OPEN ACCESS
IOP Publishing
Journal of Physics A: Mathematical and Theoretical
J. Phys. A: Math. Theor. **55** (2022) 443003 (37pp)
https://doi.org/10.1088/1751-8121/ac93cf

Topical Review

Chapter 2: An invitation to color-kinematics duality and the double copy

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Received 12 May 2022, revised 1 September 2022
Accepted for publication 21 September 2022
Published 30 November 2022

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Abstract

Advances in scattering amplitudes have exposed previously-hidden color-kinematics and double-copy structures in theories ranging from gauge and gravity theories to effective field theories such as chiral perturbation theory and the Born–Infeld model. These novel structures both simplify higher-order calculations and pose tantalizing questions related to a unified framework underlying relativistic quantum theories. This introductory mini-review article invites further exploration of these topics. After a brief introduction to color-kinematics duality and the double copy as they emerge at tree and loop-level in gauge

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and gravity theories, we present two distinct examples: (1) an introduction to the web of double-copy-constructible theories, and (2) a discussion of the application of the double copy to calculation relevant to gravitational-wave physics.

Keywords: scattering amplitudes, gauge theory, gravity

(Some figures may appear in colour only in the online journal)

Contents

1. Introduction

Gauge and gravity theories share many formal similarities even though their physical properties are distinct. Three of the known forces are described by gauge theories and give interactions between elementary particles, while gravity is a much weaker force that shapes the macroscopic evolution of the universe and spacetime itself. Nevertheless, the double-copy framework for gravity, which we outline in this chapter, exploits a direct connection between these two classes of theories, remarkably obtaining gravity directly from gauge theory. This framework provides a fresh perspective on gravity and its connection to the other forces, as well as very effective tools in the context of perturbative computations for gravity. A more comprehensive review may be found in reference [1].

Modern ideas make it much easier to calculate scattering amplitudes in perturbative quantum gravity compared to using Feynman rules. When one considers complete gauge-invariant scattering amplitudes instead of individual Feynman diagrams, which are not gauge invariant, it becomes possible to identify nontrivial structures. The double copy and the associated duality between color and kinematics [2–4] are perhaps the most remarkable of these structures, telling us that flat-space gravity scattering amplitudes can be obtained directly from gauge-theory ones. Via the unitarity method [5–10] these same ideas can be carried to loop level. The double copy is central to our ability to carry out calculations to very high loop orders in supergravity theories in Minkowski vacua and a property of all supergravities whose amplitudes have been analyzed in detail. This leads to the natural question on whether all (super)gravity theories are double copies of suitably-chosen matter-coupled gauge theories. The double copy offers a possible unification of gauge and gravity theories in the sense of providing a framework where calculations in both theories can be carried out using the same building blocks, emphasizing that the two types of theories are part of the same over-arching structure. Beyond gauge and gravity theories, double-copy relations also provide a new perspective on quantum field theories, resulting in a web of theories, linked by the same underlying building blocks (see section 3 of this review and e.g. references [11–23]).

The double copy has it origins in string theory. In the 1980s, Kawai, Lewellen, and Tye (KLT) [2] realized that open- and closed-string tree-level amplitudes both share the same fundamental gauge-invariant kinematic building blocks. They showed that closed-string tree amplitudes could be written as a sum over products of pairs of open-string tree amplitudes. In the low-energy limit, this translates directly to relations between gauge and gravity field-theory amplitudes for any number of external particles [24]. The double copy is streamlined and systematized by the introduction of the duality between color and kinematics [3]. The duality effectively states that scattering amplitudes in gauge theories—and, more generally, in theories with some continuous internal symmetry algebra—can be rearranged so that kinematic building blocks obey the same generic algebraic relations as their color factors. Via the duality, not only can we constrain the kinematic dependence of each graph, but we can also convert gauge-theory scattering amplitudes to gravity ones. This is done through the simple replacement: color $\Rightarrow$ kinematics. Such constructions have been summarized by the heuristic statement 'gravity $\sim$ (gauge theory) $\times$ (gauge theory)'.

At tree level, proofs exist [25–30] that the duality and double copy hold. At loop level, less is known, but explicit constructions show that the duality between color and kinematics and the double copy hold for a wide class of examples [4, 31–51]. A natural question is whether the double copy carries over to classical solutions beyond scattering amplitudes, especially for gravity. Scattering amplitudes in flat space are gauge invariant and independent of coordinate choices, while generic classical solutions do depend on such choices, complicating the problem of relating gravity solutions to gauge-theory ones. Nevertheless, there has been substantial progress in unraveling both the underlying principles of color-kinematics duality [22, 27, 52–62] and finding explicit examples of classical solutions related by the double-copy property [63–90]. One of the most striking applications of the double copy beyond scattering amplitudes relates to gravitational-wave physics, as highlighted by references [70, 82, 91–95].

This short review is organized as follows. In section 2 we give an overview of color-kinematics duality and the associated double copy for the simplest case of pure gauge theory. Then in section 3 we summarize the status of the web of theories linked by double copy relations, for both gravitational and non-gravitational theories. Then in section 4 we describe the application of the double copy to the problem gravitational-wave physics. Some brief comments on the outlook are given in section 5.

2. Color/kinematics duality and the double copy

2.1. Basics of color/kinematics duality

The canonical example of a theory exhibiting color-kinematics duality is a gauge theory in which all fields are in the adjoint representation of the gauge group, as considered in the original paper [3]. In any such theory, the m-point tree-level amplitudes in D dimensions may be written as

$$\mathcal{A}_m^{\text{tree}} = g^{m-2} \sum_j \frac{c_j n_j}{\prod_{i_j} d_{i_j}}, \tag{1}$$

where the sum runs over the set of distinct m-point graphs with only three-point vertices. Contributions from any diagram with quartic or higher-point vertices can be assigned to these graphs simply by multiplying and dividing by appropriate missing propagators. The color factor c_j is obtained by dressing each vertex in graph j with the relevant group-theory structure constant, $\tilde{f}^{abc} = i\sqrt{2} f^{abc} = \text{Tr}([T^a, T^b]T^c)$, where the Hermitian generators of the gauge group T^a are normalized as $\text{Tr}(T^a T^b) = \delta^{ab}$. The kinematic numerators n_j depend on momenta, polarizations, and spinors, as one would obtain using Feynman rules. The factors $1/d_{i_j}$ are ordinary scalar Feynman propagators, where i_j runs over the propagators for diagram j. We denote the gauge-theory coupling constant as g.

The nontrivial insight is that the kinematic numerators can be made to obey the same algebraic relations as the color factors [1, 3, 4, 34]. For theories with only fields in the adjoint representation there are two generic properties. The first is that they obey Jacobi relations that are inherited from the Lie algebra structure. For example, for the diagrams in figure 1 the color factors obey

$$f^{a_1 a_2 b} f^{b a_3 a_4} + f^{a_1 a_4 b i} f^{b a_2 a_3} + f^{a_1 a_3 b} f^{b a_4 a_2} = 0. \tag{2}$$

Such Lie-algebra relations are directly tied to the gauge invariance of amplitudes. For each color Jacobi identity we then demand that there be a corresponding identity for the kinematic numerators,

$$c_i + c_j + c_k = 0 \qquad \Rightarrow \qquad n_i + n_j + n_k = 0, \tag{3}$$

where i, j, and k refer to three graphs which are identical except for one internal edge. A second property is that kinematic factors should have the same antisymmetry under twists of diagrams lines as color factors

$$c_{\bar{i}} = -c_i \qquad \Rightarrow \qquad n_{\bar{i}} = -n_i, \tag{4}$$

where the graph $\bar{i}$ is graph i with twisted lines. For example, the color factor of diagram 1 of figure 1 is antisymmetric under the swap of legs 1 and 2; we then require the corresponding kinematic numerator exhibits the same antisymmetry.

The algebraic properties of color factors in gauge-theory amplitudes have important implications for kinematic numerators in equation (1). Consider a gauge-theory amplitude where we shift the numerators,

$$n_i = n'_i - \Delta_i, \tag{5}$$

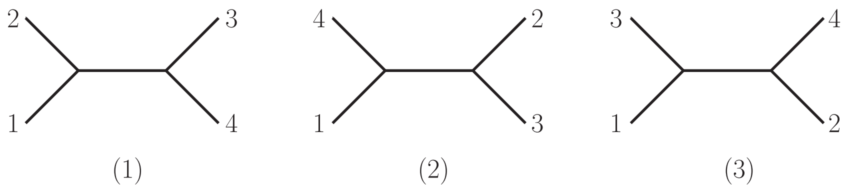

Figure 1. The three diagrams with cubic vertices describing a four-point tree amplitude.

subject to the constraint,

$$\sum_i \frac{c_i \Delta_i}{D_i} = 0, \tag{6}$$

the amplitude (1) is unchanged. Given that color factors are not independent but satisfy linear relations, nontrivial shifts of the kinematic numerators that leave the amplitudes invariant can be found. The Δ_i can be thought of as generalized gauge functions that drop out of the amplitude.

When we have numerators n_i that obey the same algebraic relations as the color factors c_i in equations (3) and (4), we can then replace

$$c_i \rightarrow n_i, \tag{7}$$

in any given formula or amplitude. Given that the algebraic properties of the kinematic numerators are the same as those of the color factors, the new amplitude that results will also satisfy a generalized gauge invariance. Remarkably, this color-to-kinematics replacement gives us gravity amplitudes,

$$\mathcal{M}_m^{\text{tree}} = i\left(\frac{\kappa}{2}\right)^{m-2} \sum_j \frac{\tilde{n}_j n_j}{D_j}, \tag{8}$$

where $\kappa^2 = 32\pi G$ with G Newton's constant, and where $\tilde{n}_j$ and n_j are the kinematic numerator factors of the two gauge-theory amplitudes. The two gauge theories can be different. Only one of the two sets of numerators needs to manifestly satisfy the duality (3) for the double-copy (8) to be gauge-invariant [4, 52].

Similar properties are conjectured to hold at loop level. Analogous to the tree level case (1), an L-loop m-point gauge theory scattering amplitude can then be organized as,

$$\mathcal{A}_m^{(L)} = i^{L-1} g^{m-2+2L} \sum_i \int \prod_{l=1}^{L} \frac{d^D \ell_l}{(2\pi)^D} \frac{1}{S_i} \frac{c_i n_i}{\prod_{i_j} d_{i_j}}, \tag{9}$$

where the sum runs over the distinct L-loop m-point diagrams with only cubic vertices. Each such diagram corresponds to a unique color factor c_i[7]. It also has an associated denominator corresponding to the product of the denominators of the Feynman propagators $\sim 1/d_{i_j}$ of each internal line of the diagram. A difference with tree level is that one needs to include symmetry factors S_i that remove internal overcount of loop diagrams; they can be computed, as for regular Feynman diagrams, by counting the number of discrete symmetries of each diagram with fixed

[7] Our conventions for the overall phase in the representations of gauge-theory and gravity amplitudes follow the one in reference [23] rather than the original Bern–Carrasco–Johansson (BCJ) papers [3, 4].

external legs. As for tree level, the representation of the amplitude in terms of cubic diagrams is trivial. The nontrivial part is to find representations of the amplitude where the duality holds so that the integrand kinematic numerators n_i satisfy the duality in equations (3) and (4). Whether this can be done in general at loop level remains a conjecture, although there is considerable evidence that such representations can be found [4, 31–51].

However, in certain cases, such as the five-loop four-point amplitude of $\mathcal{N}=4$ super-YM theory, such representations have been elusive. In other cases, such as the all-plus two-loop five-gluon amplitude in pure-YM theory, the BCJ form of the amplitude has a superficial power-count much worse than that of standard Feynman diagrams [43] leading to more complicated expressions.

Consider two m-point L-loop gauge theory amplitudes, $\mathcal{A}_m^{(L)}$ and $\widetilde{\mathcal{A}}_m^{(L)}$, and assume that they are organized as in equation (9). Furthermore, label the two sets of numerators for each amplitude n_i and $\tilde{n}_i$, respectively. If at least one of the amplitudes, say $\widetilde{\mathcal{A}}_m^{(L)}$, manifests the duality, we may now replace the color factors of the first amplitude with the duality-satisfying numerators $\tilde{n}_i$ of the second one. This gives the loop-level double-copy formula for gravitational scattering amplitudes [3, 4],

$$\mathcal{M}_m^{(L)} = \mathcal{A}_m^{(L)}\Big|_{\substack{c_i\to\tilde{n}_i \\ g\to\kappa/2}} = i^{L-1}\left(\frac{\kappa}{2}\right)^{m-2+2L}\sum_i \int \prod_{l=1}^{L} \frac{d^D\ell_l}{(2\pi)^D}\frac{1}{S_i}\frac{n_i\tilde{n}_i}{D_i}, \tag{10}$$

where the gravitational coupling $\kappa/2$ compensates for the change of engineering dimension when replacing color factors with kinematic numerators. The most challenging aspect of double-copy construction is finding a representation of the gauge-theory integrand that satisfies the duality in equations (3) and (4). For the replacement (7) to be valid under the integration symbol, it is important that the color factors not be explicitly evaluated by summing over the contracted indices. Under explicit evaluation it can turn out that certain color factors vanish, either by antisymmetry or by a special property of the group under consideration. We do not wish to impose any specific color-factor properties on the numerator factors, only generic ones.

Standard methods such as Feynman rules, on-shell recursion [96], or generalized unitarity [5, 6, 8, 9], generally do not naturally result in numerators obeying the duality. One straightforward (albeit somewhat tedious) way to find such numerators is to use an ansatz which is constrained to manifest the duality and to match the correct amplitude [32, 34]. Constructive ways to obtain numerators also exist [25–30, 97–99].

Aside from amplitudes, the duality has also been demonstrated to hold for currents with one off-shell leg [37, 45, 49, 50, 100–103]. A possible way to make the duality valid for general off-shell quantities would be to find a Lagrangian that generates Feynman rules whose diagrams automatically respect the duality. Such Lagrangians are known to a few orders in perturbation theory [52, 53, 104, 105]. An important problem is to find a useful closed form of such a Lagrangian valid to all orders.

2.2. Gauge-theory amplitude relations

The duality also implies that there are nontrivial relations between partial amplitudes, which are gauge invariant subdivisions of gauge theory scattering amplitudes. At tree level, with all particles in the adjoint representation of $SU(N_c)$, a full tree amplitude can be decomposed into partial amplitudes,

$$\mathcal{A}_n^{\text{tree}}(1,2,3,\ldots,n) = g^{n-2}\sum_{\text{noncyclic}} \text{Tr}[T^{a_1}T^{a_2}T^{a_3}\ldots T^{a_n}]A_n^{\text{tree}}(1,2,3,\ldots,n), \tag{11}$$

where A_n^{tree} is a tree-level color-ordered n-point partial amplitude. The sum is over all noncyclic permutations of legs, which is equivalent to all permutations keeping leg 1 fixed. Helicities and polarizations are suppressed. Reviews of such color decompositions are found in references [106–109].

The generalized gauge invariance (6) has an interesting consequence: it leads to nontrivial relations between gauge-theory partial amplitudes, known as BCJ amplitude relations,

$$\begin{aligned} s_{24}A_4^{\text{tree}}(1,2,4,3) &= s_{14}A_4^{\text{tree}}(1,2,3,4), \\ s_{24}A_5^{\text{tree}}(1,2,4,3,5) &= (s_{14}+s_{45})A_5^{\text{tree}}(1,2,3,4,5) + s_{14}A_5^{\text{tree}}(1,2,3,5,4), \\ s_{24}A_6^{\text{tree}}(1,2,4,3,5,6) &= (s_{14}+s_{46}+s_{45})A_6^{\text{tree}}(1,2,3,4,5,6) \\ &\quad + (s_{14}+s_{46})A_6^{\text{tree}}(1,2,3,5,4,6) + s_{14}A_6^{\text{tree}}(1,2,3,5,6,4), \end{aligned} \tag{12}$$

At tree level such relations exist for any number of external legs [3]. Progress at loop level has been more difficult, except for special kinematic configurations such the forward limit [23, 47, 110–113].

2.3. KLT formula and constructive tree-level adjoint color-kinematics duality

Double-copy relations have been known since 1985 in the form of Kawai–Lewellen–Tye relations [2]. We now review these relations from the vantage point of color-kinematics duality. At three points, the full color-dressed amplitude for Yang–Mills in D dimensions is simply given by

$$\mathcal{A}_3^{\text{tree}} = g\tilde{f}^{a_1a_2a_3}n_{123}, \tag{13}$$

where g is the gauge-theory coupling constant, $\tilde{f}^{a_1a_2a_3}$ is the suitably-normalized color structure constant for the gauge theory, and n_{123} is the on-shell Feynman three-vertex,

$$n_{123} = \sqrt{2}((\varepsilon_1\cdot\varepsilon_2)(k_2\cdot\varepsilon_3) + (\varepsilon_2\cdot\varepsilon_3)(k_3\cdot\varepsilon_1) - (\varepsilon_1\cdot\varepsilon_3)(k_3\cdot\varepsilon_2)). \tag{14}$$

The k_i and ε_j are the momenta and polarizations of the external legs. We can think of n_{123} as the kinematic numerators described above, although here there is no propagator denominators. It is straightforward to see that this is fully antisymmetric under exchange between any pair of leg labels. As this satisfies the duality between color and kinematics it can be immediately be used in the construction of a three-point gravitational amplitude,

$$-i\mathcal{M}_3^{\text{tree}} = \left(\frac{\kappa}{2}\right)n_{123}\,\tilde{n}_{123}, \tag{15}$$

where $\kappa/2$ is the gravitational coupling. Note that, in the case of three points, there is no gauge freedom. The n_{123} can be interpreted as gauge-theory ordered ('color stripped') amplitudes and we see the simplest example of the tree-level KLT relations between ordered gauge-theory amplitudes and tree-level gravitational amplitudes,

$$-i\mathcal{M}_3^{\text{tree}}(1,2,3) = \left(\frac{\kappa}{2}\right)A_3^{\text{tree}}(1,2,3)\widetilde{A}_3^{\text{tree}}(1,2,3). \tag{16}$$

The situation is more interesting at four-points. Here we have the freedom to arrive at different representations for each of the three distinct labellings n_s, n_t, n_u of the cubic graphs labeled by the Mandelstam invariant describing each graph's propagator, $s = (k_1+k_2)^2$, $t = (k_2+k_3)^2$, and $u = -s-t$. The four-point amplitude is simply

$$\mathcal{A}_4^{\text{tree}} = g^2\left(\frac{n_s c_s}{s} + \frac{n_t c_t}{t} + \frac{n_u c_u}{u}\right), \tag{17}$$

corresponding to $m = 4$ in equation (1).

We can decompose the amplitude (17) into color-ordered partial amplitudes using equation (11), which is expressed in terms of the kinematic numerators,

$$A_{st}^{\text{tree}} \equiv A_4^{\text{tree}}(1,2,3,4) = \frac{n_s}{s} - \frac{n_t}{t}, \tag{18}$$

$$A_{tu}^{\text{tree}} \equiv A_4^{\text{tree}}(1,3,2,4) = \frac{n_t}{t} - \frac{n_u}{u}, \tag{19}$$

$$A_{us}^{\text{tree}} \equiv A_4^{\text{tree}}(1,2,4,3) = \frac{n_u}{u} - \frac{n_s}{t}, \tag{20}$$

where the signs follow from antisymmetry of color factors. At first sight, it might seem that, with three kinematic numerators and three ordered amplitudes, we might be able to invert this set of linear relations to express the numerators in terms of amplitudes. However, since kinematic numerators satisfy $n_s + n_t + n_u = 0$, the matrix is singular and cannot be inverted. Reducing the linear relations, one finds that all of the ordered amplitudes are related by the BCJ relations described earlier in equation (12), which we can write in an equivalent permutation-invariant form as follows,

$$st\, A_{st} = ut\, A_{tu} = su\, A_{us}. \tag{21}$$

Using equation (18), we can solve n_u in terms of A_{st} and n_t,

$$n_u = \left(s + \frac{u}{t} n_t\right) A_{st}. \tag{22}$$

Remarkably, n_t cancels out when we substitute n_u into equation (18) and solve in terms of A_{st}. Indeed, plugging equation (22) and $n_s = -(n_t + n_u)$ into equation (18) simply produces the four-point BCJ amplitude relations (12), and doing the same to equation (17), yields the four-point amplitude in a basis of color factors,

$$\mathcal{A}_4^{\text{tree}} = g^2\left(c_s A_{st} + c_u \frac{s}{u} A_{st}\right) = g^2(c_s A_{st} + c_u A_{tu}). \tag{23}$$

By applying the above equations expressing the numerators in terms of A_{st} and n_t to the double copy in equation (8) with $m = 4$, we can thereby obtain the gravitational amplitude in terms of ordered gauge-theory amplitudes [3],

$$\begin{aligned} i\mathcal{M}_4^{\text{tree}} &= \left(\frac{\kappa}{2}\right)^2 \left(\frac{n_s \widetilde{n}_s}{s} + \frac{n_t \widetilde{n}_t}{t} + \frac{n_u \widetilde{n}_u}{u}\right) \\ &= \left(\frac{\kappa}{2}\right)^2 (st A_{st}) \left(st \widetilde{A}_{st}\right) (stu)^{-1} \\ &= \left(\frac{\kappa}{2}\right)^2 s A_{st} \widetilde{A}_{su}, \end{aligned} \tag{24}$$

where we used the BCJ amplitude relations (21) to obtain the final form. The relations (21) allow us to find many equivalent ways of expressing the four-point KLT relations. A similar exercise may be carried out at any multiplicity. Sample relations through six points are

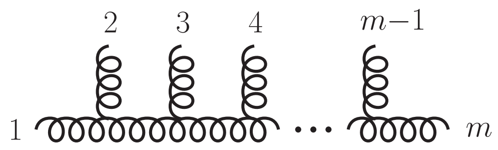

Figure 2. An m-point half-ladder tree diagram.

$$\begin{aligned}\mathcal{M}_5^{\text{tree}} &= i\left(\frac{\kappa}{2}\right)^3 \Big(s_{12}s_{45}A_5^{\text{tree}}(1,2,3,4,5)\widetilde{A}_5^{\text{tree}}(1,3,5,4,2) \\ &\quad + s_{14}s_{25}A_5^{\text{tree}}(1,4,3,2,5)\widetilde{A}_5^{\text{tree}}(1,3,5,2,4)\Big), \\ \mathcal{M}_6^{\text{tree}} &= -i\left(\frac{\kappa}{2}\right)^4 \Big(s_{12}s_{45}A_6^{\text{tree}}(1,2,3,4,5,6)\Big(s_{35}\widetilde{A}_6^{\text{tree}}(2,1,5,3,4,6) \\ &\quad + (s_{34}+s_{35})\widetilde{A}_6^{\text{tree}}(2,1,5,4,3,6)\Big) + \mathcal{P}(2,3,4)\Big),\end{aligned} \tag{25}$$

where $\mathcal{P}(i,j,k)$ represents a sum over all permutations of leg labels i, j, k. These are exactly the low-energy limit of the KLT relations [2].

These relations have an m-point generalization in terms of a basis of $(m-3)! \times (m-3)!$ ordered gauge amplitudes [24]:

$$\mathcal{M}_m^{\text{tree}} = -i\left(\frac{\kappa}{2}\right)^{m-2} \sum_{\sigma,\rho\in S_{m-3}(2,\ldots,m-2)} A_m^{\text{tree}}(1,\sigma,m-1,m)S[\sigma|\rho]\widetilde{A}_m^{\text{tree}}(1,\rho,m,m-1). \tag{26}$$

The formula makes use of a matrix $S[\sigma|\rho]$ known as the field-theory KLT or momentum kernel. This is an $(m-3)! \times (m-3)!$ matrix of kinematic polynomials that acts on the vector of $(m-3)!$ independent color-ordered amplitudes [2, 24, 25, 114, 115]:

$$S[\sigma|\rho] = \prod_{i=2}^{m-2}\left[2p_1\cdot p_{\sigma_i} + \sum_{j=2}^{i} 2p_{\sigma_i}\cdot p_{\sigma_j}\theta(\sigma_j,\sigma_i)_\rho\right], \tag{27}$$

where $\theta(\sigma_j,\sigma_i)_\rho = 1$ if σ_j is before σ_i in the permutation ρ, and zero otherwise. Compact recursive presentations of the KLT kernel have been found in references [25, 99].

There are a number of explicit constructions of the kinematic numerators that satisfy color-kinematics duality for arbitrary number of external particles. The first of these was based on matching to KLT relations [25, 116] and making use of the Del Duca–Dixon–Maltoni color-basis [117]. The result are kinematic numerators for the half-ladder (or multi-peripheral) diagrams, as depicted in figure 2, with the all remaining numerators determined by kinematic Jacobi relations. A valid specification for the half-ladder is given by the above KLT kernel,

$$\begin{aligned} n(1,\sigma(2,\ldots,m-2),m-1,m) &= -i\sum_{\rho\in S_{m-3}} S[\sigma|\rho]\widetilde{A}_m^{\text{tree}}(1,\rho,m,m-1), \\ n(1,\tau(2,\ldots,m-1),m)|_{\tau(m-1)\neq m-1} &= 0. \end{aligned} \tag{28}$$

All remaining $(2m-5)!! - (m-2)!$ numerators are determined by the Jacobi relations. Because the numerators are expressed in terms of amplitudes which are nonlocal, this representation has the disadvantage of resulting in nonlocal numerators. It also does not give manifestly

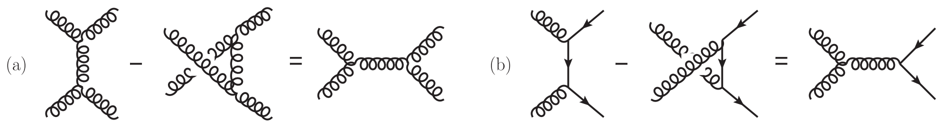

Figure 3. Graphical representation of the color-algebra relations in the adjoint (a) and some arbitrary representation (b). The curly lines represent adjoint representation states and the straight lines the arbitrary representation.

crossing-symmetric results, although the generated amplitudes do, of course, satisfy crossing. One can find crossing-symmetric kinematic numerators either by solving the Jacobi relations as functional constraints via an ansatz [118] or by appropriately symmetrizing equation (28), as in reference [119]. There are by now a number of efficient means of generating arbitrary multiplicity tree-level Yang–Mills color-dual numerators with varying degrees of manifest crossing symmetry, see e.g. references [61, 120, 121] and references therein.

2.4. Color-kinematics and double-copy construction beyond the adjoint representation

As discussed above, amplitudes with adjoint fields can manifest the duality between color and indeed lead naturally to supersymmetric theories [1, 122, 123]. What about matter fields in the fundamental or more general color representations? First we consider a gauge-theory with arbitrary gauge group and with matter particles—spin 0 or spin $\frac{1}{2}$—transforming in some matter representation of that gauge group. For simplicity, we will restrict to cases where the only color tensors appearing in amplitudes are $\tilde{f}^{abc}$ and $(T^a)_i{}^j$ which both have three free indices. Thus, all color factors can again correspond to cubic diagrams and with appropriate normalization satisfy the defining commutation relations,

$$\begin{aligned} \tilde{f}^{dae}\tilde{f}^{ebc} - \tilde{f}^{dbe}\tilde{f}^{eac} &= \tilde{f}^{abe}\tilde{f}^{ecd}, \\ (T^a)_i{}^k(T^b)_k{}^j - (T^b)_i{}^k(T^a)_k{}^j &= \tilde{f}^{abc}(T^c)_i{}^j, \end{aligned} \tag{29}$$

as depicted in figure 3. We find it convenient to introduce raised and lowered indices commonly associated with complex representations.

A difference with the pure-adjoint case is that edges of graphs now also encode the relevant representation, see e.g. figure 3. While important, many of the same ideas and approaches apply. We can still write m-point tree amplitudes in terms of cubic graphs,

$$\mathcal{A}^{\text{tree}}_{m,k} = -ig^{m-2}\sum_i \frac{c_i n_i}{D_i}, \tag{30}$$

where c_i are color factors, n_i are kinematic numerators, and D_i are denominators encoding the propagator structure of the cubic diagrams. The denominators (and numerators) may in principle contain masses, corresponding to massive propagators. The color factors c_i in equation (30) are constructed from the cubic diagrams using two building blocks: the structure constants $\tilde{f}^{abc}$ for three-gluon vertices and generators $(T^a)_i{}^j$ for quark-gluon vertices. When separating color from kinematics, the diagrammatic crossing symmetry only holds up to signs dependent on the permutation of legs. These signs are apparent in the total antisymmetry of $\tilde{f}^{abc}$. For a uniform treatment of the generic representations, it convenient to introduce a similar antisymmetry for the fundamental generators, artificially if necessary,

$$(T^a)^j{}_i \equiv -(T^a)_i{}^j \quad \Leftrightarrow \quad \tilde{f}^{cab} = -\tilde{f}^{bac}. \tag{31}$$

This allows us to introduce a compatible antisymmetry in color-ordered kinematic vertices, so that they are effectively the same as for the adjoint representation. As noted in equation (29) the color factors obey Jacobi and commutation identities. They both imply three-term color-algebraic relations of the form given in equation (3). The existence of algebraic relations between factors c_i means that the corresponding kinematic coefficients n_i/D_i need not be unique nor independently gauge-invariant.

As with the adjoint representation, one can still solve for all color factors in terms of a minimal basis exploiting relevant antisymmetry and Jacobi-like identities, equation (29). Using this basis in the full amplitude allows the identification of gauge-invariant ordered amplitudes as kinematic coefficients of the remaining color weights. These gauge-invariant ordered amplitudes will be related to each other by virtue of the fact that the kinematic weights n_i can be arranged in a color-dual fashion. A general color decomposition of tree-level amplitudes with matter representations may be found in reference [124] (see also references [125–127]). These ideas have been applied to massive scalar QCD at tree and loop level in reference [128] and to $\mathcal{N}=2$ super-QCD with N_f fermionic hypermultiplets in the fundamental through two-loops in references [129, 130]. Further discussions of massive theories are found in references [131–137].

Consider now generic single color traces and the types of algebraic structures that can describe them. Since every multiplicity could admit a symmetric term in front of each distinct color trace, we should admit symmetric color weights $d^{abc\ldots m}$. These can be understood as dressing vertices with m legs. So d^{abc} can dress cubic vertices like f^{abc}, d^{abcd} dress four-point vertices, and so on. The combination of various contractions of f^{abc} and permutation invariant d weights give rise to various algebraic structures which could have color-dual kinematic weights. These structures are rather rich, admitting rules that allow one to generate a given algebraic structure through functional composition. When an algebraic structure depends on scalar kinematics, this can admit a ladder where composition allows one to climb to higher dimension effective operators with a small number of primary building blocks without having to resort to an ansatz. At four and five points this has been shown to close, up to permutation invariants [138, 139]. Such compositional approaches have also been generalized to double-trace representations [140, 141]. Inverting the relationship between ordered amplitudes and these non-adjoint kinematic graph weights will induce distinct gauge-invariant double-copy relationships from the typical KLT formulation. When both copies can be organized into adjoint-type ordered-amplitudes satisfying KK and BCJ amplitude relations these differences can be pulled into higher-derivative corrections to a KLT-type mapping [139]. A general ansatz-based analysis of higher-derivative generalized KLT mappings has been carried out in reference [142] and its relationship to the compositional approach has been explored in reference [143].

Finally we point out the surprisingly generality of these ideas. Moving beyond the types of color structures typically found at tree- and loop-level, one can consider exotic three-dimensional color-dual Chern–Simons type theories [62]. The earliest example of such a color-dual theory is Bagger–Lambert–Gustavsson where reference [144] pointed out that, despite the color weights satisfying a three-algebra, color-dual gauge-theory numerators could be found. Fascinatingly, the amplitudes of this theory double copy to those of three-dimensional maximal supergravity theory, which can also be realized as the adjoint double-copy of the amplitudes of dimensionally-reduced maximally supersymmetric Yang-Mills theory—a point explored and clarified in references [145, 146]. Recently, topologically massive amplitudes have also been shown to be color-dual [133–135], evading consistency issues that can arise with massive gauge theories [132] that do not arise from consistent dimensional reduction of massless gauge theories [131].

3. A web of double-copy-constructible theories

Since their original formulation, color-kinematics duality and the double-copy construction have been applied to a diverse array of theories. First, they played a fundamental role in enhancing our understanding of maximal supergravity, particularly in relation to its UV behavior. From the beginning, it has also been clear that the double copy can be applied to theories that can be interpreted as consistent truncations of maximal supergravity (in some cases, with some subtleties related to the removal of undesired states, for which a variety of methods are now available [128, 147–149]). Many additional examples of double-copy-constructible theories have emerged, including non-gravitational theories, such as the Dirac–Born–Infeld (DBI) theory, and theories which presents structures that are far more involved than maximal supergravity, such as gauged supergravities. The double copy is now understood as a property of very large classes of theories, and possibly a generic feature of gravitational interactions. Seemingly-unrelated theories are now understood to share common building-blocks at the level of the underlying gauge theories entering their double-copy construction. We note that some instances of double copy connect string and superstring theories, giving a family of 'stringy' constructions. Similar programs, aiming at connecting different theories in a unified framework, have also been formulated in the contexts of the scattering-equations formalism [11], amplitude transmutation [12], and soft limits [150]. While we do not have the space to provide a comprehensive summary of all known instances of the double copy, here we aim at giving a broad overview of this web of theories, schematically portrayed in figure 4, as well as an illustration of how the new examples of double copy are connected to the original construction for maximal supergravity. For further details, we refer the reader to the more detailed review [1] and to the original literature.

3.1. Ungauged supergravities

In order to provide an overview of the available double-copy constructions, we first need to understand how to chart the space of possible gravitational theories. In the presence of supersymmetry, this is a problem that has long been studied by the supergravity community [151]. Supergravity theories can be divided into ungauged theories, Yang–Mills–Einstein (YME) theories, and gauged supergravities. The former are theories in which no field is charged under any gauge group. $\mathcal{N} = 8$ (ungauged) supergravity [152] belongs to this group (although several gauged versions are available), together with half-maximal supergravity [153, 154]. These are among the simplest examples of double-copy-constructible theories. While ungauged supergravities with $\mathcal{N} > 4$ and two-derivative actions are unique, $\mathcal{N} = 3, 4$ two-derivative supergravities are fully specified by a single parameter—the number of matter vector multiplets. If we further reduce the number of supercharges, we are in a situation in which not only different kinds of matter multiplets become possible, but additional information about their interactions is needed to fully specify the theory. This freedom is reflected by the fact that, while extended $\mathcal{N} > 2$ supersymmetry permits only a discrete set of symmetric scalar manifolds, $\mathcal{N} \leqslant 2$ supersymmetry is not as constraining. In four dimensions, supergravities with vector multiplets possess special-Kähler scalar manifolds, while the geometry is quaternionic-Kähler in the case of supergravities with hypermultiplets [151]. From the double-copy perspective, a particularly important class of theories is given by $\mathcal{N} = 2$ Maxwell–Einstein theories that can be uplifted to five dimensions. These theories are fully specified by their vector couplings in

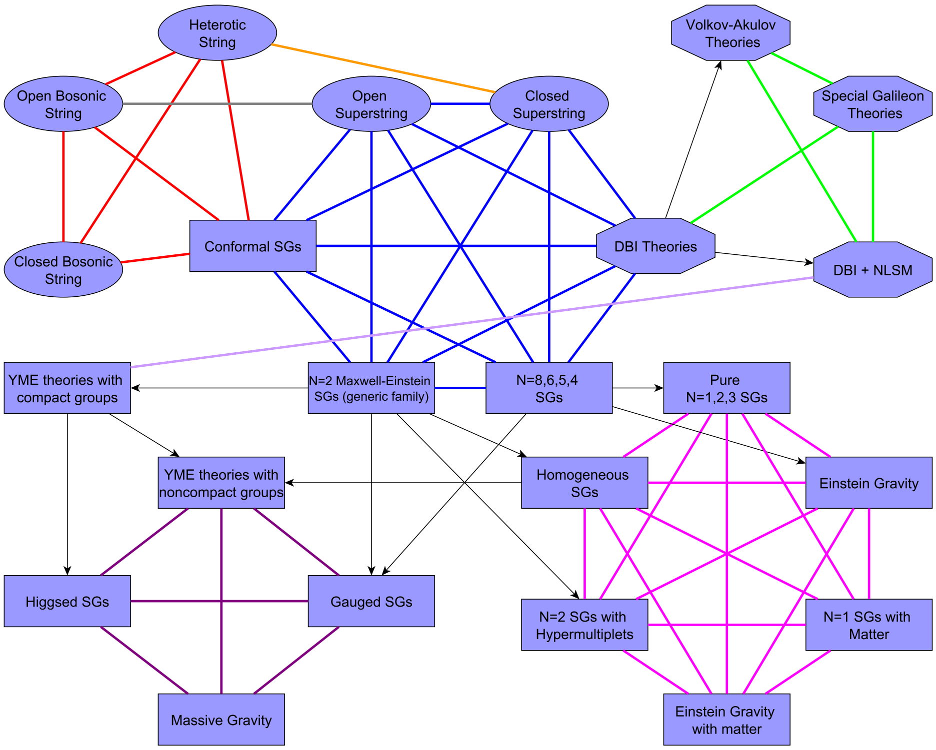

Figure 4. Web of double-copy-constructible theories. Undirected links with different colors are drawn between theories that have a common gauge-theory factor. For example, blue: pure SYM theory, red: $(DF)^2$ theory, green: NLSM, pink: (S)YM theory with massless matter, violet: spontaneously broken (S)YM theory. Directed links point toward double-copy constructions that are obtained by modifying both gauge-theory factors. Examples include: adding matter representations, assigning VEVs or truncating/projecting out some states.

five dimensions, i.e. their five-dimensional action includes a term of the form

$$\frac{1}{6\sqrt{6}} C_{IJK} \int F^I \wedge F^J \wedge A^K, \tag{32}$$

where the indices I, J, K run over the vector fields of the theory and C_{IJK} is a constant symmetric tensor. A fundamental result in supergravity states that the supergravity Lagrangian at the two-derivative level can be fully determined once the C_{IJK} tensors are given [155, 156]. In other words, this class of theories is fully specified by three-point interactions, and hence constitutes a very convenient arena for applying amplitude methods.

Double-copy constructions with $\mathcal{N} > 4$ are unique; the gauge theory factors are two super-Yang–Mills (SYM) theories with different amounts of supersymmetry [3, 4, 157–159],

$$\mathcal{N} = (N_1 + N_2)\,\text{supergravity}: \quad (\mathcal{N} = N_1\ \text{SYM}) \otimes (\mathcal{N} = N_2\ \text{SYM}).$$

When we consider $\mathcal{N}=4$ supergravity, the simplest double copy construction has one free parameter: the number of adjoint scalars in the non-supersymmetric gauge theory [3, 4, 157–159],

$$\mathcal{N}=4\,\text{supergravity}: \quad (\mathcal{N}=4\,\text{SYM})\otimes(\text{YM}+n_s\,\text{scalars}).$$

In turn, this becomes the number of vector multiplets in the outcome of the double copy. Color-kinematics duality demands that the couplings between the extra scalars be such that the theory can be regarded as the dimensional reduction of a higher-dimensional pure-YM theory. Further reducing supersymmetry, the simplest double copy for $\mathcal{N}=2$ supergravity is of the form [160]

$$\mathcal{N}=2\,\text{supergravity (generic family)}: \quad (\mathcal{N}=2\,\text{SYM})\otimes(\text{YM}+n_s\,\text{scalars}).$$

This is the double-copy construction for an infinite family of $\mathcal{N}=2$ Maxwell–Einstein theories that admit a five dimensional uplift and is known in the literature as the generic family or generic Jordan family. However, this is only one possibility and additional variants of the construction have been formulated. A very important generalization is given by adding matter (half) hypermultiplets to the supersymmetric theory, and matter fermions to the non-supersymmetric theory [161],

$$\begin{pmatrix}\mathcal{N}=2\,\text{homogeneous}\\ \text{supergravity}\end{pmatrix}: \quad \begin{pmatrix}\mathcal{N}=2\,\text{SYM}\\ +\dfrac{1}{2}\,\text{hyper}_R\end{pmatrix}\otimes\begin{pmatrix}\text{YM}+n_s\text{scalars}\\ +n_f\,\text{fermions}_R\end{pmatrix}. \qquad (33)$$

For technical reasons, the matter representation is taken to be pseudo-real, which makes it possible to introduce a single half-hypermultiplet in the supersymmetric theory. Since we are in the presence of more than one type of gauge-group representation, we need to generalize color-kinematics duality beyond the purely-adjoint case, as we have already seen in section 2.4. This is done according to the following rule:

'*Numerator factors in a CK-duality-satisfying presentation of a gauge-theory amplitude obey the same algebraic relations as the color factors. This includes those relations which stem from Jacobi identities or commutation relations of gauge group generators, as well as additional relations that are required by gauge invariance'.*

Additionally, we need to decide how different representations are combined by the double copy. To this end, we can use a simple and elegant working rule:

'*Each state in the double-copy (gravitational) theory corresponds to a gauge-invariant bilinear of gauge-theory states*'.

For this to be possible, we identify the gauge groups of the two theories entering the construction. Concretely, this rule implies that a supergravity field is obtained by combining two adjoint or two matter gauge-theory fields, but no supergravity field can originate from the double copy of one adjoint and one matter field, since this combination cannot form a gauge singlet. Because of this rule, the double copy (33) does not contain any additional gravitino multiplets, and the contribution of the extra matter fields simply yields additional vector multiplets. Furthermore, the number of matter fermions n_f is constrained by the requirement that the gauge theory should be seen as a higher-dimensional YM theory with fermions. This requirement is a consequence of color-kinematics duality, and the reader is referred to reference [161] for the full analysis. Taking this constraint into account, we have a two-parameter family of double copies which perfectly matches the classification of $\mathcal{N}=2$ Maxwell–Einstein supergravities with homogeneous scalar manifolds that has been obtained in the supergravity literature [178].

There are many more examples of double-copy constructions giving ungauged supergravities. A particularly important one is the construction for Einstein gravity. Simple counting of states shows that the double copy of two pure YM theories yields additional states beyond those of the graviton (in four dimensions, an additional complex scalar corresponding to a dilaton and an axion). An interesting way to remove the unwanted states is to introduce matter fermions in one of the two YM theories and matter ghost fields in the other [147]. These fields only double copy with each other in accordance to the rule given before. Only amplitudes with external gravitons are considered so that matter fields and ghosts appear only in loops. Reference [147] shows that the loop contributions coming from ghost fields are precisely what is needed to cancel the contribution of the unwanted axion-dilaton degrees of freedom, resulting in a double-copy construction for pure Einstein gravity. One can also use physical-state projectors to remove the unwanted states, as done in, for example, reference [95].

Given its role in the constructions outlined in this section, the reader may wonder whether SYM theory is the only purely-adjoint theory that obeys color-kinematics duality. It turns out that there is another theory with this property, which also appears in several double-copy constructions. This is the so-called $(DF)^2$ theory, which, in its simplest incarnation, is a higher-derivative version of the YM theory with a mass parameter m. It has Lagrangian

$$\mathcal{L}_{(DF)^2+\mathrm{YM}} = \frac{1}{2}(D_\mu F^{a\mu\nu})^2 - \frac{1}{4}m^2(F^a_{\mu\nu})^2. \tag{34}$$

This minimal version of the $(DF)^2$ theory enters the double-copy construction for a mass deformation of conformal supergravity,

$$(\mathrm{Mass-deformed\,minimal\,CSG}) = (\mathrm{SYM}) \otimes \left(\mathrm{minimal}(DF)^2 + \mathrm{YM}\right). \tag{35}$$

The above construction gives amplitudes in a mass-deformed minimal $\mathcal{N} = 4$ theory that interpolates between $(\mathrm{Weyl})^2$ and a Ricci-scalar term. Supersymmetry can be reduced by modifying the first gauge-theory factor. Additionally, this $(DF)^2$ theory has also a non-minimal version, containing an F^3 term together with further ghost scalars transforming in a specific matter representation. In table 1, we summarize double-copy constructions giving ungauged gravitational theories, and include references to the original literature.

3.2. Yang–Mills–Einstein and gauged supergravities with Minkowski vacua

YME theories and gauged supergravities are supergravity theories that contain gauge interactions under which some of the fields are charged. The YME theories are obtained by promoting a non-abelian subgroup of the global isometry group of a Maxwell–Einstein supergravity to a local symmetry (without touching the R symmetry and without introducing additional fields). In contrast, the defining property of gauged supergravities is that part of the R symmetry is promoted to gauge symmetry. These theories are considerably more involved than their YME relatives, exhibiting, among other things, non-trivial potentials, spontaneously-broken supersymmetry and massive gravitini. The reader interested in the relevant supergravity literature may consult references [151, 179]. Amplitudes in YME theories have been intensely investigated with a variety of methods: scattering equations [11, 180, 181], collinear limits [182], on-shell recursion [183, 184], string theory [185, 186] and ambitwistor strings [187]. From the point of view of the double-copy construction [160], non-abelian gauge interactions in the double-copy theory are generated by introducing a trilinear coupling among the adjoint scalar fields in the non-supersymmetric gauge-theory factor. These coupling are written as

Table 1. Non-exhaustive list of ungauged double-copy-constructible gravitational theories presented in the literature with references. Theories are specified in four dimensions (with the exception of the last entry).

Gravity	Gauge theories	References
$\mathcal{N} > 4$ supergravity	• $\mathcal{N} = 4$ SYM theory • SYM theory ($\mathcal{N} = 1, 2, 4$)	[3, 4, 157–159]
$\mathcal{N} = 4$ supergravity with vector multiplets	• $\mathcal{N} = 4$ SYM theory • YM-scalar theory from dimensional reduction	[3, 4, 157, 162]
$\mathcal{N} = 4$ supergravity with vector multiplets	• $\mathcal{N} = 2$ SYM theory with hypermultiplets • $\mathcal{N} = 2$ SYM theory with hypermultiplets	[3, 4, 157, 162]
Pure $\mathcal{N} < 4$ supergravity	• (S)YM theory with matter in fundamental rep. • (S)YM theory with ghosts in fundamental rep.	[147]
Einstein gravity	• YM theory with matter in fundamental rep. • YM theory with ghosts in fundamental rep.	[147]
$\mathcal{N} = 2$ Maxwell–Einstein supergravities (generic family)	• $\mathcal{N} = 2$ SYM theory • YM-scalar theory from dimensional reduction	[160]
$\mathcal{N} = 2$ Maxwell–Einstein supergravities (magical/homogeneous theories)	• $\mathcal{N} = 2$ SYM theory with half hypermultiplet in pseudoreal representation • YM-scalar theory from dimensional reduction with matter fermions in pseudo-real representation	[161, 163]
$\mathcal{N} = 2$ supergravities with hypermultiplets	• $\mathcal{N} = 2$ SYM theory with half hypermultiplet • YM-scalar theory from dimensional reduction with extra matter scalars	[161, 164]
$\mathcal{N} = 2$ supergravities with vector/ hypermultiplets	• $\mathcal{N} = 1$ SYM theory with chiral multiplets • $\mathcal{N} = 1$ SYM theory with chiral multiplets	[165–167]

(*continued on next page*)

Table 1. Continued.

Gravity	Gauge theories	References
$\mathcal{N} = 1$ supergravities with vector multiplets (truncations of generic family)	• $\mathcal{N} = 1$ SYM theory • YM-scalar theory from dimensional red	[160]
$\mathcal{N} = 1$ supergravities with vector multiplets (truncations of homogeneous theories)	• $\mathcal{N} = 1$ SYM theory with chiral multiplets in fundamental representation • YM-scalar theory with fermions in fundamental representation	[147, 165–167]
$\mathcal{N} = 1$ supergravities with chiral multiplets	• $\mathcal{N} = 1$ SYM theory with chiral multiplets in fundamental representation • YM-scalar with extra scalars in fundamental rep.	[147, 165–167]
Einstein gravity with massless matter	• YM theory with matter • YM theory with matter	[3, 147]
Einstein gravity with massive scalars	• Massive scalar QCD • Massive scalar QCD	[128, 168]
Heavy-mass effective theory	• Heavy-quark effective theory • Heavy-quark effective theory	[169, 170]
Einstein gravity with higher-derivative corrections	• YM theory with higher-derivative corrections • YM theory with higher-derivative corrections	[138, 142, 171]
Massive gravity/Kaluza–Klein gravity	• Spontaneously-broken YM theory • Spontaneously-broken YM theory	[131, 132, 172, 173]
$\mathcal{N} \leqslant 4$ conformal (super)gravity	• DF^2 theory • (S)YM theory	[174–176]
3D maximal supergravity	• BLG theory • BLG theory	[144, 146, 177]

Table 2. Gauged/YME gravities and supergravities for which a double-copy construction is presently known.

Gravity	Gauge theories	References
Unbroken $\mathcal{N} \leqslant 4$ YEM supergravities	• SYM theory • YM-scalar theory with trilinear scalar couplings	[160, 180, 188] [11, 12, 181, 183–187]
Higgsed $\mathcal{N} \leqslant 4$ YEM supergravities	• SYM theory on the Coulomb branch • YM-scalar theory with trilinear scalar couplings and extra massive scalars	[131]
$\mathcal{N} = 2$ YEM supergravities (non-compact gauge groups)	• $\mathcal{N} = 2$ SYM theory on the Coulomb branch with massive hypers • YM-scalar theory with trilinear scalar couplings and massive fermions	[189]
$U(1)_R$ gauged supergravities (with Minkowski vacua)	• SYM theory on Coulomb branch • YM theory with SUSY broken by fermion masses	[190]
Non-abelian gauged supergravities (with Minkowski vacua)	• SYM theory on the Coulomb branch • YM-scalar theory with trilinear scalar couplings and massive fermions	[191]

$$\delta\mathcal{L} = \frac{\lambda}{6!} F^{IJK}\ \mathrm{Tr}[\phi^I, \phi^J]\phi^K, \tag{36}$$

where F^{IJK} is an antisymmetric tensor with indices running over the number of scalars in the theory. The effect of these couplings is to introduce non-zero supergravity amplitudes between three vectors which are proportional to the F^{IJK} tensors. In turn, imposing color/kinematics duality on amplitudes between four scalars is equivalent to requiring that these tensors obey Jacobi relations, and hence can be thought of as the structure constants of the supergravity gauge group. This is an example of a global symmetry in a gauge-theory factor being promoted to a local symmetry by the double copy, analogous to the relation between global and local supersymmetry. The net result is a double-copy of the form [160]

$$(\text{YME supergravity}): \quad (\text{SYM theory}) \otimes \left(\text{YM} + \phi^3 \text{ theory}\right), \tag{37}$$

where, in case of $\mathcal{N} = 2$, the $\lambda \to 0$ limit will yield a theory belonging to the generic family. YME theories with spontaneously-broken gauge groups can also be constructed by taking the SYM gauge theory on its Coulomb branch and introducing extra massive scalars in the non-supersymmetric theory while making sure that color-kinematics duality is preserved [131].

Table 3. Non-gravitational local field theories constructed as double copies.

Double copy	Starting theories	References
$\mathcal{N} \leqslant 4$ DBI theory	• NLSM • (S)YM theory	[11–16, 22]
Volkov–Akulov theory	• NLSM • SYM theory (only fermions as external states)	[11, 192–195] [17, 19, 20]
Special Galileon theory	• NLSM • NLSM	[11, 12, 21] [17, 22]
$\mathcal{N} \leqslant 4$ DBI + (S)YM theory	• NLSM + ϕ^3 • (S)YM theory	[11–18, 23]
DBI + NLSM theory	• NLSM • YM + ϕ^3 theory	[11–16, 23]
3D $\mathcal{N} = 8$ DBI theory	• 3D $\mathcal{N} = 4$ Chern-Simons-matter theory • 3D $\mathcal{N} = 4$ Chern-Simons-matter theory	[62]

Gauged supergravities, even those admitting Minkowski vacua, are considerably more involved. Their double-copy construction can be thought of as a generalization of the construction for YME theories in which a spontaneously-broken YM theory is combined with a theory in which supersymmetry is broken by explicit fermionic masses[8]. As in the construction for YME theories, the appearance of trilinear scalar couplings results in non-abelian interactions in the supergravity theory, but now the F-tensors are also related to the fermionic masses by color-kinematics duality. The study of gauged supergravities in the double-copy language is still in its infancy, and the reader should consult references [190, 191] for additional details. The presently-known double-copy constructions for YEM theories and gauged supergravities are listed in table 2. Various theories without a graviton, most prominently variants of the DBI theory, have also been shown to admit such construction (see table 3 for an overview).

3.3. Stringy double copies

An important family of double-copy constructions applies to string-theory amplitudes. In this case, a fundamental ingredient is given by a set of disk integrals with punctures [196, 197],

$$Z_\sigma(\rho(1,\ldots,n)) = (2\alpha')^{n-3} \int\limits_{\sigma\{-\infty\leqslant z_1\leqslant\cdots\leqslant z_n\leqslant\infty\}} \frac{\mathrm{d}z_1\ldots\mathrm{d}z_n}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{R}))} \times \frac{\prod_{i<j}^n |z_{ij}|^{\alpha' s_{ij}}}{\rho\{z_{12}z_{23}\ldots z_{n-1,n}z_{n,1}\}}. \tag{38}$$

We use the short-hand notation $z_{ij} = z_i - z_j$, and we take care of the vol(SL$(2,\mathbb{R})$) factor by fixing three punctures as $z_i, z_j, z_k \to (0, 1, \infty)$ while introducing a Jacobian $|z_{ij}z_{ik}z_{jk}|$. The above integrals depend explicitly on two permutations $\sigma, \rho \in S_n$. They are known to satisfy

[8] The double-copy description of gauged supergravities in non-Minkowski vacua is an open problem.

Table 4. Double-copy constructions of tree-level string amplitudes with external massless states [208]. The single-valued projection sv(•) converts the disk integrals (38) to sphere integrals (42).

String $\otimes$ QFT	SYM	$(DF)^2 + \text{YM}$	$(DF)^2 + \text{YM} + \phi^3$
$Z-$ theory	Open superstring	Open bosonic string	Compactified open bosonic string
sv(open superstring)	Closed superstring	Heterotic(gravity)	Heterotic(gauge/gravity)
sv(open bosonic string)	Heterotic(gravity)	Closed bosonic string	Compactified closed bosonic string

[197] field-theory BCJ relations to all multiplicity with respect to the permutation ρ,

$$\sum_{j=2}^{n-1}(p_1\cdot p_{23\ldots j})Z_\sigma(2,3,\ldots,j,1,j+1,\ldots,n)=0, \tag{39}$$

and the so-called string-theory monodromy relations [198, 199] with respect to σ,

$$\sum_{j=1}^{n-1}\mathrm{e}^{2\mathrm{i}\pi\alpha' p_1\cdot p_{23\ldots j}}Z_{(2,3,\ldots,j,1,j+1,\ldots,n)}(\rho)=0. \tag{40}$$

Having introduced the appropriate building blocks, the open-superstring amplitudes with color-ordered massless external states can be expressed as the double copy of the Z integrals with Yang–Mills scattering amplitudes [196, 197],

$$A_{\text{OS}}^{\text{tree}}(\sigma(1,2,3,\ldots,n))=\sum_{\tau,\rho\in S_{n-3}(2,\ldots,n-2)}Z_\sigma(1,\tau,n,n-1)S[\tau|\rho]A_{\text{SYM}}(1,\rho,n-1,n), \tag{41}$$

where the field-theory KLT kernel $S[\tau|\rho]$ has been introduced in equation (27).

The Z integrals have been interpreted as the amplitudes of a scalar theory dubbed Z-theory in references [18, 99, 102]. While it is surprising that the field-theory version of the KLT kernel appears here, this may be understood from the fact that the decomposition is in terms of SYM amplitudes that obey field-theory BCJ relations. It is remarkable that in the superstring all the α' dependence is contained in the Z theory. A closed-string version of the Z-theory integrals, known to also satisfy field-theory relations to all multiplicity, is given by the following integrals on the punctured Riemann sphere [200–203],

$$\text{sv}\,Z(\tau|\sigma)=\left(\frac{2\alpha'}{\pi}\right)^{n-3}\int\frac{\mathrm{d}^2z_1\ldots\mathrm{d}^2z_n}{\text{vol}(\text{SL}(2,\mathbb{C}))}\frac{\prod_{i<j}^n|z_{ij}|^{2\alpha' s_{ij}}}{\tau\{\bar z_{12}\bar z_{23}\ldots\bar z_{n-1,n}\bar z_{n,1}\}\sigma\{z_{12}z_{23}\ldots z_{n-1,n}z_{n,1}\}}. \tag{42}$$

The notation sv$Z$ refers to the so-called single-valued projection of multiple zeta values (see references [204, 205] for details), but for us it will be simply part of the name of the building blocks we are introducing. Using these integrals, closed-superstring amplitudes are schematically given as [206, 207]

$$(\text{Closed superstring})=(\text{SYM})\otimes\text{sv}(\text{open superstring}). \tag{43}$$

The known stringy double copies are summarized in table 4. Note that the $(DF)^2$ theory we have introduced in the beginning of this section appears (in its non-minimal version) in several of the entries.

Each column in table 4 corresponds to the computation of one type of correlator. The SYM column is derived for any number of external massless states [196]. The $(DF)^2 + \text{YM}$ and the $(DF)^2 + \text{YM} + \phi^3$ columns have been explicitly checked against string amplitudes through five points, and all-multiplicity arguments were also given in reference [208]. While the discussion here focuses on tree-level amplitudes, some extensions to loop level are available in the literature [209–221]. See also references [222, 223] for a construction of string amplitudes in terms of field-theory amplitudes using the scattering-equations formalism.

4. From amplitudes to gravitational waves through the double copy

Previous sections have outlined a new approach to a wide class—perhaps even all—gravitational theories, in which they are obtained from simpler gauge theories. Applied beyond scattering amplitudes, similar procedures have been shown to relate certain classes of solutions of Einstein's equations to solutions of Maxwell's[9] equations with sources, a simple example of which is the Schwarzschild solution [64]. Since this method has been used for nontrivial calculations of supergravity ultraviolet properties up to five loops (see e.g. references [159, 162, 224–226]), it is logical to suspect that it can be useful to also advance the state of the art in gravitational-wave physics based on Einstein's general relativity by carrying out calculations that are difficult through standard methods. A good choice is high orders of two-body classical gravitational dynamics, given that it feeds into the analysis of gravitational-wave signals from the LIGO/Virgo collaborations [227] and is of interest to LIGO theorists [228].

Scattering amplitudes and associated methods enter the picture through the observation that, up to a point, scattering and bound-state motion are governed by the same equations of motion and Hamiltonian. Thus one may find the Hamiltonian from a scattering analysis and use it subsequently for analyzing bound-state motion[10]. The double copy enters very directly, because of its natural use in scattering processes. This strategy of effectively integrating out gravitons carrying momenta responsible for long-range interactions yields a two-body Hamiltonian, and can in principle be extended to the construction of n-body Hamiltonians. Such Hamiltonians can be interpreted as generating functions of classical observables.

To this end, we model the various classical bodies as point-particles, with or without spin depending on whether or not the classical bodies are spinning. This is a reasonable approximation if they are sufficiently far apart and may be systematically corrected to account for finite-size effects [230]. We begin by reviewing the kinematics, scale hierarchies, power counting, and truncation of graph structures that allow us to identify and remove the quantum contributions at the integrand level. Because of the macroscopic nature of the scattering bodies, it will turn out that loop-level amplitudes contain classical physics. The methods reviewed below lead to simplifications which we will also illustrate and are important for success at high loop orders.

4.1. Matter and graviton kinematics and the classical limit

There are several ways to extract classical physics from quantum field theory and more specifically from scattering amplitudes. We will use the correspondence principle—that is that

[9] These solutions can be thought of as being embedded in Yang–Mills solutions, by giving a nontrivial profile to the vector corresponding to a single generator of the gauge group.

[10] This philosophy requires care and possible modifications at $\mathcal{O}(G^4)$, where the Hamiltonian depends on the trajectory through the so-called tail effect [229], so a scattering-based Hamiltonian cannot be directly applied to bound-state problems.

classical physics emerges from the quantum physics in the limit of large masses and charges. Chief among them is the angular momentum: to extract the classical part of a four-point elastic amplitude we must therefore select a kinematic configuration in which the angular momentum is large in natural ($\hbar = 1$) units [91, 93, 95]. It is not difficult to see that this implies the more intuitive picture that classical physics governs processes in which the minimal inter-particle separation is much larger than the de Broglie wavelength, λ, of each particle. Indeed,

$$J \sim |\boldsymbol{b} \times \boldsymbol{p}| \gg 1 \qquad \Rightarrow \qquad |\boldsymbol{b}| \gg \lambda = \frac{1}{|\boldsymbol{p}|}. \tag{44}$$

For a scattering process we may take the impact parameter $|\boldsymbol{b}|$ as a measure of the minimal separation, while for a bound state we may take it to be the periastron or the average radius for quasi-circular orbits.

Since the impact parameter is of order of the inverse momentum transfer in a scattering process, $|\boldsymbol{b}| \sim 1/|\boldsymbol{q}|$, the classical limit implies the kinematic hierarchy[11]

$$m_1, m_2, |\boldsymbol{p}| \sim J|\boldsymbol{q}| \gg |\boldsymbol{q}|. \tag{45}$$

Classical and quantum contributions to scattering processes enter at different orders in an expansion in large J, or equivalently, in small $|\boldsymbol{q}|$. For example, since Newton's potential is classical, it follows that in the limit equation (45) any generating function of classical observables (e.g. the effective potential, the eikonal, the radial action, etc) for scalar bodies has the general form

$$V = \frac{G}{|\boldsymbol{q}|^2} c_1(\boldsymbol{p}) + \frac{1}{|\boldsymbol{q}|^3} \sum_{n \geqslant 2} (G|\boldsymbol{q}|)^n (\ln \boldsymbol{q}^2)^{n \bmod 2} c_n(\boldsymbol{p}). \tag{46}$$

For spinning bodies this expression is augmented with a dependence on scalars constructed from an equal numbers of the transferred momentum vector $\boldsymbol{q}$ and the rest frame spin $\boldsymbol{S}/m$ [231]. Quantum corrections can be systematically included by keeping terms with suitably subleading $\boldsymbol{q}$ counting.

We note that a small momentum transfer, as in equation (45), is not in contradiction with the observation that motion on a closed orbit required a change in momentum of a particle that is comparable with its initial momentum. Indeed, such long-term classical processes compound a large number of elementary two-particle interactions mediated by graviton exchange. Each such interaction transfers a momentum $|\boldsymbol{q}|$ compatible with equation (45) while the complete classical process transfers a momentum commensurate with $|\boldsymbol{p}|$. In the case of scattering, this is concretely described by the exponentiation of graviton exchange in e.g. the eikonal approximation [232]. In any case, once a potential and Hamiltonian are constructed to reproduce the scattering amplitude, they can be applied more generally to classical physics.

Having reviewed the overall kinematics of a scattering process that captures its classical limit, we proceed to detail the kinematics of the exchanged gravitons. This identifies the parts of loop amplitudes that contribute in the classical limit, thus allowing us to discard from the outset the parts that have no classical contributions. The main observation is that, in the classical regime in which the total momentum transferred $\boldsymbol{q}$ is small compared to external momenta, the momentum of each individual graviton should be of the same order. To identify the relevant

[11] This hierarchy implies that our results should not be expected to be valid for massless particles. Indeed, one can see that the classical and massless limits do not commute [95].

contributions we consider an internal graviton line with four-momentum $\ell = (\omega, \boldsymbol{\ell})$ and, following the method of regions [233, 234], we consider the possible scalings of its momentum components:

$$\begin{aligned}
\text{Hard}: &\quad (\omega, \boldsymbol{\ell}) \sim (m, m), \\
\text{Soft}: &\quad (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}|, |\boldsymbol{q}|) \sim J^{-1}(m|\boldsymbol{v}|, m|\boldsymbol{v}|), \\
\text{Potential}: &\quad (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}|) \sim J^{-1}(m|\boldsymbol{v}|^2, m|\boldsymbol{v}|), \\
\text{Radiation}: &\quad (\omega, \boldsymbol{\ell}) \sim (|\boldsymbol{q}||\boldsymbol{v}|, |\boldsymbol{q}||\boldsymbol{v}|) \sim J^{-1}(m|\boldsymbol{v}|^2, m|\boldsymbol{v}|^2),
\end{aligned} \tag{47}$$

where we take as reference scale $m = m_1 + m_2$ (or the external momentum), and we use equation (45) to arrive at the second set of scalings in the above equation. Gravitons with hard $\mathcal{O}(m) = \mathcal{O}(|\boldsymbol{p}|)$ momenta lead to quantum-mechanical contributions because their energy component is too large, causing the matter fields to fluctuate far off shell. Gravitons in the soft region mediate long-range interactions, because $|\ell| \sim |\boldsymbol{q}| \sim |\boldsymbol{b}|^{-1}$, so they can contribute to a classical potential. We use the velocity $0 \leqslant |\boldsymbol{v}| \ll 1$ to separate the soft region into potential and radiation regions. For small velocities, the modes in the potential region are off shell and carry little energy so they mediate interactions that are almost instantaneous, which is the hallmark of a classical potential. The gravitons in the radiation region can be on shell so they can be emitted in a scattering process. They can also be reabsorbed by the system and contribute to its effective potential. This is the origin of the so-called tail effect [229]. The modes in equation (47) identify the dominant contribution from each region to generic loop integrals. Each of them is computed by expanding each loop momentum about the given scaling and then integrating over the *full* phase space using dimensional regularization. To reconstruct the complete integral one simply sum over all the regions. The apparent overcount stemming from the integration over all momenta after expansion in each region is only superficial: expanding momenta in one region about another leads to scaleless integrals which vanish in dimensional regularization. For further detail on the method of regions we refer the reader to reference [234].

The above considerations, together with the observation that graviton loops are scaleless and thus vanish in the potential region, imply that the contributions of potential-region gravitons to the classical potential (46) have the following features:

(a) In all contributing diagrams, before and after reduction to a basis, the two matter lines do not intersect.
(b) Contributions where both ends of a graviton propagator attach to the same matter line are dropped.
(c) Every independent loop has at least one matter line.
(d) Terms with too high a scaling in q or ℓ are dropped because they are quantum contributions. Equation (46) implies that at L loops a for a given diagram with n_m matter propagators, n_g graviton propagators we can drop terms with more than $n_m + 2n_g - 3L - 2$ powers of loop momentum in the numerator.

The first two of these features imply that the parts of an L-loop amplitude that are relevant in the classical limit are strictly a subset of the product of two two-scalar-$(L+1)$-graviton tree amplitudes summed over the graviton states, together with scalar propagators for each of the gravitons. For example, at one loop this is a product of two gravitational Compton amplitudes summed over the graviton states and divided by $q_1^2 q_2^2$ where q_i are the momenta of the two gravitons. This is shown graphically in the left-most diagram in figure 5.

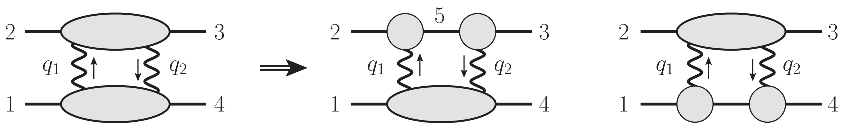

Figure 5. (Left-hand side) The part of the four-scalar one-loop amplitude that does not contain intersecting matter lines. (Right-hand side) An identification of the part of the four-scalar one-loop amplitude that do not contain intersecting matter lines and have at least one matter line in the loop. Factorization of tree- and loop-level amplitude imply that the shaded blobs are tree-level amplitudes.

The third property of the contributions to the classical limit weeds out part of the contributions appearing in the product of the two tree amplitudes and keeps only those for which each independent loop has at least one matter line before and after reduction to an integral basis. One starts with the terms having this property and in the process of reducing to a basis of integrals keeps only those contributions that continue to have this property. At one-loop level, the first step is shown on the right-hand side of the arrow in figure 5: there must be a matter line in at least one of the two Compton amplitude factors.

While identified here from the perspective of the classical limit, the contributions obtained this way have a natural interpretation in the generalized unitarity method [5, 6, 8–10, 108, 235, 236], where they are referred to as generalized cuts. The cut conditions—that is on-shell conditions for the exposed lines—prevent those lines from being canceled in the process of reduction to an integral basis. It is important to note that the cut momenta are on shell only for the amplitudes represented by the blobs; the propagators for the exposed lines are not placed on shell in this procedure.

Factorization of tree amplitudes implies that the contributions given by generalized cuts are expressed in terms of sums of products of tree amplitudes; thus, one can directly apply the KLT relations to obtain them in terms of amplitudes of scalars coupled to vector fields and thus essentially use the KLT relations to obtain higher-loop amplitudes. As an example, the expression of the first cut on the right-hand side of figure 5 is

$$\begin{aligned} C^{(\mathrm{a})}_{\mathrm{GR}} &= \sum_{h_1,h_2} M_3^{\mathrm{tree}}(3^s, q_2^{h_2}, -5^s) M_3^{\mathrm{tree}}(5^s, -q_1^{h_1}, 2^s) M_4^{\mathrm{tree}}(1^s, q_1^{-h_1}, -q_2^{-h_1}, 4^s) \\ &= \sum_{\lambda_1,\lambda_2,\tilde\lambda_1,\tilde\lambda_2} it\mathcal{P}_{h_2}\mathcal{P}_{h_2}\Big[A_3^{\mathrm{tree}}(3^s, q_2^{\lambda_2}, -5^s) A_3^{\mathrm{tree}}(5^s, -q_1^{\lambda_1}, 2^s) \\ &\quad \times A_4^{\mathrm{tree}}(1^s, q_1^{-\lambda_1}, -q_2^{-\lambda_2}, 4^s)\Big]\Big[A_3^{\mathrm{tree}}(3^s, q_2^{\tilde\lambda_2}, -5^s) A_3^{\mathrm{tree}}(5^s, -q_1^{\tilde\lambda_1}, 2^s) \\ &\quad \times A_4^{\mathrm{tree}}(4^s, q_1^{-\tilde\lambda_1}, -q_2^{-\tilde\lambda_2}, 1^s)\Big], \end{aligned} \tag{48}$$

where $h_{1,2}$ label the physical states of the graviton, $\lambda_{1,2}$ and $\tilde\lambda_{1,2}$ label the physical states of the corresponding gluons, $\mathcal{P}_{h_1,h_2}$ are projectors restricting the product of gluon states to be a graviton state (i.e. they project out the dilaton and the antisymmetric tensor) and we use the four-point BCJ amplitude relation [3] to simplify the expression. Thus, the gravity generalized cut is expressed directly in terms of the components of gauge-theory generalized cuts. In four dimensions, where physical states are labeled by their helicity, the projectors $\mathcal{P}_{h_1,h_2}$ simply correlate the helicities of the gluons, $\lambda_1 = \tilde\lambda_1$ and $\lambda_2 = \tilde\lambda_2$, and the gravity cut is expressed in terms of the four helicity configurations of the gauge-theory cut. In this way, through use of the double copy, the basic building blocks are non-abelian gauge-theory tree amplitudes.

The generalized unitarity method also provides an algorithm for assembling the various contributions obeying the properties described above while ensuring that terms that appear in several generalized cuts are counted only once. For reviews of the generalized unitarity method see references [108, 235, 236] and in the context of the classical limit of scattering amplitudes see reference [95].

4.2. Classical potential and classical observables from classical amplitudes

Assuming that amplitudes evaluated in the classical limit are known, the next task is to find a generating function of classical observables whose form is (46). This generating function is understood as part of the Wilsonian-type effective action generated by integrating out graviton configurations that contribute to conservative physics [237]. They may be potential-region gravitons [91, 93, 95, 238] or a mixture of potential and radiation region gravitons [237]. Constructing amplitudes from this effective action reveals that they exhibit classical parts, which scale in the large angular momentum limit as described in the previous section, and also 'superclassical' parts, which dominate in the large angular momentum limit over the classical ones. Thus, the task is to consistently separate the classical part. Several methods have been proposed in this direction and we briefly summarize them here in no particular order.

(a) Construct an effective two-body potential [91, 93, 95], which is then used in Hamilton's equations to generate classical observables. If the Hamiltonian is independent of the classical trajectory, as it is the case for the potential-graviton contributions, a change in boundary conditions suffices to relate open trajectory and bound orbit motion.

The effective two-body potential is obtained through by a matching calculation in which one demands [91] that the scattering amplitudes of gravitationally-coupled scalars due to potential or mixed but time-symmetric gravitons are reproduced by an action containing only the positive-energy modes of the matter fields and with instantaneous (or energy- or time-independent) interactions

$$H = A^\dagger\left(i\partial_t + \sqrt{\boldsymbol{p}^2 + m_1^2}\right)A + B^\dagger\left(i\partial_t + \sqrt{\boldsymbol{p}^2 + m_1^2}\right)B + V(\boldsymbol{p})A^\dagger A B^\dagger B, \tag{49}$$

with V in equation (46). The amplitudes following from this action are matched order by order in Newton's constant with those of the GR coupled to scalar fields of masses m_1 and m_2; at each order one more coefficient of V is determined: tree-level matching fixes $c_1(\boldsymbol{p})$, one-loop matching fixes $c_2(\boldsymbol{p})$, etc. At a loop order L, with stronger-than-classical scaling at large angular momenta are completely determined by the Hamiltonian coefficients determined through $(L-1)$-loop order. For this reason they contain no new information and are referred to as 'iteration terms'.

We note that this effective potential can be systematically extended to include quantum effects, see e.g. reference [239]; to this end one systematically keeps in the full-theory amplitude the desired quantum-suppressed terms. In particular, one may include terms subleading in the large angular momentum expansion such as graviton loops which would probe quantum gravity effects but one should not include diagrams with intersecting matter lines, as they do not contribute to long-range interactions.

(b) Other amplitudes-based approaches construct a generating function of open-orbit observables—the radial action—directly from amplitudes or evaluate open-orbit observables in terms of matrix elements of operators in the final state of the process.

The relation between the all-orders amplitude and the radial action builds on the observation that the solution to the unitarity constraint for an elastic two-particle S matrix is a phase. Inspired by the eikonal approximation [63, 240–244], the 'amplitude-radial action' relation is [237, 238]

$$i\mathcal{M}(\boldsymbol{q}) = \int_J \left(\mathrm{e}^{\mathrm{i}I_r(J)} - 1\right), \quad \tilde{I}_r(\boldsymbol{q}) = 4E|\boldsymbol{p}| \int d^{D-2}\boldsymbol{b}\mu^{-2\epsilon}\, \mathrm{e}^{\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{b}I_r(J)},$$
$$\tilde{I}_r(\boldsymbol{q}) = \frac{G}{|\boldsymbol{q}|^2}a_1(\boldsymbol{p}) + \frac{1}{|\boldsymbol{q}|^3}\sum_{n\geqslant 2}(G|\boldsymbol{q}|)^n(\ln \boldsymbol{q}^2)^{n \bmod 2}a_n(\boldsymbol{p}), \tag{50}$$

where E is the total energy, $\boldsymbol{b}$ is the impact parameter and μ is the scale of dimensional regularization. Classical observables are subsequently constructed through thermodynamic-type relations (known for closed-orbit motion as the first law of binary mechanics [245]), e.g.

$$\mathrm{d}I_r = \frac{\theta}{2\pi}\mathrm{d}J + \tau dE + \sum_a \langle z_a\rangle \mathrm{d}m_a, \tag{51}$$

where θ is the scattering angle, τ is the time delay and $\langle z\rangle$ is the averaged redshift. This has been used to systematically bypass iterated contributions [246–248]. The formalism of reference [246] makes use of a heavy mass version of the double copy [170] to produce compact expression for the amplitude. We refer the reader to the various original references for details.

The Kosower, Maybee, O'Connell (KMOC) formalism [92] constructs observables directly from amplitudes and their cuts, dressed with the appropriate operators. They are computed as the difference between the expectation values of these operators in the final and initial states,

$$\Delta\mathcal{O} = \langle f|\mathcal{O}|f\rangle - \langle i|\mathcal{O}|i\rangle \tag{52}$$

and the final and initial states are related by the S-matrix operator,

$$|f\rangle = S|i\rangle. \tag{53}$$

For example, the scattering angle is obtained from the change in momentum of matter particles. This approach will be summarized in chapter 14 of this review [249].

To illustrate the methods let us now evaluate the $\mathcal{O}(G)$ and $\mathcal{O}(G^2)$ amplitudes in the classical limit and use them to find the effective potential and radial action.

4.3. 1PM

The tree-level amplitude of two distinct massive scalars in the classical limit due to graviton exchange is simple-enough to be obtained through a Feynman graph calculation. It can also be obtained as a double copy of two massive scalar amplitude due to gluon exchange. In this second approach it is necessary to project out the dilaton-axion which is part of the double copy of two vectors and couples to massive particles. This can be done while also focusing the long-range interactions captured by this amplitude by evaluating only the pole part of the amplitude[12],

[12] We take this approach because constructing the complete four-point amplitude through the double copy requires subtracting out the dilaton exchange, which is present when the external particles are massive.

$$
\begin{aligned}
i\mathcal{M}_4^{\text{tree}}(1,2,3,4)\Big|^{\text{long range}} &= \frac{i}{q^2}\sum_h \mathcal{M}_3^{\text{tree}}(1,4,q^h)\mathcal{M}_3^{\text{tree}}(2,3,-q^{-h}) \\
&= \left(\frac{\kappa}{2}\right)^2 \frac{i}{q^2}\sum_{\lambda,\tilde{\lambda}} \mathcal{P}_h A_3^{\text{tree}}(1,4,q^{\lambda})A_3^{\text{tree}}(2,3,-q^{-\lambda}) \\
&\quad \times A_3^{\text{tree}}(1,4,q^{\tilde{\lambda}})A_3^{\text{tree}}(2,3,-q^{-\tilde{\lambda}}),
\end{aligned}
\tag{54}
$$

where $\mathcal{M}_3^{\text{tree}}(i,j,q^h)$ are two-scalar-graviton amplitudes, h tags the physical states of the graviton and we used the double-copy form (16) of $\mathcal{M}_3^{\text{tree}}(i,j,q^h)$. Particles with momenta p_1 and p_4 have mass m_1, those with momenta p_2 and p_3 have mass m_2 and the sum runs over the physical states of the exchanged graviton. The sum over the physical polarizations of the graviton gives the physical-state projector,

$$
\sum_h \varepsilon(k)_h^{\mu\nu}\varepsilon(-k)_{-h}^{\alpha\beta} = \frac{1}{2}\mathcal{P}^{\mu\alpha}\mathcal{P}^{\nu\beta} + \frac{1}{2}\mathcal{P}^{\nu\alpha}\mathcal{P}^{\mu\beta} - \frac{1}{D-2}\mathcal{P}^{\mu\nu}\mathcal{P}^{\alpha\beta},
\tag{55}
$$

where

$$
\mathcal{P}^{\mu\nu}(k) = \eta^{\mu\nu} - \frac{r^\mu k^\nu + r^\nu k^\mu}{r\cdot k},
\tag{56}
$$

and r^μ is an arbitrary null reference vector. Gauge invariance of the three-point amplitudes (54) guarantees that the reference vector drops out, so we can effectively take $\mathcal{P}^{\mu\nu}(k)\to\eta^{\mu\nu}$ and the physical-state sum (55) to be the numerator of the graviton propagator in de Donder gauge.

The two three-point amplitudes can be obtained as double-copies of the two-scalar-gluon amplitudes, as in equation (15). They are

$$
-i\mathcal{M}(1,4,q^h) = \frac{\kappa}{2}\mathcal{P}_h(\sqrt{2}\varepsilon_\mu^\lambda(q)p_1^\mu)(\sqrt{2}\varepsilon_\nu^\lambda(q)p_1^\nu) = \frac{\kappa}{2}(2\varepsilon_{\mu\nu}^h(q)p_1^\mu p_1^\nu),
\tag{57}
$$

where, as before, h labels the physical states of the graviton and $\varepsilon_{\mu\nu}^h$ is transverse. This defines the operator $\mathcal{P}_h$ used in equation (54). Using this together with (55), equation (54) then becomes

$$
i\mathcal{M}^{\text{tree, class}} = -\frac{16\pi iGm_1^2m_2^2}{q^2}(2\sigma^2-1),
\tag{58}
$$

where $m=m_1+m_2$, $\nu=m_1m_2/(m_1+m_2)^2$, and $\sigma=p_1\cdot p_2/(m_1m_2)$. Accounting for the nonrelativistic normalization of the amplitudes following from the action (49), the resulting $\mathcal{O}(G)$ potential coefficient is

$$
c_1(\boldsymbol{p}) = \frac{M^4\nu^2}{E_1E_2}(1-2\sigma^2),
\tag{59}
$$

where $M=m_1+m_2$, $\nu=m_1m_2/M^2$ and $E_{1,2}=\sqrt{\boldsymbol{p}^2+m_{1,2}^2}$ are the energies of the two incoming particles.

Similarly, comparing equation (58) with equation (50) and using references [237, 238] it follows that the leading term of the radial action is

$$
a_1(\boldsymbol{p}) = 16\pi M^4\nu^2(2\sigma^2-1).
\tag{60}
$$

Fourier-transforming to impact-parameter space[13] leads, through equation (51), to the same scattering angle as the Hamiltonian.

4.4. 2PM

The next contribution to the potential comes from the four-scalar one-loop amplitude. As we discussed, the generalized unitarity method provides an algorithmic construction for this and higher-loop amplitudes, while simultaneously seamlessly singling out the parts exhibiting the features required of the classical limit and interfacing with the double copy to organize gravity calculations in terms of simpler gauge theory ones. The one-loop amplitude however is sufficiently simple so we can construct it without making use of the details of the general approach while still avoiding explicit use of Feynman diagrammatics.

As we discussed on general grounds in section 4.1, to focus on the parts of the amplitude that do not contain intersecting matter lines it suffices to set to zero in the numerator of all contributing diagrams the squared momenta of the gravitons connecting the two matter lines—momenta q_1 and q_2 on the left-hand side of figure 5. Up to the overall factor of the two graviton propagators, this is the residue of the one-loop amplitude corresponding to the pole $q_1^2 = 0 = q_2^2$, implying that

$$\mathcal{M}_4^{\text{one-loop}} = \frac{1}{q_1^2}\frac{1}{q_2^2}\sum_{h_1,h_2} \mathcal{M}_3^{\text{tree}}(1,4,q_1^{h_1},-q_2^{h_2})\mathcal{M}_3^{\text{tree}}(2,3,-q_1^{h_1},q_2^{h_2}) + \cdots, \tag{61}$$

where $\mathcal{M}^{\text{tree}}$ are gravitational Compton amplitudes and the ellipses represent terms that are not long-range classical. This avoids discussing the details of assembling the two finer contributions to the classical amplitude shown on the right-hand side of figure 5 since equation (61) automatically contains both. At higher loops however the most efficient strategy is to use the generalized unitarity method based on tree amplitudes with the fewest numbers of legs.

The two Compton amplitude factors follow from the double copy of the dimensional reduction of higher-dimensional four-gluon amplitude, with two gluons taken in the extra dimensions. The dilaton-axion scalar is projected out from the product of each pair of intermediate gluons so the remainder are only the physical states of two gravitons. The sum over each them gives the physical-state projector (55) used in the tree-level computation. A judicious choice of polarization-stripped amplitudes [250] leads to a manifest cancellation of the reference vector. Such choices, which can involve adding terms that vanish on-shell to allow amplitudes to manifestly obey Ward identities, have been shown to always be possible [149].

The Compton amplitude may also be obtained through the KLT relation, as in equation (24)[14]. In this case $\mathcal{M}^{1-\text{loop}}$ is written as

$$\begin{aligned}\mathcal{M}_4^{\text{one-loop}} = \frac{1}{q_1^2}\frac{1}{q_2^2}(q^2)^2 \sum_{\lambda_1,\lambda_2,\tilde{\lambda}_1,\tilde{\lambda}_2} & P_{h_1}P_{h_2}A_4^{\text{tree}}(1,4,q_1^{\lambda_1},-q_2^{\lambda_2})A_4^{\text{tree}}(2,3,-q_1^{\lambda_1},q_2^{\lambda_2}) \\ & \times A_4^{\text{tree}}(1,4,-q_2^{\tilde{\lambda}_2},q_1^{\tilde{\lambda}_1})A_4^{\text{tree}}(2,3,q_2^{\tilde{\lambda}_2},-q_1^{\tilde{\lambda}_1}) + \cdots,\end{aligned} \tag{62}$$

where P_{h_1} and P_{h_2} project out the dilaton and antisymmetric tensor from the product of two gluon states. The sum over the gluon states is given by equation (56) and together with P_{h_1}

[13] Note that this is a two-dimensional Fourier-transform, because the on-shell conditions on the external states constrain the momentum transfer q to be two-dimensional.

[14] Unlike equation (54), the dilaton contribution to the four-point tree amplitudes entering $\mathcal{M}^{\text{one-loop}}$ is projected out by simply choosing the external (cut) lines to be gravitons.

and P_{h_2} gives again equation (55). In four dimensions and in spinor-helicity notation this is straightforward [95]: one simply correlates the helicities of the gluons in the two amplitude factors, $(\lambda_1, \tilde{\lambda}_1), (\lambda_2, \tilde{\lambda}_2) \in \{(+,+), (-,-)\}$, so the scalar states $\{(+,-), (-,+)\}$ never appear in the product. Four-dimensional methods continue to produce correct results at $\mathcal{O}(G^3)$ [95]; at higher orders however more caution is necessary because of subtleties with dimensional regularization [237].

The result of either of these methods is then reduced to the standard one-loop basis of scalar box, triangle and bubble integrals; during the calculation we enforce the four requirements that weed out quantum contributions. Discarded contributions are diagrams with crossing matter lines and graviton loops and the only surviving ones are the box and the triangle integrals [91, 251]

$$\frac{i\mathcal{M}^{\text{one-loop}}}{64\pi^2 G^2 m_1 m_2} = 4m_1^3 m_2^3 (2\sigma^2 - 1)^2 (I_{\text{Box}} + I_{\text{XBox}}) - 3m_1 m_2 (5\sigma^2 - 1)(m_1^2 I_{\triangle} + m_2^2 I_{\bigtriangledown}) + \cdots, \tag{63}$$

where the ellipsis stand for terms that are not long-range or classical or both, and

$$I_{\text{Box}} = \int \frac{d^d\ell}{(2\pi)^2} \frac{1}{\ell^2(\ell+q)^2((\ell+p_1)^2 + m_1^2)((\ell-p_2)^2 + m_2^2)}. \tag{64}$$

I_{XBox} is obtained by interchanging p_2 and p_3 and $I_{\triangle}$ and $I_{\bigtriangledown}$ are obtained by removing one of the matter propagators with masses m_2 and m_1, respectively. While at this order integration is quite straightforward, it becomes less so at two loops and beyond; see chapter 4 of this review [252] for modern techniques and results.

Accounting for the nonrelativistic normalization of the amplitudes following from the action (49), the resulting $\mathcal{O}(G^2)$ potential coefficient is

$$c_2(\boldsymbol{p}) = \frac{M^5\nu^2}{E_1E_2}\left(\frac{3}{4}(1-5\sigma^2) - \frac{4M\nu E}{E_1E_2}\sigma(1-2\sigma) - \frac{M^3\nu^2 E}{2E_1^2E_2^2}\left(1 - \frac{E_1E_2}{E^2}\right)(1-2\sigma)^2\right), \tag{65}$$

where $E = E_1 + E_2$. One may recognize the first term in parenthesis as the coefficient of the triangle integrals in equation (63); the other two terms originate from the subtraction of the term with stronger-than-classical scaling present in the box integral.

Separating the iteration of the tree-level radial action (60) as in reference [238], leads to the $\mathcal{O}(G^2)$ term of the radial action is

$$a_2(\boldsymbol{p}) = 6\pi^2\nu^2 M^5 (5\sigma^2 - 1). \tag{66}$$

As at $\mathcal{O}(G)$, observables following from the radial action thus derived agree with those following from the two-body Hamiltonian.

4.5. Remarks and outlook

The methods summarized above have been used to derive the two-body potential and the radial action that capture the suitably-defined [237] conservative open-orbit dynamics through $\mathcal{O}(G^4)$. An essential ingredient in these calculations has been the double-copy form of tree-level gravity amplitudes in terms of gauge-theory amplitudes. Similarly, the KMOC formalism together with the double copy as a means for deriving the necessary amplitudes has been used to derive the impulse and energy loss through $\mathcal{O}(G^3)$ [253–255]. Further progress may build on double-copy constructions with gauge-invariant kinematic numerators [170, 246] obtained from recent developments in the kinematic algebra of gauge theories [60, 256]. Spin can also be incorporated into this framework [231, 257]. Here the double-copy properties are less obvious, though at least to quadratic order in the spins the gravitational Compton amplitudes have simple double-copy relations to gauge theory, and so does the tree-level energy momentum tensor for any power of spin [231]. A double copy for massive particles with spin including quantum effects was also discussed in reference [256].

5. Conclusions

In this mini-review, we summarized the status of color-kinematics duality and the associated double-copy construction, including the basics of color-kinematics duality, the web of theories linked by the double copy, and applications to gravitational-wave physics. In recent years there has been considerable interest in color-kinematics duality and the associated double copy, especially towards finding new classical solutions where the double copy holds (see e.g. references [63–90, 258, 259]), identifying supergravity theories admitting a double-copy construction (see tables 1 and 2), and applying the double copy to physical problems such as precision gravitational-wave computations (see e.g. references [93, 95, 237, 238, 246]). There has also been important progress on basic questions such as identifying the underlying kinematic algebra behind color-kinematics duality [22, 54, 56, 58, 60–62].

There are a number of obvious future directions which have attracted recent attention, seen exciting progress, and will be interesting to investigate further:

- Identifying new classes of classical solutions where the double copy holds, especially for cases that do no rely on the special properties of Kerr–Schild form of the metric [64–66, 75, 76]. More generally, it would be important to find rules for choosing good coordinates and gauges that make double-copy relations more transparent.
- Realizing generalizations of scattering amplitudes in (A)dS that manifest the duality between color and kinematics [260–265].
- Further understanding the underlying kinematic algebra behind the duality between color and kinematics. A natural expectation is that the kinematic Jacobi identities are due to an infinite-dimensional Lie algebra [266]. Finding a complete description of such an algebra remains an open challenge, albeit with recent growing attention and progress [21, 60–62, 121, 267, 268].
- Expanding the web of theories linked by double-copy relations described in section 3. This includes finding further non-gravitational examples beyond those listed in table 3 and understanding whether all supergravity theories are necessarily double copies.
- Carrying out new state-of-the-art computations of physical or theoretical interest. Recent examples are high-order calculations in gravitational wave physics [95, 237, 238]. The recent construction of the six-loop integrand of $\mathcal{N} = 4$ SYM theory [269] suggests that analogous progress is possible for $\mathcal{N} = 8$ supergravity, with a goal of obtaining the ultraviolet behavior.
- Identifying and developing novel directions. Recent examples include finding color-kinematics duality in a non-abelian version of Navier–Stokes equation of fluid mechanics [270], Chern–Simons theory [62], quantum entanglement [271] and field-space geometry [272, 273].
- Finding new connections between the double copy and other advances in scattering amplitudes, such as the amplituhedron [274, 275], integrated high-loop results for planar $\mathcal{N} = 4$ SYM theory (see e.g. reference [276]).

The duality between color and kinematics and the associated double-copy structure offer a novel perspective on gravity theories compared to more traditional geometric approaches. They were originally formulated for flat-space perturbative scattering amplitudes, where they offer insight and tools to address a variety of problems. Based on large numbers of known examples, the double copy applies much more generally, not only to classical solutions but also to a web of interlocked gravitational and nongravitational theories. The surprisingly large web of theories included in figure 4 suggests that (quantum) field theories have new nontrivial hidden constraints, as suggested by the fact that the number of building blocks is smaller than the number of consistent theories. In the coming years, it will be fascinating to find out the reach of these ideas both on the computational and theoretical sides.

Acknowledgments

We are particularly grateful to Oliver Schlotterer for detailed feedback on the manuscript. ZB is supported by the U.S. Department of Energy (DOE) under Grant No. DE-SC0009937 and by the Mani L. Bhaumik Institute for Theoretical Physics. JJMC is grateful for the support of Northwestern University, the DOE under Contract DE-SC0021485, and by the Alfred P. Sloan Foundation. RR is supported by the U.S. Department of Energy (DOE) under Grant No. DE-SC00019066. The work of MC is supported by the Swedish Research Council under Grant 2019-05283. The research of MC and HJ is also supported by the Knut and Alice Wallenberg Foundation under grants KAW 2018.0116 (*From Scattering Amplitudes to Gravitational Waves*) and KAW 2018.0162, and the Ragnar Söderberg Foundation (Swedish Foundations' Starting Grant).

Data availability statement

No new data were created or analysed in this study.

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