations that appear in generalised geometry and from now on we will use $\rho^\pm$ rather than $\lambda^\pm$.

The bosonic "pseudo-action" takes the form

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int \sqrt{-g}\left[\mathrm{e}^{-2\phi}\left(\mathcal{R} + 4(\partial\phi)^2 - \tfrac{1}{12}H^2\right) - \tfrac{1}{4}\sum_n \tfrac{1}{n!}(F^{(B)}_{(n)})^2\right], \tag{2.5}$$

where $H = \mathrm{d}B$ and $F^{(B)}_{(n)}$ is the n-form RR field strength. Here we will use the "A-basis", where the field strengths, as sums of even or odd forms, take the form[1]

$$F^{(B)} = \sum_n F^{(B)}_{(n)} = \sum_n \mathrm{e}^B \wedge \mathrm{d}A_{(n-1)}, \tag{2.6}$$

where $\mathrm{e}^B = 1 + B + \frac{1}{2}B \wedge B + \dots$. This is a "pseudo-action" because the RR fields satisfy a self-duality relation that does not follow from varying the action, namely,

$$F^{(B)}_{(n)} = (-)^{[n/2]} * F^{(B)}_{(10-n)}, \tag{2.7}$$

where $[n]$ denotes the integer part and $*\,\omega$ denotes the Hodge dual of ω. The fermionic action, keeping only terms quadratic in the fermions, can be written after some manipulation as

$$\begin{aligned} S_{\mathrm{F}} = -\frac{1}{2\kappa^2}\int \sqrt{-g}\Big[&\mathrm{e}^{-2\phi}\Big(2\bar\psi^{+\mu}\gamma^\nu\nabla_\nu\psi^+_\mu - 4\bar\psi^{+\mu}\nabla_\mu\rho^+ - 2\bar\rho^+\nabla\!\!\!/\,\rho^+ \\ &\qquad - \tfrac{1}{2}\bar\psi^{+\mu}H\!\!\!/\,\psi^+_\mu - \bar\psi^+_\mu H^{\mu\nu\lambda}\gamma_\nu\psi^+_\lambda - \tfrac{1}{2}\rho^+ H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi^+_\lambda + \tfrac{1}{2}\rho^+ H\!\!\!/\,\rho^+\Big) \\ &+ \mathrm{e}^{-2\phi}\Big(2\bar\psi^{-\mu}\gamma^\nu\nabla_\nu\psi^-_\mu - 4\bar\psi^{-\mu}\nabla_\mu\rho^- - 2\bar\rho^-\nabla\!\!\!/\,\rho^- \\ &\qquad + \tfrac{1}{2}\bar\psi^{-\mu}H\!\!\!/\,\psi^-_\mu + \bar\psi^-_\mu H^{\mu\nu\lambda}\gamma_\nu\psi^-_\lambda + \tfrac{1}{2}\rho^- H^{\mu\nu\lambda}\gamma_{\mu\nu}\psi^-_\lambda - \tfrac{1}{2}\rho^- H\!\!\!/\,\rho^-\Big) \\ &- \tfrac{1}{4}\mathrm{e}^{-\phi}\Big(\bar\psi^+_\mu\gamma^\nu F\!\!\!/^{(B)}\gamma^\mu\psi^-_\nu + \rho^+ F\!\!\!/^{(B)}\rho^-\Big)\Big]. \end{aligned} \tag{2.8}$$

where ∇ is the Levi–Civita connection.

To match what follows it is useful to rewrite the standard equations of motion in a particular form. For the bosonic fields, with the fermions set to zero, one takes the combinations

[1] Note that in type IIA one cannot write a potential for the zero-form field strength, which must instead be added by hand in (2.6). Note also that in [84] these field strengths are denoted G.

that naturally arise from the string β-functions, namely

$$\begin{aligned}
\mathcal{R}_{\mu\nu} - \tfrac{1}{4}H_{\mu\lambda\rho}H_{\nu}{}^{\lambda\rho} + 2\nabla_{\mu}\nabla_{\nu}\phi - \tfrac{1}{4}\mathrm{e}^{2\phi}\sum_{n}\tfrac{1}{(n-1)!}F^{(B)}_{\mu\lambda_1\ldots\lambda_{n-1}}F_{\nu}^{(B)\lambda_1\ldots\lambda_{n-1}} &= 0, \\
\nabla^{\mu}\left(\mathrm{e}^{-2\phi}H_{\mu\nu\lambda}\right) - \tfrac{1}{2}\sum_{n}\tfrac{1}{(n-2)!}F^{(B)}_{\mu\nu\lambda_1\ldots\lambda_{n-2}}F^{(B)\lambda_1\ldots\lambda_{n-2}} &= 0, \\
\nabla^2\phi - (\nabla\phi)^2 + \tfrac{1}{4}\mathcal{R} - \tfrac{1}{48}H^2 &= 0, \\
\mathrm{d}F^{(B)} - H\wedge F^{(B)} &= 0,
\end{aligned} \tag{2.9}$$

where the final Bianchi identity for F follows from the definition (2.6). Keeping only terms linear in the fermions, the fermionic equations of motion read

$$\begin{aligned}
\gamma^{\nu}\left[\left(\nabla_{\nu} \mp \tfrac{1}{24}H_{\nu\lambda\rho}\gamma^{\lambda\rho} - \partial_{\nu}\phi\right)\psi^{\pm}_{\mu} \pm \tfrac{1}{2}H_{\nu\mu}{}^{\lambda}\psi^{\pm}_{\lambda}\right] - \left(\nabla_{\mu} \mp \tfrac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\rho^{\pm} \\
= \tfrac{1}{16}\mathrm{e}^{\phi}\sum_{n}(\pm)^{[(n+1)/2]}\gamma^{\nu}\not{F}^{(B)}_{(n)}\gamma_{\mu}\psi^{\mp}_{\nu}, \\
\left(\nabla_{\mu} \mp \tfrac{1}{8}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - 2\partial_{\mu}\phi\right)\psi^{\mu\pm} - \gamma^{\mu}\left(\nabla_{\mu} \mp \tfrac{1}{24}H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_{\mu}\phi\right)\rho^{\pm} \\
= \tfrac{1}{16}\mathrm{e}^{\phi}\sum_{n}(\pm)^{[(n+1)/2]}\not{F}^{(B)}_{(n)}\rho^{\mp},
\end{aligned} \tag{2.10}$$

The supersymmetry variations are parametrised by are pair of chiral spinors $\epsilon^{\pm}$ where, again, in the notation of [84], for type IIA, we have

$$\epsilon = \epsilon^{+} + \epsilon^{-} \qquad \text{where} \quad \gamma^{(10)}\epsilon^{\pm} = \mp\epsilon^{\pm}, \tag{2.11}$$

while for type IIB we have the doublet

$$\epsilon = \begin{pmatrix}\epsilon^{+} \\ \epsilon^{-}\end{pmatrix} \qquad \text{where} \quad \gamma^{(10)}\epsilon^{\pm} = \epsilon^{\pm}. \tag{2.12}$$

Again keeping only linear terms in the fermions field, the supersymmetry transformations for the bosons read

$$
\begin{aligned}
\delta e^a_\mu &= \bar{\epsilon}^+ \gamma^a \psi^+_\mu + \bar{\epsilon}^- \gamma^a \psi^-_\mu , \\
\delta B_{\mu\nu} &= 2\bar{\epsilon}^+ \gamma_{[\mu} \psi^+_{\nu]} - 2\bar{\epsilon}^- \gamma_{[\mu} \psi^-_{\nu]} , \\
\delta\phi - \tfrac{1}{4}\delta \log(-g) &= -\tfrac{1}{2}\bar{\epsilon}^+ \rho^+ - \tfrac{1}{2}\bar{\epsilon}^- \rho^- , \\
\left(\mathrm{e}^B \wedge \delta A\right)^{(n)}_{\mu_1\dots\mu_n} &= \tfrac{1}{2}\left(\mathrm{e}^{-\phi}\bar{\psi}^+_\nu \gamma_{\mu_1\dots\mu_n}\gamma^\nu \epsilon^- - \mathrm{e}^{-\phi}\bar{\epsilon}^+ \gamma_{\mu_1\dots\mu_n}\rho^-\right) \\
&\quad \mp \tfrac{1}{2}\left(\mathrm{e}^{-\phi}\bar{\epsilon}^+ \gamma^\nu \gamma_{\mu_1\dots\mu_n}\psi^-_\nu + \mathrm{e}^{-\phi}\bar{\rho}^+ \gamma_{\mu_1\dots\mu_n}\epsilon^-\right) ,
\end{aligned}
\tag{2.13}
$$

where e_μ is an orthonormal frame for $g_{\mu\nu}$ and in the last equation the upper sign refers to type IIA and the lower to type IIB. For the fermions one has

$$
\begin{aligned}
\delta\psi^\pm_\mu &= \left(\nabla_\mu \mp \tfrac{1}{8} H_{\mu\nu\lambda}\gamma^{\nu\lambda}\right)\epsilon^\pm + \tfrac{1}{16}\mathrm{e}^\phi \sum_n (\pm)^{[(n+1)/2]} \not{F}^{(B)}_{(n)} \gamma_\mu \epsilon^\mp , \\
\delta\rho^\pm &= \gamma^\mu \left(\nabla_\mu \mp \tfrac{1}{24} H_{\mu\nu\lambda}\gamma^{\nu\lambda} - \partial_\mu \phi\right)\epsilon^\pm .
\end{aligned}
\tag{2.14}
$$

2.2 Eleven-dimensional supergravity

We now briefly review the usual formulation of eleven-dimensional $N = 1$ supergravity and its restrictions to d dimensions. This will provide us with the expressions we will try to reproduce in chapter 4 based on generalised geometry.

2.2.1 $N = 1$, $D = 11$ supergravity

Let us start by reviewing the action, equations of motion and supersymmetry variations of eleven-dimensional supergravity, to leading order in the fermions, following the conventions of [85] (see also appendices A and C).

The fields are simply

$$
\{g_{MN}, \mathcal{A}_{MNP}, \psi_M\},
\tag{2.15}
$$

where g_{MN} is the metric, $\mathcal{A}_{MNP}$ the three-form potential and ψ_M is the gravitino. We use $M, N, \dots$ for eleven-dimensional coordinate indices to distinguish from the external space indices $\mu, \nu, \ldots$ we will need for the next subsection. The bosonic action is given by

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int \Big(\sqrt{-g}\, \mathcal{R} - \tfrac{1}{2} \mathcal{F} \wedge * \mathcal{F} - \tfrac{1}{6} \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F} \Big), \tag{2.16}$$

where $\mathcal{R}$ is the Ricci scalar and $\mathcal{F} = \mathrm{d}\mathcal{A}$. This leads to the equations of motion

$$\begin{aligned} \mathcal{R}_{MN} - \tfrac{1}{12} \left(\mathcal{F}_{MP_1P_2P_3} \mathcal{F}_N{}^{P_1P_2P_3} - \tfrac{1}{12} g_{MN} \mathcal{F}^2 \right) = 0, \\ \mathrm{d} * \mathcal{F} + \tfrac{1}{2} \mathcal{F} \wedge \mathcal{F} = 0, \end{aligned} \tag{2.17}$$

where $\mathcal{R}_{MN}$ is the Ricci tensor.

Taking Γ^M to be the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma matrices, the fermionic action, to quadratic order in ψ_M, is given by

$$\begin{aligned} S_{\mathrm{F}} = \frac{1}{\kappa^2} \int \sqrt{-g} \Big(& \bar{\psi}_M \Gamma^{MNP} \nabla_N \psi_P + \tfrac{1}{96} \mathcal{F}_{P_1 \ldots P_4} \bar{\psi}_M \Gamma^{MP_1 \ldots P_4 N} \psi_N \\ & + \tfrac{1}{8} \mathcal{F}_{P_1 \ldots P_4} \bar{\psi}^{P_1} \Gamma^{P_2P_3} \psi^{P_4} \Big), \end{aligned} \tag{2.18}$$

the gravitino equation of motion is

$$\Gamma^{MNP} \nabla_N \psi_P + \tfrac{1}{96} \left(\Gamma^{MNP_1 \ldots P_4} \mathcal{F}_{P_1 \ldots P_4} + 12 \mathcal{F}^{MN}{}_{P_1P_2} \Gamma^{P_1P_2} \right) \psi_N = 0. \tag{2.19}$$

The supersymmetry variations of the bosons are

$$\begin{aligned} \delta g_{MN} &= 2 \bar{\varepsilon} \Gamma_{(M} \psi_{N)}, \\ \delta \mathcal{A}_{MNP} &= -3 \bar{\varepsilon} \Gamma_{[MN} \psi_{P]}, \end{aligned} \tag{2.20}$$

while the supersymmetry variation of the gravitino is

$$\delta \psi_M = \nabla_M \varepsilon + \tfrac{1}{288} \left(\Gamma_M{}^{N_1 \ldots N_4} - 8 \delta_M{}^{N_1} \Gamma^{N_2N_3N_4} \right) \mathcal{F}_{N_1 \ldots N_4} \varepsilon, \tag{2.21}$$

where ε is the supersymmetry parameter.

2.2.2 Restricted action, equations of motion and supersymmetry

We will be interested in "restrictions" of eleven-dimensional supergravity where the space-time is assumed to be a warped product $\mathbb{R}^{10-d,1} \times M$ of Minkowski space with a d-dimensional spin manifold M, with $d \leq 7$. The metric is taken to have the form

$$\mathrm{d}s^2_{11} = \mathrm{e}^{2\Delta}\mathrm{d}s^2(\mathbb{R}^{10-d,1}) + \mathrm{d}s^2_d(M), \tag{2.22}$$

where $\mathrm{d}s^2(\mathbb{R}^{10-d,1})$ is the flat metric on $\mathbb{R}^{10-d,1}$ and $\mathrm{d}s^2_d(M)$ is a general metric on M. The warp factor Δ and all the other fields are assumed to be independent of the flat $\mathbb{R}^{10-d,1}$ space. In this sense we restrict the full eleven-dimensional theory to M. We will split the eleven-dimensional indices as external indices $\mu = 0, 1, \ldots, c-1$ and internal indices $m = 1, \ldots, d$ where $c + d = 11$.

In the restricted theory, the surviving fields include the obvious internal components of the eleven-dimensional fields (namely the metric g and three-form A) as well as the warp factor Δ. If $d = 7$, the eleven-dimensional Hodge dual of the 4-form F can have a purely internal 7-form component. This leads one to introduce, in addition, a dual six-form potential $\tilde{A}$ on M which is related to the seven-form field strength $\tilde{F}$ by

$$\tilde{F} = \mathrm{d}\tilde{A} - \tfrac{1}{2}A \wedge F. \tag{2.23}$$

The Bianchi identities satisfied by $F = \mathrm{d}A$ and $\tilde{F}$ are then

$$\begin{aligned} \mathrm{d}F &= 0, \\ \mathrm{d}\tilde{F} + \tfrac{1}{2}F \wedge F &= 0. \end{aligned} \tag{2.24}$$

With these definitions one can see that F and $\tilde{F}$ are related to the components of the eleven dimensional 4-form field strength $\mathcal{F}$ by

$$F_{m_1\ldots m_4} = \mathcal{F}_{m_1\ldots m_4}, \qquad \tilde{F}_{m_1\ldots m_7} = (*\mathcal{F})_{m_1\ldots m_7}, \tag{2.25}$$

where $*\mathcal{F}$ is the eleven-dimensional Hodge dual. The field strengths F and $\tilde{F}$ are invariant under the gauge transformations of the potentials given by

$$\begin{aligned} A' &= A + \mathrm{d}\Lambda, \\ \tilde{A}' &= \tilde{A} + \mathrm{d}\tilde{\Lambda} - \tfrac{1}{2}\mathrm{d}\Lambda \wedge A, \end{aligned} \tag{2.26}$$

for some two-form Λ and five-form $\tilde{\Lambda}$. There is an intricate hierarchy of further coupled gauge transformations of Λ and $\tilde{\Lambda}$, discussed in more detail in section 5.1 (see also [65]) and which formally defines a form of "gerbe" [86].

In order to diagonalise the kinetic terms in the fermionic Lagrangian, one introduces the standard field redefinition of the external components of the gravitino

$$\psi'_\mu = \psi_\mu + \tfrac{1}{c-2}\Gamma_\mu \Gamma^m \psi_m. \tag{2.27}$$

We then denote its trace as

$$\rho = \tfrac{c-2}{c}\Gamma^\mu \psi'_\mu, \tag{2.28}$$

and allow this to be non-zero and dependant on the internal coordinates (this is the partner of the warp factor Δ). Although the restriction to d-dimensions breaks the Lorentz symmetry to $Spin(10-d,1) \times Spin(d) \subset Spin(10,1)$, we do not make an explicit decomposition of the spinor indices under $Spin(10-d,1) \times Spin(d)$. Instead we keep expressions in terms of eleven-dimensional gamma matrices. This is helpful in what follows since it allows us to treat all dimensions in a uniform way.

In summary, the surviving degrees of freedom after the restriction to d dimensions are

$$\{g_{mn}, A_{mnp}, \tilde{A}_{m_1 \dots m_6}, \Delta; \psi_m, \rho\}. \tag{2.29}$$

One can then define the internal space bosonic action

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2} \int \sqrt{g}\, \mathrm{e}^{c\Delta} \Big(\mathcal{R} + c(c-1)(\partial\Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!}F^2 - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}^2 \Big), \tag{2.30}$$

where the associated equations of motion

$$
\begin{aligned}
\mathcal{R}_{mn} - c\nabla_m\nabla_n\Delta - c(\partial_m\Delta)(\partial_n\Delta) - \tfrac{1}{2}\tfrac{1}{4!}\big(4F_{mp_1p_2p_3}F_n{}^{p_1p_2p_3} - \tfrac{1}{3}g_{mn}F^2\big)& \\
-\tfrac{1}{2}\tfrac{1}{7!}\big(7\tilde{F}_{mp_1\dots p_6}\tilde{F}_n{}^{p_1\dots p_6} - \tfrac{2}{3}g_{mn}\tilde{F}^2\big) &= 0, \\
\mathcal{R} - 2(c-1)\nabla^2\Delta - c(c-1)(\partial\Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!}F^2 - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}^2 &= 0, \\
\mathrm{d}*(\mathrm{e}^{c\Delta}F) - \mathrm{e}^{c\Delta}F\wedge *\tilde{F} &= 0, \\
\mathrm{d}*(\mathrm{e}^{c\Delta}\tilde{F}) &= 0,
\end{aligned}
\tag{2.31}
$$

are those obtained by substituting the field ansatz into (2.17). Similarly, to quadratic order in fermions, the action for the fermion fields is

$$
\begin{aligned}
S_{\mathrm{F}} = -\frac{1}{\kappa^2(c-2)^2}\int\sqrt{g}\,\mathrm{e}^{c\Delta}\Big[&(c-4)\bar{\psi}_m\Gamma^{mnp}\nabla_n\psi_p \\
&- c(c-3)\bar{\psi}^m\Gamma^n\nabla_n\psi_m - c\left(\bar{\psi}^m\Gamma_n\nabla_m\psi^n + \bar{\psi}^m\Gamma_m\nabla_n\psi^n\right) \\
&- \tfrac{1}{4}\tfrac{1}{2!}(2c^2-5c+4)\bar{\psi}_mF^{mn}{}_{pq}\Gamma^{pq}\psi_n + \tfrac{1}{4}c(c-3)\bar{\psi}_m\not{F}\psi^m \\
&+ \tfrac{1}{2}\tfrac{1}{3!}c\bar{\psi}_mF^m{}_{pqr}\Gamma^{npqr}\psi_n + \tfrac{1}{4}\tfrac{1}{4!}(c-4)\bar{\psi}_mF_{p_1\dots p_4}\Gamma^{mnp_1\dots p_4}\psi_n \\
&- \tfrac{1}{4}\tfrac{1}{5!}(2c^2-5c+4)\bar{\psi}_m\tilde{F}^{mn}{}_{p_1\dots p_5}\Gamma^{p_1\dots p_5}\psi_n \\
&+ \tfrac{1}{4}\tfrac{1}{6!}c(c-1)\bar{\psi}_m\tilde{F}^m{}_{p_1\dots p_6}\Gamma^{np_1\dots p_6}\psi_n \\
&+ c(c-1)\big(\bar{\psi}^m\nabla_m\rho - \bar{\rho}\nabla^m\psi_m\big) + c\big(\bar{\psi}_m\Gamma^{mn}\nabla_n\rho - \bar{\rho}\Gamma^{mn}\nabla_m\psi_n\big) \\
&- c(c-1)(c-2)\bar{\psi}^m(\partial_m\Delta)\rho - c(c-2)\bar{\psi}_m\Gamma^{mn}(\partial_n\Delta)\rho \\
&+ \tfrac{1}{2}\tfrac{1}{3!}c(c-1)\bar{\rho}F^m{}_{pqr}\Gamma^{pqr}\psi_m - \tfrac{1}{2}\tfrac{1}{4!}c\bar{\rho}\Gamma^m{}_{p_1\dots p_4}F^{p_1\dots p_4}\psi_m \\
&- \tfrac{1}{2}\tfrac{1}{6!}c(c-1)\bar{\psi}_m\tilde{F}^m{}_{p_1\dots p_6}\Gamma^{p_1\dots p_6}\rho \\
&+ c(c-1)\big(\bar{\rho}\Gamma^m\nabla_m\rho + \tfrac{1}{4}\bar{\rho}\not{F}\rho - \tfrac{1}{4}\bar{\rho}\not{\tilde{F}}\rho\big)\Big].
\end{aligned}
\tag{2.32}
$$

This action leads to the equation of motion for ψ_m,

$$
\begin{aligned}
0 = &(c-4)\Gamma_m{}^{np}\left(\nabla_n + \tfrac{c}{2}\partial_n\Delta\right)\psi_p - c(c-3)\Gamma^n\left(\nabla_n + \tfrac{c}{2}\partial_n\Delta\right)\psi_m \\
&\quad - c\Gamma_n\left(\nabla_m + \tfrac{c}{2}\partial_m\Delta\right)\psi^n - c\Gamma_m\left(\nabla_n + \tfrac{c}{2}\partial_n\Delta\right)\psi^n) \\
&\quad - \tfrac{1}{4}(2c^2-5c+4)\tfrac{1}{2!}F_{mnpq}\Gamma^{pq}\psi^n + \tfrac{1}{4}c(c-3)\not{F}\psi_m \\
&\quad + \tfrac{1}{2}\tfrac{1}{3!}cF_{(m}{}^{p_1p_2p_3}\Gamma_{n)p_1p_2p_3}\psi^n + \tfrac{1}{4}\tfrac{1}{4!}(c-4)\Gamma_{mn}{}^{p_1\dots p_4}F_{p_1\dots p_4}\psi^n \\
&\quad - \tfrac{1}{4}\tfrac{1}{5!}(2c^2-5c+4)\tilde{F}_{mnp_1\dots p_5}\Gamma^{p_1\dots p_5}\psi^n + \tfrac{1}{4}\tfrac{1}{6!}c(c-1)\tilde{F}_{(m}{}^{p_1\dots p_6}\Gamma_{n)p_1\dots p_6}\psi^n \\
&+ c\Gamma_m{}^n\left(\nabla_n + \partial_n\Delta\right)\rho + c(c-1)\left(\nabla_m + \partial_m\Delta\right)\rho \\
&\quad + \tfrac{1}{4}\tfrac{1}{3!}c(c-1)F_{mp_1p_2p_3}\Gamma^{p_1p_2p_3}\rho + \tfrac{1}{4}\tfrac{1}{4!}c\Gamma_{mp_1\dots p_4}F^{p_1\dots p_4}\rho \\
&\quad - \tfrac{1}{4}\tfrac{1}{6!}c(c-1)\tilde{F}_{mn_1\dots n_6}\Gamma^{n_1\dots n_6}\rho,
\end{aligned}
\tag{2.33}
$$

and the equation of motion for ρ,

$$
\begin{aligned}
0 = &\left[\not{\nabla} + \tfrac{c}{2}(\not{\partial}\Delta) + \tfrac{1}{4}\not{F} - \tfrac{1}{4}\not{\tilde{F}}\right]\rho \\
&- \left[\nabla_m + (c-1)\partial_m\Delta\right]\psi^m - \tfrac{1}{c-1}\Gamma^{mn}\left[\nabla_m + (c-1)\partial_m\Delta\right]\psi_n \\
&+ \tfrac{1}{4}\tfrac{1}{3!}F^m{}_{p_1p_2p_3}\Gamma^{p_1p_2p_3}\psi_m - \tfrac{1}{4}\tfrac{1}{4!}\tfrac{1}{c-1}\Gamma^m{}_{p_1\dots p_4}F^{p_1\dots p_4}\psi_m \\
&+ \tfrac{1}{4}\tfrac{1}{6!}\tilde{F}^m{}_{p_1\dots p_6}\Gamma^{p_1\dots p_6}\psi_m.
\end{aligned}
\tag{2.34}
$$

Turning to the supersymmetry transformations, we find that the variations of the fermion fields are given by

$$
\begin{aligned}
\delta\rho &= \left[\not{\nabla} - \tfrac{1}{4}\not{F} - \tfrac{1}{4}\not{\tilde{F}} + \tfrac{c-2}{2}(\not{\partial}\Delta)\right]\varepsilon, \\
\delta\psi_m &= \left[\nabla_m + \tfrac{1}{288}F_{n_1\dots n_4}\left(\Gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\Gamma^{n_2n_3n_4}\right) - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{mn_1\dots n_6}\Gamma^{n_1\dots n_6}\right]\varepsilon,
\end{aligned}
\tag{2.35}
$$

and the variations of the bosons by

$$
\begin{aligned}
\delta g_{mn} &= 2\bar{\varepsilon}\Gamma_{(m}\psi_{n)}, \\
(c-2)\delta\Delta + \delta\log\sqrt{g} &= \bar{\varepsilon}\rho, \\
\delta A_{mnp} &= -3\bar{\varepsilon}\Gamma_{[mn}\psi_{p]}, \\
\delta\tilde{A}_{m_1\dots m_6} &= 6\bar{\varepsilon}\Gamma_{[m_1\dots m_5}\psi_{m_6]}.
\end{aligned}
\tag{2.36}
$$

This completes our summary of the reduced theory.

In chapter 6 the fermionic fields will be reinterpreted as representations of larger symmetry groups $\tilde{H}_d \supset Spin(d)$. To mark that distinction, the fermions that have appeared in this section will be denoted by $\varepsilon^{\text{sugra}}$, ρ^{sugra} and ψ^{sugra}. Absent this label, the fields are to be viewed as "generalised" objects transforming under $\tilde{H}_d$[2].

[2]Note that this is not necessary in the type II case as the generalised objects required there can be treated within the usual notation (the index structure, for example, is comparatively simple).

Chapter 3

$O(d, d) \times \mathbb{R}^+$ generalised geometry

We would like to define the generalised geometric analogues of each of the ingredients in the construction of the Levi–Civita connection[1]. In the first section we review the generalisations of the frame bundle, the Lie derivative, connections, torsion and curvature. In the following section we discuss the notion of a generalised metric and the analogue of the Levi–Civita connection.

One way to view generalised geometry is as a formalism for "geometrising" the bosonic structures that appear in supergravity. In the context of the NSNS sector this means first combining the symmetry algebra of diffeomorphisms and B-field gauge transformations into an algebra of "generalised" Lie derivatives. This structure is known as an "exact Courant algebroid" in the mathematics literature [87, 88] and, on a d-dimensional manifold, defines a bundle with a natural $O(d, d)$ action. Combining g, B and ϕ into a single geometrical object introduces an additional refinement of the structure, defining a generalised geometry [9, 11]. The only slight, though important, extension we will require here is to promote the $O(d, d)$ action to $O(d, d) \times \mathbb{R}^+$ [30, 17].

3.1 The $O(d, d) \times \mathbb{R}^+$ generalised tangent space

We start by recalling the generalised tangent space and defining what we will call the "generalised structure" which is the analogue of the frame bundle F in conventional geometry.

[1]These are reviewed in appendix B.

Let M be a d-dimensional spin manifold. One starts by defining the generalised tangent space E as an extension of the tangent space by the cotangent space

$$0 \longrightarrow T^*M \longrightarrow E \longrightarrow TM \longrightarrow 0, \tag{3.1}$$

which depends on a specific patching via one-forms $\Lambda_{(ij)}$. If $v_{(i)} \in \Gamma(TU_i)$ and $\lambda_{(i)} \in \Gamma(T^*U_i)$, so $V_{(i)} = v_{(i)} + \lambda_{(i)}$ is a section of E over the patch U_i, then

$$v_{(i)} + \lambda_{(i)} = v_{(j)} + \big(\lambda_{(j)} - i_{v_{(j)}}\mathrm{d}\Lambda_{(ij)}\big), \tag{3.2}$$

on the overlap $U_i \cap U_j$. Hence as defined, while the $v_{(i)}$ globally are equivalent to a choice of vector, the $\lambda_{(i)}$ do not globally define a one-form. E is in fact isomorphic to $TM \oplus T^*M$ though there is no canonical isomorphism. Instead one must choose a splitting of the sequence (3.1) which, as will be shown below, precisely reproduces the bosonic symmetries of the NSNS sector.

3.1.1 Generalised structure bundle

Crucially the definition of E is consistent with an $O(d,d)$ metric given by, for $V = v + \lambda$

$$\big\langle V, V\big\rangle = i_v\lambda, \tag{3.3}$$

since $i_{v_{(i)}}\lambda_{(i)} = i_{v_{(j)}}\lambda_{(j)}$ on $U_i \cap U_j$.

In order to describe the dilaton correctly we will actually need to consider a slight generalisation of E. We define the bundle $\tilde{E}$ weighted by $\det T^*M$ so that

$$\tilde{E} = \det T^*M \otimes E. \tag{3.4}$$

The point is that, given the metric (3.3), one can now define a natural principal bundle with fibre $O(d,d)\times\mathbb{R}^+$ in terms of bases of $\tilde{E}$. We define a *conformal basis* $\{\hat{E}_A\}$ with

$A = 1,\dots 2d$ on $\tilde{E}_x$ as one satisfying

$$\left\langle \hat{E}_A, \hat{E}_B \right\rangle = \Phi^2 \eta_{AB} \quad \text{where} \quad \eta = \frac{1}{2}\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}. \tag{3.5}$$

That is $\{\hat{E}_A\}$ is orthonormal up to a frame-dependent conformal factor $\Phi \in \Gamma(\det T^*M)$. We then define the *generalised structure bundle*

$$\tilde{F} = \left\{ (x, \{\hat{E}_A\}) : x \in M, \text{ and } \{\hat{E}_A\} \text{ is a conformal basis of } \tilde{E}_x \right\}. \tag{3.6}$$

By construction, this is a principal bundle with fibre $O(d,d)\times\mathbb{R}^+$. One can make a change of basis

$$V^A \mapsto V'^A = M^A{}_B V^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B (M^{-1})^B{}_A. \tag{3.7}$$

where $M \in O(d,d)\times\mathbb{R}^+$ so that $(M^{-1})^C{}_A (M^{-1})^D{}_B \eta_{CD} = \sigma^2 \eta_{AB}$ for some σ. The topology of $\tilde{F}$ encodes both the topology of the tangent bundle TM and of the B-field gerbe.

Given the definition (3.1) there is one natural conformal basis defined by the choice of coordinates on M, namely $\{\hat{E}_A\} = \{\partial/\partial x^\mu\} \cup \{\mathrm{d}x^\mu\}$. Given $V \in \Gamma(E)$ over the patch U_i, we have $V = v^\mu(\partial/\partial x^\mu) + \lambda_\mu \mathrm{d}x^\mu$, we will sometime denote the components of V in this frame by an index M such that

$$V^M = \begin{cases} v^\mu & \text{for } M = \mu \\ \lambda_\mu & \text{for } M = \mu + d \end{cases}. \tag{3.8}$$

3.1.2 Generalised tensors and spinors

Generalised tensors are simply sections of vector bundles constructed from different representations of $O(d,d)\times\mathbb{R}^+$, that is representations of $O(d,d)$ of definite weight under $\mathbb{R}^+$. Since the $O(d,d)$ metric gives an isomorphism between E and E^*, one has the bundle

$$E^{\otimes n}_{(p)} = (\det T^*M)^p \otimes E \otimes \cdots \otimes E. \tag{3.9}$$

for a general tensor of weight p.

One can also consider $Spin(d,d)$ spinor representations [11]. The $O(d,d)$ Clifford algebra

$$\{\Gamma_A,\Gamma_B\} = 2\eta_{AB}. \tag{3.10}$$

can be realised on each coordinate patch U_i by identifying spinors with weighted sums of forms $\Psi_{(i)} \in \Gamma((\det T^*U_i)^{1/2}\otimes\Lambda^\bullet T^*U_i)$, with the Clifford action

$$V^A\Gamma_A\Psi_{(i)} = i_v\Psi_{(i)} + \lambda_{(i)}\wedge\Psi_{(i)}. \tag{3.11}$$

The patching (3.2) then implies

$$\Psi_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)}}\wedge\Psi_{(j)}. \tag{3.12}$$

Projecting onto the chiral spinors then defines two $Spin(d,d)$ spinor bundles, isomorphic to weighted sums of odd or even forms $S^\pm(E) \simeq (\det T^*M)^{-1/2}\otimes\Lambda^{\text{even/odd}}T^*M$ where again specifying the isomorphism requires a choice of splitting.

More generally one defines $Spin(d,d)\times\mathbb{R}^+$ spinors of weight p as sections of

$$S^\pm_{(p)} = (\det T^*M)^p\otimes S^\pm(E). \tag{3.13}$$

Note that there is a natural $Spin(d,d)$ invariant bilinear on these spinor spaces given by the Mukai pairing [9, 11]. For $\Psi,\Psi'\in\Gamma(S^\pm_{(p)})$ one has

$$\big\langle\Psi,\Psi'\big\rangle = \sum_n(-)^{[(n+1)/2]}\Psi^{(d-n)}\wedge\Psi'^{(n)} \in \Gamma((\det T^*M)^{2p}), \tag{3.14}$$

where $\Psi^{(n)}$ and $\Psi'^{(n)}$ are the local weighted n-form components.

3.1.3 NSNS bosonic symmetries and split frames

Let us make a small detour and examine in more detail the symmetries of the NSNS bosonic sector. The potential B is only locally defined, so that, given an open cover $\{U_i\}$, across coordinate patches $U_i \cap U_j$ it can be patched via

$$B_{(i)} = B_{(j)} - \mathrm{d}\Lambda_{(ij)}. \tag{3.15}$$

Furthermore the one-forms $\Lambda_{(ij)}$ satisfy

$$\Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} = \mathrm{d}\Lambda_{(ijk)}, \tag{3.16}$$

on $U_i \cap U_j \cap U_k$. This makes B a "connection structure on a gerbe" [86][2].

As a side note, there is a similar patching for the sum of the RR potentials A. Given the RR field strengths (2.6) from chapter , which are globally defined, we have that, as a sum of forms[3],

$$A_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)}} \wedge A_{(j)} + \mathrm{d}\hat{\Lambda}_{(ij)}, \tag{3.17}$$

where $\hat{\Lambda}_{(ij)}$ is a sum of even or odd forms in type IIA and type IIB respectively. Note the presence of $\Lambda_{(ij)}$ in the first term, which is a consequence of us working in the "A-basis" for the RR fields.

Focusing on the NSNS sector symmetry algebra we see that, in addition to diffeomorphism invariance, we have the local bosonic gauge symmetry

$$B'_{(i)} = B_{(i)} - \mathrm{d}\lambda_{(i)}, \qquad A'_{(i)} = \mathrm{e}^{\mathrm{d}\lambda_{(i)}} A_{(i)}, \tag{3.18}$$

where the choice of sign in the gauge transformation is to match the generalised geometry conventions. Given the patching of B, the only requirement is $\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)}$ on $U_i \cap U_j$. Thus globally $\lambda_{(i)}$ is equivalent to specifying a closed two-form. The set of gauge symmetries is then the Abelian group of closed two-forms under addition $\Omega^2_{\mathrm{cl}}(M)$. The gauge transformations do not commute with the diffeomorphisms so the NSNS bosonic

[2]In supergravity, there is no requirement that the flux H is quantised. However, string theory implies the cohomological condition $H/(8\pi^2\alpha') \in H^3(M,\mathbb{Z})$ (up to torsion terms). This can be implemented in the gerbe structure by requiring $g_{ijk} = \exp(4\pi\alpha' \mathrm{i}\Lambda_{(ijk)})$ satisfy the cocycle condition $g_{jkl}g^{-1}_{ikl}g_{ijl}g^{-1}_{ijk} = 1$ on $U_i \cap U_j \cap U_k \cap U_l$. We will not consider this further restriction in the following.

[3]Note here i and j refer to the patch not the degree of the form.

symmetry group G_{NS} has a fibred structure

$$\Omega^2_{\text{cl}}(M) \longrightarrow G_{\text{NS}} \longrightarrow \text{Diff}(M), \tag{3.19}$$

sometimes written as the semi-direct product $\text{Diff}(M) \ltimes \Omega^2_{\text{cl}}(M)$.

One can see this structure infinitesimally by combining the diffeomorphism and gauge symmetries, given a vector v and one-form $\lambda_{(i)}$, into a general variation

$$\delta_{v+\lambda} g = \mathcal{L}_v g, \qquad \delta_{v+\lambda}\phi = \mathcal{L}_v \phi, \qquad \delta_{v+\lambda} B_{(i)} = \mathcal{L}_v B_{(i)} - \mathrm{d}\lambda_{(i)}, \tag{3.20}$$

where the patching (3.15) of B implies that

$$\mathrm{d}\lambda_{(i)} = \mathrm{d}\lambda_{(j)} - \mathcal{L}_v \mathrm{d}\Lambda_{(ij)}. \tag{3.21}$$

Recall that $\lambda_{(i)}$ and $\lambda_{(i)} + \mathrm{d}\phi_{(i)}$ define the same gauge transformation. One can use this ambiguity to integrate (3.21) and set

$$\lambda_{(i)} = \lambda_{(j)} - i_v \mathrm{d}\Lambda_{(ij)}, \tag{3.22}$$

on $U_i \cap U_j$.

It should by now be clear that this gerbe structure of supergravity is intimately related to the way the generalised tangent space was constructed in (3.1). Introducing a two-form B patched as in (3.15) is equivalent to specifying a map $TM \to E$ which splits the exact sequence (3.1). This defines an isomorphism $E \simeq TM \oplus T^*M$ and one is then able to identify a special class of conformal frames for $\tilde{E}$ that we call a *split frame* $\{\hat{E}_A\}$ by

$$\hat{E}_A = \begin{cases} \hat{E}_a = (\det e)\,(\hat{e}_a + i_{\hat{e}_a} B) & \text{for } A = a \\ E^a = (\det e) e^a & \text{for } A = a + d \end{cases}. \tag{3.23}$$

where $\{\hat{e}_a\}$ is a generic basis for TM and $\{e^a\}$ be the dual basis on T^*M. We immediately see that

$$\left\langle \hat{E}_A, \hat{E}_B \right\rangle = (\det e)^2 \eta_{AB}, \tag{3.24}$$

and hence the basis is conformal. Writing $V = v^a\hat{E}_a + \lambda_a E^a \in \Gamma(\tilde{E})$ we have

$$\begin{aligned} V^{(B)} &= v^a(\det e)\hat{e}_a + \lambda_a(\det e)e^a \\ &= v_{(i)} + \lambda_{(i)} - i_{v_{(i)}}B_{(i)}, \end{aligned} \tag{3.25}$$

demonstrating that the splitting defines an isomorphism $\tilde{E} \simeq (\det T^*M)\otimes(TM\oplus T^*M)$ since $\lambda_{(i)} - i_{v_{(i)}}B_{(i)} = \lambda_{(j)} - i_{v_{(j)}}B_{(j)}$.

The class of split frames defines a sub-bundle of $\tilde{F}$. Such frames are related by transformations (3.7) where M takes the form

$$M = (\det A)^{-1}\begin{pmatrix} \mathbb{1} & 0 \\ \omega & \mathbb{1}\end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & (A^{-1})^T\end{pmatrix}, \tag{3.26}$$

where $A \in GL(d,\mathbb{R})$ is the matrix transforming $\hat{e}_a \mapsto \hat{e}_b(A^{-1})^b{}_a$ while $\omega = \frac{1}{2}\omega_{ab}e^a\wedge e^b$ transforms $B \mapsto B' = B+\omega$, where ω must be closed for B' to be a splitting. This defines a parabolic subgroup $G_{\text{split}} = GL(d,\mathbb{R}) \ltimes \mathbb{R}^{d(d-1)/2} \subset O(d,d)\times\mathbb{R}^+$ and hence the set of all frames of the form (3.23) defines a G_{split} principal sub-bundle of $\tilde{F}$, that is a G_{split}-structure. This reflects the fact that the patching elements in the definition of $\tilde{E}$ lie only in this subgroup of $O(d,d)\times\mathbb{R}^+$.

In what follows it will be useful to also define a class of *conformal split frames* given by the set of split bases conformally rescaled by a function ϕ so that

$$\hat{E}_A = \begin{cases} \hat{E}_a = \mathrm{e}^{-2\phi}(\det e)\,(\hat{e}_a + i_{\hat{e}_a}B) & \text{for } A = a \\ E^a = \mathrm{e}^{-2\phi}(\det e)e^a & \text{for } A = a+d \end{cases}. \tag{3.27}$$

thus defining a $G_{\text{split}}\times\mathbb{R}^+$ sub-bundle of $\tilde{F}$. In complete analogy with the split case, the components of $V\in\Gamma(\tilde{E})$ in the conformally split frame are related to those in the coordinate basis by

$$V^{(B,\phi)} = \mathrm{e}^{2\phi}\big(v_{(i)} + \lambda_{(i)} - i_{v_{(i)}}B_{(i)}\big). \tag{3.28}$$

We can similarly write the components of generalised spinors in different frames. The relation between the coordinate and split frames implies that if $\Psi^{(B)}_{a_1\dots a_n}$ are the polyform components of $\Psi \in \Gamma(S^{\pm}_{(p)})$ in the split frame then

$$\Psi^{(B)} = \sum_n \tfrac{1}{n!}\Psi^{(B)}_{a_1\dots a_n} e^{a_1} \wedge \dots \wedge e^{a_n} = \mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)}, \tag{3.29}$$

demonstrating the isomorphism $S^{\pm}_{(p)} \simeq (\det T^*M)^{p-1/2} \otimes \Lambda^{\text{even/odd}} T^*M$, since $\mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)} = \mathrm{e}^{B_{(j)}} \wedge \Psi_{(j)}$. In the conformal split frame one similarly has

$$\Psi^{(B,\phi)} = \mathrm{e}^{p\phi} \mathrm{e}^{B_{(i)}} \wedge \Psi_{(i)}. \tag{3.30}$$

3.1.4 The Dorfman derivative, Courant bracket and exterior derivative

An important property of the generalised tangent space is that it admits a generalisation of the Lie derivative which encodes the bosonic symmetries of the NSNS sector of type II supergravity (3.20). Given $V = v + \lambda \in \Gamma(E)$, one can define an operator L_V acting on any generalised tensor, which combines the action of an infinitesimal diffeomorphisms generated by v and a B-field gauge transformations generated by λ.

Acting on $W = w + \zeta \in E_{(p)}$, we define the *Dorfman derivative*[4] or "generalised Lie derivative" as [30]

$$L_V W = \mathcal{L}_v w + \mathcal{L}_v \zeta - i_w \mathrm{d}\lambda, \tag{3.31}$$

where, since w and ζ are weighted tensors, the action of the Lie derivative is

$$\begin{aligned} \mathcal{L}_v w^\mu &= v^\nu \partial_\nu w^\mu - w^\nu \partial_\nu v^\mu + p(\partial_\nu v^\nu) w^\mu, \\ \mathcal{L}_v \zeta_\mu &= v^\nu \partial_\nu \zeta_\mu + (\partial_\mu v^\nu)\zeta_\nu + p(\partial_\nu v^\nu)\zeta^\mu. \end{aligned} \tag{3.32}$$

Defining the action on a function f as simply $L_V f = \mathcal{L}_v f$, one can then extend the notion of Dorfman derivative to any $O(d, d) \times \mathbb{R}^+$ tensor using the Leibniz property.

To see this explicitly it is useful to note that we can rewrite (3.31) in a more $O(d, d) \times \mathbb{R}^+$ covariant way, in analogy with (B.4). First note that one can embed the action of the partial derivative operator into generalised geometry using the map $T^*M \to E$. In coordinate

[4] If $p = 0$ then $L_V W$ is none other than the Dorfman bracket [89]. Since it extends to a derivation on the tensor algebra of generalised tensors, it is natural in our context to call it the "Dorfman derivative".

indices, as viewed as mapping to a section of E^*, one defines

$$\partial_M = \begin{cases} \partial_\mu & \text{for } M = \mu \\ 0 & \text{for } M = \mu + d \end{cases} . \tag{3.33}$$

One can then rewrite (5.25) in terms of generalised objects (as in [36, 37, 40])

$$L_V W^M = V^N \partial_N W^M + \left(\partial^M V^N - \partial^N V^M\right) W_N + p \left(\partial_N V^N\right) W^M, \tag{3.34}$$

where indices are contracted using the $O(d,d)$ metric (3.3), which, by definition, is constant with respect to ∂. Note that this form is exactly analogous to the conventional Lie derivative (B.4), though now with the adjoint action in $\mathfrak{o}(d,d) \oplus \mathbb{R}$ rather than $\mathfrak{gl}(d)$. Specifically the second and third terms are (minus) the action of an $\mathfrak{o}(d,d) \oplus \mathbb{R}$ element m, given by

$$m \cdot W = \begin{pmatrix} a & 0 \\ -\omega & -a^T \end{pmatrix} \begin{pmatrix} w \\ \zeta \end{pmatrix} - p \operatorname{tr} a \begin{pmatrix} w \\ \zeta \end{pmatrix}, \tag{3.35}$$

where $a^\mu{}_\nu = \partial_\nu v^\mu$ and $\omega_{\mu\nu} = \partial_\mu \lambda_\nu - \partial_\nu \lambda_\mu$. Comparing with (3.26), we see that m in fact acts in the Lie algebra of the G_{split} subgroup of $O(d,d) \times \mathbb{R}^+$.

This form can then be naturally extended to an arbitrary $O(d,d) \times \mathbb{R}^+$ tensor $\alpha \in \Gamma(E_{(p)}^{\otimes n})$ as

$$\begin{aligned} L_V \alpha^{M_1 \dots M_n} = {} & V^N \partial_N \alpha^{M_1 \dots M_n} + \left(\partial^{M_1} V^N - \partial^N V^{M_1}\right) \alpha_N{}^{M_2 \dots M_n} \\ & + \dots + \left(\partial^{M_n} V^N - \partial^N V^{M_n}\right) \alpha^{M_1 \dots M_{n-1}}{}_N + p \left(\partial_N V^N\right) \alpha^{M_1 \dots M_n}, \end{aligned} \tag{3.36}$$

again in analogy with (B.4). It similarly extends to generalised spinors $\Psi \in \Gamma(S^\pm_{(p)})$ as (see also [90])

$$L_V \Psi = V^N \partial_N \Psi + \tfrac{1}{4} \left(\partial_M V_N - \partial_N V_M\right) \Gamma^{MN} \Psi + p(\partial_M V^M) \Psi, \tag{3.37}$$

where $\Gamma_{MN} = \frac{1}{2} \left(\Gamma_M \Gamma_N - \Gamma_N \Gamma_M\right)$.

Note that when $W \in \Gamma(E)$ one can also define the antisymmetrisation of the Dorfman derivative

$$\begin{aligned}[\![V,W]\!] &= \tfrac{1}{2}\left(L_V W - L_W V\right) \\ &= [v,w] + \mathcal{L}_v\zeta - \mathcal{L}_w\lambda - \tfrac{1}{2}\mathrm{d}\left(i_v\zeta - i_w\lambda\right),\end{aligned} \tag{3.38}$$

which is known as the Courant bracket [10]. It can be rewritten in an $O(d,d)$ covariant form as

$$[\![U,V]\!]^M = U^N\partial_N V^M - V^N\partial_N U^M - \tfrac{1}{2}\left(U_N\partial^M V^N - V_N\partial^M U^N\right). \tag{3.39}$$

which follows directly from (3.34).

Finally note that since $S^{\pm}_{(1/2)} \simeq \Lambda^{\text{even/odd}}T^*M$ the Clifford action of ∂_M on $\Psi \in \Gamma(S^{\pm}_{(1/2)})$ defines a natural action of the exterior derivative. On U_i one defines $d : \Gamma(S^{\pm}_{(1/2)}) \to \Gamma(S^{\mp}_{(1/2)})$ by

$$(\mathrm{d}\Psi)_{(i)} = \tfrac{1}{2}\Gamma^M\partial_M\Psi_{(i)} = \mathrm{d}\Psi_{(i)}, \tag{3.40}$$

that is, it is simply the exterior derivative of the component p-forms. The Dorfman derivative and Courant bracket can then be regarded as derived brackets for this exterior derivative [91].

3.1.5 Generalised $O(d,d)\times\mathbb{R}^+$ connections and torsion

We now turn to the definitions of generalised connections, torsion and the possibility of defining a generalised curvature. The notion of connection on a Courant algebroid was first introduced by Alekseev and Xu [92, 88] and Gualtieri [93] (see also Ellwood [94]). At least locally, it is also essentially equivalent to the connection defined by Siegel [36, 37] and discussed in double field theory [38]. It is also very closely related to the differential operator introduced in the "stringy differential geometry" of [52].

Our definitions will follow closely those in [92, 93] though, in connecting to supergravity, it is important to extend the definitions to include the $\mathbb{R}^+$ factor in the generalised structure bundle.

Generalised connections

Here we will specifically be interested in those generalised connections that are compatible with the $O(d,d) \times \mathbb{R}^+$ structure. Following [92, 93] we can define a first-order linear differential operator D, such that, given $W \in \Gamma(\tilde{E})$, in frame indices,

$$D_M W^A = \partial_M W^A + \tilde{\Omega}_M{}^A{}_B W^B. \tag{3.41}$$

Compatibility with the $O(d,d) \times \mathbb{R}^+$ structure implies

$$\tilde{\Omega}_M{}^A{}_B = \Omega_M{}^A{}_B - \Lambda_M \delta^A{}_B, \tag{3.42}$$

where Λ is the $\mathbb{R}^+$ part of the connection and Ω the $O(d,d)$ part, so that we have

$$\Omega_M{}^{AB} = -\Omega_M{}^{BA}. \tag{3.43}$$

The action of D then extends naturally to any generalised tensor. In particular, if $\alpha \in \Gamma(E_{(p)}^{\otimes n})$ we have

$$\begin{aligned} D_M \alpha^{A_1 \dots A_n} = \partial_M \alpha^{A_1 \dots A_n} &+ \Omega_M{}^{A_1}{}_B \alpha^{B A_2 \dots A_n} \\ &+ \dots + \Omega_M{}^{A_n}{}_B \alpha^{A_1 \dots A_{n-1} B} - p \Lambda_M \alpha^{A_1 \dots A_n}. \end{aligned} \tag{3.44}$$

Similarly, if $\Psi \in \Gamma(S^{\pm}_{(p)})$ then

$$D_M \Psi = \left(\partial_M + \tfrac{1}{4} \Omega_M{}^{AB} \Gamma_{AB} - p \Lambda_M \right) \Psi. \tag{3.45}$$

Given a conventional connection ∇ and a conformal split frame of the form (3.27), one can construct the corresponding generalised connection as follows. Writing a generalised vector $W \in \Gamma(\tilde{E})$ as

$$W = W^A \hat{E}_A = w^a \hat{E}_a + \zeta_a E^a, \tag{3.46}$$

and, by construction, $w = w^a (\det e) \hat{e}_a \in \Gamma((\det T^*M) \otimes TM)$ and $\zeta = \zeta_a (\det e) e^a \in \Gamma((\det T^*M) \otimes T^*M)$, so we can define $\nabla_\mu w^a$ and $\nabla_\mu \zeta_a$. The generalised connection defined by ∇ lifted to an action on $\tilde{E}$ by the conformal split frame is then simply

$$(D^{\nabla}_M W^A)\hat{E}_A = \begin{cases} (\nabla_\mu w^a)\hat{E}_a + (\nabla_\mu \zeta_a)E^a & \text{for } M=\mu \\ 0 & \text{for } M=\mu+d \end{cases}. \tag{3.47}$$

Generalised torsion

We define the *generalised torsion* T of a generalised connection D in direct analogy to the conventional definition (B.8). Let α be any generalised tensor and $L^D_V\alpha$ be the Dorfman derivative (3.36) with ∂ replaced by D. The generalised torsion is a linear map $T:\Gamma(E)\to\Gamma(\mathrm{ad}(\tilde{F}))$ where $\mathrm{ad}(\tilde{F})\simeq\Lambda^2E\oplus\mathbb{R}$ is the $\mathfrak{o}(d,d)\oplus\mathbb{R}$ adjoint representation bundle associated to $\tilde{F}$. It is defined by

$$T(V)\cdot\alpha = L^D_V\alpha - L_V\alpha, \tag{3.48}$$

for any $V\in\Gamma(E)$ and where $T(V)$ acts via the adjoint representation on α. This definition is close to that of [93], except for the additional $\mathbb{R}^+$ action in the definition of L.

Viewed as a tensor $T\in\Gamma(E\otimes\mathrm{ad}\,\tilde{F})$, with indices such that $T(V)^M{}_N = V^P T^M{}_{PN}$, we can derive an explicit expression for T. Let $\{\hat{E}_A\}$ be a general conformal basis with $\langle\hat{E}_A,\hat{E}_B\rangle = \Phi^2\eta_{AB}$. Then $\{\Phi^{-1}\hat{E}_A\}$ is an orthonormal basis for E. Given the connection $D_MW^A = \partial_MW^A + \tilde{\Omega}_M{}^A{}_BW^B$, we have

$$T_{ABC} = -3\tilde{\Omega}_{[ABC]} + \tilde{\Omega}_D{}^D{}_B\eta_{AC} - \Phi^{-2}\langle\hat{E}_A, L_{\Phi^{-1}\hat{E}_B}\hat{E}_C\rangle, \tag{3.49}$$

where indices are lowered with η_{AB}.

Naively one might expect that $T\in\Gamma((E\otimes\Lambda^2E)\oplus E)$. However the form of the Dorfman derivative means that fewer components of $\tilde{\Omega}$ actually enter the torsion and

$$T\in\Gamma(\Lambda^3E\oplus E). \tag{3.50}$$

This can be seen most easily in the coordinate basis where the two components are

$$T^M{}_{PN} = (T_1)^M{}_{PN} - (T_2)_P\,\delta^M{}_N, \tag{3.51}$$

with

$$(T_1)_{MNP} = -3\tilde{\Omega}_{[MNP]} = -3\Omega_{[MNP]},$$
$$(T_2)_M = -\tilde{\Omega}_Q{}^Q{}_M = \Lambda_M - \Omega_Q{}^Q{}_M. \tag{3.52}$$

An immediate consequence of this definition is that for $\Psi \in \Gamma(S^{\pm}_{(1/2)})$ the Dirac operator $\Gamma^M D_M \Psi$ is determined by the torsion of the connection [92]

$$\begin{aligned}\Gamma^M D_M \Psi &= \Gamma^M(\partial_M \Psi + \tfrac{1}{4}\Omega_{MNP}\Gamma^{NP}\Psi - \tfrac{1}{2}\Lambda_M \Psi)\\ &= \Gamma^M \partial_M \Psi + \tfrac{1}{4}\Omega_{[MNP]}\Gamma^{MNP}\Psi - \tfrac{1}{2}(\Lambda_M - \Omega_N{}^N{}_M)\Gamma^M \Psi\\ &= 2\mathrm{d}\Psi - \tfrac{1}{12}(T_1)_{[MNP]}\Gamma^{MNP}\Psi - \tfrac{1}{2}(T_2)_M \Gamma^M \Psi.\end{aligned} \tag{3.53}$$

This equation could equally well be used as a definition of the torsion of a generalised connection. Note in particular that if the connection is torsion-free we see that the Dirac operator becomes equal to the exterior derivative

$$\Gamma^M D_M \Psi = 2\mathrm{d}\Psi. \tag{3.54}$$

As an example, we can calculate the torsion for the generalised connection D^∇ defined in (3.47). In general we have

$$L_{\Phi^{-1}\hat{E}_A}\hat{E}_B = \big(L_{\Phi^{-1}\hat{E}_A}\Phi\big)\,\Phi^{-1}\hat{E}_B + \Phi\big(L_{\Phi^{-1}\hat{E}_A}(\Phi^{-1}\hat{E}_B)\big), \tag{3.55}$$

where here

$$L_{\Phi^{-1}\hat{E}_A}\Phi = \begin{cases} -\mathrm{e}^{-2\phi}(\det e)\,\big(i_{\hat{e}_a} i_{\hat{e}_b}\mathrm{d}e^b + 2 i_{\hat{e}_a}\mathrm{d}\phi\big) & \text{for } A = a\\ 0 & \text{for } A = a+d\end{cases}, \tag{3.56}$$

and

$$L_{\Phi^{-1}\hat{E}_A}\Phi^{-1}\hat{E}_B = \begin{pmatrix} [\hat{e}_a,\hat{e}_b] + i_{[\hat{e}_a,\hat{e}_b]}B - i_{\hat{e}_a}i_{\hat{e}_b}H & \mathcal{L}_{\hat{e}_a}e^b\\ -\mathcal{L}_{\hat{e}_b}e^a & 0\end{pmatrix}_{AB}, \tag{3.57}$$

where $H = \mathrm{d}B$. If the conventional connection ∇ is torsion-free, the corresponding generalised torsion is given by

$$T_1 = -4H, \qquad T_2 = -4\,\mathrm{d}\phi, \tag{3.58}$$

where we are using the embedding[5] $T^*M \to E$ (and the corresponding $T^*M \to \Lambda^3 E$) to write the expressions in terms of forms. This result is most easily seen by taking $\hat{e}_a$ to be the coordinate frame, so that all but the H and $\mathrm{d}\phi$ terms in (3.56) and (3.57) vanish.

The absence of generalised curvature

Having defined torsion it is natural to ask if one can also introduce a notion of generalised curvature in analogy to the usual definition (B.9), as the commutator of two generalised connections but now using the Courant bracket (3.38) rather than the Lie bracket

$$\mathrm{R}\,(U,V,W) = [D_U, D_V]\,W - D_{[\![U,V]\!]}W. \tag{3.59}$$

However, this object is non-tensorial [93]. We can check for linearity in the arguments explicitly. Taking $U \to fU$, $V \to gV$ and $W \to hW$ for some scalar functions f, g, h, we obtain

$$\begin{aligned}&[D_{fU}, D_{gV}]\,hW - D_{[\![fU,gV]\!]}hW \\ &\qquad = fgh\left([D_U, D_V]\,W - D_{[\![U,V]\!]}W\right) - \tfrac{1}{2}h\big\langle U,V\big\rangle D_{(f\mathrm{d}g-g\mathrm{d}f)}W,\end{aligned} \tag{3.60}$$

and so the curvature is not linear in U and V.

Nonetheless, if there is additional structure, as will be relevant for supergravity, we are able to define other tensorial objects that are measures of generalised curvature. In particular, let $C_1 \subset E$ and $C_2 \subset E$ be subspaces such that $\big\langle U,V\big\rangle = 0$ for all $U \in \Gamma(C_1)$ and $V \in \Gamma(C_2)$. For such a U and V the final term in (3.60) vanishes, and so $\mathrm{R} \in \Gamma((C_1 \otimes C_2) \otimes \mathfrak{o}(d,d))$ is a tensor. A special example of this is when $C_1 = C_2$ is a null subspace of E.

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[5]Note that with our definitions we have $(\partial^A\phi)\Phi^{-1}\hat{E}_A = 2\mathrm{d}\phi$ due to the factor $\frac{1}{2}$ in η_{AB}

3.2 $O(p,q)\times O(q,p)$ structures and torsion-free connections

We now turn to constructing the generalised analogue of the Levi–Civita connection. The latter is the unique torsion-free connection that preserves the $O(d)\subset GL(d,\mathbb{R})$ structure defined by a metric g. Here we will be interested in generalised connections that preserve an $O(p,q)\times O(q,p)\subset O(d,d)\times\mathbb{R}^+$ structure on $\tilde{F}$, where $p+q=d$. We will find that, in analogy to the Levi–Civita connection, it is always possible to construct torsion-free connections of this type but there is no unique choice. Locally this is same construction that appears in Siegel [36, 37] and closely related to that of [52].

3.2.1 $O(p,q)\times O(q,p)$ structures and the generalised metric

Following closely the standard definition of the generalised metric [11], consider an $O(p,q)\times O(q,p)$ principal sub-bundle P of the generalised structure bundle $\tilde{F}$. As discussed below, this is equivalent to specifying a conventional metric g of signature (p,q), a B-field patched as in (3.15) and a dilaton ϕ. As such it clearly gives the appropriate generalised structure to capture the NSNS supergravity fields.

Geometrically, an $O(p,q)\times O(q,p)$ structure does two things. First it fixes a nowhere vanishing section of the determinant bundle which we denote $|\mathrm{vol}_G|\in\Gamma(\det T^*M)$, giving an isomorphism between weighted and unweighted generalised tangent space $\tilde{E}$ and E. Second it defines a splitting of E into two d-dimensional sub-bundles

$$E = C_+\oplus C_-\,, \tag{3.61}$$

such that the $O(d,d)$ metric (3.3) restricts to a separate metric of signature (p,q) on C_+ and a metric of signature (q,p) on C_-. (Each sub-bundle is also isomorphic to TM using the map $E\to TM$.)

In terms of $\tilde{F}$ we can identify a special set of frames defining a $O(p,q)\times O(p,q)$ sub-bundle. We define a frame $\{\hat{E}^+_a\}\cup\{\hat{E}^-_{\bar{a}}\}$ such that $\{\hat{E}^+_a\}$ form an orthonormal basis for

C_+ and $\{\hat{E}^-_{\bar{a}}\}$ form an orthonormal basis for C_-. This means they satisfy

$$\begin{aligned} \left\langle \hat{E}^+_a, \hat{E}^+_b \right\rangle &= \left|\mathrm{vol_G}\right|^2 \eta_{ab}, \\ \left\langle \hat{E}^-_{\bar{a}}, \hat{E}^-_{\bar{b}} \right\rangle &= -\left|\mathrm{vol_G}\right|^2 \eta_{\bar{a}\bar{b}}, \\ \left\langle \hat{E}^+_a, \hat{E}^-_{\bar{a}} \right\rangle &= 0. \end{aligned} \tag{3.62}$$

where $|\mathrm{vol_G}| \in \Gamma(\det T^*M)$ is now some fixed density (independent of the particular frame element) and η_{ab} and $\eta_{\bar{a}\bar{b}}$ are flat metrics with signature (p,q). There is thus a manifest $O(p,q)\times O(q,p)$ symmetry with the first factor acting on $\hat{E}^+_a$ and the second on $\hat{E}^-_{\bar{a}}$.

Note that the natural conformal frame

$$\hat{E}_A = \begin{cases} \hat{E}^+_a & \text{for } A = a \\ \hat{E}^-_{\bar{a}} & \text{for } A = \bar{a}+d \end{cases}, \tag{3.63}$$

satisfies

$$\left\langle \hat{E}_A, \hat{E}_B \right\rangle = \left|\mathrm{vol_G}\right|^2 \eta_{AB}, \quad \text{where} \quad \eta_{AB} = \begin{pmatrix} \eta_{ab} & 0 \\ 0 & -\eta_{\bar{a}\bar{b}} \end{pmatrix}, \tag{3.64}$$

where the form of η_{AB} differs from that used in (3.5). In this section, we will use this form of the metric η_{AB} throughout. It is also important to note that we will adopt the convention that we will always raise and lower the C_+ indices $a,b,c,\dots$ with η_{ab} and the C_- indices $\bar{a},\bar{b},\bar{c},\dots$ with $\eta_{\bar{a}\bar{b}}$, while we continue to raise and lower $2d$ dimensional indices $A,B,C,\dots$ with the $O(d,d)$ metric η_{AB}. Thus, for example we have

$$\hat{E}^A = \begin{cases} \hat{E}^{+a} & \text{for } A = a \\ -\hat{E}^{-\bar{a}} & \text{for } A = \bar{a}+d \end{cases}, \tag{3.65}$$

when we raise the A index on the frame.

One can write a generic $O(p,q)\times O(q,p)$ structure explicitly as

$$\begin{aligned} \hat{E}^+_a &= \mathrm{e}^{-2\phi}\sqrt{-g}\left(\hat{e}^+_a + e^+_a + i_{\hat{e}^+_a}B\right), \\ \hat{E}^-_{\bar{a}} &= \mathrm{e}^{-2\phi}\sqrt{-g}\left(\hat{e}^-_{\bar{a}} - e^-_{\bar{a}} + i_{\hat{e}^-_{\bar{a}}}B\right), \end{aligned} \tag{3.66}$$

where the fixed conformal factor in (3.62) is given by

$$|\mathrm{vol_G}| = \mathrm{e}^{-2\phi}\sqrt{|g|}, \tag{3.67}$$

and where $\{\hat{e}^+_a\}$ and $\{\hat{e}^-_{\bar{a}}\}$, and their duals $\{e^{+a}\}$ and $\{e^{-\bar{a}}\}$, are two independent orthonormal frames for the metric g, so that

$$\begin{gathered} g = \eta_{ab}e^{+a}\otimes e^{+b} = \eta_{\bar{a}\bar{b}}e^{-\bar{a}}\otimes e^{-\bar{b}}, \\ g(\hat{e}^+_a, \hat{e}^+_b) = \eta_{ab}, \qquad g(\hat{e}^-_{\bar{a}}, \hat{e}^-_{\bar{b}}) = \eta_{\bar{a}\bar{b}}. \end{gathered} \tag{3.68}$$

By this explicit construction we see that there is no topological obstruction to the existence of $O(p,q)\times O(q,p)$ structures.

In addition to the $O(p,q)\times O(q,p)$ invariant density (3.67) one can also construct the invariant *generalised metric* G [11]. It has the form

$$G = \eta^{ab}\hat{E}^+_a\otimes\hat{E}^+_b + \eta^{\bar{a}\bar{b}}\hat{E}^-_{\bar{a}}\otimes\hat{E}^-_{\bar{b}}. \tag{3.69}$$

One can also consider the rescaled $\tilde{G} = |\mathrm{vol_G}|^{-2}\,G$, which in the coordinate frame has the familiar expression

$$\tilde{G}_{MN} = \frac{1}{2}\begin{pmatrix} g - Bg^{-1}B & -Bg^{-1} \\ g^{-1}B & g^{-1}\end{pmatrix}_{MN}. \tag{3.70}$$

By construction, G parametrises the coset $(O(d,d)\times\mathbb{R}^+)/O(p,q)\times O(q,p)$ where $p+q = d$.

Finally the $O(p,q)\times O(q,p)$ structure provides two additional chirality operators $\Gamma^{\pm}$ on $Spin(d,d)\times\mathbb{R}^+$ spinors which one can define as [30, 95, 90]

$$\Gamma^{(+)} = \tfrac{1}{d!}\epsilon^{a_1\dots a_d}\Gamma_{a_1}\dots\Gamma_{a_d}, \qquad \Gamma^{(-)} = \tfrac{1}{d!}\epsilon^{\bar{a}_1\dots\bar{a}_d}\Gamma_{\bar{a}_1}\dots\Gamma_{\bar{a}_d}. \tag{3.71}$$

Using that, in the split frame, the Clifford action takes the form

$$\Gamma_a\cdot\Psi^{(B)} = i_{\hat{e}^+_a}\Psi^{(B)} + e^+_a\wedge\Psi^{(B)}, \qquad \Gamma_{\bar{a}}\cdot\Psi^{(B)} = i_{\hat{e}^-_a}\Psi^{(B)} - e^-_a\wedge\Psi^{(B)}, \tag{3.72}$$

these can be evaluated on the weighted n-form components of Ψ as

$$\Gamma^{(+)}\Psi^{(B)}_{(n)} = (-)^{[n/2]} * \Psi^{(B)}_{(n)}, \qquad \Gamma^{(-)}\Psi^{(B)}_{(n)} = (-)^{d}(-)^{[n+1/2]} * \Psi^{(B)}_{(n)}, \tag{3.73}$$

and thus we have a generalisation of the Hodge dual on $Spin(d,d)\times\mathbb{R}^+$ spinors.

Since $\tilde{G}^T\eta\tilde{G} = \eta$, the rescaled generalised metric $\tilde{G}^A{}_B$ is an element of $O(d,d)$ and one can easily check that $\tilde{G}^2 = \mathbb{1}$. Connecting to the discussion of [90], for even dimensions d, one has $\tilde{G}\in SO(d,d)$ and $\Gamma^{(-)}$ is an element of $Spin(d,d)$ satisfying

$$\Gamma^{(-)}\Gamma^A\Gamma^{(-)-1} = \tilde{G}^A{}_B\Gamma^B, \tag{3.74}$$

so that $\Gamma^{(-)}$ is a preimage of G in the double covering map $Spin(d,d)\to SO(d,d)$. In odd dimensions d, $\Gamma^{(+)}$ is an element of $Pin(d,d)$ which maps to $\tilde{G}\in O(d,d)$ under the double cover $Pin(d,d)\to O(d,d)$.

3.2.2 Torsion-free, compatible connections

A generalised connection D is compatible with the $O(p,q)\times O(q,p)$ structure $P\subset\tilde{F}$ if

$$DG = 0, \tag{3.75}$$

or equivalently, if the derivative acts only in the $O(p,q)\times O(q,p)$ sub-bundle so that for $W\in\Gamma(\tilde{E})$ given by

$$W = w^a_+\hat{E}^+_a + w^{\bar{a}}_-\hat{E}^-_{\bar{a}}, \tag{3.76}$$

we have

$$D_M W^A = \begin{cases} \partial_M w^a_+ + \Omega_M{}^a{}_b w^b_+ & \text{for } A = a \\ \partial_M w^{\bar{a}}_- + \Omega_M{}^{\bar{a}}{}_{\bar{b}} w^{\bar{b}}_- & \text{for } A = \bar{a} \end{cases}, \tag{3.77}$$

with

$$\Omega_{Mab} = -\Omega_{Mba}, \qquad \Omega_{M\bar{a}\bar{b}} = -\Omega_{M\bar{b}\bar{a}}. \tag{3.78}$$

In this subsection we will show, in analogy to the construction of the Levi–Civita connection, that

Given an $O(p,q)\times O(q,p)$ structure $P\subset\tilde{F}$ there always exists a torsion-free, compatible generalised connection D. However, it is not unique.

We can construct a compatible connection as follows. Let ∇ be the Levi–Civita connection for the metric g. In terms of the two orthonormal bases we get two gauge equivalent spin-connections, so that if $v=v^a\hat{e}^+_a=v^{\bar{a}}\hat{e}^-_{\bar{a}}\in\Gamma(TM)$ we have

$$\nabla_\mu v^\nu=\left(\partial_\mu v^a+\omega^{+a}_{\mu\ \ b}v^b\right)(\hat{e}^+_a)^\nu=\left(\partial_\mu v^{\bar{a}}+\omega^{-\bar{a}}_{\mu\ \ \bar{b}}v^{\bar{b}}\right)(\hat{e}^-_{\bar{a}})^\nu. \tag{3.79}$$

We can then define, as in (3.47)

$$D^\nabla_M W^a=\begin{cases}\nabla_\mu w^a_+ & \text{for } M=\mu\\ 0 & \text{for } M=\mu+d\end{cases},\qquad D^\nabla_M W^{\bar{a}}=\begin{cases}\nabla_\mu w^{\bar{a}}_- & \text{for } M=\mu\\ 0 & \text{for } M=\mu+d\end{cases}. \tag{3.80}$$

Since $\omega^+_{\mu ab}=-\omega^+_{\mu ba}$ and $\omega^-_{\mu\bar{a}\bar{b}}=-\omega^-_{\mu\bar{b}\bar{a}}$, by construction, this generalised connection is compatible with the $O(p,q)\times O(q,p)$ structure.

However D^∇ is not torsion-free. To see this we note that, comparing with (3.27), when we choose the two orthonormal frames to be aligned so $e^+_a=e^-_a=e_a$ we have

$$W=w^a_+\hat{E}^+_a+w^{\bar{a}}_-\hat{E}^-_{\bar{a}}=\left(w^a_++w^a_-\right)\hat{E}_a+\left(w_{+a}-w_{-a}\right)E^a, \tag{3.81}$$

and the two definitions of D^∇ in (3.47) and (3.80) agree. Hence from (3.58) we have the non-zero torsion components

$$T_1=-4H,\qquad T_2=-4\mathrm{d}\phi. \tag{3.82}$$

To construct a torsion-free compatible connection we simply modify D^∇. A generic generalised connection D can be always be written as

$$D_MW^A=D^\nabla_MW^A+\Sigma_M{}^A{}_BW^B. \tag{3.83}$$

If D is compatible with the $O(p,q) \times O(q,p)$ structure then we have $\Sigma_M{}^a{}_{\bar{b}} = \Sigma_M{}^{\bar{a}}{}_b = 0$ and

$$\Sigma_{Mab} = -\Sigma_{Mba}, \qquad \Sigma_{M\bar{a}\bar{b}} = -\Sigma_{M\bar{b}\bar{a}}. \tag{3.84}$$

By definition, the generalised torsion components of D are then given by

$$(T_1)_{ABC} = -4H_{ABC} - 3\Sigma_{[ABC]}, \qquad (T_2)_A = -4\mathrm{d}\phi_A - \Sigma_C{}^C{}_A. \tag{3.85}$$

The components H^{ABC} and $\mathrm{d}\phi^A$ are the components in frame indices of the corresponding forms under the embeddings $T^*M \hookrightarrow E$ and $\Lambda^3 T^*M \hookrightarrow \Lambda^3 E$. Given

$$\mathrm{d}x^\mu = \tfrac{1}{2}\Phi^{-1}\left(\hat{e}_a^{+\mu}\hat{E}^{+a} - \hat{e}_{\bar{a}}^{-\mu}\hat{E}^{-\bar{a}}\right), \tag{3.86}$$

we have, for instance,

$$\mathrm{d}\phi = \tfrac{1}{2}\partial_a\phi\left(\Phi^{-1}\hat{E}^{+a}\right) - \tfrac{1}{2}\partial_{\bar{a}}\phi\left(\Phi^{-1}\hat{E}^{-\bar{a}}\right). \tag{3.87}$$

where there is a similar decomposition of H under

$$\Lambda^3 T^*M \hookrightarrow \Lambda^3 E \simeq \Lambda^3 C_+ \oplus (\Lambda^2 C_+ \otimes C_-) \oplus (C_+ \otimes \Lambda^2 C_-) \oplus \Lambda^3 C_-, \tag{3.88}$$

Note also that the middle index on $\Sigma_{[ABC]}$ in equation (3.85) has also been lowered with this η_{AB} which introduces some signs. The result is that the components are

$$\mathrm{d}\phi_A = \begin{cases} \tfrac{1}{2}\partial_a\phi & A = a \\ \tfrac{1}{2}\partial_{\bar{a}}\phi & A = \bar{a}+d \end{cases}, \qquad H_{ABC} = \begin{cases} \tfrac{1}{8}H_{abc} & (A,B,C) = (a,b,c) \\ \tfrac{1}{8}H_{ab\bar{c}} & (A,B,C) = (a,b,\bar{c}+d) \\ \tfrac{1}{8}H_{a\bar{b}\bar{c}} & (A,B,C) = (a,\bar{b}+d,\bar{c}+d) \\ \tfrac{1}{8}H_{\bar{a}\bar{b}\bar{c}} & (A,B,C) = (\bar{a}+d,\bar{b}+d,\bar{c}+d) \end{cases}, \tag{3.89}$$

and that setting the torsion of D to zero is equivalent to

$$\begin{aligned}\Sigma_{[abc]} &= -\tfrac{1}{6}H_{abc}, \quad \Sigma_{\bar{a}bc} = -\tfrac{1}{2}H_{\bar{a}bc}, \quad \Sigma_a{}^a{}_b = -2\partial_b\phi, \\ \Sigma_{[\bar{a}\bar{b}\bar{c}]} &= +\tfrac{1}{6}H_{\bar{a}\bar{b}\bar{c}}, \quad \Sigma_{a\bar{b}\bar{c}} = +\tfrac{1}{2}H_{a\bar{b}\bar{c}}, \quad \Sigma_{\bar{a}}{}^{\bar{a}}{}_{\bar{b}} = -2\partial_{\bar{b}}\phi.\end{aligned} \tag{3.90}$$

Thus we can always find a torsion-free compatible connection but clearly these conditions do not determine D uniquely. Specifically, one finds

$$\begin{aligned}D_a w_+^b &= \nabla_a w_+^b - \tfrac{1}{6}H_a{}^b{}_c w_+^c - \tfrac{2}{d-1}\big(\delta_a{}^b\partial_c\phi - \eta_{ac}\partial^b\phi\big)w_+^c + Q_a^{+b}{}_c w_+^c, \\ D_{\bar{a}} w_+^b &= \nabla_{\bar{a}} w_+^b - \tfrac{1}{2}H_{\bar{a}}{}^b{}_c w_+^c, \\ D_a w_-^{\bar{b}} &= \nabla_a w_-^{\bar{b}} + \tfrac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}} w_-^{\bar{c}}, \\ D_{\bar{a}} w_-^{\bar{b}} &= \nabla_{\bar{a}} w_-^{\bar{b}} + \tfrac{1}{6}H_{\bar{a}}{}^{\bar{b}}{}_{\bar{c}} w_-^{\bar{c}} - \tfrac{2}{d-1}\big(\delta_{\bar{a}}{}^{\bar{b}}\partial_{\bar{c}}\phi - \eta_{\bar{a}\bar{c}}\partial^{\bar{b}}\phi\big)w_-^{\bar{c}} + Q_{\bar{a}}^{-\bar{b}}{}_{\bar{c}} w_-^{\bar{c}},\end{aligned} \tag{3.91}$$

where the undetermined tensors $Q^{\pm}$ satisfy

$$\begin{aligned}Q^+_{abc} &= -Q^+_{acb}, \quad Q^+_{[abc]} = 0, \quad Q_a^{+a}{}_b = 0, \\ Q^-_{\bar{a}\bar{b}\bar{c}} &= -Q^-_{\bar{a}\bar{c}\bar{b}}, \quad Q^-_{[\bar{a}\bar{b}\bar{c}]} = 0, \quad Q_{\bar{a}}^{-\bar{a}}{}_{\bar{b}} = 0,\end{aligned} \tag{3.92}$$

and hence do not contribute to the torsion.

3.2.3 Unique operators and generalised $O(p,q)\times O(q,p)$ curvatures

The fact that the $O(p,q)\times O(q,p)$ structure and torsion conditions are not sufficient to specify a unique generalised connection might raise ambiguities which could pose a problem for the applications to supergravity we are ultimately interested in. However, we will now show that it is still possible to find differential expressions that are independent of the chosen D, by forming $O(p,q)\times O(q,p)$ covariant operators which do not depend on the undetermined components $Q^{\pm}$. For example, by examining (3.91) we already see that

$$\begin{aligned}D_{\bar{a}} w_+^b &= \nabla_{\bar{a}} w_+^b - \tfrac{1}{2}H_{\bar{a}}{}^b{}_c w_+^c, \\ D_a w_-^{\bar{b}} &= \nabla_a w_-^{\bar{b}} + \tfrac{1}{2}H_a{}^{\bar{b}}{}_{\bar{c}} w_-^{\bar{c}},\end{aligned} \tag{3.93}$$

have no dependence on $Q^{\pm}$ and so are unique. We find that this is also true for

$$\begin{aligned} D_a w^a_+ &= \nabla_a w^a_+ - 2(\partial_a\phi)w^a_+, \\ D_{\bar a} w^{\bar a}_- &= \nabla_{\bar a} w^{\bar a}_- - 2(\partial_{\bar a}\phi)w^{\bar a}_-. \end{aligned} \tag{3.94}$$

Anticipating our application to supergravity, we will be especially interested in writing formulae for $Spin(p,q)$ spinors, so let us now assume that we have a $Spin(p,q)\times Spin(q,p)$ structure. If $S(C_\pm)$ are then the spinor bundles associated to the sub-bundles $C_\pm$, γ^a and $\gamma^{\bar a}$ the corresponding gamma matrices and $\epsilon^\pm \in \Gamma(S(C_\pm))$, we have that by definition a generalised connection acts as

$$\begin{aligned} D_M\epsilon^+ &= \partial_M\epsilon^+ + \tfrac14 \Omega_M{}^{ab}\gamma_{ab}\epsilon^+, \\ D_M\epsilon^- &= \partial_M\epsilon^- + \tfrac14 \Omega_M{}^{\bar a\bar b}\gamma_{\bar a\bar b}\epsilon^-. \end{aligned} \tag{3.95}$$

There are four operators which can be built out of these derivatives that are uniquely determined

$$\begin{aligned} D_{\bar a}\epsilon^+ &= \left(\nabla_{\bar a} - \tfrac18 H_{\bar a bc}\gamma^{bc}\right)\epsilon^+, \\ D_a\epsilon^- &= \left(\nabla_a + \tfrac18 H_{a\bar b\bar c}\gamma^{\bar b\bar c}\right)\epsilon^-, \\ \gamma^a D_a\epsilon^+ &= \left(\gamma^a\nabla_a - \tfrac{1}{24}H_{abc}\gamma^{abc} - \gamma^a\partial_a\phi\right)\epsilon^+, \\ \gamma^{\bar a} D_{\bar a}\epsilon^- &= \left(\gamma^{\bar a}\nabla_{\bar a} + \tfrac{1}{24}H_{\bar a\bar b\bar c}\gamma^{\bar a\bar b\bar c} - \gamma^{\bar a}\partial_{\bar a}\phi\right)\epsilon^-. \end{aligned} \tag{3.96}$$

The first two expressions follow directly from (3.93). In the final two expressions, there is an elegant cancellation from $\gamma^a\gamma^{bc} = \gamma^{abc} + \eta^{ab}\gamma^c - \eta^{ac}\gamma^b$ which removes the terms involving $Q^{\pm}$.

The restriction that expressions involving generalised connections be determined unambiguously, irrespective of the particular D, now serves as a selection criteria for constructing new generalised objects. In particular, when defining a generalised notion of curvature, we find that even though we can actually build a tensorial $O(p,q)\times O(q,p)$ generalised Riemann curvature – by following the example in section 3.1.5 and taking $C_1 = C_\pm$ and $C_2 = C_\mp$ so that the index structure would be $\left(\mathrm{R}_{a\bar b}{}^c{}_d, \mathrm{R}_{a\bar b}{}^{\bar c}{}_{\bar d}\right)$ and $\left(\mathrm{R}_{\bar a b}{}^c{}_d, \mathrm{R}_{\bar a b}{}^{\bar c}{}_{\bar d}\right)$ – it would not result in a uniquely determined object. However, we can use combinations of (3.93) and (3.94) to define the corresponding *generalised Ricci tensor* R_{AB} unambiguously. Given an $O(p,q)\times O(q,p)$ structure $P\subset\tilde{F}$, the generalised Ricci will be a section of the bundle

$$\operatorname{ad}P^{\perp}=\operatorname{ad}\tilde{F}/\operatorname{ad}P \tag{3.97}$$

which is associated to the coset $O(d,d)\times\mathbb{R}^+/O(p,q)\times O(q,p)$. It can then be most easily constructed by defining its scalar and non-scalar parts. The non-scalar is given by

$$R^0_{a\bar{b}}\,w^a_+=[D_a,D_{\bar{b}}]\,w^a_+, \tag{3.98}$$

or[6]

$$R^0_{\bar{a}b}\,w^{\bar{a}}_-=[D_{\bar{a}},D_b]\,w^{\bar{a}}_-. \tag{3.99}$$

Note that the index contractions are precisely what is needed to guarantee uniqueness.

From its index structure one clearly sees that R^0 is traceless. It turns out that to define the generalised Ricci scalar we need the help of spinors and the operators in (3.96). We can obtain the non-scalar Ricci again from either

$$\begin{aligned}\tfrac{1}{2}R^0_{a\bar{b}}\gamma^a\epsilon^+&=[\gamma^aD_a,D_{\bar{b}}]\,\epsilon^+,\\ \tfrac{1}{2}R^0_{\bar{a}b}\gamma^{\bar{a}}\epsilon^-&=[\gamma^{\bar{a}}D_{\bar{a}},D_b]\,\epsilon^-.\end{aligned} \tag{3.100}$$

However, now we also find a scalar

$$-\tfrac{1}{4}R\epsilon^+=\big(\gamma^aD_a\gamma^bD_b-D^{\bar{a}}D_{\bar{a}}\big)\epsilon^+, \tag{3.101}$$

or alternatively,

$$-\tfrac{1}{4}R\epsilon^-=\big(\gamma^{\bar{a}}D_{\bar{a}}\gamma^{\bar{b}}D_{\bar{b}}-D^aD_a\big)\epsilon^-. \tag{3.102}$$

Again, note the need to use the correct combinations of the operators in these definitions so that all the undetermined components drop out.

The fact that R is indeed a scalar and not itself an operator might not be immediately apparent, so it is useful to work out the explicit form of these curvatures. This can be done

[6] Note that naively one might expect these definitions to give distinct tensors. However one can check that compatibility with the $O(p,q)\times O(q,p)$ structure means that the two agree.

by again choosing the two orthogonal frames to be aligned, $e^+_a = e^-_a$, to find

$$R^0_{ab} = \mathcal{R}_{ab} - \tfrac{1}{4} H_{acd} H_b{}^{cd} + 2\nabla_a \nabla_b \phi + \tfrac{1}{2} \mathrm{e}^{2\phi} \nabla^c (\mathrm{e}^{-2\phi} H_{cab}), \tag{3.103}$$

and for the scalar

$$R = \mathcal{R} + 4\nabla^2 \phi - 4(\partial\phi)^2 - \tfrac{1}{12} H^2. \tag{3.104}$$

From these expressions it is clear that we have obtained genuine tensors which are uniquely determined by the torsion conditions, as desired. Furthermore, comparing with [36, 37] we see that locally these are the same tensors that appear in Siegel's formulation. The expressions (3.103) and (3.104) also appear in the discussion of [52].

Chapter 4

$O(9,1) \times O(1,9)$ generalised gravity

Having established the necessary elements of $O(d,d) \times \mathbb{R}^+$ generalised geometry in the previous chapter, we now present a full reformulation of the ten-dimensional type II supergravity presented in section 2.1 as the generalised geometrical analogues of Einstein gravity. The dynamics and supersymmetry transformations are encoded by an $O(9,1) \times O(1,9)$ structure with a compatible, torsion-free generalised connection. An outcome of this will be a formulation of type II supergravity with manifest local $O(9,1) \times O(1,9)$ symmetry.

In the following we will consider the full ten-dimensional supergraviy theory so that the relevant generalised structure is $O(10,10) \times \mathbb{R}^+$. However, one can equally well consider compactifications of theory of the form $\mathbb{R}^{9-d,1} \times M$

$$\mathrm{d}s^2_{10} = \mathrm{d}s^2(\mathbb{R}^{9-d,1}) + \mathrm{d}s^2_d, \tag{4.1}$$

where $\mathrm{d}s^2(\mathbb{R}^{9-d,1})$ is the flat metric on $\mathbb{R}^{9-d,1}$ and $\mathrm{d}s^2_d$ is a general metric on the d-dimensional manifold M. The relevant structure is then the $O(d) \times O(d) \subset O(d,d) \times \mathbb{R}^+$ generalised geometry on M. Below we will focus on the $O(10,10) \times \mathbb{R}^+$ case. The compactification case follows essentially identically.

4.1 NSNS and fermionic degrees of freedom and $O(9,1) \times O(1,9)$ structures

From the discussion of section 3.2.1 we see that an $O(9,1) \times O(1,9) \subset O(10,10) \times \mathbb{R}^+$ generalised structure is parametrised by a metric g of signature $(9,1)$, a two-form B patched as in (3.15) and a dilaton ϕ, that is, at each point $x \in M$

$$\{g, B, \phi\} \in \frac{O(10,10)}{O(9,1) \times O(1,9)} \times \mathbb{R}^+. \tag{4.2}$$

Thus it precisely captures the NSNS bosonic fields of type II theories by packaging them into the generalised metric G. As in [30], the infinitesimal bosonic symmetry transformation (3.20) is naturally encoded as the Dorfman derivative by $V = v + \lambda$

$$\delta_V G = L_V G, \tag{4.3}$$

and the algebra of these transformations is given by the Courant bracket (3.38).

The type II fermionic degrees of freedom fall into spinor and vector-spinor representations of $Spin(9,1) \times Spin(1,9)$[1]. Let $S(C_+)$ and $S(C_-)$ denote the $Spin(9,1)$ spinor bundles associated to the sub-bundles $C_\pm$ write γ^a and $\gamma^{\bar{a}}$ for the corresponding gamma matrices. Since we are in ten dimensions, we can further decompose into spinor bundles $S^\pm(C_+)$ and $S^\pm(C_-)$ of definite chirality under $\gamma^{(10)}$.

The gravitino degrees of freedom then correspond to

$$\psi^+_{\bar{a}} \in \Gamma(C_- \otimes S^\mp(C_+)), \qquad \psi^-_a \in \Gamma(C_+ \otimes S^+(C_-)), \tag{4.4}$$

where the upper sign on the chirality refers to type IIA and the lower to type IIB. Note that the vector and spinor parts of the gravitinos transform under different $Spin(9,1)$ groups.

[1]Since the underlying manifold M is assumed to possess a spin structure, we are free to promote $O(9,1) \times O(1,9)$ to $Spin(9,1) \times Spin(1,9)$. Here will ignore more complicated extended spin structures that can arise in generalised geometry as described in [64].

For the dilatino degrees of freedom one has

$$\rho^+ \in \Gamma(S^\pm(C_+)), \qquad \rho^- \in \Gamma(S^+(C_-)), \tag{4.5}$$

where again the upper and lower signs refer to IIA and IIB respectively. Similarly the supersymmetry parameters are sections

$$\epsilon^+ \in \Gamma(S^\mp(C_+)), \qquad \epsilon^- \in \Gamma(S^+(C_-)). \tag{4.6}$$

In terms of the string spectrum these gravitino and dilatino representations just correspond to the explicit left- and right-moving fermionic states of the superstring and, in a supergravity context were discussed, for example, in [96, 97].

4.2 RR fields

As is known from studying the action of T-duality, the RR field strengths transform as $Spin(10,10)$ spinors [96, 97, 1, 98, 99]. Here, the patching (3.17) of $A_{(i)}$ on $U_i \cap U_j$ implies that the polyform $F_{(i)} = \mathrm{d}A_{(i)}$ is patched as in (3.12), and hence, as generalised spinors,

$$F \in \Gamma(S^\pm_{(1/2)}), \tag{4.7}$$

where the upper sign is for type IIA and the lower for type IIB. Furthermore, we see that the RR field strengths $F^{(B)}_{(n)}$ that appear in the supergravity (2.6) are simply F expressed in a split frame as in (3.29)

$$F^{(B)} = \mathrm{e}^{B_{(i)}} \wedge F_{(i)} = \mathrm{e}^{B_{(i)}} \wedge \sum_n \mathrm{d}A^{(n-1)}_{(i)}. \tag{4.8}$$

Note that the additional gauge transformations $\mathrm{d}\hat{\Lambda}$ in (3.17) imply that $A_{(i)}$ does not globally define a section of $S^\pm_{(1/2)}$. Since $A_{(i)}$ is still locally a generalised spinor on the patch U_i we can perform the same operations on it as we do on F in the remainder of this section.

Given the generalised metric structure, we can also write F in terms of $Spin(9,1) \times Spin(1,9)$ representations. One has the decomposition $\mathrm{Cliff}(10,10;\mathbb{R}) \simeq \mathrm{Cliff}(9,1;\mathbb{R}) \otimes$

Cliff$(1,9;\mathbb{R})$ with

$$\Gamma^A = \begin{cases} \gamma^a \otimes \mathbb{1} & \text{for } A = a \\ \gamma^{(10)} \otimes \gamma^{\bar{a}}\gamma^{(10)} & \text{for } A = \bar{a} + d \end{cases}. \tag{4.9}$$

and hence we can identify[2]

$$S_{(1/2)} \simeq S(C_+) \otimes S(C_-). \tag{4.10}$$

Using the spinor norm on $S(C_-)$ we can equally well view $F \in \Gamma(S_{(1/2)})$ as a map from section of $S(C_-)$ to sections of $S(C_+)$. We denote the image under this isomorphism as

$$F_\# : S(C_-) \to S(C_+). \tag{4.11}$$

We have that $F \in \Gamma(S(C_+) \otimes S(C_-))$ naturally has spin indices $F^{\alpha\bar{\alpha}}$, while $F_\#$ naturally has indices $F^{\alpha}{}_{\bar{\alpha}}$. The isomorphism simply corresponds to lowering an index with the Cliff$(9,1;\mathbb{R})$ intertwiner $\tilde{C}_{\bar{\alpha}\bar{\beta}}$. The conjugate map, $F_\#^T : S(C_+) \to S(C_-)$, is given by

$$F_\#^T = (\tilde{C} F_\# \tilde{C}^{-1})^T, \tag{4.12}$$

which corresponds to lowering the other index on $F^{\alpha\bar{\alpha}}$ and taking the transpose.

We now give the relations between the components of the $Spin(d,d) \times \mathbb{R}^+$ spinor in all relevant frames. Note first that if the bases are aligned so that $e^+ = e^- = e$ then the $Spin(9,1) \times Spin(1,9)$ basis (3.66) is a split conformal basis and we have a $Spin(9,1) \subset Spin(9,1) \times Spin(1,9)$ structure. We can then use the isomorphism Cliff$(9,1;\mathbb{R}) \simeq \Lambda^\bullet T^* M$ to write $F^{(B,\phi)}$ as a spinor bilinear

$$\not{F}^{(B,\phi)} = \sum_n \tfrac{1}{n!} F^{(B,\phi)}_{a_1\ldots a_n} \gamma^{a_1\ldots a_n}. \tag{4.13}$$

More generally if the frames are related by Lorentz transformations $e^\pm_a = \Lambda^{\pm b}_a e_a$ and we write $\Lambda^\pm$ for the corresponding $Spin(9,1)$ transformations then we can define $F_\#$ explicitly

[2] In fact $S_{(p)} \simeq S(C_+) \otimes S(C_-)$ for any p, but here we focus on the case of interest $p = \frac{1}{2}$

as

$$F_{\#} = \Lambda^{+} \not{F}^{(B,\phi)} (\Lambda^{-})^{-1}, \tag{4.14}$$

which concretely realises the isomorphism between $F^{(B,\phi)}$ and $F_{\#}$.

This map can easily be inverted and so we can write the components of $F \in \Gamma(S_{(1/2)})$ in the coordinate frame as

$$\begin{aligned} F_{(i)} &= \mathrm{e}^{-B_{(i)}} \wedge F^{(B)} = \mathrm{e}^{-\phi} \mathrm{e}^{-B_{(i)}} \wedge F^{(B,\phi)} \\ &= \mathrm{e}^{-\phi} \mathrm{e}^{-B_{(i)}} \wedge \sum_{n} \left[\frac{1}{32(n!)} (-)^{[n/2]} \operatorname{tr} \Big(\gamma_{(n)} (\Lambda^{+})^{-1} F_{\#} \Lambda^{-} \Big) \right]. \end{aligned} \tag{4.15}$$

This chain of equalities relates the components of F in all the frames we have discussed.

Finally, we note that the self-duality conditions satisfied by the RR field strengths $F \in \Gamma(S^{\pm}_{(1/2)})$ become a chirality condition under the operator $\Gamma^{(-)}$ defined in (3.71)

$$\Gamma^{(-)} F = -F, \tag{4.16}$$

as discussed in [100, 90].

4.3 Supersymmetry algebra

We now show that the supersymmetry variations can be written in a simple, locally $Spin(9,1) \times Spin(1,9)$ covariant form using the torsion-free compatible connection D.

We start with the fermionic variations (2.14). Looking at the expressions (3.96), we see that the uniquely determined spinor operators allow us to write the supersymmetry variations compactly as

$$\begin{aligned} \delta\psi^{+}_{\bar{a}} &= D_{\bar{a}} \epsilon^{+} + \tfrac{1}{16} F_{\#} \gamma_{\bar{a}} \epsilon^{-}, \\ \delta\psi^{-}_{a} &= D_{a} \epsilon^{-} + \tfrac{1}{16} F_{\#}^{T} \gamma_{a} \epsilon^{+}, \\ \delta\rho^{+} &= \gamma^{a} D_{a} \epsilon^{+}, \\ \delta\rho^{-} &= \gamma^{\bar{a}} D_{\bar{a}} \epsilon^{-}, \end{aligned} \tag{4.17}$$

where we have also used the results from the previous section to add the RR field strengths to the gravitino variations.

For the bosonic fields, we need the variation of a generic $Spin(9,1) \times Spin(1,9)$ frame (3.66). Note that this means defining the variation of a pair of orthonormal bases $\{e^{+a}\}$ and $\{e^{-\bar{a}}\}$ whereas the conventional supersymmetry variations (2.13) are given in terms of a single basis $\{e^{a}\}$. The only possibility, compatible with the $Spin(9,1) \times Spin(1,9)$ representations of the fermions, is to take

$$\begin{aligned} \tilde{\delta}\hat{E}^{+}_{a} &= (\delta \log|\mathrm{vol_G}|)\hat{E}^{+}_{a} + (\delta\Lambda^{+}_{a\bar{b}})\hat{E}^{-\bar{b}}, \\ \tilde{\delta}\hat{E}^{-}_{\bar{a}} &= (\delta \log|\mathrm{vol_G}|)\hat{E}^{-}_{\bar{a}} + (\delta\Lambda^{-}_{\bar{a}b})\hat{E}^{+b}, \end{aligned} \tag{4.18}$$

where

$$\begin{aligned} \delta\Lambda^{+}_{a\bar{a}} &= \bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}, \\ \delta\Lambda^{-}_{\bar{a}a} &= \bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}, \end{aligned} \tag{4.19}$$

and

$$\delta \log|\mathrm{vol_G}| = -2\delta\phi + \tfrac{1}{2}\delta\log(-g) = \bar{\epsilon}^{+}\rho^{+} + \bar{\epsilon}^{-}\rho^{-}. \tag{4.20}$$

Note that the variation of the basis (4.18) is by construction orthogonal to the $Spin(9,1) \times Spin(1,9)$ action. This is because it is impossible to construct an $Spin(9,1) \times Spin(1,9)$ tensor linear in $\psi^{+}_{\bar{a}}$ and ψ^{-}_{a} with two indices of the same type, that is L^{+}_{ab} or $L^{-}_{\bar{a}\bar{b}}$.

The corresponding variations of the frames $\hat{e}^{\pm}$ are

$$\begin{aligned} \tilde{\delta}e^{+a}_{\mu} &= \bar{\epsilon}^{+}\gamma_{\mu}\psi^{+a} + \bar{\epsilon}^{-}\gamma^{a}\psi^{-}_{\mu}, \\ \tilde{\delta}e^{-\bar{a}}_{\mu} &= \bar{\epsilon}^{+}\gamma^{\bar{a}}\psi^{+}_{\mu} + \bar{\epsilon}^{-}\gamma_{\mu}\psi^{-\bar{a}}, \end{aligned} \tag{4.21}$$

which both give

$$\tilde{\delta}g_{\mu\nu} = 2\bar{\epsilon}^{+}\gamma_{(\mu}\psi^{+}_{\nu)} + 2\bar{\epsilon}^{-}\gamma_{(\mu}\psi^{-}_{\nu)}, \tag{4.22}$$

as required, but, when setting the frames equal so $e^{+a} = e^{a}$ and $e^{-\bar{a}} = e^{\bar{a}}$, differ by Lorentz transformations from the standard form (2.13)

$$\begin{aligned} \tilde{\delta}e^{+a}_{\mu} &= \delta e^{+a}_{\mu} - \left(\bar{\epsilon}^{+}\gamma^{a}\psi^{+b} - \bar{\epsilon}^{+}\gamma^{b}\psi^{+a}\right)e^{+}_{\mu b}, \\ \tilde{\delta}e^{-\bar{a}}_{\mu} &= \delta e^{+\bar{a}}_{\mu} - \left(\bar{\epsilon}^{-}\gamma^{\bar{a}}\psi^{-\bar{b}} - \bar{\epsilon}^{-}\gamma^{\bar{b}}\psi^{-\bar{a}}\right)e^{-}_{\mu\bar{b}}. \end{aligned} \tag{4.23}$$

This can also be expressed in terms of the generalised metric G_{AB} as

$$\delta G_{a\bar{a}} = \delta G_{\bar{a}a} = 2\left(\bar{\epsilon}^{+}\rho^{+} + \bar{\epsilon}^{-}\rho^{-}\right) G_{a\bar{a}} + 2\left|\mathrm{vol_G}\right|^{2}\left(\bar{\epsilon}^{+}\gamma_{a}\psi^{+}_{\bar{a}} + \bar{\epsilon}^{-}\gamma_{\bar{a}}\psi^{-}_{a}\right). \tag{4.24}$$

The variation of the RR potential A can be written as a bispinor

$$\tfrac{1}{16}(\delta A_{\#}) = \left(\gamma^{a}\epsilon^{+}\bar{\psi}^{-}_{a} - \rho^{+}\bar{\epsilon}^{-}\right) \mp \left(\psi^{+}_{\bar{a}}\bar{\epsilon}^{-}\gamma^{\bar{a}} + \epsilon^{+}\bar{\rho}^{-}\right), \tag{4.25}$$

where the upper sign is for type IIA and the lower for type IIB.

4.4 Equations of motion

Finally, we rewrite the supergravity equations of motion (2.9) and (2.10) with local $Spin(9,1)\times Spin(1,9)$ covariance, using the generalised notions of curvature obtained in section 3.2.3.

From the generalised Ricci tensor (3.103) we find that the equations of motion for g and B can be written as

$$R^{0}_{a\bar{b}} + \tfrac{1}{16}\left|\mathrm{vol_G}\right|^{-1}\left\langle F, \Gamma_{a\bar{b}}F\right\rangle = 0, \tag{4.26}$$

where we have made use of the Mukai pairing defined in (3.14)[3] to introduce the RR fields in a $Spin(9,1)\times Spin(1,9)$ covariant manner.

The equation of motion for ϕ does not involve the RR fields, so it is simply given by the generalised scalar curvature (3.104)

$$R = 0. \tag{4.27}$$

Using definition (3.40) and equation (3.54) we can write the equation of motion for the RR fields in the familiar form

$$\tfrac{1}{2}\Gamma^{A}D_{A}F = \mathrm{d}F = 0, \tag{4.28}$$

[3]Note that $\left\langle F, \Gamma_{a\bar{b}}F\right\rangle \in \Gamma((\det T^{*}M)\otimes C_{+}\otimes C_{-})$ so $\left|\mathrm{vol_G}\right|^{-1}\left\langle F, \Gamma_{a\bar{b}}F\right\rangle \in \Gamma(C_{+}\otimes C_{-})$

where the first equality serves as a reminder that this definition of the exterior derivative is fully covariant under $Spin(d,d) \times \mathbb{R}^+$.

We also have the bosonic pseudo-action (2.5) which takes the simple form[4]

$$S_B = \frac{1}{2\kappa^2} \int \left(|\mathrm{vol_G}|\ R + \tfrac{1}{4} \langle F, \Gamma^{(-)} F \rangle \right), \tag{4.29}$$

using the density $|\mathrm{vol_G}|$. Note that the Mukai pairing is a top-form which can be directly integrated.

The fermionic action (2.8) is given by

$$\begin{aligned} S_F = -\frac{1}{2\kappa^2} \int 2\,|\mathrm{vol_G}| \Big[& \bar{\psi}^{+\bar{a}} \gamma^b D_b \psi^+_{\bar{a}} + \bar{\psi}^{-a} \gamma^{\bar{b}} D_{\bar{b}} \psi^-_a \\ & + 2\bar{\rho}^+ D_{\bar{a}} \psi^{+\bar{a}} + 2\bar{\rho}^- D_a \psi^{-a} \\ & - \bar{\rho}^+ \gamma^a D_a \rho^+ - \bar{\rho}^- \gamma^{\bar{a}} D_{\bar{a}} \rho^- \\ & - \tfrac{1}{8} \Big(\bar{\rho}^+ F_\# \rho^- + \bar{\psi}^+_{\bar{a}} \gamma^a F_\# \gamma^{\bar{a}} \psi^-_a \Big) \Big]. \end{aligned} \tag{4.30}$$

Varying this with respect to the fermionic fields leads to the generalised geometry version of (2.10)

$$\begin{aligned} \gamma^b D_b \psi^+_{\bar{a}} - D_{\bar{a}} \rho^+ &= +\tfrac{1}{16} \gamma^b F_\# \gamma_{\bar{a}} \psi^-_b, \\ \gamma^{\bar{b}} D_{\bar{b}} \psi^-_a - D_a \rho^- &= +\tfrac{1}{16} \gamma^{\bar{b}} F^T_\# \gamma_a \psi^+_{\bar{b}}, \\ \gamma^a D_a \rho^+ - D^{\bar{a}} \psi^+_{\bar{a}} &= -\tfrac{1}{16} F_\# \rho^-, \\ \gamma^{\bar{a}} D_{\bar{a}} \rho^- - D^a \psi^-_a &= -\tfrac{1}{16} F^T_\# \rho^+, \end{aligned} \tag{4.31}$$

and it is straightforward to verify that by applying a supersymmetry variation (4.17) we recover the bosonic equations of motion (4.26)-(4.28).

We have thus rewritten all the supergravity equations from section 2.1 in terms of torsion free generalised connections and therefore as manifestly covariant under local $Spin(9,1)\times Spin(1,9)$ transformations.

[4]Up to integration by parts of the $\nabla^2\phi$ term

Chapter 5

$E_{d(d)} \times \mathbb{R}^+$ generalised geometry

Following closely the construction given in chapter 3, here we introduce the generalised geometry versions of the tangent space, frame bundle, Lie derivative, connections and torsion, now in the more subtle context of an $E_{d(d)} \times \mathbb{R}^+$ structure. The $E_{d(d)}$ generalised tangent space was first developed in [64] and independently in [65], where the exceptional Courant bracket was also given for the first time. We slightly generalise those notions by introducing an $\mathbb{R}^+$ factor, known as the "trombone symmetry" [101], as it allows one to specify the isomorphism between the generalised tangent space and a sum of vectors and forms. Physically, it is known to be related to the "warp factor" of warped supergravity reductions. The need for this extra factor in the context of $E_{7(7)}$ geometries has already been identified in [74, 81, 82, 102].

5.1 The $E_{d(d)} \times \mathbb{R}^+$ generalised tangent space

We start by recalling the definition of the generalised tangent space for $E_{d(d)} \times \mathbb{R}^+$ generalised geometry [64, 65] and defining what is meant by the "generalised structure".

Let M be a d-dimensional spin manifold with $d \leq 7$[1]. The generalised tangent space is

[1] We actually only consider $4 \leq d \leq 7$, as for lower dimensions the relevant structures simplify to a point that generalised geometry has little to add to the usual Riemannian description.

isomorphic to a sum of tensor bundles

$$E \simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M), \tag{5.1}$$

where for $d < 7$ some of these terms will of course be absent. The isomorphism is not unique. The bundle is actually described using a specific patching. If we write

$$\begin{aligned} V_{(i)} &= v_{(i)} + \omega_{(i)} + \sigma_{(i)} + \tau_{(i)} \\ &\in \Gamma(TU_i \oplus \Lambda^2 T^*U_i \oplus \Lambda^5 T^*U_i \oplus (T^*U_i \otimes \Lambda^7 T^*U_i)), \end{aligned} \tag{5.2}$$

for a section of E over the patch U_i, then

$$V_{(i)} = \mathrm{e}^{\mathrm{d}\Lambda_{(ij)} + \mathrm{d}\tilde{\Lambda}_{(ij)}} V_{(j)}, \tag{5.3}$$

on the overlap $U_i \cap U_j$ where $\Lambda_{(ij)}$ and $\tilde{\Lambda}_{(ij)}$ are locally two- and five-forms respectively. The exponentiated action is given by

$$\begin{aligned} v_{(i)} &= v_{(j)}, \\ \omega_{(i)} &= \omega_{(j)} + i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}, \\ \sigma_{(i)} &= \sigma_{(j)} + \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \tfrac{1}{2} \mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)} + i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)}, \\ \tau_{(i)} &= \tau_{(j)} + j\mathrm{d}\Lambda_{(ij)} \wedge \sigma_{(j)} - j\mathrm{d}\tilde{\Lambda}_{(ij)} \wedge \omega_{(j)} + j\mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\tilde{\Lambda}_{(ij)} \\ &\qquad + \tfrac{1}{2} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge \omega_{(j)} + \tfrac{1}{6} j\mathrm{d}\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(ij)} \wedge i_{v_{(j)}} \mathrm{d}\Lambda_{(ij)}, \end{aligned} \tag{5.4}$$

where we are using the notation of (A.5). Technically this defines E as a result of a series of extensions

$$\begin{aligned} 0 \longrightarrow \Lambda^2 T^*M \longrightarrow E'' \longrightarrow TM \longrightarrow 0, \\ 0 \longrightarrow \Lambda^5 T^*M \longrightarrow E' \longrightarrow E'' \longrightarrow 0, \\ 0 \longrightarrow T^*M \otimes \Lambda^7 T^*M \longrightarrow E \longrightarrow E' \longrightarrow 0. \end{aligned} \tag{5.5}$$

Note that while the $v_{(i)}$ globally are equivalent to a choice of vector, the $\omega_{(i)}$, $\sigma_{(i)}$ and $\tau_{(i)}$ are not globally tensors.

Note that globally the collection $\Lambda_{(ij)}$ formally define a "connective structures on gerbe" (for a review see, for example, [86]). This essentially means there is a hierarchy of successive gauge transformations on the multiple intersections

$$\begin{aligned} \Lambda_{(ij)} + \Lambda_{(jk)} + \Lambda_{(ki)} &= \mathrm{d}\Lambda_{(ijk)} && \text{on } U_i \cap U_j \cap U_k, \\ \Lambda_{(jkl)} - \Lambda_{(ikl)} + \Lambda_{(ijl)} - \Lambda_{(ijk)} &= \mathrm{d}\Lambda_{(ijkl)} && \text{on } U_i \cap U_j \cap U_k \cap U_l. \end{aligned} \tag{5.6}$$

If the supergravity flux is quantised, we will have $g_{(ijkl)} = \mathrm{e}^{\mathrm{i}\Lambda_{(ijkl)}} \in U(1)$ with the cocycle condition

$$g_{(jklm)} g^{-1}_{(iklm)} g_{(ijlm)} g^{-1}_{(ijkm)} g_{(ijkl)} = 1, \tag{5.7}$$

on $U_i \cap \cdots \cap U_m$. For $\tilde{\Lambda}_{(ij)}$ there is a similar set of structures,

$$\begin{aligned} &\tilde{\Lambda}_{(ij)} - \tilde{\Lambda}_{(ik)} + \tilde{\Lambda}_{(jk)} \\ &\qquad = \mathrm{d}\tilde{\Lambda}_{(ijk)} + \tfrac{1}{2}\tfrac{1}{3!}\big(\Lambda_{(ij)} \wedge \mathrm{d}\Lambda_{(jk)} + \text{antisymmetrisation in } [ijk]\big) \\ &\qquad\quad \text{on } U_i \cap U_j \cap U_k, \\ &\tilde{\Lambda}_{(ijk)} - \tilde{\Lambda}_{(ijl)} + \tilde{\Lambda}_{(ikl)} - \tilde{\Lambda}_{(jkl)} \\ &\qquad = \mathrm{d}\tilde{\Lambda}_{(ijkl)} + \tfrac{1}{2}\tfrac{1}{4!}\big(\Lambda_{(ijk)} \wedge \mathrm{d}\Lambda_{(kl)} + \text{antisymmetrisation in } [ijkl]\big) \\ &\qquad\quad \text{on } U_i \cap U_j \cap U_k \cap U_l, \end{aligned} \tag{5.8}$$

etc.

with the final cocycle condition defined on a octuple intersection $U_{i_1} \cap \cdots \cap U_{i_8}$. Note that this does not give a gerbe structure, but a kind of "gerbe twisted by a gerbe".

The bundle E encodes all the topological information of the supergravity background: the twisting of the tangent space TM as well as that of the gerbes, which encode the topology of the supergravity form-field potentials.

5.1.1 Generalised $E_{d(d)} \times \mathbb{R}^+$ structure bundle and split frames

In all dimensions $d \leq 7$ the fibre E_x of the generalised vector bundle at $x \in M$ forms a representation space of $E_{d(d)} \times \mathbb{R}^+$. These are listed in table 5.1. As we discuss below, the explicit action is defined using the $GL(d, \mathbb{R})$ subgroup that acts on the component spaces

T_xM, $\Lambda^2 T^*_x M$, $\Lambda^5 T^*_x M$ and $T^*_x M \otimes \Lambda^7 T^*_x M$. Note that without the additional $\mathbb{R}^+$ action, sections of E would transform as tensors weighted by a power of $\det T^*M$. Thus it is key to extend the action to $E_{d(d)} \times \mathbb{R}^+$ in order to define E directly as the extension (5.5).

$E_{d(d)}$ group	$E_{d(d)} \times \mathbb{R}^+$ rep.
$E_{7(7)}$	$\mathbf{56_1}$
$E_{6(6)}$	$\mathbf{27'_1}$
$E_{5(5)} \simeq Spin(5,5)$	$\mathbf{16^c_1}$
$E_{4(4)} \simeq SL(5,\mathbb{R})$	$\mathbf{10'_1}$

Table 5.1: Generalised tangent space and frame bundle representations where the subscript denotes the $\mathbb{R}^+$ weight, where $\mathbf{1_1} \simeq (\det T^*M)^{1/(9-d)}$

Crucially, the patching defined in (5.3) is compatible with this $E_{d(d)} \times \mathbb{R}^+$ action. This means that one can define a generalised structure bundle as a sub-bundle of the frame bundle F for E. Let $\{\hat{E}_A\}$ be a basis for E_x, where the label A runs over the dimension n of the generalised tangent space as listed in table 5.1. The frame bundle F formed from all such bases is, by construction, a $GL(n,\mathbb{R})$ principal bundle. We can then define the generalised structure bundle as the natural $E_{d(d)} \times \mathbb{R}^+$ principal sub-bundle of F compatible with the patching (5.3) as follows.

Let $\hat{e}_a$ be a basis for T_xM and e^a the dual basis for T^*_xM. We can use these to construct an explicit basis of E_x as

$$\{\hat{E}_A\} = \{\hat{e}_a\} \cup \{e^{ab}\} \cup \{e^{a_1\dots a_5}\} \cup \{e^{a,a_1\dots a_7}\}, \tag{5.9}$$

where $e^{a_1\dots a_p} = e^{a_1} \wedge \cdots \wedge e^{a_p}$ and $e^{a,a_1\dots a_7} = e^a \otimes e^{a_1} \wedge \cdots \wedge e^{a_7}$. A generic section of E at $x \in U_i$ takes the form

$$V = V^A \hat{E}_A = v^a \hat{e}_a + \tfrac{1}{2}\omega_{ab} e^{ab} + \tfrac{1}{5!}\sigma_{a_1\dots a_5} e^{a_1\dots a_5} + \tfrac{1}{7!}\tau_{a,a_1\dots a_7} e^{a,a_1\dots a_7}. \tag{5.10}$$

As usual, a choice of coordinates on U_i defines a particular such basis where $\{\hat{E}_A\} = \{\partial/\partial x^m\} \cup \{dx^m \wedge dx^n\} + \dots$. We will denote the components of V in such a coordinate frame by an index M, namely $V^M = (v^m, \omega_{mn}, \sigma_{m_1\dots m_5}, \tau_{m,m_1\dots m_7})$.

We then define a $E_{d(d)} \times \mathbb{R}^+$ *basis* as one related to (5.9) by an $E_{d(d)} \times \mathbb{R}^+$ transformation

$$V^A \mapsto V'^A = M^A{}_B V^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B (M^{-1})^B{}_A, \tag{5.11}$$

where the explicit action of M is defined in appendix E. The action has a $GL(d,\mathbb{R})$ subgroup that acts in a conventional way on the bases $\hat{e}_a$, e^{ab} etc, and includes the patching transformation (5.3)[2].

The fact that the definition of the $E_{d(d)} \times\mathbb{R}^+$ action is compatible with the patching means that we can then define the *generalised $E_{d(d)} \times\mathbb{R}^+$ structure bundle* $\tilde{F}$ as a subbundle of the frame bundle for E given by

$$\tilde{F} = \big\{(x, \{\hat{E}_A\}) : x \in M, \text{and } \{\hat{E}_A\} \text{ is an } E_{d(d)} \times\mathbb{R}^+ \text{ basis of } E_x\big\}. \tag{5.12}$$

By construction, this is a principal bundle with fibre $E_{d(d)} \times\mathbb{R}^+$. The bundle $\tilde{F}$ is the direct analogue of the frame bundle of conventional differential geometry, with $E_{d(d)} \times\mathbb{R}^+$ playing the role of $GL(d,\mathbb{R})$.

A special class of $E_{d(d)} \times\mathbb{R}^+$ frames are those defined by a splitting of the generalised tangent space E, that is, an isomorphism of the form (5.1). Let A and $\tilde{A}$ be three- and six-form (gerbe) connections patched on $U_i \cap U_j$ by

$$\begin{aligned} A_{(i)} &= A_{(j)} + \mathrm{d}\Lambda_{(ij)}, \\ \tilde{A}_{(i)} &= \tilde{A}_{(j)} + \mathrm{d}\tilde{\Lambda}_{(ij)} - \tfrac{1}{2}\mathrm{d}\Lambda_{(ij)} \wedge A_{(j)}. \end{aligned} \tag{5.13}$$

Note that from these one can construct the globally defined field strengths

$$\begin{aligned} F &= \mathrm{d}A_{(i)}, \\ \tilde{F} &= \mathrm{d}\tilde{A}_{(i)} - \tfrac{1}{2}A_{(i)} \wedge F. \end{aligned} \tag{5.14}$$

Given a generic basis $\{\hat{e}_a\}$ for TM with $\{e^a\}$ the dual basis on T^*M and a scalar function

[2] In analogy to the definitions for $O(d,d) \times \mathbb{R}^+$ generalised geometry in chapter 3, we could equivalently define an $E_{d(d)} \times\mathbb{R}^+$ basis using invariants constructed from sections of E. For example, in $d = 7$ there is a natural symplectic pairing and symmetric quartic invariant that can be used to define $E_{7(7)}$ (in the context of generalised geometry see [65]). However, these invariants differ in different dimension d so it is more useful here to define $E_{d(d)}$ by an explicit action.

Δ, we define a *conformal split frame* $\{\hat{E}_A\}$ for E by

$$\begin{aligned}
\hat{E}_a &= \mathrm{e}^{\Delta}\Big(\hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \tfrac{1}{2}A\wedge i_{\hat{e}_a}A \\
&\qquad + jA\wedge i_{\hat{e}_a}\tilde{A} + \tfrac{1}{6}jA\wedge A\wedge i_{\hat{e}_a}A\Big), \\
\hat{E}^{ab} &= \mathrm{e}^{\Delta}\left(e^{ab} + A\wedge e^{ab} - j\tilde{A}\wedge e^{ab} + \tfrac{1}{2}jA\wedge A\wedge e^{ab}\right), \\
\hat{E}^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(e^{a_1\dots a_5} + jA\wedge e^{a_1\dots a_5}\right), \\
\hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}e^{a,a_1\dots a_7},
\end{aligned} \tag{5.15}$$

while a *split frame* has the same form but with $\Delta = 0$. To see that A and $\tilde{A}$ define an isomorphism (5.1) note that, in the conformal split frame,

$$\begin{aligned}
V^{(A,\tilde{A},\Delta)} &= \mathrm{e}^{-\Delta}\mathrm{e}^{-A_{(i)}-\tilde{A}_{(i)}}V_{(i)} \\
&= v^a\hat{e}_a + \tfrac{1}{2}\omega_{ab}e^{ab} + \tfrac{1}{5!}\sigma_{a_1\dots a_5}e^{a_1\dots a_5} + \tfrac{1}{7!}\tau_{a,a_1\dots a_7}e^{a,a_1\dots a_7} \\
&\in \Gamma(TM\oplus\Lambda^2T^*M\oplus\Lambda^5T^*M\oplus(T^*M\otimes\Lambda^7T^*M)),
\end{aligned} \tag{5.16}$$

since the patching implies $\mathrm{e}^{-A_{(i)}-\tilde{A}_{(i)}}V_{(i)} = \mathrm{e}^{-A_{(j)}-\tilde{A}_{(j)}}V_{(j)}$ on $U_i\cap U_j$.

The class of split frames defines a sub-bundle of $\tilde{F}$

$$P_{\text{split}} = \big\{(x,\{\hat{E}_A\}) : x\in M, \text{ and } \{\hat{E}_A\} \text{ is split frame}\big\} \subset \tilde{F}. \tag{5.17}$$

Split frames are related by transformations (5.11) where M takes the form $M = \mathrm{e}^{a+\tilde{a}}m$ with $m\in GL(d,\mathbb{R})$. The action of $a+\tilde{a}$ shifts $A\mapsto A+a$ and $\tilde{A}\mapsto\tilde{A}+\tilde{a}$. This forms a parabolic subgroup $G_{\text{split}} = GL(d,\mathbb{R})\ltimes(a+\tilde{a})$-shifts $\subset E_{d(d)}\times\mathbb{R}^+$ where $(a+\tilde{a})$-shifts is the nilpotent group of order two formed of elements $M = \mathrm{e}^{a+\tilde{a}}$. Hence P_{split} is a G_{split} principal sub-bundle of $\tilde{F}$, that is a G_{split}-structure. This reflects the fact that the patching elements in the definition of E lie only in this subgroup of $E_{d(d)}\times\mathbb{R}^+$.

5.1.2 Generalised tensors

Generalised tensors are simply sections of vector bundles constructed from the generalised structure bundle using different representations of $E_{d(d)}\times\mathbb{R}^+$. We have already discussed the generalised tangent space E. There are four other vector bundles which will be of particular importance in the following. The relevant representations are summarised in table 5.2.

dimension	E^*	$\mathrm{ad}\,\tilde{F} \subset E \otimes E^*$	$N \subset S^2E$	$K \subset E^* \otimes \mathrm{ad}\,\tilde{F}$
7	$\mathbf{56_{-1}}$	$\mathbf{133_0} + \mathbf{1_0}$	$\mathbf{133_{+2}}$	$\mathbf{912_{-1}}$
6	$\mathbf{27_{-1}}$	$\mathbf{78_0} + \mathbf{1_0}$	$\mathbf{27'_{+2}}$	$\mathbf{351'_{-1}}$
5	$\mathbf{16^c_{-1}}$	$\mathbf{45_0} + \mathbf{1_0}$	$\mathbf{10_{+2}}$	$\mathbf{144^c_{-1}}$
4	$\mathbf{10_{-1}}$	$\mathbf{24_0} + \mathbf{1_0}$	$\mathbf{5'_{+2}}$	$\mathbf{40_{-1}} + \mathbf{15'_{-1}}$

Table 5.2: Some generalised tensor bundles

The first is the dual generalised tangent space

$$E^* \simeq T^*M \oplus \Lambda^2 TM \oplus \Lambda^5 TM \oplus (TM \otimes \Lambda^7 TM). \tag{5.18}$$

Given a basis $\{\hat{E}_A\}$ for E we have a dual basis $\{E^A\}$ on E^* and sections of E^* can be written as $Z = Z_A E^A$.

Next we then have the adjoint bundle $\mathrm{ad}\,\tilde{F}$ associated with the $E_{d(d)} \times \mathbb{R}^+$ principal bundle $\tilde{F}$

$$\mathrm{ad}\,\tilde{F} \simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM. \tag{5.19}$$

By construction $\mathrm{ad}\,\tilde{F} \subset E \otimes E^*$ and hence we can write sections as $R = R^A{}_B \hat{E}_A \otimes E^B$. We write the projection on the adjoint representation as

$$\times_{\mathrm{ad}} : E^* \otimes E \to \mathrm{ad}\,\tilde{F}. \tag{5.20}$$

It is given explicitly in (E.13).

We also consider the sub-bundle of the symmetric product of two generalised tangent bundles $N \subset S^2 E$,

$$\begin{aligned} N \simeq T^*M &\oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \\ &\oplus (\Lambda^3 T^*M \otimes \Lambda^7 T^*M) \oplus (\Lambda^6 T^*M \otimes \Lambda^7 T^*M). \end{aligned} \tag{5.21}$$

We can write sections as $Y = Y^{AB}\hat{E}_A \otimes \hat{E}_B$ with the projection

$$\times_N : E \otimes E \to N. \tag{5.22}$$

It is given explicitly in (E.15).

Finally, we also need the higher dimensional representation $K \subset E^* \otimes \operatorname{ad}\tilde{F}$ listed in the last column of table 5.2. Decomposing under $GL(d,\mathbb{R})$ one has

$$\begin{aligned} K \simeq\ & T^*M \oplus S^2TM \oplus \Lambda^2TM \oplus (\Lambda^2T^*M \otimes TM)_0 \oplus (\Lambda^3TM \otimes T^*M)_0 \\ & \oplus \Lambda^4T^*M \oplus (\Lambda^4TM \otimes TM)_0 \oplus \Lambda^5TM \oplus (\Lambda^2TM \otimes \Lambda^6TM)_0 \\ & \oplus \Lambda^7T^*M \oplus (TM \otimes \Lambda^7TM) \oplus (\Lambda^7TM \otimes \Lambda^7TM) \\ & \oplus (S^2T^*M \otimes \Lambda^7TM) \oplus (\Lambda^4TM \otimes \Lambda^7TM), \end{aligned} \tag{5.23}$$

where, in fact, the Λ^5TM term is absent when $d = 5$. Note also that the zero subscripts are defined such that

$$\begin{aligned} a_{mn}{}^{n} &= 0, \qquad \text{if } a \in \Gamma((\Lambda^2T^*M \otimes TM)_0), \\ a^{mnp}{}_{p} &= 0, \qquad \text{if } a \in \Gamma((\Lambda^3TM \otimes T^*M)_0), \\ a^{[m_1m_2m_3m_4,m_5]} &= 0, \qquad \text{if } a \in \Gamma((\Lambda^4TM \otimes TM)_0), \\ a^{m[n_1,m_2\dots,n_7]} &= 0, \qquad \text{if } a \in \Gamma((\Lambda^2TM \otimes \Lambda^6TM)_0). \end{aligned} \tag{5.24}$$

Since $K \subset E^* \otimes \operatorname{ad}\tilde{F}$ we can write sections as $T = T_A{}^B{}_C E^A \otimes \hat{E}_B \otimes E^C$.

5.1.3 The Dorfman derivative and Courant bracket

An important property of the generalised tangent space is that it admits a generalisation of the Lie derivative which encodes the bosonic symmetries of the supergravity. Given $V = v + \omega + \sigma + \tau \in \Gamma(E)$, one can define an operator L_V acting on any generalised tensor, which combines the action of an infinitesimal diffeomorphism generated by v and A- and $\tilde{A}$-field gauge transformations generated by ω and σ. Formally this gives E the structure of a "Leibniz algebroid" [102].

Acting on $V' = v' + \omega' + \sigma' + \tau' \in \Gamma(E)$, one defines the *Dorfman derivative*[3] or "generalised Lie derivative"

$$\begin{aligned} L_V V' = \mathcal{L}_v v' + (\mathcal{L}_v \omega' - i_{v'} \mathrm{d}\omega) + (\mathcal{L}_v \sigma' - i_{v'} \mathrm{d}\sigma - \omega' \wedge \mathrm{d}\omega) \\ + (\mathcal{L}_v \tau' - j\sigma' \wedge \mathrm{d}\omega - j\omega' \wedge \mathrm{d}\sigma) \,. \end{aligned} \tag{5.25}$$

Defining the action on a function f as simply $L_V f = \mathcal{L}_v f$, one can then extend the notion of Dorfman derivative to a derivative on the space of $E_{d(d)} \times \mathbb{R}^+$ tensors using the Leibniz property.

To see this, first note that we can rewrite (5.25) in a more $E_{d(d)} \times \mathbb{R}^+$ covariant way, in analogy with the corresponding expressions for the conventional Lie derivative and the Dorfman derivative in $O(d,d) \times \mathbb{R}^+$ generalised geometry (3.31). One can embed the action of the partial derivative operator via the map $T^*M \to E^*$ defined by the dual of the exact sequences (5.5). In coordinate indices M, as viewed as mapping to a section of E^*, one defines

$$\partial_M = \begin{cases} \partial_m & \text{for } M = m \\ 0 & \text{otherwise} \end{cases} . \tag{5.26}$$

Such an embedding has the property that under the projection onto N^* we have

$$\partial f \times_{N^*} \partial g = 0, \tag{5.27}$$

for arbitrary functions f, g. We will comment on this observation in section 5.1.5.

One can then rewrite (5.25) in terms of generalised objects as

$$L_V V'^M = V^N \partial_N V'^M - (\partial \times_{\text{ad}} V)^M{}_N V'^N, \tag{5.28}$$

where $\times_{\text{ad}}$ denotes the projection onto $\operatorname{ad} \tilde{F}$ given in (5.20). Concretely, from (E.13) we have

$$\partial \times_{\text{ad}} V = r + a + \tilde{a}, \tag{5.29}$$

[3] We are following [102] in keeping the same nomenclature for this object as the one we used for the corresponding derivative in a Courant algebroid (3.34)

where $r^m{}_n = \partial_n v^m$, $a = \mathrm{d}\omega$ and $\tilde{a} = \mathrm{d}\sigma$. We see that the action actually lies in the adjoint of the $G_{\text{split}} \subset E_{d(d)} \times \mathbb{R}^+$ group. This form of the Dorfman derivative can then be naturally extended to an arbitrary $E_{d(d)} \times \mathbb{R}^+$ tensor by taking that appropriate adjoint action on the $E_{d(d)} \times \mathbb{R}^+$ representation.

Note that we can also define a bracket by taking the antisymmetrisation of the Dorfman derivative. This was originally given in [65] where it was called the "exceptional Courant bracket", and re-derived in [102]. It is given by

$$\begin{aligned}
[\![V, V']\!] &= \tfrac{1}{2}\left(L_V V' - L_{V'} V\right) \\
&= [v, v'] + \mathcal{L}_v \omega' - \mathcal{L}_{v'} \omega - \tfrac{1}{2}\mathrm{d}\left(i_v \omega' - i_{v'} \omega\right) \\
&\quad + \mathcal{L}_v \sigma' - \mathcal{L}_{v'} \sigma - \tfrac{1}{2}\mathrm{d}\left(i_v \sigma' - i_{v'} \sigma\right) + \tfrac{1}{2}\omega \wedge \mathrm{d}\omega' - \tfrac{1}{2}\omega' \wedge \mathrm{d}\omega \\
&\quad + \tfrac{1}{2}\mathcal{L}_v \tau' - \tfrac{1}{2}\mathcal{L}_{v'} \tau + \tfrac{1}{2}\big(j\omega \wedge \mathrm{d}\sigma' - j\sigma' \wedge \mathrm{d}\omega\big) - \tfrac{1}{2}\big(j\omega' \wedge \mathrm{d}\sigma - j\sigma \wedge \mathrm{d}\omega'\big).
\end{aligned} \tag{5.30}$$

Note that the group generated by closed A and $\tilde{A}$ shifts is a semi-direct product $\Omega^3_{\text{cl}}(M) \ltimes \Omega^6_{\text{cl}}(M)$ and corresponds to the symmetry group of gauge transformations in the supergravity. The full automorphism group of the exceptional Courant bracket is then the local symmetry group of the supergravity $G_{\text{sugra}} = \mathrm{Diff}(M) \ltimes (\Omega^3_{\text{cl}}(M) \ltimes \Omega^6_{\text{cl}}(M))$.

For $U, V, W \in \Gamma(E)$, the Dorfman derivative also satisfies the Leibniz identity

$$L_U(L_V W) - L_V(L_U W) = L_{L_U V} W, \tag{5.31}$$

and hence E is a "Leibniz algebroid". On first inspection, one might expect that the bracket of $[\![U, V]\!]$ should appear on the RHS. However, the statement is correct since one can show that

$$L_{[\![U,V]\!]} W = L_{L_U V} W, \tag{5.32}$$

so that the RHS is automatically antisymmetric in U and V.

5.1.4 Generalised $E_{d(d)} \times \mathbb{R}^+$ connections and torsion

We now turn to the definitions of generalised connections and torsion. Generalised connections on algebroids were first introduced by Alekseev and Xu [92, 88]. To study the dynamics of $E_{7(7)}$ geometries with an eleven-dimensional supergravity origin and supersymmetric backgrounds, related notions were also developed by [81, 82, 103, 104]. Here, for the $E_{d(d)} \times \mathbb{R}^+$ case, we follow much the same procedure and conventions as we did for $O(d,d) \times \mathbb{R}^+$ in chapter 3.

Generalised connections

We first define generalised connections that are compatible with the $E_{d(d)} \times \mathbb{R}^+$ structure. These are first-order linear differential operators D, such that, given $W \in E$, in frame indices,

$$D_M W^A = \partial_M W^A + \Omega_M{}^A{}_B W^B. \tag{5.33}$$

where Ω is a section of E^* (denoted by the M index) taking values in $E_{d(d)} \times \mathbb{R}^+$ (denoted by the A and B frame indices), and as such, the action of D then extends naturally to any generalised $E_{d(d)} \times \mathbb{R}^+$ tensor.

Given a conventional connection ∇ and a conformal split frame of the form (5.15), one can construct the corresponding generalised connection as follows. Given the isomorphism (5.16), by construction $v^a \hat{e}_a \in \Gamma(TM)$, $\frac{1}{2}\omega_{ab} e^{ab} \in \Gamma(\Lambda^2 T^* M)$ etc and hence $\nabla_m v^a$ and $\nabla_m \omega_{ab}$ are well-defined. The generalised connection defined by ∇ lifted to an action on E by the conformal split frame is then simply

$$D^{\nabla}_M V = \begin{cases} (\nabla_m v^a)\hat{E}_a + \frac{1}{2}(\nabla_m \omega_{ab})\hat{E}^{ab} & \\ \quad + \frac{1}{5!}(\nabla_m \sigma_{a_1\dots a_5})\hat{E}^{a_1\dots a_5} + \frac{1}{7!}(\nabla_m \tau_{a,a_1\dots a_7})\hat{E}^{a,a_1\dots a_7} & \text{for } M = m, \\ 0 & \text{otherwise.} \end{cases} \tag{5.34}$$

Generalised torsion

We define the *generalised torsion* T of a generalised connection D in direct analogy to the conventional definition and to the one we defined in the $O(d,d) \times \mathbb{R}^+$ case.

Let α be any generalised $E_{d(d)} \times\mathbb{R}^+$ tensor and let $L^D_V\alpha$ be the Dorfman derivative (5.28) with ∂ replaced by D. The generalised torsion is a linear map $T : \Gamma(E) \to \Gamma(\mathrm{ad}(\tilde{F}))$ defined by

$$T(V) \cdot \alpha = L^D_V\alpha - L_V\alpha, \tag{5.35}$$

for any $V \in \Gamma(E)$ and where $T(V)$ acts via the adjoint representation on α. Let $\{\hat{E}_A\}$ be an $E_{d(d)} \times\mathbb{R}^+$ frame for E and $\{E^A\}$ be the dual frame for E^* satisfying $E^A(\hat{E}_B) = \delta^A{}_B$. We then have the explicit expression

$$T(V) = V^C \left[\Omega_C{}^A{}_B - \Omega_B{}^A{}_C - E^A(L_{\hat{E}_C}\hat{E}_B)\right] \hat{E}_A \times_{\text{ad}} E^B. \tag{5.36}$$

Note that we are projecting onto the adjoint representation on the A and B indices. Note also that in a coordinate frame the last term vanishes.

Viewed as a generalised $E_{d(d)} \times\mathbb{R}^+$ tensor we have $T \in \Gamma(E^* \otimes \mathrm{ad}\,\tilde{F})$. However, the form of the Dorfman derivative means that fewer components actually survive and we find

$$T \in \Gamma(K \oplus E^*), \tag{5.37}$$

where K was defined in table 5.2. Note that these representations are exactly the same ones that appear in the embedding tensor formulation of gauged supergravities [105, 106], including gaugings [107] of the so-called “trombone” symmetry [101].

As an example, we can calculate the torsion of the generalised connection D^∇ defined by a conventional connection ∇ and a conformal split frame as given in (5.34). Assuming ∇ is torsion-free we find

$$T(V) = \mathrm{e}^\Delta \left(-i_v\mathrm{d}\Delta + v \otimes \mathrm{d}\Delta - i_vF + \mathrm{d}\Delta \wedge \omega - i_v\tilde{F} + \omega \wedge F + \mathrm{d}\Delta \wedge \sigma\right), \tag{5.38}$$

where we are using the isomorphism (5.19), and F and $\tilde{F}$ are the (globally defined) field strengths of the potentials A and $\tilde{A}$ given by (5.14).

5.1.5 The "section condition", Jacobi identity and the absence of generalised curvature

Restricting our analysis to $d \leq 6$, we find that the bundle N given in (5.21) measures the failure of the generalised tangent bundle to satisfy the properties of a Lie algebroid. This follows from the observation that the difference between the Dorfman derivative and the exceptional Courant bracket (that is, the symmetric part of the Dorfman derivative), for $V, V' \in \Gamma(E)$, is precisely given by[4]

$$L_V V' - [\![V, V']\!] = \tfrac{1}{2}\mathrm{d}\left(i_v\omega' + i_{v'}\omega - i_v\sigma' - i_{v'}\sigma + \omega \wedge \omega'\right) = \partial \times_E (V \times_N V'), \quad (5.39)$$

where the last equality stresses the $E_{d(d)} \times \mathbb{R}^+$ covariant form of the exact term. Therefore, while the Dorfman derivative satisfies a sort of Jacobi identity via the Leibniz identity (5.31), the Jacobiator of the exceptional Courant bracket, like that of the $O(d,d)$ Courant bracket, does not vanish in general. In fact, it can be shown that

$$\mathrm{Jac}(U, V, W) = [\![[\![U, V]\!], W]\!] + \text{c.p.} = \tfrac{1}{3}\partial \times_E \left([\![U, V]\!] \times_N W + \text{c.p.}\right), \quad (5.40)$$

where $U, V, W \in \Gamma(E)$ and c.p. denotes cyclic permutations in U, V and W. We see that both the failure of the exceptional Courant bracket to be Jacobi and the Dorfman derivative to be antisymmetric is measured by an exact term given by the $\times_N$ projection. The proof is essentially the same as the one for the $O(d,d)$ case, see for example [11], section **3.2**[5].

Similarly, and as was the case with $O(d,d) \times \mathbb{R}^+$ generalised connections, for notions of generalised curvature one finds the naive definition $[D_U, D_V]\, W - D_{[\![U,V]\!]}W$ is not a tensor and its failure to be covariant is measured by the projection of the first two arguments to N. Explicitly, taking $U \to fU$, $V \to gV$ and $W \to hW$ for some scalar functions f, g, h,

[4]For $d \geq 7$ the RHS can no longer be written covariantly as a derivative of an $E_{d(d)} \times \mathbb{R}^+$ tensor built from U and V. Similar complications occur in the discussion of the curvature below. This is the reason for the restriction to $d \leq 6$ in this section.

[5]Note that $N \simeq \mathbb{R}$ in the $O(d,d)$ case.

we obtain

$$
\begin{aligned}
&[D_{fU}, D_{gV}]\, hW - D_{[\![fU,gV]\!]} hW \\
&\qquad = fgh \big([D_U, D_V]\, W - D_{[\![U,V]\!]} W \big) - \tfrac{1}{2} h D_{(f\partial g - g\partial f) \times_E (U \times_N V)} W.
\end{aligned}
\tag{5.41}
$$

Note, however, that it is still possible to define analogues of the Ricci tensor and scalar when there is additional structure on the generalised tangent space, as we see in the following section.

Finally, we note that from the point of view of "double field theory"-like geometries [35, 74, 36], the equation

$$\partial f \times_{N^*} \partial g = 0, \tag{5.42}$$

for any functions f and g acquires a special interpretation. In these theories, one starts by enlarging the spacetime manifold so that its dimension matches that of the generalised tangent space. The partial derivative $\partial_M f$ is then generically non-zero for all M. However, the corresponding Dorfman derivative does not then satisfy the Leibniz property, nor is the action for the generalised metric invariant. One must instead impose a "section condition" or "strong constraint". In the original $O(d, d)$ double field theory the condition takes the form $(\partial^A f)(\partial_A g) = 0$. It implies that, in fact, the fields only depend on half the coordinates. For exceptional geometries, the $d = 4$ case was thoroughly analysed in [79], and is given by (5.42). Again it implies that the fields depend on only d of the coordinates.

It is in fact easy to show that satisfying (5.42) always implies the Leibniz property. Thus it gives the section condition in general dimension. In generalised geometry it is satisfied identically by taking ∂_M of the form (5.26). However given the $E_{d(d)} \times \mathbb{R}^+$ covariant form of the Dorfman derivative (5.28), any subspace of E^* in the same orbit under $E_{d(d)} \times \mathbb{R}^+$ will also satisfy the Leibniz condition. Note further that any such subspace, like T^*, is invariant under an action of the parabolic subgroup G_{split}.

5.1.6 Generalised G structures

In what follows we will be interested in further refinements of the generalised frame bundle $\tilde{F}$. We define a *generalised G structure P* as a $G \subset E_{d(d)} \times \mathbb{R}^+$ principle sub-bundle of the generalised structure bundle $\tilde{F}$, that is

$$P \subset \tilde{F} \text{ with fibre } G. \tag{5.43}$$

It picks out a special subset of frames that are related by G transformations. Typically one can also define P by giving a set of nowhere vanishing generalised tensors $\{K_{(a)}\}$, invariant under the action of G. By definition, the invariant tensors parametrise, at each point $x \in M$, an element of the coset

$$\{K_{(a)}\}\big|_x \in \frac{E_{d(d)} \times \mathbb{R}^+}{G}. \tag{5.44}$$

A generalised connection D is said to be compatible with the G structure P if it preserves all the invariant tensors

$$DK_{(a)} = 0 \tag{5.45}$$

or, equivalently, if the derivative acts only in the G sub-bundle P.

A special class of generalised G structures are those characterised by the maximal compact subgroup H_d of $E_{d(d)}$.

5.2 H_d structures and torsion-free connections

We now turn to the construction of the analogue of the Levi–Civita connection by considering additional structure on the generalised tangent space.

We consider H_d structures on E where H_d is the maximally compact[6] subgroup of $E_{d(d)}$. These, along with their double covers[7] $\tilde{H}_d$ are listed in table 5.3. We will then

[6] Note that one could equally consider the non-compact versions of H_d by switching the signature of the metric in appendix F so that it defines an $SO(p,q)$ subgroup of $GL(d,\mathbb{R})$, and the corresponding results then follow identically. For instance, if in $d = 7$ one chooses the $SO(6,1)$ signature, one would obtain the non-compact $SU^*(8)$ subgroup of $E_{7(7)} \times \mathbb{R}^+$, which would be relevant for discussing timelike reductions of 11-dimensional supergravity [108].

[7] We give the double covers of the maximally compact group, since we will be interested in the analogues of spinor representations. A necessary and sufficient condition for the existence of the double cover is the vanishing of the 2nd Stiefel-Whitney class of the generalised tangent bundle [64]. As the underlying manifold is spin by assumption, this is automatically satisfied.

be interested in generalised connections D that preserve the H_d structure. We find it is always possible to construct torsion-free connections of this type but they are not unique. Nonetheless we show that, using the H_d structure, one can construct unique projections of D, and that these can be used to define analogues of the Ricci tensor and scalar curvatures with a local H_d symmetry.

$E_{d(d)}$	H_d	$\tilde{H}_d$	$E \simeq E^*$	$\operatorname{ad} P^\perp$
$E_{7(7)}$	$SU(8)/\mathbb{Z}_2$	$SU(8)$	$\mathbf{28}+\bar{\mathbf{28}}$	$\mathbf{35}+\bar{\mathbf{35}}+\mathbf{1}$
$E_{6(6)}$	$USp(8)/\mathbb{Z}_2$	$USp(8)$	$\mathbf{27}$	$\mathbf{42}+\mathbf{1}$
$E_{5(5)}$	$Spin(5)\times Spin(5)/\mathbb{Z}_2$	$Spin(5)\times Spin(5)$	$(\mathbf{4},\mathbf{4})$	$(\mathbf{5},\mathbf{5})+(\mathbf{1},\mathbf{1})$
$E_{4(4)}$	$SO(5)$	$Spin(5)$	$\mathbf{10}$	$\mathbf{14}+\mathbf{1}$

Table 5.3: Maximal compact subgroups H_d of $E_{d(d)}$, their double covers $\tilde{H}_d$, and H_d representations of the generalised tangent spaces and coset bundles $\operatorname{ad} P^\perp = \operatorname{ad}\tilde{F}/\operatorname{ad} P$ in various d dimensions

5.2.1 H_d structures and the generalised metric

An H_d structure on the generalised tangent space is the direct analogue of metric structure, where one considers the set of orthonormal frames related by $O(d)$ transformations. As we saw in section 5.1.6, it formally defines an H_d principal sub-bundle of the generalised structure bundle $\tilde{F}$, that is

$$P \subset \tilde{F} \text{ with fibre } H_d. \tag{5.46}$$

which is parametrised by an element of the coset $(E_{d(d)} \times\mathbb{R}^+)/\, H_d$ at each point on the manifold. The corresponding representations are listed in table 5.3. Note that there is always a singlet corresponding to the $\mathbb{R}^+$ factor.

One can construct elements of P concretely, that is, identify the analogues of "orthonormal" frames, in the following way. Given an H_d structure, it is always possible to put the

H_d frame in a conformal split form, namely,

$$\begin{aligned}\hat{E}_a &= \mathrm{e}^{\Delta}\Big(\hat{e}_a + i_{\hat{e}_a}A + i_{\hat{e}_a}\tilde{A} + \tfrac{1}{2}A\wedge i_{\hat{e}_a}A \\ &\qquad\qquad + jA\wedge i_{\hat{e}_a}\tilde{A} + \tfrac{1}{6}jA\wedge A\wedge i_{\hat{e}_a}A\Big), \\ \hat{E}^{ab} &= \mathrm{e}^{\Delta}\left(e^{ab} + A\wedge e^{ab} - j\tilde{A}\wedge e^{ab} + \tfrac{1}{2}jA\wedge A\wedge e^{ab}\right), \\ \hat{E}^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(e^{a_1\dots a_5} + jA\wedge e^{a_1\dots a_5}\right), \\ \hat{E}^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}e^{a,a_1\dots a_7}.\end{aligned} \tag{5.47}$$

Any other frame is then related by an H_d transformation of the form given in appendix F. Concretely given $V = V^A\hat{E}_A \in \Gamma(E)$ expanded in such a frame, different frames are related by

$$V^A \mapsto V'^A = H^A{}_B V^B, \qquad \hat{E}_A \mapsto \hat{E}'_A = \hat{E}_B(H^{-1})^B{}_A, \tag{5.48}$$

where H is defined in (F.4). Note that the $O(d) \subset H_d$ action simply rotates the $\hat{e}_a$ basis, defining a set of orthonormal frames for a conventional metric g. It also keeps the frame in the conformal split form. Thus the set of conformal split H_d frames actually forms an $O(d)$ structure on E, that is

$$(P \cap P_{\text{split}}) \subset \tilde{F} \text{ with fibre } O(d). \tag{5.49}$$

One can also define the generalised metric acting on $V = V^A\hat{E}_A \in \Gamma(E)$, expanded in an H_d basis, one defines

$$G(V,V) = v^2 + \tfrac{1}{2!}\omega^2 + \tfrac{1}{5!}\sigma^2 + \tfrac{1}{7!}\tau^2, \tag{5.50}$$

where $v^2 = v_a v^a$, $\omega^2 = \omega_{ab}\omega^{ab}$, $\sigma^2 = \sigma_{a_1\dots a_5}\sigma^{a_1\dots a_5}$, $\tau^2 = \tau_{a,a_1\dots a_7}\tau^{a,a_1\dots a_7}$, and indices are contracted using the flat frame metric δ_{ab} (as used to define the H_d subgroup in appendix F). Note that G allows us to identify $E \simeq E^*$. Since, by definition, this is independent of the choice of H_d frame, it can be evaluated in the conformal split representative (5.47). Hence one sees explicitly that the metric is defined by the fields g, A, $\tilde{A}$ and Δ that determine the coset element. Explicit expressions for the generalised metric in terms of the supergravity fields in the coordinate frame have been worked out, for example, in [73].

Note that the H_d structure embeds as $H_d \subset E_{d(d)} \subset E_{d(d)} \times \mathbb{R}^+$. This mirrors the chain of embeddings in Riemannian geometry $SO(d) \subset SL(d,\mathbb{R}) \subset GL(d,\mathbb{R})$ which allows one to define a $\det T^*M$ density that is $SO(d)$ invariant, $\sqrt{g}$. Likewise, here we can define a density that is H_d (and $E_{d(d)}$) invariant, corresponding to the choice of $\mathbb{R}^+$ factor which, in terms of the conformal split frame, is given by[8]

$$|\mathrm{vol}_{\mathrm{G}}| = \sqrt{g}\,\mathrm{e}^{(9-d)\Delta}, \tag{5.51}$$

as can be seen from appendix E.1.

5.2.2 Torsion-free, compatible connections

A generalised connection D is compatible with the H_d structure $P \subset \tilde{F}$ if

$$DG = 0, \tag{5.52}$$

or, equivalently, if the derivative acts only in the H_d sub-bundle. In this subsection we will show, in analogy to the construction of the Levi–Civita connection, that

> *Given an H_d structure $P \subset \tilde{F}$ there always exists a torsion-free, compatible generalised connection D. However, it is not unique.*

We construct the compatible connection explicitly by working in the conformal split H_d frame (5.47). However the connection is H_d covariant, so the form in any another frame simply follows from an H_d transformation.

Let ∇ be the Levi–Civita connection for the metric g. We can lift the connection to an action on $V \in \Gamma(E)$ by defining, as in (5.34),

$$D^{\nabla}_M V = \begin{cases} (\nabla_m v^a)\hat{E}_a + \frac{1}{2}(\nabla_m \omega_{ab})\hat{E}^{ab} & \\ \quad + \frac{1}{5!}(\nabla_m \sigma_{a_1\dots a_5})\hat{E}^{a_1\dots a_5} + \frac{1}{7!}(\nabla_m \tau_{a,a_1\dots a_7})\hat{E}^{a,a_1\dots a_7} & \text{for } M = m, \\ 0 & \text{otherwise.} \end{cases} \tag{5.53}$$

[8]In general, $|\mathrm{vol}_{\mathrm{G}}|$ can be related to the determinant of the metric by $\det G = |\mathrm{vol}_{\mathrm{G}}|^{-\dim E/(9-d)}$.

Since ∇ is compatible with the $O(d) \subset H_d$ subgroup, it is necessarily an H_d-compatible connection. However, D^∇ is not torsion-free. From (5.38), since ∇ is torsion-free (in the conventional sense), we have

$$T(V) = \mathrm{e}^{\Delta} \left(-i_v \mathrm{d}\Delta + v \otimes \mathrm{d}\Delta - i_v F + \mathrm{d}\Delta \wedge \omega - i_v \tilde{F} + \omega \wedge F + \mathrm{d}\Delta \wedge \sigma \right). \quad (5.54)$$

To construct a torsion-free compatible connection we simply modify D^∇. A generic generalised connection D can always be written as

$$D_M W^A = D^\nabla_M W^A + \Sigma_M{}^A{}_B W^B. \quad (5.55)$$

If D is compatible with the H_d structure then

$$\Sigma \in \Gamma(E^* \otimes \operatorname{ad} P), \quad (5.56)$$

that is, it is a generalised covector taking values in the adjoint of H_d. The problem is then to find a suitable Σ such that the torsion of D vanishes. Fortunately, decomposing under H_d one finds that all the representations that appear in the torsion are already contained in Σ. Thus a solution always exists, but is not unique. The relevant representations are listed in table 5.4. As H_d tensor bundles one has

$$E^* \otimes \operatorname{ad} P \simeq (K \oplus E^*) \oplus U, \quad (5.57)$$

so that the torsion $T \in \Gamma(K \oplus E^*)$ and the unconstrained part of Σ is a section of U.

dimension	$K \oplus E^*$	$U \simeq (E^* \otimes \operatorname{ad} P)/(K \oplus E^*)$
7	$\mathbf{28} + \bar{\mathbf{28}} + \mathbf{36} + \bar{\mathbf{36}} + \mathbf{420} + \bar{\mathbf{420}}$	$\mathbf{1280} + \bar{\mathbf{1280}}$
6	$\mathbf{27} + \mathbf{36} + \mathbf{315}$	$\mathbf{594}$
5	$(\mathbf{4}, \mathbf{4}) + (\mathbf{4}, \mathbf{4}) + (\mathbf{16}, \mathbf{4}) + (\mathbf{4}, \mathbf{16})$	$(\mathbf{20}, \mathbf{4}) + (\mathbf{4}, \mathbf{20})$
4	$\mathbf{1} + \mathbf{5} + \mathbf{10} + \mathbf{14} + \mathbf{35}'$	$\mathbf{35}$

Table 5.4: Components of the connection Σ that are constrained by the torsion, T, and the unconstrained ones, U, as H_d representations

The solution for Σ can be written very explicitly as follows. Contracting with $V \in \Gamma(E)$ so $\Sigma(V) \in \operatorname{ad} P$ and using the basis for the adjoint of H_d given in (F.2) and (F.3) we have

$$\begin{aligned} \Sigma(V)_{ab} &= \mathrm{e}^{\Delta}\left(2\left(\tfrac{7-d}{d-1}\right)v_{[a}\partial_{b]}\Delta + \tfrac{1}{4!}\omega_{cd}F^{cd}{}_{ab} + \tfrac{1}{7!}\sigma_{c_1\dots c_5}\tilde{F}^{c_1\dots c_5}{}_{ab} + C(V)_{ab}\right), \\ \Sigma(V)_{abc} &= \mathrm{e}^{\Delta}\left(\tfrac{6}{(d-1)(d-2)}(\mathrm{d}\Delta\wedge\omega)_{abc} + \tfrac{1}{4}v^{d}F_{dabc} + C(V)_{abc}\right), \\ \Sigma(V)_{a_1\dots a_6} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{7}v^{b}\tilde{F}_{ba_1\dots a_6} + C(V)_{a_1\dots a_6}\right), \end{aligned} \tag{5.58}$$

where the ambiguous part of the connection $Q \in \Gamma(E^* \otimes \operatorname{ad} P)$ projects to zero under the map to the torsion representation $K \oplus E^*$, that is $Q \in \Gamma(U)$. Using the embedding of $\tilde{H}_d$ in $\mathrm{Cliff}(d;\mathbb{R})$ given in (F.8) we can thus write the full connection as

$$\begin{aligned} D_a &= \mathrm{e}^{\Delta}\left(\nabla_a + \tfrac{1}{2}\left(\tfrac{7-d}{d-1}\right)(\partial_b\Delta)\gamma_a{}^{b} - \tfrac{1}{2}\tfrac{1}{4!}F_{ab_1b_2b_3}\gamma^{b_1b_2b_3} - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}_{ab_1\dots b_6}\gamma^{b_1\dots b_6} + \not{Q}_a\right), \\ D^{a_1a_2} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{4}\tfrac{2!}{4!}F^{a_1a_2}{}_{b_1b_2}\gamma^{b_1b_2} - \tfrac{3}{(d-1)(d-2)}(\partial_b\Delta)\gamma^{a_1a_2b} + \not{Q}^{a_1a_2}\right), \\ D^{a_1\dots a_5} &= \mathrm{e}^{\Delta}\left(\tfrac{1}{4}\tfrac{5!}{7!}\tilde{F}^{a_1\dots a_5}{}_{b_1b_2}\gamma^{b_1b_2} + \not{Q}^{a_1\dots a_5}\right), \\ D^{a,a_1\dots a_7} &= \mathrm{e}^{\Delta}\left(\not{Q}^{a,a_1\dots a_7}\right), \end{aligned} \tag{5.59}$$

where

$$\begin{aligned} \not{Q}_m &= \tfrac{1}{2}\left(\tfrac{1}{2!}Q_{m,ab}\gamma^{ab} - \tfrac{1}{3!}Q_{m,a_1a_2a_3}\gamma^{a_1a_2a_3} - \tfrac{1}{6!}Q_{m,a_1\dots a_6}\gamma^{a_1\dots a_6}\right), \\ \not{Q}^{m_1m_2} &= \tfrac{1}{2}\left(\tfrac{1}{2!}Q^{m_1m_2}{}_{ab}\gamma^{ab} - \tfrac{1}{3!}Q^{m_1m_2}{}_{a_1a_2a_3}\gamma^{a_1a_2a_3} - \tfrac{1}{6!}Q^{m_1m_2}{}_{a_1\dots a_6}\gamma^{a_1\dots a_6}\right), \\ &\text{etc.} \end{aligned} \tag{5.60}$$

is the embedding of the ambiguous part of the connection.

5.2.3 Unique operators and generalised H_d curvatures

We now turn to the construction of unique operators and curvatures from the torsion-free and $\tilde{H}_d$-compatible connection D constructed in the previous section. To keep the $\tilde{H}_d$ covariance manifest in all dimensions, we will necessarily have to maintain the discussion in this section fairly abstract. However, once we reach the construction of the supergravity in chapter 6 it will be possible to make the concepts discussed here much more concrete.

Given a bundle X transforming as some representation of $\tilde{H}_d$, we define the map

$$\mathcal{Q}_X : U \otimes X \longrightarrow E^* \otimes X, \tag{5.61}$$

via the embedding $U \subset E^* \otimes \mathrm{ad}\, P$ and the adjoint action of $\mathrm{ad}\, P$ on X. We then have the projection

$$\mathcal{P}_X : E^* \otimes X \longrightarrow \frac{E^* \otimes X}{\mathrm{Im}\mathcal{Q}_X}. \tag{5.62}$$

Recall that the ambiguous part Q of the connection D is a section of U, which acts on X via the map $\mathcal{Q}_X$. If $\alpha \in \Gamma(X)$, then, by construction, $\mathcal{P}_X(D \otimes \alpha)$ is uniquely defined, independent of Q.

We can construct explicit examples of such operators as follows. Consider two real $\tilde{H}_d$ bundles S and J, which we refer to as the "spinor" bundle and the "gravitino" bundle respectively, since, as we will see in the following chapter, the supersymmetry parameter and the gravitino field in supergravity are sections of them. The relevant $\tilde{H}_d$ representations are listed in table 5.5. Note that the spinor representation is simply the $\mathrm{Cliff}(d;\mathbb{R})$ spinor

$\tilde{H}_d$	S	J
$SU(8)$	$\mathbf{8} + \bar{\mathbf{8}}$	$\mathbf{56} + \bar{\mathbf{56}}$
$USp(8)$	$\mathbf{8}$	$\mathbf{48}$
$USp(4) \times USp(4)$	$(\mathbf{4},\mathbf{1}) + (\mathbf{1},\mathbf{4})$	$(\mathbf{4},\mathbf{5}) + (\mathbf{5},\mathbf{4})$
$USp(4)$	$\mathbf{4}$	$\mathbf{16}$

Table 5.5: Spinor and gravitino representations in each dimension

representation using the embedding (F.8).

One finds that under the projection $\mathcal{P}_X$ we have

$$\begin{aligned} \mathcal{P}_S(E^* \otimes S) &\simeq S \oplus J, \\ \mathcal{P}_J(E^* \otimes J) &\simeq S \oplus J. \end{aligned} \tag{5.63}$$

Therefore, for any $\varepsilon \in \Gamma(S)$ and $\psi \in \Gamma(J)$, one has that the following are unique for any torsion-free connection

$$\begin{aligned} &D \times_J \varepsilon, \qquad D \times_S \varepsilon, \\ &D \times_J \psi, \qquad D \times_S \psi, \end{aligned} \tag{5.64}$$

where $\times_X$ denotes the projection onto the X bundle.

We would now like to define measures of generalised curvature. As was mentioned in section 5.1.5, the natural definition of a Riemann curvature does not result in a tensor. Nonetheless, for a torsion-free, $\tilde{H}_d$-compatible connection D there does exist a generalised Ricci tensor R_{AB}, and it is a section of the bundle

$$\mathrm{ad}\, P^{\perp} = \mathrm{ad}\, \tilde{F} / \mathrm{ad}\, P \subset E^* \otimes E^*, \tag{5.65}$$

where the last relation follows because, as representations of H_d, $E \simeq E^*$. It is not immediately apparent that we can make such a definition, but R_{AB} can in fact be constructed from compositions of the unique operators (5.64) as

$$\begin{aligned} &D \times_J (D \times_J \varepsilon) + D \times_J (D \times_S \varepsilon) = R^0 \cdot \varepsilon, \\ &D \times_S (D \times_J \varepsilon) + D \times_S (D \times_S \varepsilon) = R\, \varepsilon, \end{aligned} \tag{5.66}$$

where R and R^0_{AB} provide the scalar and non-scalar parts of R_{AB} respectively[9]. The existence of expressions of this type is a non-trivial statement. By computing in the split frame, it can be shown that the LHS is linear in ε, and since ε and the LHS are manifestly covariant, these expressions define a tensor. We will write the components explicitly in equation (6.16). This calculation further provides the non-trivial result that R_{AB} is restricted to be a section of $\mathrm{ad}\, P^{\perp}$, rather than a more general section of $(S \otimes J) \oplus \mathbb{R}$. In the context of supergravity, this calculation exactly corresponds to the closure of the supersymmetry algebra on the fermionic equations of motion, as will be discussed further in section 6.2. Finally, since it is built from unique operators, the generalised curvature is automatically unique for a torsion-free compatible connection.

The expressions (5.66) can be written with a different sequence of projections. This

[9] Note that $\mathrm{ad}\, P^{\perp} \subset (S \otimes J) \oplus \mathbb{R}$ and the $\tilde{H}_d$ structure gives an isomorphism $S \simeq S^*$ and $J \simeq J^*$. Thus, as in the first line of (5.66), we can also view R^0 as a map from S to J.

helps elucidates the nature of the curvature in terms of certain second-order differential operators. In conventional differential geometry the commutator of two connections $[\nabla_m, \nabla_n]$ has no second-derivative term simply because the partial derivatives commute. This is a necessary condition for the curvature to be tensorial. In $E_{d(d)}$ indices one can similarly write the commutator of two generalised derivatives formally as $(D \wedge D)_{AB} = [D_A, D_B]$. More precisely, acting on an $E_{d(d)} \times\mathbb{R}^+$ vector bundle X we have

$$(D \wedge D) : X \to \Lambda^2 E^* \otimes X. \tag{5.67}$$

Since again the partial derivatives commute this operator contains no second-order derivative term, and so can potentially be used to construct a curvature tensor. However, in $E_{d(d)} \times\mathbb{R}^+$ generalised geometry we also have $\partial f \times_{N^*} \partial g = 0$ for any f and g, and so we can take the projection to the bundle N^* defined earlier, giving a similar operator

$$(D \times_{N^*} D) : X \to N^* \otimes X, \tag{5.68}$$

which will again contain no second-order derivatives. One thus expects that these two operators, which can be defined for an arbitrary $E_{d(d)} \times\mathbb{R}^+$ connection, should appear in any definition of generalised curvature. Given an $\tilde{H}_d$ structure and a torsion-free compatible connection D, they indeed enter the definition of R_{AB}. Using $\tilde{H}_d$ covariant projections one finds

$$\begin{aligned} (D \wedge D) \times_J \varepsilon + (D \times_{N^*} D) \times_J \varepsilon &= R^0 \cdot \varepsilon, \\ (D \wedge D) \times_S \varepsilon + (D \times_{N^*} D) \times_S \varepsilon &= R\varepsilon. \end{aligned} \tag{5.69}$$

This structure suggests there will be similar definitions of curvature in terms of the operators $(D \wedge D)$ and $(D \times_{N^*} D)$ independent of the representation on which they act, and potentially without the need for additional structure.

Chapter 6

H_d generalised gravity

We are now able to give a complete rewriting in the language of generalised geometry of the restricted eleven-dimensional supergravity from section 2.2. This will result in a unified formulation which has the larger bosonic symmetries of the theory manifest. Specifically, the local symmetry of the theory is $Spin(10-d,1) \times \tilde{H}_d$ where $\tilde{H}_d$ is the double-cover of the maximal compact subgroup of $E_{d(d)}$.

6.1 Supergravity degrees of freedom

Bosons

Consider then the maximally compact subgroup of $E_{d(d)}$, H_d. As we saw in section 5.2.1, the choice of such a structure is parametrised, at each point on the manifold, by a Riemannian metric g, a three-form A and a six-form $\tilde{A}$ gauge fields, and a scalar Δ, that is

$$\{g, A, \tilde{A}, \Delta\} \in \frac{E_{d(d)} \times \mathbb{R}^+}{H_d}. \tag{6.1}$$

These are precisely the set of bosonic fields in the restricted theory. We thus have that all the bosonic fields get unified in the generalised metric G defined in (5.50).

As in [30], the infinitesimal bosonic symmetry transformation is naturally encoded as the Dorfman derivative by $V \in \Gamma(E)$

$$\delta_V G = L_V G, \tag{6.2}$$

and the algebra of these transformations is given by $[L_U, L_V] = L_{L_U V} = -L_{L_V U} = L_{[\![U,V]\!]}$ where the bracket $[\![U, V]\!]$ is the antisymmetrisation of the Dorfman derivative (5.30).

Fermions

The fermionic degrees of freedom form spinor representations of $\tilde{H}_d$, the double cover[1] of H_d [59, 60, 109]. Let S and J denote the bundles associated to the representations of $\tilde{H}_d$ listed in table 5.5. The fermion fields ψ, ρ and the supersymmetry parameter ε of the restricted theory are sections

$$\psi \in \Gamma(J), \qquad \rho \in \Gamma(S), \qquad \varepsilon \in \Gamma(S). \tag{6.3}$$

However, the restricted fermions also transform as spinors of the flat $\mathbb{R}^{10-d,1}$ space. As discussed in section 2.2.2, the simplest formulation is to view them as eleven-dimensional spinors and use the embedding $Spin(10-d,1) \times \tilde{H}_d \subset \text{Cliff}(10,1;\mathbb{R})$ described in appendix F.3. This will allow us to write expressions directly comparable to the ones in section 2.2.2. There is, however, a price to pay, as there are actually two distinct ways of realising the action of $\tilde{H}_d$ on the $\text{Cliff}(10,1;\mathbb{R})$ spinor bundle $\hat{S}$, related by a change of sign of the gamma matrices. Given $\chi^\pm \in \Gamma(\hat{S})$ and $N \in \Gamma(\text{ad}\, P)$ we have the two actions

$$N \cdot \hat{\chi}^\pm = \tfrac{1}{2}\big(\tfrac{1}{2!} n_{ab}\Gamma^{ab} \pm \tfrac{1}{3!} b_{abc}\Gamma^{abc} - \tfrac{1}{6!}\tilde{b}_{a_1\dots a_6}\Gamma^{a_1\dots a_6}\big)\hat{\chi}^\pm. \tag{6.4}$$

If one denotes as $\hat{S}^\pm$ the bundle of spinors transforming under the two actions, one finds, for even d, that the two representations are equivalent, and $\hat{S} \simeq \hat{S}^+ \simeq \hat{S}^-$. However for odd d they are distinct and the spinor bundle decomposes $\hat{S} \simeq \hat{S}^+ \oplus \hat{S}^-$. The same applies to spin-$\frac{3}{2}$ bundles $\hat{J}^\pm$. The $Spin(10-d,1) \times \tilde{H}_d$ representations of the corresponding four

[1] Note that, as discussed in appendix C.1, $\tilde{H}_d$ can be defined abstractly for all $d \leq 8$ as the subgroup of $\text{Cliff}(d;\mathbb{R})$ preserving a particular involution of the algebra.

bundles listed in table 6.1 (see also [110]).

d	$\hat{S}^-$	$\hat{S}^+$	$\hat{J}^-$	$\hat{J}^+$
7	$(\mathbf{2},\mathbf{8})+(\bar{\mathbf{2}},\bar{\mathbf{8}})$	$(\mathbf{2},\bar{\mathbf{8}})+(\bar{\mathbf{2}},\mathbf{8})$	$(\mathbf{2},\mathbf{56})+(\bar{\mathbf{2}},\bar{\mathbf{56}})$	$(\mathbf{2},\bar{\mathbf{56}})+(\bar{\mathbf{2}},\mathbf{56})$
6	$(\mathbf{4},\mathbf{8})$	$(\mathbf{4},\mathbf{8})$	$(\mathbf{4},\mathbf{48})$	$(\mathbf{4},\mathbf{48})$
5	$(\mathbf{4},\mathbf{4},\mathbf{1})+(\bar{\mathbf{4}},\mathbf{1},\mathbf{4})$	$(\mathbf{4},\mathbf{1},\mathbf{4})+(\bar{\mathbf{4}},\mathbf{4},\mathbf{1})$	$(\mathbf{4},\mathbf{4},\mathbf{5})+(\bar{\mathbf{4}},\mathbf{5},\mathbf{4})$	$(\mathbf{4},\mathbf{5},\mathbf{4})+(\bar{\mathbf{4}},\mathbf{4},\mathbf{5})$
4	$(\mathbf{8},\mathbf{4})$	$(\mathbf{8},\mathbf{4})$	$(\mathbf{8},\mathbf{16})$	$(\mathbf{8},\mathbf{16})$

Table 6.1: Spinor and gravitino as $Spin(10-d,1)\times\tilde{H}_d$ representations. Note that when d is even the positive and negative representations are actually equivalent.

Finally, we find that the supergravity fields of section 2.2.2 can be identified as follows,

$$\begin{aligned} \hat{\varepsilon}^- &= \mathrm{e}^{-\Delta/2}\,\varepsilon^{\text{sugra}} && \in \Gamma(\hat{S}^-), \\ \hat{\rho}^+ &= \mathrm{e}^{\Delta/2}\,\rho^{\text{sugra}} && \in \Gamma(\hat{S}^+), \\ \hat{\psi}_a^- &= \mathrm{e}^{\Delta/2}\,\psi_a^{\text{sugra}} && \in \Gamma(\hat{J}^-). \end{aligned} \tag{6.5}$$

Note that, due to the warping of the metric, the precise maps between the fermion fields as viewed in the geometry and in the supergravity description involve a conformal rescaling. This is of course purely conventional, since one could just as easily perform field redefinitions at the supergravity level. We chose, however, to maintain the conventions in section 2.2.2 as familiar as possible and make the identification at this point.

6.2 Supergravity operators

The differential operators present in the supergravity equations will be built out of generalised connections D which are simultaneously torsion-free and H_d compatible. As we saw in chapter 5, there always exists such a torsion-free, metric compatible connection but, unlike the Levi–Civita connection, it is not unique. Instead, we were led to define projectors which result in unique operators when applied to D. We identified four such maps in section 5.2.3, and they turned out to be directly related to the representations of the fermion fields. Since we are interested in comparing with the supergravity expressions, we can take the embedding (F.12) and consider the natural action of D on the $Spin(10-d,1)\times\tilde{H}_d$

representations listed in table 6.1. The four projections then split into eight

$$\begin{aligned} D\times_{\hat{S}^{\mp}} &: \hat{S}^{\pm}\to\hat{S}^{\mp}, & D\times_{\hat{J}^{\pm}} &: \hat{S}^{\pm}\to\hat{J}^{\pm}, \\ D\times_{\hat{J}^{\mp}} &: \hat{J}^{\pm}\to\hat{J}^{\mp}, & D\times_{\hat{S}^{\pm}} &: \hat{J}^{\pm}\to\hat{S}^{\pm}. \end{aligned} \tag{6.6}$$

In section 5.2.3 we kept the discussion fairly abstract, but now we can check explicitly that the projected derivatives are indeed independent of the undetermined components of the connection Q, by decomposing under $Spin(d)\subset\tilde{H}_d$ and taking the torsion-free connection (5.59). Using the formulae for the projections given in (F.18) and (F.19), and already applying the operators to the supersymmetry parameter $\hat{\varepsilon}^-$ in (6.5), we then find for the first two

$$\begin{aligned} D\times_{\hat{S}^+}\hat{\varepsilon}^- &= \mathrm{e}^{\Delta/2}\Big(\nabla\!\!\!\!/ + \tfrac{9-d}{2}(\partial\!\!\!/\Delta) - \tfrac{1}{4}F\!\!\!\!/ - \tfrac{1}{4}\tilde{F\!\!\!\!/}\Big)\varepsilon^{\text{sugra}}, \\ (D\times_{\hat{J}^-}\hat{\varepsilon}^-)_a &= \mathrm{e}^{\Delta/2}\Big(\nabla_a + \tfrac{1}{288}(\Gamma_a{}^{b_1\dots b_4} - 8\delta_a{}^{b_1}\Gamma^{b_2b_3b_4})F_{b_1\dots b_4} \\ &\qquad - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{ab_1\dots b_6}\Gamma^{b_1\dots b_6}\Big)\varepsilon^{\text{sugra}}. \end{aligned} \tag{6.7}$$

From derivatives of elements $\Gamma(\hat{J}^{\pm})$ we obtain the second set of unique operators which using (F.20) and (F.21) as applied to $\hat{\psi}^-$ of (6.5), take the form

$$\begin{aligned} D\times_{\hat{S}^-}\hat{\psi}^- &= \mathrm{e}^{3\Delta/2}\Big[\nabla^a - \tfrac{1}{10-d}\Gamma^{ab}\nabla_b + (10-d)\partial^a\Delta - \Gamma^{ab}\partial_b\Delta \\ &\qquad - \tfrac{1}{4}\tfrac{1}{3!}F^a{}_{b_1b_2b_3}\Gamma^{b_1b_2b_3} + \tfrac{1}{4}\tfrac{1}{10-d}\tfrac{1}{4!}\Gamma^a{}_{b_1\dots b_4}F^{b_1\dots b_4} \\ &\qquad - \tfrac{1}{4}\tfrac{1}{6!}\tilde{F}^a{}_{b_1\dots b_6}\Gamma^{b_1\dots b_6}\Big]\psi_a^{\text{sugra}}, \\ (D\times_{\hat{J}^+}\hat{\psi}^-)_a &= -\mathrm{e}^{3\Delta/2}\Big[\Gamma^c(\nabla_c + \tfrac{11-d}{2}\partial_c\Delta)\delta_a{}^b + \tfrac{2}{9-d}\Gamma^b(\nabla_a + \tfrac{11-d}{2}\partial_a\Delta) \\ &\qquad - \tfrac{1}{12}(3+\tfrac{2}{9-d})F\!\!\!\!/\,\delta_a{}^b + \tfrac{1}{3}\tfrac{10-d}{9-d}\tfrac{1}{2!}F_a{}^b{}_{cd}\Gamma^{cd} \\ &\qquad - \tfrac{1}{3}\tfrac{1}{9-d}\tfrac{1}{3!}F_a{}^{c_1\dots c_3}\Gamma^b{}_{c_1\dots c_3} + \tfrac{1}{6}\tfrac{10-d}{9-d}\tfrac{1}{3!}F^{bc_1\dots c_3}\Gamma_{ac_1\dots c_3} \\ &\qquad - \tfrac{1}{6}\tfrac{1}{9-d}\tfrac{1}{4!}F_{c_1\dots c_4}\Gamma_a{}^{bc_1\dots c_4} + \tfrac{1}{4}\tfrac{1}{5!}\tilde{F}_a{}^b{}_{c_1\dots c_5}\Gamma^{c_1\dots c_5}\Big]\psi_b^{\text{sugra}}. \end{aligned} \tag{6.8}$$

These four operators, all constructed from the same connection, will now enable us to rewrite all the supergravity equations of section 2.2.2.

6.3 Supersymmetry algebra

Comparing with (2.35), we immediately see that the operators (6.6) give precisely the supersymmetry variations of the two fermion fields

$$\begin{aligned}\delta\hat{\psi}^- &= D \times_{\hat{J}^-} \hat{\varepsilon}^-,\\ \delta\hat{\rho}^+ &= D \times_{\hat{S}^+} \hat{\varepsilon}^-.\end{aligned} \tag{6.9}$$

Since the bosons arrange themselves into the generalised metric, one expects that their supersymmetry variations (2.36) are given by the variation of G. In fact, the most convenient object to consider is $G^{-1}\delta G$ which is naturally a section of the bundle $\mathrm{ad}(P)^\perp$, listed in table 5.3. One has the isomorphism (F.5)

$$\mathrm{ad}(P)^\perp \simeq \mathbb{R} \oplus S^2T^*M \oplus \Lambda^3T^*M \oplus \Lambda^6T^*M \tag{6.10}$$

and we can identify the component variations of the generalised metric, as written in the split frame, as

$$\begin{aligned}(G^{-1}\delta G) &= -2\delta\Delta,\\ (G^{-1}\delta G)_{ab} &= \delta g_{ab}\\ (G^{-1}\delta G)_{abc} &= -\delta A_{abc},\\ (G^{-1}\delta G)_{a_1\dots a_6} &= -\delta\tilde{A}_{a_1\dots a_6}.\end{aligned} \tag{6.11}$$

One finds that the supersymmetry variations of the bosons (2.36) can be written in the $\tilde{H}_d$ covariant form

$$G^{-1}\delta G = (\hat{\psi}^- \times_{\mathrm{ad}\,P^\perp} \hat{\varepsilon}^-) + (\hat{\rho}^+ \times_{\mathrm{ad}\,P^\perp} \hat{\varepsilon}^-), \tag{6.12}$$

where $\times_{\mathrm{ad}\,P^\perp}$ denotes the projection to $\mathrm{ad}(P)^\perp$ given in (F.15) and (F.16).

6.4 Generalised Curvatures and the Equations of Motion

To realise the fermionic equations of motion one uses the unique projections (6.6). We can then formulate the two equations (2.33) and (2.34) as, respectively,

$$\begin{aligned} -D \times_{\hat{J}^+} \hat{\psi}^- - \tfrac{11-d}{9-d} D \times_{\hat{J}^+} \hat{\rho}^+ &= 0, \\ -D \times_{\hat{S}^-} \hat{\rho}^+ - D \times_{\hat{S}^-} \hat{\psi}^- &= 0. \end{aligned} \tag{6.13}$$

Note that $\hat{\rho}^+$ is embedded with a different conformal factor to $\hat{\varepsilon}^-$ and also is a section of $\hat{S}^+$ rather than $\hat{S}^-$. This means we have

$$\begin{aligned} D \times_{\hat{S}^-} \hat{\rho}^+ &= -\mathrm{e}^{3\Delta/2} \Big(\nabla\!\!\!/ + \tfrac{11-d}{2} (\partial\!\!\!/ \Delta) + \tfrac{1}{4} F\!\!\!/ - \tfrac{1}{4} \tilde{F}\!\!\!/ \Big) \rho^{\text{sugra}} \\ (D \times_{\hat{J}^+} \hat{\rho}^+)_a &= \mathrm{e}^{3\Delta/2} \Big[(\nabla_a + \partial_a \Delta) - \tfrac{1}{288} ({\Gamma_a}^{b_1 \dots b_4} - 8 {\delta_a}^{b_1} \Gamma^{b_2 b_3 b_4}) F_{b_1 \dots b_4} \\ &\qquad - \tfrac{1}{12} \tfrac{1}{6!} \tilde{F}_{a b_1 \dots b_6} \Gamma^{b_1 \dots b_6} \Big] \rho^{\text{sugra}} \end{aligned} \tag{6.14}$$

From these we can now find explicitly the generalised Ricci tensor R_{AB} we defined in (5.66). Recall that the supersymmetric variation of the fermionic equations of motion vanishes up to the bosonic equations of motion (6.13). Anticipating that the bosonic equations of motion will correspond to $R_{AB} = 0$, one way to define generalised Ricci tensor is via the variation of (6.13) under (6.9). By construction this gives R_{AB} as a section of $\operatorname{ad} P^\perp \subset E^* \otimes E^*$, the same space as variations of the generalised metric δG, in complete analogy to the conventional metric and Ricci tensor. Defining R_{AB} as an $\tilde{H}_d$ tensor we write

$$\begin{aligned} -D \times_{\hat{J}^+} (D \times_{\hat{J}^-} \hat{\varepsilon}^-) - \tfrac{11-d}{9-d} D \times_{\hat{J}^+} (D \times_{\hat{S}^+} \hat{\varepsilon}^-) &= R^0 \cdot \hat{\varepsilon}^-, \\ D \times_{\hat{S}^-} (D \times_{\hat{J}^-} \hat{\varepsilon}^-) + D \times_{\hat{S}^-} (D \times_{\hat{S}^+} \hat{\varepsilon}^-) &= R \, \hat{\varepsilon}^-, \end{aligned} \tag{6.15}$$

for any $\hat{\varepsilon}^- \in \Gamma(\hat{S}^-)$ and where R and R^0_{AB} are the scalar and non-scalar parts of R_{AB} respectively. The action of R^0_{AB} on $\hat{\varepsilon}^-$ that appears of the right-hand side of (6.15) is given explicitly in (F.11).

In components, using the notation of (F.5), we find

$$\begin{aligned} R &= \mathrm{e}^{2\Delta}\Big[\mathcal{R} - 2(c-1)\nabla^2\Delta - c(c-1)(\partial\Delta)^2 - \tfrac{1}{2}\tfrac{1}{4!}F^2 - \tfrac{1}{2}\tfrac{1}{7!}\tilde{F}^2\Big] \\ R_{ab} &= \mathrm{e}^{2\Delta}\Big[\mathcal{R}_{ab} - c\nabla_a\nabla_b\Delta - c(\partial_a\Delta)(\partial_b\Delta) \\ &\qquad - \tfrac{1}{2}\tfrac{1}{4!}\Big(4F_{ac_1c_2c_3}F_b{}^{c_1c_2c_3} - \tfrac{1}{3}g_{ab}F^2\Big) \\ &\qquad - \tfrac{1}{2}\tfrac{1}{7!}\Big(7\tilde{F}_{ac_1\dots c_6}\tilde{F}_b{}^{c_1\dots c_6} - \tfrac{2}{3}g_{ab}\tilde{F}^2\Big)\Big], \\ R_{abc} &= \tfrac{1}{2}\mathrm{e}^{2\Delta} * \Big[\mathrm{e}^{-c\Delta}\mathrm{d} * (\mathrm{e}^{c\Delta}F) - F\wedge *\tilde{F}\Big]_{abc}, \\ R_{a_1\dots a_6} &= \tfrac{1}{2}\mathrm{e}^{2\Delta} * \Big[\mathrm{e}^{-c\Delta}\mathrm{d} * (\mathrm{e}^{c\Delta}\tilde{F})\Big]_{a_1\dots a_6}, \end{aligned} \tag{6.16}$$

where $c = 11 - d$. The generalised Ricci tensor is manifestly uniquely determined and comparing with (2.31) we see that the bosonic equations of motion become simply

$$R_{AB} = 0. \tag{6.17}$$

The bosonic action (2.30) is given by the generalised curvature scalar, integrated with the volume form (5.51)

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int |\mathrm{vol}_{\mathrm{G}}|\, R. \tag{6.18}$$

Finally, the fermionic action can be written using the natural invariant pairings of the terms in (6.13) with the fermionic fields. Using the expressions (F.14) and (F.13) for the spinor bilinears, we find that (2.32) can be rewritten as

$$\begin{aligned} S_{\mathrm{F}} = \frac{1}{\kappa^2}\int |\mathrm{vol}_{\mathrm{G}}|\,\Big[&- \langle\hat{\psi}^-, D\times_{\hat{J}^+}\hat{\psi}^-\rangle - \tfrac{c}{c-2}\langle\hat{\psi}^-, D\times_{\hat{J}^+}\hat{\rho}^+\rangle \\ &+ \tfrac{c(c-1)}{(c-2)^2}\langle\hat{\rho}^+, D\times_{\hat{S}^-}\hat{\psi}^-\rangle + \tfrac{c(c-1)}{(c-2)^2}\langle\hat{\rho}^+, D\times_{\hat{S}^-}\hat{\rho}^+\rangle\Big]. \end{aligned} \tag{6.19}$$

6.5 Explicit H_d constructions

In the previous section, we gave the generic construction of the supergravity in terms of generalised geometry, valid in all $d \leq 7$. The theory has a local $\tilde{H}_d$ symmetry, however this was not explicit since we used a $\mathrm{Cliff}(10,1;\mathbb{R})$ formulation for the fermionic fields.

For completeness, we now work out explicitly two examples, in $d = 4$ and $d = 7$, where the local $Spin(5)$ and $SU(8)$ symmetries are manifest. Correspondingly, in this section we treat the fermions slightly differently from the previous ones. Whereas before we kept all spinors as $\mathrm{Cliff}(10,1;\mathbb{R})$ objects, we now want to make their $\tilde{H}_d$ nature more explicit. In order to make this possible, one has to decompose the eleven-dimensional spinors following the procedures outlined in appendix D and embed the $\mathrm{Cliff}(d;\mathbb{R})$ expressions into $\tilde{H}_d$ representations, according to appendix F.2. We will then keep the external spinor indices of the fermion fields hidden and treat them as sections of the genuine $\tilde{H}_d$ bundles S and J.

6.5.1 $d = 4$ and $\tilde{H}_4 = Spin(5)$

$GL^+(5,\mathbb{R})$ generalised geometry

In four dimensions, we have $E_{4(4)} \times \mathbb{R}^+ \simeq SL(5,\mathbb{R}) \times \mathbb{R}^+ \simeq GL^+(5,\mathbb{R})$. We can then write the generalised geometry explicitly in terms of indices $i, j, k, \dots = 1, \dots, 5$ transforming under $GL^+(5,\mathbb{R})$.

Generalised vectors V transform in the antisymmetric **10** representation. We can introduce a basis $\{\hat{E}_{ii'}\}$ (locally a section of the generalised structure bundle $\tilde{F}$) transforming under $GL^+(5,\mathbb{R})$ so that

$$V = \tfrac{1}{2} V^{ii'} \hat{E}_{ii'}. \tag{6.20}$$

In the conformal split frame (5.15), we can identify [64, 79]

$$\begin{aligned} \hat{E}_{a5} &= \mathrm{e}^{\Delta} \left(\hat{e}_a + i_{\hat{e}_a} A \right), \\ \hat{E}_{ab} &= \tfrac{1}{2} \mathrm{e}^{\Delta} \epsilon_{abcd} e^{cd}, \end{aligned} \tag{6.21}$$

where ϵ is the numerical totally antisymmetric symbol. Equivalently

$$\begin{aligned} V^{a5} &= v^a, \\ V^{ab} &= \tfrac{1}{2} \epsilon^{abcd} \omega_{cd}, \end{aligned} \tag{6.22}$$

where v^a and ω_{ab} are as in (5.16). In this frame the partial derivative (5.26) $\partial_{ii'}$ has the form

$$\begin{aligned} \partial_{a5} &= \tfrac{1}{2}\mathrm{e}^{\Delta}\partial_a, \\ \partial_{ab} &= 0. \end{aligned} \tag{6.23}$$

Note that there is also a generalised tensor bundle W which transforms in the fundamental **5** representation of $GL^+(5,\mathbb{R})$. One finds

$$W \simeq (\det T^*M)^{1/2} \otimes (TM \oplus \det TM)\,, \tag{6.24}$$

and a choice of basis $\{\hat{E}_{ii'}\}$ defines a basis $\{\hat{E}_i\}$ of W where $K = K^i\hat{E}_i \in \Gamma(W)$, such that

$$\hat{E}_{ii'} = \hat{E}_i \wedge \hat{E}_{i'}, \tag{6.25}$$

since $E \simeq \Lambda^2 W$, and where we use the four-dimensional isomorphism $\det T^*M \otimes \Lambda^2 TM \simeq \Lambda^2 T^*M$.

With this notation we can then use the $GL^+(5,\mathbb{R})$ adjoint action explicitly to write the Dorfman derivative (5.28) of a generalised vector. It takes its simplest form in the coordinate frame, where it reads

$$L_V W^{ij} = V^{kk'}\partial_{kk'}W^{ij} + 4\left(\partial_{kk'}V^{k[i}\right)W^{j]k'} + \left(\partial_{kk'}V^{kk'}\right)W^{ij}. \tag{6.26}$$

This form of the $d = 4$ Dorfman derivative was given, without the $\mathbb{R}^+$ action, in [79][2]. We can then write a generic generalised connection as

$$D_{ii'}V^{jj'} = \partial_{ii'}V^{jj'} + \Omega_{ii'}{}^{j}{}_{k}V^{kj'} + \Omega_{ii'}{}^{j'}{}_{k}V^{jk}, \tag{6.27}$$

where the j and k indices of $\Omega_{ii'}{}^{j}{}_{k}$ parametrise an element of the adjoint of $GL^+(5,\mathbb{R})$.

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[2]For the antisymmetrisation of $L_V W$ (which is simply the Courant bracket for two-forms [9]) in $SL(5,\mathbb{R})$ indices see also [111].

$Spin(5)$ structures and supergravity

In four dimensions $H_d \simeq SO(5)$ and we define the sub-bundle $P \subset \tilde{F}$ of $SO(5)$ frames as the set of frames where the generalised metric (5.50) can be written as

$$G(V, W) = \tfrac{1}{2}\delta_{ij}\delta_{i'j'}V^{ii'}W^{jj'}, \tag{6.28}$$

where δ_{ij} is the flat $SO(5)$ metric with which we can raise and lower indices frame indices. Equivalently we can think of the generalised metric as defining orthonormal frames on the $\mathbf{5}$-representation bundle W.

Upon decomposing the fermionic fields of the supergravity according to D.1, one finds that they embed into the spinor and traceless vector-spinor representations of $Spin(5)$. Our conventions regarding $\mathrm{Cliff}(4;\mathbb{R})$ and $\mathrm{Cliff}(5;\mathbb{R})$ algebras are given in appendix C.5 and we leave $Spin(5)$ spinor indices implicit throughout. We define

$$\begin{aligned} \varepsilon &= \mathrm{e}^{-\Delta/2}\varepsilon^{\text{sugra}} && \in \Gamma(S), \\ \rho &= \mathrm{e}^{\Delta/2}\gamma^{(4)}\rho^{\text{sugra}} && \in \Gamma(S), \\ \psi_i &= \begin{cases} \mathrm{e}^{\Delta/2}\gamma^{(4)}\left(\delta^b{}_a - \tfrac{2}{5}\gamma_a\gamma^b\right)\psi_b^{\text{sugra}} & \text{for } i = a \\ -\tfrac{3}{5}\mathrm{e}^{\Delta/2}\gamma^a\psi_a^{\text{sugra}} & \text{for } i = 5 \end{cases} && \in \Gamma(J). \end{aligned} \tag{6.29}$$

Crucially, note the appearance of conformal factors in the definitions, in similar fashion to (6.5). Recall also that in four dimensions we have $S \simeq S^+ \simeq S^-$, where the action by $\gamma^{(4)}$ in the second line of (6.29) realises the second isomorphism.

A generalised connection is compatible with the generalised metric (6.28) if $DG = 0$. In terms of the connection (6.27) in frame indices this implies

$$\Omega_{ii'\,jj'} = -\Omega_{ii'\,j'j} \tag{6.30}$$

where indices are lowered using the $SO(5)$ metric δ_{ij}. For such $SO(5)$-connections, we can define the generalised spinor derivative, given $\chi \in \Gamma(S)$

$$D_{ii'}\chi = \left(\partial_{ii'} + \tfrac{1}{4}\Omega_{ii'jj'}\hat{\gamma}^{jj'}\right)\chi. \tag{6.31}$$

An example of such a generalised connection is the one (5.53) defined by the Levi–Civita connection ∇, where, acting on $\chi \in \Gamma(S)$, we have

$$D^{\nabla}_{ii'}\chi = \begin{cases} \frac{1}{2}\mathrm{e}^{\Delta}\left(\partial_a + \frac{1}{4}\omega_{a\,bc}\hat{\gamma}^{bc}\right)\chi & \text{if } i = a \text{ and } i' = 5 \\ 0 & \text{if } i = a \text{ and } i' = b \end{cases}, \tag{6.32}$$

where $\omega_{a\,bc}$ is the usual spin-connection.

We can construct a torsion-free compatible connection D, by shifting D^{∇} by an additional connection piece $\Sigma_{[ii'][jj']}$, such that its action on $\chi \in \Gamma(S)$ is given by

$$D_{ii'}\chi = D^{\nabla}_{ii'}\chi + \tfrac{1}{4}\Sigma_{ii'jj'}\hat{\gamma}^{jj'}\chi. \tag{6.33}$$

The connection is torsion-free if

$$\Sigma_{ii'jj'} = \tfrac{1}{2}\left(\delta_{j[i}\Sigma_{i']j'} - \delta_{j'[i}\Sigma_{i']j}\right) + Q_{ii'jj'}, \tag{6.34}$$

where $Q_{ii'jj'}$ is the undetermined part – traceless and symmetric under exchange of pairs of indices, so it transforms in the **35** of $SO(5)$, see table 5.4 – and

$$\begin{aligned} \Sigma_{a5} &= -\Sigma_{5a} = -2\mathrm{e}^{\Delta}\partial_a\Delta, \\ \Sigma_{ab} &= -\tfrac{1}{12}\mathrm{e}^{\Delta}F\delta_{ab}, \\ \Sigma_{55} &= \tfrac{7}{12}\mathrm{e}^{\Delta}F, \end{aligned} \tag{6.35}$$

with $F = \frac{1}{4!}\epsilon^{abcd}F_{abcd}$. The projections (6.6) can be written in $Spin(5)$ indices as

$$\begin{aligned} D \times_S \varepsilon &= -\hat{\gamma}^{ij}D_{ij}\varepsilon, \\ (D \times_J \varepsilon)_i &= 2(\hat{\gamma}^j D_{ij}\varepsilon - \tfrac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\varepsilon), \\ (D \times_J \psi)_i &= -\hat{\gamma}^{jj'}D_{jj'}\psi_i + \tfrac{12}{5}D_{ij}\psi^j - \tfrac{8}{5}\hat{\gamma}_i{}^j D_{jj'}\psi^{j'}, \\ D \times_S \psi &= -\tfrac{5}{3}\hat{\gamma}^i D_{ij}\psi^j. \end{aligned} \tag{6.36}$$

and are unique, independent of $Q_{ii'jj'}$.

The supersymmetry variations of the fermions (6.9) can then be written in a manifestly

$Spin(5)$ covariant form

$$\begin{aligned}\delta\psi_i &= (D\times_J\varepsilon)_i = 2(\hat{\gamma}^j D_{ij}\varepsilon - \tfrac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\varepsilon),\\ \delta\rho &= D\times_S\varepsilon = -\hat{\gamma}^{ij}D_{ij}\varepsilon,\end{aligned} \tag{6.37}$$

whereas the variation of the bosons (6.12) is given by

$$\delta G_{[ii'][jj']} = \tfrac{1}{2}\big(\delta H_{i[j}\delta_{j']i'} - \delta H_{i'[j}\delta_{j']i}\big), \tag{6.38}$$

with

$$\delta H_{ij} = -2\bar{\varepsilon}\hat{\gamma}_{(i}\psi_{j)} - \tfrac{1}{5}\delta_{ij}\bar{\varepsilon}\rho. \tag{6.39}$$

Turning to the equations of motion, from (6.13), we find that the fermionic equations take the form

$$\begin{aligned}-\tfrac{14}{5}(\hat{\gamma}^j D_{ij}\rho - \tfrac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\rho) - \hat{\gamma}^{jj'}D_{jj'}\psi_i + \tfrac{12}{5}D_{ij}\psi^j - \tfrac{8}{5}\hat{\gamma}_i{}^j D_{jj'}\psi^{j'} &= 0,\\ \hat{\gamma}^{ij}D_{ij}\rho + \tfrac{5}{3}\hat{\gamma}^i D_{ij}\psi^j &= 0.\end{aligned} \tag{6.40}$$

The generalised Ricci tensor (6.15), after some rearrangement and gamma matrix algebra, can be written as

$$\begin{aligned}R^0_{ij}\hat{\gamma}^j\varepsilon &= \tfrac{4}{5}\hat{\gamma}^j\left[D_{ik}, D_j{}^k\right]\varepsilon - 2\hat{\gamma}^{jkl}\left[D_{ij}, D_{kl}\right]\varepsilon - \tfrac{56}{25}\hat{\gamma}_i{}^{jk}\left[D_{jl}, D_k{}^l\right]\varepsilon\\ &\quad - \tfrac{16}{5}\hat{\gamma}^{jkl}D_{[ij}D_{kl]}\varepsilon + \tfrac{8}{5}\hat{\gamma}_i{}^{j_1\dots j_4}D_{[j_1j_2}D_{j_3j_4]}\varepsilon,\\ \tfrac{5}{24}R\varepsilon &= \tfrac{5}{3}\hat{\gamma}^{ii'jj'}D_{ii'}D_{jj'}\varepsilon - \tfrac{5}{3}\hat{\gamma}^{ij}[D_{ik}, D_j{}^k]\varepsilon.\end{aligned} \tag{6.41}$$

Note that in this form one can clearly see that the curvatures cannot be obtained simply from the commutator of two generalised covariant derivatives. Instead, one must consider additional terms resulting from a specific symmetric projection of the connections, as observed in section 5.2.3.

The bosonic action (6.18) is

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int |\mathrm{vol}_{\mathrm{G}}|\, R. \tag{6.42}$$

While the fermionic action (6.19) can be written as

$$S_{\rm F} = \frac{1}{\kappa^2}\int |{\rm vol_G}| \Big(-\bar{\psi}^i(-\hat{\gamma}^{jk}D_{jk}\psi_i + \tfrac{12}{5}D_{ij}\psi^j - \tfrac{8}{5}\hat{\gamma}_i{}^j D_{jk}\psi^k) - \tfrac{14}{5}\bar{\psi}^i(\hat{\gamma}^j D_{ij}\rho - \tfrac{1}{5}\hat{\gamma}_i\hat{\gamma}^{jj'}D_{jj'}\rho) - \tfrac{14}{5}(\bar{\rho}\hat{\gamma}^i D_{ij}\psi^j) - \tfrac{42}{25}(\bar{\rho}\hat{\gamma}^{ij}D_{ij}\rho)\Big), \tag{6.43}$$

where we use the $Spin(5)$ covariant spinor conjugate. It is also important to note that there are two sets of suppressed indices on the spinors in this expression. These are the $SU(2)$ indices for the five-dimensional symplectic Majorana spinors and the external $Spin(6,1)$ indices, which must be summed over. For full details of the spinor conventions used, see appendices C.5 and D.1.

We have now rewritten all of the supergravity equations with manifest $Spin(5)$ symmetry.

6.5.2 $d = 7$ and $\tilde{H}_7 = SU(8)$

$E_{7(7)} \times \mathbb{R}^+$ generalised geometry

We follow the standard approach [59] of describing $E_{7(7)}$ in terms of its $SL(8,\mathbb{R})$ subgroup, following the notation of [65][3]. We denote indices transforming under $SL(8,\mathbb{R})$ by $i, j, k, \dots = 1, \dots, 8$.

Generalised vectors transform in the $\mathbf{56}$ representation of $E_{7(7)}$, which under $SL(8,\mathbb{R})$ decomposes into the sum $\mathbf{28} + \mathbf{28}'$ of bivectors and two-forms. We can introduce a basis $\{\hat{E}_{ii'}, \check{E}^{ii'}\}$ transforming under $E_{7(7)}$ and write a generalised vector as

$$V = \tfrac{1}{2}V^{ii'}\hat{E}_{ii'} + \tfrac{1}{2}\tilde{V}_{ii'}\check{E}^{ii'}. \tag{6.44}$$

[3] Note however that when it comes to spinors, here we take instead $\gamma^{(7)} = -\mathrm{i}$, the opposite choice to that in [65], and we also use a different normalisation of our $SU(8)$ indices.

In the conformal split frame (5.15), we can identify

$$\begin{aligned} V^{a8} &= v^a, & V^{ab} &= \tfrac{1}{5!}\epsilon^{abc_1\dots c_5}\sigma_{c_1\dots c_5}, \\ \tilde{V}_{a8} &= \tfrac{1}{7!}\epsilon^{b_1\dots b_7}\tau_{a,b_1\dots b_7}, & \tilde{V}_{ab} &= \omega_{ab}, \end{aligned} \tag{6.45}$$

where v^a, ω_{ab}, etc. are as in (5.16), with the obvious corresponding identification of $\hat{E}_{a8}$ etc. The partial derivative ∂_μ is lifted into E^*, with a conformal factor due to the form of the conformal split frame, as

$$\partial_{a8} = \tfrac{1}{2}\mathrm{e}^{\Delta}\partial_a, \qquad \partial_{ab} = 0, \qquad \tilde{\partial}^{ii'} = 0. \tag{6.46}$$

In this notation, the Dorfman derivative (5.28), the antisymmetrisation of which is the "exceptional Courant bracket" of [65], can then be written in the coordinate frame as

$$\begin{aligned} (L_V W)^{ii'} &= V^{jj'}\partial_{jj'}W^{ii'} + 4W^{j[i}\partial_{jj'}V^{i']j'} \\ &\qquad + W^{ii'}\partial_{jj'}V^{jj'} - \tfrac{1}{4}\epsilon^{ii'jj'kk'll'}\tilde{W}_{jj'}\partial_{kk'}\tilde{V}_{ll'}, \\ (L_V W)_{ii'} &= V^{jj'}\partial_{jj'}\tilde{W}_{ii'} - 4\tilde{W}_{j[i}\partial_{i']j'}V^{jj'} - 6W^{jj'}\partial_{[jj'}\tilde{V}_{ii']}, \end{aligned} \tag{6.47}$$

where $\epsilon^{i_1\dots i_8}$ is the totally antisymmetric symbol preserved by $SL(8,\mathbb{R})$.

A generic $E_{7(7)}\times\mathbb{R}^+$ generalised connection $D = (D_{ii'}, \tilde{D}^{ii'})$ acting on $V \in \Gamma(E)$ takes the form

$$\begin{aligned} D_{ii'}V^{jj'} &= \partial_{ii'}V^{jj'} + \Omega_{ii'}{}^{j}{}_{k}V^{kj'} + \Omega_{ii'}{}^{j'}{}_{k}V^{jk} + {*\Omega_{ii'}}^{jj'kk'}\tilde{V}_{kk'}, \\ D_{ii'}\tilde{V}_{jj'} &= \partial_{ii'}\tilde{V}_{jj'} - \Omega_{ii'}{}^{k}{}_{j}\tilde{V}_{kj'} - \Omega_{ii'}{}^{k}{}_{j'}\tilde{V}_{jk} + \Omega_{ii'\,jj'kk'}V^{kk'}, \\ \tilde{D}^{ii'}V^{jj'} &= \tilde{\partial}^{ii'}V^{jj'} + \tilde{\Omega}^{ii'\,j}{}_{k}V^{kj'} + \tilde{\Omega}^{ii'\,j'}{}_{k}V^{jk} + {*\tilde{\Omega}}^{ii'\,jj'kk'}\tilde{V}_{kk'}, \\ \tilde{D}^{ii'}\tilde{V}_{jj'} &= \tilde{\partial}^{ii'}\tilde{V}_{jj'} - \tilde{\Omega}^{ii'\,k}{}_{j}\tilde{V}_{kj'} - \tilde{\Omega}^{ii'\,k}{}_{j'}\tilde{V}_{jk} + \tilde{\Omega}^{ii'}{}_{jj'kk'}V^{kk'}, \end{aligned} \tag{6.48}$$

where ${*\Omega_{ii'}}^{jj'kk'} = \frac{1}{4}\epsilon^{jj'kk'll'mm'}\Omega_{ii'\,ll'mm'}$ and similarly for ${*\tilde{\Omega}}^{ii'jj'kk'}$.

$SU(8)$ structures and supergravity

In seven dimensions $H_d = SU(8)/\mathbb{Z}_2$ and the common subgroup of H_d and the $SL(8,\mathbb{R})$ subgroup that we used to define $E_{7(7)}$ is $SO(8)$. We define the sub-bundle $P \subset \tilde{F}$ of $SU(8)/\mathbb{Z}_2$ frames as the set of frames where the generalised metric (5.50) can be written as

$$G(V,W) = \tfrac{1}{2}\big(\delta_{ij}\delta_{i'j'}V^{ii'}W^{jj'} + \delta^{ij}\delta^{i'j'}\tilde{V}_{ii'}\tilde{W}_{jj'}\big), \tag{6.49}$$

where δ_{ij} is the flat $SO(8)$ metric. To write sections of E with manifest $SU(8)$ indices $\alpha, \beta, \gamma, \ldots = 1, \ldots, 8$ one uses the $SO(8)$ gamma matrices

$$\begin{aligned} V^{\alpha\beta} &= \mathrm{i}(\hat{\gamma}_{ij})^{\alpha\beta}\big(V^{ij} + \mathrm{i}\tilde{V}^{ij}\big), \\ \bar{V}_{\alpha\beta} &= -\mathrm{i}(\hat{\gamma}^{ij})_{\alpha\beta}\big(V_{ij} - \mathrm{i}\tilde{V}_{ij}\big). \end{aligned} \tag{6.50}$$

where, $\hat{\gamma}^{ij}$ are defined in (C.27) and, when restricted to the $Spin(8)$ subgroup $\alpha, \beta, \ldots$ indices are raised and lowered using the intertwiner $\tilde{C}$ (see appendix C.2).

The eleven-dimensional supergravity fermion fields can be decomposed into complex seven-dimensional spinors following the discussion in D.2. Using the embedding $Spin(7) \subset Spin(8) \subset SU(8)$, discussed in detail in appendix C.6, they can be identified as $SU(8)$ representations as follows. For the spinors we simply have

$$\begin{aligned} \varepsilon^{\alpha} &= \mathrm{e}^{-\Delta/2}(\varepsilon^{\text{sugra}})^{\alpha} && \in \Gamma(S^{-}), \\ \bar{\rho}_{\alpha} &= \mathrm{i}\mathrm{e}^{\Delta/2}\tilde{C}_{\alpha\beta}(\gamma^{(7)}\rho^{\text{sugra}})^{\beta} && \in \Gamma(S^{+}). \end{aligned} \tag{6.51}$$

Note the need to include the conformal factors in the definitions and also that, though we write $\bar{\rho}$ since it is embedded into the $\bar{\mathbf{8}}$ representation of $SU(8)$, $\bar{\rho}_{\alpha}$ is defined in terms of the un-conjugated ρ^{sugra}. The $\mathbf{8}$ and $\bar{\mathbf{8}}$ representations are simply the fundamental and anti-fundamental so are related by conjugation so that $\bar{\varepsilon}_{\alpha} = (\varepsilon^{\beta})^{*}A_{\dot{\beta}\alpha}$, using the $SU(8)$-invariant intertwiner A (see appendix C.2).

For the 56-dimensional vector-spinor we proceed in two steps, first embedding into

$Spin(8)$ by writing

$$\begin{aligned} \psi^{Spin(8)}_{a8} &= \tfrac{1}{4}\mathrm{e}^{\Delta/2}\big(\delta^b{}_a + \tfrac{1}{2}\gamma_a\gamma^b\big)\psi^{\text{sugra}}_b, \\ \psi^{Spin(8)}_{ab} &= -\tfrac{1}{2}\mathrm{e}^{\Delta/2}\gamma^{(7)}\big(\gamma_{[a}\delta_{b]}{}^c - \tfrac{1}{4}\gamma_{ab}\gamma^c\big)\psi^{\text{sugra}}_c, \end{aligned} \tag{6.52}$$

and then into $SU(8)$ as

$$\psi^{\alpha\beta\gamma} = \tfrac{1}{3}\mathrm{i}(\hat{\gamma}^{ii'})^{[\alpha\beta}(\psi^{Spin(8)}_{ii'})^{\gamma]} \in \Gamma(J^-). \tag{6.53}$$

A generalised connection is compatible with the generalised metric (6.49) if $DG = 0$. For such connections, we can define the generalised spinor derivative via the adjoint action of $SU(8)$ given in [65]. Acting on $\chi \in \Gamma(S^-)$ we have

$$\begin{aligned} D_{ii'}\chi &= \partial_{ii'}\chi + \tfrac{1}{4}\Omega_{ii'jj'}\hat{\gamma}^{jj'}\chi - \tfrac{1}{48}\mathrm{i}\Omega_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi, \\ \tilde{D}_{ii'}\chi &= \tilde{\partial}_{ii'}\chi + \tfrac{1}{4}\tilde{\Omega}_{ii'jj'}\hat{\gamma}^{jj'}\chi - \tfrac{1}{48}\mathrm{i}\tilde{\Omega}_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi. \end{aligned} \tag{6.54}$$

where we have used the $SO(8)$ metric δ_{ij} to lower indices. An example of such a generalised connection is the one (5.53) defined by the Levi–Civita connection ∇

$$\begin{aligned} D^{\nabla}_{ii'}\chi &= \begin{cases} \tfrac{1}{2}\mathrm{e}^{\Delta}\left(\partial_a + \tfrac{1}{4}\omega_{a\,bc}\hat{\gamma}^{bc}\right)\chi & \text{if } i = a \text{ and } i' = 8 \\ 0 & \text{if } i = a \text{ and } i' = b \end{cases}, \\ \tilde{D}^{\nabla}_{ii'}\chi &= 0. \end{aligned} \tag{6.55}$$

where $\omega_{a\,bc}$ is the usual spin-connection.

We can construct a torsion-free compatible connection D, by shifting D^{∇} by an additional connection piece Σ, such that its action on $\chi \in \Gamma(S^-)$ is given by

$$\begin{aligned} D_{ii'}\chi &= D^{\nabla}_{ii'}\chi + \tfrac{1}{4}\Sigma_{ii'jj'}\hat{\gamma}^{jj'}\chi - \tfrac{1}{48}\mathrm{i}\Sigma_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi, \\ \tilde{D}_{ii'}\chi &= \tilde{D}^{\nabla}_{ii'}\chi + \tfrac{1}{4}\tilde{\Sigma}_{ii'jj'}\hat{\gamma}^{jj'}\chi - \tfrac{1}{48}\mathrm{i}\tilde{\Sigma}_{ii'k_1\dots k_4}\hat{\gamma}^{k_1\dots k_4}\chi. \end{aligned} \tag{6.56}$$

where, in the conformal split frame,

$$\begin{aligned}
\Sigma_{ii'jj'} &= -\tfrac{1}{3}\mathrm{e}^{\Delta}\delta_{ij}\tilde{K}_{i'j'} + \tfrac{1}{42}\mathrm{e}^{\Delta}\tilde{F}\delta_{ij}\delta_{i'j'} - \mathrm{e}^{\Delta}\delta_{ij}\partial_{i'j'}\Delta + Q_{ii'jj'}, \\
\tilde{\Sigma}_{ii'jj'} &= \tfrac{1}{3}\mathrm{e}^{\Delta}K_{ii'jj'} - \tfrac{1}{6}\mathrm{e}^{\Delta}K_{jj'ii'} + \tilde{Q}_{ii'jj'}, \\
\Sigma_{i_1\dots i_6} &= Q_{i_1\dots i_6}, \\
\tilde{\Sigma}_{i_1\dots i_6} &= \tilde{Q}_{i_1\dots i_6}.
\end{aligned} \tag{6.57}$$

In this expression primed and unprimed indices are antisymmetrised implicitly, $(Q,\tilde{Q})$ are the undetermined components[4], $\tilde{F} = \frac{1}{7!}\epsilon^{a_1\dots a_7}\tilde{F}_{a_1\dots a_7}$ and

$$\begin{aligned}
K_{ii'jj'} &= \begin{cases} (*F)_{abc} & \text{for } (i,i,'j,j') = (a,b,c,8) \\ 0 & \text{otherwise} \end{cases}, \\
\tilde{K}_{ij} &= \begin{cases} \tilde{F} & \text{for } (i,j) = (8,8) \\ 0 & \text{otherwise} \end{cases},
\end{aligned} \tag{6.58}$$

give the embedding of the supergravity fluxes. The connection can be rewritten in $SU(8)$ indices through

$$\begin{aligned}
D^{\alpha\beta} &= \mathrm{i}(\hat{\gamma}^{ij})^{\alpha\beta}\big(D_{ij} + \mathrm{i}\tilde{D}_{ij}\big), \\
\bar{D}_{\alpha\beta} &= -\mathrm{i}(\hat{\gamma}_{ij})_{\alpha\beta}\big(D^{ij} - \mathrm{i}\tilde{D}^{ij}\big).
\end{aligned} \tag{6.59}$$

With these definitions, we can now give the explicit form of the unique operators (6.6) in $SU(8)$ indices

$$\begin{aligned}
(D \times_{J^-} \varepsilon)^{\alpha\beta\gamma} &= D^{[\alpha\beta}\varepsilon^{\gamma]}, \\
(D \times_{S^+} \varepsilon)_{\alpha} &= -\bar{D}_{\alpha\beta}\varepsilon^{\beta}, \\
(D \times_{J^+} \psi)_{\alpha\beta\gamma} &= -\tfrac{1}{12}\,\epsilon_{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3}D^{\delta\delta'}\psi^{\theta_1\theta_2\theta_3}, \\
(D \times_{S^-} \psi)^{\alpha} &= \tfrac{1}{2}\bar{D}_{\beta\gamma}\psi^{\alpha\beta\gamma},
\end{aligned} \tag{6.60}$$

where $\epsilon_{\alpha_1\dots\alpha_8}$ is the totally antisymmetric symbol preserved by $SU(8)$.

[4]From section 5.2.2 and table 5.4, these are sections of the $\mathbf{1280} + \bar{\mathbf{1280}}$ representations of $SU(8)$.

From the first two we can immediately read off the supersymmetry variations of the fermions (6.9)

$$\delta\psi^{\alpha\beta\gamma} = D^{[\alpha\beta}\varepsilon^{\gamma]}, \qquad \delta\bar{\rho}_\alpha = -\bar{D}_{\alpha\beta}\varepsilon^\beta, \tag{6.61}$$

while the variations of the bosons (6.12) can be packaged as

$$\delta G_{AB} = \begin{pmatrix} \delta G_{\alpha\beta\gamma\delta} & \delta G_{\alpha\beta}{}^{\gamma\delta} \\ \delta G^{\alpha\beta}{}_{\gamma\delta} & \delta G^{\alpha\beta\gamma\delta} \end{pmatrix} = \frac{1}{|\mathrm{vol_G}|}\begin{pmatrix} \delta\bar{H}_{\alpha\beta\gamma\delta} & 0 \\ 0 & \delta H^{\alpha\beta\gamma\delta} \end{pmatrix} - G_{AB}\,\delta\log|\mathrm{vol_G}| \tag{6.62}$$

with

$$\begin{aligned} \delta H^{\alpha\beta\gamma\delta} &= -\tfrac{3}{16}\big(\varepsilon^{[\alpha}\psi^{\beta\gamma\delta]} + \tfrac{1}{4!}\epsilon^{\alpha\beta\gamma\delta\alpha'\beta'\gamma'\delta'}\bar{\varepsilon}_{\alpha'}\bar{\psi}_{\beta'\gamma'\delta'}\big) \\ \delta\log|\mathrm{vol_G}| &= \bar{\rho}_\alpha\varepsilon^\alpha + \rho^\alpha\bar{\varepsilon}_\alpha \end{aligned} \tag{6.63}$$

The fermion equations of motion (6.13) are

$$\begin{aligned} -\tfrac{1}{12}\,\epsilon_{\alpha\beta\gamma\delta\delta'\theta_1\theta_2\theta_3}D^{\delta\delta'}\psi^{\theta_1\theta_2\theta_3} + 2\bar{D}_{[\alpha\beta}\bar{\rho}_{\gamma]} &= 0, \\ D^{\alpha\beta}\bar{\rho}_\beta - \tfrac{1}{2}\bar{D}_{\beta\gamma}\psi^{\alpha\beta\gamma} &= 0. \end{aligned} \tag{6.64}$$

As before, the curvatures can be obtained by taking the supersymmetry variations of the fermion equations of motion and after some algebra one obtains the expressions

$$\begin{aligned} R^0_{\alpha\beta\gamma\delta}\varepsilon^\delta &= -2\big(\bar{D}_{[\alpha\beta}\bar{D}_{\gamma\delta]} + \tfrac{1}{4!}\,\epsilon_{\alpha\beta\gamma\delta\epsilon\epsilon'\theta\theta'}D^{\epsilon\epsilon'}D^{\theta\theta'}\big)\varepsilon^\delta - \big[\bar{D}_{[\alpha\beta}, \bar{D}_{\gamma]\delta}\big]\,\varepsilon^\delta, \\ \tfrac{1}{6}R\varepsilon^\alpha &= \tfrac{2}{3}\left(\{D^{\alpha\gamma}, \bar{D}_{\beta\gamma}\} - \tfrac{1}{8}\delta^\alpha{}_\beta\{D^{\gamma\delta}, \bar{D}_{\gamma\delta}\}\right)\varepsilon^\beta \\ &\quad - \tfrac{1}{3}\left([\bar{D}_{\beta\gamma}, D^{\alpha\gamma}] - \tfrac{1}{8}\delta^\alpha{}_\beta[\bar{D}_{\gamma\delta}, D^{\gamma\delta}]\right)\varepsilon^\beta + \tfrac{1}{8}\left[D^{\beta\gamma}, \bar{D}_{\beta\gamma}\right]\varepsilon^\alpha. \end{aligned} \tag{6.65}$$

The vanishing of these then corresponds to the bosonic equations of motion (6.17). As for $d = 4$, we again observe that the curvatures contain terms symmetric in the two connections, in the representations of the bundle N identified in section 5.2.3.

The bosonic action (6.18) takes the form

$$S_{\mathrm{B}} = \frac{1}{2\kappa^2}\int |\mathrm{vol_G}|\,R, \tag{6.66}$$

while the fermion action (6.19) is

$$S_{\mathrm{F}} = \frac{3}{2\kappa^2} \int |\mathrm{vol}_{\mathrm{G}}| \Big(\tfrac{1}{4!} \epsilon_{\alpha_1\alpha_2\alpha_3\beta\beta'\gamma_1\gamma_2\gamma_3} \psi^{\alpha_1\alpha_2\alpha_3} D^{\beta\beta'} \psi^{\gamma_1\gamma_2\gamma_3} \\ + \bar{\rho}_\alpha \bar{D}_{\beta\gamma} \psi^{\alpha\beta\gamma} - \psi^{\alpha\beta\gamma} \bar{D}_{\alpha\beta} \bar{\rho}_\gamma - 2\bar{\rho}_\alpha D^{\alpha\beta} \bar{\rho}_\beta + \mathrm{cc} \Big). \tag{6.67}$$

This completes the rewriting of the seven-dimensional theory with explicit local $SU(8)$ symmetry.

Chapter 7

Conclusion

We have provided a reformulation of type II and eleven-dimensional supergravity, including the fermions to leading order, such that their larger bosonic symmetries are manifest. This was accomplished by writing down the natural analogue of Einstein gravity for generalised geometry. In the type II case we geometrised the NSNS sector in terms of $O(10,10) \times \mathbb{R}^+$ generalised geometry, and showed that the both the RR fields and the fermions embedded directly into representations of the local symmetry group, $Spin(9,1) \times Spin(1,9)$. For eleven-dimensional supergravity we showed how $E_{d(d)} \times \mathbb{R}^+$ generalised geometry encompasses all the bosonic symmetries and that the fermion fields fill out representations of the local $\tilde{H}_d$ group. To summarise, in both cases the supergravity is described by a simple set of equations which are manifestly covariant not just under gauge transformations and diffeomorphisms, but also under the action of the larger local groups. In the abstract language of section 5.2.3 these are

$$\begin{array}{cc} \text{Equations of Motion} & \text{Supersymmetry} \\ \begin{cases} (D \times_J \psi) + (D \times_J \rho) = 0, \\ (D \times_S \psi) + (D \times_S \rho) = 0, \\ R_{AB} = 0, \end{cases} & \begin{cases} \delta\psi = D \times_J \varepsilon, \\ \delta\rho = D \times_S \varepsilon, \\ \delta G = (\psi \times_{\mathrm{ad}\,P^\perp} \varepsilon) + (\rho \times_{\mathrm{ad}\,P^\perp} \varepsilon). \end{cases} \end{array} \tag{7.1}$$

The simplicity of these expressions is all the more remarkable given how naturally they arise – the generalised connection D and generalised metric G are the direct analogues of the Levi-Civita ∇ and metric g of Riemannian geometry. The representations S and J in which the fermionic fields transform can be uniquely identified just by examining the $\tilde{H}_d$ decomposition of torsion-free, metric compatible generalised connections, as was explained in section 5.2.3. We believe that this provides compelling evidence that generalised geometry is a natural framework with which to formulate supergravity.

A surprising outcome of our work is the observation that, despite the fact that the geometric construction is entirely bosonic, supersymmetry is deeply integrated in the formalism – torsion-free, metric compatible connections describe the variation of the fermions and the equations of motion of the fermions close under supersymmetry on the bosonic generalised curvatures. One problem that generalised geometry is thus particularly well suited to tackle is that of describing supersymmetric vacua with flux [104, 103, 112, 113]. In this context, our formalism can in a sense be viewed as an expansion of the ideas of generalised complex structures, as it provides a way of unifying all the possible structures that describe supersymmetric backgrounds in a single generalised G-structure (from section 5.1.6), where $G \subset \tilde{H}_d$, at each level of preserved supersymmetry. Furthermore, it turns out that the Killing spinor equations can be shown to be equivalent to integrability conditions on the generalised connection – supersymmetric backgrounds are in one-to-one correspondence with manifolds with generalised special holonomy.

Let us see how this works in a particularly important case.

7.1 $d = 7$ backgrounds with $N = 1$ supersymmetry

Recall that a background of M theory is said to be supersymmetric if there exists a nowhere-vanishing choice of supersymmetry parameter on the manifold, the Killing spinor, such that the supersymmetry variations of all of the background fields vanish. For classical supergravity solutions the background fermionic fields are zero. The variations of the bosonic fields always have a fermionic factor, so these are automatically zero. Therefore, the non-trivial condition for supersymmetry is the vanishing of the variations of the fermionic fields, and we need only consider the lowest order terms in fermions.

Let us then consider the $d = 7$ case, using the same field ansatz from section 2.2.2. If one has an $N = 1$ vacuum, then there exists a spinor field ϵ globally defined on the background manifold M which satisfies the Killing spinor equations

$$\begin{aligned} 0 &= \gamma^m \nabla_m \epsilon + \gamma^m (\partial_m \Delta)\epsilon - \tfrac{1}{96}\gamma^{m_1\dots m_4} F_{m_1\dots m_4}\epsilon \\ &\quad - \tfrac{1}{4}\tfrac{1}{7!}\gamma^{m_1\dots m_7}\tilde{F}_{m_1\dots m_7}\epsilon, \\ 0 &= \nabla_m \epsilon + \tfrac{1}{288}(\gamma_m{}^{n_1\dots n_4} - 8\delta_m{}^{n_1}\gamma^{n_2 n_3 n_4})F_{n_1\dots n_4}\epsilon \\ &\quad - \tfrac{1}{12}\tfrac{1}{6!}\tilde{F}_{m n_1\dots n_6}\gamma^{n_1\dots n_6}\epsilon, \end{aligned} \tag{7.2}$$

The Killing spinor defines a G-structure on the tangent space of M, which for non-vanishing fluxes will have some intrinsic torsion (see appendix B for a quick review on the notion of intrinsic torsion). Thus, in ordinary geometry one finds that generic supersymmetric flux backgrounds are not integrable. However, from the point of view of generalised geometry, the larger symmetries of the generalised tangent space allow us to use the spinor to define generalised G-structures, which, as we will now see, are integrable precisely if and only if the Killing spinor equations hold.

We start by encoding the equations (7.2) in the $\tilde{H}_d$-covariant form

$$D \times_J \epsilon = 0, \qquad D \times_S \epsilon = 0. \tag{7.3}$$

Since ϵ is a globally non-vanishing section of S, its components are stabilised by transition functions in some subgroup G of $\tilde{H}_d$. In other words, those $\tilde{H}_d$ frames in which the components of ϵ are fixed define a generalised G-structure P. The condition for a generalised connection to be compatible with this structure is then $D\epsilon = 0$.

Now, the equations (7.3), which hold for any torsion-free $\tilde{H}_d$ compatible D, appear weaker than the compatibility condition, as they constrain only two of the irreducible parts of $D\epsilon$. If the generalised G-structure has vanishing generalised intrinsic torsion we need to show that one can still find a torsion-free compatible connection $\hat{D}$ such that the full compatibility condition $\hat{D}\epsilon = 0$ is satisfied[1]. We will prove this by calculating the repre-

[1]Note that this is a different definition of generalised holonomy to the one in [110, 114], as we are considering the full generalised connection D as opposed to just the projected derivatives which appear in the

sentation in which the intrinsic torsion transforms and demonstrating that the Killing spinor equations precisely annihilate this representation. Note that will we study only the linear algebra involved at a single point in the manifold so that we may discuss representations rather than bundles and so, for the sake of readability, we will use a slight abuse of notation and not distinguish between the two.

The argument goes as follows. Denote the vector space of $\tilde{H}_d$ connection pieces at the point by C and the space of G-compatible connections (where $G \subset \tilde{H}_d$) by C_P. These are (reducible) representations of $\tilde{H}_d$ and G respectively and split into two (also in general reducible) representations

$$\begin{aligned} C &= E^* \otimes \text{ad}\,\tilde{H}_d = K \oplus U, \\ C_P &= E^* \otimes \text{ad}\,G = K_P \oplus U_P, \end{aligned} \tag{7.4}$$

where K and K_P are the components constrained by the torsion and U and U_P are unconstrained by the torsion. Clearly $C_P \subset C$, $K_P \subset K$ and $U_P \subset U$.

Decomposing the $\tilde{H}_d$ representations under G, we have

$$K = K_P \oplus K_{\text{int}}, \tag{7.5}$$

where $K_{\text{int}} \simeq K/K_P$ is the (reducible) representation of G under which the intrinsic torsion transforms. This can easily be found as we know K and $K_P = C_P \cap K$.

The Killing spinor equations transform under the representation $S \oplus J$ of $\tilde{H}_d$ and the projection which give rise to them from the generalised connection becomes

$$\begin{aligned} \mathcal{P} : C &\to S \oplus J, \\ \Sigma &\mapsto (\Sigma \times_S \epsilon) + (\Sigma \times_J \epsilon). \end{aligned} \tag{7.6}$$

Recall that this projection is such that the Killing spinor equations are uniquely determined for torsion-free connections, i.e. components of the connection lying in U do not contribute to them. Now, G is the stabiliser of ϵ so for $\Sigma \in C_P$ we have $\Sigma \cdot \epsilon = 0$. We have the

fermionic supersymmetry variations.

decomposition $C = K_P \oplus K_{\text{int}} \oplus U$ and the projection depends only on torsion components of Σ. Suppose the kernel of the mapping $\mathcal{P}$ were equal to $K_P \oplus U$, as the above observations suggest, then we would have an isomorphism

$$\mathcal{P}|_{K_{\text{int}}} : K_{\text{int}} \to S \oplus J, \tag{7.7}$$

and this would demonstrate that the vanishing of the intrinsic torsion was equivalent to the Killing spinor equations. We assert that this is the case for now, and one can check this by explicit calculation of the dependences.

In the $d = 7$ case for $N = 1$ backgrounds, the internal complex spinor ϵ is a section of the generalised spinor bundle S. The fibre of this bundle is the representation $\mathbf{8} + \bar{\mathbf{8}}$ of $SU(8)$ where the two parts are related by complex conjugation, see table 5.5. Therefore ϵ is stabilised by $SU(7) \subset SU(8)$ and so defines an $SU(7)$ structure. This statement unifies all of the different subgroups of $Spin(7)$ which can stabilise both the real and imaginary parts of ϵ [65].

We can then explicitly compute the generalised intrinsic torsion of this $SU(7)$ structure. We have the $SU(8)$ representations

$$\begin{aligned} S \oplus J &= \left[\mathbf{8} + \bar{\mathbf{8}}\right] + \left[\mathbf{56} + \bar{\mathbf{56}}\right], \\ K &= \left[\mathbf{28} + \bar{\mathbf{28}}\right] + \left[\mathbf{36} + \bar{\mathbf{36}}\right] + \left[\mathbf{420} + \bar{\mathbf{420}}\right]. \\ U &= \left[\mathbf{1280} + \bar{\mathbf{1280}}\right]. \end{aligned} \tag{7.8}$$

The next step is to calculate their $SU(7)$ decompositions

$$\begin{aligned} S \oplus J &= 2 \times \mathbf{1} + \left[\mathbf{7} + \bar{\mathbf{7}}\right] + \left[\mathbf{21} + \bar{\mathbf{21}}\right] + \left[\mathbf{35} + \bar{\mathbf{35}}\right], \\ K &= 2 \times \left(\mathbf{1} + \left[\mathbf{7} + \bar{\mathbf{7}}\right] + \left[\mathbf{21} + \bar{\mathbf{21}}\right]\right) \\ &\quad + \left[\mathbf{28} + \bar{\mathbf{28}}\right] + \left[\mathbf{35} + \bar{\mathbf{35}}\right] + \left[\mathbf{140} + \bar{\mathbf{140}}\right] + \left[\mathbf{224} + \bar{\mathbf{224}}\right]. \\ U &= \left[\mathbf{7} + \bar{\mathbf{7}}\right] + \left[\mathbf{21} + \bar{\mathbf{21}}\right] + \left[\mathbf{28} + \bar{\mathbf{28}}\right] + \left[\mathbf{112} + \bar{\mathbf{112}}\right] + \left[\bar{\mathbf{140}} + \mathbf{140}\right] \\ &\quad + \left[\mathbf{189} + \bar{\mathbf{189}}\right] + \left[\mathbf{735} + \bar{\mathbf{735}}\right] + 2 \times \mathbf{48}. \end{aligned} \tag{7.9}$$

We have that generalised connections compatible with the $SU(7)$ structure fill out the vector space

$$\begin{aligned} C_P &= E \otimes \mathrm{ad}(SU(7)) \\ &= \big[\mathbf{7} + \bar{\mathbf{7}}\big] + \big[\mathbf{21} + \bar{\mathbf{21}}\big] + \big[\mathbf{28} + \bar{\mathbf{28}}\big] + \big[\mathbf{140} + \bar{\mathbf{140}}\big] + \big[\mathbf{224} + \bar{\mathbf{224}}\big] \\ &\quad + \big[\mathbf{189} + \bar{\mathbf{189}}\big] + \big[\mathbf{735} + \bar{\mathbf{735}}\big]. \end{aligned} \tag{7.10}$$

Now we must find the intersection $K_P = C_P \cap K$. Clearly the last two terms will not contribute, and by checking against the decomposition of U we immediately see that the $\mathbf{224} + \bar{\mathbf{224}}$ must be in the intersection. The remaining components require a more careful analysis, but it is not too difficult to check explicitly that none of them vanish under the torsion map $C_P \to K_P$. Therefore, we have that

$$K_P = \big[\mathbf{7} + \bar{\mathbf{7}}\big] + \big[\mathbf{21} + \bar{\mathbf{21}}\big] + \big[\mathbf{28} + \bar{\mathbf{28}}\big] + \big[\mathbf{140} + \bar{\mathbf{140}}\big] + \big[\mathbf{224} + \bar{\mathbf{224}}\big], \tag{7.11}$$

and so conclude

$$\begin{aligned} K_{\text{int}} &= 2 \times \mathbf{1} + \big[\mathbf{7} + \bar{\mathbf{7}}\big] + \big[\mathbf{21} + \bar{\mathbf{21}}\big] + \big[\mathbf{35} + \bar{\mathbf{35}}\big] \\ &= S \oplus J. \end{aligned} \tag{7.12}$$

Shur's lemma then guarantees that the restricted map (7.7) is thus either zero or an isomorphism on each of the irreducible parts of $K_{\text{int}} = S \oplus J$. We assert that it is in fact an isomorphism, which proves that the Killing spinor equations have precisely the necessary degrees of freedom to set the intrinsic torsion of the $SU(7)$ structure to zero.

It is worth mentioning that the solution corresponding to an NS5-brane wrapped on a Kahler 2-cycle in a Calabai-Yau manifold [115] is included in this framework, even though it falls outside the classification of [116] (as this is an $N = 1$ background with vanishing RR fields). Reducing from M theory to type II, we have that the decomposition of the $SU(8)$ Killing spinor under $SU(4) \times SU(4)$ is $\epsilon = \epsilon_1 + \epsilon_2 \in (\mathbf{4} + \bar{\mathbf{4}}, \mathbf{1}) + (\mathbf{1}, \mathbf{4} + \bar{\mathbf{4}})$. For this solution, one sets the second spinor ϵ_2 to zero, so the pure spinors of [116] vanish identically. However, in our formalism, which does not involve the tensor product of the spinors, this supersymmetry parameter still gives a non-vanishing section of $S \sim (\mathbf{4} + \bar{\mathbf{4}}, \mathbf{1}) + (\mathbf{1}, \mathbf{4} + \bar{\mathbf{4}})$, describing an $SU(3) \times SU(4) \subset SU(7)$ structure.

7.2 Future Outlook

We have thus shown that $N = 1$ backgrounds in $d = 7$ are described by manifolds with generalised $SU(7)$ special holonomy. Though we will not elaborate on it further here, it can actually be shown that similar results hold for higher amounts of preserved supersymmetry N in any dimension, and in the $O(d, d)$ formalism as well if the RR fields vanish. An obvious next step is then to apply these results in order to find new background solutions for both type II and M theory. We note that if this proves successful, then the study of the moduli space and the phenomenological implications of these new background classes should benefit from the fact that our formalism is able to unify all the bosonic fields in the internal manifold while keeping the larger symmetries manifest. Additionally, it should not be too difficult to expand the formalism to, for example, describe AdS reductions as opposed to just the Minkowski ones we considered here. This could be extremely rewarding as it would then have a direct application to the gravity side of the AdS/CFT correspondence.

On the subject of string compactifications, a somewhat related problem is the study of non-geometric backgrounds. There have been a few recent developments in this field using generalised geometry and Double Field Theory (for example [30, 117, 118, 119]). The formalism we developed here allows us to clarify some of these results and their scope of applicability. For instance, the action given in [117] can be interpreted in the context of the $O(9, 1) \times O(1, 9)$ generalised gravity of chapter 4. Given a bivector $\beta \in \Gamma(\Lambda^2 TM)$, it amounts to evaluating the generalised Ricci scalar R in a different frame from the standard split frame, namely

$$\hat{E}_A = \begin{cases} \hat{E}_a = \mathrm{e}^{-2\phi}(\det e)\hat{e}_a & \text{for } A = a \\ E^a = \mathrm{e}^{-2\phi}(\det e)\,(e^a + \beta\hat{e}^a) & \text{for } A = a + d \end{cases} . \tag{7.13}$$

which is related to the conformal split frame by an $O(10, 10)$ transformation outside the geometric subgroup G_{split}. Locally, from a generalised geometrical perspective, these are equivalent, as expected from [30]. However, given the patching (3.2), the new frame is not, generically, globally defined in a conventional generalised geometry. The suggestion though is that on a non-geometrical background (patched for instance by a T-duality) it may be possible to make some global notion of such a frame. So it seems clear that while generalised geometry can provide some new insights into non-geometric backgrounds, one will need to move beyond it in order to obtain the full picture.

On the other hand, the formalism we developed here might already be directly applicable to the ongoing problem in string perturbation theory of finding corrections to the effective action which are higher order in derivatives. Not only should the calculations benefit from the more manifest symmetries, but the fact that defining curvature tensors in generalised geometry is a very a non-trivial question, as we saw in section 3.1.5, could be very significant. The simple requirement that the higher order Lagrangians should be built out of generalised connections provides one with extra constraints which might be key in finding the corrections.

The full extent of the relation between generalised geometries and supersymmetry is also something that clearly warrants further exploration. One might try to formulate other supergravities, such as six-dimensional $N = (1, 0)$, which should provide further evidence of this connection. These will likely require the use of other "duality groups" for the generalised structure – for instance a formalism based on $B_{d(d)} = SO(d+1, d)$ [102] seems to describe heterotic coupled to a $U(1)$ gauge field [120]. Alternatively, one could attempt to extend the geometry to make supersymmetry manifest, perhaps by adapting some of the tools from superspace formalism.

A potentially harder problem can be posed if one tries to extend the $E_{d(d)} \times \mathbb{R}^+$ structure to higher dimensions. Here one has to deal with $E_{8(8)}$ and its infinite dimensional Kac-Moody extensions, as is the case in West's E_{11} [66] programme. The basic obstruction, even for $d = 8$, is that although the generalised tangent space exists one cannot write a covariant expression for the Dorfman derivative in the form (5.28). This is a consequence of the presence of tensors in the generalised tangent space which are not forms, and as such one cannot act upon them with the de Rham differential. Indeed, in $d = 7$ one already has $\tau \in \Gamma(T^*M \otimes \Lambda^7 T^*M)$ in E, but this gets projected to zero in the Dorfman derivative – we have no such luck in higher dimensions. Furthermore, this term already meant that in $d = 7$ we could not write the symmetric component of the Dorfman derivative (5.39) in an $E_{d(d)} \times \mathbb{R}^+$ covariant form. Physically extra terms of this sort in the generalised tangent space correspond to Kaluza–Klein monopole charges in the U-duality algebra and should be associated to the symmetries of "dual gravitons". Ultimately, what is needed is a derived bracket construction for the $E_{d(d)}$ case, with an appropriate generalisation of the de Rahm differential that can act on these higher spin fields. It seems likely that creating a complete theory that describes these large exceptional geometries would be a tremendous step forward in our understanding of string and M theory.

Appendix A

Conventions

A.1 Lorentzian signature

We take the g metric with the mostly plus signature $(- + + \cdots +)$. We use the indices $\mu, \nu, \lambda \ldots$ as the spacetime coordinate indices and $a, b, c \ldots$ for the tangent space indices. We take symmetrisation of indices with weight one. Give forms $\lambda \in \Lambda^k T^* M$, in D dimensions, our conventions are

$$\begin{aligned} \lambda &= \tfrac{1}{k!}\lambda_{\mu_1\ldots\mu_k}\mathrm{d}x^{\mu_1} \wedge \ldots \lambda \mathrm{d}x^{\mu_k}, \\ \lambda \wedge \lambda' &= \tfrac{1}{(k+k')!}\left(\tfrac{(k+k')!}{k!\,k!}\lambda_{[\mu_1\ldots\mu_k}\lambda_{\mu_{k+1}\ldots\mu_{k+k'}]}\right)\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{k+k'}}, \\ *\lambda &= \tfrac{1}{(10-k)!}\left(\tfrac{1}{k!}\sqrt{-g}\epsilon_{\mu_1\ldots\mu_{D-k}\nu_1\ldots\nu_k}\lambda^{\nu_1\ldots\nu_k}\right)\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_{D-k}}, \end{aligned} \tag{A.1}$$

where $\epsilon_{01\ldots D-1} = -\epsilon^{01\ldots D-1} = +1$.

A.2 Euclidean signature

We use the indices $m, n, p, \ldots$ as the coordinate indices and $a, b, c \ldots$ for the tangent space indices. We take symmetrisation of indices with weight one. Given a polyvector $w \in$

$\Lambda^p TM$ and a form $\lambda \in \Lambda^q T^*M$, we write in components

$$\begin{aligned} w &= \frac{1}{p!} w^{m_1 \dots m_p} \frac{\partial}{\partial x^{m_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{m_p}}, \\ \lambda &= \frac{1}{q!} \lambda_{m_1 \dots m_q} \mathrm{d}x^{m_1} \wedge \dots \wedge \mathrm{d}x^{m_q}, \end{aligned} \tag{A.2}$$

so that wedge products and contractions are given by

$$\begin{aligned} (w \wedge w')^{m_1 \dots m_{p+p'}} &= \frac{(p+p')!}{p!p'!} w^{[m_1 \dots m_p} w'^{m_{p+1} \dots m_{p+p'}]}, \\ (\lambda \wedge \lambda')_{m_1 \dots m_{q+q'}} &= \frac{(q+q')!}{q!q'!} \lambda_{[m_1 \dots m_q} \lambda'_{m_{p+1} \dots m_{q+q'}]}, \\ (w \lrcorner \lambda)_{a_1 \dots a_{q-p}} &:= \frac{1}{p!} w^{c_1 \dots c_p} \lambda_{c_1 \dots c_p a_1 \dots a_{q-p}} \qquad \text{if } p \leq q, \\ (w \lrcorner \lambda)^{a_1 \dots a_{p-q}} &:= \frac{1}{q!} w^{a_1 \dots a_{p-q} c_1 \dots c_q} \lambda_{c_1 \dots c_q} \qquad \text{if } p \geq q. \end{aligned} \tag{A.3}$$

Given the tensors $t \in TM \otimes \Lambda^7 TM$, $\tau \in T^*M \otimes \Lambda^7 T^*M$ and $a \in TM \otimes T^*M$ with components

$$\begin{aligned} t &= \frac{1}{7!} w^{m, m_1 \dots m_7} \frac{\partial}{\partial x^m} \otimes \frac{\partial}{\partial x^{m_1}} \wedge \dots \wedge \frac{\partial}{\partial x^{m_7}}, \\ \tau &= \frac{1}{7!} \tau_{m, m_1 \dots m_7} \mathrm{d}x^m \otimes \mathrm{d}x^{m_1} \wedge \dots \wedge \mathrm{d}x^{m_q}, \\ a &= a^m{}_n \frac{\partial}{\partial x^m} \otimes \mathrm{d}x^n, \end{aligned} \tag{A.4}$$

and also a form $\sigma \in \Lambda^5 T^*M$ and a vector $v \in TM$, we use the "j-notation" from [65], defining

$$
\begin{aligned}
(w \lrcorner \tau)_{a_1\ldots a_{8-p}} &:= \frac{1}{(p-1)!} w^{c_1\ldots c_p} \tau_{c_1,c_2\ldots c_p a_1\ldots a_{8-p}}, \\
(t \lrcorner \lambda)^{a_1\ldots a_{8-q}} &:= \frac{1}{(q-1)!} t^{c_1,c_2\ldots c_q a_1\ldots a_{8-q}} \lambda_{c_1\ldots c_q}, \\
(t \lrcorner \tau) &:= \frac{1}{7!} t^{a,b_1\ldots b_7} \tau_{a,b_1\ldots b_7}, \\
(jw \wedge w')^{a,a_1\ldots a_7} &:= \frac{7!}{(p-1)!(8-p)!} w^{a[a_1\ldots a_{p-1}} w'^{a_p\ldots a_7]}, \\
(j\lambda \wedge \lambda')_{a,a_1\ldots a_7} &:= \frac{7!}{(q-1)!(8-q)!} \lambda_{a[a_1\ldots a_{q-1}} \lambda'_{a_q\ldots a_7]}, \\
(jw \lrcorner j\lambda)^a{}_b &:= \frac{1}{(p-1)!} w^{ac_1\ldots c_{p-1}} \lambda_{bc_1\ldots c_{p-1}}, \\
(jt \lrcorner j\tau)^a{}_b &:= \frac{1}{7!} t^{a,c_1\ldots c_7} \tau_{b,c_1\ldots c_7}, \\
\left(j^{p+1}\lambda \wedge \tau\right)_{a_1\ldots a_{p+1},b_1\ldots b_7} &:= (p+1)\lambda_{[a_1\ldots} \tau_{a_{p+1}],b_1\ldots b_7}, \\
\left(j^3\sigma \wedge \sigma'\right)_{a_1\ldots a_3,b_1\ldots b_7} &:= \tfrac{7!}{5!\cdot 2!} \sigma_{a_1\ldots a_3[b_1 b_2} \sigma'_{\ldots b_7]}, \\
(v \lrcorner j\tau)_{mn_1\ldots n_6} &:= v^p \tau_{m,pn_1\ldots n_6}.
\end{aligned}
\tag{A.5}
$$

The d dimensional metric g is always positive definite. We define the orientation, $\epsilon_{1\ldots d} = \epsilon^{1\ldots d} = +1$, and use the conventions

$$
\begin{aligned}
*\lambda_{m_1\ldots m_{d-q}} &= \tfrac{1}{q!}\sqrt{|g|}\epsilon_{m_1\ldots m_{d-k} n_1\ldots n_q} \lambda^{n_1\ldots n_q}, \\
\lambda^2 &= \lambda_{m_1\ldots m_q} \lambda^{m_1\ldots m_q}.
\end{aligned}
\tag{A.6}
$$

Appendix B

Metric structures, torsion and the Levi–Civita connection

In this appendix we briefly review the basic geometry that goes into the construction of the Levi–Civita connection, as context for the corresponding generalised geometrical analogues.

Let M be a d-dimensional manifold. We write $\{\hat{e}_a\}$ for a basis of the tangent space T_xM at $x \in M$ and $\{e^a\}$ be the dual basis of T^*_xM satisfying $i_{\hat{e}_a} e^b = \delta_a{}^b$. Recall that the frame bundle F is the bundle of all bases $\{\hat{e}^a\}$ over M,

$$F = \{(x, \{\hat{e}_a\}) : \; x \in M \text{ and } \{\hat{e}_a\} \text{ is a basis for } T_xM\} . \tag{B.1}$$

On each fibre of F there is an action of $A^a{}_b \in GL(d, \mathbb{R})$, given $v = v^a \hat{e}_a \in \Gamma(T_xM)$,

$$v^a \mapsto v'^a = A^a{}_b v^b, \qquad \hat{e}_a \mapsto \hat{e}'_a = \hat{e}_b (A^{-1})^b{}_a. \tag{B.2}$$

giving F the structure of a $GL(d, \mathbb{R})$ principal bundle.

The Lie derivative $\mathcal{L}_v$ encodes the effect of an infinitesimal diffeomorphism. On a vector field w it is equal to the Lie bracket

$$\mathcal{L}_v w = -\mathcal{L}_w v = [v, w] , \tag{B.3}$$

while on a general tensor field α one has, in coordinate indices,

$$\begin{aligned}\mathcal{L}_v \alpha^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} = v^\mu \partial_\mu \alpha^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} \\ + (\partial_\mu v^{\mu_1})\, \alpha^{\mu\mu_2\ldots\mu_p}_{\nu_1\ldots\nu_q} + \cdots + (\partial_\mu v^{\mu_p})\, \alpha^{\mu_1\ldots\mu_{p-1}\mu}_{\nu_1\ldots\nu_q} \\ - (\partial_{\nu_1} v^\mu)\, \alpha^{\mu_1\ldots\mu_p}_{\mu\nu_2\ldots\nu_q} - \cdots - \left(\partial_{\nu_q} v^\mu\right) \alpha^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_{q-1}\mu}.\end{aligned} \tag{B.4}$$

Note that the terms on the second and third lines can be viewed as the adjoint action of the $\mathfrak{gl}(d,\mathbb{R})$ matrix $a^\mu{}_\nu = \partial_\nu v^\mu$ on the particular tensor field α. This form will have an analogous expression when we come to generalised geometry.

Let $\nabla_\mu v^\nu = \partial_\mu v^\nu + \omega_\mu{}^\nu{}_\lambda v^\lambda$ be a general connection on TM. The torsion $T \in \Gamma(TM \otimes \Lambda^2 T^*M)$ of ∇ is defined by

$$T(v,w) = \nabla_v w - \nabla_w v - [v,w]\,. \tag{B.5}$$

or concretely, in coordinate indices,

$$T^\mu{}_{\nu\lambda} = \omega_\nu{}^\mu{}_\lambda - \omega_\lambda{}^\mu{}_\nu, \tag{B.6}$$

while, in a general basis where $\nabla_\mu v^a = \partial_\mu v^a + \omega_\mu{}^a{}_b v^b$, one has

$$T^a{}_{bc} = \omega_b{}^a{}_c - \omega_c{}^a{}_b + [\hat{e}_b, \hat{e}_c]^a\,. \tag{B.7}$$

Since again it has a natural generalised geometric analogue, it is useful to equivalently define the torsion in terms of the Lie derivative. If $\mathcal{L}^\nabla_v \alpha$ is the analogue of the Lie derivative (B.4) but with ∂ replaced by ∇, and $(i_v T)^\mu{}_\nu = v^\lambda T^\mu{}_{\lambda\nu}$ then

$$(i_v T)\alpha = \mathcal{L}^\nabla_v \alpha - \mathcal{L}_v \alpha, \tag{B.8}$$

where we view $i_v T$ as a section of the $\mathfrak{gl}(d,\mathbb{R})$ adjoint bundle acting on the given tensor field α.

The curvature of a connection ∇ is given by the Riemann tensor $\mathcal{R} \in \Gamma(\Lambda^2 T^*M \otimes$

$TM \otimes T^*M)$, defined by

$$
\begin{aligned}
\mathcal{R}(u,v)w &= [\nabla_u, \nabla_v]w - \nabla_{[u,v]}w, \\
\mathcal{R}_{\mu\nu}{}^{\lambda}{}_{\rho}v^{\rho} &= [\nabla_\mu, \nabla_\nu]v^\lambda - T^{\rho}{}_{\mu\nu}\nabla_\rho v^\lambda.
\end{aligned}
\tag{B.9}
$$

The Ricci tensor is the trace of the Riemann curvature

$$
\mathcal{R}_{\mu\nu} = \mathcal{R}_{\lambda\mu}{}^{\lambda}{}_{\nu}. \tag{B.10}
$$

If the manifold admits a metric g then the Ricci scalar is defined by

$$
\mathcal{R} = g^{\mu\nu}\mathcal{R}_{\mu\nu}. \tag{B.11}
$$

A G-structure is a principal sub-bundle $P \subset F$ with fibre G. In the case of the metric g, the $G = O(d)$ sub-bundle is formed by the set of orthonormal bases

$$
P = \{(x, \{\hat{e}_a\}) \in F : g(\hat{e}_a, \hat{e}_b) = \delta_{ab}\}\,, \tag{B.12}
$$

related by an $O(d) \subset GL(d,\mathbb{R})$ action. (A Lorentzian metric defines a $O(d-1,1)$-structure and δ_{ab} is replaced by η_{ab}.) At each point $x \in M$, the metric defines a point in the coset space

$$
g|_x \in GL(d,\mathbb{R})/O(d). \tag{B.13}
$$

In general the existence of a G-structure can impose topological conditions on the manifold, since it implies that the tangent space can be patched using only $G \subset GL(d,\mathbb{R})$ transition functions. (For example, for even d, if $G = GL(d/2,\mathbb{C})$, the manifold must admit an almost complex structure, while for $G = SL(d,\mathbb{R})$ it must be orientable.) However, for $O(d)$ there is no such restriction.

A connection ∇ is compatible with a G-structure $P \subset F$ if the corresponding connection of the principal bundle F reduces to a connection on P. This means that, given a basis $\{\hat{e}_a\}$, one has a set of connection one-forms $\omega^a{}_b$ taking values in the adjoint representation of G given by

$$
\nabla_{\partial/\partial x^\mu}\hat{e}_a = \omega_\mu{}^b{}_a\hat{e}_b. \tag{B.14}
$$

For a metric structure this is equivalent to the condition $\nabla g = 0$. A compatible connection can then be locally represented as a section

$$\omega \in \Gamma(T^*M \otimes \mathrm{ad}(P)). \tag{B.15}$$

The torsion of ∇ is a section of the bundle $TM \otimes \Lambda^2 T^*M$, and in general both of these bundles can be decomposed into irreducible parts under G.

The intrinsic torsion of P can then be defined as follows. Consider two such connections ∇ and ∇', both compatible with the structure P, and let $T(\nabla)$ and $T(\nabla')$ be their respective torsions. The difference of these $\Delta T = T(\nabla') - T(\nabla)$ is a section of $W := TM \otimes \Lambda^2 T^*M$. However, it can happen that, varying ∇' for fixed ∇, ΔT fills out only a subspace of the full space of torsions. Let $\Sigma \in \Gamma(T^*M \otimes \mathrm{ad}(P))$ be the difference of the two connections, which is a tensor such that for $v \in \Gamma(TM)$

$$\Sigma_v = \nabla'_v - \nabla_v. \tag{B.16}$$

ΔT is a linear function of Σ. Therefore, if the dimension of $T^*M \otimes \mathrm{ad}(P)$ is less than the dimension of $TM \otimes \Lambda^2 T^*M$, it is clear that ΔT must be restricted to a subspace. Label the image of the torsion map on $T^*M \otimes \mathrm{ad}(P)$ as W_P, then we can define the bundle

$$K_{\text{int}} = \frac{W}{W_P}. \tag{B.17}$$

Now, given any compatible connection ∇ on P, its torsion defines an element of W_I, which is independent of which connection one chooses. This element of W_I is the intrinsic torsion of P, and if it is non-zero, then there does not exist a torsion-free connection which is compatible with P. G-structures with vanishing intrinsic torsion are said to be torsion-free or equivalently integrable (to first order).

In general, the vanishing of the intrinsic torsion is a first-order differential constraint on the structure. Suppose the structure is defined by a G-invariant tensor Φ, and let $\nabla' = \nabla + \Sigma$, where this time ∇ is torsion-free and ∇' is assumed to be torsion-free and compatible. This implies that

$$0 = \nabla'\Phi = \nabla\Phi + \Sigma \cdot \Phi. \tag{B.18}$$

We must therefore be able to solve the equation $\nabla\Phi = -\Sigma \cdot \Phi$ for Σ, subject to the constraint that $T(\Sigma) = 0$, and in general this constrains which irreducible parts of $\nabla\Phi$ can be non-zero. Thus we have first-order differential constraints on the invariant tensor Φ which defines the structure.

Appendix C

Clifford algebras

C.1 Clifford algebras, involutions and $\tilde{H}_d$

The real Clifford algebras $\mathrm{Cliff}(p,q;\mathbb{R})$ are generated by gamma matrices satisfying

$$\{\gamma^m, \gamma^n\} = 2g^{mn}, \qquad \gamma^{m_1\dots m_k} = \gamma^{[m_1}\dots\gamma^{m_k]}, \tag{C.1}$$

where g is a d-dimensional metric of signature (p,q). Here we will be primarily interested in $\mathrm{Cliff}(d;\mathbb{R}) = \mathrm{Cliff}(d,0;\mathbb{R})$ and $\mathrm{Cliff}(d-1,1;\mathbb{R})$. The top gamma matrix is defined as

$$\gamma^{(d)} = \tfrac{1}{d!}\epsilon_{m_1\dots m_d}\gamma^{m_1\dots m_d} = \begin{cases}\gamma^0\gamma^1\dots\gamma^{d-1} & \text{for } \mathrm{Cliff}(d-1,1;\mathbb{R})\\ \gamma^1\dots\gamma^d & \text{for } \mathrm{Cliff}(d;\mathbb{R})\end{cases}, \tag{C.2}$$

and one has $\left[\gamma^{(d)}, \gamma^m\right] = 0$ if d is odd, while $\{\gamma^{(d)}, \gamma^m\} = 0$ if d is even, and

$$(\gamma^{(d)})^2 = \begin{cases}1 & \text{if } p-q = 0,1 \pmod 4\\ -1 & \text{if } p-q = 2,3 \pmod 4\end{cases} \tag{C.3}$$

We also use Dirac slash notation with weight one so that for $\omega \in \Gamma(\Lambda^k T^*M)$

$$\not{\omega} = \tfrac{1}{k!}\omega_{m_1\dots m_k}\gamma^{m_1\dots m_k}. \tag{C.4}$$

The real Clifford algebras are isomorphic to matrix algebras over $\mathbb{R}$, $\mathbb{C}$ or the quaternions $\mathbb{H}$. These are listed in table C.1. Note that in odd dimensions the pair $\{1, \gamma^{(d)}\}$ generate the centre of the algebra, which is isomorphic to $\mathbb{R} \oplus \mathbb{R}$ if $p - q = 1 \pmod 4$ and $\mathbb{C}$ if $p - q = 3 \pmod 4$. In the first case $\mathrm{Cliff}(p, q; \mathbb{R})$ splits into two pieces with $\gamma^{(d)}$ eigenvalues of ± 1. In the second case $\gamma^{(d)}$ plays the role of i under the isomorphism with $GL(2^{[d/2]}, \mathbb{C})$.

$p - q \pmod 8$	$\mathrm{Cliff}(p, q; \mathbb{R})$
0, 2	$GL(2^{d/2}, \mathbb{R})$
1	$GL(2^{[d/2]}, \mathbb{R}) \oplus GL(2^{[d/2]}, \mathbb{R})$
3, 7	$GL(2^{[d/2]}, \mathbb{C})$
4, 6	$GL(2^{d/2-1}, \mathbb{H})$
5	$GL(2^{[d/2]-1}, \mathbb{H}) \oplus GL(2^{[d/2]-1}, \mathbb{H})$

Table C.1: Real Clifford algebras

There are three involutions of the algebra given by

$$\begin{aligned} \gamma^{m_1 \dots m_k} &\mapsto (-)^k \gamma^{m_1 \dots m_k}, \\ \gamma^{m_1 \dots m_k} &\mapsto \gamma^{m_k \dots m_1}, \\ \gamma^{m_1 \dots m_k} &\mapsto (-)^k \gamma^{m_k \dots m_1}, \end{aligned} \tag{C.5}$$

usually called "reflection", "reversal" and "Clifford conjugation". The first is an automorphism of the algebra, the other two are anti-automorphisms. The reflection involution gives a grading of $\mathrm{Cliff}(p, q; \mathbb{R}) = \mathrm{Cliff}^+(p, q; \mathbb{R}) \oplus \mathrm{Cliff}^-(p, q; \mathbb{R})$ into odd and even powers of γ^m. The group $Spin(p, q)$ lies in $\mathrm{Cliff}^+(p, q; \mathbb{R})$.

The involutions can be used to define other subgroups of the Clifford algebra. In particular one has

$$\tilde{H}_{p,q} = \{g \in \mathrm{Cliff}(p, q; \mathbb{R}) : g^t g = 1\} \tag{C.6}$$

g^t is the image of g under the reversal involution. For the corresponding Lie alegbra we require $a^t + a = 0$, and so the algebra is generated by elements in the negative eigenspace of the involution. For $d \leq 8$, this is the set $\{\gamma^{mn}, \gamma^{mnp}, \gamma^{m_1 \dots m_6}, \gamma^{m_1 \dots m_7}\}$. We see that the maximally compact subgroups $\tilde{H}_d \subset E_{d(d)}$ are given by

$$\tilde{H}_d = \tilde{H}_{d,0} \tag{C.7}$$

for the $\mathrm{Cliff}(d;\mathbb{R})$ algebras[1].

C.2 Representations of $\mathrm{Cliff}(p,q;\mathbb{R})$ and intertwiners

It is usual to consider irreducible complex representations of the gamma matrices acting on spinors. When d is even there is only one such representation. There are then three intertwiners realising the involutions discussed above, namely,

$$\begin{aligned} \gamma_{(d)}\gamma^m\gamma_{(d)}^{-1} &= -\gamma^m, \\ C\gamma^m C^{-1} &= (\gamma^m)^T, \\ \tilde{C}\gamma^m\tilde{C}^{-1} &= -(\gamma^m)^T, \end{aligned} \tag{C.8}$$

where $\tilde{C} = C\gamma^{(d)}$. There are four further intertwiners, not all independent, giving

$$\begin{aligned} A\gamma^m A^{-1} &= (\gamma^m)^\dagger, \qquad & D\gamma^m D^{-1} &= (\gamma^m)^*, \\ \tilde{A}\gamma^m\tilde{A}^{-1} &= -(\gamma^m)^\dagger, \qquad & \tilde{D}\gamma^m\tilde{D}^{-1} &= -(\gamma^m)^*. \end{aligned} \tag{C.9}$$

By construction we see that $\tilde{H}_d$ is the group preserving C.

When d is odd there are two inequivalent irreducible representations with either $\gamma^{(d)} = \pm 1$ when $p - q = 1 \pmod 4$ or $\gamma^{(d)} = \pm\mathrm{i}$ when $p - q = 3 \pmod 4$. Since here $\gamma^{(d)}$ is odd under the reflection, this involution exchanges the two representations. Thus only half of the possible intertwiners exist on each. One has

$$\begin{aligned} C\gamma^m C^{-1} &= (\gamma^m)^T, \qquad & &\text{if } d = 1 \pmod 4, \\ \tilde{C}\gamma^m\tilde{C}^{-1} &= -(\gamma^m)^T, \qquad & &\text{if } d = 3 \pmod 4. \end{aligned} \tag{C.10}$$

[1] Note that $\tilde{H}_{7,0}$ is strictly $U(8)$. Dropping the $\gamma^{(7)}$ generator one gets $\tilde{H}_7 = SU(8)$.

while

$$\begin{aligned} A\gamma^m A^{-1} &= (\gamma^m)^\dagger, && \text{if } p \text{ is odd,} \\ \tilde{A}\gamma^m \tilde{A}^{-1} &= -(\gamma^m)^\dagger, && \text{if } p \text{ is even,} \\ D\gamma^m D^{-1} &= (\gamma^m)^*, && \text{if } p-q=1 \pmod 4, \\ \tilde{D}\gamma^m \tilde{D}^{-1} &= -(\gamma^m)^*, && \text{if } p-q=3 \pmod 4. \end{aligned} \tag{C.11}$$

Note that under reversal $(\gamma^{(d)})^t = (-)^{d(d-1)/2}\gamma^{(d)}$ so when $d = 3 \pmod 4$ the involution exchanges representations and we have no C intertwiner. In particular for $\mathrm{Cliff}(d;\mathbb{R})$ it maps $\gamma^{(d)} = \mathrm{i}$ to $\gamma^{(d)} = -\mathrm{i}$. However, this map can also be realised on each representation separately by the adjoint $A\gamma^m A^{-1} = (\gamma^m)^\dagger$. Hence for $d = 3 \pmod 4$ we can instead define $\tilde{H}_d$ as the group preserving A.

The conjugate intertwiners allow us to define Majorana and symplectic Majorana representations when there is an isomorphism to real and quaternionic matrix algebras respectively. Thus when $p - q = 0, 1, 2 \pmod 8$ one has $DD^* = 1$ and can define a reality condition on the spinors

$$\chi = (D\chi)^*. \tag{C.12}$$

When $p - q = 4, 5, 6 \pmod 8$ one has $DD^* = -1$ one can define a symplectic reality condition. Introducing a pair of $SU(2)$ indices $A, B, \ldots = 1, 2$ on the spinors with the convention for raising and lowering these indices

$$\chi_A = \epsilon_{AB}\chi^B, \qquad \chi^A = \epsilon^{AB}\chi_B, \tag{C.13}$$

the symplectic Majorana condition is

$$\eta^A = \epsilon^{AB}(D\eta^B)^*. \tag{C.14}$$

Note that for $p - q = 0, 6, 7 \pmod 8$ and $p - q = 2, 3, 4 \pmod 8$ one can also define analogous Majorana and symplectic Majorana conditions respectively using $\tilde{D}$.

C.3 Cliff$(10,1;\mathbb{R})$

For Cliff$(10,1;\mathbb{R}) \simeq GL(32,\mathbb{R}) \oplus GL(32,\mathbb{R})$, following the conventions of [85] we take the representation with

$$\Gamma^{(11)} = \Gamma^0\Gamma^1 \dots \Gamma^{10} = -1. \tag{C.15}$$

The D intertwiner defines Majorana spinors, while $\tilde{C} = -\tilde{C}^T$ defines the conjugate

$$\varepsilon = (D\varepsilon)^*, \qquad \bar{\varepsilon} = \varepsilon^T \tilde{C}. \tag{C.16}$$

such that

$$\overline{\Gamma^{M_1\dots M_k}\varepsilon} = (-1)^{[(k+1)/2]}\bar{\varepsilon}\Gamma^{M_1\dots M_k}. \tag{C.17}$$

C.4 Cliff$(9,1;\mathbb{R})$

The Cliff$(9,1;\mathbb{R})$ gamma matrices are defined as

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}, \qquad \gamma^{\mu_1\dots\mu_k} = \gamma^{[\mu_1} \dots \gamma^{\mu_k]}, \tag{C.18}$$

and we use the anti-symmetric transpose intertwiner

$$\tilde{C}\gamma^\mu\tilde{C}^{-1} = -(\gamma^\mu)^T, \qquad \tilde{C}^T = -\tilde{C}, \tag{C.19}$$

to define the Majorana conjugate as $\bar{\epsilon} = \epsilon^T \tilde{C}$. This leads to the formulae

$$\begin{aligned} \tilde{C}\gamma^{\mu_1\dots\mu_k}\tilde{C}^{-1} &= (-)^{[(k+1)/2]}(\gamma^{\mu_1\dots\mu_k})^T, \\ \bar{\epsilon}\gamma^{\mu_1\dots\mu_k}\chi &= (-)^{[(k+1)/2]}\bar{\chi}\gamma^{\mu_1\dots\mu_k}\epsilon, \end{aligned} \tag{C.20}$$

where in the second equation the spinors ϵ and χ are anti-commuting. The top gamma is defined as

$$\gamma^{(10)} = \gamma^0\gamma^1 \dots \gamma^9 = \tfrac{1}{10!}\epsilon_{\mu_1\dots\mu_{10}}\gamma^{\mu_1\dots\mu_{10}}, \tag{C.21}$$

and this gives rise to the equation

$$\gamma_{\mu_1\dots\mu_k}\gamma^{(10)} = (-)^{[k/2]}\tfrac{1}{(10-k)!}\sqrt{-g}\epsilon_{\mu_1\dots\mu_k\nu_1\dots\nu_{10-k}}\gamma^{\nu_1\dots\nu_{10-k}}, \tag{C.22}$$

which is also commonly written as

$$\gamma^{(k)}\gamma^{(10)} = (-)^{[k/2]} * \gamma^{(10-k)}. \tag{C.23}$$

C.5 $\mathrm{Cliff}(4;\mathbb{R})$ and $Spin(5)$

For $\mathrm{Cliff}(4;\mathbb{R}) \simeq GL(2,\mathbb{H})$, $D^*D = -1$ and we can use this to introduce symplectic Majorana spinors, while we use $\tilde{C}$ to define the conjugate spinor

$$\chi^A = \epsilon^{AB}(D\chi^B)^*, \qquad \bar{\chi}_A = \epsilon_{AB}(\chi^B)^T\tilde{C} \tag{C.24}$$

The other intertwiner $C = \tilde{C}\gamma^{(4)}$ provides a symplectic inner product on spinors, which is preserved by $\{\gamma^{mn}, \gamma^{mnp}\}$, i.e. the $\tilde{H}_4 \cong Spin(5)$ algebra. The $Spin(5)$ gamma matrix algebra can be realised explicitly by setting

$$\hat{\gamma}^i = \begin{cases} \gamma^a & i = a \\ \gamma^{(4)} & i = 5 \end{cases}, \tag{C.25}$$

and identifying $\gamma^{mnp} = -\epsilon^{mnpq}\gamma_q\gamma^{(4)}$. The same gamma matrices give a representation of $\mathrm{Cliff}(5;\mathbb{R})$ (with $\gamma^{(5)} = +1$).

C.6 $\mathrm{Cliff}(7;\mathbb{R})$ and $Spin(8)$

For $\mathrm{Cliff}(7;\mathbb{R})$ we take the representation with $\gamma^{(7)} = -\mathrm{i}$ and define conjugate spinors

$$\bar{\varepsilon} = \varepsilon^\dagger A. \tag{C.26}$$

This provides a hermitian inner product on spinors, which is preserved by $\tilde{H}_7 \cong SU(8)$, generated by $\{\gamma^{mn}, \gamma^{mnp}, \gamma^{m_1\dots m_6}\}$. The intertwiner $\tilde{C} = \tilde{C}^T$ is preserved by a $Spin(8) \subset$

$SU(8)$ subgroup. The corresponding generators can be written as

$$\hat{\gamma}^{ij} = \begin{cases} \gamma^{ab} & i=a, j=b \\ +\gamma^{a}\gamma^{(7)} & i=a, j=8 \\ -\gamma^{b}\gamma^{(7)} & i=8, j=b \end{cases}, \tag{C.27}$$

This representation has negative chirality in the sense that

$$\hat{\gamma}^{i_1\ldots i_8} = -\epsilon^{i_1\ldots i_8}. \tag{C.28}$$

We have the useful completeness relations, reflecting $SO(8)$ triality,

$$\hat{\gamma}^{ij}{}_{\alpha\beta}\hat{\gamma}_{ij}{}^{\gamma\delta} = 16\delta^{\gamma\delta}_{\alpha\beta}, \qquad \hat{\gamma}^{ij}{}_{\alpha\beta}\hat{\gamma}_{kl}{}^{\alpha\beta} = 16\delta^{ij}_{kl},$$

where we have used $\tilde{C}$ to raise and lower spinor indices, and Fierz identity, which also serves as our definition of $\epsilon_{\alpha_1\ldots\alpha_8}$,

$$\tfrac{1}{4!}\epsilon_{\alpha\alpha'\beta\beta'\gamma\gamma'\delta\delta'}\hat{\gamma}^{ij\gamma\gamma'}\hat{\gamma}^{kl\delta\delta'} = 2\hat{\gamma}^{[ij}{}_{[\alpha\alpha'}\hat{\gamma}^{kl]}{}_{\beta\beta']} - \hat{\gamma}^{ij}{}_{[\alpha\alpha'}\hat{\gamma}^{kl}{}_{\beta\beta']}. \tag{C.29}$$

Note that as a representation of the $Spin(8)$ algebra we can impose a reality condition on the spinors $\chi = (D\chi)^*$ using the intertwiner $\tilde{D}$ with $\tilde{D}^*\tilde{D} = +1$. For such a real spinor the two possible definitions of spinor conjugate coincide $\bar{\chi} = \chi^T\tilde{C} = \chi^\dagger A$. In fact there exists a $GL(8,\mathbb{R})$ family of purely imaginary bases of gamma matrices such that $\tilde{D} = 1$ and $A = \tilde{C}$. In such a basis we have $\bar{\varepsilon} = \varepsilon^\dagger\tilde{C} = \varepsilon^\dagger A$ for a general spinor $\varepsilon = \chi_1 + \mathrm{i}\chi_2$. Many of our $SU(8)$ equations are written under a $Spin(8) = SU(8) \cap SL(8,\mathbb{R})$ decomposition in such an imaginary basis, and thus it is natural to raise and lower spinor indices with the $Spin(8)$ invariant $\tilde{C}$.

Appendix D

Spinor Decompositions

D.1 $(10,1) \to (6,1) + (4,0)$

We can decompose the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma matrices as

$$\Gamma^\mu = \gamma^\mu \otimes \gamma^{(4)}, \qquad \Gamma^m = 1 \otimes \gamma^m, \tag{D.1}$$

and the eleven-dimensional intertwiners as

$$\tilde{C} = \tilde{C}_{(6,1)} \otimes \tilde{C}_{(4)}, \qquad D = D_{(6,1)} \otimes D_{(4)}. \tag{D.2}$$

Introducing a basis of seven dimensional symplectic Majorana spinors $\{\eta_I^A\}$, we can then decompose a general eleven-dimensional Majorana spinor as

$$\varepsilon = \epsilon_{AB} \left(\eta_I^A \otimes \chi^{BI} \right), \tag{D.3}$$

where $\{\chi^{AI}\}$ are some collection of four dimensional symplectic Majorana spinors. All of the data of the eleven dimensional spinor is now contained in χ^{AI}, the extra index I serving as the external $Spin(6,1)$ index.

The eleven dimensional spinor conjugate can be realised in terms of the four dimensional spinors χ^{AI} by setting

$$\bar{\chi}_{AI} = \epsilon_{AIBJ}(\chi^{BJ})^T \tilde{C}_{(4)}, \tag{D.4}$$

where $\epsilon_{AIBJ} = (\eta_{AI})^T \tilde{C}_{(6,1)} \eta_{BJ}$.

Clearly from the decomposition (D.1) the action of the internal eleven dimensional gamma matrices is simply

$$\Gamma^m \varepsilon \leftrightarrow \gamma^m \chi^{AI}, \tag{D.5}$$

and any eleven dimensional equation with only internal gamma matrices takes the same form in terms of χ^{AI}. Thus, supressing the extra indices on χ, the supergravity equations with fermions in section 2.2.2 take exactly the same form when written in terms of the four-dimensional spinors.

D.2 $(10,1) \to (3,1)+(7,0)$

We can use a complex decomposition of the $\mathrm{Cliff}(10,1;\mathbb{R})$ gamma matrices as

$$\Gamma^\mu = \gamma^\mu \otimes 1, \qquad \Gamma^m = \mathrm{i}\gamma^{(4)} \otimes \gamma^m, \tag{D.6}$$

and the eleven-dimensional intertwiners as

$$\tilde{C} = \tilde{C}_{(3,1)} \otimes \tilde{C}_{(7)}, \qquad D = D_{(3,1)} \otimes \tilde{D}_{(7)}. \tag{D.7}$$

We take a chiral decomposition of an eleven-dimensional Majorana spinor

$$\varepsilon = \left(\eta_I^+ \otimes \chi^I\right) + (D_{(3,1)}\eta_I^+)^* \otimes (\tilde{D}_{(7)}\chi^I)^*, \tag{D.8}$$

where $\gamma^{(4)}\eta_I^+ = -\mathrm{i}\eta_I^+$ so that $\{\eta_I^+\}$ are a basis of complex Weyl spinors in the external space. The Majorana condition on ε is automatic with no additional constraint on χ^I, which is complex. Again the extra index I on χ provides an external $Spin(3,1)$ index.

The Clifford action of the internal eleven-dimensional gamma matrices then reduces to the action of the seven-dimensional gamma matrices on χ

$$\Gamma^m \varepsilon = \eta_I^+ \otimes (\gamma^m \chi^I) + (D_{(3,1)}\eta_I^+)^* \otimes (\tilde{D}_{(7)}\gamma^m \chi^I)^*. \tag{D.9}$$

To see how to write eleven-dimensional spinor bilinears in this language, we expand

$$\begin{aligned}\bar{\varepsilon}\,\Gamma^{m_1\dots m_k}\varepsilon' &= \Big((\eta_I^+)^T \tilde{C}_{(3,1)}\eta_J^+\Big)\Big((\chi^I)^T \tilde{C}_{(7)}\gamma^{m_1\dots m_k}\chi'^J\Big) \\ &\quad + \Big((\eta_I^+)^T D_{(3,1)}^T \tilde{C}_{(3,1)} D_{(3,1)}\eta_J^+\Big)^*\Big((\chi^I)^T \tilde{D}_{(7)}^T \tilde{C}_{(7)} \tilde{D}_{(7)}\gamma^{m_1\dots m_k}\chi'^J\Big)^* \\ &= \Big(\bar{\chi}_I \gamma^{m_1\dots m_k}\chi'^I\Big) + \Big(\text{cc}\Big),\end{aligned} \tag{D.10}$$

where we have made the definition

$$\bar{\chi}_I = \epsilon_{IJ}(\chi^J)^T \tilde{C}_{(7)}, \tag{D.11}$$

with $\epsilon_{IJ} = -(\eta_I^+)^T \tilde{C}_{(3,1)}\eta_J^+$.

With these definitions, the equations linear in spinors in section 2.2.2 take the same form when written in terms of χ^I, while the spinor bilinear expressions take the same form with a complex conjugate piece added to them.

Appendix E

$E_{d(d)} \times \mathbb{R}^+$

In this appendix we give an explicit construction of $E_{d(d)} \times \mathbb{R}^+$ for $d \leq 7$ based on the $GL(d, \mathbb{R})$ subgroup. We will describe the action directly in terms of the bundles that appear in the generalised geometry.

E.1 Construction of $E_{d(d)} \times \mathbb{R}^+$ from $GL(d, \mathbb{R})$

We have

$$\begin{aligned} E &\simeq TM \oplus \Lambda^2 T^*M \oplus \Lambda^5 T^*M \oplus (T^*M \otimes \Lambda^7 T^*M), \\ E^* &\simeq T^*M \oplus \Lambda^2 TM \oplus \Lambda^5 TM \oplus (TM \otimes \Lambda^7 TM), \\ \mathrm{ad}\, \tilde{F} &\simeq \mathbb{R} \oplus (TM \otimes T^*M) \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M \oplus \Lambda^3 TM \oplus \Lambda^6 TM. \end{aligned} \tag{E.1}$$

The corresponding $E_{d(d)} \times \mathbb{R}^+$ representations are listed in Table 5.1. We write sections as

$$\begin{aligned} V &= v + \omega + \sigma + \tau && \in E, \\ Z &= \zeta + u + s + t && \in E^*, \\ R &= c + r + a + \tilde{a} + \alpha + \tilde{\alpha} && \in \mathrm{ad}\, \tilde{F}, \end{aligned} \tag{E.2}$$

so that $v \in TM$, $\omega \in \Lambda^2 T^*M$, $\zeta \in T^*M$, $c \in \mathbb{R}$ etc. If $\{\hat{e}_a\}$ is a basis for TM with a dual basis $\{e^a\}$ on T^*M then there is a natural $gl(d, \mathbb{R})$ action on each tensor component. For instance

$$(r \cdot v)^a = r^a{}_b v^b, \qquad (r \cdot \omega)_{ab} = -r^c{}_a \omega_{cb} - r^c{}_b \omega_{ac}, \qquad \text{etc.} \tag{E.3}$$

Writing $V' = R \cdot V$ for the adjoint $E_{d(d)} \times\mathbb{R}^+$ action of $R \in \operatorname{ad}\tilde{F}$ on $V \in E$, the components of V', using the notation of appendix A.2, are given by

$$\begin{aligned}
v' &= cv + r \cdot v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma, \\
\omega' &= c\omega + r \cdot \omega + v \lrcorner a + \alpha \lrcorner \sigma + \tilde{\alpha} \lrcorner \tau, \\
\sigma' &= c\sigma + r \cdot \sigma + v \lrcorner \tilde{a} + a \wedge \omega + \alpha \lrcorner \tau, \\
\tau' &= c\tau + r \cdot \tau - j\tilde{a} \wedge \omega + ja \wedge \sigma.
\end{aligned} \tag{E.4}$$

Note that, the $E_{d(d)}$ sub-algebra is generated by setting $c = \frac{1}{(9-d)} r^a{}_a$. Similarly, given $Z \in E^*$ we have

$$\begin{aligned}
\zeta' &= -c\zeta + r \cdot \zeta - u \lrcorner a + s \lrcorner \tilde{a}, \\
u' &= -cu + r \cdot u - \alpha \lrcorner \zeta - s \lrcorner a + t \lrcorner \tilde{a}, \\
s' &= -cs + r \cdot s - \tilde{\alpha} \lrcorner \zeta - \alpha \wedge u - t \lrcorner a, \\
t' &= -ct + r \cdot t - j\alpha \wedge s - j\tilde{\alpha} \wedge u.
\end{aligned} \tag{E.5}$$

Finally the adjoint commutator

$$R'' = [R, R'] \tag{E.6}$$

has components

$$\begin{aligned}
c'' &= \tfrac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a) + \tfrac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}'), \\
r'' &= [r, r'] + j\alpha \lrcorner ja' - j\alpha' \lrcorner ja - \tfrac{1}{3}(\alpha \lrcorner a' - \alpha' \lrcorner a)\mathbb{1} \\
&\qquad + j\tilde{\alpha}' \lrcorner j\tilde{a} - j\tilde{\alpha} \lrcorner j\tilde{a}' - \tfrac{2}{3}(\tilde{\alpha}' \lrcorner \tilde{a} - \tilde{\alpha} \lrcorner \tilde{a}')\mathbb{1}, \\
a'' &= r \cdot a' - r' \cdot a + \alpha' \lrcorner \tilde{a} - \alpha \lrcorner \tilde{a}', \\
\tilde{a}'' &= r \cdot \tilde{a}' - r' \cdot \tilde{a} - a \wedge a', \\
\alpha'' &= r \cdot \alpha' - r' \cdot \alpha + \tilde{\alpha}' \lrcorner a - \tilde{\alpha} \lrcorner a', \\
\tilde{\alpha}'' &= r \cdot \tilde{\alpha}' - r' \cdot \tilde{\alpha} - \alpha \wedge \alpha'
\end{aligned} \tag{E.7}$$

Here we have $c'' = \frac{1}{9-d} r''^a{}_a$, as R'' lies in the $E_{d(d)}$ sub-algebra.

The $E_{d(d)} \times \mathbb{R}^+$ Lie group can then be constructed starting with $GL(d, \mathbb{R})$ and using the exponentiated action of a, $\tilde{a}$, α and $\tilde{\alpha}$. The $GL(d, \mathbb{R})$ action by an element m is standard so

$$(m \cdot v)^a = m^a{}_b v^b, \qquad (m \cdot \omega)_{ab} = (m^{-1})^c{}_a (m^{-1})^d{}_b \omega_{cd}, \qquad \text{etc.} \tag{E.8}$$

The action of a and $\tilde{a}$ form a nilpotent subgroup of nilpotency class two. One has

$$\begin{aligned} \mathrm{e}^{a+\tilde{a}} V = v &+ (\omega + i_v a) \\ &+ \big(\sigma + a \wedge \omega + \tfrac{1}{2} a \wedge i_v a + i_v \tilde{a}\big) \\ &+ \big(\tau + j a \wedge \sigma - j\tilde{a} \wedge \omega + \tfrac{1}{2} j a \wedge a \wedge \omega \\ &\qquad + \tfrac{1}{2} j a \wedge i_v \tilde{a} - \tfrac{1}{2} j \tilde{a} \wedge i_v a + \tfrac{1}{6} j a \wedge a \wedge i_v a\big), \end{aligned} \tag{E.9}$$

with no terms higher than cubic in the expansion. The action of α and $\tilde{\alpha}$ form a similar nilpotent subgroup of nilpotency class two with

$$\begin{aligned} \mathrm{e}^{\alpha+\tilde{\alpha}} V = &\big(v + \alpha \lrcorner \omega - \tilde{\alpha} \lrcorner \sigma + \tfrac{1}{2} \alpha \lrcorner \alpha \lrcorner \sigma \\ &\qquad + \tfrac{1}{2} \alpha \lrcorner \tilde{\alpha} \lrcorner \tau + \tfrac{1}{2} \tilde{\alpha} \lrcorner \alpha \lrcorner \tau + \tfrac{1}{6} \alpha \lrcorner \alpha \lrcorner \alpha \lrcorner \tau\big) \\ &+ (\omega + \alpha \lrcorner \sigma + \tilde{\alpha} \lrcorner \tau + \alpha \lrcorner \alpha \lrcorner \sigma) \\ &+ (\sigma + \alpha \lrcorner \tau) + \tau. \end{aligned} \tag{E.10}$$

A general element of $E_{d(d)} \times \mathbb{R}^+$ then has the form

$$M \cdot V = \mathrm{e}^{\lambda}\, \mathrm{e}^{\alpha+\tilde{\alpha}}\, \mathrm{e}^{a+\tilde{a}}\, m \cdot V, \tag{E.11}$$

where e^{λ} with $\lambda \in \mathbb{R}$ is included to give a general $\mathbb{R}^+$ scaling.

E.2 Some tensor bundle products

We also define two tensor bundle products. We have the map into the adjoint bundle

$$\times_{\mathrm{ad}} : E^* \otimes E \to \mathrm{ad}\, \tilde{F}. \tag{E.12}$$

Writing $R = Z \times_{\text{ad}} V$ we have

$$
\begin{aligned}
c &= -\tfrac{1}{3} u \lrcorner \omega - \tfrac{2}{3} s \lrcorner \sigma - t \lrcorner \tau, \\
m &= v \otimes \zeta - ju \lrcorner j\omega + \tfrac{1}{3}(u \lrcorner \omega)\mathbb{1} - js \lrcorner j\sigma + \tfrac{2}{3}(s \lrcorner \sigma)\mathbb{1} - jt \lrcorner j\tau, \\
\alpha &= v \wedge u + s \lrcorner \omega + t \lrcorner \sigma, \\
\tilde{\alpha} &= -v \wedge s - t \lrcorner \omega, \\
a &= \zeta \wedge \omega + u \lrcorner \sigma + s \lrcorner \tau, \\
\tilde{a} &= \zeta \wedge \sigma + u \lrcorner \tau.
\end{aligned}
\tag{E.13}
$$

We can also consider the bundle N as given in table 5.2. Taking

$$
\begin{aligned}
N &\simeq T^*M \oplus \Lambda^4 T^*M \oplus (T^*M \otimes \Lambda^6 T^*M) \oplus (\Lambda^3 T^*M \otimes \Lambda^7 T^*M) \oplus (\Lambda^6 T^*M \otimes \Lambda^7 T^*M). \\
Y &= \lambda + \kappa + \mu + \nu + \pi,
\end{aligned}
\tag{E.14}
$$

we have that the symmetric map $E \otimes E \to N$ is

$$
\begin{aligned}
\lambda &= v \lrcorner \omega' + v' \lrcorner \omega, \\
\kappa &= v \lrcorner \sigma' + v' \lrcorner \sigma - \omega \wedge \omega', \\
\mu &= (j\omega \wedge \sigma' + j\omega' \wedge \sigma) - \tfrac{1}{4}(\sigma \wedge \omega' + \sigma' \wedge \omega) \\
&\qquad + (v \lrcorner j\tau) + (v \lrcorner j\tau') - \tfrac{1}{4}(v \lrcorner \tau' + v' \lrcorner \tau), \\
\nu &= j^3\omega \wedge \tau' + j^3\omega' \wedge \tau - j^3\sigma \wedge \sigma', \\
\pi &= j^6\sigma \wedge \tau' + j^6\sigma' \wedge \tau,
\end{aligned}
\tag{E.15}
$$

Appendix F

H_d and $\tilde{H}_d$

We now turn to the analogous description of H_d in $SO(d)$ representations. We then give a detailed description of the spinor representations of H_d and provide several important projections of tensor products in this language.

F.1 H_d and $SO(d)$

Given a positive definite metric g on TM, which for convenience we take to be in standard form δ_{ab} in frame indices, we can define a metric on E by

$$G(V,V) = v^2 + \tfrac{1}{2!}\omega^2 + \tfrac{1}{5!}\sigma^2 + \tfrac{1}{7!}\tau^2, \tag{F.1}$$

where $v^2 = v_a v^a$, $\omega^2 = \omega_{ab}\omega^{ab}$, etc as in (A.6). Note that this metric allows us to identify $E \simeq E^*$.

The subgroup of $E_{d(d)} \times \mathbb{R}^+$ that leaves the metric is invariant is H_d, the maximal compact subgroup of $E_{d(d)}$ (see table 5.3). Geometrically it defines a generalised H_d structure, that is an H_d sub-bundle P of the generalised structure bundle $\tilde{F}$. The corresponding Lie algebra bundle is parametrised by

$$\begin{aligned} \operatorname{ad} P &\simeq \Lambda^2 T^*M \oplus \Lambda^3 T^*M \oplus \Lambda^6 T^*M, \\ N &= n + b + \tilde{b}, \end{aligned} \tag{F.2}$$

and embeds in $\mathrm{ad}\,\tilde{F}$ as

$$\begin{aligned} c &= 0, \\ m_{ab} &= n_{ab}, \\ a_{abc} &= -\alpha_{abc} = b_{abc}, \\ \tilde{a}_{a_1\dots a_6} &= \tilde{\alpha}_{a_1\dots a_6} = \tilde{b}_{a_1\dots a_6}, \end{aligned} \tag{F.3}$$

where indices are lowered with the metric g. Note that n_{ab} generates the $O(d) \subset GL(d,\mathbb{R})$ subgroup that preserves g. Concretely a general group element can be written as

$$H \cdot V = \mathrm{e}^{\alpha+\tilde{\alpha}}\,\mathrm{e}^{a+\tilde{a}}\,h \cdot V, \tag{F.4}$$

where $h \in O(d)$ and a and α and $\tilde{a}$ and $\tilde{\alpha}$ are related as in (F.3).

The generalised tangent space $E \simeq E^*$ forms an irreducible H_d bundle, where the action of H_d just follows from (E.4). The corresponding representations are listed in table 5.3.

Another important representation of H_d is the complement of the adjoint of H_d in $E_{d(d)} \times \mathbb{R}^+$, which we denote as $H^\perp$ (see table 5.3). An element of $H^\perp$ is represented as

$$\begin{aligned} H^\perp &\simeq \mathbb{R} \oplus S^2F^* \oplus \Lambda^3F^* \oplus \Lambda^6F^*, \\ Q &= c + h + q + \tilde{q} \end{aligned} \tag{F.5}$$

and it embeds in $\mathrm{ad}\,\tilde{F}$

$$\begin{aligned} c &= c, \\ m_{ab} &= h_{ab}, \\ a_{abc} &= \alpha_{abc} = q_{abc}, \\ \tilde{a}_{a_1\dots a_6} &= -\tilde{\alpha}_{a_1\dots a_6} = \tilde{q}_{a_1\dots a_6}. \end{aligned} \tag{F.6}$$

The action of H_d on this representation is given by $E_{d(d)} \times \mathbb{R}^+$ Lie algebra. Writing $Q' =$

$N \cdot Q$ we have

$$\begin{aligned} c' &= -\tfrac{2}{3} b \lrcorner q - \tfrac{4}{3} \tilde{b} \lrcorner \tilde{q}, \\ h' &= n \cdot h - jb \lrcorner jq - jq \lrcorner jb - j\tilde{b} \lrcorner j\tilde{q} - j\tilde{q} \lrcorner j\tilde{b} + \left(\tfrac{2}{3} b \lrcorner q + \tfrac{4}{3} \tilde{b} \lrcorner \tilde{q}\right)\mathbb{1}, \\ q' &= n \cdot q - h \cdot b + b \lrcorner \tilde{q} + q \lrcorner \tilde{b}, \\ \tilde{q}' &= n \cdot \tilde{q} - h \cdot \tilde{b} - b \wedge q, \end{aligned} \tag{F.7}$$

where we are using the $GL(d,\mathbb{R})$ adjoint action of h on $\Lambda^3 T^* M$ and $\Lambda^6 T^* M$. The H_d invariant scalar part of Q is given by $c - \frac{1}{9-d} h^a{}_a$, while the remaining irreducible component has $c = \frac{1}{9-d} h^a{}_a$.

F.2 $\tilde{H}_d$ and $\text{Cliff}(d;\mathbb{R})$

The double cover $\tilde{H}_d$ of H_d has a realisation in terms of the Clifford algebra $\text{Cliff}(d;\mathbb{R})$. Let S be the bundle of $\text{Cliff}(d;\mathbb{R})$ spinors. We can identify sections of S as $\tilde{H}_d$ bundles in two different ways, which we denote $S^{\pm}$. Specifically $\chi^{\pm} \in S^{\pm}$ if

$$N \cdot \chi^{\pm} = \tfrac{1}{2}\left(\tfrac{1}{2!} n_{ab} \gamma^{ab} \pm \tfrac{1}{3!} b_{abc} \gamma^{abc} - \tfrac{1}{6!} \tilde{b}_{a_1 \dots a_6} \gamma^{a_1 \dots a_6}\right) \chi^{\pm}, \tag{F.8}$$

for $N \in \text{ad}\, P$. As expected, in both cases n generates the $Spin(d)$ subgroup of $\tilde{H}_d$. The two representations are mapped into each other by $\gamma^a \to -\gamma^a$. As such, they are inequivalent in odd dimensions. However, in even dimensions, since $-\gamma^a = \gamma^{(d)} \gamma^a (\gamma^{(d)})^{-1}$, they are equivalent and one can identify $\chi^- = \gamma^{(d)} \chi^+$. Thus one finds

$$\begin{aligned} S &\simeq S^+ \oplus S^- \quad \text{if } d \text{ is odd}, \\ S &\simeq S^+ \simeq S^- \quad \text{if } d \text{ is even}. \end{aligned} \tag{F.9}$$

The different $\tilde{H}_d$ representations are listed explicitly in table 5.5.

The $Spin(d)$ vector-spinor bundle J also forms representations of $\tilde{H}_d$. Again we can identify two different actions. If $\varphi_a^\pm \in J^\pm$ we have[1]

$$\begin{aligned} N \cdot \varphi_a^\pm = {} & \tfrac{1}{2}\big(\tfrac{1}{2!}n_{bc}\gamma^{bc} \pm \tfrac{1}{3!}b_{bcd}\gamma^{bcd} - \tfrac{1}{6!}\tilde{b}_{b_1\dots b_6}\gamma^{b_1\dots b_6}\big)\varphi_a^\pm - n^b{}_a\varphi_b^\pm \\ & \mp \tfrac{2}{3}b_a{}^b{}_c\gamma^c\varphi_b^\pm \mp \tfrac{1}{3}\tfrac{1}{2!}b^b{}_{cd}\gamma_a{}^{cd}\varphi_b^\pm \\ & + \tfrac{1}{3}\tfrac{1}{4!}\tilde{b}_a{}^b{}_{c_1\dots c_4}\gamma^{c_1\dots c_4}\varphi_b^\pm + \tfrac{2}{3}\tfrac{1}{5!}\tilde{b}^b{}_{c_1\dots c_5}\gamma_a{}^{c_1\dots c_5}\varphi_b^\pm . \end{aligned} \tag{F.10}$$

Again in even dimension $J^+ \simeq J^-$. The $\tilde{H}_d$ representations are listed explicitly in table 5.5.

Finally will also need the projections $H^\perp \otimes S^\pm \to J^\mp$, which, for $Q \in H^\perp$ and $\chi^\pm \in S^\pm$, is given by

$$\begin{aligned} (Q \times_{J^\mp} \chi^\pm)_a = {} & \tfrac{1}{2}h_{ab}\gamma^b\chi^\pm \mp \tfrac{1}{3}\tfrac{1}{2!}q_{abc}\gamma^{bc}\chi^\pm \pm \tfrac{1}{6}\tfrac{1}{3!}q_{bcd}\gamma_a{}^{bcd}\chi^\pm \\ & + \tfrac{1}{6}\tfrac{1}{5!}\tilde{q}_{ab_1\dots b_5}\gamma^{b_1\dots b_5}\chi^\pm - \tfrac{1}{3}\tfrac{1}{6!}\tilde{q}_{c_1\dots c_6}\gamma_a{}^{c_1\dots c_6}\chi^\pm . \end{aligned} \tag{F.11}$$

F.3 $\tilde{H}_d$ and $\mathrm{Cliff}(10,1;\mathbb{R})$

To describe the reformulation of $D = 11$ supergravity restricted to d dimensions it is very useful to use the embedding of $\tilde{H}_d$ in $\mathrm{Cliff}(10,1;\mathbb{R})$. This identifies the same action of $\tilde{H}_d$ on spinors given in (F.8) but now using the internal spacelike gamma matrices Γ^a for $a = 1, \dots, d$. Combined with the external spin generators $\Gamma^{\mu\nu}$, this actually gives an action of $Spin(10-d,1) \times \tilde{H}_d$ on eleven-dimensional spinors. As before the action of $\tilde{H}_d$ can be embedded in two different ways. We write $\hat{\chi}^\pm \in \hat{S}^\pm$ with

$$N \cdot \hat{\chi}^\pm = \tfrac{1}{2}\big(\tfrac{1}{2!}n_{ab}\Gamma^{ab} \pm \tfrac{1}{3!}b_{abc}\Gamma^{abc} - \tfrac{1}{6!}\tilde{b}_{a_1\dots a_6}\Gamma^{a_1\dots a_6}\big)\hat{\chi}^\pm . \tag{F.12}$$

Since the algebra of the $\{\Gamma^a\}$ is the same as $\mathrm{Cliff}(d;\mathbb{R})$ all the equations of the previous section translate directly to this presentation of $\tilde{H}_d$. The advantage of the direct action on eleven-dimensional spinors is that it allows us to write $\tilde{H}_d$ covariant spinor equations in a dimension independent way.

As before we can also identify two realisations $\hat{J}^\pm$ of $\tilde{H}_d$ on the representations with one

[1]The formula given here matches those found in [121, 122] for levels 0, 1 and 2 of $K(E_{10})$. A similar formula also appears in the context of E_{11} in [123].

eleven-dimensional spinor index and one internal vector index which transform as (F.10) (with Γ^a in place of γ^a). The $Spin(d-1,1)\times\tilde{H}_d$ representations for $\hat{S}^\pm$ and $\hat{J}^\pm$ are listed explicitly in table 6.1.

In addition to the projection $H^\perp\otimes\hat{S}^\pm\to\hat{J}^\mp$ given by (F.11) (with Γ^a in place of γ^a) we can identify various other tensor products. We have the singlet projections $\langle\cdot,\cdot\rangle$: $\hat{S}^\mp\otimes\hat{S}^\pm\to\mathbb{1}$ given by the conventional $\mathrm{Cliff}(10,1;\mathbb{R})$ bilinear, defined using (C.16), so

$$\langle\hat{\chi}^-,\hat{\chi}^+\rangle=\bar{\hat{\chi}}^-\chi^+, \tag{F.13}$$

where $\hat{\chi}^\pm\in\hat{S}^\pm$. There is a similar singlet projection $\langle\cdot,\cdot\rangle:\hat{J}^\mp\otimes\hat{J}^\pm\to\mathbb{1}$ given by[2]

$$\langle\hat{\varphi}^\mp,\hat{\varphi}^\pm\rangle=\bar{\hat{\varphi}}^\mp_a\big(\delta^{ab}+\tfrac{1}{9-d}\Gamma^a\Gamma^b\big)\hat{\varphi}^\pm_b, \tag{F.14}$$

where $\hat{\varphi}^\pm\in\hat{J}^\pm$.

We also have projections from $\hat{S}^\pm\otimes\hat{J}^\pm$ and $\hat{S}^\pm\otimes\hat{S}^\mp$ to $H^\perp$. Given $\hat{\chi}^+\in\hat{S}^-$ and $\hat{\varphi}^\pm\in\hat{J}^\pm$ we have, using the decomposition (F.5),

$$\begin{aligned}(\hat{\chi}^\pm\times_{H^\perp}\hat{\varphi}^\pm)&=\tfrac{2}{9-d}\bar{\hat{\chi}}^\pm\Gamma^a\hat{\varphi}^\pm_a,\\(\hat{\chi}^\pm\times_{H^\perp}\hat{\varphi}^\pm)_{ab}&=2\bar{\hat{\chi}}^\pm\Gamma_{(a}\hat{\varphi}^\pm_{b)}\\(\hat{\chi}^\pm\times_{H^\perp}\hat{\varphi}^\pm)_{abc}&=\mp3\bar{\hat{\chi}}^\pm\Gamma_{[ab}\hat{\varphi}^\pm_{c]},\\(\hat{\chi}^\pm\times_{H^\perp}\hat{\varphi}^\pm)_{a_1\dots a_6}&=-6\bar{\hat{\chi}}^\pm\Gamma_{[a_1\dots a_5}\hat{\varphi}^\pm_{a_6]},\end{aligned} \tag{F.15}$$

Note that the image of this projection does not include the $\tilde{H}_d$ scalar part of $H^\perp$, since, from the first two components, $c-\frac{1}{9-d}h^a{}_a=0$. We also have

$$(\hat{\chi}^+\times_{H^\perp}\hat{\chi}^-)=\tfrac{2}{9-d}\bar{\hat{\chi}}^-\hat{\chi}^+, \tag{F.16}$$

and all other components of $H^\perp$ are set to zero. We see that the image of this map is in the $\tilde{H}_d$ scalar part of $H^\perp$.

Finally, we also need the $\tilde{H}_d$ projections for $E\simeq E^*$ acting on $S^\pm$ and $J^\pm$. Given

[2] Setting $d=10$ in this reproduces the corresponding inner product in [121].

$V \in E$, $\hat{\chi}^\pm \in \hat{S}^\pm$ and $\hat{\varphi}^\pm_a \in \hat{J}^\pm$ it is useful to introduce the notation

$$\begin{aligned} &V \times_{\hat{S}^\mp} \hat{\chi}^\pm, \qquad V \times_{\hat{J}^\pm} \hat{\chi}^\pm, \\ &V \times_{\hat{S}^\pm} \hat{\varphi}^\pm, \qquad V \times_{\hat{J}^\mp} \hat{\varphi}^\pm, \end{aligned} \tag{F.17}$$

which are given explicitly by

$$(V \times_{\hat{S}^\pm} \hat{\chi}^\mp) = \big(\pm v^a \Gamma_a + \tfrac{1}{2!}\omega_{ab}\Gamma^{ab} \pm \tfrac{1}{5!}\sigma_{a_1\dots a_5}\Gamma^{a_1\dots a_5} + \tfrac{1}{6!}\tau^b{}_{ba_1\dots a_6}\Gamma^{a_1\dots a_6}\big)\hat{\chi}^\pm, \tag{F.18}$$

and

$$\begin{aligned} (V \times_{\hat{J}^\pm} \hat{\chi}^\pm)_a = \big(&v_a \pm \tfrac{2}{3}\Gamma^b\omega_{ab} \mp \tfrac{1}{3}\tfrac{1}{2!}\Gamma_a{}^{cd}\omega_{cd} - \tfrac{1}{3}\tfrac{1}{4!}\Gamma^{c_1\dots c_4}\sigma_{ac_1\dots c_4} \\ &+ \tfrac{2}{3}\tfrac{1}{5!}\Gamma_a{}^{c_1\dots c_5}\sigma_{c_1\dots c_5} \pm \tfrac{1}{7!}\Gamma^{c_1\dots c_7}\tau_{a,c_1\dots c_7}\big)\hat{\chi}^\pm, \end{aligned} \tag{F.19}$$

while

$$\begin{aligned} (V \times_{\hat{S}^\pm} \hat{\varphi}^\pm) = v^a\hat{\varphi}_a &+ \tfrac{1}{10-d}v_a\Gamma^{ab}\hat{\varphi}_b \pm \tfrac{1}{10-d}\tfrac{1}{2!}\omega_{bc}\Gamma^{abc}\hat{\varphi}^+_a \pm \tfrac{8-d}{10-d}\omega^a{}_b\Gamma^b\hat{\varphi}^+_a \\ &- \tfrac{1}{10-d}\tfrac{1}{5!}\sigma^{b_1\dots b_5}\Gamma^a{}_{b_1\dots b_5}\hat{\varphi}^+_a - \tfrac{8-d}{10-d}\tfrac{1}{4!}\sigma^a{}_{b_1\dots b_4}\Gamma^{b_1\dots b_4}\hat{\varphi}^+_a \\ &\mp \tfrac{1}{7!}\tau^a{}_{,b_1\dots b_7}\Gamma^{b_1\dots b_7}\hat{\varphi}^+_a \mp \tfrac{1}{3}\tfrac{1}{5!}\tau^c{}_{,c}{}^a{}_{b_1\dots b_5}\Gamma^{b_1\dots b_5}\hat{\varphi}^+_a, \end{aligned} \tag{F.20}$$

and finally

$$\begin{aligned} (V \times_{\hat{J}^\mp} \hat{\varphi}^\pm)_a = \pm v^c\Gamma_c\hat{\varphi}^+_a &\pm \tfrac{2}{9-d}\Gamma^c v_a\hat{\varphi}^+_c - \tfrac{1}{2!}\omega_{cd}\Gamma^{cd}\hat{\varphi}^+_a + \tfrac{4}{3}\omega_a{}^b\hat{\varphi}^+_b \\ &- \tfrac{2}{3}\omega_{cd}\Gamma_a{}^c\hat{\varphi}^{+d} - \tfrac{4}{3}\tfrac{1}{9-d}\omega_{ab}\Gamma^b\Gamma^c\hat{\varphi}^+_c + \tfrac{2}{3}\tfrac{1}{9-d}\tfrac{1}{2!}\omega_{bc}\Gamma_a{}^{bc}\Gamma^d\hat{\varphi}^+_d \\ &\pm \tfrac{1}{5!}\sigma_{c_1\dots c_5}\Gamma^{c_1\dots c_5}\hat{\varphi}^+_a \mp \tfrac{2}{3}\tfrac{1}{3!}\sigma_a{}^b{}_{c_1c_2c_3}\Gamma^{c_1c_2c_3}\hat{\varphi}^+_b \mp \tfrac{4}{3}\tfrac{1}{4!}\sigma^b{}_{c_1\dots c_4}\Gamma_a{}^{c_1\dots c_4}\hat{\varphi}^+_b \\ &\mp \tfrac{2}{3}\tfrac{1}{9-d}\tfrac{1}{4!}\sigma_{ac_1\dots c_4}\Gamma^{c_1\dots c_4}\Gamma^d\hat{\varphi}^+_d \pm \tfrac{4}{3}\tfrac{1}{9-d}\tfrac{1}{5!}\sigma_{c_1\dots c_5}\Gamma_a{}^{c_1\dots c_5}\Gamma^d\hat{\varphi}^+_d \\ &+ \tfrac{1}{7!}\tau_{c,d_1\dots d_7}\Gamma^c\Gamma^{d_1\dots d_7}\hat{\varphi}^+_a + \tfrac{1}{7!}\tau_{a,c_1\dots c_7}\Gamma^{c_1\dots c_7}\Gamma^d\hat{\varphi}^+_d. \end{aligned} \tag{F.21}$$

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