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Article

Coupled Quintessence Inspired by Warm Inflation

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Abstract: We investigate a coupled quintessence cosmological model in which a dark-energy scalar field with an exponential potential interacts directly with a dark-matter fluid through a dissipative term inspired by warm inflation. The evolution equations of this model give rise to a three-dimensional dynamical system for which a thorough qualitative analysis is performed for all values of the relevant parameters. We find that the model is able to replicate the observed sequence of late-time cosmological eras, namely, a long enough matter-dominated era followed by a present era of accelerated expansion. In situations where there is a significant transfer of energy from dark energy to dark matter, temporary scaling-type solutions may arise, but, asymptotically, all solutions are dominated by dark energy.

Keywords: coupled quintessence; dark energy; dark matter; cosmology

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1. Introduction

One of the most remarkable developments in modern cosmology was the discovery, in the late 1990s, that the Universe is undergoing a period of accelerated expansion [1,2] due to an unknown form of energy called dark energy. Today, this acceleration is firmly established by several other independent precise cosmological observations [3]. In particular, recent measurements of baryon acoustic oscillations from the first year of observations from the Dark Energy Spectroscopic Instrument (DESI) confirm, with unprecedented precision, the accelerated expansion of the Universe [4].

The simplest candidate for dark energy is a cosmological constant Λ, accounting for about 68% of the current total energy density of the Universe [5]. The other components of the standard cosmological model—dark matter, baryonic matter, and radiation—account for about 27%, 5%, and 0.005% of the total energy density, respectively [5,6].

A theoretically appealing alternative is to consider that the role of dark energy is played not by a cosmological constant but rather by a dynamic scalar field whose potential energy dominates the current phase of the evolution of the Universe [7], inducing the observed cosmic acceleration. Such a possibility seems to be preferred—albeit moderately—by DESI's first-year results when they are combined with data from other studies [4]. This quintessence scalar field can in principle be directly coupled to dark matter—whose precise nature is also unknown [8]—giving rise to coupled quintessence [9].

If the nature of dark energy and dark matter are currently unknown, much more so is the form of an eventual non-gravitational coupling between them. This circumstance has allowed for great freedom in the construction of coupled quintessence models (also known as interacting dark energy models) [9–38].

A popular choice in the literature identifies dark matter with a pressureless non-relativistic perfect fluid and considers the potential of the dark-energy scalar field ϕ to be of the exponential type and the interaction term between the dark components to be of the form $Q \propto \rho_{\mathrm{DM}}\dot{\phi}$, where ρ_{DM} is the energy density of the dark-matter fluid and an overdot denotes a derivative with respect to cosmic time t. Such an interaction term, which has been motivated by scalar–tensor theories, allows for the existence of late-time accelerated scaling solutions [10–12]. Because of its potential to address the cosmological coincidence problem [39] and the Hubble constant tension [40,41], this coupled quintessence model has attracted substantial attention over the years.

More recently, a generalized interaction term of the form $Q \propto \rho_{\mathtt{DM}}\dot{\phi}C(\phi)$, where $C(\phi) \propto \phi^n$ and n is a positive integer, was considered [35]. With such a coupling between dark energy and dark matter, the quintessence model no longer admits scaling attractor solutions, but, interestingly, for certain values of a relevant parameter, during the approach to the dark-energy-dominated final state, the solution mimics an accelerated scaling solution. Naturally, this result raises the question of how common accelerated scaling solutions are in coupled quintessence models. We deem this issue to deserve further investigation.

In this article, we investigate a coupled quintessence model with an interaction term Q, inspired by warm inflation.

According to the warm-inflation paradigm [42], energy is continuously transferred from the inflaton field ψ to a radiation bath, thus ensuring that the energy density of the latter, $\rho_{\mathtt{R}}$, is not diluted during the inflationary expansion and that a smooth transition to a radiation-dominated era can occur without the need for a separate post-inflationary reheating phase (for recent reviews on warm inflation, see [43,44]). As a result, during the inflationary period, the evolution equations for the inflaton field and radiation require an extra dissipative term, becoming

$$\ddot{\psi} + 3H\dot{\psi} + \frac{\partial V}{\partial \psi} = -\Gamma\dot{\psi}, \tag{1}$$

$$\dot{\rho}_{\mathtt{R}} + 4H\rho_{\mathtt{R}} = \Gamma\dot{\psi}^2, \tag{2}$$

where H is the Hubble parameter, $V = V(\psi)$ the potential of the inflaton field, and Γ the so-called dissipation coefficient, which, in general, is a function of the inflaton field and the temperature of the radiation bath, $\Gamma = \Gamma(\psi, T)$.

The above-described warm-inflation paradigm can be realized in realistic cosmological models, yielding results for the scalar spectral index n_s and the tensor-to-scalar ratio r, consistent with Planck observations (see [43,44] and references therein). Furthermore, it has been shown that warm inflation is favored by the de Sitter swampland conjectures [45–48].

Warm-inflation-type dissipation processes can be present at later stages of the Universe's evolution, giving rise, in particular, to a direct coupling between dark energy and dark matter. Such a possibility was recently implemented in the context of a steep-potential quintessence inflationary model, allowing for the unification of early and late stages of the evolution of the Universe through dissipative effects [49,50]. Here, we consider a coupled quintessence model in which, at late times, a dark-energy scalar field ϕ interacts directly with a dark-matter fluid through a dissipative term of the type $Q = \Gamma\dot{\phi}^2$. As a first approach, we choose the dissipation coefficient Γ to be constant, leaving more complex cases for future publications.

This article is organized as follows. In Section 2, we present our coupled quintessence cosmological model and write the corresponding evolution equations as a three-dimensional dynamical system. In Section 3, we investigate the stability properties of this dynamical system, identify the global attractors, and describe the trajectories that correspond to the relevant late-time cosmological solutions. Finally, in Section 4, we present our conclusions.

2. The Coupled Quintessence Cosmological Model

Assuming a flat Friedmann–Lemaître–Robertson–Walker metric,

$$ds^2 = -dt^2 + a^2(t)d\Sigma^2, \tag{3}$$

where $a(t)$ is the scale factor and $d\Sigma^2$ is the metric of the three-dimensional Euclidean space, the evolution equations of the coupled quintessence cosmological model are given by

$$H^2 = \frac{\kappa^2}{3}\left(\frac{\dot{\phi}^2}{2} + V + \rho_{\mathrm{DM}}\right), \tag{4}$$

$$\dot{H} = -\frac{\kappa^2}{2}\left(\dot{\phi}^2 + \rho_{\mathrm{DM}}\right), \tag{5}$$

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\partial V}{\partial \phi} = \frac{Q}{\dot{\phi}}, \tag{6}$$

$$\dot{\rho}_{\mathrm{DM}} + 3H\rho_{\mathrm{DM}} = -Q. \tag{7}$$

In the above equations, ϕ is the quintessence dark-energy scalar field with potential $V(\phi)$, ρ_{DM} is the energy density of a pressureless dark-matter fluid, $H \equiv \dot{a}/a$ is the Hubble parameter, Q is the interaction term between dark energy and dark matter, overdots denote derivatives with respect to cosmic time t, and we use the notation $\kappa \equiv \sqrt{8\pi G} = \sqrt{8\pi}/m_{\mathrm{P}}$, where G is the gravitational constant and m_{P} is the Planck mass.

Defining the energy density and pressure of the scalar field ϕ as

$$\rho_\phi = \frac{\dot{\phi}^2}{2} + V(\phi) \quad \text{and} \quad p_\phi = \frac{\dot{\phi}^2}{2} - V(\phi), \tag{8}$$

Equation (6) can be written as

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = Q. \tag{9}$$

In the evolution Equations (4)–(7), for the sake of simplicity, ordinary baryonic matter was neglected since it makes a small contribution to the total energy density of matter. Additionally, radiation was also excluded since we are only interested in late-time cosmological solutions.

At present, no fundamental underlying theory specifies the exact form of the interaction term Q between dark energy and dark matter. Hence, any approach to this problem is necessarily phenomenological, and the selection of the most suitable interaction model will be decided, ultimately, by precise cosmological observations.

In this spirit, one could consider the energy transfer between the dark components to be dictated by non-local quantities, for instance, by the Universe's expansion rate H, yielding an interaction term of the form $Q \propto Hf(\rho_\phi, \rho_{\mathrm{DM}})$, where f is some function of ρ_ϕ and ρ_{DM}. Such an approach has been rather popular in the literature (for a review, see Refs. [51,52].

Alternatively, we could relate this energy transfer to local dissipative effects as in the warm inflationary scenario [43,44]. This is our option in this article. More specifically, we choose the coupling between dark energy and dark matter to be of the form

$$Q = \Gamma\dot{\phi}^2, \tag{10}$$

where Γ is a dissipation coefficient determined only by local properties of the dark-sector interactions. Here, we are assuming that the dissipative processes occurring in the early Universe are also present at later stages of evolution, namely, during the matter-dominated era and the current era of accelerated expansion driven by the potential energy of the quintessence field. This is a natural assumption. Indeed, if seemingly disparate phenomena, like inflation, dark matter, and dark energy, can be unified under the same theoretical framework (for such a triple unification, see, for instance, Ref. [30], one could also expect the interactions between the underlying particles/fields to somehow remain active during all stages of the Universe's evolution. These interactions would, then, give rise to the interaction term (10), with a dissipation coefficient Γ, depending both on the scalar field ϕ and the dark-matter energy density ρ_{DM} [49].

In this article, as a first approach, we choose Γ to be a constant (with dimension of mass), leaving more general dissipative coefficients for future work.

Note that, with an interaction term of the form (10), the evolution equation for the quintessence scalar field (6) can be written as

$$\ddot{\phi} + (3H - \Gamma)\dot{\phi} + \frac{\partial V}{\partial \phi} = 0, \tag{11}$$

revealing that the direct interaction between the dark components modifies the so-called "Hubble friction" term. In what follows, we will consider both possibilities that this term is enhanced or diminished due to the direct interaction with the dark-matter fluid.

In what concerns the potential of the dark-energy scalar field, we choose it to be of the exponential form,

$$V(\phi) = V_a e^{-\mu\kappa\phi}, \tag{12}$$

where V_a is a positive constant with dimension $(\text{mass})^4$ and μ is a dimensionless constant.

To study the cosmological solutions of the system of Equations (4)–(7), we resort to the powerful method of qualitative analysis of dynamical systems.

To this end, we introduce the dimensionless variables

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H}, \quad y = \frac{\kappa\sqrt{V}}{\sqrt{3}H}, \quad \text{and} \quad z = \frac{H_*}{H\Omega_{\mathtt{DM}} + H_*}, \tag{13}$$

as well as a new time variable τ, defined as

$$\frac{d\tau}{dt} = \frac{H}{1-z}, \tag{14}$$

where $\Omega_{\mathtt{DM}} \equiv \kappa^2\rho_{\mathtt{DM}}/(3H^2)$ is the dark-matter density parameter and H_* is a positive constant with dimension of mass (the dimensionless variable z should not be confused with the cosmological redshift, defined as $a_0/a(t) - 1$, where a_0 is the present-time value of the scale factor). Note that, from Equation (4), it immediately follows

$$\Omega_{\mathtt{DM}} = 1 - x^2 - y^2. \tag{15}$$

Before proceeding, some comments are in order about the choice of the new variables z and τ (the choice of x and y is the usual one [53]). The interaction term between dark energy and dark matter, given by Equation (10), cannot be written as a function of the variables x and y only; therefore, an extra variable z is required to close the dynamical system. In choosing it, we must ensure that the surface $x^2 + y^2 = 1$ becomes an invariant manifold of the dynamical system, so that no trajectory can cross this surface and enter the region of (unphysical) negative values of $\Omega_{\mathtt{DM}}$. In addition, we also want to compactify the phase space in the z direction, say between $z = 0$, corresponding to $H = +\infty$, and $z = 1$, corresponding to $H = 0$. The simplest choice of z that satisfies these requirements is given in Equation (13). However, with such z, the interaction term appearing in the evolution equation for x becomes proportional to $(1-z)^{-1}$ and, consequently, diverges as $z \to 1$. We could study the properties of the dynamical system on the $z = 1$ plane by just considering the divergent term and neglecting the others. Instead, we opt to remove the singularity by choosing $d\tau$ proportional to $(1-z)^{-1}$, which amounts to multiplying the right-hand side of the evolution equations by $1 - z$. This procedure allows us to study the dynamical system on the $z = 1$ plane (without any divergent terms) and simultaneously preserves the stability properties in other regions of phase space.

In the variables x, y, z, and τ, the evolution Equations (4)–(7) of the coupled quintessence cosmological model reduce to a three-dimensional dynamical system, namely,

$$x_\tau = \left[-3x + \frac{\sqrt{6}}{2}\mu y^2 + \frac{3}{2}x(1 + x^2 - y^2) \right](1 - z) + \alpha x(1 - x^2 - y^2)z, \tag{16a}$$

$$y_\tau = \left[-\frac{\sqrt{6}}{2}\mu x + \frac{3}{2}(1 + x^2 - y^2) \right] y(1 - z), \tag{16b}$$

$$z_\tau = \left[\frac{3}{2}(1 - x^2 + y^2)(1 - z) + 2\alpha x^2 z \right] z(1 - z), \tag{16c}$$

where $\alpha = \Gamma/H_*$ is a dimensionless constant parameterizing the energy exchange between dark matter and dark energy.

Inspection of this dynamical system reveals that the surfaces $y = 0$, $z = 0$, and $z = 1$ are invariant manifolds. Furthermore, from the evolution equation for the dark-matter density parameter, obtained from Equations (15), (16a) and (16b),

$$\Omega_{\mathtt{DM},\tau} = \Omega_{\mathtt{DM}}\left[3(x^2 - y^2)(1 - z) - 2\alpha x^2 z \right], \tag{17}$$

we conclude that the surface $x^2 + y^2 = 1$ ($\Omega_{\mathtt{DM}} = 0$) is also an invariant manifold. Taking into account that $\Omega_{\mathtt{DM}}$ is, by definition, non-negative, and restricting ourselves to expanding cosmologies, the phase space of the dynamical system (16) is then the half-cylinder $\{(x, y, z) \,|\, x^2 + y^2 \leq 1, y \geq 0, 0 \leq z \leq 1\}$.

In what concerns the parameter space, let us point out that the dynamical system (16) is invariant under the transformation $x \to -x$ and $\mu \to -\mu$, implying that, without any loss of generality, the parameter μ can be assumed to be positive. Furthermore, we assume that the parameter α can be either positive (energy is transferred from the dark-matter fluid to the dark-energy scalar field) or negative (energy is transferred in the opposite direction). Altogether, this means that the parameter space of our coupled quintessence model is $\{(\alpha, \mu) \,|\, \alpha \neq 0, \mu > 0\}$.

In the dimensionless variables x, y, and z, the dark-energy density parameter and the effective equation-of-state parameter are given by

$$\Omega_\phi \equiv \frac{\kappa^2}{3H^2}\rho_\phi = x^2 + y^2 \tag{18}$$

and

$$w_{\mathtt{eff}} \equiv \frac{p_\phi}{\rho_{\mathtt{DM}} + \rho_\phi} = x^2 - y^2, \tag{19}$$

respectively, where ρ_ϕ and p_ϕ are defined in Equation (8). The latter quantity may take values in the range $[-1, 1]$. When the evolution of the Universe is completely dominated by the potential $V(\phi)$, the effective equation-of-state parameter equals -1, and one recovers the cosmological constant case. When the dynamics is dominated by the dark-matter fluid or the scalar-field kinetic term, one obtains $w_{\mathtt{eff}} = 0$ or $w_{\mathtt{eff}} = 1$, respectively. Expansion is accelerated if $-1 \leq w_{\mathtt{eff}} < -1/3$.

The coupled phantom cosmological model inspired by warm inflation, in which $w_{\mathtt{eff}}$ may be less than -1, was studied in Ref. [54].

3. Cosmological Solutions

The dynamical system (16) has two critical lines and, depending on the values of the parameters α and μ, up to six critical points. Table 1 summarizes their properties, namely, the conditions for their existence and the physical behavior of the solutions in their vicinity, while Table 2 summarizes the stability properties of the critical points and lines.

Table 1. Properties of the critical points and critical lines of the dynamical system (16). Ω_ϕ, $\Omega_{\mathtt{DM}}$, and $w_{\mathtt{eff}}$ are, respectively, the dark-energy density parameter, the dark-matter density parameter, and the effective equation-of-state parameter. Accelerated expansion of the Universe occurs for $w_{\mathtt{eff}} < -1/3$.

Critical Point/Line	Existence	Ω_ϕ	$\Omega_{\mathtt{DM}}$	$w_{\mathtt{eff}}$	Acceleration
$A(1,0,0)$	$\mu > 0, \alpha \neq 0$	1	0	1	no
$B(-1,0,0)$	$\mu > 0, \alpha \neq 0$	1	0	1	no
$C(0,0,0)$	$\mu > 0, \alpha \neq 0$	0	1	0	no
$D\left(\frac{\mu}{\sqrt{6}}, \sqrt{1-\frac{\mu^2}{6}}, 0\right)$	$0 < \mu \leq \sqrt{6}, \alpha \neq 0$	1	0	$-1+\frac{\mu^2}{3}$	$0 < \mu < \sqrt{2}$
$E\left(\frac{\sqrt{6}}{2\mu}, \frac{\sqrt{6}}{2\mu}, 0\right)$	$\mu \geq \sqrt{3}, \alpha \neq 0$	$\frac{3}{\mu^2}$	$1-\frac{3}{\mu^2}$	0	no
$F\left(\frac{\mu}{\sqrt{6}}, \sqrt{1-\frac{\mu^2}{6}}, \frac{3(\mu^2-6)}{(2\alpha+3)\mu^2-18}\right)$	$0 < \mu < \sqrt{6}, \alpha < 0$ or $\mu = \sqrt{6}, \alpha \neq 0$	1	0	$-1+\frac{\mu^2}{3}$	$0 < \mu < \sqrt{2}$
$G(0,y,1)$	$\mu > 0, \alpha \neq 0, 0 \leq y \leq 1$	y^2	$1-y^2$	$-y^2$	$\frac{1}{\sqrt{3}} < y \leq 1$
$H\left(x, \sqrt{1-x^2}, 1\right)$	$\mu > 0, \alpha \neq 0, -1 \leq x \leq 1$	1	0	$-1+2x^2$	$\lvert x\rvert < \frac{1}{\sqrt{3}}$

Table 2. The stability of the critical points and critical lines of the dynamical system (16). None of the critical points A to F are stable. For positive α (transfer of energy from dark matter to dark energy), the critical line $H\left(x, \sqrt{1-x^2}, 1\right)$ is an attractor, while for negative α (transfer of energy from dark energy to dark matter), the attractor is the point $G(0,1,1) = H(0,1,1)$, lying at the intersection of the critical lines.

Critical Point/Line	Eigenvalues	Stability
$A(1,0,0)$	$\left\{3, 0, 3-\frac{\sqrt{6}}{2}\mu\right\}$	no
$B(-1,0,0)$	$\left\{3, 0, 3+\frac{\sqrt{6}}{2}\mu\right\}$	no
$C(0,0,0)$	$\left\{-\frac{3}{2}, \frac{3}{2}, \frac{3}{2}\right\}$	no
$D\left(\frac{\mu}{\sqrt{6}}, \sqrt{1-\frac{\mu^2}{6}}, 0\right)$	$\left\{3-\frac{\mu^2}{2}, -3+\frac{\mu^2}{2}, -3+\mu^2\right\}$	no
$E\left(\frac{\sqrt{6}}{2\mu}, \frac{\sqrt{6}}{2\mu}, 0\right)$	$\left\{\frac{3}{2}, -\frac{3}{4}\left(1-\sqrt{\frac{24}{\mu^2}-7}\right), -\frac{3}{4}\left(1+\sqrt{\frac{24}{\mu^2}-7}\right)\right\}$	no
$F\left(\frac{\mu}{\sqrt{6}}, \sqrt{1-\frac{\mu^2}{6}}, \frac{3(\mu^2-6)}{(2\alpha+3)\mu^2-18}\right)$	$\left\{\frac{\alpha\mu^4}{(2\alpha+3)\mu^2-18}, \frac{\alpha\mu^2(\mu^2-6)}{(2\alpha+3)\mu^2-18}, \frac{\alpha\mu^2(\mu^2-6)}{(2\alpha+3)\mu^2-18}\right\}$	no
$G(0,y,1)$	$\{0, 0, \alpha(1-y^2)\}$	$\mu > 0, \alpha < 0, y = 1$
$H\left(x, \sqrt{1-x^2}, 1\right)$	$\{0, -2\alpha x^2, -2\alpha x^2\}$	$\mu > 0, \alpha > 0, \lvert x\rvert \leq 1$ or $\mu > 0, \alpha < 0, x = 0$

The critical points A, B, C, D, and E, lying on the plane $z = 0$, are those already present in the uncoupled quintessence model. This follows from the fact that for $z = 0$, Equations (16a) and (16b) decouple from Equation (16c), yielding the two-dimensional dynamical system of Ref. [53], which describes a quintessence scalar field interacting only through gravity with a pressureless fluid. The critical point F and the critical lines G and H are new; they arise due to the introduction of a direct interaction term between dark energy and dark matter.

Inspection of the eigenvalues shown in Table 2 for all possible values of the parameters α and μ belonging to the parameter space reveals that none of the critical points A to F can be an attractor, because at least one of the corresponding eigenvalues is positive. The same applies to the critical lines G for $\alpha > 0$ ($y \neq 1$) and H for $\alpha < 0$ ($x \neq 0$). Thus, we conclude that the attractors can only be the critical lines G and H for $\alpha < 0$ and for $\alpha > 0$, respectively. Let us analyze these two cases in detail.

The critical points belonging to the critical line $H(x, \sqrt{1-x^2}, 1)$ have just one eigenvalue equal to zero (for $x \neq 0$) and, therefore, they are normally hyperbolic, meaning that the linear stability theory suffices to assess the behavior of the trajectories in their vicinity. Since for $\alpha > 0$, the other two eigenvalues are negative, we conclude that, for such values of the parameter α and $x \neq 0$, these critical points are stable, i.e., they attract the trajectories along the noncritical directions. By continuity arguments, the critical point $H(0, 1, 1)$ is also an attractor for $\alpha > 0$.

The stability of the critical points belonging to the critical line $G(0, y, 1)$ is not so straightforward to assess. Indeed, because at least two eigenvalues are equal to zero, the linear theory does not suffice, and one has to resort to alternative methods to investigate stability. In Appendix A, we show using the center manifold theory that, for $\alpha < 0$, the trajectories, when approaching the critical line G, drift in the y-direction, converging to the critical point $G(0, 1, 1) = H(0, 1, 1)$, which lies at the intersection of the critical lines G and H. Therefore, this point is a global attractor for $\alpha < 0$.

In summary, for $\alpha > 0$ the critical line $H(x, \sqrt{1-x^2}, 1)$ is the global attractor, while for $\alpha < 0$, all trajectories are attracted to the critical point $G(0, 1, 1) = H(0, 1, 1)$.

Taking into account that, at the critical lines of G and H, the dark-energy density parameter is $\Omega_\phi = y^2$ and $\Omega_\phi = 1$, respectively, (see Table 1), from the above results on the stability of the critical points and lines, it immediately follows that, asymptotically, all cosmological solutions describe a Universe completely dominated by dark energy, irrespective of the value of α, which parameterizes the energy exchange between the dark components of the Universe.

This result is somewhat unexpected since, at first sight, a significant transfer of energy from the dark-energy scalar field to the dark-matter fluid should favor the appearance of scaling solutions, i.e., solutions for which the ratio between the density parameters of dark matter and dark energy is nonzero. However, as shown above, there are no attractors corresponding to such scaling solutions, and the asymptotic dominance of dark energy is an unavoidable feature of the cosmological model under consideration. Still, there is a subtlety here; we will clarify it below after presenting a detailed description of the solutions of cosmological relevance.

Note that the critical point $G(0, 1, 1) = H(0, 1, 1)$ is deep inside the region of the phase space in which expansion is accelerated ($x^2 - y^2 < -1/3$), while the critical line $H(x, \sqrt{1-x^2}, 1)$ is only partially inside this region. This means that, for $\alpha < 0$, the attractor always corresponds to a Universe with accelerated expansion, while for $\alpha > 0$, the attractor corresponds to accelerated expansion only if $|x| < 1/\sqrt{3}$.

Agreement with observations requires the present era of accelerated expansion of the Universe to be preceded by a matter-dominated era, long enough to allow for structure formation. This sequence of cosmological eras corresponds to those trajectories of the dynamical system (16) that pass near the critical point C before proceeding to the global attractor at $G(0, 1, 1) = H(0, 1, 1)$ for $\alpha < 0$, or $H(x, \sqrt{1-x^2}, 1)$ for $\alpha > 0$ and $|x| < 1/\sqrt{3}$. In what follows, we will focus our attention on such trajectories since they correspond to solutions that agree, at least qualitatively, with cosmological observations.

Each set of values of the parameters α and μ corresponds to a specific trajectory in the phase space of the dynamical system (16). Let us recall that α parameterizes the direct transfer of energy between the two dark components of the model (from dark matter to dark energy if $\alpha > 0$, in the opposite direction if $\alpha < 0$) and μ parameterizes the steepness of the scalar-field potential.

For $\alpha > 0$, we can restrict our analysis to $0 < \mu < \sqrt{2}$ because only for these values of the parameter μ does the final state of evolution correspond to accelerated expansion. In this case, the trajectories we are interested in pass close to the critical point C (close enough to guarantee a matter-dominated era of appropriate duration), proceed to the vicinity of the critical point D, and then climb vertically in the direction of the $z = 1$ plane, converging asymptotically to the point with coordinates $x = \mu/\sqrt{6}$, $y = \sqrt{1 - \mu^2/6}$, and

$z = 1$, belonging to the critical line H. At this final state, the effective equation-of-state parameter is given by $w_{\texttt{eff}} = -1 + 2x^2 \simeq -1 + \mu^2/3$ (see Table 1).

The evolution of the dark-energy density parameter Ω_ϕ, the dark-matter density parameter $\Omega_{\texttt{DM}}$, and the effective equation-of-state parameter $w_{\texttt{eff}}$ are illustrated in Figure 1 for the case $\alpha = 1$ and $\mu = 0.3$. Initial conditions $x_i = y_i = 9.21 \times 10^{-6}$ and $z_i = 10^{-7}$ were chosen to guarantee a long enough matter-dominated era, stretching from $\tau = -8$ to $\tau = -0.4$, which corresponds to a redshift between 3000 and 0.5, and also to guarantee that, at the present time $\tau = 0$, $\Omega_\phi(0) = 0.68$. The effective equation-of-state parameter tends, asymptotically, to -0.97. In Figure 2, the evolution of the effective equation-of-state parameter $w_{\texttt{eff}}$ is shown for different values of μ. For comparison, we also show the curve corresponding to the ΛCDM concordance model, obtained by choosing $\alpha = 0$, $\mu = 0$, and $x_i = 0$.

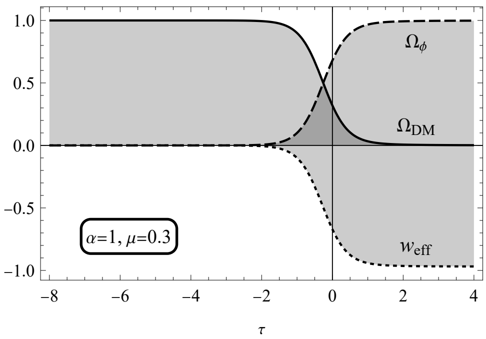

Figure 1. Evolution of Ω_ϕ, $\Omega_{\texttt{DM}}$, and $w_{\texttt{eff}}$ for the case $\alpha = 1$ and $\mu = 0.3$. A matter-dominated era, long enough to allow for structure formation, is followed by an era of dark-energy domination, during which the expansion of the Universe is accelerated.

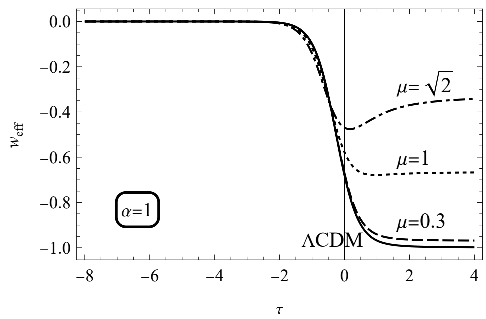

Figure 2. Evolution of the effective equation-of-state parameter $w_{\texttt{eff}}$ for $\alpha = 1$ and different values of the parameter μ. For comparison, the curve corresponding to the ΛCDM concordance model is also shown. For small values of μ, the evolution of the coupled quintessence cosmological model is quite similar to ΛCDM.

Let us now turn to the case $\alpha < 0$. For $0 < \mu \leq \sqrt{3}$, the behavior of the trajectories is similar, initially, to the case $\alpha > 0$; they pass near C and D, but then they climb to the vicinity of the critical point F (lying above D), and, from there, they proceed to the final state at $G(0,1,1) = H(0,1,1)$. For $\sqrt{3} < \mu < \sqrt{6}$, the trajectories, after passing near C, proceed to E (instead of D), then approach F, before heading to the attractor at $G(0,1,1) = H(0,1,1)$. Finally, for $\mu \geq \sqrt{6}$, the trajectories pass near C, then approach E, from where they proceed directly to the attractor at $G(0,1,1) = H(0,1,1)$.

In all cosmological solutions with $\alpha < 0$, at the final state, the effective equation-of-state parameter is given by $w_{\texttt{eff}} = -y^2 = -1$ (see Table 1). This corresponds to accelerated expansion irrespective of the value of the parameter μ. This result is illustrated in Figure 3, where $w_{\texttt{eff}}$ is seen to asymptotically converge to -1 for different values of the parameter μ. Note, however, that this convergence proceeds at an exceedingly slow rate. Meanwhile, the solution mimics a scaling solution, for which the energy density of dark matter is a significant fraction of the total energy density, as shown in Figure 4. In both Figures 3 and 4, initial conditions were chosen to guarantee a long enough matter-dominated era and also $\Omega_\phi(0) = 0.68$, namely, $x_i = y_i = 9.21 \times 10^{-6}$ for $\mu = 0.3$, $x_i = y_i = 1.024 \times 10^{-5}$ for $\mu = 1$, $x_i = y_i = 1.455 \times 10^{-5}$ for $\mu = \sqrt{3}$, and $x_i = y_i = 2.04 \times 10^{-5}$ for $\mu = 2$ ($z_i = 10^{-7}$ in all cases).

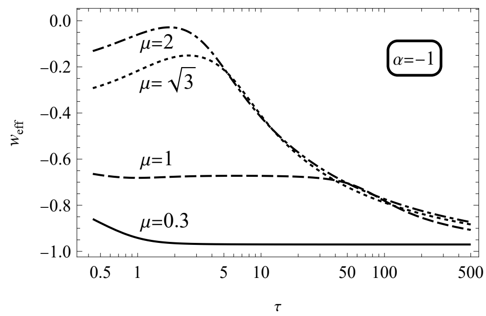

Figure 3. Evolution of the effective equation-of-state parameter $w_{\texttt{eff}}$ for $\alpha = -1$ and different values of the parameter μ. For negative α, $w_{\texttt{eff}}$ asymptotically converge to -1 irrespective of μ.

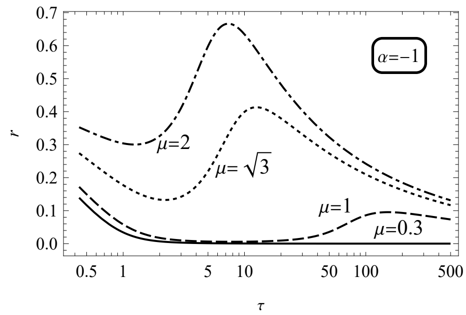

Figure 4. Evolution of the ratio $r \equiv \Omega_{\texttt{DM}}/\Omega_\phi$ for $\alpha = -1$ and different values of the parameter μ. For negative α, the convergence of r to zero proceeds at an exceedingly slow rate, meaning that, in the near future, the solutions effectively behave as scaling solutions.

The asymptotic behavior of the above-described cosmological solutions can be better understood by analyzing the evolution of the interaction term Q, which, in terms of the variables x, y, and z, reads

$$Q = \frac{6\alpha H_*^3}{\kappa^2} \frac{(1-z)^2 x^2}{(1-x^2-y^2)^2 z^2}, \tag{20}$$

where H_* is the positive constant introduced in Equation (13).

As shown in Figure 5, for $\alpha > 0$, Q rapidly converges to a constant positive value, which guarantees a steady transfer of energy from the dark-matter fluid to the dark-energy scalar field and, consequently, reinforces the dominance of the latter in the evolution of the Universe. For $\alpha < 0$, there is initially a significant transfer of energy from dark energy to dark matter, keeping the energy density of the latter at an expressive level—a situation that mimics the behavior of a scaling solution—but, as time goes on, this transfer of energy approaches zero, the energy density of the dark-matter fluid becomes more and more negligible, and the Universe finally completes its transition to an era of total dark-energy domination. In Figure 5, initial conditions were chosen to guarantee a long enough matter-dominated era and also $\Omega_\phi(0) = 0.68$, namely, $x_i = y_i = 1.024 \times 10^{-5}$ and $z_i = 10^{-7}$.

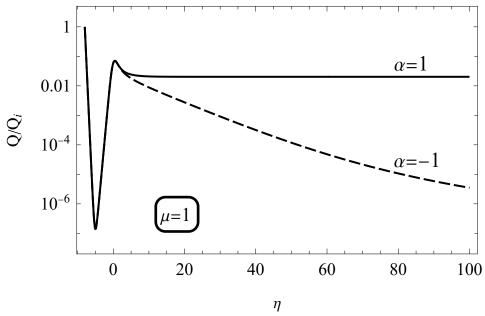

Figure 5. Evolution of the interaction term Q (divided by its value at the beginning of the numerical integration Q_i). For $\alpha > 0$, a steady transfer of energy from dark matter to dark energy reinforces the asymptotic dominance of the latter. For $\alpha < 0$, an initial significant transfer of energy from dark energy to dark matter helps the latter to be maintained at an expressive level, a situation that mimics the behavior of a scaling solution; however, as time passes, this energy transfer tends to zero and the Universe is allowed to complete its transition to an era of total dark-energy domination.

4. Conclusions

In this article, we have investigated a coupled quintessence cosmological model with an interaction term inspired by warm inflation.

With an appropriate choice of dimensionless variables, the evolution equations of the coupled quintessence model can be written as a three-dimensional dynamical system.

A stability analysis of the dynamical system's critical points and lines shows that, asymptotically, all cosmological solutions describe a Universe completely dominated by dark energy, irrespective of the values of the parameters α and μ.

However, a thorough study of the phase-space trajectories for different α and μ reveals that, in the case of negative α (corresponding to a transfer of energy from the dark-energy scalar field to the dark-matter fluid), for any value of μ, the approach to the final state of dark-energy dominance is so slow that, in the near future, the cosmological solutions effectively behave as scaling solutions.

Such behavior had already been observed previously in another coupled quintessence model [35] with an interaction term between dark energy and dark matter of the form $Q \propto \rho_{\mathtt{DM}}\dot{\phi}\phi^n$, where n is a positive integer. In that study, as in the present one, strictly speaking, there are no scaling attractor solutions; however, for certain values of a relevant parameter, there are solutions that, during the approach to the final state of cosmic evolution, behave, for all practical purposes, as accelerated scaling solutions. It seems thus that, in coupled quintessence cosmological models, scaling solutions are more ubiquitous than one might expect from a simple stability analysis of the critical points of the corresponding dynamical system.

The analysis of the dynamical system (16) also reveals the existence of a family of phase-space trajectories corresponding to an appropriate sequence of cosmic eras, namely, a matter-dominated era, long enough to allow for structure formation, followed by a present era of accelerated expansion. Such cosmologically relevant trajectories exist for both positive and negative values of the parameter α (for $\alpha > 0$, accelerated expansion at the current time requires $\mu < \sqrt{2}$). Thus, irrespective of the direction of energy transfer between dark matter and dark energy, the coupled quintessence cosmological model given by Equations (4)–(7) is able to replicate the observed late-time stages of the evolution of the Universe.

Within warm inflationary models, a variety of forms have been adopted for the dissipation coefficient, from the simplest, based on general phenomenological considerations, to the more elaborate ones, derived from quantum field theory. As a first approach to the study of coupled quintessence models with an interaction term inspired by warm inflation, we have considered Γ in Equation (10) to be constant. Dissipation coefficients depending on both the scalar field ϕ and the dark-matter energy density $\rho_{\mathtt{DM}}$ will be considered in future work.

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Abbreviations

The following abbreviations are used in this manuscript:

DESI	Dark Energy Spectroscopic Instrument
ΛCDM	Lambda Cold Dark Matter

Appendix A. Stability Analysis of the Critical Line $G(0, y, 1)$

Consider a specific point $G(0, y_c, 1)$ of the critical line $G(0, y, 1)$, where $y_c \neq 1$ is a constant.

The eigenvalues of the stability matrix of the dynamical system (16), evaluated at this point, are $\lambda_{1,2} = 0$ and $\lambda_3 = \alpha(1 - y_c^2)$. Because two eigenvalues are zero, the stability properties of the critical point cannot be determined within the linear theory; we must resort to alternative methods. Here, we will use the center manifold theory to assess the stability of the critical point $G(0, y_c, 1)$ (see Refs. [32,35] for examples of this theory's application in cosmological contexts similar to the present article).

Introducing new variables

$$u = x + \frac{\sqrt{6}\mu y_c^2}{2\alpha(y_c^2 - 1)}(z - 1), \quad v = y - y_c, \quad w = z - 1, \tag{A1}$$

the dynamical system (16) is brought to the form

$$u_\tau = \alpha(1 - y_c^2)u + f_1(u, v, w), \tag{A2a}$$

$$v_\tau = -\frac{3}{2}y_c(1 - y_c^2)w + f_2(u, v, w), \tag{A2b}$$

$$w_\tau = f_3(u, v, w), \tag{A2c}$$

where $f_i = \mathcal{O}(u^2, v^2, w^2, uv, uw, vw)$, $i = 1, 2, 3$.

The center manifold $u = h(v, w)$, which is a solution of the partial differential equation

$$\frac{\partial h}{\partial v}\left[-\frac{3}{2}y_c(1 - y_c^2)w + f_2\big(h(v, w), v, w\big)\right] + \frac{\partial h}{\partial w}f_3\big(h(v, w), v, w\big) - \alpha(1 - y_c^2)h(v, w) - f_1\big(h(v, w), v, w\big) = 0, \tag{A3}$$

is given by

$$h(v, w) = \frac{\sqrt{6}\mu}{2\alpha}v^2 w + \mathcal{O}(v^4, v^3 w, v^2 w^2, v w^3, w^4), \tag{A4}$$

for $y_c = 0$, and

$$h(v, w) = \frac{\sqrt{6}\mu y_c}{\alpha(1 - y_c^2)^2}vw - \frac{\sqrt{6}\mu y_c^2[3 + \alpha(1 - y_c^2)]}{2\alpha^2(1 - y_c^2)^2}w^2 + \mathcal{O}(v^3, v^2 w, v w^2, w^3), \tag{A5}$$

for $0 < y_c < 1$.

The flow on the center manifold is determined by the differential equations

$$v_\tau = -\frac{3}{2}vw \quad \text{and} \quad w_\tau = \frac{3}{2}w^2, \tag{A6}$$

for $y_c = 0$, and

$$v_\tau = -\frac{3}{2}y_c(1-y_c^2)w \quad \text{and} \quad w_\tau = \frac{3}{2}(1+y_c^2)w^2, \tag{A7}$$

for $0 < y_c < 1$. In the u-direction the flow is given by Equation (A2a).

Taking into account that in the neighborhood of the critical point $w < 0$ (also $v > 0$ for the case $y_c = 0$), we conclude that both v_τ and w_τ are positive, implying that the trajectories approach the critical point along the w-direction and move away from it along the v-direction. Along the u-direction, the trajectories approach the critical point if $\alpha < 0$ and move away from it if $\alpha > 0$.

In terms of the original variables x, y, and z, the above results mean that, for $\alpha < 0$, the trajectories, when approaching the critical line $G(0, y, 1)$, drift in the y-direction, asymptotically converging to the critical point $G(0, 1, 1) = H(0, 1, 1)$, which lies in the intersection of the critical lines G and H. Hence, this point is a global attractor for $\alpha < 0$.

References

1. Riess, A.G. et al. [Supernova Search Team] Observational evidence from supernovae for an accelerating Universe and a cosmological constant. *Astron. J.* **1998**, *116*, 1009–1038. [CrossRef]
2. Perlmutter, S. et al. [Supernova Cosmology Project] Measurements of Ω and Λ from 42 high-redshift supernovae. *Astrophys. J.* **1999**, *517*, 565–586. [CrossRef]
3. Astier, P.; Pain, R. Observational evidence of the accelerated expansion of the Universe. *C. R. Phys.* **2012**, *13*, 521–538. [CrossRef]
4. Adame, A.G. et al. [DESI Collaboration] DESI 2024 VI: Cosmological Constraints from the Measurements of Baryon Acoustic Oscillations. *arXiv* **2024**, arXiv:2404.03002
5. Aghanim, N. et al. [Planck Collaboration] Planck 2018 results VI. Cosmological parameters. *Astron. Astrophys.* **2020**, *641*, A6. [CrossRef]
6. Fixsen, D.J. The temperature of the cosmic microwave background. *Astrophys. J.* **2009**, *707*, 916–920. [CrossRef]
7. Caldwell, R.R.; Dave, R.; Steinhardt, P.J. Cosmological imprint of an energy component with general equation of state. *Phys. Rev. Lett.* **1998**, *80*, 1582–1585. [CrossRef]
8. Bertone, G.; Tait, T.M.P. A new era in the search for dark matter. *Nature* **2018**, *562*, 51–56. [CrossRef] [PubMed]
9. Amendola, L. Coupled quintessence. *Phys. Rev. D* **2000**, *62*, 043511. [CrossRef]
10. Amendola, L. Scaling solutions in general nonminimal coupling theories. *Phys. Rev. D* **1999**, *60*, 043501. [CrossRef]
11. Holden, D.J.; Wands, D. Self-similar cosmological solutions with nonminimally coupled scalar field. *Phys. Rev. D* **2000**, *61*, 043506. [CrossRef]
12. Billyard, A.P.; Coley, A.A. Interactions in scalar field cosmology. *Phys. Rev. D* **2000**, *61*, 083503. [CrossRef]
13. Zimdahl, W.; Pavón, D.; Chimento, L.P. Interacting quintessence. *Phys. Lett. B* **2001**, *521*, 133–138. [CrossRef]
14. Tocchini-Valentini, D.; Amendola, L. Stationary dark energy with a baryon-dominated era: Solving the coincidence problem with a linear coupling. *Phys. Rev. D* **2002**, *65*, 063508. [CrossRef]
15. Gumjudpai, B.; Naskar, T.; Sami, M.; Tsujikawa, S. Coupled dark energy: Towards a general description of the dynamics. *J. Cosmol. Astropart. Phys.* **2005**, *06*, 007. [CrossRef]
16. Zhang, X. Coupled Quintessence in a power-law case and the cosmic coincidence problem. *Mod. Phys. Lett. A* **2005**, *20*, 2575–2582. [CrossRef]
17. Capozziello, S.; Nojiri, S.; Odintsov, S.D. Unified phantom cosmology: Inflation, dark energy and dark matter under the same standard. *Phys. Lett. B* **2006**, *632*, 597–604. [CrossRef]
18. Böhmer, C.G.; Caldera-Cabral, G.; Lazkoz, R.; Maartens, R. Dynamics of dark energy with a coupling to dark matter. *Phys. Rev. D* **2008**, *78*, 023505. [CrossRef]
19. Henriques, A.B.; Potting, R.; Sá, P.M. Unification of inflation, dark energy, and dark matter within the Salam-Sezgin cosmological model. *Phys. Rev. D* **2009**, *79*, 103522. [CrossRef]
20. Lopez Honorez, L.; Mena, O.; Panotopoulos, G. Higher-order coupled quintessence. *Phys. Rev. D* **2010**, *82*, 123525. [CrossRef]
21. Bertolami, O.; Carrilho, P.; Páramos, J. Two-scalar-field model for the interaction of dark energy and dark matter. *Phys. Rev. D* **2012**, *86*, 103522. [CrossRef]
22. Amendola, L.; Barreiro, T.; Nunes, N.J. Multifield coupled quintessence. *Phys. Rev. D* **2014**, *90*, 083508. [CrossRef]
23. Tzanni, K.; Miritzis, J. Coupled quintessence with double exponential potentials. *Phys. Rev. D* **2014**, *89*, 103540. [CrossRef]
24. van de Bruck, C.; Mifsud, J.; Mimoso, J.P.; Nunes, N.J. Generalized dark energy interactions with multiple fluids. *J. Cosmol. Astropart. Phys.* **2016**, *11*, 031. [CrossRef]
25. Nunes, R.C.; Pan, S.; Saridakis, E.N. New constraints on interacting dark energy from cosmic chronometers. *Phys. Rev. D* **2016**, *94*, 023508. [CrossRef]
26. Barros, B.J. Kinetically coupled dark energy. *Phys. Rev. D* **2019**, *99*, 064051. [CrossRef]
27. Odintsov, S.D.; Oikonomou, V.K. Unification of inflation with dark energy in $f(R)$ gravity and axion dark matter. *Phys. Rev. D* **2019**, *99*, 104070. [CrossRef]

28. Pan, S.; Yang, W.; Di Valentino, E.; Saridakis, E.N.; Chakraborty, S. Interacting scenarios with dynamical dark energy: Observational constraints and alleviation of the H_0 tension. *Phys. Rev. D* **2019**, *100*, 103520. [CrossRef]
29. Sá, P.M. Unified description of dark energy and dark matter within the generalized hybrid metric-Palatini theory of gravity. *Universe* **2020**, *6*, 78. [CrossRef]
30. Sá, P.M. Triple unification of inflation, dark energy, and dark matter in two-scalar-field cosmology. *Phys. Rev. D* **2020**, *102*, 103519. [CrossRef]
31. Oikonomou, V.K. Unifying inflation with early and late dark energy epochs in axion $F(R)$ gravity. *Phys. Rev. D* **2021**, *103*, 044036. [CrossRef]
32. Sá, P.M. Late-time evolution of the Universe within a two-scalar-field cosmological model. *Phys. Rev. D* **2021**, *103*, 123517. [CrossRef]
33. Waeming, A.; Klangburam, T.; Pongkitivanichkul, C.; Samart, D. Dark matter and dark energy from a Kaluza–Klein inspired Brans–Dicke gravity with barotropic fluid. *Eur. Phys. J. C* **2022**, *82*, 409. [CrossRef]
34. da Fonseca, V.; Barreiro, T.; Nunes, N.J. A simple parametrisation for coupled dark energy. *Phys. Dark Universe* **2022**, *35*, 100940. [CrossRef]
35. Potting, R.; Sá, P.M. Coupled quintessence with a generalized interaction term. *Int. J. Mod. Phys. D* **2022**, *31*, 2250111. [CrossRef]
36. D'Agostino, R.; Luongo, O. Cosmological viability of a double field unified model from warm inflation. *Phys. Lett. B* **2022**, *829*, 137070. [CrossRef]
37. van de Bruck, C.; Poulot, G.; Teixeira, E.M. Scalar field dark matter and dark energy: A hybrid model for the dark sector. *J. Cosmol. Astropart. Phys.* **2023**, *07*, 019. [CrossRef]
38. Yang, W.; Pan, S.; Mena, O.; Di Valentino, E. On the dynamics of a dark sector coupling. *J. High Energy Astrophys.* **2023**, *40*, 19–40. [CrossRef]
39. Steinhardt, P.J. *Critical Problems in Physics*; Fitch, V.L., Marlow, D.R., Dementi, M.A.E., Eds.; Princeton University Press: Princeton, NJ, USA, 1997.
40. Di Valentino, E.; Mena, O.; Pan, S.; Visinelli, L.; Yang, W.; Melchiorri, A.; Mota, D.F.; Riess, A.G.; Silk, J. In the realm of the Hubble tension—A review of solutions. *Class. Quantum Grav.* **2021**, *38*, 153001. [CrossRef]
41. Riess, A.G.; Breuval, L. The Local Value of H_0. In Proceedings of the IAU Symposium 376, Budapest, Hungary, 17–21 April 2023; arXiv:2308.10954.
42. Berera, A. Warm inflation. *Phys. Rev. Lett.* **1995**, *75*, 3218–3221. [CrossRef]
43. Kamali, V.; Motaharfar, M.; Ramos, R.O. Recent Developments in Warm Inflation. *Universe* **2023**, *9*, 124. [CrossRef]
44. Berera, A. The Warm Inflation Story. *Universe* **2023**, *9*, 272. [CrossRef]
45. Motaharfar, M.; Kamali, V.; Ramos, R.O. Warm inflation as a way out of the swampland. *Phys. Rev. D* **2019**, *99*, 063513. [CrossRef]
46. Das, S. Warm inflation in the light of swampland criteria. *Phys. Rev. D* **2019**, *99*, 063514. [CrossRef]
47. Das, S. Note on single-field inflation and the swampland criteria. *Phys. Rev. D* **2019**, *99*, 083510. [CrossRef]
48. Bertolami, O.; Sá, P.M. Multi-field cold and warm inflation and the de Sitter swampland conjectures. *J. Cosmol. Astropart. Phys.* **2022**, *09*, 001. [CrossRef]
49. Lima, G.B.F.; Ramos, R.O. Unified early and late Universe cosmology through dissipative effects in steep quintessential inflation potential models. *Phys. Rev. D* **2019**, *100*, 123529. [CrossRef]
50. Das, S.; Hussain, S.; Nandi, D.; Ramos, R.O.; Silva, R. Stability analysis of warm quintessential dark energy model. *Phys. Rev. D* **2023**, *108*, 083517. [CrossRef]
51. Bolotin, Y.L.; Kostenko, A.; Lemets, O.A.; Yerokhin, D.A. Cosmological evolution with interaction between dark energy and dark matter. *Int. J. Mod. Phys. D* **2015**, *24*, 1530007. [CrossRef]
52. Wang, B.; Abdalla, E.; Atrio-Barandela, F.; Pavón, D. Dark matter and dark energy interactions: Theoretical challenges, cosmological implications and observational signatures. *Rep. Prog. Phys.* **2016**, *79*, 096901. [CrossRef]
53. Copeland, E.J.; Liddle, A.R.; Wands, D. Exponential potentials and cosmological scaling solutions. *Phys. Rev. D* **1998**, *57*, 4686–4690. [CrossRef]
54. Halder, S.; Pan, S.; Sá, P.M.; Saha, T. Coupled phantom cosmological model motivated by the warm inflationary paradigm. *arXiv* **2024**, arXiv:2407.15804.