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Scattering Amplitudes in Supersymmetric Quantum Chromodynamics and Gravity

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Abstract

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Quantum field theory is a theoretical framework for the description of nature in terms of fundamental particles, fields and their interactions. In the quantum regime, elementary scattering processes are observables in many experiments and studied in theoretical physics. The theoretical understanding of scattering amplitudes is often based on a perturbative analysis in powers of the coupling strength of the fundamental forces. Whereas the computation of scattering amplitudes has been dominated by Feynman diagram constructions for a long time, new methods have lead to a multitude of novel results in the last 20-30 years. Thereafter discoveries of new representations, dualities and construction methods have enormously increased our understanding of the mathematical structure of scattering amplitudes.

In this thesis we focus on a particular structure of gauge theory amplitudes known as the *color-kinematics duality*. Closely tied to this duality is the *double copy* construction of gravitational amplitudes, and a set of identities among basic building blocks of the gauge theory, the *BCJ identities*. Using methods developed for the study of this duality, we obtain new results for scattering amplitudes in non-maximal supersymmetric Yang-Mills coupled to massless fundamental matter at one and two loops. We immediately construct amplitudes in supergravity theories via the double copy. Furthermore, we include methods and results for the integration of gauge theory amplitudes and the ultraviolet structure of supergravity amplitudes.

In a second part we present ideas related to the identification of basic building blocks that underlie the construction of scattering amplitudes. A decomposition of gauge theory amplitudes into color- and kinematic-dependent contributions exposes a set of primitive objects. Relations among these objects allow us to identify a minimal set of independent kinematic building blocks.

Keywords: Scattering Amplitudes, Supersymmetric Gauge Theory, Supergravity, Quantum Chromodynamics

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*To everyone who brought
some color into my life*

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I H. Johansson, G. Kälin and G. Mogull, *Two-loop supersymmetric QCD and half-maximal supergravity amplitudes*, JHEP **09** (2018) 019, arXiv:1706.09381 [hep-th].

II G. Kälin, *Cyclic Mario worlds – color-decomposition for one-loop QCD*, JHEP **04** (2018) 141, arXiv:1712.03539 [hep-th].

III G. Kälin, G. Mogull and A. Ochirov, *Two-loop $\mathcal{N}=2$ SQCD amplitudes with external matter from iterated cuts*, arXiv:1811.09604 [hep-th]. Submitted to JHEP.

IV C. Duhr, H. Johansson, G. Kälin, G. Mogull and B. Verbeek, *The Full-Color Two-Loop Four-Gluon Amplitude in $\mathcal{N}=2$ Super-QCD*, arXiv:1904.05299 [hep-th]. Submitted to PRL.

Contents

1. Introduction

The theoretical framework describing the most fundamental objects in nature and their interactions is known as quantum field theory (QFT). A central realization of a QFT is the standard model (SM) of particle physics, unifying three out of the four fundamental forces. It has been extremely successful in predicting the outcome of experiments and explaining natural phenomena in the quantum regime. A commonly studied type of observables are scattering amplitudes (or the S-matrix). They describe the outcome of scattering events involving two or more fundamental particles.

The last missing piece for the SM, the Higgs boson, has been experimentally confirmed in 2012 by a joint effort of experiments at the Large Hadron Collider (LHC) at CERN [1, 2]. In contrast, it is next to impossible to observe any effects due to a quantization of the fourth force: gravity. Quantum effects occur at small scales, which requires particle collisions at high energy to capture them. Due to the relatively weak strength of the gravitational interaction the required energy for experiments to uncover gravitational quantum effects is out of our reach.

From a theoretical point of view, the determination of many quantities in a QFT is plagued by infinities at high energies, called ultraviolet (UV) divergences. For the theory described by the SM, a gauge theory, these unphysical divergences can be systematically removed by counterterms. This allows the SM to do accurate predictions for observables. An equivalent mechanism for quantum gravity is not known as the procedure would require infinitely many counterterms leading to a formalism without any predictive power. A complete understanding of the underlying structure of UV divergences of extended models of gravity is lacking until today.

The perturbative study of scattering amplitudes in general gauge and gravity theories has seen an increased interest since the end of the last century. New computational techniques, methods and representations [3–15] have led to a better understanding of the mathematical structures encoding amplitudes. This led to many novel results for seemingly untractable problems. One of these structures was discovered in 2008 as a duality between color and kinematics [16]. It induces a novel representation of gravity amplitudes as a *double copy* of two gauge theory amplitudes [17]. A similar relation between open and closed string amplitudes was discovered by Kawai, Lewellen and Tye (KLT) [18] already in 1986. KLT relations are known at leading order of the perturbative expansion in the coupling, and at one-loop level in the field theory description [19,20]. The aforementioned color-kinematics duality and double copy by Bern, Carrasco, and Johansson (BCJ) has a conjectured realization for any order. The duality organizes the amplitude in terms of trivalent graphs, similar to Feynman diagrams, and splits the contribution for each graph into a color and a kinematic part. These two parts are related by the duality in the sense that they obey the same algebraic identities. The perturbative expansion is then realized order by order in the number of loops in the involved diagrams — or equivalently in powers of the coupling constant.

An extremely powerful consequence of the duality is the construction of gravity amplitudes via simpler gauge theory building blocks. The simplest case of a gravity theory in four spacetime dimensions is the maximally supersymmetric extension of Einstein gravity, $\mathcal{N} = 8$ supergravity (SG) [21–25]. Therein the double copy construction has resulted in an impressive computation of the UV behavior at five loops [26] in general spacetime dimension. Symmetry arguments exclude a divergence by the absence of a valid counterterm up to a critical dimension $D_c < 24/5$ [27,28] at five loops. The divergence has been confirmed — and the explicit expression in the critical dimension has been presented. Even though a particular UV divergence has been predicted to vanish in four dimensions by symmetry arguments [27–33], the explicit computation has revealed new patterns relating the divergence structure of different loop orders.

There are cases with a vanishing UV divergence, where a valid counterterm exists and cannot be ruled out by any known (symmetry) argument. This effect has been named *enhanced cancellation* [34–40]. The cancellations leading to a vanishing divergence are highly non-trivial from the point of view of currently known amplitude representations, i.e. it requires a conspiracy between contributions coming from different diagrammatic structures.

This thesis is centered around the BCJ construction and discusses various aspects thereof for lower degree or even in absence of supersymmetry. Supersymmetric theories form a clean testing ground for new methods and ideas and allow us to do explicit computations at perturbative orders that are not yet accessible for the SM or Einstein gravity. The basic building blocks are purely *kinematic* numerator factors associated to trivalent graphs of a gauge theory. These numerators — counterparts of the color objects through the duality — are far from being unique. They are fixed by the duality up to a residual generalized gauge transformation, which leaves room for many different representations of the same amplitude. The search for an underlying kinematic Lie algebra [41, 42] that would allow for a direct construction of such numerators has as of yet not been successful.

The main idea followed here is based upon an Ansatz construction for the building blocks of the gauge theory. Apart from the duality property of these numerators, a larger set of constraints manifesting symmetries or various other properties have been identified in publications enclosed in this thesis. The resulting representations of numerators are often conveniently expressed as a single trace over contractions of momenta with Dirac matrices. This form of the integrand is especially useful when it comes to integration or assembling of gravitational amplitudes via the double copy.

If the goal is to only compute amplitudes in a gauge theory, seeking a color-kinematics dual representation might seem unnecessary. However, it is clear from publication III that such a representation may be easy to obtain in some cases, and exposes novel properties. For example, the amplitudes obtained in publication I and III are manifestly local, and exhibit an infrared (IR) behavior that manifests collinear and soft limits. Most of the simplicity is coming from a reduced number of independent kinematic quantities and novel properties that fix the residual generalized gauge freedom.

Amplitude computations at higher loop orders become intractable to a brute-force approach even with modern computers. The complexity of amplitude computations tend to grow severely with each additional external parton or order in the loop expansion. For example the number of graphs in a Feynman computation grows factorially and integration may be intractable for even a simple scalar theory at higher orders. Some of the new ideas in article I and III arose from the need to cope with this growth of complexity.

A Feynman diagram, or a numerator in a BCJ construction, is not a measurable quantity and generically is gauge dependent. Color-ordered amplitudes are another type of building blocks, built out of a special set of diagrams using modified Feynman rules. They are gauge-independent objects that naturally appear after a so called *color decomposition*. The amplitude is split into purely color-dependent factors and kinematics-dependent objects — exactly these color-ordered amplitudes. For certain (planar) theories, color-ordered amplitudes may be recursively constructed, removing the dependence on a diagrammatic description entirely. Their number is still growing factorially but at a slower rate. For planar Yang-Mills theory, the so called Kleiss-Kuijf (KK) relations [43] — and corresponding loop-level generalizations — together with the color-kinematics duality (BCJ relations) reduce the number of independent color-ordered amplitudes.

Removing redundancies among the color-ordered amplitudes and the color factors leads to a *minimal* color decomposition. Different types of (minimal) color decompositions, for example for pure Yang-Mills by Del Duca, Dixon, and Maltoni (DDM) [44, 45] or for quantum chromodynamics (QCD) by Johansson and Ochirov [46], are well understood at tree level. At loop level much less is known and even the definition of primitive amplitudes for non-planar contributions is far from being understood. Publication II discusses a color decomposition at one-loop level for a general QCD amplitude for an arbitrary multiplicity of external partons.

The thesis is organized into four main parts. In part I, we start by discussing general background material and review the core subjects. It contains an introduction to computational tools like the *spinor helicity* formalism and *generalized unitarity*. We discuss the (supersymmetric) quantum field theories of interest and the color-kinematics duality together with the double copy and BCJ relations on a formal level.

The second part builds up the main discussion of the thesis. It contains a review of the general methods and explicit computations for amplitudes in (supersymmetric) gauge and gravity theories. The focus will lie on the construction of additional matter states on both sides of the double copy. Introducing fundamental matter allows us to reach a larger set of (super)gravity theories via an extension of the usual double copy prescription. A major part will focus on $\mathcal{N} = 2$ supersymmetric QCD (SQCD), a supersymmetric extension of QCD. This theory is interesting as it has features resembling the well-studied $\mathcal{N} = 4$ super Yang-Mills (SYM) theory as well as ordinary QCD. It is a first stepping stone introducing the complications of QCD into the simpler structures of maximally supersymmetric YM. We summarize the study of the integrated form of the two-loop $\mathcal{N} = 2$ SQCD amplitude and its transcendentality properties presented in article IV. Via the double copy of (S)QCD a plethora of different (super)gravity theories — with or without the inclusion of matter — can be reached. Most interestingly, we discuss general methods for the extraction of UV divergences, followed by an explicit computation thereof for half-maximal supergravity at two loops.

Part III focuses on color decompositions for QCD or supersymmetric extensions thereof. We review a decomposition at tree level and the results from article II, attached to this thesis, for the one-loop case.

The final part collects several technical methods and ideas that have been used for the various amplitude computations discussed in this thesis. A first chapter includes algorithms for the reduction of tensor integrals to basic scalar integrals. In a second chapter we present the use of finite field methods that can significantly speed up many computations arising in the broader field of scattering amplitudes.

We conclude with a short summary of the thesis and an outlook into future developments.

Part I: Background Material and Review

Or: *How to Stand on The Shoulder of Giants*

The perturbative computation of scattering amplitudes in quantum field theories has seen an immense progress due to a plethora of new methods developed during the last three decades. Not only has the traditional diagrammatic technique prescribed by Feynman been mostly superseded, but also our understanding of the structure of scattering amplitudes and their building blocks has greatly changed.

This first part of the thesis summarizes some basic tools like the spinor helicity formalism (sec. 2) and unitarity cuts (sec. 5). We furthermore review necessary background material mostly centered around the color-kinematics duality and the double copy (sec. 3). A more extensive discussion of many of these tools can for example be found in [47–49]. In section 4 we introduce an on-shell superfield formalism to conveniently describe supersymmetric gauge and gravity theories and their scattering amplitudes. As many of these tools are fairly standard we keep the discussions short and only introduce concepts insofar needed for the rest of the thesis.

2. Spinor Helicity Formalism

Kinematic building blocks appearing in the integrand of scattering amplitudes are Lorentz-invariant objects built out of external data. For us, the information consists of momenta and polarization vectors that are contracted with invariant objects — for example the metric $\eta^{\mu\nu}$ and the totally antisymmetric Levi-Civita symbol $\epsilon^{\mu\nu\rho\sigma}$. The external data for a physical process encoded inside an amplitude is constrained, for example, by on-shell criteria and momentum convervation. Such constraints in general lead to non-linear relations between certain building blocks. Hence, there exist many representations of the same integrand which differ significantly in structure and size. Now, the challenge is to identify building blocks and find algorithms that bring a given amplitudes into a "useful" form, where the definition of usefulness is intentionally left open. Some common criteria are the size of the algebraic expression, locality of the assembled amplitude, removal of redundancies (comparability) or manifestation of a certain (symmetry) property. The spinor helicity formalism discussed in this section is useful in the sense that it trivializes the on-shellness of massless particles.

All processes considered in this thesis include exclusively massless states. If p_i denotes the momentum of an external parton with label i the massless condition reads

$$p_i^2 = p_{i,\mu} p_i^{\mu} = 0. \tag{2.1}$$

Furthermore, the overall momentum is conserved:

$$\sum_i p_i = 0, \tag{2.2}$$

where we assume that all particles have outgoing momenta. In general, a solution to these constraints contains square rootse and leads to blown-up expressions once resubstituted into an amplitude. The spinor helicity formalism instead introduces a set of variables that trivializes the masslessness condition for the system. A short enlightning introduction to the four dimensional spinor helicity formalism can for example be found in [9].

2.1 Four Dimensions

Following the notation in [48], we first note that in four spacetime dimensions the Lorentz group is isomorphic to $\mathrm{SL}(2)_{\mathrm{L}} \times \mathrm{SL}(2)_{\mathrm{R}}$. The finite-dimensional representations of SL(2) are classified by half-integers. Objects transforming in the two-dimensional representation $(1/2,0)$ and $(0,1/2)$ are called left- and right-handed chiral spinors respectively. We denote left-handed chiral spinors by $|p]_a$, $a=1,2$, and right-handed chiral spinors by $|p\rangle^{\dot{a}}$, $\dot{a}=1,2$. We will later on identify p as an on-shell momentum. Lorentz invariant objects are formed using the antisymmetric tensor ϵ_{ab}, $\epsilon_{12}=+1$, which is used to raise or lower indices. A useful shorthand is

$$[12] \equiv [p_1|^a|p_2]_a = \epsilon^{ab}|p_1]_b|p_2]_a\,, \quad \langle 12\rangle \equiv \langle p_1|_{\dot{a}}|p_2\rangle^{\dot{a}} = \epsilon_{\dot{a}\dot{b}}|p_1\rangle^{\dot{b}}|p_2\rangle^{\dot{a}}\,. \tag{2.3}$$

Consider a four-momentum vector p_μ transforming under the vector $(1/2,1/2)$ representation of the Lorentz group. By the above isomorphism, it is possible to map this vector to an object with both types of chiral indices $p_{a\dot{a}}$. The explicit form of the map uses the Pauli matrices

$$p_{a\dot{a}} = \sigma^\mu_{a\dot{a}} p_\mu\,. \tag{2.4}$$

Coming back to the on-shellness redundancy from before we observe that for massless momenta p

$$p_\mu p^\mu = \det(p_{a\dot{a}}) = 0\,. \tag{2.5}$$

This implies that $p_{a\dot{a}}$ is a 2×2 matrix with non-maximal rank and can be written as

$$p_{a\dot{a}} = -|p]_a\langle p|_{\dot{a}}\,, \tag{2.6}$$

where $|p]_a$ and $\langle p|_{\dot{a}}$ are chiral spinors as introduced above. These two spinors are not uniquely determined since there is a rescaling freedom

$$|p] \to t|p], \qquad\qquad |p\rangle \to \frac{1}{t}|p\rangle\,, \tag{2.7}$$

with a non-zero complex number t. This corresponds to a little group scaling, i.e. a transformation in the subgroup of the Lorentz group that leaves the momentum invariant. Some useful properties and identities for spinor products are collected in the appendix of [48].

By a slight abuse of notation we can insert a Dirac γ-matrix between two spinors

$$\langle p|P|q] = \langle p|\gamma^\mu|q]P_\mu \equiv \begin{pmatrix}0, & \langle p|_{\dot{a}}\end{pmatrix} \begin{pmatrix} 0 & (\sigma^\mu)_{a\dot{a}} \\ (\bar{\sigma}^m u)^{a\dot{a}} & 0 \end{pmatrix} \begin{pmatrix} |q]_a \\ 0 \end{pmatrix} P_\mu\,, \tag{2.8}$$

where P is an arbitrary (not necessarily light-like) vector. Polarization vectors for massless spin-1 vectors can then be expressed in spinor helicity notation using an arbitrary reference spinor q

$$\varepsilon^\mu_-(p;q) = -\frac{\langle p|\gamma^\mu|q]}{\sqrt{2}[qp]}\,, \quad \varepsilon^\mu_+(p;q) = -\frac{\langle q|\gamma^\mu|p]}{\sqrt{2}\langle qp\rangle}\,. \tag{2.9}$$

It can easily be checked that these polarization vectors fulfill the necessary properties, i.e. they are null and orthogonal to the corresponding momentum p

$$\varepsilon_{\pm}(p;q)^2 = 0\,, \qquad\qquad \varepsilon^{\mu}_{\pm}(p;q)p_{\mu} = 0\,. \tag{2.10}$$

The non-localities introduced by polarization vectors will lead to seemingly unphysical poles in amplitudes discussed in later sections. The unitary cut formalism developed in publication III extracts these poles and absorbs them into special state-configuration objects.

Even though we discuss amplitudes in arbitrary dimensions — mainly for the purpose of dimensional regularization — we use four-dimensional notation and the spinor helicity formalism for the external objects. One can see this as an embedding in a four-dimensional subspace. To obtain information for the extra-dimensional part of dimensionality $D-4$, a six-dimensional spinor-helicity formalism introduced by Cheung and O'Connell [13] can be used. We will not discuss the six-dimensional computation in this thesis. Publications I and III contain an extended discussion for the use of six-dimensional spinor-helicity for unitarity cuts.

3. Color-Kinematics Duality and the Double Copy

The amplitude computations in article I and III use a construction introduced by Bern, Carrasco and Johansson (BCJ). The two main ideas — the *color-kinematics duality* for gauge theories and the *double copy construction* for gravity amplitudes [16, 17] — are summarized as:

- There exists a duality between the color algebra and kinematic objects, the numerator factors, of gauge theory amplitudes. A representation of a gauge theory amplitude that manifestly implements this duality is called *color-kinematics dual.*
- Certain gravity amplitudes can be obtained by a procedure which acts on a color-kinematics dual form of a gauge theory amplitude. The prescription is to replace color objects by their kinematic counterparts.

This construction has been used to obtain gravitational amplitudes from the simpler gauge theory expressions. Gravitational amplitudes are extremely difficult, if not impossible, to compute with a classical Feynman graph approach even with today's computing power. Once a color-dual form of the gauge theory amplitude has been found, its double copy necessarily is invariant under linearized diffeomorphism. Thus it is a valid candidate for a gravitational amplitude (of some gravity theory) [50]. There is no complete criterion for the existence of a color-dual form at general loop level. The existence has been proven for specific cases at tree level and even for some cases at higher orders [51–55]. The only way to check the duality is thus by finding an explicit color-kinematics dual representation.

Another open question concerns the set of (super)gravity theories that are constructible via a double copy. We call a gravity theory *factorizable* if its spectrum can be written as a tensor product of the full spectrum of two (not necessarily equal) gauge theories, schematically

$$(\text{gravity theory}) = (\text{gauge theory}) \otimes (\text{gauge theory})' . \tag{3.1}$$

Factorizable theories are not the only ones that are BCJ-constructible. In [46], Johansson and Ochirov developed an extension of the usual double copy construction by adding matter multiplets to the gauge theory. This allows for the construction of a larger set of gravity theories. Publication I uses this construction at two-loop level and verifies the existence of the duality for this specific case. In publication III we constructed gauge theory amplitudes with matter on external legs. This allows for matter-matter or matter-graviton scattering on the double copy side.

In this section we review the basics of the color-kinematics duality and the double copy as far needed for part II. The last subsection will also briefly introduce BCJ-relations, which motivate the story told in part III of this thesis. For an extensive review see for example [49].

3.1 A Duality between Color and Kinematics

To expose the duality, we start from a diagrammatic representation of a gauge theory amplitude. The color factor of a four-point vertex can be expressed as a sum of products of two three-point color-factors. Therefore, every L-loop amplitude admits a representation in terms of a sum over trivalent diagrams $\Gamma_n^{(L)}$ with n external legs

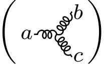
$$\mathcal{A}_n^{(L)} = i^{L-1} g^{n+2L-2} \sum_{i \in \Gamma_n^{(L)}} \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i c_i}{D_i} \,. \tag{3.2}$$

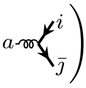
Each summand, corresponding to a graph, consists of four building blocks,

- the symmetry factor S_i of the graph,
- the denominator D_i built as a product of inverse Feynman propagators,
- the color factors c_i according to usual Yang-Mills Feynman rules,
- and numerators n_i capturing the leftover kinematical dependence of the amplitude. Note that these numerators are not necessarily local as they might contain spurious poles.

For the gauge theories considered in the thesis — massless QCD with gauge group $\mathrm{SU}(N_c)$ or supersymmetric extensions thereof — color factors of trivalent graphs are built out of two basic building blocks: the structure constants $\tilde{f}^{abc} = \mathrm{tr}([T^a, T^b]T^c)$ and generators $T^a_{i\bar{\jmath}}$ of the gauge group, normalized as $\mathrm{tr}(T^a T^b) = \delta^{ab}$. We use the convention that T^a are hermitian and f_{abc} are imaginary. Graphically, we represent them as

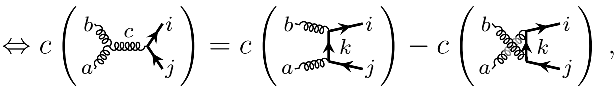
$$\tilde{f}^{abc} = c\left(\quad\right), \qquad T^a_{i\bar{\jmath}} = c\left(\quad\right). \tag{3.3}$$



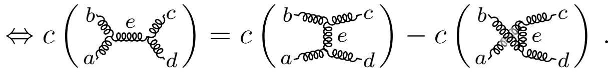


By defining $T^a_{\bar{\imath}j} \equiv -T^a_{j\bar{\imath}}$, we observe that both building blocks are antisymmetric under a flip of legs since also $\tilde{f}^{abc} = -\tilde{f}^{acb} = -\tilde{f}^{bac}$. Furthermore, properties of the gauge algebra include similar identities for these building blocks: Color factors of a triplet of four-point subgraphs fulfill the commutation relation and the Jacobi identity respectively:

$$\tilde{f}^{abc}T^{c}_{i\bar{j}} = T^{b}_{i\bar{k}}T^{a}_{k\bar{j}} - T^{a}_{i\bar{k}}T^{b}_{k\bar{j}} = [T^a, T^b]_{i\bar{j}}$$



$$\tilde{f}^{abe}\tilde{f}^{cde} = \tilde{f}^{dae}\tilde{f}^{bce} - \tilde{f}^{dbe}\tilde{f}^{ace} \tag{3.4}$$



Combined with the antisymmetry property, this leads to a system of relations among the color factors in eq. (3.2) of the schematic form

$$c_i = c_j - c_k\,, \qquad\qquad c_i = -c_j\,. \tag{3.5}$$

Whereas the symmetry factors, denominators, and color factors are uniquely determined by the graph, the numerator factors are not unique and admit a *generalized gauge freedom.*

Now, the duality between color and kinematics is exposed by a set of numerators n_i, which fulfill the same identities as the corresponding color factors

$$\begin{aligned} c_i &= c_j - c_k \quad \Leftrightarrow \quad n_i = n_j - n_k\,,\\ c_i &= -c_j \qquad \Leftrightarrow \quad n_i = -n_j\,. \end{aligned} \tag{3.6}$$

These constraints do still not uniquely determine n_i for most cases. This leaves us some residual generalized gauge freedom.

Apart from the double copy prescription described in the next subsection, this representation has the advantage of manifestly reducing the number of independent building blocks compared to a standard color decomposition. The identities between the kinematical building blocks n_i expose new relations between color-ordered amplitudes. These identities will be discussed in the last section of this chapter.

3.2 Gravity Amplitudes as Squares

An important application of the color-kinematics duality is the double-copy construction for gravitational amplitudes. Assume we have found a color-kinematics dual representation (3.2) of two gauge theory amplitudes at L loops with numerators n_i and n_i' respectively. From these gauge theory building blocks we compute the following object

$$\mathcal{M}_n^{(L)} = i^{L-1}\left(\frac{\kappa}{2}\right)^{n+2L-2} \sum_{i\in\Gamma_n^{(L)}} \int \frac{\mathrm{d}LD\ell}{(2\pi)^{LD}} \frac{1}{S_i} \frac{n_i n_i'}{D_i}\,, \tag{3.7}$$

where we replaced the YM coupling constant g by the gravitational coupling κ and the color factors c_i by the second set of numerators n'_i. This object is invariant under linearized diffeomorphisms and has the correct dimensions to be a valid candidate for a gravitational scattering amplitude. An extended discussion, why this object is a gravitational scattering amplitudes can, for example, be found in [56] for $\mathcal{N}=8$ at tree level or in [49] more generally.

Two comments are in order: Firstly, the second copy of numerator factors do not need to come from the same gauge theory. Secondly, only one of the two sets of numerators needs to be in color-kinematics dual form. The other set can be in any (trivalent) representation. We elaborate more on these properties in the rest of this section, following the discussion in [49, 50].

A gauge invariant action implies amplitudes that are invariant under linearized gauge transformations. Explicitly, the amplitude is invariant under the shift of the polarization vector ε_μ of an external gluon by the corresponding momentum $\varepsilon_\mu \to \varepsilon_\mu + p_\mu$. The polarization vector ε_μ fulfills $\varepsilon \cdot \varepsilon = 0$. In terms of the integrand we write this as $n_i \to n_i + \delta_i$, where $\delta_i = n_i|_{\varepsilon \to p}$. Gauge invariance then implies

$$\sum_i \frac{c_i \delta_i}{D_i} = 0\,. \tag{3.8}$$

Similarly, from a diffeomorphism invariant action follows the invariance of amplitudes under linearized diffeomorphisms

$$\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + p_{(\mu} q_{\nu)}\,, \tag{3.9}$$

for $\varepsilon_{\mu\nu}$ the polarization tensor of an external graviton and an auxiliary vector q that obeys $p \cdot q = 0$. This transformation preserves transversality $\varepsilon_{\mu\nu} p^\nu = 0$ and tracelessness $\varepsilon_{\mu\nu}\eta^{\mu\nu} = 0$ of the polarization tensor.

In the double copy, the graviton polarization vector is expressed in terms of two gluon polarization vectors from both factors of the double copy as the symmetric traceless product

$$\varepsilon_{\mu\nu} = \varepsilon_{((\mu}\tilde{\varepsilon}_{\nu))} = \varepsilon_{(\mu}\tilde{\varepsilon}_{\nu)} - \eta^{\mu\nu}\varepsilon_\mu \tilde{\varepsilon}_\nu\,. \tag{3.10}$$

Linearized diffeomorphisms (3.9) are then realized by a linearized gauge transformation $\varepsilon_\mu \to \varepsilon_\mu + p_\mu$ and replacing $\tilde{\varepsilon}_\mu \to q_\mu$ or vice versa. The integrand of the gravity amplitude (3.7) finally changes by a factor proportional to

$$\sum_i \frac{\delta_i \tilde{n}_i|_{\tilde{\varepsilon}\to q}}{D_i} + \sum_i \frac{n_i|_{\varepsilon \to q}\tilde{\delta}_i}{D_i}\,. \tag{3.11}$$

Both these terms vanish by eq. (3.8) as the numerator factors fulfill the same algebraic identities as the color factors and we conclude that the amplitude obtained via a double copy is invariant under linearized diffeomorphisms. However, we have assumed that both numerator factors are color-kinematics dual.

The generalized gauge freedom mentioned in the previous section allows a deformation of numerator factors $n_i \to n_i + \Delta_i$ such that the complete amplitude (integrand) is unchanged, i.e.

$$\sum_i \frac{\Delta_i c_i}{D_i} = 0\,. \tag{3.12}$$

The main point is the observation that this constraint holds due to the algebraic properties of the color factors and not their explicit values. As color-kinematics dual numerators n_i fulfill the same algebraic properties we immediately conclude that

$$\sum_i \frac{\Delta_i n_i}{D_i} = 0\,. \tag{3.13}$$

This shows that if the second set of numerators is shifted by $n'_i \to n'_i + \Delta i$ the graviton amplitude (3.7) is unchanged — as long as the first set of numerators remains in a color-kinematics dual form. This proves that only one set of numerators needs to be color-kinematics dual.

3.3 Relations for Color-Ordered Amplitudes

In a color-kinematics dual construction the separation of color and kinematics dependent objects is governed on a diagrammatic level. The traditional way of separating color and kinematics is done on a higher level by identifying a set of independent color objects $\{c_i\}$. A gauge theory amplitude can in general be written as

$$\mathcal{A} = \sum_i c_i A_i\,, \tag{3.14}$$

where c_i solely depend on structure constants $\tilde{f}^{abc}$ and generators $T^a_{i\bar{\jmath}}$. The factors A_i collect all other (kinematic) dependence, including propagators.

Such a decomposition is useful as long as the kinematical objects A_i are simpler to compute than the whole amplitude. Consider for example a *trace-based* decomposition of a Yang-Mills tree level amplitude [57–61]

$$\mathcal{A}_n^{(0)} = g^{n-2} \sum_{\sigma \in S_{n-1}} \mathrm{tr}\left(T^{a_1} T^{a_{\sigma(2)}} \cdots T^{a_{\sigma(n)}}\right) A_n^{(0)}(1, \sigma)\,, \tag{3.15}$$

where the sum is over a permutation σ of the gluon labels $2, \ldots, n$. The kinematical factors A_n are called *color-ordered* or *partial amplitudes* and can be computed diagrammatically using color-ordered Feynman rules [47] and obtained recursively for higher points (e.g. BCFW [11, 12]).

The color-ordered amplitudes fulfill a number of identities.

- From the cyclicity of the trace, cyclicity of A_n follows

$$A_n(1,2,\ldots,n) = A_n(2,\ldots,n,1)\,. \tag{3.16}$$

- A reversal of its arguments is compensated by a sign flip for odd n

$$A_n(1,2,\ldots,n) = (-1)^n A_n(n,\ldots,2,1)\,. \tag{3.17}$$

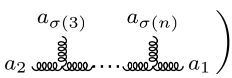
- The Kleiss-Kuijf (KK) relations [43] reduce the number of independent color-ordered amplitudes to $(n-2)!$

$$A_n(1,\alpha,n,\beta) = (-1)^{|\beta|} \sum_{\sigma\in\alpha\shuffle\beta} A_n(1,\sigma,n)\,. \tag{3.18}$$

  The symbol $\shuffle$ denotes the shuffle product which is defined as the set of permutations of the set $\alpha\cup\beta$ that leaves the relative ordering of α and β intact.

The KK relations motivate a color-decomposition that removes this redundancy and reduces the sum to $(n-2)!$ terms. This is explicitly realized through a construction by Del Duca, Dixon, and Maltoni (DDM) [44, 45]

$$\mathcal{A}_n^{(0)} = g^{n-2} \sum_{\sigma\in S_{n-2}} \tilde{f}^{a_2 a_{\sigma(3)} b_1} \tilde{f}^{b_1 a_{\sigma(4)} b_2} \cdots \tilde{f}^{b_{n-3} a_{\sigma(n)} a_1} A_n^{(0)}(1,2,\sigma)$$

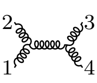
$$= g^{n-2} \sum_{\sigma\in S_{n-2}} c\Bigg($$


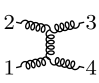
$$\Bigg) A_n^{(0)}(1,2,\sigma)\,, \tag{3.19}$$

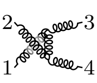
where the first argument of each summand is a color factor of a *half-ladder* graph. We will further discuss this problem for amplitudes including matter in a non-adjoint representation in part III.

Generically, there exist no further linear relations between color-ordered tree amplitudes over the field of rational numbers. Allowing for kinematics-dependent coefficients, though, leads to the so-called *BCJ relations*. These are — as the name suggests — closely related to the color-kinematics duality. It is instructive to obtain these identities through a simple example.

The BCJ representation of a four-point YM amplitude consists of three terms corresponding to the diagrams



$$= \frac{c_s n_s}{s}\,,$$


$$= \frac{c_t n_t}{t}\,,$$


$$= \frac{c_u n_u}{u}\,. \tag{3.20}$$

We labeled the terms according to their pole structure expressed in the three Mandelstam invariants $s = (p_1 + p_2)^2$, $t = (p_2 + p_3)^2$ and $u =$

$(p_3+p_1)^2$. For massless particles they fulfill the identity $s+t+u=0$. The color factors are given by

$$c_s = \tilde{f}^{a_1 a_2 b}\tilde{f}^{b a_3 a_4}\,, \qquad c_t = \tilde{f}^{a_4 a_1 b}\tilde{f}^{b a_2 a_3}\,, \qquad c_u = \tilde{f}^{a_4 a_2 b}\tilde{f}^{b a_1 a_3}\,. \tag{3.21}$$

The DDM color decomposition for this amplitudes is expressed as

$$\begin{aligned} \mathcal{A}_4^{(0)} &= c_t A_4^{(0)}(1,2,3,4) + c_u A_4^{(0)}(1,2,4,3) \\ &= c_t\left(\frac{n_s}{s}+\frac{n_t}{t}\right) + c_u\left(-\frac{n_s}{s}+\frac{n_u}{u}\right)\,. \end{aligned} \tag{3.22}$$

The color-kinematics duality is based on a single relation (ignoring sign flip identities)

$$c_s = c_t - c_u \quad \Leftrightarrow \quad n_s = n_t - n_u\,. \tag{3.23}$$

Choosing n_s and n_u as a basis of independent building blocks for the kinematic numerators we can express the two color-ordered amplitudes in the DDM decomposition as

$$\begin{pmatrix} A_4^{(0)}(1,2,3,4) \\ A_4^{(0)}(1,2,3,4) \end{pmatrix} = \Theta \begin{pmatrix} n_s \\ n_u \end{pmatrix} \qquad \Theta = \begin{pmatrix} \frac{1}{s}+\frac{1}{t} & \frac{1}{t} \\ -\frac{1}{s} & \frac{1}{s+t} \end{pmatrix}\,. \tag{3.24}$$

Since the matrix Θ has rank 1, it induces a relation between the two amplitudes

$$t\,A_4^{(0)}(1,2,3,4) + u\,A_4^{(0)}(1,2,4,3) = 0\,. \tag{3.25}$$

For general multiplicity n, the matrix has rank $(n-3)!$ and the simplest type of relations has the form [16]

$$\sum_{i=3}^{n}\left(\sum_{j=3}^{i} s_{2j}\right) A_n^{(0)}(1,3,4,\ldots,i,2,i+1,\ldots,n) = 0\,. \tag{3.26}$$

The variables $s_{ij}=(p_i+p_j)^2$ are the Mandelstam invariants for higher points. We conclude that there are maximally $(n-3)!$ independent color-ordered amplitudes at tree level. An extension to QCD, adding color factors for fundamental matter, is discussed in [46].

At loop level much less is known about relations among partial amplitudes, as the kinematic structure is clouded by loop momentum labeling ambiguities. The discussion of some features of higher loop color decompositions and relations among color-ordered objects is continued in part III.

4. Supersymmetric Gauge and Gravity Theories

This thesis discusses scattering amplitudes in supersymmetrics gauge and gravity theories in four and six dimensions. In this chapter we introduce the theories of interest with their particle content in terms of on-shell multiplets. The main focus will lie on supersymmetric Yang-Mills theories (SYM) coupled to massless fundamental matter. The basic building blocks that enter the computations in the following chapters are the tree-level amplitudes of these theories.

4.1 Supersymmetric Yang Mills Coupled to Matter

On-shell tree-level amplitudes in maximally supersymmetric Yang-Mills can be derived from symmetry arguments and recursion relations without the need of specifying a Lagrangian [62]. Tree-level superamplitudes for theories with less supersymmetry can subsequently be projected out from the maximally symmetric amplitudes. Everything needed is thus the supermultiplet for the maximally supersymmetric theory and an understanding of the projection mechanism for a lower amount of supersymmetry. We start by discussing the particle content of maximally supersymmetric Yang-Mills, followed by a reduction procedure for their non-maximally supersymmetric counterparts in four dimensions. Via an uplift to six dimensions, we can obtain amplitudes in a dimensionally regulated theory. Hence, we continue the discussion with the corresponding six dimensional theories and their relation to their four dimensional counterpart.

4.1.1 Four-Dimensional SYM

The on-shell particle content for a supersymmetric theory is conveniently described by a superspace formalism. Using Grassmann variables η^A, $A = 1, \cdots, \mathcal{N}$, introduced by Ferber [63], the on-shell particle content of the $\mathcal{N} = 4$ vector supermultiplet is organized as

$$\mathcal{V}_{\mathcal{N}=4} = A^+ + \eta^A \psi^+_A + \frac{1}{2}\eta^A\eta^B\phi_{AB} + \frac{1}{3!}\eta^A\eta^B\eta^C\psi^-_{ABC} + \eta^1\eta^2\eta^3\eta^4 A^- \,, \quad (4.1)$$

where $A^{\pm}$ denote gluons, ψ^{+}_{A} and $\psi^{-}_{ABC} \equiv -\epsilon_{ABCD}\psi^{D}_{-}$ are the two helicity states of the four gluinos, and $\phi_{AB} \equiv \epsilon_{ABCD}\phi^{CD}$ describe three complex scalars.

The tree-level color-ordered amplitude in the maximally helicity violating (MHV) sector, containing the gluonic amplitude with configuration $(- - + \cdots +)$, is especially simple. It is given by the supersymmetric generalization of the Parke-Taylor formula [64, 65]

$$A_n^{(0)[\mathcal{N}=4]} = \frac{\delta^8(Q)}{\langle 12\rangle\langle 23\rangle\cdots\langle n1\rangle}\,, \tag{4.2}$$

where $\delta^8(Q)$ represents the supermomentum conserving delta function. Using the on-shell superspace formalism it is given by

$$\delta^{2\mathcal{N}}(Q) = \prod_{A=1}^{\mathcal{N}} \sum_{i<j}^{n} \eta_i^A \langle ij\rangle \eta_j^A\,. \tag{4.3}$$

We note that there is an asymmetry in our description as we could have equivalently written

$$\mathcal{V}_{\mathcal{N}=4} = \bar{\eta}_1\bar{\eta}_2\bar{\eta}_3\bar{\eta}_4 A^+ + \frac{1}{3!}\bar{\eta}_A\bar{\eta}_B\bar{\eta}_C\psi_+^{ABC} + \frac{1}{2}\bar{\eta}_A\bar{\eta}_B\phi^{AB} + \bar{\eta}_A\psi_-^A + A^-\,, \tag{4.4}$$

using a conjugated set of Grassmann variables $\bar{\eta}_A$. It is the natural set of variables to describe an amplitude in the $\overline{\text{MHV}}$ sector, for example for the gluon configuration $(+ + - \cdots -)$. We refer to the former as the chiral and the latter as the antichiral on-shell superspace.

For certain applications discussed further on, it will be useful to have the ability to switch between the two formulations or even mix them. The explicit transformation between the two superfields is given by a Grassmann Fourier transform [66]

$$\mathcal{V}_{\mathcal{N}=4}(\eta) = \int \mathrm{d}\bar{\eta}_1\,\mathrm{d}\bar{\eta}_2\,\mathrm{d}\bar{\eta}_3\,\mathrm{d}\bar{\eta}_4 e^{\bar{\eta}_1\eta^1+\bar{\eta}_2\eta^2+\bar{\eta}_3\eta^3+\bar{\eta}_4\eta^4}\mathcal{V}_{\mathcal{N}=4}(\bar{\eta})\,. \tag{4.5}$$

Expressions for on-shell vector and matter multiplets of non-maximally supersymmetric Yang-Mills theories are obtained by projecting onto the wanted states inside $\mathcal{V}_{\mathcal{N}=4}$. We will write down the multiplets using the chiral superspace. As for $\mathcal{N}=4$, there exists an equivalent description using antichiral Grassmann variables. For $\mathcal{N}=2$, we have the vector and a pair of conjugate (half-)hyper multiplets

$$\begin{aligned}
\mathcal{V}_{\mathcal{N}=2} &= A^+ + \eta^A\psi_A^+ + \eta^1\eta^2\phi_{12} + \eta^3\eta^4\phi_{34} + \eta^A\eta^3\eta^4\psi^-_{A34} + \eta^1\eta^2\eta^3\eta^4 A^-\,,\\
\Phi_{\mathcal{N}=2} &= \eta^3\psi_3^+ + \eta^A\eta^3\phi_{A3} + \eta^1\eta^2\eta^3\psi^-_{123}\,,\\
\overline{\Phi}_{\mathcal{N}=2} &= \eta^4\psi_4^+ + \eta^A\eta^4\phi_{A4} + \eta^1\eta^2\eta^4\psi^-_{124}\,,
\end{aligned} \tag{4.6}$$

where $A = 1, 2$.

The three supermultiplets $\mathcal{V}_{\mathcal{N}=2}$, $\Phi_{\mathcal{N}=2}$, and $\overline{\Phi}_{\mathcal{N}=2}$ form a complete supersymmetric decomposition of $\mathcal{V}_{\mathcal{N}=4}$:

$$\mathcal{V}_{\mathcal{N}=4} = \mathcal{V}_{\mathcal{N}=2} + \Phi_{\mathcal{N}=2} + \overline{\Phi}_{\mathcal{N}=2} \,. \tag{4.7}$$

This decomposition has useful consequences for the computation of amplitudes at loop-level. One can demand a similar decomposition on the level of numerators for the integrand.

A further decomposition of $\mathcal{V}_{\mathcal{N}=2}$ provides the on-shell multiplets of $\mathcal{N} = 1$ SYM theory with additional chiral matter:

$$\begin{aligned}
\mathcal{V}_{\mathcal{N}=1} &= A^+ + \eta^1\psi_1^+ + \eta^2\eta^3\eta^4\psi_{234}^- + \eta^1\eta^2\eta^3\eta^4 A^- \,, \\
\Phi_{\mathcal{N}=1} &= \eta^2\psi_2^+ + \eta^1\eta^2\phi_{12} \,, \\
\overline{\Phi}_{\mathcal{N}=1} &= \eta^3\eta^4\phi_{34} + \eta^1\eta^3\eta^4\psi_{134}^- \,.
\end{aligned} \tag{4.8}$$

The hypermultiplet of $\mathcal{N} = 2$ can also be decomposed to describe another two sets of chiral-antichiral pairs of matter multiplets

$$\begin{aligned}
\Phi'_{\mathcal{N}=1} &= \eta^3\psi_3^+ + \eta^1\eta^3\phi_{13} \,, \\
\overline{\Phi}'_{\mathcal{N}=1} &= \eta^2\eta^3\phi_{23} + \eta^1\eta^2\eta^3\psi_{123}^- \,, \\
\Phi''_{\mathcal{N}=1} &= \eta^4\psi_4^+ + \eta^1\eta^4\phi_{14} \,, \\
\overline{\Phi}''_{\mathcal{N}=1} &= \eta^2\eta^4\phi_{24} + \eta^1\eta^2\eta^4\psi_{124}^- \,.
\end{aligned} \tag{4.9}$$

The three sets of chiral matter Φ, Φ' and Φ'' and their antichiral partners may transform in a subgroup of the underlying $\mathcal{N} = 4$ R-symmetry group.

These supermultiplets can finally be split into their non-supersymmetric constituents, parts of which form the on-shell states of massless quantum chromodynamics (QCD).

The $\mathcal{N} = 1$ or $\mathcal{N} = 2$ theory with

- a vector multiplet $\mathcal{V}$, transforming under the adjoint representation of the gauge group $\mathrm{SU}(N_c)$, coupled to
- hyper/chiral multiplets Φ and $\overline{\Phi}$, transforming under the fundamental representation of the gauge group,

is called supersymmetric QCD (SQCD). The name stems from its similarity to (massless) QCD. $\mathcal{V}$ can be seen as a supersymmetric version of a gluon and Φ and $\overline{\Phi}$ as supersymmetric generalizations of quarks.

4.1.2 Six-Dimensional SYM

A six dimensional on-shell superspace has been introduced by Dennen, Huang and Siegel (DHS) in [67]. We mostly follow the notation of [66] to define the superfields and a map to the corresponding four dimensional theory. Supermultiplets in maximal $(\mathcal{N}, \widetilde{\mathcal{N}}) = (1,1)$ SYM in six dimensions can be defined using two sets of Grassmann variables η^{aI} and $\tilde{\eta}_{\dot{a}\dot{I}}$. The indices a and $\dot{a}$ belong to the little group $\mathrm{SU}(2) \times \mathrm{SU}(2)$; I and $\dot{I}$ are R-symmetry indices, which is $\mathrm{USp}(2) \times \mathrm{USp}(2)$ in this case [67, 68].

Formally, η^{aI} and $\tilde{\eta}_{\dot{a}\dot{I}}$ form the fermionic part of supertwistors transforming under the supergroup $\mathrm{OSp}^*(8|4)$. Due to the self-conjugate property

$$\{\eta^{aI}, \eta^{bJ}\} = \epsilon^{ab}\Omega^{IJ}, \tag{4.10}$$

where Ω^{IJ} is the metric of the fermionic part of the supergroup $\mathrm{USp}(4)$, one needs to remove half of the degrees of freedom in order to arrive at a consistent superfield formalism. The choices are to either break manifest R-symmetry or little group covariance. The former has been successfully applied in amplitude computations [69, 70], whereas the latter is more natural from an ambitwistor string point of view [71, 72].

We choose to break the R symmetry covariance using the two independent Grassmann variables η^a and $\tilde{\eta}_{\dot{a}}$. For maximally supersymmetric YM, the vector multiplet is then written as

$$\begin{aligned}\mathcal{V}_{\mathcal{N}=(1,1)} =& \eta^a \tilde{\eta}_{\dot{a}} g_a{}^{\dot{a}} + \eta^a \chi_a + \tilde{\eta}_{\dot{a}} \tilde{\chi}^{\dot{a}} + \frac{1}{2} \tilde{\eta}^{\dot{a}} \tilde{\eta}_{\dot{a}} \eta^a \bar{\chi}_a + \frac{1}{2} \eta^a \eta_a \tilde{\eta}_{\dot{a}} \bar{\tilde{\chi}}^{\dot{a}} \\ &+ \phi + \frac{1}{2}\eta^a \eta_a \phi' + \frac{1}{2} \tilde{\eta}_{\dot{a}} \tilde{\eta}^{\dot{a}} \phi'' + \frac{1}{4} \eta^a \eta_a \tilde{\eta}_{\dot{a}} \tilde{\eta}^{\dot{a}} \phi''' \,,\end{aligned} \tag{4.11}$$

where the four vector states are encoded in $g_a{}^{\dot{a}}$; χ_a and $\tilde{\chi}^{\dot{a}}$ together with their conjugates are Weyl fermions; and ϕ, ϕ', ϕ'', and ϕ''' are the four scalar states of the theory.

Projecting out half of the states leads to the supermultiplets for $\mathcal{N} = (1,0)$ SQCD:

$$\begin{aligned}\mathcal{V}_{\mathcal{N}=(1,0)} &= \eta^a \tilde{\eta}_{\dot{a}} g_a{}^{\dot{a}} + \tilde{\eta}_{\dot{a}} \tilde{\chi}^{\dot{a}} + \frac{1}{2} \eta^a \eta_a \tilde{\eta}_{\dot{a}} \bar{\tilde{\chi}}^{\dot{a}} \,, \\ \Phi_{\mathcal{N}=(1,0)} &= \phi + \eta^a \chi_a + \frac{1}{2} \eta^a \eta_a \phi' \,, \\ \overline{\Phi}_{\mathcal{N}=(1,0)} &= \frac{1}{2} \tilde{\eta}_{\dot{a}} \tilde{\eta}^{\dot{a}} \phi'' + \frac{1}{2} \tilde{\eta}_{\dot{a}} \tilde{\eta}^{\dot{a}} \eta^a \bar{\chi}_a + \frac{1}{4} \eta^a \eta_a \tilde{\eta}_{\dot{a}} \tilde{\eta}^{\dot{a}} \phi''' \,,\end{aligned} \tag{4.12}$$

and similarly for $\mathcal{N} = (0,1)$:

$$\begin{aligned}\mathcal{V}_{\mathcal{N}=(0,1)} &= \eta^a \tilde{\eta}_{\dot{a}} g_a{}^{\dot{a}} + \eta^a \chi_a + \frac{1}{2} \tilde{\eta}^{\dot{a}} \tilde{\eta}_{\dot{a}} \eta^a \bar{\chi}_a \,, \\ \Phi_{\mathcal{N}=(0,1)} &= \phi + \tilde{\eta}_{\dot{a}} \tilde{\chi}^{\dot{a}} + \frac{1}{2} \tilde{\eta}_{\dot{a}} \tilde{\eta}^{\dot{a}} \phi'' \,, \\ \overline{\Phi}_{\mathcal{N}=(0,1)} &= \frac{1}{2} \eta^a \eta_a \phi' + \frac{1}{2} \eta^a \eta_a \tilde{\eta}_{\dot{a}} \bar{\tilde{\chi}}^{\dot{a}} + \frac{1}{4} \eta^a \eta_a \tilde{\eta}_{\dot{a}} \tilde{\eta}^{\dot{a}} \phi''' \,.\end{aligned} \tag{4.13}$$

Since $\mathcal{N} = 4$ SYM in four and $\mathcal{N} = (1,1)$ in six dimensions have the same on-shell degrees of freedom one can identify states through a one-to-one map. A Grassmann Fourier transform brings the four dimensional vector multiplet into a form resembling its six dimensional counterpart

$$\begin{aligned}\int \mathrm{d}\eta^2\,\mathrm{d}\eta^3 e^{\eta^2\bar{\eta}_2+\eta^3\bar{\eta}_3}\mathcal{V}_{\mathcal{N}=4} = \\ &- \bar{\eta}_2\bar{\eta}_3 A^+ + \eta^1\eta^4 A^- + \eta^4\bar{\eta}_3\phi_{12} + \eta^4\bar{\eta}_2\phi_{34} \\ &+ \eta^1\psi^-_{123} - \bar{\eta}_2\psi^+_3 + \bar{\eta}_3\psi^+_2 + \eta^4\psi^-_{234} \\ &- \eta^1\eta^4\bar{\eta}_3\psi^-_{124} - \eta^4\bar{\eta}_2\bar{\eta}_3\psi^+_4 - \eta^1\bar{\eta}_2\bar{\eta}_3\psi^+_1 + \eta^1\eta^4\bar{\eta}_2\psi^-_{134} \\ &+ \phi_{23} - \eta^1\bar{\eta}_2\phi_{13} - \eta^4\bar{\eta}_3\phi_{24} - \eta^1\eta_4\bar{\eta}_2\bar{\eta}_3\phi_{14}\,.\end{aligned} \tag{4.14}$$

Identifying ${}^{(4)}\eta^1 \leftrightarrow {}^{(6)}\eta^1$, ${}^{(4)}\bar{\eta}_2 \leftrightarrow {}^{(6)}\eta^2$, ${}^{(4)}\bar{\eta}_3 \leftrightarrow {}^{(6)}\tilde{\eta}_{\dot{1}}$ and ${}^{(4)}\eta^4 \leftrightarrow {}^{(6)}\tilde{\eta}_{\dot{2}}$ leads to the following map between the states

$$\begin{array}{llll} A^+ \leftrightarrow -{g_2}^{\dot{1}}\,, & A^- \leftrightarrow {g_1}^{\dot{2}}\,, & \phi_{12} \leftrightarrow {g_1}^{\dot{1}}\,, & \phi_{34} \leftrightarrow -{g_2}^{\dot{2}}\,, \\ \psi^-_{123} \leftrightarrow \chi_1\,, & \psi^+_3 \leftrightarrow -\chi_2\,, & \psi^+_2 \leftrightarrow \tilde{\chi}^{\dot{1}}\,, & \psi^-_{234} \leftrightarrow \tilde{\chi}^{\dot{2}}\,, \\ \psi^-_{124} \leftrightarrow \bar{\chi}_1\,, & \psi^+_4 \leftrightarrow -\bar{\chi}_2\,, & \psi^+_1 \leftrightarrow -\bar{\tilde{\chi}}^{\dot{1}}\,, & \psi^-_{134} \leftrightarrow -\bar{\tilde{\chi}}^{\dot{2}}\,, \\ \phi_{23} \leftrightarrow \phi\,, & \phi_{13} \leftrightarrow -\phi'\,, & \phi_{24} \leftrightarrow \phi''\,, & \phi_{14} \leftrightarrow -\phi'''\,. \end{array} \tag{4.15}$$

This map is also valid for finding a one-to-one relationship between the states of four dimensional $\mathcal{N} = 2$ and six dimensional $\mathcal{N} = (1, 0)$ or $\mathcal{N} = (0, 1)$ SQCD

4.2 Supergravity

Instead of repeating the analysis as in the previous section, we directly construct the spectrum of various supergravity theories as a double copy of the spectrum of two supersymmetric gauge theories. This construction, targeted to amplitude computations with the inclusion of matter multiplets, has first been presented in [46] for four dimensions and in paper I for six dimensions. Here we focus on supergravity theories with $\mathcal{N} = 8$ and $\mathcal{N} = 4$ in four dimensions and the corresponding theories $\mathcal{N} = (2, 2)$, $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ in six dimensions. The spectrum for $\mathcal{N} = 6, 5, 3, 2, 1, 0$ and their six dimensional cousins, for the cases they exist, can be obtained through a similar treatment. We summarize the various double-copied multiplets for $\mathcal{N} = 0, 1, 2, 4, 6, 8$ in four dimensions and $\mathcal{N} = (0, 0), (1, 0), (2, 0), (1, 1), (2, 1), (2, 2)$ in six dimensions respectively in appendix B.

$\mathcal{N} = 8$ *in* $D = 4$

We start with maximal supersymmetric gravity in four spacetime dimensions. By supercharge counting, we can only reach it as a double copy of ($\mathcal{N} = 4$ SYM) $\times$ ($\mathcal{N} = 4$ SYM). Hence, there exists a unique graviton supermultiplet

$$\mathcal{V}_{\mathcal{N}=4} \otimes \mathcal{V}_{\mathcal{N}=4} = \mathcal{H}_{\mathcal{N}=8}\,, \tag{4.16}$$

as a double copy of two vector multiplets of $\mathcal{N} = 4$ super Yang-Mills. Upon appropriate renaming of the Grassmann variables η^A in each of the vector multiplets, one obtains an explicit expression for the on-shell superfield of the form

$$\begin{aligned}\mathcal{H}_{\mathcal{N}=8} = h^{++} + \eta^A \chi_A^+ + \cdots + \eta^A\eta^B\eta^C\eta^D\eta^E\eta^F\eta^G\chi^-_{ABCDEFG} \\ + \eta^1\eta^2\eta^3\eta^4\eta^5\eta^6\eta^7\eta^8 h^{--}\,,\end{aligned} \tag{4.17}$$

in the chiral superspace formalism. Here h^{++} and h^{--} denote the two graviton states and $\chi^{\pm}$ are the two states of the 16 gravitini. It is implied that states with spin one and smaller are suppressed.

$\mathcal{N} = 4$ in $D = 4$

For half-maximal supergravity, there exist two different useful constructions. Inspecting the double copy construction of the vector multiplets of ($\mathcal{N} = 4$ SYM) $\times$ YM, one finds that pure $\mathcal{N} = 4$ supergravity is factorizable

$$\mathcal{V}_{\mathcal{N}=4} \otimes \mathcal{V}_{\mathcal{N}=0} = \mathcal{H}_{\mathcal{N}=4}\,, \tag{4.18}$$

where the on-shell vector multiplet of pure Yang-Mills consists simply of the two gluon states

$$\mathcal{V}_{\mathcal{N}=0} = A^+ + \eta^1\eta^2\eta^3\eta^4 A^-\,. \tag{4.19}$$

A construction via ($\mathcal{N} = 2$ SYM) $\times$ ($\mathcal{N} = 2$ SYM) leads to additional states on the supergravity side:

$$\mathcal{V}_{\mathcal{N}=2} \otimes \mathcal{V}_{\mathcal{N}=2} = \mathcal{H}_{\mathcal{N}=4} \oplus 2\mathcal{V}_{\mathcal{N}=4}\,. \tag{4.20}$$

A double copy construction of this form leads thus to amplitudes in a supergravity theory coupled to two vector multiplets.

Adding matter to the gauge theory, one can for example consider a double copy of ($\mathcal{N} = 2$ SQCD) $\times$ ($\mathcal{N} = 2$ SQCD). A tensor product of the hypermultiplets leads to a vector supermultiplet on the gravity side

$$\begin{aligned}\Phi_{\mathcal{N}=2} \otimes \overline{\Phi}_{\mathcal{N}=2} = \mathcal{V}_{\mathcal{N}=4}\,, \\ \overline{\Phi}_{\mathcal{N}=2} \otimes \Phi_{\mathcal{N}=2} = \mathcal{V}_{\mathcal{N}=4}\,.\end{aligned} \tag{4.21}$$

Of course, the same can be achieved by adding scalars to the pure YM side in the former double copy

$$\mathcal{V}_{\mathcal{N}=4} \otimes \phi = \mathcal{V}_{\mathcal{N}=4}\,. \tag{4.22}$$

More exotic massive matter content on the gravity side can be obtained by double copies of "cross-terms" like $\mathcal{V} \otimes \Phi$, see e.g. [73]. They are of less interest for the purpose of this thesis and we will not discuss them.

An interesting observation is that the double copy of two hypermultiplets exactly gives the additional states that render the graviton multiplet non-factorizable into two copies of $\mathcal{N} = 2$. We will use this to generalize the double copy construction discussed in the previous section to also allow for a construction of pure supergravity via $(\mathcal{N} = 2) \times (\mathcal{N} = 2)$.

$\mathcal{N} = (2, 2)$ *in* $D = 6$

The particle content of maximal supergravity in six dimensions is equivalent to $\mathcal{N} = 4$ in four dimensions, stemming from the equivalence of the corresponding SYM vector multiplets. We directly obtain the graviton on-shell multiplet via

$$\mathcal{V}_{\mathcal{N}=(1,1)} \otimes \mathcal{V}_{\mathcal{N}=(1,1)} = \mathcal{H}_{\mathcal{N}=(2,2)} \,. \tag{4.23}$$

A one-to-one map between four and six dimensions is found by a double copy of the gauge theory spectra, for which the correspondence was given in eq. (4.15).

$\mathcal{N} = (1, 1)$ *and* $\mathcal{N} = (2, 0)$ *in* $D = 6$

The lift of $\mathcal{N} = 4$ supergravity to six dimensions is related to $\mathcal{N} = (1, 1)$ and $\mathcal{N} = (2, 0)$ supergravity. Both of these theories can be double-copied from gauge theory building blocks. The $\mathcal{N} = (1, 1)$ graviton multiplet is factorizable in two ways

$$\begin{aligned} \mathcal{V}_{\mathcal{N}=(1,0)} \otimes \mathcal{V}_{\mathcal{N}=(0,1)} &= \mathcal{H}_{\mathcal{N}=(1,1)} \,, \\ \mathcal{V}_{\mathcal{N}=(1,1)} \otimes \mathcal{V}_{\mathcal{N}=(0,0)} &= \mathcal{H}_{\mathcal{N}=(1,1)} \,. \end{aligned} \tag{4.24}$$

In contrast, the graviton multiplet in the chiral $\mathcal{N} = (2, 0)$ theory is non-factorizable

$$\mathcal{V}_{\mathcal{N}=(1,0)} \otimes \mathcal{V}_{\mathcal{N}=(1,0)} = \mathcal{H}_{\mathcal{N}=(2,0)} \oplus \mathcal{T}_{\mathcal{N}=(2,0)} \,, \tag{4.25}$$

where $\mathcal{T}$ denotes a tensor multiplet [74].

The double copy of the hypermultiplets leads to vector and tensor supermultiplets respectively

$$\begin{aligned} \Phi_{\mathcal{N}=(1,0)} \otimes \overline{\Phi}_{\mathcal{N}=(0,1)} &= \mathcal{V}_{\mathcal{N}=(1,1)} \,, \\ \overline{\Phi}_{\mathcal{N}=(1,0)} \otimes \Phi_{\mathcal{N}=(0,1)} &= \mathcal{V}_{\mathcal{N}=(1,1)} \,, \\ \Phi_{\mathcal{N}=(1,0)} \otimes \overline{\Phi}_{\mathcal{N}=(1,0)} &= \mathcal{T}_{\mathcal{N}=(2,0)} \,. \end{aligned} \tag{4.26}$$

We observe once more that the double copy of matter multiplets leads to gravitational matter appearing in the double copy of vector supermultiplets.

5. Generalized Unitarity

There exist a wide variety of analytic and numerical recursion relations for tree level amplitudes. The computations at loop level have been dominated by the unitarity method. To demonstrate the usefulness of this method consider the following non-exhaustive list of references related to new technologies and results obtained through unitarity [6, 7, 75–111]. The generalized unitarity method [6, 7] constructs loop level integrands from information solely contained in tree level amplitudes. Tree level amplitudes contain all information necessary to compute any loop level amplitude — they are the only building blocks needed for all the computations discussed in this thesis. Here, we present only the basics in order to give the necessary background for the cut construction methods contained in later chapters. For a more complete discussion and extensions of the method see e.g. [112–115].

Consider an L-loop amplitude in arbitrary D dimensions. It can be expressed as a single integral over a sum of Feynman diagrams

$$I = \int \mathrm{d}^{LD}\ell \sum_i \mathcal{J}_i(\ell)\,. \tag{5.1}$$

The integrand at a subregion of the integration domain given by constraints of the form

$$\ell_1^2 = 0\,, \qquad (\ell_1 + k_1 + k_2)^2 = 0\,, \qquad \ell_2^2 = 0\,, \qquad \ldots\,, \tag{5.2}$$

is singular since some of the propagator denominators of some Feynman diagram expressions $\mathcal{J}_i$ vanish — or equivalently some momenta of internal legs of Feynman diagrams become on-shell. These denominator factors are reduced to $i\epsilon$ coming from the Feynman prescription. The residue at each singularity must then, in general, be given by a product of lower-loop on-shell amplitudes (at possibly higher points). The cut propagator legs appear as external states thereof. This procedure for all different on-shell subregions gives us information about the integrand to often completely reconstruct it.

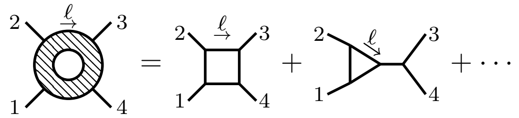
In traditional treatment, the unitarity of the S-matrix, $S^\dagger S = 1$, leads to an identity for its interacting part:

$$-i(T - T^\dagger) = T^\dagger T\,, \tag{5.3}$$

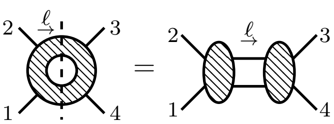
where T is defined by $S = 1 + iT$. Order by order in a perturbative expansion, this leads to equalities between the imaginary part of an L-loop amplitude and a sum over products of on-shell lower-loop amplitudes. Only physical states can cross the cut. Compared to a brute-force Feynman graph computation, a clear advantage is that all involved objects are on-shell and there is no need for Fadeev-Popov ghosts.

Graphically, these constraints can be interpreted as cuts into the diagrams building up the amplitude. Let us do a simple one-loop example by cutting through a color-ordered four-point amplitude. We use shaded blobs in a Feynman-like notation to represent amplitudes

$$A_4^{(1)}(1,2,3,4;\ell) = \tag{5.4}$$



Cutting through the loop



$$= \sum_{\text{states}} A_4^{(0)}(1,2,\ell,-\ell-p_{12}) \times A_4^{(0)}(3,4,\ell+p_{12},-\ell) \tag{5.5}$$

leads to a factorization of the residue into tree-level on-shell amplitudes. The sum includes all physical states that are allowed to cross the cut.

In practice, one often starts with an Ansatz for the amplitude, either in terms of a set of master integrals or as a sum over trivalent diagrams if one seeks a color-dual representation. The residue of the Ansatz on a given cut kinematics is simple to compute and can then be compared to the cut expression to constrain the free parameters.

The method of *generalized unitarity* [6, 7] systematizes this approach by identifying the necessary cuts. We call a minimal set of cuts required for a complete reconstruction *spanning cuts*. For massless theories, the complete integrand is reconstructible via cuts.

Part II: Amplitudes in SQCD and Non-Maximal Supergravity

> *"All our wisdom is stored in the trees."*
>
> Santosh Kalwar

This main part of the thesis is based around amplitude computations in gauge and gravity theories via the color-kinematics duality and the double copy. Even though most methods discussed here are applicable to a general gauge theory, they are developed in the simpler context of half-maximal SYM and supergravity. The following five sections discuss the complete process for the computation of color-kinematics dual gauge theory integrands, their double copy, and the extraction of UV divergences. Furthermore, we present an integrated form of a $\mathcal{N} = 2$ two-loop SQCD amplitude. The example we will use throughout the thesis is the four-point two-loop amplitude for half-maximal SCQD and pure supergravity in the MHV sector.

We start in chapter 6 with the discussion of closed formulae for a certain type of cuts: iterated two-particle cuts. These cut expressions have properties that make them well-suited for the computation of local integrands. The general procedure summarizing all required steps for the calculation on the gauge theory side is then given in chapter 7. In chapter 8, we specify to $\mathcal{N} = 1$ and $\mathcal{N} = 2$ SQCD at one and two loops and discuss the specific details of these theories. The amplitude representations resulting from this procedure have an especially nice and compact form revealing new structures and symmetries.

We present the integrated four-point amplitude of $\mathcal{N} = 2$ SQCD at two-loops and inspect its transcendentality structure in chapter 9. The theory has a conformal point if the number of hypermultiplets N_f equals $2N_c$ in the case of $\mathrm{SU}(N_c)$ gauge group. At this point, the amplitude has an unexpectedly clean form, parts of which will be presented here.

Finally, we conclude this part with a double copy construction of a supergravity amplitude and its UV structure in chapter 10.

6. Supersums and Cut Combinatorics

One difficulty for the computation of unitarity cuts in supersymmetric theories lies in performing the sum over all on-shell states propagating on cut lines. This sum is referred to as the *supersum*. With the on-shell superspace formalism discussed in chapter 4, this sum becomes an integral over the Grassmann variables η^A in four dimensions or their six dimensional counterparts respectively. Inspired by ideas in [91, 116, 117] and the so-called *rung rule* [118, 119], we found manifestly local expressions for iterated two-particle cuts for $\mathcal{N} = 2$ SQCD, published in paper III. Local formulae for unitarity cuts are not only efficient for organizing the cuts but can, for some cases, be lifted off-shell — directly giving us expressions for integrands.

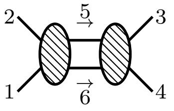
Combined with the color-kinematics duality, one can simply try to read off expressions for a small set of master numerators from the cuts. If the resulting amplitude is consistent on all cuts and if the Jacobi identities are fulfilled, one has obtained a color-kinematics dual representation. This thesis goes a step further and presents the general form of iterated two-particle MHV cuts (at four points) for any amount of supersymmetry in four dimensions.

For the following discussion, it will be essential to present tree-level building blocks in a favorable form for supersums. An important development in paper III was the observation that unitarity cuts in four-dimensional $\mathcal{N} = 0, 1, 2, 4$ SYM can be performed in a locality sensitive way. We keep track of the physical poles and consistently remove spurious poles from cut expressions.

Thus, we will first discuss the pole structure of the tree-level amplitudes and the importance of the given representations in sec. 6.1. In sec. 6.2 and 6.3 the general result is presented. Finally we discuss some simplifications for $\mathcal{N} = 2$ and examples thereof in sec. 6.4.

6.1 Notation and Tree-Level Amplitudes

Consider the following two-particle cut



$$\propto \int \mathrm{d}^{\mathcal{N}}\eta_5\, \mathrm{d}^{\mathcal{N}}\eta_6 A_{\mathrm{L}}(1,2,5,6) A_{\mathrm{R}}(3,4,-6,-5)\,, \tag{6.1}$$

where $A_{\mathrm{L/R}}$ are color-ordered four-point MHV tree-level superamplitudes. The explicit form of this amplitude for $\mathcal{N}=4$ is given by a supersymmetric generalization of the Parke-Taylor formula (4.2). For less supersymmetry, we use the projection from $\mathcal{N}=4$ supermultiplets onto the wanted states discussed in section 4.1.1.

We introduce a combinatorial notation for tree-level amplitudes to treat all possible configurations of supermultiplets on external and internal legs. The object that keeps track of the Grassmann variables, i.e. that represents the state configuration is

$$\kappa_{(i_1 j_1)\cdots(i_{\tilde{\mathcal{N}}} j_{\tilde{\mathcal{N}}})} \equiv \frac{[12][34]}{\langle 12\rangle\langle 34\rangle}\delta^{2\mathcal{N}}(Q)\eta_{i_1}^{\mathcal{N}+1}\langle i_1 j_1\rangle\eta_{j_1}^{\mathcal{N}+1}\cdots\eta_{i_{\tilde{\mathcal{N}}}}^{4}\langle i_{\tilde{\mathcal{N}}} j_{\tilde{\mathcal{N}}}\rangle\eta_{j_{\tilde{\mathcal{N}}}}^{4}\,, \tag{6.2}$$

where $\tilde{\mathcal{N}}=4-\mathcal{N}$. The rational prefactor of spinor helicity products captures the correct overall helicity weight — at four points there is only one independent quantity that has the correct weight. The $\delta^{2\mathcal{N}}(Q)$ is the supermomentum conserving delta function introduced in eq. (4.3). The remaining objects are combinations of Grassmann η and spinor angle brackets accounting for different state configurations and their helicity weight. For future convenience, we introduce a set of collected indices $\underline{i}\equiv(i_1,\ldots,i_{\tilde{\mathcal{N}}})$, such that the above object is written as $\kappa_{(\underline{i}\,\underline{j})}$.

The correspondence between κ and a configuration of external legs is found by comparison with the supermultiplets in sec. 4.1.1. For example, the color-ordered SYM amplitudes are

$$\begin{aligned}
A_{\mathcal{N}=4}(\mathcal{V}_1\mathcal{V}_2\mathcal{V}_3\mathcal{V}_4) &= -\frac{i}{st}\kappa\,,\\
A_{\mathcal{N}=2}(V_1^-V_2^+V_3^-V_4^+) &= -\frac{i}{st}\kappa_{(13)(13)}\,,\\
A_{\mathcal{N}=2}(V_1^-V_2\Phi_3\overline{\Phi}_4) &= -\frac{i}{st}\kappa_{(13)(14)}\,,\\
A_{\mathcal{N}=1}(\Phi_1\overline{\Phi}_2\Phi_3\overline{\Phi}_4) &= -\frac{i}{st}\kappa_{(13)(24)(24)}\,,\\
A_{\mathcal{N}=0}(A_1^-A_2^+\Psi_3^+\Psi_4^-) &= -\frac{i}{st}\kappa_{(13)(14)(14)(14)}\,,
\end{aligned} \tag{6.3}$$

where $V^{\pm}$ stands for the negative- and positive-helicity part of the vector multiplet

$$\mathcal{V}=V^++V^-\,, \tag{6.4}$$

e.g.

$$V^+_{\mathcal{N}=2}=A^++\eta^A\psi_A^++\eta_1\eta_2\phi_{12}\,, \tag{6.5}$$

for $\mathcal{N}=2$ SYM.

For higher point MHV and $\overline{\mathrm{MHV}}$ tree level amplitudes, there is a generalization of these objects — using chiral superspace variables η for MHV and antichiral $\bar{\eta}$ for $\overline{\text{MHV}}$— ignoring the helicity factor. At multiplicity four, MHV and $\overline{\text{MHV}}$ are equivalent. As such there is an equivalent representation of the four-point amplitude using the antichiral superspace. To relate these equivalent formulations, we introduce a *complement operation* acting on two legs of a four-point amplitude or diagram. Let $\{1,2,3,4\}$ be the set of legs and $\{i,j\}$ a choice of two different elements thereof. We define $\{\overline{i,j}\} \equiv \{1,2,3,4\} \setminus \{i,j\}$, where the overline always goes over two elements. The object carrying the state configuration in the antichiral superspace is

$$\bar{\kappa}_{(\underline{i}\,\underline{j})} = \int \mathrm{d}^{4\times 4}\eta e^{\eta\cdot\bar{\eta}} \kappa_{(\overline{\underline{i}\,\underline{j}})} = \kappa_{(\underline{i}\,\underline{j})}\Big|_{[ij]\leftrightarrow\langle ij\rangle,\eta\leftrightarrow\bar{\eta},Q\leftrightarrow\bar{Q}} . \tag{6.6}$$

We define some shorthand notations using collective indices to express the final result in a compact form:

$$\begin{aligned} [\underline{i}\underline{j}] &\equiv [i_1 j_1]\cdots[i_{\tilde{\mathcal{N}}} j_{\tilde{\mathcal{N}}}] \,, \\ \langle \underline{i}\underline{j}\rangle &\equiv \langle i_1 j_1\rangle\cdots\langle i_{\tilde{\mathcal{N}}} j_{\tilde{\mathcal{N}}}\rangle \,, \\ s_{\underline{i}\,\underline{j}} &\equiv s_{i_1 j_1}\cdots s_{i_{\tilde{\mathcal{N}}} j_{\tilde{\mathcal{N}}}} \,. \end{aligned} \tag{6.7}$$

This completes the set of kinematic objects appearing in the final formula for the two-particle cut.

6.2 General Two-Particle Four-Point Supersum

For the supersum (6.1), the full η dependence is contained in $\kappa_{(\underline{i}\underline{j})}$ and $\kappa_{(\underline{k}\underline{l})}$. Thus, we only consider a supersum over these objects. All other factors are contributing to the cut via simple multiplication. The general result for this supersum is

$$\begin{aligned} I_{(\underline{i}\underline{j})(\underline{k}\underline{l})} &\equiv \int \mathrm{d}^{\mathcal{N}}\eta_5\, \mathrm{d}^{\mathcal{N}}\eta_6 \kappa_{(\underline{i}\underline{j})}(1,2,5,6)\kappa_{(\underline{k}\underline{l})}(3,4,-6,-5) \\ &= (-1)^{\mathrm{sign}(\underline{i},\underline{j},\underline{k},\underline{l})} \frac{[56]^{\tilde{\mathcal{N}}}\langle\underline{i}\underline{j}\rangle\langle\underline{k}\underline{l}\rangle[\underline{q}\underline{r}]}{s_{56}^{2-\mathcal{N}}} \hat{\kappa}_{(\underline{q}\underline{r})} \,, \end{aligned} \tag{6.8}$$

where the collective indices $\underline{q}$ and $\underline{r}$ are determined by $\{q_m, r_m, 5, 6\} = \{i_m, j_m, k_m, l_m\}$ for $m = 1, \ldots, \tilde{\mathcal{N}}$, i.e. they encode the external state configuration of the overall cut. The sign of an ordered four-tuple is determined by the signature of its permutation (i_m, j_m, k_m, l_m) with respect to $(q_m, r_m, 5, 6)$. The overall sign is then given by the sum $\mathrm{sign}(\underline{i},\underline{j},\underline{k},\underline{l}) = \sum_m \mathrm{sign}(i_m, j_m, k_m, l_m)$. And finally we absorb some poles by defining

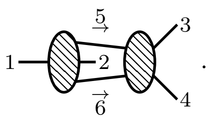
$$\hat{\kappa}_{\underline{i}\underline{j}} \equiv \frac{\kappa_{\underline{i}\underline{j}}}{s_{\underline{i}\underline{j}}^{\tilde{\mathcal{N}}}} \,. \tag{6.9}$$

With this notation, a local integrand representation is simply built out of $\hat{\kappa}$, local polynomials in the numerator, and physical poles in the denominator.

One can equivalently write the formula using the antichiral superspace

$$\bar{I}_{(\overline{\underline{ij}})(\overline{\underline{kl}})} = \int \mathrm{d}^{4\times 4}\eta e^{\eta\cdot\bar{\eta}} I_{(\underline{ij})(\underline{kl})} = (-1)^{\mathrm{sign}(\underline{i},\underline{j},\overline{\underline{kl}})} \frac{\langle 56\rangle^{\tilde{\mathcal{N}}}[\overline{\underline{ij}}][\overline{\underline{kl}}]\langle\overline{\underline{qr}}\rangle}{s_{56}^{2-\mathcal{N}}}\hat{\bar{\kappa}}_{\overline{\underline{qr}}} \,. \tag{6.10}$$

This form has the advantage that it can be iterated. We can glue another four-point blob into our cut and reuse (6.8) or (6.10) to perform the supersum over the two newly glued legs. Furthermore, since the helicity prefactor and the supermomentum-conserving delta function are permutation invariant, the same formula also holds for a non-planar cut, e.g.



. (6.11)

This non-planar cut then only differs from its planar cousin by the poles and other prefactors of κ that we dropped in this computation.

The formulae (6.8) and (6.10) have further useful properties apart from their recursive structure. The spinor-helicity factors in the numerator can naturally be combined into helicity Dirac traces. Ultimately, the resulting expressions are manifestly built out of Lorentz-invariant objects. Our conventions for the definition of these traces are given in appendix A.

The cut formula introduces two types of (unphysical) poles. For $\mathcal{N} < 2$ there is a factor of $s_{56}^{2-\mathcal{N}}$ in the denominator. This factor can always be removed by spinor-helicity identities on the cut kinematics as we will momentarily show. The second factor is the denominator that we absorbed into $\hat{\kappa}$. These factors are canceled in an iterated application of the supersum formula as we will discuss in the next sections. Evidently, only an overall factor of $1/s_{\underline{qr}}^{\tilde{\mathcal{N}}}$, where $(\underline{qr})$ denotes the overall external state configuration, will be present in any iterated two-particle cut. This factor is an artifact of our use of the spinor helicity formalism. More concretely, these poles are coming from denominators introduced by polarization vectors expressed in the spinor helicity formalism as defined in eq. (2.9). Hence, we obtain a formula for the cuts that is free of unwanted unphysical poles. As the Grassmann Fourier transform that relates I and $\bar{I}$ only acts on κ and $\bar{\kappa}$ respectively — these two objects are exactly related by a Fourier transform — we conclude that the coefficients of these two objects in equations (6.8) and (6.10) are the same. Using that $\langle 56\rangle[65] = s_{56}$, the cut is equivalently expressed in a more symmetric form as

$$I_{(\underline{ij})(\underline{kl})} = \left(s_{56}^{\mathcal{N}}\langle\underline{ij}\rangle[\overline{\underline{ij}}]\langle\underline{kl}\rangle[\overline{\underline{kl}}][\underline{qr}]\langle\overline{\underline{qr}}\rangle\right)^{\frac{1}{2}}\hat{\kappa}_{(\underline{qr})} \,, \tag{6.12}$$

trading the unwanted pole for a square root.

For $\mathcal{N}=4$ and $\mathcal{N}=2$, the square root can be easily removed as will be discussed in the following sections. For less supersymmetry, the square root does not pose a problem, but some more work is required to get rid of it. It requires the use of kinematic spinor-helicity identities to write its argument as a square. We also note that there is an ambiguity of the overall sign for the cut. In practice this sign can be inferred through other consistency criteria.

The supersum formula (6.8) for $\mathcal{N}=4$ has already been obtained in [76]; for $\mathcal{N}=2$, proof was given in paper III. It is shown by a direct computation of the supersum and applications of spinor helicity identities on the cut kinematics. The general proof is a minor generalization thereof and gives no further insight into the structure of the answer and we skip it here.

6.3 Iteration and Graphical Rules

Formula (6.12) is well suited for iteration, i.e. we can iteratively glue four-point blobs into the cut and construct higher loop cuts in the *Mondrian* (box-like) family [120]. Consider a two-loop cut constructed by gluing another four-point tree into the above cut. The two newly glued legs are numbered 7 and 8. The supersum takes the form

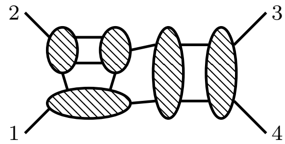
$$\begin{aligned} I_{(\underline{i}j)(\underline{k}l)(\underline{mn})} &\equiv \int \mathrm{d}^{\mathcal{N}}\eta_5\, \mathrm{d}^{\mathcal{N}}\eta_6\, \mathrm{d}^{\mathcal{N}}\eta_7\, \mathrm{d}^{\mathcal{N}}\eta_8 \kappa_{(\underline{i}j)}\kappa_{(\underline{k}l)}\kappa_{(\underline{mn})} \\ &= \int \mathrm{d}^{\mathcal{N}}\eta_7\, \mathrm{d}^{\mathcal{N}}\eta_8 \left(s_{56}^{\mathcal{N}} \langle \underline{i}j\rangle [\overline{\underline{i}j}] \langle \underline{k}l\rangle [\overline{\underline{k}l}] [\underline{q}r] \langle \overline{\underline{q}r}\rangle \right)^{\frac{1}{2}} \hat{\kappa}_{(\underline{q}r)}\kappa_{(\underline{mn})}\,, \end{aligned} \tag{6.13}$$

where we plugged in the above result for the integration over η_5 and η_6 to arrive at the second line. Using formula (6.8) a second time for the integration over η_7 and η_8 leads to

$$\begin{aligned} &I_{(\underline{i}j)(\underline{k}l)(\underline{mn})} \\ &= \frac{\left(s_{56}^{\mathcal{N}} s_{78}^{\mathcal{N}} \langle \underline{i}j\rangle [\overline{\underline{i}j}] \langle \underline{k}l\rangle [\overline{\underline{k}l}] [\underline{q}r] \langle \overline{\underline{q}r}\rangle \langle \underline{mn}\rangle [\overline{\underline{mn}}] \langle \underline{q}r\rangle [\overline{\underline{q}r}] [\underline{s}t] \langle \overline{\underline{s}t}\rangle \right)^{\frac{1}{2}}}{s_{(\underline{q}r)}} \hat{\kappa}_{(\underline{s}t)} \\ &= \left(s_{56}^{\mathcal{N}} s_{78}^{\mathcal{N}} \langle \underline{i}j\rangle [\overline{\underline{i}j}] \langle \underline{k}l\rangle [\overline{\underline{k}l}] \langle \underline{mn}\rangle [\overline{\underline{mn}}] [\underline{s}t] \langle \overline{\underline{s}t}\rangle \right)^{\frac{1}{2}} \hat{\kappa}_{(\underline{s}t)}\,, \end{aligned} \tag{6.14}$$

where we have used that $\langle \underline{q}r\rangle [r\underline{q}] = \langle \overline{\underline{q}r}\rangle [\overline{r\underline{q}}] = s_{(\underline{q}r)}$. The overall state configuration is given by $(\underline{s}t)$. As promised, the intermediate unphysical pole $1/s_{(\underline{q}r)}$ is canceled out. Apart from the pole sitting inside $\hat{\kappa}$ — an artifact from the spinor-helicity formalism as noted before — there are no further unphysical poles present.

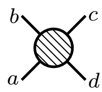
Furthermore, we observe a factorization into several building blocks coming from different parts of the cut. It is possible to assemble a cut of this form directly from graphical rules. In the following, we provide graphical rules to obtain the full analytic expression for any two-particle iterated cut with any combination of vector- and mattermultiplet on external and internal lines. Consider, for example, a cut of the form



$$, \tag{6.15}$$

where each line can represent any (half-)supermultiplet in a gauge theory described in section 4.1.1. The basic building blocks are the tree-level amplitudes described above. We distinguish between the positive- and negative-helicity part of the vector multiplet V^+ and V^- and between Φ and $\overline{\Phi}$ for matter multiplets. This graph can, with the rules described in this section, directly be translated into a closed mathematical expression for the cut.

From the form of the tree level four-point amplitude, one can immediately obtain a rule for the overall pole factors. Each tree-level blob contributes with a factor



$$\to -\frac{i}{s_{ab}s_{bc}} \,, \tag{6.16}$$

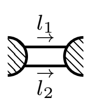
independently from the state configuration — a, b, c and d denote particle labels.

From eq. (6.12), a blob with a tree level contribution proportional to $\kappa_{(\underline{i}\bar{j})}$ comes with a factor of $(\langle \underline{i}\underline{j}\rangle[\overline{\underline{i}\underline{j}}])^{\frac{1}{2}}$, graphically

$$\to (\langle \underline{i}\underline{j}\rangle[\overline{\underline{i}\underline{j}}])^{\frac{1}{2}} \,. \tag{6.17}$$

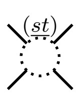
The ordering of the legs is irrelevant for this rule as it only depends on the state configuration.

The factors $s_{ij}^{\mathcal{N}}$ under the square root can be seen as a rule for the simultaneous gluing of two legs



$$\to \left(s_{l_1 l_2}^{\mathcal{N}}\right)^{\frac{1}{2}} \,. \tag{6.18}$$

Finally, there is a rule associated to the external configuration $(\underline{st})$ of the overall cut



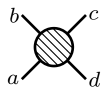
$$\to ([\underline{st}]\langle\overline{\underline{st}}\rangle)^{\frac{1}{2}}\hat{\kappa}_{(\underline{st})} \,. \tag{6.19}$$

Also this rule is agnostic of the ordering of the legs and solely depends on the state configuration. The four rules (6.16)-(6.19) completely determine a cut. For $\mathcal{N} = 4$ and $\mathcal{N} = 2$, we discuss a prescription to rewrite these rules free of square root terms.

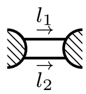
6.4 Cuts for $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SQCD

The details for the maximal and half-maximal case have been worked out in detail in paper III. We review the results here.

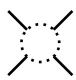
For $\mathcal{N} = 4$ SYM the rule (6.17) is obsolete as $\langle \underline{ij} \rangle = [\underline{ij}] = 1$. The remaining three rules furthermore simplify to



$$\rightarrow -\frac{i}{s_{ab}s_{bc}} \,,$$



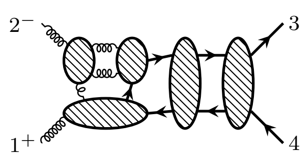
$$\rightarrow s_{l_1 l_2}^2 \,, \tag{6.20}$$



$$\rightarrow \kappa \,,$$

where we dropped any state configuration indices as there is only a single vector multiplet. This reproduced the rung rule prescription [118, 119].

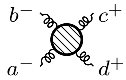
For $\mathcal{N} = 2$, we specify the blob rule (6.17) and the external rule (6.19) for the three different non-zero combinations of vector and hyper multiplets on external legs. Graphically, we represent the vector multiplet (for both chiralities) as a coiled line and the hypermultiplet Φ and $\overline{\Phi}$ as arrows going in different directions. Consider as an example the cut from above with an explicit choice of multiplets assigned to each propagator line



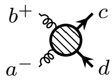
$$, \tag{6.21}$$

with two helicity vector states V_1^+ and V_2^- and two matter constituents Φ_3 and $\overline{\Phi}_4$. This cut construction also requires us to specify the helicities for internal vector lines. It is then necessary to sum over all possible configurations of helicity assignments for internal vector lines to assemble the full cut.

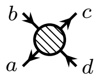
The rule for the pole factors (6.16) is independent of the state configuration and remains unchanged for $\mathcal{N} = 2$. The internal rule for tree-level numerator factors can be simplified for three non-vanishing cases to



$$\to \langle ab\rangle [cd] \,, \tag{6.22a}$$



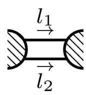
$$\to \langle a|c|b] \,, \tag{6.22b}$$



$$\to s_{ac} = s_{bd} \,. \tag{6.22c}$$

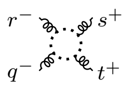
Note that these rules are still agnostic to the ordering of the external legs.

The gluing rule (6.18) is only simplified insofar the square root and the square cancel (ignoring any overall sign issues)

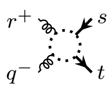


$$\to s_{l_1 l_2} \,. \tag{6.23}$$

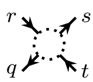
Last but not least, the external rule (6.19) is specified for the three non-trivial cases in a similar manner as the above rule for internal blobs



$$\to [qr]\langle st\rangle \hat{\kappa}_{(qr)(qr)} \,, \tag{6.24a}$$



$$\to [q|s|r\rangle \hat{\kappa}_{(qs)(qt)} \,, \tag{6.24b}$$



$$\to s_{rt}\hat{\kappa}_{(qs)(rt)} = s_{qs}\hat{\kappa}_{(qs)(rt)} \,, \tag{6.24c}$$

where we have explicitly written out the full subscript of $\hat{\kappa}$. As before these formulae are independent of the ordering of the legs.

7. Construction of Color-Kinematics Dual Amplitudes

The construction of a color-kinematics dual representation, see chapter 3, is often done via an Ansatz construction. A first step is to identify a minimal set of numerators independent under Jacobi identities/commutation relations. We call this set *master* numerators or simply *masters*. After having identified the basic Lorentz-invariant kinematic objects, we can write down an Ansatz for each master. The requirements from the color-kinematics duality and physical requirements constrain the parameters in the Ansatz. For us, the latter information is coming from unitarity cuts. In general, any object fulfilling these requirements is a valid color-kinematics dual representation and can be used for a double copy construction. Since the resulting expressions may still have additional freedom, one can choose to implement further constraints. For example, we can manifest certain useful properties to improve the presentation for further processing (e.g. integration or double copy).

The main content of the first two parts of this section is a presentation of various useful constraints that have been identified in [46] and publications I and III attached to this thesis. The third part concerns the idea of rendering the Ansatz approach obsolete by trying to lift expressions for unitarity cuts off-shell and directly construct numerators for trivalent graphs. This discussion is based on the ideas of paper III.

7.1 Master Numerators and the Ansatz

The system of equations built out of the Jacobi identities and commutation relations for color-kinematics dual numerators

$$n_i + n_j + n_k = 0 \tag{7.1}$$

is in general hard to solve (i.e. finding a set of master numerators). It grows significantly for each additional external leg or additional loop, see for example [121]. However, one possible algorithm is the following:

1. pick a numerator that has not yet been solved for and add it to the set of masters;
2. identify the set of numerators that can be expressed in terms of the masters and those that cannot;

3. if all numerators are expressible in terms of the masters, the algorithm terminates, otherwise continue from step 1.

Depending on the choice in step 1, the set of master numerators may notably change. Possible purely heuristic criteria are given in the following list. One can choose master numerators such that

- its associated graph has a large amount of symmetries (to simplify the symmetry constraints of the system),
- its associated graph corresponds to a maximal cut; i.e. it is the only numerator contributing to the cut,
- the associated graph is planar,
- the number of master numerators is minimal,
- the average number of masters appearing in expressions for derived numerators is minimized.

Depending on the situation, some of the criteria can be more useful than others. It can, for example, be useful to increase the number of master numerators if in turn other constraint equations simplify.

The idea of master numerators can be extended to not only include constraints from the color-kinematics duality but also any other kind of functional relations among numerators. This in general reduces the number of masters and as such the size of the Ansatz (number of free parameters). It comes with the risk that several constraints are mutually incompatible and no representation can be found. This problem might either be overcome by dropping the constraints or by increasing the size of the Ansatz. We will discuss some functional constraints in the following section. This system of equations is not necessarily trivially solved in terms of the master numerators as there might be different "paths" how a derived numerator is reached from the masters. The equations that are not trivially solved need to be implemented as consistency constraints on the Ansatz.

Once the master numerators are identified, we make a uniform Ansatz for each of them. From the discussion of unitarity cuts in sec. 6 and explicit computations in [46] and papers I and III, we see that the state configuration object $\hat{\kappa}$ captures the correct helicity weights and non-localities. Thus, we expect the numerators to be of the form (poly) $\times\,\hat{\kappa}$, where (poly) is a *Lorentz-invariant* polynomial in the kinematics. Most importantly, it is local.

For a dimensionally regulated theory living in $D = 4 - 2\epsilon$ dimensions, we build the polynomial from a collection of objects:

- Lorentz products between internal and external momenta such as
$$p_i \cdot p_j,\; p_i \cdot \ell_j,\; \ell_i \cdot \ell_j \tag{7.2}$$
where p_i denotes four-dimensional external momenta and ℓ_i are D-dimensional internal loop momenta. Since the p_i can be embedded in a four-dimensional subspace of the full D-dimensional space, the

product between an external and a loop momentum is equvalently written as $p_i \cdot \ell_j = p_i \cdot \bar{\ell}_j$, where $\bar{\ell}_i$ denotes the four-dimensional part of the loop momentum.

- The contraction of the Levi-Civita tensor with external momenta and the four-dimensional part of loop momenta

$$\epsilon(k_1, k_2, k_3, k_4) \equiv \det(k_{i\mu}), \quad \text{where } \mu = 1, \ldots, 4\,. \tag{7.3}$$

- We can decompose $\ell_i = \bar{\ell}_i + \mu_i$. Objects denoted by $\mu_{ij} = -\mu_i \cdot \mu_j$ need to be included in the Ansatz as well. The minus sign is coming from our use of the mostly-minus metric.
- We use the six-dimensional spinor-helicity formalism to obtain information about the amplitude in D dimensions. To encode this information, we need additional extra-dimensional antisymmetric objects, e.g. for two loops there is a single object:

$$\epsilon(\mu_1, \mu_2) \equiv \frac{\epsilon^{(6)}(p_1, p_2, p_3, p_4, \ell_1, \ell_2)}{\epsilon(p_1, p_2, p_3, p_4)} = \det(\mu_1, \mu_2)\,. \tag{7.4}$$

The Lorentz products, μ_{ij}, and $\epsilon(\mu_1, \mu_2)$ have dimension (mass)^2, the four-dimensional antisymmetric objects have twice this dimension (mass)^4. We define $M^{(N)}$ as the set of (linearly independent) monomials built out of the above objects with dimension (mass)^{2N}. Note that independence of these objects is not a strict requirement; redundancy may even help to simplify the algebraic representation of the final numerators.

For a numerator of a four-point L-loop amplitude for $\mathcal{N}$ supersymmetries, a possible Ansatz reads

$$n\left(1, 2, 3, 4, \{\ell_i\}_{i=1}^{L}\right) = \sum_{(\underline{ij})} \hat{\kappa}_{(\underline{ij})} \sum_k a_{(\underline{ij});k} M_k^{(3-\mathcal{N}-L)}\,, \tag{7.5}$$

where $a_{(\underline{ij});k}$ are constant rational coefficients to be determined. Depending on possible external state configurations of the corresponding graph, one includes a certain set of $(\underline{ij})$ in this sum.

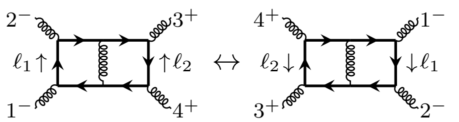
This Ansatz has been successfully used for the calculation of a color-kinematics dual form of all one-loop amplitudes with four external gluons in $\mathcal{N} = 0, 1, 2$ (S)QCD [46] and of all one- and two-loop amplitudes in $\mathcal{N} = 2$ SDCQ with four arbitrary external states in publications I and III.

7.2 A Plethora of Constraints

The necessary constraints for a color-kinematics dual representation come from Jacobi identities and commutation relations for the numerator factors and the physical unitarity cuts. Generically, the Ansatz (7.5) still has a large number of free parameters after fixing these constraints. In principle it is possible to solve the constraints for an arbitrary subset of parameters $\{a_{(\underline{ij});k}\}$ and set the remaining free parameters to zero (or to any other arbitrary value). Expressions obtained this way are in general algebraically large and badly suited for integration or a double copy. Furthermore, for a better understanding of the mathematical structure of integrands, it can be advantageous to manifest symmetries or other properties.

We present here a collection of such constraints that have been useful for the computations in the referenced papers.

Crossing symmetry
A powerful set of constraints comes from the symmetry of diagrams. The idea is to enforce a relabeling symmetry of a diagram on the corresponding numerator expression. An instructive example is:



$$\Leftrightarrow n(1^-,2^-,3^+,4^+;\ell_1,\ell_2) = n\left(3^+,4^+,1^-,2^-;-\ell_2,-\ell_1\right)\ . \tag{7.6}$$

This identity relates different state configurations and can directly be used to decrease the number of master numerators. In general, these constraints have to be implemented on the Ansatz by constraining the coefficients. Computationally, this can be implemented rather efficiently for symmetries of master numerators but unfortunately, it becomes significantly harder the further away a numerator is from the masters.

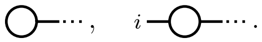
CPT conjugation
CPT invariance of the theory can be manifested in the numerator expressions. In terms of the basic building blocks, CPT conjugation acts by exchanging $|i\rangle \leftrightarrow |i]$ and $\eta_i^A \leftrightarrow \bar\eta_{i,A}$. For each diagram, this is achieved by swapping the helicity of external vector states and reversing the arrow for hypermultiplets. In terms of the kinematical objects in the Ansatz, this transformation corresponds to replacing the state configuration $(\underline{ij})$ with its complement $(\overline{\underline{ij}})$ and flipping the sign of parity-odd terms. Schematically,

$$n(1,2,3,4,\{\ell_i\}) = \bar n(1,2,3,4,\{\ell_i\})|_{\hat\kappa_{(\underline{ij})}\to\hat\kappa_{(\overline{\underline{ij}})},|i\rangle\leftrightarrow|i]}\ , \tag{7.7}$$

where $\bar n$ denotes the numerator corresponding to the graph with reversed matter lines. From experience, the CPT constraints are not very strong and often automatically fulfilled through unitarity cuts.

Power counting
Heuristically, numerators with m_i propagators carrying the loop momentum ℓ_i in the corresponding diagram permit a representation with maximally $m_i - \mathcal{N}$ powers of ℓ_i. This constraint is straightforward to implement, since terms with too high power-counting are simply excluded from the Ansatz. Power counting constraints can be extremely powerful but have a tendency to clash with other constraints discussed below.

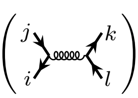
Tadpoles and external bubbles
In principle, a color-kinematics dual representation is allowed to include diagrams with massless tadpole subdiagrams or bubbles on external legs, diagrammatically

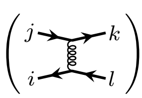


(7.8)

Since these diagrams never appear in physical unitarity cuts the corresponding numerators can be chosen to vanish as long as it is consistent with the color-kinematics duality (or other constraints). It is of course also possible to remove any other type of diagram as long as it is consistent with all the other constraints. Since these numerators tend to be far away from the masters, i.e. they are expressed in terms of a large number of master numerators, these constraints tend to be computationally expensive to implement on an Ansatz. Implemented as functional constraint, they can reduce the number of master numerators and simplify the system in general. An example for this simplification has been seen in paper I, where a two-loop pentagon-triangle diagrams has been chosen to vanish as its maximal cut was zero.

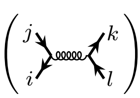
Two-term identities
In the spirit of the color-kinematics three-term identities (3.4), one might want to consider an extension of the numerator constraints from the Jacobi identities and commutation relations of the dual color factors (3.4). Even though

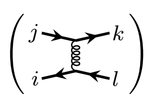
c 

$\overset{?}{=} c$ 

(7.9)

is not true for a generic gauge group, the corresponding identity for numerators,

n 

$= n$ 

, (7.10)

turns out to be useful as it allows diagrams with several closed matter loops to be expressed in terms of diagrams with a single closed matter loop. Thus, these identities are powerful when implemented in a functional way. They significantly reduce the number of masters, but also give further constraints when imposed on an Ansatz.

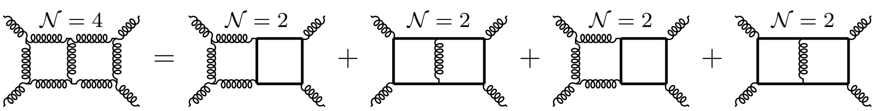
For $\mathcal{N} = 2$ SQCD, the two-term identity can be justified by the cut rule (6.22c) from the supersum discussion. The factor from this internal rule can be decomposed as $s_{ac} = -s_{ab} - s_{bc}$. These terms can be used to cancel the s and t-pole respectively of the corresponding tree level amplitude. So, it seems possible to assign these two contributions to the two graphs that are exactly related by a two-term identity. This idea is closely tied to an off-shell lift construction discussed in the following section.

Supersymmetric decomposition

From the discussion in sec. 4 it is clear that the on-shell state content of $\mathcal{N} = 4$ SYM and $\mathcal{N} = 2$ SQCD with a single hypermultiplet are equivalent

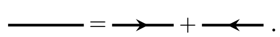
$$\mathcal{V}_{\mathcal{N}=4} = \mathcal{V}_{\mathcal{N}=2} + \Phi_{\mathcal{N}=2} + \overline{\Phi}_{\mathcal{N}=2}\,. \tag{7.11}$$

From an amplitudes point of view, these two theories only differ by the gauge group representation of the hypermultiplet. Since the numerators are color-independent, one can imagine that there exists a (color-kinematics dual) representation of the amplitude fulfilling relations of the sort



(7.12)

where a bold line stands for both parts of the hypermultiplet added up:



(7.13)

We also dropped the $n(\dots)$ notation to denote numerators — from this point on a diagram represents the corresponding numerator factor.

As amplitudes with higher degree of supersymmetry are simpler to compute, we can recycle them as known expressions into our system. Already the color-kinematics dual representation of the amplitude in the simpler theory might not be unique. Its choice influences the representation of the amplitude in the decomposed theory.

Equations of this form are most conveniently solved for the diagram with purely vectorial content in the decomposed theory as this diagram only appears once in the equation (while other diagrams might appear with different labelings which are related by crossing symmetry).

Minimal denominator solution

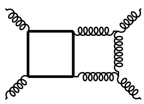
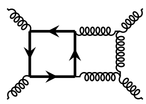
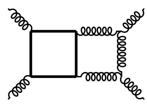
A brute force way of simplifying numerator expressions is by implementing an extremization criterion on the numerical expressions of the coefficients. One criterion that can be efficiently implemented is via a so-called *minimal denominator solution* [122]. The idea is simply to find a solution to the collected set of constraint equations — a linear system — that has a minimal denominator for the coefficients $\{a_{(\underline{i}j);k}\}$ in the Ansatz (7.5) and minimizes the norm of the coefficient vector

$$\left(\sum_{(\underline{i}j),k} (a_{(\underline{i}j);k})^2\right)^{\frac{1}{2}} . \tag{7.14}$$

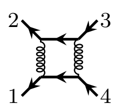
The explicit form of the norm can also be changed and might lead to different results.

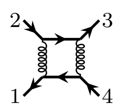
An implementation is for example included in the *integer matrix library* [122].

Matter reversal symmetry
Restricting to $\mathcal{N}=2$, it has been observed in [46] that both halves of the on-shell hypermultiplet Φ and $\overline{\Phi}$ contain the same states, interacting in the same way with the vector multiplet. A possible constraint arising from this is the following identity:

 $=$  $= \frac{1}{2}$  , (7.15)

which in general is the statement that diagrams with matter can be related to the diagram with (individually) reversed matter arrows. For matter on external legs, one also needs to take the state configuration into account, e.g.



$= -$


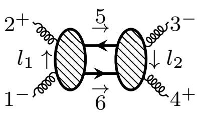
$\Big|_{\eta_2^4\to\eta_2^3,\,\eta_3^3\to\eta_3^4}$. (7.16)

$\mathcal{N}=2$ SYM permits the hypers to transform in a pseudoreal representation of the gauge group. Since in such a representation the arrow has no meaning, the above identity is justified. This symmetry can be implemented functionally as well as imposed on the Ansatz.

7.3 Candidate Numerators from Cuts

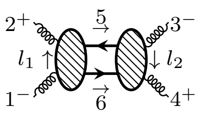
The iterated two-particle cuts for $\mathcal{N}=2$ SQCD discussed in chapter 6 are manifestly local and handle the propagator poles in a transparent manner. An idea from paper III is to massage the cut expressions into a form such that each term cancels either the s or t pole of all contributing tree level amplitudes and as such can be interpreted as the contribution of the graph with the corresponding remaining propagator poles. These expressions obey the cut kinematics and might not necessary correspond to a valid representation of the amplitude with off-shell kinematics. Nevertheless, this procedure has been successfully applied to various one- and two-loop amplitudes for $\mathcal{N}=2$ SQCD.

As a simple example, consider the one-loop cut



$$\propto -\frac{\langle 1|l_1|2]}{s l_1^2} \times s_{56} \times \frac{\langle 3|l_2|4]}{s l_2^2} \times [13]\langle 24\rangle$$
$$= \frac{s_{56}\,\mathrm{tr}_-(1l_1 24 l_2 3)}{s^2 l_1^2 l_2^2}\,, \tag{7.17}$$

where we have dropped an overall factor of $\hat{\kappa}_{(13)(13)}$. The goal of the following transformations is to use the Clifford algebra anti-commutation relation (see appendix A for conventions) to split this expression into several summands. We demand that each of them has either the s or l_1^2 pole from the left tree and either the s or l_1^2 poles from the right tree removed. We start by moving the l_1 and l_2 towards the middle



$$\propto \frac{s_{56}\left(l_1^2\,\mathrm{tr}_-(14l_2 3) - l_2^2\,\mathrm{tr}_-(12l_1 3) + \mathrm{tr}_-(12 l_1 l_2 43)\right)}{s^2 l_1^2 l_2^2}\,. \tag{7.18}$$

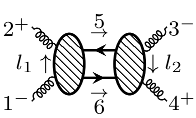
Using momentum conservation (and more anti-commutations), we rewrite the first two traces as

$$\begin{aligned} \mathrm{tr}_-(14l_2 3) &= -\,\mathrm{tr}_-(1463) = -l_2^2 u + \mathrm{tr}_-(1653)\,,\\ \mathrm{tr}_-(12l_1 3) &= \mathrm{tr}_-(1253) = -\,\mathrm{tr}_-(1653)\,, \end{aligned} \tag{7.19}$$

and the six-trace as

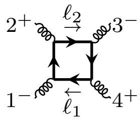
$$\mathrm{tr}_-(12l_1 l_2 43) = -\,\mathrm{tr}_+(2l_1 l_2 421) = -s\,\mathrm{tr}_+(2l_1 l_2 4) = -s\,\mathrm{tr}_-(1l_1 l_2 3)\,. \tag{7.20}$$

Canceling propagator poles leads then to

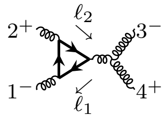


$$\propto -\frac{u s_{56}}{s^2} + \frac{\mathrm{tr}_-(1653)}{s l_2^2} + \frac{\mathrm{tr}_-(1653)}{s l_1^2} - \frac{\mathrm{tr}_-(1l_1 l_2 3)}{l_1^2 l_2^2}\,. \tag{7.21}$$

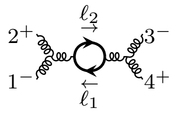
Now, these four summands carry exactly the poles of the four contributing graphs, whereof two are related by a crossing symmetry. Furthermore, the color-kinematics duality for the numerator factors are fulfilled if these expressions are interpreted off-shell in the following way



$$= \mathrm{tr}_-(1\ell_1\ell_2 3)\hat{\kappa}_{(13)(13)}\,,$$



$$= \mathrm{tr}_-(1\ell_1\ell_2 3)\hat{\kappa}_{(13)(13)}\,, \tag{7.22}$$



$$= -2u\,\ell_1\cdot\ell_2\hat{\kappa}_{(13)(13)}\,.$$

By a similar treatment for the other external configurations and with the help of color-kinematics duality, all other numerators are obtained. Note that these are results in four dimensions. For dimensionally reduced expressions, one can supplement the construction with cuts in six dimensions to obtain the missing information for the $4-2\epsilon$ dimensional part. For the complete solution see paper III.

The box numerator in eq. (7.22), together with further external configurations thereof, can be chosen as the only master for this one-loop amplitudes — assuming that a supersymmetric decomposition as discussed in section 7.2 holds. Hence, the full solution can be encoded via a single numerator (including several external configurations). This provides an efficient way of obtaining color-kinematic dual representations. Many examples for one and two loops have been worked out in paper III and we will discuss their structure in the next section.

It appears that it is a coincidence that the expression for the derived numerators via Jacobi identities/commutation relations and their expression through the off-shell lift of the cut agree. Even without seeking a color-kinematics dual representation, it is not clear how one should massage each cut expression to get a consistent off-shell lift from all cuts. For some cases with external matter states at two-loop level, only a subset of the master numerators has been constructed in this way. For these examples, it seems that a simple off-shell lift is incompatible with the color-kinematics duality. Hence, one would need to work much harder in order to bring the cut expressions into a form that upon an off-shell lift respects all diagram symmetries and the color-kinematics duality.

8. All One- and Two-Loop Four-Point Amplitudes in $\mathcal{N} = 2$ SQCD

In paper I and III, the methods from the previous sections have been applied to compute all four-point amplitudes for the MHV sector at one and two loops for $\mathcal{N} = 2$ SQCD in a color-kinematics dual representation. Also some examples of indegrands in the all-chiral sector have been computed, that we will not discuss here. This includes three distinct cases for each loop level: Four external vector states, two vector and two matter multiplets, and finally four matter states on external legs. We will not present explicit expressions here, but rather discuss the exact method and the properties of the color-kinematics dual numerators.

8.1 The Method

All of the amplitudes in question can be obtained by the following "brute-force" approach:

1. Identify all trivalent graphs for the given external configuration and as such all numerators required for a color-kinematics dual representation. See section 3.1.
2. Combine the constraints from color-kinematics duality, i.e. the Jacobi-identities and commutation relations, with the following set of additional constraints from section 7.2: crossing symmetry, CPT conjugation, two-term identity, supersymmetric decomposition (with exception of the two-loop four-matter amplitude) and matter reversal symmetry. Identify a set of master numerators for this functional system of equations according to section 7.1.
3. Make an Ansatz for each of the master numerators. Collect the constraints from above that have not yet been trivially solved by the choice of master numerators as a linear system in the coefficients.
4. Extend the system by constraints from unitarity cuts, see section 5. A good strategy is to separate the purely four-dimensional part from the extra-dimensional contributions. We can use the simple cuts worked out in sec. 6 for the four-dimensional part. The missing contributions are obtained after a subtraction of the four-dimensional result from six-dimensional cuts. In the attached publications we have used the six-dimensional spinor-helicity formalism to compute these cuts.

	Two-term	Manifest CPT	Matter reversal	$\mathcal{N}=4$
1-loop vectors	✓	✓	✓	✓
1-loop mixed	✓	✓	✓	✓
1-loop matter	✓	✓	✓	✓
2-loop vectors	✓	✓	✓	✓
2-loop mixed	✓	✓	✓	✓
2-loop matter	✓	✓	$\times^*$	$\times$

Table 8.1. *Properties of the various solutions summarized: two-term identities, manifest CPT invariance, matter-reversal symmetry, and the supersymmetric decomposition, i.e. adding up to $\mathcal{N}=4$. *Matter-reversal symmetry works for all numerators except for certain matter tadpoles. The symmetry can still be used to reduce the set of masters for all other topologies. (This table is partially extracted from paper III.)*

5. Solve the linear system, using for example *finite field* methods as will be discussed in chapter 14. Any solution is a valid color-kinematics dual representation of the amplitude. Optionally, clean up the solution and fix the residual freedom by constructing a *minimal denominator solution*, see sec. 7.2.

The workload is significantly reduced by off-shell lifts of cut expressions as discussed in section 7.3. All one-loop amplitudes and the two-loop amplitude with four external vector multiplets, have been demonstrated to be completely constructible from an off-shell lift. For the two-loop amplitude with four external hypermultiplets the situation is more complicated. One out of two master numerators suggests two different natural off-shell lifts and an Ansatz solely consisting of these two terms has been proposed — effectively reducing the number of free parameters from several hundreds to two. The second master appears in more complicated cuts and no simplified Ansatz was constructed. The brute force algorithm above has then been applied to resolve the residual freedom.

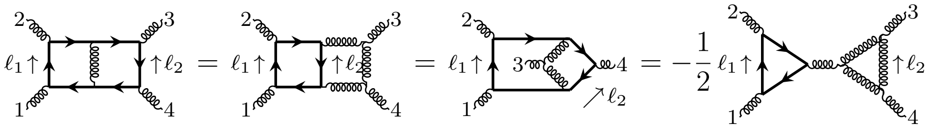
Finally, for the mixed case of two external vectors and two external matter multiplets two of the eight master numerators have been obtained directly from the cuts, whereas an Ansatz construction for the rest was necessary.

8.2 Properties

The representations of the amplitude obtained this way have a variety of properties manifested. Table 8.1 summarizes the properties of each solution as discussed in sec. 7.2.

More interestingly, some of the representations transparently expose the infrared (IR) behavior of single diagrams. The trace structure that naturally appears through our cut construction regulates regions of small or collinear loop momenta. Let us discuss these features with an explicit example, the two-loop all-vector amplitude.

This solution contains a subset of numerators that are all equal:



$$
\begin{aligned}
&= \mathrm{tr}_-(1\ell_1 24\ell_2 3)\hat{\kappa}_{13} + \mathrm{tr}_+(1\ell_1 24\ell_2 3)\hat{\kappa}_{24} \\
&\quad + \mathrm{tr}_-(1\ell_1 23\ell 24)\hat{\kappa}_{14} + \mathrm{tr}_+(1\ell_1 23\ell_2 4)\hat{\kappa}_{23} \\
&\quad - s\mu_{12}(s(\hat{\kappa}_{12} + \hat{\kappa}_{34}) + t(\hat{\kappa}_{23} + \hat{\kappa}_{14}) + u(\hat{\kappa}_{13} + \hat{\kappa}_{24})) \\
&\quad + i\,\epsilon(\mu_1, \mu_2)s^2(\hat{\kappa}_{12} - \hat{\kappa}_{34})\,,
\end{aligned}
\tag{8.1}
$$

where we used the shorthand $\hat{\kappa}_{ij} \equiv \hat{\kappa}_{(ij)(ij)}$.

These numerators have a supressed soft behavior for hyper legs: They manifestly vanish whenever the momentum of an internal matter leg goes to zero. This can be directly seen from the trace representation. Since $\mathrm{tr}(\cdots pp \cdots) = 0$ for an on-shell momentum p, we can move the loop momentum label across an external leg, e.g. for the kinematics of the first double box we have:

$$
\begin{aligned}
\mathrm{tr}(1\ell_1 24\ell_2 3) &= \mathrm{tr}(1(\ell_1 + p_1)24\ell_2 3) = \mathrm{tr}(1(\ell_1 - p_2)24\ell_2 3) \\
&= \mathrm{tr}(1\ell_1 24(\ell_2 - p_3)3) = \mathrm{tr}(1\ell_1 24(\ell_2 + p_4)3)\,.
\end{aligned}
\tag{8.2}
$$

This trace hence vanishes whenever one of the six matter legs carries a zero momentum. Similar equalities hold for the other diagrams.

In fact, this trace numerator equipped with propagators from the double-box diagram has already been studied in [123] in the context of IR-finite integral bases. A paper studying the IR properties of these amplitudes in more detail is in preparation [124].

9. Complete SQCD Amplitude at Two-Loops

The analytic form and the transcendentality proporties of amplitudes in $\mathcal{N}=4$ SYM have been studied to high loop orders and multiplicity [118, 120, 125–146]. The integrated expressions for massless amplitudes are commonly expressed in terms of multiple polylogarithms (MPL) [147,148]. These functions can be classified by the number of integrations in its definition, called the *transcendental weight*. The transcendentality of four-dimensional L-loop amplitudes has been observed to be bounded from above by $2L$. Heuristically, $\mathcal{N}=4$ SYM amplitudes are solely built out of MPLs of maximal transcendental weight $2L$. A uniform weight property allows for bootstrapping methods that determine the amplitude solely from its kinematical limits [129–139, 149]

For non-supersymmetric QCD terms of different transcendental weight appear. Hence, the function space for bootstrapping techniques is much bigger. An understanding of the transcendental structure of integrated amplitudes might improve such methods significantly. Publication IV discusses the fully integrated four-gluon amplitude for $\mathcal{N}=2$ SQCD — a testing ground between $\mathcal{N}=4$ SYM and QCD — at two loops and its transcendental properties. We consider only the case for a $\mathrm{SU}(N_c)$ gauge group here. An interesting part of the discussion is about the amplitude at the superconformal phase of the theory, where the number of hypermultiplets N_f equals twice the number of colors N_c, i.e. $N_f = 2N_c$. This theory is called $\mathcal{N}=2$ superconformal quantum chromodynamics (SCQCD). Its integrated amplitudes has a simpler transcendental structure and its expression fits into only a few lines.

We start by briefly discussing the computational methods used to obtain the integrated answer before discussing the transcendentality structure. The color-kinematics dual representation of this amplitude, found in papers I and III, is well suited for integration due to its simplicity coming from the low number of independent building blocks. Modern integration techniques such as tensor reduction (see chapter 13), integration by parts identities (IBP) [3,4] or the differential equations method [5,8,150] are easily available in various computer implementations. For the specific amplitude discussed here the master integrals are known [151–154]. For conventions regarding the UV renormalization see publication IV attached to the thesis.

Due to our use of a supersymmetric decomposition (7.11), it is convenient to present the result as the difference to the two-loop $\mathcal{N}=4$ SYM amplitude $\mathcal{A}_4^{(2)[\mathcal{N}=4]}$. Furthermore, we split the remaining amplitude into parts that either do or do not contribute at the superconformal point

$$\mathcal{A}_4^(2) = \mathcal{A}_4^{(2)[\mathcal{N}=4]} + \mathcal{R}_4^{(2)} + (2N_c - N_f)\mathcal{S}_4^{(2)}\,. \tag{9.1}$$

The remainder function $\mathcal{R}_4^{(2)}$ captures the full dependence (together with the $\mathcal{N} = 4$ part) in the conformal case. The completion to the full amplitude is collected in $\mathcal{S}_4^{(2)}$.

The properties and relations among the numerators in the integrand representation allow us to reduce the number of independent objects that actually need to be integrated. For the conformal contribution in $\mathcal{R}_4^{(2)}$ it even turns out that the leading color part is encoded in a single double-box integral of one numerator (and permutations thereof) combined with the known $\mathcal{N} = 4$ SYM result.

Through the formalism by Catani [155] — which relies on the known IR pole structure — we can further split the remainder function into a part that is reproduced by the application of the one-loop Catani operator $\mathbb{I}^{(1)}(\epsilon)$ on the corresponding one-loop remainder

$$\mathcal{R}_4^{(2)} = \mathbb{I}^{(1)}(\epsilon)\mathcal{R}_4^{(1)} + \mathcal{R}_4^{(2)\mathrm{fin}}\,, \tag{9.2}$$

where $\mathcal{R}_4^{(2)}$fin is IR finite. We will discuss this finite remainder for the superconformal theory ($N_f = 2N_c$) in the rest of this section.

There are two independent helicity configurations for the leading-color contributions, i.e. terms proportional to $N_c^2\,\mathrm{tr}(T^{a_1}T^{a_2}T^{a_3}T^{a_4})$. We denote the color-independent coefficient of this object by $R^{(2)[2]}_{(1234)}$. The two contributions, expressed in classical $\mathrm{Li}_n(z)$ and Nielsen generalized polylogarithms $S_{n,\rho}(z)$, are

$$\begin{aligned}
R^{(2)[2]}_{--++} =& 12\zeta_3 + \frac{\tau}{6}\big[48\,\mathrm{Li}_4(\tau) - 24(T+U)\,\mathrm{Li}_3(\tau) - 24T\,\mathrm{Li}_3(v) \\
&- 24S_{2,2}(\tau) + 24\,\mathrm{Li}_2(\tau)(\zeta_2 + TU) + 24TU\,\mathrm{Li}_2(v) + T^4 - 4T^3U \\
&+ 18T^2U^2 - 12\zeta_2T^2 + 24\zeta_2TU + 24\zeta_3U - 168\zeta_4 \\
&- 4i\pi\big(6\,\mathrm{Li}_3(\tau) + 6\,\mathrm{Li}_3(v) - 6U\,\mathrm{Li}_2(\tau) - 6U\,\mathrm{Li}_2(v) - T^3 \\
&+ 3T^2U - 6TU^2 - 6\zeta_2T + 6\zeta_2U)\big] + \mathcal{O}(\epsilon)\,, \\
R^{(2)[2]}_{-+-+} =& 12\zeta_3 + \frac{\tau}{6v^2}T^2(T+2i\pi)^2 + \mathcal{O}(\epsilon)\,,
\end{aligned} \tag{9.3}$$

where we have used the shorthand notations $\tau = -t/s$, $v = -u/s$, $T = \log(\tau)$, and $U = \log(v)$ such that all appearing logarithms are real. Furthermore, we dropped an overall factor of the tree level color-ordered amplitude of the corresponding helicity configuration $A_4^{(0)} = -i/(st)\kappa_{ij}$. Also, we did not specify the finite superscript as these contributions are IR finite and represent the full color-independent remainder function.

The only term that breaks uniform transcendentality is proportional to ζ_3, confirming previous results [156, 157]. This deviation might be a hint for a simple underlying structure in the non-uniform transcendental terms. The subleading color contributation of the remainder are of equal complexity. One of the two independent parts for different helicity combinations is even uniform transcendental, requiring non-trivial cancellations to happen among the different contributions.

These expressions are unexpectedly short and include only a restricted set of MPLs compared to the non-conformal amplitude $\mathcal{S}_4^{(2)}$. For gauge groups different from $\mathrm{SU}(N_c)$ there exist two cases where uniform transcendentality is completely restored: There is only a single diagram contributing to the abelian $\mathrm{U}(1)$ theory which contains the above ζ_3 term. Summing over permutations exactly kills this term and the full amplitude has transcendentality 4. Even more interestingly, for $\mathrm{SO}(3)$ the $\mathcal{N} = 2$ SQCD is exactly the same as for $\mathcal{N} = 4$ with the same gauge group, and as such it is also uniform transcendental.

10. The UV Structure of Half-Maximal Supergravity Amplitudes

After having obtained a color-kinematics dual form of a gauge theory amplitude, it is a simple task to obtain the gravitational amplitude of the double-copied theory. Some gravitational theories are obtained from the double copy in a more intricate way, this includes the non-factorizable ones. We discussed the factorizabilty of a gravity multiplet for some cases in section 4.2 and a more extensive summary table is given in appendix B. Consider, for example, the double copy of two pure Yang-Mills states. The product of two on-shell vector states

$$A^\mu \otimes A^\mu = h^{\mu\nu} + a + \varphi\,, \tag{10.1}$$

not only produces two graviton states but also an axion a and dilaton state φ. If one is interested in amplitudes of pure Einstein gravity this direct construction does not work.

In [46], it has been observed that the double copy construction can be supplemented by fundamental matter states that allow the addition or subtraction (ghost statistics) of the corresponding states.

We will first discuss this general construction. Afterwards, we will specify to a double copy of ($\mathcal{N}=2$ SQCD)$\otimes$($\mathcal{N}=2$ SQCD) at two loops and show how the construction can be used to obtained pure supergravity amplitudes for $\mathcal{N}=4$. Finally, we discuss how the UV divergence of this amplitude can be extracted and present explicit expressions obtained in paper I.

10.1 Double Copy with Ghosts

The double copy of two vector multiplets with supersymmetry $\mathcal{N},\mathcal{M}\leq 2$ leads to the decomposition

$$\mathcal{V}_{\mathcal{N}} \otimes \mathcal{V}'_{\mathcal{M}} = \mathcal{H}_{\mathcal{N}+\mathcal{M}} \oplus X_{\mathcal{N}+\mathcal{M}} \oplus \overline{X}_{\mathcal{N}+\mathcal{M}}\,, \tag{10.2}$$

which contains additional matter multiplets. In order to remove the matter states, one needs to consider the double copy of fundamental matter multiplets

$$\Phi_{\mathcal{N}} \otimes \overline{\Phi}'_{\mathcal{M}} = X_{\mathcal{N}+\mathcal{M}}\,, \quad \overline{\Phi}_{\mathcal{N}+\mathcal{M}} \otimes \Phi'_{\mathcal{M}} = \overline{X}_{\mathcal{N}+\mathcal{M}}\,. \tag{10.3}$$

Assigning ghosts statistics to the latter double-copied matter states will formally cancel out the matter states in the former double copy. Effectively the ghost statistics is assigned to one side of the double copy, say $\Phi_{\mathcal{N}}$ but not $\Phi'_{\mathcal{M}}$.

We could have ignored this problem for external states as the unwanted axion and dilaton states can simply be projected out. But we want to suppress matter states from propagating in the loop. This is achieved by modifying the double copy prescription formula (3.7). In the case where we want to cancel them out, we can introduce a ghost factor by replacing

$$c_i \to (-1)^{|i|}\bar{n}'_i\,, \tag{10.4}$$

where $|i|$ counts the number of closed matter loops in the diagram corresponding to the color factor c_i. More generally, if we want to add matter, one can introduce an integer quantity N_f counting the number of matter multiplets in one of the gauge theory factors. The replacement rule then takes the form

$$c_i \to (N_f)^{|i|}\bar{n}'_i\,. \tag{10.5}$$

This amounts to a total number $N_X = 1 + N_f$ of matter multiplets on the gravity side. For consistency, we set $0^0 = 1$ for the case we choose to not add any additional matter states, i.e. $N_f = 0$.

The double copy formula (3.7) takes then the form

$$\mathcal{M}_n^{(L)} = i^{L-1}\left(\frac{\kappa}{2}\right)^{n+2L-2}\sum_{i\in\Gamma_n^{(L)}}\int\frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD}}\frac{(N_f)^{|i|}}{S_i}\frac{n_i\bar{n}'_i}{D_i}\,, \tag{10.6}$$

where $\bar{n}'_i$ denotes the numerator of the graph with reversed matter arrows with respect to n_i. This construction can be done for any number of supersymmetries and in any spacetime dimension. Note, however, that it is essential that the double copy of matter exactly produces the unwanted states. It is currently known how to get pure theories in $D = 4$, and in some cases in D=6 as we discuss below. A complete picture of pure theories in any dimension is not known.

10.2 Pure $\mathcal{N} = 4$ Supergravity Amplitudes

The amplitude construction via ($\mathcal{N} = 2$ SQCD) $\otimes$ ($\mathcal{M} = 2$ SQCD) at one and two loops has been discussed in [46] and paper I respectively. We will focus on the two-loop computation. The double copy of two vector multiplets leads to unwanted vector states on the supergravity side:

$$\mathcal{V}_{\mathcal{N}=2}\otimes\mathcal{V}_{\mathcal{N}=2} = \mathcal{H}_{\mathcal{N}=4}\oplus 2\mathcal{V}_{\mathcal{N}=4}\,, \tag{10.7}$$

which can be remove by the double copy of hypermultiplets on both sides

$$\Phi_{\mathcal{N}=2} \otimes \overline{\Phi}_{\mathcal{N}=2} = \overline{\Phi}_{\mathcal{N}=2} \otimes \Phi_{\mathcal{N}=2} = 2\mathcal{V}_{\mathcal{N}=4}\,. \tag{10.8}$$

In order to obtain dimensionally regulated amplitudes we have used the six-dimensional uplift

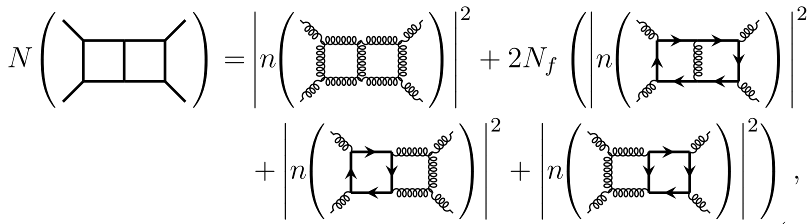
$$\mathcal{V}_{\mathcal{N}=(1,0)} \otimes \mathcal{V}_{\mathcal{N}=(0,1)} = \mathcal{H}_{\mathcal{N}=(1,1)} \tag{10.9}$$

to compute the extra-dimensional part of the answer. The map between four- and six-dimensional states is one-to-one, see eq. (4.15). For simplicity we stick to four-dimensional notation in the rest of this section.

Instead of directly applying the double copy formula (10.6), we group contributions into supergravity numerators if they share the same denominator D_i:

$$N_i = \sum_{D_j = D_i} (N_f)^{|i|} n_j \bar{n}_j\,. \tag{10.10}$$

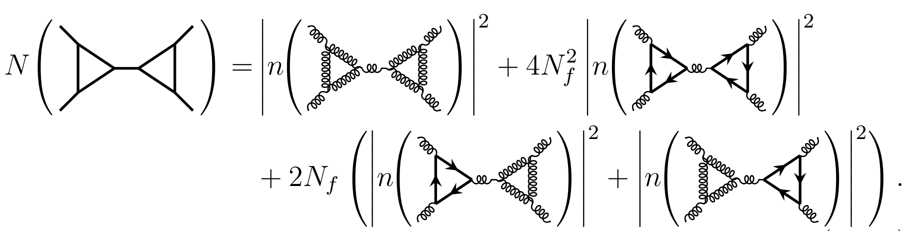
Diagrammatically, this leads, for example, to the double box supergravity numerator



(10.11)

where we have used the fact that the SQCD numerators obey a matter reversal symmetry which allowed us to add twice only one direction of the hyper loop. The modulus square represents the product of a numerator with its barred version $n_j \bar{n}_j$. We have also dropped the external and loop momentum labels. For a correct statement these will need to be inserted at the same position for each of the diagrams.

The same procedure applies to each topology. For topologies allowing for two matter loops there are also terms proportional to N_f^2, e.g.



(10.12)

The flavor counting parameter N_f can be eliminated using the relation $N_V = 2(1 + N_f)$, where N_V counts the number of vector multiplets in the four-dimensional gravity theory. The results can be interpreted in $D = 4, 5, 6$ due to the six-dimensional construction (with external momenta always living in a four-dimensional subspace).

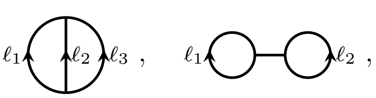
There is one further interesting detail for the six-dimensional construction. Four-dimensional $\mathcal{N} = 2$ SQCD can either be mapped to $\mathcal{N} = (1, 0)$ or $\mathcal{N} = (0, 1)$ in six dimensions. Hence, we can do two inequivalent double copies, reaching either $\mathcal{N} = (2, 0)$ or $\mathcal{N} = (1, 1)$ supergravity. The pure $\mathcal{N} = (1, 1)$ supergraviton multiplet is factorizable. The double copy for $\mathcal{N} = (2, 0)$ produces an additional tensor multiplet, which can be removed. More details on these constructions can be found in appendix B and paper I.

10.3 UV Divergences in Supergravity

An interesting aspect of a supergravity amplitude is its ultraviolet (UV) structure. We review a method to extract the UV divergence from a supergravity integrand, starting from a double copy construction as discussed in the previous section. This method has been successfully applied to the supergravity amplitude of $\mathcal{N} = 4$ supergravity at two loops. The results have also been published in paper I, which we will discuss at the end of this section.

There is no UV divergence for any supersymmetric theory at two loops in four dimensions. The corresponding counterterms are schematically of the form R^3, where R represents the Riemann tensor. Supersymmetry rules out these counterterms and any amplitude is manifestly UV finite. Hence, we will mostly consider amplitudes in five dimensions, where a supersymmetric counterterm exists.

A complete integration of the supergravity integrand followed by an expansion in small ϵ is complicated and leads to a mix up of UV and IR divergences. A better approach, which is standard since long [91], is to isolate integrand contributions that diverge in the UV. The basic idea is to first expand the numerator for small external momenta, leading to an expression of vacuum tensor integrals. At two loops, there are two types of vacuum integrals, that are diagrammatically represented as



(10.13)

where we allow for propagator denominator factors of arbitrary power. Formally, the second vacuum diagram is then a special case of the first one. Mathematically, we define

$$
\begin{aligned}
&I^D[n(p,\ell),\{\nu_1,\nu_2,\nu_3\}] = \\
&\quad\int \frac{\mathrm{d}^D\ell_1\,\mathrm{d}^D\ell_2}{(i\pi^{D/2})^2}\frac{n(p,\ell)}{(-\ell_1^2+m^2)^{\nu_1}(-\ell_2^2+m^2)^{\nu_2}(-\ell_3^2+m^2)^{\nu_3}}\,,
\end{aligned}
\tag{10.14}
$$

where $n(p,\ell)$ is a numerator factor depending on external and loop momenta. We have also introduced a uniform mass regulator to handle possible infrared (IR) divergences [34]. In the presence of subdivergences, e.g. at two loops in even spacetime dimensions, it is necessary to introduce the uniform mass regulator before the expansion in small external momenta. For a vanishing or negative value of ν_i the integral can be reduced to a product of two one-loop integrals (see for example [158]).

However, we choose another route by first performing a tensor reduction of ℓ-dependent factors in the numerator. A tensor reduction for vacuum integrals can be implemented efficiently using methods that will be discussed in sec. 13.2. The reduction to a basis of scalar integrals is most conveniently done via integration by parts (IBP) identities for which there exist completely automated implementations (for example LiteRed [159, 160] and FIRE6 [161]). If the basis of scalar integrals is UV finite, the whole UV divergence is captured in the coefficients of the integrals. This has the advantage that we do not need an explicit form of the integrals to check for cancellations among different contributions.

Let us turn to an example: the UV divergence of $\mathcal{N}=4$ supergravity via the double copy of ($\mathcal{N}=2$ SQCD)$\otimes$($\mathcal{N}=2$ SQCD) in five dimensions. A two-loop UV divergence has been shown to be absent via a double copy of ($\mathcal{N}=4$ SYM) $\otimes$ (YM) [36]. This effect is an example of an *enhanced cancellation* [35, 37] as there is no known argument that would forbid a corresponding counterterm. The cancellation is highly non-trivial since it only occurs for the assembled amplitude as a sum over all the individual diagrams and their permutations of external legs.

A choice of a basis of scalar integrals in five dimensions consists of the two UV finite integrals

$$
I^{(5-2\epsilon)}[1,\{1,2,3\}]\,,\quad I^{(5-2\epsilon)}[1,\{2,2,2\}]\,. \tag{10.15}
$$

In four dimensions, a basis of finite integrals is

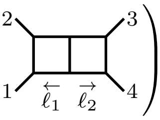
$$
I^{(4-2\epsilon)}[1,\{1,2,2\}]\,,\quad I^{(4-2\epsilon)}[1,\{2,2,2\}]\,. \tag{10.16}
$$

Explicit expressions for these integrals in terms of generalized hypergeometric functions are known [158].

For the example of $\mathcal{N}=4$ supergravity in $D=5-2\epsilon$, there are in total five diagrams that do not vanish upon integration. For example, the

UV divergence of the double box is

$$(4\pi)^5 \int \frac{\mathrm{d}^{LD}\ell}{(2\pi)^{LD} D_{\mathrm{db}}} N \left(\right.$$



$$\left. \right) \Bigg|_{\mathrm{div}} = - \frac{(2+N_V)\pi}{70\epsilon}(\kappa_{12}^2 + \kappa_{34}^2) - \frac{(29N_V - 26)\pi}{210\epsilon}(\kappa_{13}^2 + \kappa_{14}^2 + \kappa_{23}^2 + \kappa_{24}^2)\,, \tag{10.17}$$

where D_{db} is the propagator denominator factor of the double box topology. Similar expressions for the other four topologies have been given in publication I. The sum over all permutations of the five topologies, including symmetry factors, reproduces the enhanced cancellation for the MHV sector in $D = 5$ $\mathcal{N} = 4$ supergravity.

Part III: Color Decomposition for QCD

"It's-a me, Mario!"

Mario

We will now return to a more basic problem, the decomposition of an amplitude in terms of independent color factors. The concept of color decomposition was already introduced in section 3.3. The color decomposition of YM amplitudes by DDM [44,45] was extended by Johansson and Ochirov in [162] to all QCD tree amplitudes. In publication II a closed form of a one-loop color decomposition was presented.

The main motivation is to find basic *gauge invariant* kinematic building blocks that allow for a *minimal* color decomposition. Minimality in this context means that the number of color-dependent objects form an independent basis of the set of all color structures for a given amplitude.

In this minimality lies the difficulty of the problem. The system of relations among color factors is generated by the Jacobi identity and the commutation relation, see eq. (3.4). A brute-force reduction to an independent set of color factors leads to a complicated representation and corresponding kinematic building blocks that are hard to compute. The usefulness of a color decomposition is measured by the properties of the associated kinematic building blocks. For example, color-ordered tree-level amplitudes have natural symmetry properties, they can be computed via color-ordered Feynman rules, and have a recursive structure via BCFW [11,12] or CSW [10].

The hope is to find a closed formula for a color decomposition for a generic gauge theory with states in the adjoint or fundamental representation at higher loops. The kinematic building blocks obtained in such a way may be good candidates for non-planar primitive amplitudes and lead to a gauge-invariant definition of the full non-planar integrand. Furthermore, a basis for all kinematic objects, the primitive amplitudes, form a set of basic kinematic building blocks. This basis might be further reduced using BCJ relations. An example at tree level has been discussed in section 3.3.

We review the formulae at tree level in chapter 11. The extension to any multiplicity at one-loop level is presented afterwards in chapter 12.

11. Tree Level Review

A tree-level DDM-like color-decomposition for QCD [162] is completely understood for any multiplicity and has been proven in [163]. Here we will mostly rely on diagrammatic explanations and intuitive notations, and refer to the original papers for more mathematically rigorous statements.

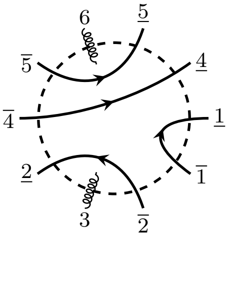
As discussed in section 3.3, the DDM construction realizes a decomposition of pure Yang-Mill amplitudes into a minimal basis of color factors. With the inclusion of quarks, the tree level color decomposition for any number and type of external legs has been worked out in [162]. Let us introduce some notation before we present the final formula.

We consider the case for mutually different quark flavors for each quark-antiquark pair and internal closed quark loop. The equal-flavor case is obtained by summing over all pairings of quarks and antiquarks of the same flavor [164]. A valid configuration of external legs for a non-vanishing primitive tree-level amplitude in QCD is represented by a *Dyck word* [165, 166].

Consider a disk where the external legs of a diagram are cyclically fixed on the boundary. The color-ordered amplitude associated with this setup is computed by all graphs that can be drawn on this disk without any crossing of legs — using color-ordered Feynman rules [47] to obtain a mathematical expression. Since the quark lines connect a quark with the corresponding antiquark of same flavor, it cannot intersect another quark line. Mathematically, this means that the external leg configuration has the same structure as a "valid" combination of opening and closing parentheses, i.e. a Dyck word. For example, '()()(())' is such a valid configuration. Each pair of opening and closing bracket is identified with a quark-antiquark pair. Gluons do not have any such restriction and can be inserted at any point.

Using the notation that quarks are labeled by underlined numbers $\underline{i}$ and the corresponding antiquark with the same number overlined $\overline{i}$, we can assign explicit particle labels to a Dyck word. Diagrammatically,

$$()()(()) \to A^{(0)}(\underline{1},\overline{1},\overline{2},3,\underline{2},\overline{4},\overline{5},6,\underline{5},\underline{4}) \; = $$



, (11.1)

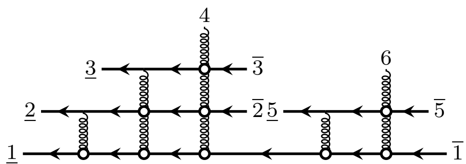
where we have randomly chosen the direction of each quark line and inserted gluons with label 3 and 6 at certain positions. We collect all possible particle assignments of all Dyck words of length k for a total number of n particles in the set $\mathrm{Dyck}_{n,k}$. We additionally demand that a quark comes before the corresponding antiquark with the same flavor, i.e. opening parentheses are quarks and closing parentheses are antiquarks. Accordingly, k denotes the number of quark-antiquark pairs and $n-2k$ gluons have to be additionally inserted to obtain the correct state counting.

With this setup, we can write down the color decomposition for general n and k

$$\mathcal{A}^{(0)}_{n,k} = g^{n-2} \sum_{\sigma \in \mathrm{Dyck}_{n-2,k-1}} C^{(0)}(\underline{1}, \sigma, \overline{1}) A^{(0)}(\underline{1}, \sigma, \overline{1})\,. \tag{11.2}$$

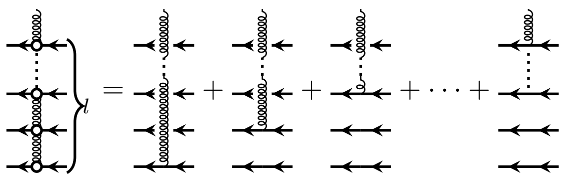
The color factors $C^{(0)}$ are most conveniently defined diagrammatically, using the definition of colorful objects in eq. (3.3). A color factor is obtained by fixing the quark line of legs $\underline{1}$ and $\overline{1}$ at the base of a diagram. All other legs are cyclically added and the following 'Mario world' structure is built:

$$C^{(0)}(\underline{1}, \underline{2}, \underline{3}, 4, \overline{3}, \overline{2}, \underline{5}, 6, \overline{5}, \overline{1}) =$$



$$. \tag{11.3}$$

The operator that connects gluonic and fermionic lines is defined via



$$\tag{11.4}$$

A mathematically rigorous definition can be found in [162]. That the primitive amplitudes appearing in this decomposition form a basis of kinematic objects has already been observed earlier [167, 168]

$$\mathcal{B}^{(0)}_{n,k} = \left\{ A^{(0)}(\underline{1}, \sigma, \overline{1}) \mid \sigma \in \mathrm{Dyck}_{n-2,k-1} \right\}\,. \tag{11.5}$$

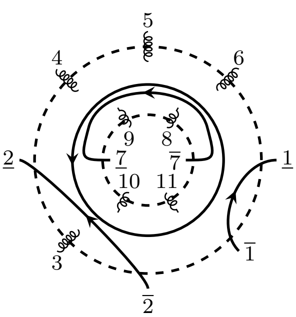
The construction is based on a generalization of the KK relations (3.18) for fundamental matter, i.e. relations between primitive amplitudes including quarks. The basis is formally defined as a maximal set of amplitudes independent under these relations. The size of the basis is $(n-2)!/k!$.

12. A Decomposition at One-Loop Level

Paper II presents a conjecture for a one-loop color decomposition for QCD for any multiplicity and any combination of external particles. The validity of the presented formulae has been explicitly checked for up to ten partons in some cases.

Extending the notation from tree level, we want to consider primitive amplitudes on an annulus, i.e. two boundaries where external legs can live. The bounderies are separated by either a closed quark or a mixed/purely gluonic loop. Consider as an example the following primitive amplitude:

$$A^q(\underline{1},\overline{1},\overline{2},3,\underline{2},4,5,6|\overline{7},8,9,\underline{7},10,11) \to$$



$$. \quad (12.1)$$

We have to distinguish different routings of quark lines on this annulus. The quark line connecting $\overline{2} \to \underline{2}$ could also go around counter-clockwise instead, or we could reverse the direction of the loop. In principle, one needs to specify the routing a quark line takes with respect to all other quark lines. However, often less information is sufficient. It might be clear which routing a quark line has, since a different configuration would lead to a vanishing amplitude (i.e. it is not possible to draw a single graph with the specified routing). A notation for routings has been introduced in paper II, but for simplicity we will rely on purely graphical statements here.

For a gluonic loop it is also possible that a quark line stretches between two boundaries. Effectively, it is possible to find a basis of primitive amplitudes that have no external parton on the inner boundary. If there is no parton present on the inner boundary, we drop the vertical bar in the argument of the primitive amplitude. The primitive amplitudes are computed using color-ordered Feynman rules as before.

To extend the color decomposition from tree-level for the case of a closed quark loop, one can imagine to close the lowest quark line $\underline{1} \to \overline{1}$

to form a loop. A good guess would then be to consider the same set of Dyck words that we have already seen at tree level and remove all cyclically equivalent configurations. Denote this set by $\mathrm{Dyck}^{\circlearrowleft}_{n,k}$. The proposed basis of primitive amplitudes is then given by

$$\mathcal{B}^q_{n,k} = \left\{ A^q(\sigma) \mid \sigma \in \mathrm{Dyck}^{\circlearrowleft}_{n,k} \right\} , \tag{12.2}$$

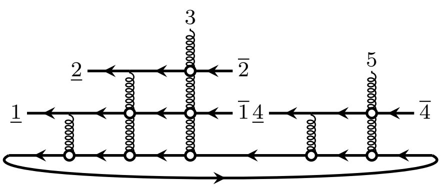
where we additionally assume a routing where the quark loop goes counter-clockwise around the hole of the annulus and lies to the left of each external quark line. Any other convention would work equally well. Note that there is no leg on the inner boundary of the annulus.

The color decomposition is then simply given as a sum over the basis of kinematic objects combined with the corresponding color factors,

$$\mathcal{A}^q_{n,k} = g^n \sum_{\sigma \in \mathrm{Dyck}^{\circlearrowleft}_{n,k}} C^q(\sigma) A^q(\sigma) \,. \tag{12.3}$$

The color factors $C^q(\sigma)$ are defined by fixing the quark loop at the base of a 'Mario world' structure and building up the top of it the same way as in the tree-level case, e.g.

$C^q(\underline{1}\underline{2}3\overline{2}\overline{1}\underline{4}5\overline{4}) =$



. (12.4)

The size of the basis is $|\mathrm{Dyck}^{\circlearrowleft}_{n,k}| = (n-1)!/k!$.

In the case of a gluonic or mixed loop, we need to introduce some more notation regarding Dyck words. So far, we have always assigned quarks before antiquarks, i.e. we identify opening parentheses with quarks and closing parentheses with antiquarks. We define a modified set of Dyck words with a second type of brackets '[]', where the antiquark is assigned to the opening and the quark to the closing bracket. For example '()[()][[]]' is a valid modified Dyck word. The set $\mathrm{mDyck}^{\circlearrowleft}_{n,k}$ is then given by the set of all modified Dyck words via the same cyclic equivalence as before and demanding that the first bracket in the word is not square. The color decomposition is written down as

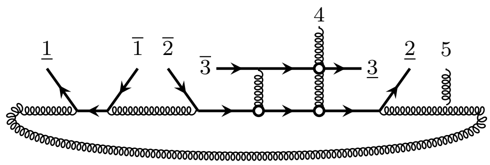
$$\mathcal{A}^g_{n,k>0} = g^n \sum_{\sigma \in \mathrm{mDyck}^{\circlearrowleft}_{n,k}} C^g(\sigma) A^g(\sigma) \,. \tag{12.5}$$

Accordingly, the corresponding basis of primitive amplitudes is given by

$$\mathcal{B}^g_{n,k>0} = \left\{ A^g(\sigma) \mid \sigma \in \mathrm{mDyck}^{\circlearrowleft}_{n,k} \right\} , \tag{12.6}$$

where we demand that the loop lies to the left of a quark line assigned to round brackets and to the right of a quark line assigned to square brackets. This fixes all other relative routings of quark lines. The color factors take a slightly more complicated form. As an example, consider

$$C^g(\underline{1}\overline{1}\overline{2}\overline{3}4\underline{3}\underline{2}5) = \qquad . \qquad (12.7)$$



The proper mathematical definition can be found in paper II. The size of the basis is given by $|\mathrm{mDyck}^{\circlearrowleft}_{n,k}| = 2^{2k-1}(n-1)!k!/(2k)!$. The special case of a purely gluonic amplitude has already been discussed in [45]. It requires to additionally remove primitive amplitudes that are equivalent under a reflection of the arguments. This reflection redundancy is broken for the general case by requiring the first bracket in the modified Dyck word to be round.

Part IV: Computational Tools

"Work smarter, not harder!"

The Internet

The results summarized in this thesis heavily rely on efficient algorithms and implementations. This final part of the thesis presents some of the tools we have found most useful and are of general interest to perturbative amplitude computations.

For the integration of the two-loop $\mathcal{N} = 2$ SQCD amplitudes in paper IV and the computation of UV divergences in various supergravity theories we have studied various tensor reduction schemes. We present the two most promising ones is chapter 13.

Using *finite fields* for the reduction of huge linear systems gave us a significant speed up in our computations. A technical discussion and an explicit implementation is presented in chapter 14.

13. Tensor Reduction

Following ideas in [169, 170], we present an efficient algorithm for an all-loop all-multiplicity tensor reduction in arbitrary dimension D.

In the second subsection we describe a strategy for an efficient tensor reduction in the case of vacuum diagrams that commonly appear in the computation of UV divergences.

In dimensional regularization it is often advantageous to express numerators of Feynman integrals using the extra dimensional objects μ_{ij} defined in sec. 7.1. Writing these objects as $\mu_{ij} = \tilde{\eta}_{\mu\nu}\ell_i^\mu \ell_j^\nu$ with the extra-dimensional part of the metric

$$\tilde{\eta}_{\mu\nu} = \mathrm{diag}(0, \dots, 0, -, -, \dots)\,, \tag{13.1}$$

allows the application of the tensor reduction formulae also for these terms.

13.1 General Reduction via Schwinger Parametrization

The basic idea is to perform a Schwinger parametrization and absorb terms appearing from tensor structures into the Schwinger parametrized form of the integrand. Ultimately, we obtain a reduction to scalar integrals with shifted numerator powers and dimension.

13.1.1 Gaussian Form of a Schwinger Parametrization

Let us start with some preparations by inspecting the Schwinger parametrized form of a general Feynman denominator

$$\begin{aligned} \frac{1}{D_1^{\nu_1}\cdots D_m^{\nu_m}} &= \prod_{i=1}^{m} \frac{(-1)^{\nu_i}}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i x_i^{\nu_i - 1} \exp\left(\sum_{j=1}^{m} x_j D_j\right) \\ &\equiv \int \mathcal{D}x \exp\left(\sum_{j=1}^{m} x_j D_j\right)\,. \end{aligned} \tag{13.2}$$

Each denominator factor is of the general form $D_i = (\sum_j \#_j \ell_j + p_i)^2 - m_i^2$, where ℓ_j denote loop momenta, p_i are an arbitrary vectors, and m_i are scalar mass terms. The hash stands for an arbitrary numerical factor. The exponent in (13.2) accordingly has the structure

$$\sum_{j=1}^{m} x_j D_j = \sum_{i,j=1}^{L} b_{ij} \ell_i \cdot \ell_j + \sum_{i=1}^{L} 2c_i \cdot \ell_i + d\,, \tag{13.3}$$

where b_{ij}, c_i and d are scalar/vector functions depending on x_i, masses, and momenta. The number L counts the total number of loop momenta. We also set $b_{ij} = b_{ji}$. Performing a change of variables

$$\ell_i^\mu = \sum_{j=1}^{L} x_{ij} k_j^\mu + y_i^\mu \qquad \text{s.t.} \qquad \sum_{j=1}^{m} x_j D_j = \sum_{i=1}^{L} A_i k_i^2 + D\,, \tag{13.4}$$

will bring the integration over the loop momenta into a Gaussian form.

We observe that the unknowns x_{ij} and y_i have a simple form if we assume that $x_{ij} = 0\ \forall i > j$ and $x_{ii} = 1$. To show this we start by defining some useful objects. Let $B_{(\beta)}$ be the $(\beta-1)\times(\beta-1)$-matrix with entries b_{ij}. Furthermore we define the vector quantities

$$b_{(\beta)} \equiv \begin{pmatrix} b_{1\beta} \\ \vdots \\ b_{\beta-1,\beta} \end{pmatrix}, \quad x_{(\beta)} \equiv \begin{pmatrix} x_{1\beta} \\ \vdots \\ x_{\beta-1,\beta} \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ \vdots \\ y_L \end{pmatrix}, \quad c = \begin{pmatrix} c_1 \\ \vdots \\ c_L \end{pmatrix}. \tag{13.5}$$

By a direct computation it can be checked that

$$x_{(\beta)} \equiv -B_{(\beta)}^{-1} \cdot b_{(\beta)} \qquad\qquad y \equiv -B_{(L+1)}^{-1} \cdot c \tag{13.6}$$

fulfills (13.4). This leads to an explicit expression for the exponent in Gaussian form in terms of the parameters b_{ij}, c_i and d

$$\begin{aligned} A_i &= \frac{\det B_{(i+1)}}{\det B_{(i)}}\,, \\ D &= d - c^\intercal \cdot B_{(L+1)}^{-1} \cdot c = \frac{d \det B_{(L+1)} - c^\intercal \cdot \operatorname{adj} B_{(L+1)} \cdot c}{\det B_{(L+1)}}\,, \end{aligned} \tag{13.7}$$

where we have explicitly exposed the denominator by expressing the inverse matrix through its adjugate matrix $B_{(L+1)}^{-1} = \operatorname{adj} B_{(L+1)} / \det B_{(L+1)}$. In summary, the argument of the exponent is given by

$$\sum_{j=1}^{m} x_j D_j = \sum_{i=1}^{L} \frac{P_{i+1}}{P_i} k_i^2 + \frac{Q_{L+1}}{P_{L+1}}, \tag{13.8}$$

where we used the shortcuts

$$\begin{aligned} P_i &\equiv \det B_{(i)}\,, \\ Q_{L+1} &\equiv d P_{L+1} - c^\intercal \cdot \operatorname{adj} B_{(L+1)} \cdot c\,. \end{aligned} \tag{13.9}$$

Note that for the boundary case, $P_1 = \det B_{(1)} = 1$.

The expression for x_{ij} can be rewritten such that one finds the substitution in a form

$$l_i^\mu = k_i^\mu + \sum_{j=i+1}^{L} \frac{(-1)^{i+j} P_{j+1}^{ij}}{P_j} k_j^\mu - \frac{a_{(L+1)}^{[i]} \cdot c^\mu}{P_{L+1}}, \tag{13.10}$$

where $a_{(L+1)} \equiv \operatorname{adj}(B_{(L+1)})$ and the superscript $[i]$ denotes the i'th row. Furthermore, A^{ij} denotes the matrix A with row i and column j removed. We are now prepared to perform the tensor reduction.

13.1.2 The Reduction Formula

Our goal is to express the general L-loop tensor Feynman integral

$$I^D\left[\{\ell_i^{\mu_{i,s}}\},\{\nu_i\}\right] \equiv \int \underbrace{\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{D/2}}}_{\equiv \mathcal{D}\ell} \prod_{s=1}^{r_i} \ell_i^{\mu_{i,s}} \frac{1}{D_1^{\nu_1}\cdots D_m^{\nu_m}}, \tag{13.11}$$

as a sum over tensor structures built out of $\eta^{\mu\nu}$ and scalar integrals. Using eq. (13.2) to Schwinger parametrize the denominator factors, the above integral takes the form

$$I^D\left[\{\ell_i^{\mu_{i,s}}\},\{\nu_i\}\right] = \int \mathcal{D}x\mathcal{D}\ell \prod_{s=1}^{r_i} \ell_i^{\mu_{i,s}} \exp\left(\sum_{i=1}^{m} x_i D_i\right). \tag{13.12}$$

We replace factors of $\ell_i^{\mu_{i,s}}$ in eq. (13.12) with a derivative with respect to $c_i^{\mu_{i,s}}$ sitting inside the exponent (13.3)

$$I^D\left[\{\ell_i^{\mu_{i,s}}\},\{\nu_i\}\right] = \int \mathcal{D}x\mathcal{D}\ell \prod_{s=1}^{r_i} \frac{1}{2}\frac{\partial}{\partial c_i^{\mu_{i,s}}} \exp\left(\sum_{i=1}^{m} x_i D_i\right). \tag{13.13}$$

Pulling the derivatives out of the inner integral and performing the substitution (13.10) leads to

$$I^D\left[\{\ell_i^{\mu_{i,s}}\},\{\nu_i\}\right] = \int \mathcal{D}x \prod_{i=1}^{L}\prod_{s=1}^{r_i} \frac{1}{2}\frac{\partial}{\partial c_i^{\mu_{i,s}}} \int \mathcal{D}k \exp\left(\sum_{i=1}^{L} \frac{P_{i+1}}{P_i} k_i^2 + \frac{Q_{L+1}}{P_{L+1}}\right). \tag{13.14}$$

The integral over the loop momenta k is in a Gaussian form. It is conveniently rewritten as

$$\begin{aligned} I^D\left[\{\ell_i^{\mu_{i,s}}\},\{\nu_i\}\right] &= \int \mathcal{D}x \prod_{i=1}^{L}\prod_{s=1}^{r_i} \frac{1}{2}\frac{\partial}{\partial c_i^{\mu_{i,s}}} \prod_{i=1}^{L}\left(\frac{P_i}{P_{i+1}}\right)^{D/2} \exp\left(\frac{Q_{L+1}}{P_{L+1}}\right) \\ &= \int \mathcal{D}x \frac{1}{P_{L+1}^{D/2}} \prod_{i=1}^{L}\prod_{s=1}^{r_i} \frac{1}{2}\frac{\partial}{\partial c_i^{\mu_{i,s}}} \exp\left(\frac{Q_{L+1}}{P_{L+1}}\right). \end{aligned} \tag{13.15}$$

The derivatives acting on the exponential will increase the powers of P_{L+1} which can be absorbed into the denominator factor by increasing the dimension. Since $P_{L+1} = \det B_{(L+1)}$ is independent of c_i and

$$\frac{\partial}{\partial c_i^{\mu_{i,s}}} Q_{L+1} = -2(\operatorname{adj} B_{L+1} \cdot c^{\mu_{i,s}})_i, \tag{13.16}$$

$$\frac{\partial^2}{\partial c_i^{\mu_{i,s}} \partial c_j^{\mu_{j,t}}} Q_{L+1} = -2(\operatorname{adj} B_{L+1})_{ij} \eta^{\mu_{i,s}\mu_{j,t}}, \tag{13.17}$$

we see that the numerator of the integrand will be a polynomial in x_i. These can be absorbed into $\mathcal{D}x$ by shifting the power of the corresponding propagator. This relates the given tensor integral to a sum of scalar integrals with a tensor structure that is given by the derivatives acting on Q_{L+1}.

Formula (13.15) is well suited for an automatized computer implementation since the calculation of determinants is already efficiently implemented in many libraries and derivatives can be performed fast.

13.2 A Reduction for Vacuum Integrals

The tensor reductions for vacuum integrals can be done in a more efficient way, since any object of the form $\ell_i \cdot \ell_j$ can be absorbed into a (inverse) propagator. Consider an L-loop vacuum integral of the form

$$\int \underbrace{\prod_{i=1}^{L} \frac{\mathrm{d}^D \ell_i}{i\pi^{D/2}}}_{\equiv \mathcal{D}\ell} \frac{\ell_{i_1}^{\mu_1} \cdots \ell_{i_n}^{\mu_n}}{D}, \tag{13.18}$$

where the denominator is a product of inverse vacuum Feynman propagators $(\sum_j \#_j \ell_j)^2 - m^2$. The hash stands for an arbitrary numerical factor.

To identify all possible tensor structures, we define $\mathbb{P}_n$ as the set of all inequivalent ways of combining objects of a permutation into pairs

$$\mathbb{P}_n = \{(\sigma_1\sigma_2)\cdots(\sigma_{n-1}\sigma_n) | \sigma \in S_n\}/\text{equivalent configurations}, \tag{13.19}$$

where two configurations are equivalent upon swapping the two members of a pair and a permutation of its set of pairs. For $\sigma \in \mathbb{P}_n$ we set

$$\begin{aligned} \eta^\sigma &\equiv \eta^{\mu_{\sigma_1}\mu_{\sigma_2}} \cdots \eta^{\mu_{\sigma_{n-1}}\mu_{\sigma_n}}, \\ \ell_\sigma &\equiv \ell_{i_{\sigma_1}} \cdot \ell_{i_{\sigma_2}} \cdots \ell_{i_{\sigma_{n-1}}} \cdot \ell_{i_{\sigma_n}}, \end{aligned} \tag{13.20}$$

for a given set of spacetime indices $\{\mu_j\}_{j\in J}$ and loop momentum labels $\{i_k\}_{k\in K}$. The tensor structure of the general vacuum integral (13.18) is expressible in terms of tensor structures in $\mathbb{P}_n$ multiplied by Lorentz products of loop momenta $\ell_i \cdot \ell_j$. We make the Ansatz

$$\int \mathcal{D}\ell \frac{\ell_{i_1}^{\mu_1} \cdots \ell_{i_n}^{\mu_n}}{D} = \sum_{\sigma,\rho \in \mathbb{P}_n} c_\sigma{}^\rho \int \mathcal{D}\ell \frac{\ell_\rho \eta^\sigma}{D}, \tag{13.21}$$

where $c_\sigma{}^\rho$ are coefficients that we will determine in what follows. In order to solve this equation for $c_\sigma{}^\rho$, we drop the integral and assume — by a slight abuse of notation — an implicit summation over the indices $\sigma, \rho \in \mathbb{P}_n$. Contracting both sides with an arbitrary η_π, $\pi \in \mathbb{P}_n$ leads to

$$\ell_\pi = (\eta_\pi \eta^\sigma) c_\sigma{}^\rho \ell_\rho\,. \tag{13.22}$$

Implicit spacetime indices in $(\eta_\pi \eta^\sigma)$ can be contracted and lead, using $\eta_\mu{}^\mu = D$, to an interpretation of $(\eta_\pi \eta^\sigma)$ as a matrix with scalar entries. The rows and columns of this matrix are labeled by π and σ. Interpreting this statement as a matrix equation and choosing $\pi = \rho$, we can solve for

$$c_\sigma{}^\rho = (\eta_\sigma \eta^\rho)^{-1}. \tag{13.23}$$

To conclude, a tensor reduction for vacuum integrals can be done by replacing the tensor structure via

$$\ell_{i_1}^{\mu_1} \cdots \ell_{i_n}^{\mu_n} \to \eta^\sigma c_\sigma{}^\rho \ell_\rho \quad \text{(implicit summation)}, \tag{13.24}$$

and absorbing factors containing loop momenta, i.e. ℓ_ρ, into the propagators.

For massive propagators one needs to introduce appropriate factors of the masses in the numerator to perform this absorption. In general, this procedure leads to negative power propagators. Integrals containing propagators with vanishing or negative power can be factorize into lower loop vacuum integrals. This factorization procedure for the two-loop sunset diagram with different masses is for example described in [158].

13.2.1 Implementation strategies

Contracting the free spacetime indices after a tensor reduction back into a full amplitude expression is computationally costly. Using formula (13.24), the contraction can be done locally with each element of η^σ separately. The resulting vector is then simply dotted into the vector $c_\sigma{}^\rho \ell_\rho$, which can be hard-coded.

It turns out that this procedure, in general, does not give us the shortest possible expressions, and the object $c_\sigma{}^\rho$ is a large matrix that takes a lot of time to compute and uses a lot of space in memory. A useful representation of the matrix is found by an eigenvalue decomposition. Since the matrix is symmetric, there exists a decomposition

$$c = PDP^{\intercal}, \tag{13.25}$$

where D is a diagonal matrix containing the eigenvalues and P is a matrix of (orthogonalized) eigenvectors. By inspecting the cases for $n = 2, 4, 6, 8, 10$, it turns out that the eigenvalues tend to have a high multiplicity and the eigenvectors are numerical, i.e. they do not depend on the spacetime dimension D. Let E_i be the i'th eigenvalue and $\{p_{i;j}\}_j$ a basis of corresponding orthonormal eigenvectors. We can then write

$$c = \sum_i E_i \sum_j p_{i;j} p_{i;j}^{\intercal}. \tag{13.26}$$

Using slightly schematic notation, this leads to a partially factorized form of the answer

$$\ell_{i_1}^{\mu_1} \cdots \ell_{i_n}^{\mu_n} \to \sum_i E_i \sum_j (\eta \cdot p_{i;j})(\ell \cdot p_{i;j}), \tag{13.27}$$

where η can be thought of as already having been contracted with external vectors. Not only are the resulting expressions relatively short, but also the matrix c represented as a list of eigenvalues and eigenvectors takes significantly less space compared to a brute force implementation of formula (13.24).

14. Finite Fields

One of the bottlenecks of the computations discussed in part II is the solving of the constraint equations for the coefficients in the Ansatz. First of all, the full system tends to be quite large with a lot of redundant equations. For example, the Ansatz for $\mathcal{N} = 1$ SQCD at two loops contains around 10000 free parameters for solely the four-dimensional part. The various constraints discussed in sec. 7.2 lead to a linear system with over 200000 equations.

A further complication comes from the property of this system that tends to produce intermediate expressions with enormous rational numbers — making the memory requirements and CPU time explode. In contrast, the final expressions tend to contain only very simple numerical coefficients. Both of these purely computational complications can be overcome by the use of finite fields.

The basic idea is to solve, or more precisely row reduce, the numerical linear system over different finite fields (most conveniently chosen as integers modulo some prime number) and reconstruct the rational result afterwards. This prevents intermediate expressions to blow up. Since the final expressions often have a tame behavior, not many runs over different finite fields are required.

Finite fields have been successfully applied to many other problems in mathematics, physics and computational science, for example polynomial factorization, the computation of the greatest common divisor (GCD), or integration by parts (IBP) techniques [171].

We follow Peraro [172] to review the most important formulae and algorithms. All the algorithms referred to here can be found there.

Most conveniently, we will use finite fields of integers modulo a prime p denoted by $\mathbb{Z}_p$. The operation of taking inverses is defined via

$$(aa^{-1}) \mod p = 1\,. \tag{14.1}$$

This uniquely defines the element $a^{-1} \in \mathbb{Z}_p$ for all non-zero $a \in \mathbb{Z}_p$. The inverse can be efficiently computed via the extended Euclidean algorithm. Together with the definition of addition, subtraction and multiplication modulo p, this defines all required rational operations. We can map any rational number $q = a/b$ into the finite field via

$$q \mod p = (ab^{-1}) \mod p\,. \tag{14.2}$$

Once the system is translated into the finite field, one can use special purpose libraries for the row reduction. An implementation in *C++* is for example the SpaSM (Sparse Solver Module p) library [173].

After having reduced the system, we can apply rational reconstruction methods to obtain the row reduced system over the full set of rational numbers. A variation of the extended Euclidean algorithm allows one to make a guess for a number a mapped back into the rational numbers $q \in \mathbb{Q}$. The guess is, in general, correct if the numerator and denominator of q are much smaller than the prime p. Since integer numbers on computers are restricted in size for efficient implementations, this simple procedure will fail for systems with representationally large rational coefficients.

The Chinese remainder theorem saves the day. It allows a unique reconstruction of a number $a \in \mathbb{Z}_n$ from its images in $\mathbb{Z}_{p_i}$, where $n = p_1 \cdots p_k$ and all the p_i are pairwise co-prime. Thus, we can solve the system several times over different finite fields $\mathbb{Z}_{p_i}$ to increase the probability of correctly reconstructing the rational number. In practice, one can add more and more primes p_i and check if the result is correct by resubstituting it back into the original system. Heuristically, a faster method is to check if for a reconstructed $q = a/b$ the numerator and denominator are below the threshold $\sqrt{n}$, i.e. $a < \sqrt{n}$ and $b < \sqrt{n}$.

15. Epilogue

The ideas and methods summarized in this thesis are small steps towards a deeper understanding of the mathematical structure of scattering amplitudes and their fundamental building blocks. As the complexity for each additional external parton or loop grows factorially, we are challenged to improve our technology for every new calculation. We summarize the main findings and ideas and have a first look into interesting future directions and challenges awaiting us.

A supersymmetric reduction leads to a natural organization of on-shell states in terms of the on-shell superfield formulation for the maximally supersymmetric theory — which extends even to the non-supersymmetric theories. This organization allowed us to find general formulae for cuts involving vector, hyper, chiral, fermionic or scalar partons on the same footing. Combined with the color-kinematics duality and new types of constraints, the number of independent building blocks can be significantly reduced. This in turn simplifies the computation of the complete amplitude and shrinks the algebraic size of its representation. The amplitudes for $\mathcal{N} = 2$ SQCD were shown to have an especially simple structure at one and two loops, manifesting, among others, IR properties in terms of collinear and soft limits.

A further step towards pure (massless) QCD is its supersymmetric extension with $\mathcal{N} = 1$. This theory contains purely chiral matter multiplets. The chirality is expected to introduce new structures implementing the chiral nature in the scattering amplitudes. Compared to $\mathcal{N} = 1$ SQCD, the non-supersymmetric theory has no fundamentally new properties. The same methods will be applicable for a BCJ construction, just the size of the Ansatz will be increased.

Seeking a color-kinematics dual representation might seem unnecessary for a computation solely in a gauge theory. For some examples at one and two loops we have demonstrated that the duality can also help simplifying this computation. The reason is, once more, the significant reduction in the number of independent building blocks. The properties of our integrand representation that relate all terms to a small set of master numerators also lead to a reduction of independent integrals. This immensely simplifies the integration process for the full amplitude. The integrated representation of the four-point two-loop SQCD amplitude was performed to analyze its transcendental properties and general structure. We have identified the source of the non-maximal transcendental terms for the theory at the conformal point.

Equipped with amplitudes for $\mathcal{N} = 0$ QCD through $\mathcal{N} = 4$ SYM, the double copy immediately opens the construction of amplitudes in Einstein gravity (with arbitrary matter) up to maximal supergravity. We have discussed how matter states on the gravity side can be added or removed via a double copy of the matter states in the corresponding gauge theories. This BCJ construction is currently the only method that will allow us to cover new terrain at higher multiplicity or loop order in the study of (supersymmetric) gravity amplitudes and their UV properties. Explicit results obtained in publications attached to this thesis have reproduced an enhanced UV cancellation of half-maximal supergravity in five dimensions. Further cases of such enhanced cancellations are expected for lower degrees of supersymmetry and at higher multiplicity. Their realization in an amplitude representation might lead to novel insights into the UV structure of gravity theories.

A different approach of presenting an amplitude was discussed in the third part. Color decompositions factorize an amplitude into kinematic and color parts. Both, color and kinematic parts, are in the optimal case built out of an independent set of basic building blocks. At tree and one-loop level, minimal decompositions for (S)QCD are known at any multiplicity. The goal of this line of research is to identify a basis of kinematic building blocks for higher loops and, most challenging, for non-planar contributions. A first interesting project would involve the study of identities among primitive amplitudes, i.e. a generalization of Kleiss-Kuijf relations to higher loop amplitudes including fundamental matter.

In the final part of the thesis we have presented purely computational methods that efficiently solve a specific problem. We discussed two methods for the tensor reduction of Feynman integrals and the use of finite fields for general amplitude computations.

It is expected that the methods presented here will be instrumental for further unraveling the properties of gauge and gravity theories with reduced supersymmetry. At two loops four points, any SQCD and supergravity amplitude should be within computational reach. Further refinements of the methods should open up the possibility of performing detailed three-loop studies of SQCD and related supergravity theories.

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Svensk Sammanfattning

Kvantfältteori är ett generellt teoretisk ramverk för beskrivningen av naturens grundläggande partiklar och krafter. En synnerligen viktig kvantfältteori utgörs av standardmodellen inom partikelfysiken (SM). Den beskriver framgångsrikt tre av de fyra elementära krafterna på kvantnivån: elektromagnetism, den svaga samt starka växelverkan. Experimentella utfall av partikelkollisioner, som bland annat beskrivs av spridningsamplituder, kan förutsägas med hjälp av störningsteori i dessa teorier med en extremt hög precision. Den fjärde fundamentala kraften, gravitation, kan inte förenas med SM i ett gemensamt kvantmekaniskt ramverk med vår nuvarande kunskap. Einsteins teori av gravitationen har problem med oändligheter som dyker upp vid beräkningar som involverar höga energier, så kallade ultravioletta (UV) divergenser.

Oberoende av teori så är beräkningar av spridningsamplituder med hjälp av Feynmandiagram mycket komplicerade. Efter första ordningen i störningsserien så växer antalet diagram faktoriellt, och varje diagram är ett komplicerat algebraiskt uttryck som ska integreras över ett flerdimensionellt rum. Nydanande metoder och nya representationer av amplituder, som har utvecklats under de senaste tre årtionden, har märkbart förbättrat effektiviteten av dessa beräkningar och lett oss till en ökad förståelse av deras matematiska struktur.

År 2008 fann Bern, Carrasco och Johansson (BCJ) en ny dualitet inom kvantfältteorier som allmänt kallas för gaugeteorier, där standardmodellens krafter ingår. Dualiteten är mellan kvantiteter som beskriver färgladdning — den starka kraftens laddning — och kinematiska byggstenar som beskriver växelverkningar likt Feynmandiagrammen. Det här leder till en ny representation av spridningsamplituder där färgladdning och kinematik har liknande egenskaper. Dessutom leder denna dualitet till nya relationer mellan olika partiella amplituder (så kallade BCJ relationer).

En enastående konsekvens av dualiteten är den så kallade *dubbelkopian*. Med avstamp av en representation av en amplitud, som uppfyller dualiteten mellan färgladdning och kinematik, kan man konstruera gravitationkraftens spridningsamplituder genom att byta ut färgladdning mot en ytterligare kopia av kinematiska byggstenar. Schematiskt är dubbelkopian

$$\text{gravitation} = (\text{gaugeteori}) \times (\text{gaugeteori}')\,. \tag{15.1}$$

För att testa och studera dualiteten och dubbelkopian kan vi använda olika varianter av gaugeteorier där spridningsamplituderna är av en enklare typ. Supersymmetri innebär ytterligare relationer mellan kraftbärare och materia. Denna symmetry förenklar markant beräkningarna av spridningsamplituder och deras algebraiska uttryck.

I denna avhandling studeras spridningsamplituder i supersymmetrisk kvantkromodynamik (SQCD), som är en supersymmetrisk variant av teorin som beskriver den starka växelverkan. Förutom att beakta färg-kinematik dualitet som den ursprungligen formulerades, så studerar vi en utvidgning utav dualiteten genom att lägga till materia utan vilomassa. Vi diskuterar flera nya metoder och idéer som hjälper till göra beräkningar mer effektiva samt exponerar egenskaper som är av stor nytta. En generalisering av dubbelkopian ger oss en metod för att beräkna spridningsamplituder i supergravitation utan extra materia, samt ger oss tillfälle att studera dess UV struktur.

Representationer som presenteras för $\mathcal{N} = 2$ SQCD i första och andra ordningen av störningsserien har en speciell och enkel form. Varje kinematisk byggsten uttrycks via ett spår över gamma-matriser sammandraget med rörelsemängder. Detta förenklar märkbart integreringen av både gaugeamplituderna och dubbelkopiorna. Det förenklar även identifiering av källor av icke-likformigt transcendenta termer i uttrycken för gaugeamplituderna — samt UV divergenser i dubbelkopian för $\mathcal{N} = 4$ supergravitationen.

Avhandlingen diskuterar även hur man bäst presenterar en gaugeamplitud genom uppdelning av färgkomponenter. Idéen är att man separerar faktorer som endast beror på färgladdning och faktorer som endast beror på kinematik. Målet är att hitta en linjärt oberoende bas av färgkomponenter så att representationen av amplituden är både unik och så enkel som möjligt.

Slutligen presenteras specifika tekniska detaljer för de matematiska metoder som användes för beräkningarna av spridningsamplituderna i denna avhandling.

Appendices

A. Dirac Traces

A classical amplitude computation via Feynman diagrams naturally leads to expressions with traces of Dirac matrices contracted to internal or external momenta. The cut construction discussed in chapter 6 also leads to similar trace structures via combinations of four-dimensional spinor-helicity objects.

We adopt the Weyl basis of gamma matrices for the Clifford algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\,, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}\,, \quad \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -\mathbb{1} & 0 \\ 0 & \mathbb{1} \end{pmatrix}\,, \tag{A.1}$$

where σ^μ are the Pauli matrices. The Dirac traces fulfill the following identities:

$$\mathrm{tr}(\gamma^\mu) = 0\,, \qquad \mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}\,, \qquad \mathrm{tr}(\gamma^\mu\gamma^\nu\gamma_5) = 0\,. \tag{A.2}$$

Using the relation

$$\gamma^\mu\gamma^\nu\gamma^\rho = \eta^{\mu\nu}\gamma^\rho - \eta^{\mu\rho}\gamma^\nu + \eta^{\nu\rho}\gamma^\mu - i\epsilon^{\sigma\mu\nu\rho}\gamma_\sigma\gamma_5\,, \tag{A.3}$$

we can recursively reduce traces of gamma matrices with a larger number of arguments. To get from traces of γ matrices to traces of Pauli matrices we use

$$\begin{aligned} \mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}) &= \mathrm{tr}(\bar{\sigma}^{\mu_1}\sigma^{\mu_2}\cdots\sigma^{\mu_{2n}}) + \mathrm{tr}(\sigma^{\mu_1}\bar{\sigma}^{\mu_2}\cdots\bar{\sigma}^{\mu_{2n}})\,, \\ \mathrm{tr}(\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}) &= \mathrm{tr}(\bar{\sigma}^{\mu_1}\sigma^{\mu_2}\cdots\sigma^{\mu_{2n}}) - \mathrm{tr}(\sigma^{\mu_1}\bar{\sigma}^{\mu_2}\cdots\bar{\sigma}^{\mu_{2n}})\,, \end{aligned} \tag{A.4}$$

which is justified by direct computation using the Weyl basis. This implies

$$\begin{aligned} \mathrm{tr}(\bar{\sigma}^{\mu_1}\sigma^{\mu_2}\cdots\sigma^{\mu_{2n}}) &= \frac{1}{2}\left(\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}) + \mathrm{tr}(\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}})\right)\,, \\ \mathrm{tr}(\sigma^{\mu_1}\bar{\sigma}^{\mu_2}\cdots\bar{\sigma}^{\mu_{2n}}) &= \frac{1}{2}\left(\mathrm{tr}(\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}}) - \mathrm{tr}(\gamma_5\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_{2n}})\right)\,, \end{aligned} \tag{A.5}$$

and the expressions on the right-hand side coincide with the usual definition of tr_+ and tr_- respectively. This directly leads to

$$\begin{aligned} [i_1 i_2]\langle i_2 i_3\rangle\cdots[i_{k-1}i_k]\langle i_k i_1\rangle &= (p_{i_1})_{\mu_1}(p_{i_2})_{\mu_2}\cdots(p_{i_k})_{\mu_k}\,\mathrm{tr}(\bar{\sigma}^{\mu_1}\sigma^{\mu_2}\cdots\bar{\sigma}^{\mu_k}) \\ &\equiv \mathrm{tr}_+(i_1 i_2\cdots i_k)\,, \\ \langle i_1 i_2\rangle[i_2 i_3]\cdots\langle i_{k-1}i_k\rangle[i_k i_1] &= (p_{i_1})_{\mu_1}(p_{i_2})_{\mu_2}\cdots(p_{i_k})_{\mu_k}\,\mathrm{tr}(\sigma^{\mu_1}\bar{\sigma}^{\mu_2}\cdots\sigma^{\mu_k}) \\ &\equiv \mathrm{tr}_-(i_1 i_2\cdots i_k)\,, \end{aligned} \tag{A.6}$$

which we use throughout this thesis to assemble four-dimensional traces.

B. Summary of On-Shell Supermultiplets and their Double Copy

We review the particle content of four and six dimensional on-shell supermultiplets in terms of its representation theory of the little/helicity and R-symmetry group. Furthermore, we summarize how the tensor product represents the particle content of double copied theories.

The little (helicity) group of massless particles in four dimensions is given by U(1). We denote a representation thereof by its helicity $\pm$ together with the spin. Representations in the R-symmetry group $\mathrm{U}(\mathcal{N})$ are given by its integer dimension. Table B.1 summarizes all multiplets in $\mathcal{N} = 8, 6, 4, 2, 1$ up to spin 2, see e.g. [174] for the construction of this table.

For six (and higher) dimensional theories, the supermultiplets for gravity and gauge theories have been nicely reviewed in [175]. We summarize these results here. Six-dimensional states are categorized by their little group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ and R-symmetry group $\mathrm{USp}(2\mathcal{N}) \times \mathrm{USp}(2\widetilde{\mathcal{N}})$ representations. There exist six different types of states categorized by the little group. Graviton states transform under the $(3, 3)$ representation, gravitini either under $(2, 3)$ or $(3, 2)$. For spin one states there are the usual vectors transforming under $(2, 2)$, but we also include tensor states transforming under $(1, 3)$ or $(3, 1)$. Corresponding representations for gauginos are $(1, 2)$ and $(2, 1)$. Scalars finally trivially transform under a $(1, 1)$ representation of the little group. The graviton $\mathcal{H}$, vector $\mathcal{V}$, tensor $\mathcal{T}$ and matter Φ on-shell supermultiplets are summarized in table B.2.

The double copy can be used to construct supergravity on-shell multiplets (or sums thereof) as explained in section 4.2. We summarize construction of gravity supermultiplets as tensor products of vector and matter multiplets for $\mathcal{N} = 0, 1, 2, 4, 6, 8$ supergravity in table B.3.

The same construction for various six dimensional supergravity theories is given in table B.4, see also paper I. It can be derived by computing the tensor product of the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ little group representations and fitting them into multiplets in table B.2.

$\mathcal{N}$		representation/counting							
8	$\mathcal{H}$	$(\pm 2, 1)$	$\oplus$	$(\pm\frac{3}{2}, 8)$	$\oplus$	$(\pm 1, 28)$	$\oplus$	$(\pm\frac{1}{2}, 56)$	$\oplus$
		2	$+$	16	$+$	56	$+$	128	$+$
		$(0, 70)$							
		70	$=$	256					
6	$\mathcal{H}$	$(\pm 2, 1)$	$\oplus$	$(\pm\frac{3}{2}, 6)$	$\oplus$	$(\pm 1, 15+1)$	$\oplus$	$(\pm\frac{1}{2}, 20+6)$	$\oplus$
		2	$+$	12	$+$	32	$+$	52	$+$
		$(0, 15+15)$							
		30	$=$	128					
4	$\mathcal{H}$	$(\pm 2, 1)$	$\oplus$	$(\pm\frac{3}{2}, 4)$	$\oplus$	$(\pm 1, 6)$	$\oplus$	$(\pm\frac{1}{2}, 4)$	$\oplus$
		2	$+$	8	$+$	12	$+$	8	$+$
		$(0, 1+1)$							
		2	$=$	32					
	$\mathcal{V}$	$(\pm 1, 1)$	$\oplus$	$(\pm\frac{1}{2}, 4)$	$\oplus$	$(0, 6)$			
		2	$+$	8	$+$	6	$=$	16	
2	$\mathcal{H}$	$(\pm 2, 1)$	$\oplus$	$(\pm\frac{3}{2}, 2)$	$\oplus$	$(\pm 1, 1)$			
		2	$+$	4	$+$	2	$=$	8	
	$\mathcal{V}$	$(\pm 1, 1)$	$\oplus$	$(\pm\frac{1}{2}, 2)$	$\oplus$	$(0, 2)$			
		2	$+$	4	$+$	2	$=$	8	
	Φ	$(\pm\frac{1}{2}, 1)$	$\oplus$	$(0, 2)$					
		2	$+$	2	$=$	4			
		$(\pm\frac{1}{2}, 1)$	$\oplus$	$(0, 2)$					
		2	$+$	2	$=$	4			
1	$\mathcal{H}$	$(\pm 2, 1)$	$\oplus$	$(\pm\frac{3}{2}, 1)$					
		2	$+$	2	$=$	4			
	$\mathcal{V}$	$(\pm 1, 1)$	$\oplus$	$(\pm\frac{1}{2}, 1)$					
		2	$+$	2	$=$	4			
	Φ	$(+\frac{1}{2}, 1)$	$\oplus$	$(0, 1)$					
		1	$+$	1	$=$	2			
		$(0, 1)$	$\oplus$	$(-\frac{1}{2}, 1)$					
		1	$+$	1	$=$	2			

Table B.1. *This table summarizes the particle content of various four dimensional on-shell supermultiplet. The first row gives the helicity with the spin and the dimensionality of the R-symmetry representation. The second row counts the number of states.*

$(\mathcal{N},\widetilde{\mathcal{N}})$		representation/counting								
$(2,2)$	$\mathcal{H}$	$(3,3;1,1)$	$\oplus$	$(3,2;4,1)$	$\oplus$	$(2,3;1,4)$	$\oplus$	$(2,2;4,4)$	$\oplus$	
		9	$+$	$6\cdot 4$	$+$	$6\cdot 4$	$+$	$4\cdot 16$	$+$	
		$(3,1;1,5)$	$\oplus$	$(1,3;5,1)$	$\oplus$	$(2,1;4,5)$	$\oplus$	$(1,2;5,4)$	$\oplus$	
		$3\cdot 5$	$+$	$3\cdot 5$	$+$	$2\cdot 20$	$+$	$2\cdot 20$	$+$	
		$(1,1;5,5)$								
		25	$=$	256						
$(2,1)$	$\mathcal{H}$	$(3,3;1,1)$	$\oplus$	$(3,2;1,2)$	$\oplus$	$(2,3;4,1)$	$\oplus$	$(2,2;4,2)$	$\oplus$	
		9	$+$	$6\cdot 2$	$+$	$6\cdot 4$	$+$	$4\cdot 8$	$+$	
		$(1,3;5,1)$	$\oplus$	$(3,1;1,1)$	$\oplus$	$(1,2;5,2)$	$\oplus$	$(2,1;4,1)$	$\oplus$	
		$3\cdot 5$	$+$	3	$+$	$2\cdot 10$	$+$	$2\cdot 4$	$+$	
		$(1,1;5,1)$								
		5	$=$	128						
$(1,1)$	$\mathcal{H}$	$(3,3;1,1)$	$\oplus$	$(3,2;1,2)$	$\oplus$	$(2,3;2,1)$	$\oplus$	$(2,2;2,2)$	$\oplus$	
		9	$+$	$6\cdot 2$	$+$	$6\cdot 2$	$+$	$4\cdot 4$	$+$	
		$(1,3;1,1)$	$\oplus$	$(3,1;1,1)$	$\oplus$	$(1,2;1,2)$	$\oplus$	$(2,1;2,1)$	$\oplus$	
		3	$+$	3	$+$	$2\cdot 2$	$+$	$2\cdot 2$	$+$	
		$(1,1;1,1)$								
		1	$=$	64						
	$\mathcal{V}$	$(2,2;1,1)$	$\oplus$	$(2,1;1,2)$	$\oplus$	$(1,2;2,1)$	$\oplus$	$(1,1;2,2)$		
		4	$+$	$2\cdot 2$	$+$	$2\cdot 2$	$+$	$1\cdot 4$		$=16$
$(2,0)$	$\mathcal{H}$	$(3,3;1,1)$	$\oplus$	$(2,3;4,1)$	$\oplus$	$(1,3;5,1)$				
		9	$+$	$6\cdot 4$	$+$	$3\cdot 5$	$=$	48		
	$\mathcal{T}$	$(3,1;1,1)$	$\oplus$	$(2,1;4,1)$	$\oplus$	$(1,1;5,1)$				
		3	$+$	$2\cdot 4$	$+$	5	$=$	16		
$(1,0)$	$\mathcal{H}$	$(3,3;1,1)$	$\oplus$	$(2,3;2,1)$	$\oplus$	$(1,3;1,1)$				
		9	$+$	$6\cdot 2$	$+$	3	$=$	24		
	$\mathcal{V}$	$(2,2;1,1)$	$\oplus$	$(1,2;2,1)$						
		4	$+$	$2\cdot 2$	$=$	8				
	$\mathcal{T}$	$(3,1;1,1)$	$\oplus$	$(1,2;2,1)$	$\oplus$	$(1,1;1,1)$				
		3	$+$	$2\cdot 2$	$+$	1	$=$	8		
	Φ	$(2,1;1,1)$	$\oplus$	$(1,1;2,1)$						
		2	$+$	$1\cdot 2$	$=$	2				
		$(1,2;1,1)$	$\oplus$	$(1,1;2,1)$						
		2	$+$	$1\cdot 2$	$=$	2				

Table B.2. *This table summarizes the particle content of various six dimensional supermultiplet. The first row gives the dimensionalities $(m,n;\tilde{m},\tilde{n})$ of representations of the little and R-symmetry group $SU(2)\times SU(2)\times USp(2\mathcal{N})\times USp(2\widetilde{\mathcal{N}})$. The second row counts the number of states.*

$\mathcal{N}=0+0$	$A^\mu \otimes A^\mu = h^{\mu\nu} \oplus \phi \oplus \overline{\phi}$ $\psi^+ \otimes \psi^- = \overline{\phi}$ $\psi^- \otimes \psi^+ = \phi$
$\mathcal{N}=1+0$	$\mathcal{V}_{\mathcal{N}=1} \otimes A^\mu = \mathcal{H}_{\mathcal{N}=1} \oplus \Phi_{\mathcal{N}=1} \oplus \overline{\Phi}_{\mathcal{N}=1}$ $\Phi_{\mathcal{N}=1} \otimes \psi^- = \overline{\Phi}_{\mathcal{N}=1}$ $\overline{\Phi}_{\mathcal{N}=1} \otimes \psi_+ = \Phi_{\mathcal{N}=1}$
$\mathcal{N}=2+0$	$\mathcal{V}_{\mathcal{N}=2} \otimes A^\mu = \mathcal{H}_{\mathcal{N}=2} \oplus \mathcal{V}_{\mathcal{N}=2}$ $\Phi_{\mathcal{N}=2} \otimes \psi_- = \overline{V}_{\mathcal{N}=2}$ $\overline{\Phi}_{\mathcal{N}=2} \otimes \psi_+ = V_{\mathcal{N}=2}$
$\mathcal{N}=1+1$	$\mathcal{V}_{\mathcal{N}=1} \otimes \mathcal{V}_{\mathcal{N}=1} = \mathcal{H}_{\mathcal{N}=2} \oplus \Phi_{\mathcal{N}=2} \oplus \overline{\Phi}_{\mathcal{N}=2}$ $\Phi_{\mathcal{N}=1} \otimes \overline{\Phi}_{\mathcal{N}=1} = \overline{\Phi}_{\mathcal{N}=2}$ $\overline{\Phi}_{\mathcal{N}=1} \otimes \Phi_{\mathcal{N}=1} = \Phi_{\mathcal{N}=2}$
$\mathcal{N}=4+0$	$\mathcal{V}_{\mathcal{N}=4} \otimes A^\mu = \mathcal{H}_{\mathcal{N}=4}$
$\mathcal{N}=2+2$	$\mathcal{V}_{\mathcal{N}=2} \otimes \mathcal{V}_{\mathcal{N}=2} = \mathcal{H}_{\mathcal{N}=4} \oplus \mathcal{V}_{\mathcal{N}=4} \oplus \overline{\mathcal{V}}_{\mathcal{N}=4}$ $\Phi_{\mathcal{N}=2} \otimes \overline{\Phi}_{\mathcal{N}=2} = \overline{\mathcal{V}}_{\mathcal{N}=4}$ $\overline{\Phi}_{\mathcal{N}=2} \otimes \Phi_{\mathcal{N}=2} = \mathcal{V}_{\mathcal{N}=4}$
$\mathcal{N}=4+2$	$\mathcal{V}_{\mathcal{N}=4} \otimes \mathcal{V}_{\mathcal{N}=2} = \mathcal{H}_{\mathcal{N}=6}$
$\mathcal{N}=4+4$	$\mathcal{V}_{\mathcal{N}=4} \otimes \mathcal{V}_{\mathcal{N}=4} = \mathcal{H}_{\mathcal{N}=8}$

Table B.3. *This table summarizes the double copy construction for various on-shell supermultiplets in* $\mathcal{N}=0,1,2,4,6,8$ *supergravity in four dimensions. For the notation of the various multiplets see chapter 4.*

$\mathcal{N}=(0,0)\otimes(0,0)$	$A^a{}_{\dot a}\otimes A^b{}_{\dot b} = h^{ab}{}_{\dot a\dot b}\oplus B^{ab}\oplus B_{\dot a\dot b}\oplus\phi$ $\chi^a\otimes\tilde\chi_{\dot a} = A^a{}_{\dot a}$ $\tilde\chi_{\dot a}\otimes\chi^a = A^a{}_{\dot a}$ $\chi^a\otimes\chi^b = B^{ab}\oplus\phi$ $\tilde\chi_{\dot a}\otimes\tilde\chi_{\dot b} = B_{\dot a\dot b}\oplus\phi$
$\mathcal{N}=(1,0)\otimes(0,0)$	$\mathcal{V}_{\mathcal{N}=(1,0)}\otimes A^a{}_{\dot a} = \mathcal{H}_{\mathcal{N}=(1,0)}\oplus\mathcal{T}_{\mathcal{N}=(1,0)}$ $\Phi_{\mathcal{N}=(1,0)}\otimes\chi^a = \mathcal{T}_{\mathcal{N}=(1,0)}$ $\Phi_{\mathcal{N}=(1,0)}\otimes\tilde\chi_{\dot a} = \mathcal{V}_{\mathcal{N}=(1,0)}$ $\overline{\Phi}_{\mathcal{N}=(1,0)}\otimes\chi^a = \mathcal{V}_{\mathcal{N}=(1,0)}$
$\mathcal{N}=(1,0)\otimes(1,0)$	$\mathcal{V}_{\mathcal{N}=(1,0)}\otimes\mathcal{V}_{\mathcal{N}=(1,0)} = \mathcal{H}_{\mathcal{N}=(2,0)}\oplus\mathcal{T}_{\mathcal{N}=(2,0)}$ $\Phi_{\mathcal{N}=(1,0)}\otimes\Phi_{\mathcal{N}=(1,0)} = \mathcal{T}_{\mathcal{N}=(2,0)}$
$\mathcal{N}=(1,1)\otimes(0,0)$	$\mathcal{V}_{\mathcal{N}=(1,1)}\otimes A^a{}_{\dot a} = \mathcal{H}_{\mathcal{N}=(1,1)}$
$\mathcal{N}=(1,0)\otimes(0,1)$	$\mathcal{V}_{\mathcal{N}=(1,0)}\otimes\mathcal{V}_{\mathcal{N}=(0,1)} = \mathcal{H}_{\mathcal{N}=(1,1)}$ $\Phi_{\mathcal{N}=(1,0)}\otimes\overline{\Phi}_{\mathcal{N}=(0,1)} = \mathcal{V}_{\mathcal{N}=(1,1)}$ $\overline{\Phi}_{\mathcal{N}=(1,0)}\otimes\Phi_{\mathcal{N}=(0,1)} = \mathcal{V}_{\mathcal{N}=(1,1)}$
$\mathcal{N}=(1,1)\otimes(1,0)$	$\mathcal{V}_{\mathcal{N}=(1,1)}\otimes\mathcal{V}_{\mathcal{N}=(1,0)} = \mathcal{H}_{\mathcal{N}=(2,1)}$
$\mathcal{N}=(1,1)\otimes(1,1)$	$\mathcal{V}_{\mathcal{N}=(1,1)}\otimes\mathcal{V}_{\mathcal{N}=(1,1)} = \mathcal{H}_{\mathcal{N}=(2,2)}$

Table B.4. *This table summarizes the double copy construction for various on-shell supermultiplets in $\mathcal{N}=(0,0),(1,0),(2,0),(1,1),(2,1),(2,2)$ supergravity in six dimensions. For the notation of the various multiplets see chapter 4.*

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