Some classical features of polynomial higher derivative gravities

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In this talk we discuss some classical aspects of general polynomial higher-derivative gravity. In particular, we describe the behaviour of the weak-field solutions associated to a point-like mass at small distances and provide necessary and sufficient conditions for the metric to be regular. We also consider the metric for a collapsing thick null shell, and verify that it is regular if the aforementioned conditions are valid.

Keywords: Higher-derivative gravity; Lee-Wick gravity; Spacetime singularities.

1. Introduction

Higher-derivative theories of gravity possess interesting features from both quantum and classical point of views. Indeed, the inclusion of higher-derivative terms in the Lagrangian can make the theory perturbatively renormalizable,[1,2] while they can also smooth out classical singularities.[3–6] Owed to these good properties, higher derivatives are often considered in the search for a fundamental theory of gravity. One of the main difficulties of this approach to quantum gravity, nonetheless, is the presence of ghost-like degrees of freedom, which classically generate instabilities in the solutions and, in the quantum perspective, violate unitarity. For example, consider the model defined by the action[2]

$$S = \frac{1}{4\kappa} \int d^4x \sqrt{-g} \left(2R + RF_1(\Box)R + R_{\mu\nu}F_2(\Box)R^{\mu\nu} \right), \tag{1}$$

where $F_j(\Box)$ is a real polynomial of degree δ_j of the d'Alembert operator. The propagator associated this model, in the Landau gauge, is given by

$$G_{\mu\nu,\alpha\beta}(k) = \frac{P^{(2)}_{\mu\nu,\alpha\beta}}{k^2 f_2(-k^2)} - \frac{P^{(0-s)}_{\mu\nu,\alpha\beta}}{2k^2 f_0(-k^2)}, \tag{2}$$

where f_0 and f_2 are polynomial functions of degree $d_0 = \max\{\delta_1, \delta_2\} + 1$ and $d_2 = \delta_2 + 1$, respectively, defined as

$$f_0(\Box) = 1 - [F_2(\Box) + 3F_1(\Box)]\,\Box, \quad f_2(\Box) = 1 + \frac{1}{2}\,F_2(\Box)\,\Box. \tag{3}$$

Hence, if the polynomial $f_s(z)$ has N_s distinct real roots $z = -m^2_{(s)i}$ with multiplicity $n_{(s)i}$, then the propagator has $N_0 + N_2$ massive poles. In Ref. 2 it was shown that half of these excitations correspond to ghost modes.

Recently there have been some proposals for dealing with these ghosts still in the framework of higher-derivative gravity (HDG). For example, in the Lee-Wick HDG[7,8] the ghost-like poles of the propagator are associated to complex masses. Such modes can then appear as virtual states only, yielding an unitary scattering matrix.[7] Another approach to avoid ghosts is to make a non-local extension of the polynomial HDG by using non-polynomial functions F_j such that the propagator has no other degrees of freedom besides the graviton.[9–13]

In this talk we present some classical aspects of the general polynomial-derivative gravity model (1), namely, we discuss the avoidance of Newtonian singularities. We remark that, in the weak field approximation, the local model (1) is the most general one with higher derivatives, and it contains the case of Lee-Wick gravity as particular case. The original results presented here were published in Ref. 14. Using similar arguments it is possible to extend considerations to some classes of non-local gravity theories, as carried out in a more general manner in Ref. 15.

2. Singularities in the Newtonian limit

In order to evaluate the field generated by a point-like source in the static non-relativistic weak-field approximation we consider metric fluctuations around the Minkowski spacetime, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, sourced by the energy-momentum tensor $T_{\mu\nu} = M\,\delta^3(\mathbf{r})\,\delta^0_\mu\,\delta^0_\nu$. In this case the metric can be written in the isotropic form

$$ds^2 = -\left[1 + \frac{2}{3}(2\chi_2 + \chi_0)\right] dt^2 + \left[1 - \frac{2}{3}(\chi_2 - \chi_0)\right](dx^2 + dy^2 + dz^2) \qquad (4)$$

and one can show that the (linearised) equations of motion for the potentials χ_s ($s = 0, 2$) are equivalent to solving[14]

$$f_s(\Delta)\Delta\chi_s = \kappa_s\, M\, \delta^3(\mathbf{r})\,, \qquad (5)$$

with $\kappa_0 = -\kappa/2$, $\kappa_2 = \kappa$. We remark that the decomposition of the usual Newtonian potentials into $\varphi = \frac{1}{3}(2\chi_2 + \chi_0)$ and $\psi = \frac{1}{3}(\chi_2 - \chi_0)$ allows a great simplification in the notation and considerations, as it splits the contribution of the scalar and spin-2 modes, through χ_0 and χ_2, respectively.

The solution of (5) can be obtained by the Laplace[14,16] or Fourier[15] transform technique and reads

$$\chi_s(r) = -\frac{\kappa_s M}{4\pi r} + \frac{\kappa_s M}{4\pi^{3/2}} \sum_{i=1}^{N_s} \sum_{j=1}^{n_{(s)i}} \frac{a_{(s)i,j}}{(j-1)!} \left(\frac{r}{2m_{(s)i}}\right)^{j-\frac{3}{2}} K_{j-\frac{3}{2}}(m_{(s)i} r)\,, \qquad (6)$$

where K_ν is the modified Bessel function of the second kind. Also, $z = -m^2_{(s)i}$ is one of the N_s distinct roots of the equation $f_s(-z) = 0$ and $n_{(s)i}$ is its multiplicity. Of course, if d_s is the degree of $f_s(z)$, then $\sum_i n_{(s)i} = d_s$. The coefficient $a_{(s)i,j}$ can be obtained by the Heaviside residue method, and follows from the partial fraction decomposition of $[zf_s(-z)]^{-1}$.

The solution above can be expanded in power series around $r = 0$, which gives

$$\chi_s(r) = -\frac{\kappa_s M}{4\pi r}\left(1 - S_s^{(0)}\right) + c_s + \frac{\kappa_s M}{8\pi}\sum_{i=1}^{N_s}\left(S_s^{(1)} - S_s^{(2)}\right) r + O(r^2)\,, \tag{7}$$

where c_s is a constant and

$$S_s^{(0)} = \sum_{i=1}^{N_s} a_{(s)i,1}\,, \qquad S_s^{(1)} = \sum_{i=1}^{N_s} m_{(s)i}^2 a_{(s)i,1}\,, \qquad S_s^{(2)} = \sum_{i=1}^{N_s} a_{(s)i,2}. \tag{8}$$

It is possible to show[5,14] that $S_s^{(0)} = 1$ for any $d_s \geqslant 1$, which means that all the theories which have at least four derivatives in the spin-s sector have a finite potential $\chi_s(r)$ at $r = 0$. Moreover, one can prove[14,15] that $S_s^{(1)} = S_s^{(2)}$ if and only if $d_s \geqslant 2$ — in other words, $\chi_s'(0) = 0$ in all theories with more than four derivatives in the spin-s sector. It can be shown that, for these models, the condition $\chi_0'(0) = \chi_2'(0) = 0$ is necessary and sufficient to have regular curvature invariants.[14,16] Thus, it follows that the metric (4) is regular in all theories which have at least six derivatives in both spin-0 and spin-2 sectors.[14,15]

3. Singularities in the ultrarelativistic limit

The static Newtonian solution presented in the previous section can be used to construct the metric of a non-spinning gyraton (see, *e.g.*, Refs. 17, 18). The general idea is to apply a boost to the non-relativistic metric and then take the Penrose limit. With this solution one can consider a homogeneous spherical shell distribution of gyratons with total mass M imploding towards its centre, and analyse the occurrence of singularities and the formation of mini black holes.[16,17,19]

In Ref. 14 it was shown that the Kretschmann scalar associated to the $\mathcal{I}$ domain[a] of the collapsing thick null shell in a general polynomial theory is

$$R^2_{\mu\nu\alpha\beta} = \frac{32G^2M^2}{3\tau^2}\left[4\Delta^2(\ln r)^2 + c'\Delta\ln r + c''\Delta + \left(S_2^{(0)} - S_2^{(1)}\right)^2\right] + O(r^2)\,, \tag{9}$$

where $\tau > 0$ is the thickness of the shell, $\Delta \equiv S_2^{(1)} - S_2^{(2)}$ and c' and c'' are constants which depend on the massive parameters of the theory. Inasmuch as $\Delta = 0$ for the theories with more than four derivatives in the spin-2 sector, the collapse of the thick null shell does not generate a singularity. On the other hand, the Kretschmann invariant diverges for models with less than six derivatives.

[a]The $\mathcal{I}$ domain, defined by the locus of the spacetime points for which $r+|t| < \tau/2$, is characterized by the intersection of the in-coming and the out-coming fluxes of null fluid. In this domain the shell assumes its highest density, favouring the mini black hole formation and the emergence of singularities.

The solution for the collapsing shell can also be used to verify the existence of mass gap for the formation of mini black holes. This is related to the occurrence of apparent horizons in the solution, *i.e.*, regions such that $g \equiv (\nabla \varrho)^2 = 0$, where ϱ is the component $g_{\theta\theta}$ of the spherically symmetric metric. In fact, one can show that the invariant g reads

$$g(r) \,=\, 1 + \frac{2GM(S_2^{(0)} - S_2^{(2)})r^2}{3\tau} \,+\, O(r^4) \tag{10}$$

for the collapsing thick null shell in any polynomial gravity theory.[14] Since $r < \tau$ on the domain $\mathcal{I}$, it follows that

$$\frac{2GM|S_2^{(0)} - S_2^{(2)}|r^2}{3\tau} < \frac{2GM|S_2^{(0)} - S_2^{(2)}|\tau}{3}\,. \tag{11}$$

In other words, given any τ it is possible to avoid the existence of an apparent horizon on $\mathcal{I}$ provided that the total mass M of the shell is sufficiently small. This result was obtained for the first time in Ref. 19 for the theory with only four derivatives; in Refs. 17 and 16 it was extended for the ghost-free gravity and polynomial gravity with only real and non-degenerate masses, while in Ref. 14 general polynomial theories were considered, including the case of Lee-Wick gravity.

4. Conclusions

The results presented here show that there is a significant difference of the HDG theories with four and more derivatives. Even though the modified Newtonian potential is finite in both cases (if there are at least four derivatives in the spin-2 and spin-0 sectors), the former always contains curvature singularities in the solution associated to a point-like mass. On the other hand, these singularities are regularized in the models with more than four derivatives. The same situation occurs for the collapsing thick null shell, for which curvature singularities are avoided only in the models with six or more derivatives.

These results bring more motivations for further studies on the occurrence of singularities in the spherically symmetric static solutions in the full non-linear regime of HDG models. For example, the numerical searches for solutions in theories with 6, 8 and 10 derivatives reported in Ref. 20 only found regular solutions. In what concerns the fourth-derivative gravity, it is known that singularities are present in both regimes.[3,21,22] This subject certainly deserves more investigation.

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