INTERNATIONAL SOLVAY INSTITUTES FOR PHYSICS AND CHEMISTRY

Proceedings of the **Symposium Henri Poincaré** (Brussels, 8-9 October 2004)

Poincaré, Relativity, Billiards and Symmetry

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Abstract. This review is made of two parts which are related to Poincaré in different ways. The first part reviews the work of Poincaré on the Theory of (Special) Relativity. One emphasizes both the remarkable achievements of Poincaré, and the fact that he never came close to what is the essential conceptual achievement of Einstein: changing the concept of time. The second part reviews a topic which probably would have appealed to Poincaré because it involves several mathematical structures he worked on: chaotic dynamics, discrete reflection groups, and Lobachevskii space. This topic is the hidden role of Kac-Moody algebras in the billiard description of the asymptotic behaviour of certain Einstein-matter systems near a cosmological singularity. Of particular interest are the Einstein-matter systems arising in the low-energy limit of superstring theory. These systems seem to exhibit the highest-rank hyperbolic Kac-Moody algebras, and notably E_{10}, as hidden symmetries.

1 POINCARÉ and RELATIVITY

1.1 Some significant biographical dates

Starting in 1886, Poincaré holds the chair of "Physique mathématique et calcul des probabilités" at the Sorbonne. His 1888 lectures are about Maxwell's theories and the electromagnetic theory of light. In 1889, 1892 and 1899, he lectures on the works by Helmholtz, Hertz, Larmor and, especially, Lorentz. His lectures give him the matter of four books in which he expounds all the different theories. This pedagogical work makes him fully aware of the state of the art in modern electromagnetic theories, and establishes him as a renowned expert in mathematical physics. He exchanges a correspondence with Lorentz (who keeps him informed of his work), and starts writing some research papers on electromagnetic theory.

In 1893, Poincaré becomes a member of the "Bureau des Longitudes". It is the time where one starts to use telegraphic signals for synchronizing clocks (Bréguet, 1857; Le Verrier, 1875,...) and for measuring longitudes (see [1]).

In 1900, the "Congrès International de Physique" takes place in Paris, and Poincaré gives an invited review talk on the "Relations entre la physique expérimentale et la physique mathématique". Four years later, on September 24, 1904, during the "Congrès international des Arts et des Sciences" of Saint-Louis, Missouri (USA), Poincaré gives an invited review talk on "L'état actuel et l'avenir de la physique mathématique".

In 1902, Poincaré publishes his popular book "La Science et l'hypothèse"; more than 16000 copies of this book have been sold. In 1905, he publishes his second popular book: "La valeur de la Science". Two other popular books shall come out later: "Science et Méthode", in 1908, and "Dernières Pensées", posthumously, in 1913.

1.2 Selected citations and contributions of Poincaré to Relativity

In 1898, in a paper entitled "La Mesure du temps" [2], he writes: "Nous n'avons pas l'intuition directe de la simultanéité, pas plus que celle de l'égalité de deux durées." He discusses in detail the fact that, in order to define time and simultaneity, one must admit, as postulates, some "rules", e.g. that the velocity of light is constant and isotropic, and then that one must correct for the non-zero transmission times when using telegraphic signals to synchronize faraway clocks.

In the paper "La théorie de Lorentz et le principe de réaction" [3], written in 1900 at the occasion of the 25th anniversary of Lorentz's thesis, Poincaré discusses (as emphasized in [4]) the effect of an overall translation, at some speed v, on the synchronization of clocks by the exchange of electromagnetic signals. More precisely, he works only to *first order* in v, and notes that, if moving observers synchronize their watches by exchanging optical signals, and if they correct these signals by the transmission time under the (incorrect) assumption that the signals travel at the same speed in both directions, their watches will indicate not the "real time", but the "apparent time", say

$$\tau = t - \frac{vx}{c^2} + O(v^2). \tag{1.1}$$

His main point is that the "apparent time" τ coincides with the formal mathematical variable $t' \equiv t - \frac{vx}{c^2} + O(v^2)$ introduced by Lorentz in 1895 under the name of "local time" (and used by him to show the invariance of Maxwell theory under uniform translations, to first order in v).

In the book "La Science et l'hypothèse", dated from 1902[1], Poincaré writes suggestive sentences such as:

- "Il n'y a pas d'espace absolu et nous ne concevons que des mouvements relatifs."
- "Il n'y a pas de temps absolu."
- "Nous n'avons pas l'intuition directe de la simultanéité de deux événements."
- " $\cdots$ je ne crois pas, malgré Lorentz, que des observations plus précises puissent jamais mettre en évidence autre chose que les déplacements relatifs des corps matériels." [In fact, this is reprinted from his 1900 talk at the Congrès International de Physique.]

Poincaré recalls that when experiments testing effects of first order in v all came out negative, Lorentz found a general explanation at this order $O(v^1)$. When experiments testing the order v^2 again gave negative results (Michelson-Morley 1887), one found an ad hoc hypothesis (namely the Lorentz-FitzGerald contraction). For Poincaré, this is clearly unsatisfactory:

"Non, il faut trouver une même explication pour les uns et pour les autres et alors tout nous porte à penser que cette explication vaudra également pour tous les termes d'ordre supérieurs et que la destruction mutuelle de ces termes sera rigoureuse et absolue".

On Mai 27, 1904, Lorentz publishes his crucial paper: "Electromagnetic phenomena in a system moving with any velocity smaller than that of light". This paper contains the full "Lorentz transformations" linking variables associated to a moving frame, say (t', x', y', z'), to the "true, absolute" time and space coordinates (t, x, y, z). [Lorentz considers (t', x', y', z') only as convenient mathematical variables.]

On September 24, 1904, at the Saint-Louis "Congrès International", and later in his 1905 popular book "La Valeur de la Science", Poincaré mentions the "Principe de Relativité" among a list of the basic principles of physics.

Then he mentions that Lorentz's "local time" is the (apparent) time indicated by moving clocks (say, A and B) when they are synchronized by exchanging light signals and by ('wrongly' but conventionally) assuming the isotropy of the speed of light in the moving frame, *i.e.* the equality between the transmission times during the two exchanges $A \to B$ and $B \to A$. [However, he does not write down any equations, so that it is not clear whether he is alluding to his previous *first order in v* result, (1.1), or to an all order result (see below)].

He also mentions the existence of a "mécanique nouvelle" where inertia goes to ∞ as $v \to c$, and therefore, c is a limiting speed. [Note, however, that this was a feature common to all the current electron dynamics: Lorentz's, Abraham's, etc.]

[1]This seems to be the only work of Henri Poincaré read by Einstein before 1905.

On June 5, 1905[2], Poincaré submits to the Comptes Rendus of the Académie des Sciences the short Note: "Sur la dynamique de l'électron" [5]. This is followed by a more detailed article [6] (received on July 23, 1905). In these papers:

- he admits "sans restriction" le "Postulat de Relativité" and explores its consequences
- he modifies and completes Lorentz's 1904 paper (by giving the correct transformation laws for the electromagnetic field, and source, quantities).
- he explains "dynamically" why each "electron" undergoes the Lorentz contraction by introducing a (negative) internal pressure holding the electron against its electric self-repulsion
- he proves the group structure of "Lorentz transformations" [a name that he introduces in these papers]
- he proves the invariance of

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2\,(\Delta t)^2 \tag{1.2}$$

- he introduces $\ell = ict$ and makes the identification of Lorentz transformations with rotations in a 4-dimensional euclidean space
- he proves the addition law for velocity parameters

$$w = \frac{u+v}{1+\frac{uv}{c^2}} \tag{1.3}$$

- he discusses the invariants and covariants of the Lorentz group, e.g.

$$\vec{E}^{\,2} - \vec{B}^{\,2} = \text{invariant} \qquad , \qquad x^\mu \sim J^\mu \tag{1.4}$$

- he demands that the Principle of Relativity apply to gravitation
- he discusses possible relativistic laws of gravity (of an action-at-a-distance type, *i.e.* without assuming any explicit field content)
- he mentions that relativistic "retarded" gravitational interactions propagating with velocity $c_g = c$ are in agreement with existing observational limits on c_g (due to Laplace) because they all differ from Newton's law only at the order $O(v^2/c^2)$
- he speaks of "ondes gravifiques" both in the sense of retarded interaction and of emission of radiation
- he concludes about the necessity of a more detailed discussion of $O(v^2/c^2)$ deviations from Newtonian gravity.

In 1906-1907, Poincaré's Sorbonne lectures [7] (published in 1953 !) are about "Les limites de la loi de Newton":

[2] Notice that Einstein's paper on Relativity was received by the *Annalen der Physik* on June 30, 1905.

- In them Poincaré sets a limit $|\frac{\Delta a}{a}| \leq 2 \times 10^{-8}$ on the ratio between the gravitational mass and the inertial mass, $m_{gravit}/m_{inertia}$, by updating Laplace's work on the polarization of the Earth-Moon orbit by the Sun. [This effect has been rediscovered, in a different context, by Nordtvedt in 1968.]

- he discusses observational consequences of (among many others possible modifications of Newton's law) some selected "relativistic" laws of gravity and shows that their main observational effect is an additional advance of the perihelion of Mercury: e.g. he mentions that an electromagnetic-like gravitational law ("spin-1 exchange") yields an additional perihelion advance of 7" per century[3], instead of an observed value that he quotes as $\sim$ 38"[4] at the time)

- he works out the synchronization of moving clocks, by the method he had already mentioned in 1900-1904, to all orders in v/c and *seems* to conclude (see, however, below) that the result is exactly the (all orders) "local time" t' introduced by Lorentz in 1904

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} = \sqrt{1 - v^2/c^2}\left[t - \frac{v}{c^2 - v^2}(x - vt)\right] \tag{1.5}$$

- he determines how one must modify the fundamental law of dynamics $F = ma$ so that the principle of relativity holds

In 1908, in a paper entitled "La dynamique de l'électron" [8] (which is essentially reprinted in his 1908 book "Science et Méthode"), Poincaré speaks more about gravitational waves. More precisely, he mentions that the main observable effect of the "onde gravifique" emitted at infinity by an orbiting system ("onde d'accélération") will be, because of radiation reaction in the source, a secular acceleration of the orbital frequency, *i.e.* a negative value of the orbital period P: $\dot{P} < 0$. This observable effect is exactly what has been measured in binary pulsars, such as PRS 1913+16, which provided the first direct proofs that gravity propagates with the velocity of light (see, e.g., [9]).

1.3 Assessment of Poincaré's contributions to Relativity.

The above list of statements and results is certainly an impressive list of achievements! Some people (have) claim(ed) that Poincaré should share, with Einstein, the credit for discovering the "Theory of Relativity". When discussing this matter one should carefully distinguish various aspects of Poincaré's contributions.

Technically, it is true that Poincaré made important new contributions related to what is now called Special Relativity. Notably, the action of the Lorentz group on electromagnetic variables $A_\mu, F_{\mu\nu}, J_\mu$; the group structure of Lorentz transformations; the invariance of the spacetime interval $\Delta s^2 =$

[3] This had been already derived by Lorentz in 1900. Poincaré knew, however, at this stage that Lorentz's electromagnetic-like gravitational law was just one possibility among many "relativistic laws".

[4] It is amusing to speculate about what would have happened if Poincaré had used the better value (obtained by the american astronomer Simon Newcomb at the end of the 19th century) of $\sim$ 43", and had noticed that the various relativistic results he was deriving were all *integer* submultiples of the observed value (1/6 in the case of spin-1 exchange).

$(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - c^2\,(\Delta t)^2$; and the proposal to consider the "Principle of Relativity" as a general principle of physics applying, for instance, to gravitation and thereby restricting possible relativistic generalizations of Newton's law. For all those technical achievements, it would be quite reasonable to imagine that, if a Nobel prize had been given for Special Relativity before the death of Poincaré in 1912, the prize could have been shared between Einstein, Lorentz and Poincaré.

However, at the *conceptual* level, it seems to me (in spite of contrary claims in the literature) that Einstein was the first one to make what is the crucial step of Special Relativity, namely proposing a revolutionary[5] change in the concept ot *time*. Moreover, as we shall see below, Poincaré resisted till his death such a change in the concept of time.

This conceptual revolution in the notion of time is encapsulated in the "twin paradox", *i.e.* in time dilation effects, much more than in any change of synchronization conventions. Indeed, it was the idea that the variable t' was "time, pure and simple" which led Einstein, for the first time, to think and predict that, independently of any synchronization convention, a clock moving away and then coming back will not mark the same time when it reconvenes with its "sister clock" that remained in inertial motion. It is true that Poincaré's discussion of synchronization in a moving frame seems close to Einstein's synchronization process, but, when looking more carefully at what Poincaré actually wrote, one finds that there is a world of difference between the two.

First, let us mention that all the papers of Poincaré dealing with clock synchronization and published before Einstein's 1905 work on Special Relativity either dealt only with $O(v^1)$ effects (at which order there are no time dilation effects), or contained no explicit formulas [as in his Saint-Louis, September 1904 lecture]. The only explicit work of Poincaré on clock synchronization which keeps all orders in v/c is *posterior* to Einstein's 1905 paper on Relativity. It is contained in his 1906-7 Sorbonne lectures (published only in 1953 [7]) or in his 1908 paper [8].

- When looking in detail at the results actually derived by Poincaré, both in [7] and (consistently) in [8], one finds that Poincaré actually derives the following expression for what he calls the "apparent time" ("temps apparent") marked moving clocks in the way he advocates:

$$\tau_{(\text{Poincaré})} = t - \frac{v}{c^2 - v^2}(x - vt) \equiv \frac{1}{\sqrt{1 - v^2/c^2}} t' \tag{1.6}$$

  with t' given by (1.5). In other words, t' is the result of Einstein for the "time" in the moving frame (previously introduced by Lorentz as a mathematically auxiliary "local time" variable).

  The crucial point is that Poincaré's synchronized time τ,(1.6), *differs from Einstein's* "time, pure and simple", t', in the moving frame *precisely by*

[5] Let us note that Max Planck was the first scientist who understood the revolutionary nature of the new einsteinian conceptual setup. In 1910 he wrote that "This principle [of Relativity] has brought about a revolution in our physical picture of the world, which, in extent and depth, can only be compared to that produced by the introduction of the Copernican world system." He also wrote that "In boldness it [Special Relativity] probably surpasses anything so far achieved in speculative natural science, and indeed in philosophical cognition theory; non-Euclidean geometry is child's play in comparison."[10].

the time-dilation factor $\gamma = 1/\sqrt{1-v^2/c^2}$. In other words, though moving clocks marking the Poincaré time τ are desynchronized among themselves with respect to the absolute time ("temps vrai") t, (because of the $(x-vt)$-term in (1.6)), they do beat the same "absolute time" as a clock at rest, $d\tau = dt$, and do not exhibit any "twin paradox". [This is consistent with Poincaré's statements, both in his 1904 Saint-Louis lecture, and in "La Valeur de la Science", that, among two moving clocks "l'une d'elles retardera sur l'autre". The context shows that he does not speak of Einstein's time retardation effects linked to the factor $\gamma = 1/\sqrt{1-v^2/c^2}$ but of the constant offset $\propto -\frac{v}{c^2-v^2}\Delta(x-vt)$ between the indications of two clocks moving with the same, uniform velocity v.] It is true that in a subsequent paragraph of [7] Poincaré seems to identify the result τ,(1.6), of his explicit calculation with the full Lorentz "local time" t'. However, it seems clear to me that, in doing that, Poincaré has missed "thinking" the crucial einsteinian revolutionary step. *Mathematically*, Poincaré knew that the variable with good properties was t' (and this is the "time variable" he uses in his important technical papers of 1905 [5, 6]), but *physically* he never thought, nor proposed, that a moving clock will mark the time t' (and will therefore exhibit a "twin paradox").

Additional evidence for this limitation of the horizon of thought of Poincaré comes from other statements of his:

- Poincaré always distinguishes "le temps vrai", t, from "le temps apparent", τ or t', and, similarly, he always thinks in terms of "absolute space"
- Poincaré kept asking for a deeper (dynamical ?) reason behind the "relativity postulate", and the appearance of the velocity of light (c_{light}) in possible relativistic laws of gravity
- Poincaré had no firm theoretical a priori conviction in the "relativity principle"; e.g.

- In his 1908 paper [8] (and in his 1908 book on "Science et Méthode"[6]), commenting Kaufmann's early experiments (that did not seem to quite agree with relativistic dynamics), he expresses doubts about the exact validity of the relativity principle: He writes that the latest experiments of Kaufmann

"*ont donné raison à la théorie d'Abraham.* Le Principe de Relativité n'aurait donc pas la valeur rigoureuse qu'on était tenté de lui attribuer; $\cdots$"

By contrast, Einstein, commenting in 1907 the *same* experimental results, states that the agreement with relativity is rather good in first approximation, and that the deviations are probably due to systematic errors. Indeed, Einstein writes that it is a priori more probable that relativistic dynamics be correct, rather than Abraham's dynamics, because the former is based on a general principle having wide ranging consequences for physics as a whole.

- in his 1906-1907 lectures, Poincaré concludes that the most probable explanation for the anomaly in Mercury's perihelion advance is the existence of an infra-mercurial ring of matter. He has not enough trust in any of the possible relativistic gravitational theories he had introduced in 1905 to propose their

[6] Though he added a last minute footnote stating that the more recent experiments of Bucherer agreed with relativistic dynamics.

v^2/c^2 effects as a likely explanation for it. [As we said above, had he noticed that they all gave an *integer* submultiple of the observed anomaly, he might have suspected that one of them might give the correct explanation.]

- in 1904 (Saint-Louis), 1905 ("La valeur de la science"), and in 1908 (*i.e.* several years after Einstein's famous September 1905 paper on $E = mc^2$), Poincaré speaks of the recoil in reaction to the emission of electromagnetic waves and says : "Ce que nous avons envoyé au loin, ce n'est plus un projectile matériel, c'est de l'énergie et l'énergie n'a pas de masse", *i.e.* "energy has no mass" ! ("Science et méthode", livre III, chapter II, 1908)

- in 1912, a few months before his death, Poincaré writes [11] some sentences that have been quoted as evidence for Poincaré's role in conceptualizing (or at least accepting the Einstein-Minkowski) spacetime as a physical structure. For instance, he writes: "Tout se passe comme si le temps était une quatrième dimension de l'espace; et comme si l'espace à quatre dimensions résultant de la combinaison de l'espace ordinaire et du temps pouvait tourner non seulement autour d'un axe de l'espace ordinaire [$\cdots$] mais autour d'un axe quelconque. [$\cdots$] dans la nouvelle conception l'espace et le temps ne sont plus deux entités entièrement distinctes et que l'on puisse envisager séparément, mais deux parties d'un même tout [$\cdots$] qui sont comme étroitement enlaçées $\cdots$ ". However, if one reads the full text, one realizes that Poincaré explains here a conception proposed by "some physicists", and that he is not at all ready to accept this new conception (or "convention"). Indeed, he ends his text by writing:

"Aujourd'hui certains physiciens veulent adopter une convention nouvelle [$\cdots$] ceux qui ne sont pas de cet avis peuvent légitimement conserver l'ancienne pour ne pas troubler leurs vieilles habitudes. Je crois, entre nous, que c'est ce qu'ils feront encore longtemps."

This last sentence, which constitutes the last words written by Poincaré on Special Relativity, shows clearly that Poincaré never believed in the physical relevance of the conceptual revolution brought by Einstein in the concept of time (and extended by Minkowski to a revolutionary view of the physical meaning of spacetime).

1.4 Poincaré on Einstein, concerning Relativity

- Poincaré never mentioned Einstein's work on relativity (neither in his papers or books, nor, as far as I know, in his letters). Poincaré seemed to be unaware of Einstein's work during the years 1906-1909. His attention was probably brought to the work of Einstein and Minkowski only in the spring of 1909. [In April 1909 Poincaré gave some lectures in Göttingen, notably on "la mécanique nouvelle", without mentioning the names of Einstein or Minkowski.]

- Maybe he thought that

  - technically, there was nothing new in Einstein's work on Relativity

  - conceptually, Einstein was "cheating"[7] because he simply assumed (kinematically) what had to be proven (dynamically) [as Lorentz thought, and as Poincaré's electron-pressure model contributed to proving]

[7] I thank David Gross for a useful discussion on this point.

- Einstein's real contribution remained physically obscure to him, because Poincaré always thought that "apparent time" should be different from "real time" (while Einstein summarized his main contribution as being "the realization that an auxiliary term introduced by H. A. Lorentz and called by him 'local time' could be defined as 'time pure and simple'.").

- As a final vista on the conceptual difference between Poincaré and Einstein, let us mention the following revealing anecdote. Poincaré and Einstein met only once, at the Solvay meeting of 1911. Maurice de Broglie (who was one of the secretaries of this first Solvay meeting) wrote (in 1954 [12]) the following:

  "Je me rappelle qu'un jour à Bruxelles, comme Einstein exposait ses idées [sur la "mécanique nouvelle" c'est-à-dire relativiste], Poincaré lui demanda: ' Quelle mécanique adoptez-vous dans vos raisonnements ?' Einstein lui répondit :'Aucune mécanique', ce qui parut surprendre son interlocuteur."

  This conversation on "relativistic mechanics" (which contrasts the "dynamical approach" of Poincaré to the "kinematical" one of Einstein) is not reported in the official proceedings of the 1911 Solvay meeting which concerned the (old) theory of quanta.

2 RELATIVITY, BILLIARDS and SYMMETRY

2.1 Introduction and overview

A remarkable connection between the asymptotic behavior of certain Einstein-matter systems near a cosmological singularity and billiard motions in the Weyl chambers of some corresponding Lorentzian Kac–Moody algebras was uncovered in a series of works [13, 14, 15, 16, 17, 18, 19]. This simultaneous appearance of *billiards* (with *chaotic* properties in important physical cases) and of an underlying *symmetry* structure (infinite-dimensional Lie algebra) is an interesting fact, which deserves to be studied in depth. This topic would have pleased Poincaré because it involves several mathematical structures dear to his heart: notably, discrete reflection groups (and their fundamental polytope), Lobachevskii space and chaotic dynamics. Before explaining the techniques that have been used to uncover this connection, we will start by reviewing previous related works, and by stating the main results of this billiard/symmetry connection.

The simplest example of this connection concerns the pure Einstein system in $D = 3 + 1$-dimensional space-time. The Einstein's equations requiring the vanishing of the Ricci tensor ($R_{\mu\nu}(g_{\alpha\beta}) = 0$) are non-linear PDE's for the metric components. Near a cosmological spacelike singularity, here chosen as $t = 0$, the spatial gradients are expected to become negligible compared to time derivatives ($\frac{\partial}{\partial x^i} << \frac{\partial}{\partial t}$); this then suggests the decoupling of spatial points and allows for an approximate treatment in which one replaces the above partial differential equations by (a three-dimensional family) of ordinary differential equations. Within this simplified context, Belinskii, Khalatnikov and Lifshitz (BKL) gave a description [20, 21, 22] of the asymptotic behavior of the general solution of the Einstein's equations, close to the singularity, and showed that it can be described as a chaotic [23, 24] sequence of generalized Kasner solutions. The

Kasner metric is of the type

$$g_{\alpha\beta}(t)dx^\alpha dx^\beta = -N^2 dt^2 + A_1 t^{2p_1} dx_1^2 + A_2 t^{2p_2} dx_2^2 + A_3 t^{2p_3} dx_3^2 \qquad (2.7)$$

where the constants p_i obey[8]

$$\overrightarrow{p}^2 = p_1^2 + p_2^2 + p_3^2 - (p_1 + p_2 + p_3)^2 = 0. \qquad (2.8)$$

An exact Kasner solution, with a given set of A_i's and p_i's, can be represented by a null line in a 3-dimensional auxiliary Lorentz space with coordinates p_1, p_2, p_3 equipped with the metric given by the quadratic form $\overrightarrow{p}^2$ above. The auxiliary Lorentz space can be radially projected on the unit hyperboloid or further on the Poincaré disk (*i.e.* on the hyperbolic plane H_2): the projection of a null line is a geodesic on the hyperbolic plane. See Figure 1.

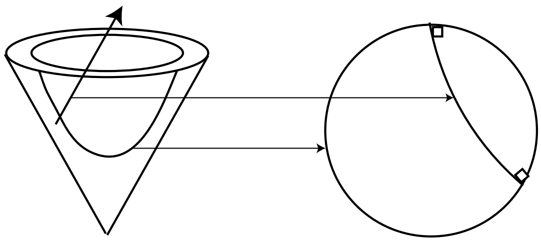


Figure 1: Lorentz space and projection on Poincaré disk.

BKL showed that, because of non-linearities in Einstein's equations, the generic solution behaves as a succession of Kasner epochs, i.e., as a broken null line in the auxiliary Lorentz space, or (after projection) a broken geodesic on the Poincaré disk. This broken geodesic motion is a "billiard motion" (seen either in Lorentzian space or in hyperbolic space). See Figure 2.

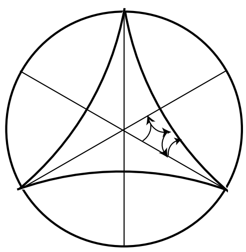


Figure 2: Picture of chaotic cosmological behavior

The billiard picture naturally follows from the Hamiltonian approach to cosmological behavior and was first obtained in the homogeneous (Bianchi IX) four-dimensional case [25, 26] and then extended to higher space-time dimensions with p-forms and dilatons [27, 28, 29, 30, 31, 15, 18, 32, 19]. Recent work [19] has improved the derivation of the billiard picture by using the Iwasawa decomposition of the spatial metric. Combining this decomposition with the Arnowitt-Deser-Misner [33] Hamiltonian formalism highlights the mechanism

[8]In the $N = 1$ gauge, they also obey $p_1 + p_2 + p_3 = 1$.

by which all variables except the scale factors and the dilatons get asymptotically frozen. The non-frozen variables (logarithms of scale factors and dilatons) then undergo a billiard motion. This billiard motion can be seen either in a $(D-1+n)$-dimensional Lorentzian space, or, after radial projection, on $(D-2+n)$-dimensional hyperbolic space. Here, D is the spacetime dimension and n the number of dilaton fields (see below for details). The Figures 1 and 2 correspond to the case $D=4$, $n=0$.

A remarkable connection was also established [13, 14, 15, 16, 17, 18, 19] between certain specific Einstein-matter systems and Lorentzian Kac-Moody (KM) algebras [34]. In the leading asymptotic approximation, this connection is simply that the Lorentzian billiard table within which the motion is confined can be identified with the Weyl chamber of some corresponding Lorentzian KM algebra. This can happen only when many conditions are met: in particular, (i) the billiard table must be a Coxeter polyhedron (the dihedral angles between adacent walls must be integer submultiples of π) and ii) the billiard must be a simplex. Surprisingly, this occurs in many physically interesting Einstein-matter systems. For instance, pure Einstein gravity in D dimensional space-time corresponds to the Lorentzian KM algebra AE_{D-1} [16] which is the overextension of the finite Lie algebra A_{D-3}: for $D=4$, the algebra is AE_3 the Cartan matrix of which is given by

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \qquad (2.9)$$

Chaotic billiard tables have finite volume in hyperbolic space, while non-chaotic ones have infinite volume; as a consequence, chaotic billiards are associated with *hyperbolic* KM algebras; this happens to be the case for pure gravity when $D \leq 10$.

Another connection between physically interesting Einstein-matter systems and KM algebras concerns the low-energy bosonic effective actions arising in string and M theories. Bosonic string theory in any space-time dimension D is related to the Lorentzian KM algebra DE_D [15, 17]. The latter algebra is the canonical Lorentzian extension of the finite-dimensional algebra D_{D-2}. The various superstring theories (in the critical dimension $D=10$) and M-theory have been found [15] to be related either to E_{10} (when there are two supersymmetries in $D=10$, i.e. for type IIA, type IIB and M-theory) or to BE_{10} (when there is only one supersymmetry in $D=10$, i.e. for type I and II heterotic theories), see the table below. A construction of the Einstein-matter systems related to the canonical Lorentzian extensions of *all* finite-dimensional Lie algebras A_n, B_n, C_n, D_n, G_2, F_4, E_6, E_7 and E_8 (in the above "billiard" sense) is presented in Ref. [17]. See also Ref. [35] for the identification of all hyperbolic KM algebras whose Weyl chambers are Einstein billiards.

The correspondence between the specific Einstein–three-form system (including a Chern–Simons term) describing the bosonic sector of 11-dimensional supergravity (also known as the "low-energy limit of M-theory") and the hyperbolic KM group E_{10} was studied in more detail in [18]. Reference [18] introduces a formal expansion of the field equations in terms of positive roots, i.e. combinations $\alpha = \Sigma_i \, n^i \, \alpha_i$ of simple roots of E_{10}, α_i, $i = 1, \ldots, 10$, where the n^i's are integers ≥ 0. It is then useful to *order* this expansion according to the *height* of the positive root $\alpha = \Sigma_i n^i \alpha_i$, defined as $\mathrm{ht}(\alpha) = \Sigma_i n^i$. The correspondence discussed above between the *leading* asymptotic evolution near a cosmological singularity (described by a billiard) and Weyl chambers of KM algebras involves only the terms in the field equation whose height is $\mathrm{ht}(\alpha) \leq 1$. By contrast, the authors of Ref. [18] managed to show, by explicit calculation, that there exists a way to define, at each spatial point x, a correspondence between the field variables $g_{\mu\nu}(t,x)$, $A_{\mu\nu\lambda}(t,x)$ (and their gradients), and a (finite) subset of the parameters defining an element of the (infinite-dimensional) coset space $E_{10}/K(E_{10})$ where $K(E_{10})$ denotes the maximal compact subgroup of E_{10}, such that the (PDE) field equations of supergravity get mapped onto the (ODE) equations describing a null geodesic in $E_{10}/K(E_{10})$ *up to terms of height* 30. A complementary check of the correspondence between 11-dimensional supergravity and the $E_{10}/K(E_{10})$ σ-model has been obtained in [37]. This result was further extended to the correspondence between the $E_{10}/K(E_{10})$ σ-model and, both, *massive* 10-dimensional IIA supergravity [38], and 10-dimensional IIB supergravity [39].

These tantalizing results suggest that the infinite-dimensional hyperbolic Kac–Moody group E_{10} may be a "hidden symmetry" of supergravity in the sense of mapping solutions onto solutions (the idea that E_{10} might be a symmetry of supergravity was first raised by Julia long ago [36, 40]). Note that the conjecture here is that the *continuous* group $E_{10}(\mathbb{R})$ be a hidden symmetry group of *classical* supergravity. At the *quantum* level, i.e. for M theory, one expects only a discrete version of E_{10}, say $E_{10}(\mathbb{Z})$, to be a quantum symmetry. See [41] for recent work on extending the identification of [18] between roots of E_{10} and symmetries of supergravity/M-theory beyond height 30, and for references about previous suggestions of a possible role for E_{10}. For earlier appearances of the Weyl groups of the E series in the context of U-duality see [42, 43, 44]. A series of recent papers [45, 46, 47, 48, 49] has also explored the possible role of E_{11} (a nonhyperbolic extension of E_{10}) as a hidden symmetry of M theory.

It is also tempting to assume that the KM groups underlying the other (special) Einstein-matter systems discussed above might be hidden (solution-generating) symmetries. For instance, in the case of pure Einstein gravity in $D = 4$ space-time, the conjecture is that AE_3 be such a symmetry of Einstein gravity. This case, and the correspondence between the field variables and the coset ones is further discussed in [19].

Note that rigorous mathematical proofs [31, 50, 32, 51] concerning the PDE/billiard connection are only available for 'non chaotic' billiards.

In the remainder of this paper, we will outline various arguments explaining the above results; a more complete derivation can be found in [19].

2.2 General Models

The general systems considered here are of the following form

$$S[\mathrm{g}_{MN}, \phi, A^{(p)}] = \int d^D x \sqrt{-\mathrm{g}} \left[R(\mathrm{g}) - \partial_M \phi \partial^M \phi \right. \\ \left. - \frac{1}{2} \sum_p \frac{1}{(p+1)!} e^{\lambda_p \phi} F^{(p)}_{M_1 \cdots M_{p+1}} F^{(p)\, M_1 \cdots M_{p+1}} \right] + \dots \quad (2.10)$$

Units are chosen such that $16\pi G_N = 1$, G_N is Newton's constant and the spacetime dimension $D \equiv d+1$ is left unspecified. Besides the standard Einstein–Hilbert term the above Lagrangian contains a dilaton[9] field ϕ and a number of p-form fields $A^{(p)}_{M_1\cdots M_p}$ (for $p \geq 0$). The p-form field strengths $F^{(p)} = dA^{(p)}$ are normalized as

$$F^{(p)}_{M_1\cdots M_{p+1}} = (p+1)\partial_{[M_1} A^{(p)}_{M_2\cdots M_{p+1}]} \equiv \partial_{M_1} A^{(p)}_{M_2\cdots M_{p+1}} \pm p \text{ permutations} . \tag{2.11}$$

As a convenient common formulation we adopt the Einstein conformal frame and normalize the kinetic term of the dilaton ϕ with weight one with respect to the Ricci scalar. The Einstein metric g_{MN} has Lorentz signature $(-+\cdots+)$ and is used to lower or raise the indices; its determinant is denoted by g. The dots in the action (2.10) above indicate possible modifications of the field strength by additional Yang–Mills or Chapline–Manton-type couplings [52, 53]. The real parameter λ_p measures the strength of the coupling of $A^{(p)}$ to the dilaton. When $p = 0$, we assume that $\lambda_0 \neq 0$ so that there is only one dilaton.

2.3 Dynamics in the vicinity of a spacelike singularity

The main technical points that will be reviewed here are the following

- near the singularity, $t \to 0$, due to the decoupling of space points, Einstein's PDE equations become ODE's with respect to time.
- The study of these ODE's near $t \to 0$, shows that the $d \equiv D-1$ diagonal spatial metric components "g_{ii}" and the dilaton ϕ move on a billiard in an auxiliary $d+1 \equiv D$ dimensional Lorentz space.
- All the other field variables $(g_{ij}, i \neq j, A_{i_1\dots i_p}, \pi^{i_1\cdots i_p})$ freeze as $t \to 0$.
- In many interesting cases, the billiard tables can be identified with the fundamental Weyl chamber of an hyperbolic KM algebra.
- For SUGRA$_{11}$, the KM algebra is E_{10}. Moreover, the PDE's are equivalent to the equations of a null geodesic on the coset space $E_{10}/K(E_{10})$, up to height 30.

2.3.1 Arnowitt-Deser-Misner Hamiltonian formalism

To focus on the features relevant to the billiard picture, we assume here that there are no Chern–Simons and no Chapline–Manton terms and that the curvatures $F^{(p)}$ are abelian, $F^{(p)} = dA^{(p)}$. That such additional terms do not alter the analysis has been proven in [19]. In any pseudo-Gaussian gauge and in the temporal gauge ($g_{0i} = 0$ and $A_{0i_2\dots i_p} = 0$, $\forall p$), the Arnowitt-Deser-Misner Hamiltonian action [33] reads

$$\begin{aligned} &S\left[g_{ij}, \pi^{ij}, \phi, \pi_\phi, A^{(p)}_{j_1\cdots j_p}, \pi^{j_1\cdots j_p}_{(p)}\right] = \\ &\qquad \int dx^0 \int d^d x \left(\pi^{ij}\dot{g_{ij}} + \pi_\phi \dot{\phi} + \frac{1}{p!}\sum_p \pi^{j_1\cdots j_p}_{(p)} \dot{A}^{(p)}_{j_1\cdots j_p} - H \right) \end{aligned} \tag{2.12}$$

[9]The generalization to any number of dilatons is straightforward.

where the Hamiltonian density H is

$$
\begin{aligned}
H &\equiv \tilde{N}\mathcal{H}\,, && (2.13)\\
\mathcal{H} &= \mathcal{K}+\mathcal{M}\,, && (2.14)\\
\mathcal{K} &= \pi^{ij}\pi_{ij}-\frac{1}{d-1}\pi^{i}{}_{i}\pi^{j}{}_{j}+\frac{1}{4}\pi_\phi^2+\sum_p\frac{e^{-\lambda_p\phi}}{2\,p!}\,\pi_{(p)}^{j_1\cdots j_p}\pi_{(p)\,j_1\cdots j_p}\,, && (2.15)\\
\mathcal{M} &= -gR+gg^{ij}\partial_i\phi\partial_j\phi+\sum_p\frac{e^{\lambda_p\phi}}{2\,(p+1)!}\,g\,F^{(p)}_{j_1\cdots j_{p+1}}F^{(p)\,j_1\cdots j_{p+1}}\,, && (2.16)
\end{aligned}
$$

and R is the spatial curvature scalar. $\tilde{N}=N/\sqrt{g^{(d)}}$ is the rescaled lapse. The dynamical equations of motion are obtained by varying the above action with respect to the spatial metric components, the dilaton, the spatial p-form components and their conjugate momenta. In addition, there are constraints on the dynamical variables,

$$
\begin{aligned}
\mathcal{H} &\approx 0 && (\text{“Hamiltonian constraint”}), && (2.17)\\
\mathcal{H}_i &\approx 0 && (\text{“momentum constraint”}), && (2.18)\\
\varphi_{(p)}^{j_1\cdots j_{p-1}} &\approx 0 && (\text{“Gauss law” for each } p\text{-form}), && (2.19)
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{H}_i &= -2\pi^{j}{}_{i|j}+\pi_\phi\partial_i\phi+\sum_p\frac{1}{p!}\,\pi_{(p)}^{j_1\cdots j_p}F^{(p)}_{ij_1\cdots j_p}\,, && (2.20)\\
\varphi_{(p)}^{j_1\cdots j_{p-1}} &= \pi_{(p)}^{j_1\cdots j_{p-1}j_p}{}_{|j_p}\,, && (2.21)
\end{aligned}
$$

where the subscript $|j$ stands for spatially covariant derivative.

2.3.2 Iwasawa decomposition of the spatial metric

We systematically use the Iwasawa decomposition of the spatial metric g_{ij} and write

$$g_{ij}=\sum_{a=1}^{d}e^{-2\beta^a}\mathcal{N}^a{}_i\mathcal{N}^a{}_j \tag{2.22}$$

where $\mathcal{N}$ is an upper triangular matrix with 1's on the diagonal. We will also need the Iwasawa coframe $\{\theta^a\}$,

$$\theta^a=\mathcal{N}^a{}_i\,dx^i\,, \tag{2.23}$$

as well as the vectorial frame $\{e_a\}$ dual to the coframe $\{\theta^a\}$,

$$e_a=\mathcal{N}^i{}_a\frac{\partial}{\partial x^i} \tag{2.24}$$

where the matrix $\mathcal{N}^i{}_a$ is the inverse of $\mathcal{N}^a{}_i$, i.e., $\mathcal{N}^a{}_i\mathcal{N}^i{}_b = \delta^a_b$. It is again an upper triangular matrix with 1's on the diagonal. Let us now examine how the Hamiltonian action gets transformed when one performs, at each spatial point, the Iwasawa decomposition (2.22) of the spatial metric. The kinetic terms of the metric and of the dilaton in the Lagrangian (2.10) are given by the quadratic form

$$G_{\mu\nu}d\beta^\mu d\beta^\nu = \sum_{a=1}^{d}(d\beta^a)^2 - \left(\sum_{a=1}^{d} d\beta^a\right)^2 + d\phi^2, \quad \beta^\mu = (\beta^a, \phi). \tag{2.25}$$

The change of variables $(g_{ij} \to \beta^a, \mathcal{N}^a{}_i)$ corresponds to a point transformation and can be extended to the momenta as a canonical transformation in the standard way via

$$\pi^{ij}\dot{g}_{ij} \equiv \sum_a \pi_a \dot{\beta}^a + \sum_a P^i{}_a \dot{\mathcal{N}}^a{}_i \ . \tag{2.26}$$

Note that the momenta

$$P^i{}_a = \frac{\partial \mathcal{L}}{\partial \dot{\mathcal{N}}^a{}_i} = \sum_b e^{2(\beta^b - \beta^a)} \dot{\mathcal{N}}^a{}_j \mathcal{N}^j{}_b \mathcal{N}^i{}_b \tag{2.27}$$

conjugate to the nonconstant off-diagonal Iwasawa components $\mathcal{N}^a{}_i$ are only defined for $a < i$; hence the second sum in (2.26) receives only contributions from $a < i$.

2.3.3 Splitting of the Hamiltonian

We next split the Hamiltonian density $\mathcal{H}$ (2.13) in two parts: $\mathcal{H}_0$, which is the kinetic term for the local scale factors and the dilaton $\beta^\mu = (\beta^a, \phi)$, and $\mathcal{V}$, a "potential density" (of weight 2) , which contains everything else. Our analysis below will show why it makes sense to group the kinetic terms of both the off-diagonal metric components and the p-forms with the usual potential terms, i.e. the term $\mathcal{M}$ in (2.14). Thus, we write

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{V} \tag{2.28}$$

with the kinetic term of the β variables

$$\mathcal{H}_0 = \frac{1}{4} G^{\mu\nu}\pi_\mu\pi_\nu \, , \tag{2.29}$$

where $G^{\mu\nu}$ denotes the inverse of the metric $G_{\mu\nu}$ of Eq. (2.25). In other words, the right hand side of Eq. (2.29) is defined by

$$G^{\mu\nu}\pi_\mu\pi_\nu \equiv \sum_{a=1}^{d} \pi_a^2 - \frac{1}{d-1}\left(\sum_{a=1}^{d}\pi_a\right)^2 + \pi_\phi^2 \, , \tag{2.30}$$

where $\pi_\mu \equiv (\pi_a, \pi_\phi)$ are the momenta conjugate to β^a and ϕ, respectively, i.e.

$$\pi_\mu = 2\tilde{N}^{-1} G_{\mu\nu}\dot{\beta}^\nu = 2G_{\mu\nu}\frac{d\beta^\nu}{d\tau} \, . \tag{2.31}$$

The total (weight 2) potential density,

$$\mathcal{V} = \mathcal{V}_S + \mathcal{V}_G + \sum_p \mathcal{V}_p + \mathcal{V}_\phi \,, \tag{2.32}$$

is naturally split into a "centrifugal" part $\mathcal{V}_S$ linked to the kinetic energy of the off-diagonal components (the index S referring to "symmetry,"), a "gravitational" part $\mathcal{V}_G$, a term from the p-forms, $\sum_p \mathcal{V}_p$, which is a sum of an "electric" and a "magnetic" contribution and also a contribution to the potential coming from the spatial gradients of the dilaton $\mathcal{V}_\phi$.

- "centrifugal" potential

$$\mathcal{V}_S = \frac{1}{2}\sum_{a<b} e^{-2(\beta^b-\beta^a)} \left(P^j{}_b \mathcal{N}^a{}_j\right)^2 , \tag{2.33}$$

- "gravitational" (or "curvature") potential

$$\mathcal{V}_G = -gR = \frac{1}{4}\sum_{a\neq b\neq c} e^{-2\alpha_{abc}(\beta)}(C^a{}_{bc})^2 - \sum_a e^{-2\mu_a(\beta)}F_a \,, \tag{2.34}$$

  where

$$\alpha_{abc}(\beta) \equiv \sum_e \beta^e + \beta^a - \beta^b - \beta^c,\ a\neq b, b\neq c, c\neq a \tag{2.35}$$

  and

$$d\theta^a = -\frac{1}{2}C^a{}_{bc}\theta^b \wedge \theta^c \tag{2.36}$$

  while F_a is a polynomial of degree two in the first derivatives $\partial\beta$ and of degree one in the second derivatives $\partial^2\beta$.

- p-form potential

$$\mathcal{V}_{(p)} = \mathcal{V}^{el}_{(p)} + \mathcal{V}^{magn}_{(p)} \,, \tag{2.37}$$

  which is a sum of an "electric" $\mathcal{V}^{el}_{(p)}$ and a "magnetic" $\mathcal{V}^{magn}_{(p)}$ contribution. The "electric" contribution can be written as

$$\begin{aligned} \mathcal{V}^{el}_{(p)} &= \frac{e^{-\lambda_p\phi}}{2\,p!}\pi^{j_1\cdots j_p}_{(p)}\pi_{(p)\,j_1\cdots j_p} \\ &= \frac{1}{2\,p!}\sum_{a_1,a_2,\cdots,a_p} e^{-2e_{a_1\cdots a_p}(\beta)}(\mathcal{E}^{a_1\cdots a_p})^2 \,, \end{aligned} \tag{2.38}$$

  where $\mathcal{E}^{a_1\cdots a_p} \equiv \mathcal{N}^{a_1}{}_{j_1}\mathcal{N}^{a_2}{}_{j_2}\cdots\mathcal{N}^{a_p}{}_{j_p}\pi^{j_1\cdots j_p}$, and $e_{a_1\cdots a_p}(\beta)$ are the "electric wall" forms,

$$e_{a_1\cdots a_p}(\beta) = \beta^{a_1} + \cdots + \beta^{a_p} + \frac{\lambda_p}{2}\phi \,. \tag{2.39}$$

  And the "magnetic" contribution reads,

$$\begin{aligned}\mathcal{V}^{magn}_{(p)} &= \frac{e^{\lambda_p \phi}}{2\,(p+1)!}\, g\, F^{(p)}_{j_1\cdots j_{p+1}} F^{(p)\, j_1\cdots j_{p+1}} \\ &= \frac{1}{2\,(p+1)!} \sum_{a_1,a_2,\cdots,a_{p+1}} e^{-2m_{a_1\cdots a_{p+1}}(\beta)} (\mathcal{F}_{a_1\cdots a_{p+1}})^2 \,. \qquad (2.40)\end{aligned}$$

where $\mathcal{F}_{a_1\cdots a_{p+1}} = \mathcal{N}^{j_1}{}_{a_1} \cdots \mathcal{N}^{j_{p+1}}{}_{a_{p+1}} F_{j_1\cdots j_{p+1}}$ and the $m_{a_1\cdots a_{p+1}}(\beta)$ are the magnetic linear forms

$$m_{a_1\cdots a_{p+1}}(\beta) = \sum_{b\notin\{a_1,a_2,\cdots a_{p+1}\}} \beta^b - \frac{\lambda_p}{2}\,\phi\,, \qquad (2.41)$$

- dilaton potential

$$\begin{aligned}\mathcal{V}_\phi &= g g^{ij} \partial_i\phi \partial_j \phi & (2.42)\\ &= \sum_a e^{-\mu_a(\beta)} (\mathcal{N}_a{}^i \partial_i \phi)^2,. & (2.43)\end{aligned}$$

where

$$\mu_a(\beta) = \sum_e \beta^e - \beta^a \qquad (2.44)$$

2.3.4 Appearance of sharp walls in the BKL limit

In the decomposition of the hamiltonian as $\mathcal{H} = \mathcal{H}_0 + \mathcal{V}$, $\mathcal{H}_0$ is the kinetic term for the β^μ's while all other variables now only appear through the potential $\mathcal{V}$ which is schematically of the form

$$\mathcal{V}(\beta^\mu, \partial_x\beta^\mu, P, Q) = \sum_A c_A(\partial_x \beta^\mu, P, Q) \exp\big(-2w_A(\beta)\big)\,, \qquad (2.45)$$

where $(P,Q) = (\mathcal{N}^a{}_i, P^i{}_a, \mathcal{E}^{a_1\cdots a_p}, \mathcal{F}_{a_1\cdots a_{p+1}})$. Here $w_A(\beta) = w_{A\mu}\beta^\mu$ are the linear wall forms already introduced above:

$$\begin{aligned}
\text{symmetry walls} &: \; w^S_{ab} \equiv \beta^b - \beta^a; \quad a<b \\
\text{gravitational walls} &: \; \alpha_{abc}(\beta) \equiv \sum_e \beta^e + \beta^a - \beta^b - \beta^c,\; a\neq b, b\neq c, c\neq a \\
&\quad\; \mu_a(\beta) \equiv \sum_e \beta^e - \beta^a, \\
\text{electric walls} &: \; e_{a_1\cdots a_p}(\beta) \equiv \beta^{a_1} + \ldots + \beta^{a_p} + \frac{1}{2}\lambda_p\phi, \\
\text{magnetic walls} &: \; m_{a_1\cdots a_{p+1}}(\beta) \equiv \sum_e \beta^e - \beta^{a_1} - \ldots - \beta^{a_{p+1}} - \frac{1}{2}\lambda_p \phi.
\end{aligned}$$

In order to take the limit $t\to 0$ which corresponds to β^μ tending to future time-like infinity, we decompose β^μ into hyperbolic polar coordinates (ρ, γ^μ), i.e.

$$\beta^\mu = \rho\gamma^\mu \qquad (2.46)$$

where γ^μ are coordinates on the future sheet of the unit hyperboloid which are constrained by

$$G_{\mu\nu}\gamma^\mu\gamma^\nu \equiv \gamma^\mu\gamma_\mu = -1 \tag{2.47}$$

and ρ is the time-like variable defined by

$$\rho^2 \equiv -G_{\mu\nu}\beta^\mu\beta^\nu \equiv -\beta_\mu\beta^\mu > 0, \tag{2.48}$$

which behaves like $\rho \sim -\ln t \to +\infty$ at the BKL limit. In terms of these variables, the potential term looks like

$$\sum_A c_A(\partial_x\beta^\mu, P, Q)\rho^2 \exp\big(-2\rho w_A(\gamma)\big)\,. \tag{2.49}$$

The essential point now is that, since $\rho \to +\infty$, each term $\rho^2 \exp\big(-2\rho w_A(\gamma)\big)$ becomes a *sharp wall potential*, i.e. a function of $w_A(\gamma)$ which is zero when $w_A(\gamma) > 0$, and $+\infty$ when $w_A(\gamma) < 0$. To formalize this behavior we define the sharp wall Θ-function [10] as

$$\Theta(x) := \begin{cases} 0 & \text{if } x < 0\,, \\ +\infty & \text{if } x > 0\,. \end{cases} \tag{2.50}$$

A basic formal property of this Θ-function is its invariance under multiplication by a positive quantity. Because all the relevant prefactors $c_A(\partial_x\beta^\mu, P, Q)$ are generically *positive* near each leading wall, we can formally write

$$\begin{aligned}\lim_{\rho\to\infty} \quad & \Big[c_A(\partial_x\beta^\mu, Q, P)\rho^2 \exp\big(-\rho w_A(\gamma)\big)\Big] = c_A(Q,P)\Theta\big(-2w_A(\gamma)\big) \\ \equiv \quad & \Theta\big(-2w_A(\gamma)\big)\end{aligned} \tag{2.51}$$

valid in spite of the increasing of the spatial gradients [19]. Therefore, the limiting dynamics is equivalent to a free motion in the β-space interrupted by reflections against hyperplanes in this β-space given by $w_A(\beta) = 0$ which correspond to a potential described by infinitely high step functions

$$\mathcal{V}(\beta, P, Q) = \sum_A \Theta\big(-2w_A(\gamma)\big) \tag{2.52}$$

The other dynamical variables (all variables but the β^μ's) completely disappear from this limiting Hamiltonian and therefore they all get frozen as $t \to 0$.

2.3.5 Cosmological singularities and Kac–Moody algebras

Two kinds of motion are possible according to the volume of the billiard table on which it takes place, i.e. the volume of the region where $\mathcal{V} = 0$ for $t \to 0$, also characterized by the conditions,

[10] One should more properly write $\Theta_\infty(x)$, but since this is the only step function encountered here, we use the simpler notation $\Theta(x)$.

$$w_A(\beta) > 0 \quad \forall A. \tag{2.53}$$

Depending on the fields present in the lagrangian, on their dilaton-couplings and on the spacetime dimension, the (projected) billiard volume is either of finite or infinite. The finite volume case corresponds to never-ending, chaotic oscillations for the β's while in the infinite volume case, after a finite number of reflections off the walls, they tend to an asymptotically monotonic Kasner-like behavior, see Figure 3:

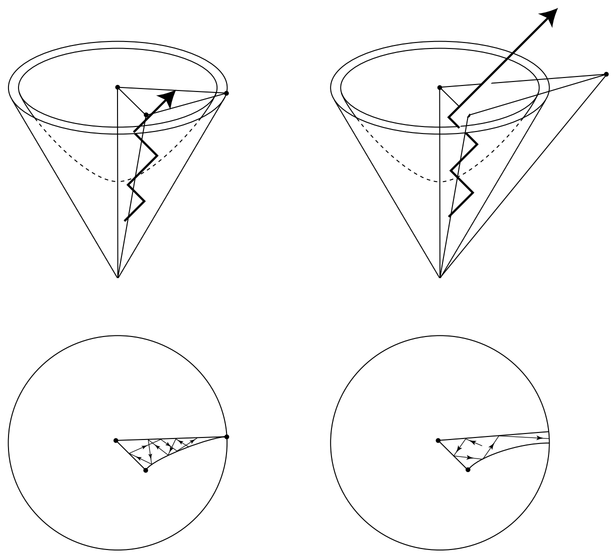


Figure 3: Sketch of billiard tables describing the asymptotic cosmological behavior of Einstein-matter systems.

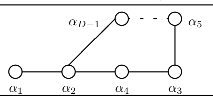
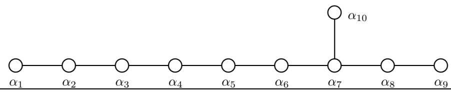
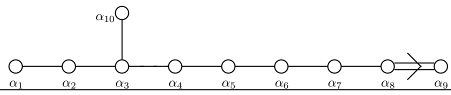
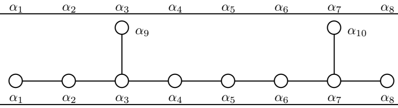
In Figure 3 the upper panels are drawn in the Lorentzian space spanned by $(\beta^\mu) = (\beta^a, \phi)$. The billiard tables are represented as "wedges" in $(d+1)$–dimensional (or d-dimensional, if there are no dilatons) β-space, bounded by hyperplanar walls $w_A(\beta) = 0$ on which the billiard ball undergoes specular reflections. The upper left panel is a (critical) "chaotic" billiard table (contained within the β-space future light cone), while the upper right one is a (subcritical) "nonchaotic" one (extending beyond the light cone). The lower panels represent the corresponding billiard tables (and billiard motions) after projection onto hyperbolic space H_d (H_{d-1} if there are no dilatons). The latter projection is defined in the text by central projection onto γ-space (i.e. the unit hyperboloid $G_{\mu\nu}\,\gamma^\mu\,\gamma^\nu = -1$, see the upper panels), and is represented in the lower panels by its image in the Poincaré ball (disk).

In fact, not all the walls are relevant for determining the billiard table. Some of the walls stay behind the others and are not met by the billiard ball. Only a subset of the walls $w_A(\beta)$, called dominant walls and here denoted $\{w_i(\beta)\}$ are needed to delimit the hyperbolic domain. Once the dominant walls $\{w_i(\beta)\}$ are found, one can compute the following matrix

$$A_{ij} \equiv 2\frac{w_i.w_j}{w_i.w_i} \tag{2.54}$$

where $w_i.w_j = G^{\mu\nu} w_{i\mu} w_{j\nu}$. By definition, the diagonal elements are all equal to 2. Moreover, in many interesting cases, the off-diagonal elements happen to be non positive integers. These are precisely the characteristics of a generalized Cartan matrix, namely that of an infinite KM algebra (see appendix A). As recalled in the introduction, for pure gravity in D space-time dimensions, there are $D-1$ dominant walls and the matrix A_{ij} is exactly the generalized Cartan matrix of the hyperbolic KM algebra $AE_{D-1} \equiv A^{\wedge\wedge}_{D-3} \equiv A^{++}_{D-3}$ which is hyperbolic for $D \leq 10$. More generally, bosonic string theory in D space-time dimensions is related to the Lorentzian KM algebra DE_D [15, 17] which is the canonical Lorentzian extension of the finite-dimensional Lie algebra D_{D-2}. The various superstring theories, in the critical dimension $D = 10$, and M-theory have been found [15] to be related either to E_{10} (when there are two supersymmetries, i.e. for type IIA, type IIB and M-theory) or to BE_{10} (when there is only one supersymmetry, i.e. for type I and II heterotic theories), see the table.

The hyperbolic KM algebras are those relevant for chaotic billiards since their fundamental Weyl chamber has a finite volume.

Theory	Corresponding Hyperbolic KM algebra
Pure gravity in $D \leq 10$	
M-theory, IIA and IIB Strings	
type I and heterotic Strings	
closed bosonic string in $D = 10$	

This table displays the Coxeter–Dynkin diagrams which encode the geometry of the billiard tables describing the asymptotic cosmological behavior of General Relativity and of three blocks of string theories: $\mathcal{B}_2 = \{M$-theory, type IIA and type IIB superstring theories$\}$, $\mathcal{B}_1 = \{$type I and the two heterotic superstring theories$\}$, and $\mathcal{B}_0 = \{$closed bosonic string theory in $D = 10\}$. Each node of the diagrams represents a dominant wall of the cosmological billiard. Each Coxeter diagram of a billiard table corresponds to the Dynkin diagram of a (hyperbolic) KM algebra: E_{10}, BE_{10} and DE_{10} .

The precise links between a chaotic billiard and its corresponding Kac–Moody algebra can be summarized as follows

- the scale factors β^μ parametrize a Cartan element $h = \sum_{\mu=1}^{r} \beta^\mu h_\mu$,
- the dominant walls $w_i(\beta), (i = 1, ..., r)$ correspond to the simple roots α_i of the KM algebra,

- the group of reflections in the cosmological billiard is the Weyl group of the KM algebra, and
- the billiard table can be identified with the Weyl chamber of the KM algebra.

2.3.6 E_{10} and a "small tension" limit of SUGRA$_{11}$

The main feature of the gravitational billiards that can be associated with the KM algebras is that there exists a group theoretical interpretation of the billiard motion: the asymptotic BKL dynamics is equivalent (in a sense to be made precise below), at each spatial point, to the asymptotic dynamics of a one-dimensional nonlinear σ-model based on a certain infinite-dimensional coset space G/K, where the KM group G and its maximal compact subgroup K depend on the specific model. As we have seen, the walls that determine the billiards are the *dominant walls*. For the KM billiards, they correspond to the *simple roots* of the KM algebra. As we discuss below, some of the subdominant walls also have an algebraic interpretation in terms of higher-height positive roots. This enables one to go beyond the BKL limit and to see the beginnings of a possible identification of the dynamics of the scale factors *and* of all the remaining variables with that of a nonlinear σ-model defined on the cosets of the KM group divided by its maximal compact subgroup [18, 19].

For concreteness, we will only consider one specific example here: the relation between the cosmological evolution of $D = 11$ supergravity and a null geodesic on $E_{10}/K(E_{10})$ [18] where KE_{10} is the maximally compact subgroup of E_{10}. The σ-model is formulated in terms of a one-parameter dependent group element $\mathcal{V} = \mathcal{V}(t) \in E_{10}$ and its Lie algebra value derivative

$$v(t) := \frac{d\mathcal{V}}{dt}\mathcal{V}^{-1}(t) \in e_{10}. \tag{2.55}$$

The action is

$$S_1^{E_{10}} = \int \frac{dt}{n(t)} < v_{sym}(t)|v_{sym}(t) > \tag{2.56}$$

with a lapse function $n(t)$ whose variation gives rise to the Hamiltonian constraint ensuring that the trajectory is a null geodesic. The symmetric projection

$$v_{sym} := \frac{1}{2}(v + v^T) \tag{2.57}$$

is introduced in order to define an evolution on the coset space. Here $< .|. >$ is the standard invariant bilinear form on E_{10} ; v^T is the "transpose" of v defined with the Chevalley involution[11] as $v^T = -\omega(v)$. This action is invariant under E_{10}:

$$\mathcal{V}(t) \to k(t)\mathcal{V}(t)g \qquad \text{where} \qquad k \in KE_{10}\ g \in E_{10} \tag{2.58}$$

Making use of the explicit Iwasawa parametrization of the generic E_{10} group element $\mathcal{V} = KAN$ together with the gauge choice $K = 1$ (Borel gauge), one can write

$$\mathcal{V}(t) = \exp X_h(t) \cdot \exp X_A(t)$$

[11] The Chevalley involution is defined by $\omega(h_i) = -h_i;\ \omega(e_i) = -f_i;\ \omega(f_i) = -e_i$

with $X_h(t) = h^a{}_b K^b{}_a$ and

$$X_A(t) = \tfrac{1}{3!} A_{abc} E^{abc} + \tfrac{1}{6!} A_{a_1\dots a_6} E^{a_1\dots a_6} + \tfrac{1}{9!} A_{a_0|a_1\dots a_8} E^{a_0|a_1\dots a_8} + \dots .$$

Using the E_{10} commutation relations in $GL(10)$ form (see [38, 39] for other decompositions of the E_{10} algebra) together with the bilinear form for E_{10}, one obtains up to height 30 [12] ,

$$\begin{aligned} n\mathcal{L} &= \tfrac{1}{4}(g^{ac}g^{bd} - g^{ab}g^{cd})\dot{g}_{ab}\dot{g}_{cd} + \tfrac{1}{2}\tfrac{1}{3!} DA_{a_1a_2a_3} DA^{a_1a_2a_3} \\ &\quad + \tfrac{1}{2}\tfrac{1}{6!} DA_{a_1\dots a_6} DA^{a_1\dots a_6} + \tfrac{1}{2}\tfrac{1}{9!} DA_{a_0|a_1\dots a_8} DA^{a_0|a_1\dots a_8} , \end{aligned} \tag{2.59}$$

where $g^{ab} = e^a{}_c e^b{}_c$ with

$$e^a{}_b \equiv (\exp h)^a{}_b ,$$

and all "contravariant indices" have been raised by g^{ab}. The "covariant" time derivatives are defined by (with $\partial A \equiv \dot{A}$)

$$\begin{aligned} DA_{a_1a_2a_3} &:= \partial A_{a_1a_2a_3} , \\ DA_{a_1\dots a_6} &:= \partial A_{a_1\dots a_6} + 10 A_{[a_1a_2a_3}\partial A_{a_4a_5a_6]} , \\ DA_{a_1|a_2\dots a_9} &:= \partial A_{a_1|a_2\dots a_9} + 42 A_{\langle a_1a_2a_3}\partial A_{a_4\dots a_9\rangle} \\ &\quad -42 \partial A_{\langle a_1a_2a_3} A_{a_4\dots a_9\rangle} + 280 A_{\langle a_1a_2a_3} A_{a_4a_5a_6} \partial A_{a_7a_8a_9\rangle} . \end{aligned} \tag{2.60}$$

Here antisymmetrization $[\dots]$, and projection on the $\ell = 3$ representation $\langle \dots \rangle$, are normalized with strength one (e.g. $[[\dots]] = [\dots]$). Modulo field redefinitions, all numerical coefficients in (2.59) and in (2.60) are uniquely fixed by the structure of E_{10}.

In order to compare the above coset model results with those of the bosonic part of $D = 11$ supergravity, we recall the action

$$\begin{aligned} S_{11}^{sugra} &= \int d^{11}x \Big[\sqrt{-\mathrm{G}}\, R(\mathrm{G}) - \frac{\sqrt{-\mathrm{G}}}{48}\, \mathcal{F}_{\alpha\beta\gamma\delta}\, \mathcal{F}^{\alpha\beta\gamma\delta} \\ &\quad + \frac{1}{(12)^4}\, \varepsilon^{\alpha_1\dots\alpha_{11}}\, \mathcal{F}_{\alpha_1\dots\alpha_4}\, \mathcal{F}_{\alpha_5\dots\alpha_8}\, \mathcal{A}_{\alpha_9\alpha_{10}\alpha_{11}} \Big] . \end{aligned} \tag{2.61}$$

The space-time indices $\alpha, \beta, \dots$ take the values $0, 1, \dots, 10$; $\varepsilon^{01\dots 10} = +1$, and the four-form $\mathcal{F}$ is the exterior derivative of $\mathcal{A}$, $\mathcal{F} = d\mathcal{A}$. Note the presence of the Chern–Simons term $\mathcal{F} \wedge \mathcal{F} \wedge \mathcal{A}$ in the action (2.61). Introducing a zero-shift slicing ($N^i = 0$) of the eleven-dimensional space-time, and a *time-independent* spatial zehnbein $\theta^a(x) \equiv E^a{}_i(x) dx^i$, the metric and four-form $\mathcal{F} = d\mathcal{A}$ become

$$ds^2 = \mathrm{G}_{\alpha\beta}\, dx^\alpha\, dx^\beta = -N^2(dx^0)^2 + G_{ab}\theta^a\theta^b \tag{2.62}$$

$$\mathcal{F} = \frac{1}{3!}\mathcal{F}_{0abc}\, dx^0 \wedge \theta^a \wedge \theta^b \wedge \theta^c + \frac{1}{4!}\mathcal{F}_{abcd}\, \theta^a \wedge \theta^b \wedge \theta^c \wedge \theta^d .$$

[12] We keep only the generators E^{abc}, $E^{a_1\dots a_6}$ and $E^{a_0|a_1\dots a_8}$ corresponding to the E_{10} roots $\alpha = \sum n_i\alpha_i$ with height $\sum_i n_i \leq 29$ (α_i are simple roots and n_i integers)

We choose the time coordinate x^0 so that the lapse $N = \sqrt{G}$, with $G := \det G_{ab}$ (note that x^0 is not the proper time[13] $T = \int N dx^0$; rather, $x^0 \to \infty$ as $T \to 0$). In this frame the complete evolution equations of $D = 11$ supergravity read

$$
\begin{aligned}
\partial_0\big(G^{ac}\partial_0 G_{cb}\big) &= \tfrac{1}{6} G\mathcal{F}^{a\beta\gamma\delta}\mathcal{F}_{b\beta\gamma\delta} - \tfrac{1}{72} G\mathcal{F}^{\alpha\beta\gamma\delta}\mathcal{F}_{\alpha\beta\gamma\delta}\delta^a_b - 2GR^a{}_b(\Gamma, C)\,, \\
\partial_0\big(G\mathcal{F}^{0abc}\big) &= \tfrac{1}{144}\varepsilon^{abca_1a_2a_3b_1b_2b_3b_4}\mathcal{F}_{0a_1a_2a_3}\mathcal{F}_{b_1b_2b_3b_4} \\
&\quad + \tfrac{3}{2} G\mathcal{F}^{de[ab}C^{c]}{}_{de} - GC^e{}_{de}\mathcal{F}^{dabc} - \partial_d\big(G\mathcal{F}^{dabc}\big)\,, \\
\partial_0 \mathcal{F}_{abcd} &= 6\mathcal{F}_{0e[ab}C^e{}_{cd]} + 4\partial_{[a}\mathcal{F}_{0bcd]}\,,
\end{aligned}
\tag{2.63}
$$

where $a, b \in \{1, \ldots, 10\}$ and $\alpha, \beta \in \{0, 1, \ldots, 10\}$, and $R_{ab}(\Gamma, C)$ denotes the spatial Ricci tensor; the (frame) connection components are given by $2G_{ad}\Gamma^d{}_{bc} = C_{abc} + C_{bca} - C_{cab} + \partial_b G_{ca} + \partial_c G_{ab} - \partial_a G_{bc}$ with $C^a{}_{bc} \equiv G^{ad}C_{dbc}$ being the structure coefficients of the zehnbein $d\theta^a = \frac{1}{2}C^a{}_{bc}\theta^b \wedge \theta^c$. (Note the change in sign convention here compared to above.) The frame derivative is $\partial_a \equiv E^i{}_a(x)\partial_i$ (with $E^a{}_i E^i{}_b = \delta^a_b$). To determine the solution at any *given* spatial point x requires knowledge of an infinite tower of spatial gradients; one should thus augment (2.63) by evolution equations for $\partial_a G_{bc}, \partial_a \mathcal{F}_{0bcd}, \partial_a \mathcal{F}_{bcde}$, etc., which in turn would involve higher and higher spatial gradients.

The main result of concern here is the following: there exists a *map* between geometrical quantities constructed at a given spatial point x from the supergravity fields $G_{\mu\nu}(x^0, x)$ and $\mathcal{A}_{\mu\nu\rho}(x^0, x)$ and the one-parameter-dependent quantities $g_{ab}(t), A_{abc}(t), \ldots$ entering the coset Lagrangian (2.59), under which the supergravity equations of motion (2.63) become *equivalent, up to 30th order in height*, to the Euler-Lagrange equations of (2.59). In the gauge (2.62) this map (or "dictionary") is defined by $t = x^0 \equiv \int dT/\sqrt{G}$ and

$$
\begin{aligned}
g_{ab}(t) &= G_{ab}(t, x)\,, \\
DA_{a_1a_2a_3}(t) &= \mathcal{F}_{0a_1a_2a_3}(t, x)\,, \\
DA^{a_1\ldots a_6}(t) &= -\tfrac{1}{4!}\varepsilon^{a_1\ldots a_6 b_1b_2b_3b_4}\mathcal{F}_{b_1b_2b_3b_4}(t, x)\,, \\
DA^{b|a_1\ldots a_8}(t) &= \tfrac{3}{2}\varepsilon^{a_1\ldots a_8 b_1 b_2}\big(C^b{}_{b_1b_2}(x) + \tfrac{2}{9}\delta^b_{[b_1}C^c{}_{b_2]c}(x)\big)\,.
\end{aligned}
\tag{2.64}
$$

2.4 Conclusions

We have reviewed the finding that the general solution of many physically relevant (bosonic) Einstein-matter systems, in the vicinity of a space-like singularity, exhibits a remarkable mixture of chaos and symmetry. Near the singularity, the behavior of the general solution is describable, at each (generic) spatial point, as a billiard motion in an auxiliary Lorentzian space or, after a suitable "radial" projection, as a billiard motion on hyperbolic space. This motion appears to be chaotic in many physically interesting cases involving pure Einstein gravity in any space-time dimension $D \leq 10$ and the particular Einstein-matter systems arising in string theory. Also, for these cases, the billiard tables can be identified with the Weyl chambers of some Lorentzian Kac–Moody algebras. In the case of

[13] In this section, the proper time is denoted by T while the variable t denotes the parameter of the one-dimensional σ-model introduced above.

the bosonic sector of supergravity in 11-dimensional space-time the underlying Lorentzian algebra is that of the hyperbolic Kac–Moody group E_{10}, and there exists some evidence of a correspondence between the general solution of the Einstein-three-form system and a null geodesic in the infinite-dimensional coset space $E_{10}/K(E_{10})$, where $K(E_{10})$ is the maximal compact subgroup of E_{10}.

Acknowledgement
It is a pleasure to thank Sophie de Buyl and Christiane Schomblond for their essential help in preparing these notes.

A Kac-Moody algebras

A KM algebra $\mathcal{G}(A)$ can be constructed out of a generalized Cartan matrix A, (i.e. an $r \times r$ matrix such that $A_{ii} = 2, i = 1, ..., r$, ii) $-A_{ij} \in \mathbb{N}$ for $i \neq j$ and iii) $A_{ij} = 0$ implies $A_{ji} = 0$) according to the following rules for the Chevalley generators $\{h_i, e_i, f_i\}, i = 1, ..., r$:

$$\begin{aligned}
[e_i, f_j] &= \delta_{ij} h_i \\
[h_i, e_j] &= A_{ij} e_j \\
[h_i, f_j] &= -A_{ij} f_j \\
[h_i, h_j] &= 0.
\end{aligned}$$

The generators must also obey the Serre's relations, namely

$$\begin{aligned}
(\mathrm{ad}\, e_i)^{1-A_{ij}} e_j &= 0 \\
(\mathrm{ad}\, f_i)^{1-A_{ij}} f_j &= 0
\end{aligned}$$

and the Jacobi identity. $\mathcal{G}(A)$ admits a triangular decomposition

$$\mathcal{G}(A) = n_- \oplus h \oplus n_+ \tag{A.65}$$

where n_- is generated by the multicommutators of the form $[f_{i_1}, [f_{i_2}, ...]]$, n_+ by the multicommutators of the form $[e_{i_1}, [e_{i_2}, ...]]$ and h is the Cartan subalgebra. The algebras $\mathcal{G}(\mathcal{A})$ build on a symmetrizable Cartan matrix A have been classified according to properties of their eigenvalues

- if A is positive definite, $\mathcal{G}(\mathcal{A})$ is a finite dimensional Lie algebra;
- if A admits one null eigenvalue and the others are all strictly positive, $\mathcal{G}(\mathcal{A})$ is an Affine KM algebra;
- if A admits one negative eigenvalue and all the others are strictly positive, $\mathcal{G}(\mathcal{A})$ is a Lorentzian KM algebra.

A KM algebra such that the deletion of one node from its Dynkin diagram gives a sum of finite or affine algebras is called an *hyperbolic* KM algebra. These algebras are all known; in particular, there exists no hyperbolic algebra with rang higher than 10.

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