Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

Noncommutative Geometry and Spacetime: A Historical Reconstruction

Enrico Maresca[1,2]

[1]Department of Humanities and Philosophy, University of Florence, via della Pergola 60, 50121, Florence, Italy
[2]Department of Civilisations and Forms of Knowledge, University of Pisa, via Pasquale Paoli 15, 56126, Pisa, Italy

E-mail: enrico.maresca@phd.unipi.it

Abstract. Noncommutative geometry (NCG) is a branch of pure mathematics with a wide range of applications to spacetime physics. Stemming from the divergence problem in QFT, modern contributions conjecture that the fundamental structure of spacetime is noncommutative. This seemingly homogeneous picture is the result of almost a century of discontinuous interest in noncommutative spacetime (NCST). In this paper, I present a three-phase division of the development of NCST approaches. The initial phase (1930s–1950s) introduced noncommutativity as a means of addressing the divergence problem while maintaining Lorentz-invariance. The second phase (1950s–early 1990s) initially dismissed noncommutativity and focused on the pressing problem of localisation at high energies. In this context, independent alternative approaches to QG identified the value of a fundamental length scale, which ultimately contributed to a reconsideration of the conjecture of spacetime noncommutativity. This was despite the fact that it undermined the original operationalist methodology. Finally, a third phase (1990s–today) has seen a renewed interest in NCST. I argue that this resurgence can be attributed to the discovery of new mathematics in the 1980s. Furthermore, I demonstrate how the third phase builds upon and continues instances from the previous attempts, but redirects the research question. In light of this, I finally argue that the history of NCG in physics can be understood as an example of "interlaced convergence." This instance of theory convergence may prove useful as a case study for the more general problem of theory building in QG.

1 Introduction

Noncommutative geometry (NCG) has been a fruitful area of research in both mathematics and physics over the past few decades. Applications span fields as varied as operator theory, differential geometry, particle physics, and cosmology.[1] Emphasis has been put on the relevance of noncommutativity to the construction of a theory of quantum gravity (QG). The primary contributions to this field of study conjecture that the fundamental structure of spacetime is noncommutative in appropriate regimes of applicability, that are typically taken to be near the Planck length. Despite numerous publications in mathematics and physics journals, there remains a considerable scope for further exploration by philosophers and historians.

[1]For a representative sample of the extensive literature on this topic, see [1, 2, 3, 4, 5, 6].

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

An important attempt to reconstruct the principal applications of NCG to spacetime was made by Heller, Lambert and Madore [7]. According to their reconstruction, noncommutative approaches in QG emerge from the appearance of divergences in field-theoretic calculations. The pioneers of NCG conjectured that continuous four-dimensional spacetime could be replaced by a fuzzy, quantum structure that holds at the Planck length. The introduction of a fundamental scale led to the development of NCG as a programme to reformulate existing geometric concepts in algebraic terms and extend them to the noncommutative case. This strategy applies to physics insofar as the gravitational field can indeed be treated as a noncommutative structure. NCG provides the foundation for developing an alternative theory of gravity that is consistent with general relativity (GR) at the appropriate scale.

Heller, Lambert and Madore's reconstruction is incomplete in that it omits many contextual elements. Furthermore, their work calls for an update that takes into account the numerous and more recent developments that have occurred over the last decades. Indeed, a significant increase in interest in NCST can be observed from the early 1990s onwards, with a notable acceleration in the early 2000s. However, the underlying reasons for this "rebirth" remain largely unexplored.

It is not my intention to argue in favour of NCG as a successful theory of QG in contrast to other approaches, such as loop quantum gravity and string theory. Similarly, the purpose of this work is not to argue in support of the *viability* of NCG as a concrete theory of QG. Two main reasons render this type of evaluation inappropriate. First, as it will be demonstrated, the development of NCG is considerably different and arguably more intricate than that of the other approaches, and so it necessitates some caution when making comparisons. Second, noncommutative geometric structures also emerge in the other approaches, which indicates that accepting the latter does not preclude the acceptance of the former. This paper, therefore, aims to present how physicists have attempted to render NCG a physical theory and to identify the epistemological strategies that have been employed in order to achieve this goal. This is crucially different from assessing whether these strategies have proved successful or not. Furthermore, this paper identifies some challenges that a NCST theory must overcome in order to achieve its goal. Not only is NCG an interesting topic of philosophical investigation in its own right, but it also raises new perspectives on questions and challenges that appear throughout the diversified literature on QG.

In this respect, this paper has three main goals. The first is to reconstruct the evolution of the noncommutative approach to spacetime. The reconstruction identifies a family of "ingredients" (principles, methodologies, conjectures, etc.) that are derived from neighbouring research areas and initially developed largely independently of each other. From the 1930s to the 2000s, a variety of combinations of these factors characterised the attempts to study NCST. From a diachronic perspective, the shifts from one set of ingredients to another can be explained by reference to the growing independent research in mathematics that occurred during the same period. Two driving problems characterise the initial and the final periods of the development of NCST theories (with a phase of stagnation occurring between the two). These are the divergence problem (first phase) and the breaking of Lorentz-invariance in spacetime models with a fundamental length scale (third phase). For the sake of presentation, I will be primarily concerned with the examination of these two. In fact, these issues demonstrate the existence of lines of continuity between the various proposals, thereby allowing an insight into an overall convergent development over the past decades. Further problems raised by NCG, which will not be extensively discussed in this paper due to length and focus constraints, include: the IR/UV mixing; the relationship between noncommutativity and curvature in noncommutative theories of gravity; the cosmological implications of NCST models.[2]

The second goal is to provide a systematic overview of the main approaches to the noncommutative structure of spacetime. The presentation of these results is functional to the assessment of the applications of NCG to physics as a research programme and the epistemological strategies employed to this end. In light of this, a detailed examination of the methodologies and underlying assumptions of NCST approaches reveals a shared fundamental structure inherited from previous results. The main points of divergence concern the mathematical techniques employed to model NCST and the relative priority to be given to metatheoretical elements drawn from standard physics. As a result, I argue that this programme can be understood as a case of "interlaced convergence." In other words, its development can be explained by pointing to recurrent methodologies and conjectures which brought together independent and often diverging attempts.

The third goal is to elucidate interesting dynamics in the history and construction of theories in modern physics, with NCG serving as a case study.[3] This entails incorporating sociological elements,

[2]Indications regarding these topics are briefly indicated in Section 4.2.3.

[3]For some recent work on theory-building in more discussed approaches to QG, see e.g. [8, 9, 10, 11].

such as the dissemination of pertinent research between academic communities and shifts in physicists' attitudes towards the noncommutativity of the spacetime structure. In particular, NCG offers the case of a theory with solid mathematical foundations, which however struggles to attach to physical reality. The reconstruction reveals that its mathematical usefulness often exceeds the physical applications due to an unclear physical interpretation.

To these ends, I will argue that the seemingly homogeneous picture of the development of NCST approaches is, in fact, the result of nearly a century of discontinuous interest in the concept of NCST and varying development of ideas. This development can be divided into three phases.

The first phase, spanning from the 1930s to the 1950s, originates from the problem of divergences in QFT and ultimately culminates in the debate on the localisation of physical systems at high energies (Section 2). In this context, orthodox quantum mechanics (QM) is regarded as a source of analogies for the introduction of a minimal length in spacetime models and the construction of the first proposal for a noncommutative algebra of spacetime coordinates.

The second phase, spanning from the 1950s to the early 1990s, focuses on the problem of localisation (Section 3). It is characterised by a generalised stagnation of the idea of a noncommutative structure of spacetime and by several attempts to solve the main issues raised by the first proposals.

The third phase, from the early 1990s to today, sees a rebirth of interest in NCST approaches (Section 4). I argue that the research conducted in the previous phase highlighted some ideas that naturally led to noncommutativity. Moreover, results of independent mathematical research in operator theory and quantisation contributed to this shift back to the notion of NCST. Consequently, the new research trends in noncommutative physics are concerned with the extension of the Standard Model of particles and the development of alternative theories of gravity.

Finally, I present the reconstruction as a case for theory building in QG (Section 5). NCST approaches originate from a particular treatment of the physical scales. Spacetime is conceived as a scale-dependent object, and the energy at which it is probed is crucial for defining the structure of the corresponding coordinate and field algebras. The reorientation of research from solving the divergence problem to incorporating (effective) Lorentz-invariance exemplifies the different attitude of physicists engaged in this field from the 1930s to the present day.

2 Phase 1 (1930s–1950s): Divergences and the origins of noncommutative geometry

Noncommutativity is a pervasive feature of significant parts of modern mathematics and physics. This is primarily a feature of composition laws: given two objects f and g (let them be functions or else) and an operation $\otimes$ between them, we say that $f \otimes g \neq g \otimes f$. In other words, the order of composition matters.

QM provides a particularly famous illustration of a physical theory that is inherently noncommutative. Soon after its discovery, physicists realised that extending this theory to relativistic regimes gave rise to significant concerns. In the context of the Lorentz-Abraham theory, the description of the interaction between the electron and its radiation field required the introduction of a coupling term to elucidate the manner in which the radiation field interacts with the particle that acts as its source. The result is known as the *divergence of the self-energy of the electron*: the energy of the electromagnetic field diverges in the vicinity of the source particle, leading to unphysical predictions. The appearance of infinities posed a threat to the consistency of the theory (see e.g. [12]).

NCST was initially motivated by the programme of removing the divergences from QFT.[4] The strategy was to identify a suitable spacetime structure that would allow a natural cutoff to be imposed on the field equations. The heuristic core was to emulate the construction of quantum phase space as closely as possible on three levels. Firstly, spacetime is described in quantum terms by means of Hermitian coordinate operators at high energies, and reduced to its special relativistic structure at appropriate scales.[5] Secondly, uncertainty relations regulate the coordinate operators, as for incompatible observables in QM. Finally, a fundamental scale exists which imposes a lower bound on the localisation of states in spacetime, as the Planck constant does in quantum phase space.

Heisenberg was the first to propose a modification of the structure of spacetime along these lines, by changing it from a continuous object to a lattice [17]. The aforementioned analogy suggested the division of physical space into cells of finite size as a means of constraining the infinities associated with

[4]See [13, 14, 15, 16] for other attempts. For a detailed analysis of the relation between NCST theories and the divergence problem, I direct the reader to Section 5.2.

[5]The original NCST models addressed the issue from the point of view of SR. This implies that the theory is expected to recover Minkowski spacetime in the commutative limit, in order to ensure its compatibility with SR in overlapping regimes. The investigation of the noncommutative solutions to Einstein field equations is more recent, due to the intricate relationship between noncommutativity and curvature. On the NCST approach to GR, see Section 4.2.3.

the electron's self-energy. Each cell had linear dimension equal to $\lambda = \hbar/mc$, where m is the mass of the proton. The introduction of cells of non-zero size precluded the possibility of electron wave packets peaking within arbitrarily small subregions. However, this solution was found to be overly simplistic. Not only did it violate the conservation of energy, momentum and charge. It also resulted in a breakdown of Lorentz-invariance, as Lorentzian boosts were shown to be incompatible with the introduction of a fundamental length.[6]

The initial interest in NCST is motivated by the desire to address the challenges posed by QFT, by examining how specific spacetime structures influence the behaviour of fields. The objective was to adapt the existing quantum mechanical construction while circumventing the shortcomings of Heisenberg lattice solution, which was deemed too inflexible to accommodate special relativity (SR). Rather than offering a genuine solution to the divergence problem, NCST assumed an *instrumental role* in reconciling potential solutions (especially those involving a fundamental length scale) with core physical principles, such as Lorentz-invariance. It is worth noting that, despite their initial popularity, these attempts to solve the divergence problem by introducing a fundamental length scale ultimately proved unsuccessful. In contrast, it can be argued that NCST, as a theoretical construction, retained a degree of independence that enabled it to be detached from its original research context in QFT and re-discovered in response to new research questions (see Section 3.3).

A significant challenge in the first phase was to identify a suitable candidate for the fundamental length. It was conjectured that the current description of the physical world would break down beyond this length. In this sense, the fundamental length represented both an epistemic boundary for well-established theories and a source of new mathematical research and developments in fundamental physics. By taking the cutoff to coincide with the fundamental length (in position space), the alleged arbitrariness associated with regularisation techniques was eliminated. However, positing the existence of a fundamental length (or energy or mass) as a postulate of the new theory gave rise to two interrelated issues: (i) introducing Lorentz-invariance into the new theory, and (ii) determining the exact value of the fundamental length.

Stemming from a deep dissatisfaction with standard regularisation methods ("distasteful" and "arbitrary"), in 1947, Hartland Snyder conjectured that the appearance of the divergences could be traced back to the assumption of spacetime continuity [18]. Consequently, he proposed that a modification of the concept of spacetime should provide new guidelines for their elimination.

Snyder's core idea was to maintain the Lorentz-invariance of the spacetime distance, while promoting spacetime coordinates to Hermitian operators (in analogy with QM) and relinquishing the continuity of their spectra. In general, Lorentz-invariance and commutativity imply that the coordinate operators possess continuous spectra. However, the introduction of a minimal unit of length constrains the capacity of localising physical systems within spacetime regions of smaller size, and so necessitates that the spectra of coordinate operators exhibit some degree of discreteness. Consequently, in order to preserve the principle of Lorentz-invariance, it is necessary to abandon the commutativity of the algebra of spacetime coordinate operators, as the continuity of spacetime is to be broken by the introduction of a minimal length scale.

The introduction of a fundamental scale precludes arbitrary displacements along the four spacetime axes. It is necessary that spatial and temporal displacements be of distances greater than the fundamental length. Momentum operators act on position states as generators of these displacements; thus, their expression is inversely proportional to the minimal length. Consequently, the conventional commutators between coordinate and momentum operators are deformed with respect to QM by an additional factor with quadratic dependence on the fundamental unit of length. To illustrate, the introduction of noncommutativity implies that the new commutators are:

$$[x^i, p_i] = i\hbar\Big[1 + \Big(\frac{\lambda}{\hbar}\Big)^2 p_i^2\Big] \quad , \quad [x^0, p_0] = i\hbar\Big[1 + \Big(\frac{\lambda}{\hbar c}\Big)^2 p_0^2\Big]. \tag{1}$$

For large values of momentum, these commutators differ from those posited in QM. This implies that a theory on NCST will reduce to a standard theory on Minkowski spacetime for low energies or small momenta, and diverge from the standard scenario for high energies or large momenta. *Born-reciprocity* then permits the reconstruction of Maxwell's equations in NCST, by imposing compatibility between the differential calculus and the underlying spacetime structure [19]. In other words, the reconstruction

[6] At that time, the prevailing operationalism required that Heisenberg's uncertainty relations be Lorentz-invariant, if they were to govern the behaviour of experimental devices. As a consequence of length contraction, a boosted observer measures a fundamental length that is shorter than that measured by another observer in inertial motion. As the fundamental length is assumed to be a universal, observer-independent property of spacetime, the application of Lorentzian boosts violates fundamentality: see [12].

employs a conjectured symmetry between spacetime coordinates and momenta, whereby a function $f(x, p)$ can be rewritten as $f(p, -x)$ in an equivalent manner.[7]

While noncommutativity was a natural strategy for implementing a bound on spatiotemporal localisation and interactions with a point-like source, it also generated two types of problems. Specific issues with Snyder's work included the unbounded spectra of the coordinate operators and the energy operator. These required the development of more sophisticated quantisation techniques, such as Weyl quantisation, which were not contemplated or sufficiently explored at the time. Additionally, Snyder did not specify how the algebra of spacetime coordinates acts on physical states, i.e., how the algebra is represented on the Hilbert space. Conversely, general issues included the physical interpretation of the new fundamental scale and the determination of its precise value.[8]

3 Phase 2 (1950s–1990s): Localisation and stagnation

Despite the limited immediate impact of Snyder's proposal, the general questions it raised extended beyond the specific case of NCG. The conjecture of a discrete spacetime structure gave rise to the issue of examining the consistency between localisation techniques within arbitrarily small regions and a fundamental length scale. Put differently, it was conceptualised as the attempt to answer to the question: Is it possible to indicate an operational context to localise a target system with arbitrary precision within a spacetime region of arbitrarily small size? Several physicists relied on an operationalist methodology: a system is sharply localisable if there is a localisation procedure with optimal accuracy. Consequently, these thought experiments typically involved a probe, whose initial state was well known, and a target, located somewhere within a region of finite size. The localisation procedure is typically described as a collision experiment between massless point particles, in accordance with the Bohr-Rosenfeld analysis [23]. After the interaction, information mediated by the detected probe should suffice to locate the target with arbitrary precision within that region. It is noteworthy that QM also precludes the sharp localisation of states in phase space due to the uncertainty relations. However, it does permit pure position states, albeit at the cost of maximum momentum uncertainty.

In contrast, a quantum spacetime structure was thought to eliminate even this possibility entirely, thereby precluding the existence of pure position states. This is, in fact, a geometric structure in which events cannot be sharply localised at spacetime points. The relevance of both general relativistic and quantum effects at the relevant energy scales implies that it is not possible to accept an infinite-mass limit as a method to increase the precision of the localisation procedure in quantum spacetime (see e.g. [24]).

A series of thought experiments emphasising the incompatibility between QFT and GR supported this disconcerting conclusion in the context of high-energy regimes. Therefore, despite the stagnation of research on NCST between 1948 and 1994, it is noteworthy that other physicists began addressing the two general problems above, independently of the noncommutative research programme. These attempts presented new challenges to physicists working on NCG, compelling them to re-examine pivotal elements of a prospective fundamental theory of NCST.

3.1 Bronstein's paradox and the localisation problem

Matvei Bronstein was among the first to investigate the conjecture that the spacetime structure can display quantum behaviour [25]. To substantiate his claim, he put forth a *breakdown* argument for the measurement of the Christoffel symbols of a classical gravitational field using a quantum probe. The experiment is based on two key assumptions: firstly, that a suitable probe exists; and secondly, that the domain of GR can be extended to high energies.

In the experiment, the total position uncertainty of the probe is expressed as a sum of two terms. The first is the Heisenberg uncertainty resulting from the measurement of the momentum of the probe, whereas the second is the gravitational uncertainty, which is given by the recoil of the probe after its interaction with the gravitational field. In order to minimise these uncertainties, it is necessary to increase the gravitational mass beyond a certain threshold. However, GR prohibits the mass to go beyond this

[7] Born-reciprocity is the equivalence between two plane wave expressions of the state of a free quantum particle in position and momentum spaces. If position operators are represented in diagonal form, then by reciprocity the momentum operators are derivatives with respect to them and vice versa, up to a factor. It can thus be concluded that any wave equation in position space can be transformed into another equation in momentum space. See [20, 21].

[8] Flint proposes the Compton wavelength as a fundamental bound, based on an operationalist interpretation of Snyder [22]. This scale represents the limit to the resolution of particle localisation. Beyond this limit, another particle is created and the procedure inevitably fails due to the interaction between the two. Nevertheless, Flint's proposal was met with considerable controversy and received only limited consideration.

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

threshold. In fact, it is required that the Schwarzschild radius of the probe do not exceed the linear dimension of the target region. For V volume of the region and ρ mass density of the probe, GR thus imposes a bound to the density:

$$\rho \lesssim \frac{c^2}{GV^{2/3}} \tag{2}$$

According to Bronstein, this bound must be violated if the theory intends to account for the sharp measurement of the Christoffel symbols within the target region at the relevant energies. Conversely, if the bound is preserved, the thought experiment concludes that there is a restriction on the possible measurements within a strong gravitational regime with quantum effects.

Bronstein's experiment demonstrates an incompatibility between the general relativistic description of classical spacetime (where events can be sharply localised and the Christoffel symbols precisely measured) and the quantum nature of the probe in a high-energy regime. In particular, the two theories are *operationally incompatible*.[9] In Bronstein's interpretation, high-energy regimes exclude GR and calls for a new quantum theory of spacetime. The fundamental quantity identified by GR provides a bound for its regime of applicability under suitable conditions.

For historical reasons, Bronstein's argument was largely overlooked by the literature.[10] However, it raised the issue of defining the limits of spatiotemporal localisation procedures in high-energy regimes and established a recurring template for proposals from the 1950s to the early 1990s.

It should be noted that no quantum gravitational scale trivially follows from the bound fixed in equation (2). This implies that a specific physical scale, which marks the regime where quantum effects become significant, must still be identified. The discovery of the existence of a fundamental length might rely on Bronstein's argument, but the latter is insufficient to determine its exact value unless further hypotheses or data are provided.

The problem of localisation was given significant attention by Mead in 1964 [27]. As many scholars before him, Mead identified the root of the divergence problem in the point-like picture of particles. A fundamental length, l, would eliminate the problem by imposing that no particle could be localised in a subregion of smaller size, i.e. with greater precision than the corresponding bound. This solution circumvents the divergence of the self-energy of the electron by precluding the interaction between the electromagnetic field and its point-like source. The intuition is that, if the notion of a point-like source particle is proved incompatible with the existence of a fundamental length scale, then the physical process which is deemed to cause the divergence is preempted. Mead proposes that "the physical content of a fundamental length postulate is that successive measurements of the position of a body, or of the reading of a clock, will show fluctuations at least of the order of l" [27, p. B850]. Consequently, the uncertainties introduced by the postulation of the fundamental bound preclude the possibility of sharp measurements of the respective noncommuting quantities. The value of the fundamental length can be derived from two sources: firstly, it can be calculated as a multiple of the size of the regions over which the field quantities of the theory are averaged; secondly, it can be determined by considering the gravitational effects.

Pursuing the second method, Mead's thought experiment considers a Heisenberg microscope structure and a probe used to locate a target within a region of radius r. Upon collision with the probe, the target emits light radiation, which is detected by a lens or detection surface. The frequency of the photons emitted provides information about the location of the target within the region of interest. The position will be affected by the following uncertainty. For simplicity, consider the x-direction in a nonrelativistic regime. Let ν be the frequency of light emission, m the rest mass of the target particle, r the radius of the localisation region, and ϵ the opening angle of the microscope. Then the position uncertainty in the x-direction is:

$$\Delta x \gtrsim \Big(\frac{r}{m} + G\Big)\nu \sin \epsilon. \tag{3}$$

[9]Indeed, the bound corresponds to the Schwarzshild radius of a black hole. Should the bound be violated, a black hole will form, with the event horizon preventing the escape of information and its subsequent reaching of the observer. Therefore, the operationalist constraint cannot be satisfied. For a similar mechanism, see below the experiment described by Doplicher, Fredenhagen and Roberts.

[10]Bronstein was indeed executed by Stalin's police shortly after the publication and his paper was not available in translation for several decades. Alexander Adrianov was the first to draw Bronstein's paper to the attention of Fedele Lizzi, who subsequently conveyed it to Sergio Doplicher (private communication). This occurred after the publication of Doplicher's seminal paper from 1995, in collaboration with Fredenhagen and Roberts (see below). Nevertheless, a comparison between the two arguments reveals striking parallels in their line of reasoning. It appears reasonable to trace a connection going from Bronstein to the early 1990s. This is not due to direct inspiration, but rather as a recurring approach to the same localisation problem, with minor modifications. The general strategy of this class of thought experiments is analysed in [26].

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

An analogous experiment can be designed for general relativistic regimes: see [27, pp. B853–B855].

In order to optimise localisation by decreasing the position uncertainty, we must increase the energy of the photons. Consequently, the gravitational pull of the target in the direction of the probe increases the uncertainty of the target's position, thereby defeating the objective of the procedure, namely to sharply localise the target. The argument can be generalised to derive the Planck length (λ_P) as a minimal bound for the position uncertainty:

$$\lambda_P = \sqrt{\frac{\hbar G_N}{c^3}} \approx 1.6 \times 10^{-33} cm, \tag{4}$$

where $\hbar$ is Planck's constant, G_N Newton's constant, and c is the speed of light.

The generalisation to strong gravitational fields demonstrates that spacetime regions of size smaller than λ_P are inaccessible due to a property inherent to the gravitational field itself. In Mead's words, "the equivalence between fundamental length and gravitational field fluctuations [...] indicates that any fundamental length theory is likely to involve gravitational effects of some kind in an important way." As the precision of the measurements increases, the region of spacetime over which these quantities are averaged must also expand. "Hence," Mead concludes, "a theory in which the gravitational field is quantized in the usual way [...] can at best be an approximate theory" [27, p. B860].

3.2 Geons and strings

The issue of fixing the exact value of the fundamental scale is not solely confined to the matter of localisation. Indeed, it was also a central issue in other approaches to quantum spacetime that did not rely on an argument similar to those proposed by Bronstein and Mead. These lines of independent research were undoubtedly well known to physicists in the early 1990s and contributed to the general acceptance of their results as the basis for the new NCST approaches.

Prior to Mead's contribution, John Wheeler had already attempted to address this question within the context of his theory of geons [28]. Inspired by his discussions with Misner in Leiden, Wheeler conjectured that fluctuations in the gravitational field, originated by high-energy quantum effects, would become significant at scales of the order of the Planck length. Since classical geons are much larger than this scale, spacetime fluctuations have no appreciable effect on low-energy experiments and are therefore neglected.[11]

Another suggestion came from early string theory. In this context, the S-matrix approach to the strong interaction demonstrated that poles of non-analyticity in the expansion of wave amplitudes for angular momenta could be identified by examining the (Regge) trajectories, which were all approximately linear with a common slope $\alpha \approx 1.0 GeV^{-2}$. By 1974, it had been established that the spectrum of closed strings contained a particle of spin 2. Building on a collaboration with Neveu, Sherk and Schwarz therefore proposed to interpret string theory as a quantum theory of gravity. In this approach, the string scale was identified as a natural candidate for deriving Newton's constant in four dimensions. This entailed an increase in the string tension (proportional to the reciprocal of α) by 38 orders of magnitude, resulting in a reduction in the string size (see [30]).

Fifteen years later, Amati, Ciafaloni and Veneziano demonstrated that the examination of scattering processes in the vicinity of the Planck energy results in the identification of a breakdown scale for the theory of the order of the string size [31]. At the Planck energy, the string size dynamically generates a new scale, the gravitational radius (hereafter R). If R is greater than the string size, gravitational instabilities occur and lead to the formation of a black hole. This result offers a suitable framework for the examination of the relation between the short-distance behaviour of strings and the scattering angles.

The authors identify a critical angle θ_M that separates two distinct regimes, characterised by different scattering angles θ. When $\theta < \theta_M$, the scattering amplitudes are classical and R is smaller than the string size. Conversely, if $\theta > \theta_M$, the strings undergo a "stretching" during the scattering process due to non-classical behaviour. Consequently, the usual uncertainty relations are inadmissible and we are left with the open problem of how to study this new regime. In particular, in this regime R is greater than the string size, thereby causing gravitational collapse. The authors state: "in no instance is the resolution smaller than the string length. This suggests that below the Planck scale the very concept of space-time changes meaning" [31, p. 41]. They conclude that θ_M marks the threshold for the change of the distance-angle relation from one regime to the other. This result substantiates the derivation of the string size as a minimal length due to the occurrence of gravitational instabilities in specific regimes.

[11] See also [29, pp. 247–248].

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

3.3 Fundamental length and noncommutative spacetime

A third type of localisation argument was put forth by Doplicher, Fredenhagen and Roberts (henceforth, DFR) between 1994 and 1995 [32, 33]. The objective of this *black-hole-production thought experiment* is to demonstrate the inconsistency between the operational tenet of sharp spatiotemporal localisation and the notion of quantum spacetime. The goal is to localise a target system within a spacetime region of radius r with a classical probe. The experiment is similar in design and execution to Bronstein's, while the emphasis on the limit of spacetime localisation is reminiscent of Mead's work.

DFR highlight that an increase in the energy of the probe serves to suppress the gravitational uncertainty (as in equation (2)), but it curves spacetime at the scattering region containing the target. As a result, a mini-black hole is formed, and information regarding the location of the target is unable to escape the event horizon and reach the observer. Consequently, the operationalist requirement is not satisfied. It is worth noting that the authors also demonstrate that the Schwarzschild radius of the black hole (which Bronstein had previously derived decades earlier) is of the order of the Planck length. This approach leads to the conclusion that operational meaning cannot be assigned to spacetime at high energies in terms of localisation procedures via scattering. In particular, DFR trace the limit of sharp localisation back to the effects of the gravitational collapse induced by the increase of the energy of the probe. It is precisely the localisation procedure that causes the black hole to form, thereby establishing a causal link between the two. This conclusion is compatible with the aforementioned results derived from alternative research programmes in QG, as it establishes the bound for operationalist accounts of spacetime at the Planck length.

DFR posit that this *quantum spacetime* structure can be characterised as a *noncommutative spacetime manifold*. This is a differentiable manifold with a noncommutative algebra $\mathcal{E}$ of complex-valued distributions, where the points of the manifold are the pure states of $\mathcal{E}$. By applying the framework of algebraic QFT to spacetime, they reduce the problem of defining this structure to that of appropriately characterising the noncommutative algebra of complex-valued, tempered distributions acting on it. This algebra $\mathcal{E}$ is generated by four Hermitian operators $q_0, \ldots, q_3$ which describe the coordinates in quantum spacetime. The algebraic structure is defined by the commutators

$$[q_\mu, q_\nu] = i\theta_{\mu\nu}(q), \tag{5}$$

for $\mu, \nu = 0, \ldots, 3$ and $\theta_{\mu\nu}$ a second-rank tensor. Further specification of the tensor leads to the definition of specific quantum spacetime models. It is crucial to encode the degree of spacetime noncommutativity directly in the algebraic structure. In particular, DFR interpret the noncommutativity of the algebra as the inability to sharply localise events in spacetime.

The thought experiment informs the formalism in that, under appropriate conditions, the localisation states are constrained by *generalised uncertainty relations* on the generators of the spacetime coordinate algebras, which are in turn interpreted as possible localisation procedures. Most importantly, DFR demonstrate how different choices of the commutators of this algebra, hence different uncertainty relations, characterise inequivalent NCST models. Nevertheless, all models must be reduced to their *commutative limit* in order to recover Minkowski spacetime (a further extension to curved spacetime is merely sketched in [33, p. 213]. This limit is calculated using a set of mathematical procedures which eliminate the effects of noncommutativity and recover a semi-classical (semi-commutative) formulation of the standard equations. To illustrate, in the context of DFR's work, if the noncommutativity is encoded in the noncommutative parameter q within $\mathcal{E}$, e.g. as $f \otimes g - g \otimes f = q$ for $f, g \in \mathcal{E}$, then the commutative limit can be obtained by expanding the product $f \otimes g$ into a series around $q < 1$ and truncating the series at the appropriate order. This mathematical procedure is consistent with the idea that higher-order effects, namely those induced by higher degrees of noncommutativity, are negligible and screened off at low energies. This allows for the recovery of an algebra that is appropriate for the description of Minkowski spacetime at low energies.

Furthermore, DFR propose that the algebra of localisation operators be Poincaré-invariant. Subsequently, standard procedures of QFT can be employed to define a field theory on NCST, despite the tension with the fundamental length scale. One notable example is Wigner's concept of elementary particle as being characterised by the irreducible representations of the universal covering of the Poincaré group. In other words, Poincaré-invariance is motivated by the ability to use some of the typical techniques of QFT to derive a mathematical description of free field theories on NCST.[12]

At the end of this second phase, it is evident that the identification of a minimal length scale, which bounds the regime of GR (or, alternatively, operationally meaningful localisation procedures), promoted

[12] Wigner classification has been later modified in [34] in order to fit the underlying structure of those NCST models which violate the Poincaré-invariance.

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

the recovery of NCG from its stagnation. As previously suggested by Heisenberg (see Section 2), the noncommutativity of the theory offers a natural formalism to introduce a minimal length scale. The commutators of an algebra of spacetime coordinate operators are shown to depend on powers of a constant quantity, q, which characterises the scale described by the theory. This quantity is identified with the noncommutative parameter and serves to delineate the regime of applicability of the noncommutative theory.

Moreover, q provides a lower bound on the minimal uncertainty for the localisation of events in spacetime at high energies. In Hellund and Tanaka's words, "[t]he noncommutativity of operators representing configuration space means that simultaneous measurements of space and time are interfering operations" [35, p. 192].[13] Consequently, the generalised uncertainty relations restrict the set of admissible states and exclude sharply localisable ones (corresponding to classical spacetime events) as *unphysical*. In the most favourable case, the algebra allows for *optimal localisation states*, whose spacetime uncertainty is in accordance with both the noncommutativity scale and the energy scale of the theory.

4 Phase 3 (1990s–today): New approaches

The second phase demonstrated the limitations of the operationalist methodology in quantum gravitational regimes, necessitating either its refinement or abandonment. Nevertheless, this also provided new impetus for NCST approaches in the 1990s, as noncommutativity was identified as a natural candidate for modelling the introduction of a fundamental length in the formalism, as the noncommutative parameter of the spacetime coordinate algebra.

In contrast to Snyder's work, DFR demonstrated that there are *numerous inequivalent* candidates structures for modelling NCST. Indeed, consider the following mathematical argument. Let $q_0, \ldots, q_3$ be coordinate operators for flat quantum spacetime. Their structure is specified by an antisymmetric, second-rank tensor $\theta_{\mu\nu}$ as

$$[q_\mu, q_\nu] = i\theta_{\mu\nu}(q). \tag{6}$$

To specify a model, call λ the fundamental length and impose that the commutative limit (i.e. sending λ to zero) recovers Minkowski spacetime. One possible strategy is to expand $\theta_{\mu\nu}$ in a neighbourhood of λ and to keep only those terms that depend on it (and so vanish in the limit). Up to the second order,

$$[q_\mu, q_\nu] = i\theta_{\mu\nu}(q) = i(q_\mu q_\nu + \lambda q_\sigma \gamma^\sigma_{\mu\nu} + \lambda^2 \Theta_{\mu\nu}) = i(\lambda q_\sigma \gamma^\sigma_{\mu\nu} + \lambda^2 \Theta_{\mu\nu}). \tag{7}$$

for $\gamma^\sigma_{\mu\nu}$ and $\Theta_{\mu\nu}$ to be further specified. From this identity, taking $\gamma^\sigma_{\mu\nu} = 0$ implies that

$$[q_\mu, q_\nu] = i\lambda^2 \Theta_{\mu\nu}, \tag{8}$$

which defines the *canonical noncommutative spacetime*. By contrast, suppose that $\gamma^\sigma_{\mu\nu}$ is coordinate-independent. If $\Theta_{\mu\nu} = 0$ for all $\mu, \nu = 0, \ldots, 3$, then a suitable choice of the components of $\gamma^\sigma_{\mu\nu}$ gives

$$[q_0, q_j] = i\lambda q_j = \frac{i}{\kappa} q_j \quad , \quad [q_i, q_j] = 0, \tag{9}$$

which is the structure of the *κ-Minkowski spacetime*. This second model has a Lie-algebra-type noncommutativity: the commutation relations between spacetime coordinates are non-zero only for the time-space product and have a linear dependence (in this case through $\gamma^\sigma_{\mu\nu}$) similar to that of a Lie algebra. Other modifications of such models, e.g. a version of κ-Minkowski spacetime in polar coordinates (ρ-Minkowski), have also been suggested in the recent literature, but are not discussed in this paper due to length constraints: see for example [36].

This construction emerges from the debate on spatiotemporal localisation at high energies. An examination of the literature reveals that, since the 1990s, Snyder's general questions had arguably already been addressed. This was not the case, however, with regard to his specific questions. In particular, it was unclear how to precisely formalise noncommutativity at the field level and to conceptualise NCST in an appropriate manner. In this context, I argue that the development of new areas of mathematics in the 1980s represents a crucial missing piece of the puzzle. These new techniques enabled the precise formulation of preceding ideas, such as the introduction of bounded spectra for Snyder's coordinate operators. Furthermore, they helped clarifying the physical structures involved and encouraged the re-examination of ideas from the first phase, while contributing to a shift in the overall direction of research. As outlined

[13] This condition bounds the admissible physical states of NCST. In particular, the reduction of the noncommutative structure to the standard one depends on q either linearly or quadratically, depending on the models.

in Section 4.2, the new approaches often employed the new tools to relegate Lorentz-invariance to the commutative limit. Additionally, they shifted the research question to focus on the development of noncommutative extensions of QFT, while maintaining the dependence of field quantities on the underlying NCST structure.

4.1 The contribution of new mathematics

The 1980s marked the culmination of extensive research in pure mathematics that had been ongoing since the 1930s. For the purposes of this presentation, the principal areas of development are quantisation theory, quantum group theory and algebraic geometry (briefly discussed in Section 4.2.1).[14]

At this juncture, I briefly illustrate the development of these techniques from a mathematical standpoint. Their main physical applications of NCG depend on the specific approaches employed and will be presented in the following section. For additional applications, the reader is directed to the respective references.

The idea of canonical quantisation originated at the advent of QM. Indeed, physicists were already familiar with the procedure of promoting a special set of functions on classical phase space to operators acting on a Hilbert space. However, only a restricted subset of classical observables could be quantised. In 1931, Weyl put forth a more comprehensive approach to quantisation, whereby bounded observables were associated with otherwise unbounded counterparts of certain classical observables, such as the position and momentum of a particle [40]. The *Weyl map* was defined by means of the Fourier transform of the function in question, with the result dependeding on a specific ordering of the product of noncommuting operators. A second map, discovered by Wigner, returned the operators to their classical counterparts by calculating their trace [41].

Moyal built upon Weyl's work, developing a statistical interpretation of the quantum mechanical wavefunction [42]. He proved that the new theory of phase space distributions was equivalent to a theory of functions constituted by noncommuting operators. The noncommutative law of composition was subsequently designated the *Moyal* or *star product*.[15]

These findings promoted a new method, called *deformation quantisation* [44, 45]. Indeed, the utilisation of the star product was expanded to encompass the case of *quantum theories as deformations of their classical counterparts*. Deformation quantisation developed an idea that had "remained hidden 'in the back of the minds' for a long time" [37, p. 2]. Building on the previous attempt of geometrical quantisation in the 1970s and on the notion of *contraction* [46, 47],[16] a *deformation* of an algebra is defined as an extension where the product depends on an additional parameter (q). The newly introduced q-dependent terms in the equations of a deformed theory represent additional contributions to the standard, undeformed theory. If we take q to represent a physical parameter, these contributions become relevant at the appropriate high-energy scale. In theory space, they serve to identify trajectories that converge to the undeformed theory at low energies. Subsequently, contraction entails a semi-classical analysis of the deformed theory of interest. In the simplest cases, this methodology will align with the calculation of a limit by sending the deformation parameter to an accumulation point. In a noncommutative deformation, the contraction constitutes a crucial aspect of the commutative limit. While the deformation preserves associativity and Lie-structure, it also introduces non-trivial cases. Among these are bialgebras, which are associative algebras with an additional coproduct and compatibility relations. They admit deformation of both algebraic and dual sectors (see e.g. [48, pp. 6–10]).[17]

Deformation quantisation aligns with the operational approach, which had proven successful in numerous applications of the Weyl quantisation map. Moreover, it facilitated a more rigorous mathematical construction of QFTs. In particular, the star product was proved equivalent to a deformation of the point-wise product of smooth, complex-valued functions on the manifolds of classical phase space, with deformation parameter equal to $\hbar$ (see, in particular, [49]).

[14] This section largely follows [37] in the presentation of the results leading to deformation quantisation. See also [38]. For a more precise reconstruction of the development of quantum groups, see [39].

[15] It should be noted that Groenewold also discovered a similar formula just one year later [43]. The product is therefore sometimes called the *Groenewold-Moyal product*.

[16] Building on further work by Segal, Wigner and Inonü from the 1950s, the basic idea of contraction is to derive a mathematical structure by turning a fixed value into a variable parameter and sending it to zero. To illustrate, the Galilei group can be derived from the Poincaré group through a mathematical contraction, whereby the former can be obtained by taking c^{-1} as a variable parameter in the latter and computing the limit for $c^{-1} \to 0$. Conversely, changing the contraction procedure modifies the result: if the Poincaré group is contracted by sending c, instead of c^{-1}, to zero, the new structure is the Carroll group.

[17] In other words, they admit a deformation that introduces a dependence on q of the coproduct while maintaining the standard algebraic product intact. Quantum groups (see below) serve as a paradigmatic example of bialgebras.

A notable example of deformation, both historically and mathematically, is that of *quantum groups*,[18] which represent deformations of associative algebras. *Hopf algebras* are a special case where the structure is derived from an associative algebra and its dual co-algebra. These two sectors are connected by compatibility relations (as in a bialgebra) and the co-sector possesses a non-trivial co-product.[19] These objects were initially studied by three independent research groups: the Leningrad school for its applications to inverse methods ([50]; see also the research of Drienfel'd [51, 52] in Moscow); the Kyoto school for the quantisation of the Lie algebra $\mathfrak{sl}(2)$ [53]; the Warsaw school for the differential calculus in NCG, based on a deformation of $\mathfrak{sl}(2)$ [54, 55].[20] The connections with NCG were further explored once it was realised that quantum groups are special cases of algebraic structures with star product, and that all Lie groups admit a quantum version. Further research demonstrated their suitability for use in field theories requiring more advanced geometry, in particular, super-space (see e.g. [56]).

Finally, in two seminal papers [57, 58], Majid suggested to construct quantum co-algebras as extensions of physically significant groups, with a particular focus on the Poincaré group. This resulted in the quantisation of the relativistic transformation laws from one reference frame to another. The deformation parameter κ, which occurs in the deformed group, was indeed interpreted as a scale with dimensions of mass [59]. Subsequent developments included the construction of field theories on spaces with this new symmetry group, named the κ-Poincaré group.[21] κ-Poincaré is a Hopf algebra, specifically the bicrossproduct between the Lorentz algebra and a deformed translation coalgebra with a deformed action on the momentum sector. To illustrate, the commutator structure of κ-Poincaré in its bicrossproduct basis is as follows:

$$\begin{gathered}
[P^R_\mu, P^R_\nu] = 0, \qquad [M_j, M_k] = i\epsilon^s_{jk} M_s, \qquad [\mathcal{N}_j, \mathcal{N}_k] = -i\epsilon^s_{jk} M_s, \\
[M_j, P^R_0] = 0, \qquad [M_j, P^R_k] = i\epsilon^s_{jk} P^R_s, \qquad [\mathcal{N}_j, M_k] = i\epsilon^s_{jk} \mathcal{N}_s, \\
[\mathcal{N}_j, P^R_0] = iP^R_j, \qquad [\mathcal{N}_j, P^R_k] = i\delta_{jk}\left[\frac{\kappa}{2}\left(1 - e^{2P^R_0/\kappa}\right) + \frac{1}{2\kappa}\vec{P^R}^2\right] - \frac{i}{2\kappa} P^R_j P^R_j,
\end{gathered} \tag{10}$$

where P^R is used to indicate the spatial and temporal translations, M the spatial rotations, $\mathcal{N}$ the deformed boosts, and the R superscript indicates the dependence of the generators on the chosen basis.

It is evident that the commutator between deformed boosts and spatial translations is very dissimilar from that of the Poincaré algebra. The deformation of the action of boosts on field quantities in NCST is crucial for resolving the invariance problems presented in Section 2 (see Section 4.2.2). Moreover, it allows for the derivation of a deformed mass Casimir. Consequently, in a theory constructed on a NCST model with underlying κ-Poincaré symmetry, the dispersion relation is also deformed and ultimately dependent on the noncommutative parameter of the model. A considerable amount of studies into these theories with a κ-deformed dispersion relation have been conducted from the point of view of phenomenology. For example, this mathematical result is interpreted as an indication that such theories allow for the wavelength-dependence of the speed of light proportional to the noncommutative parameter or significant shifts in relevant astrophysical thresholds of the order of $1/\kappa$.[22]

Furthermore, the duality of the two algebraic sectors of a Hopf algebra with each other gives rise to the notion of *self-duality*, as noted by Majid and Ruegg ([66], but see also [48, pp. 10–11]). From a mathematical perspective, this implies that the arrows of the algebraic sector can be reversed into those of the coalgebraic sector, and vice versa, thereby deriving an equivalent Hopf algebra. The feature is significant because it implies that the noncommutative deformation of the coalgebraic sector induces a corresponding modification in the algebraic sector, and vice versa. This correspondence between sectors allows for the reconstruction of the underlying NCST structure from the modified action of translations, which generates trajectories in the corresponding phase space. To illustrate, consider the deformed

[18] The name should raise suspicion. In fact, they are not technically groups in the standard sense; rather, they are particular bialgebras.

[19] The compatibility is expressed by the fact that there is a homomorphism from the deformed coproduct structure to the deformed commutator structure of the algebraic sector. A special class of Hopf algebras, called triangular quantum groups, has been extensively studied in the physics of scattering processes. A relaxation of the commutativity of the co-product results in quasitriangular Hopf algebras which satisfy the Yang-Baxter equation (see e.g. [48, ch. 2]). The study of these groups, however, is beyond the scope of this paper.

[20] The enthusiasm for this new research area is evident, for example, in Drienfel'd: "It is the author's opinion that Hopf algebras should be regarded not only as a technical tool for the construction of solutions of QYBE [i.e. quantum Yang-Baxter equations] but also as a natural language for the quantum method of the inverse problem" [52, p. 274].

[21] See e.g. [60, 61, 62, 63, 64] to name a few.

[22] See e.g. [65] and references therein.

subalgebra of translations in (10). The coproduct of a Hopf algebra $\mathcal{C}$ is defined as $\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$, $\Delta(a) = a_i^{(1)} \otimes a_i^{(2)}$. It can be shown that the coproduct of the translation generators of κ-Poincaré is:

$$\Delta(P_0^R) = P_0^R \otimes \mathbb{I} + \mathbb{I} \otimes P_0^R, \qquad \Delta(P_i^R) = P_i^R \otimes \mathbb{I} + e^{-P_0^R/\kappa} \otimes P_i^R. \tag{11}$$

These provide a composition law for momentum space that is not trivial. The resulting structure of the momentum sector can be dualised to give rise to the κ-Minkowski algebra (9). The noncommutativity of the characteristic algebra of κ-Minkowski is motivated by the non-cocommutative relations between the generators of the dual space. Consequently, the duality between coproduct structure and commutator structure links the subalgebra of deformed translations to an underlying NCST model, on which κ-Poincaré acts covariantly. This new *quantum Born-reciprocity* (as the authors also describe it) justifies the interpretation of quantum groups as symmetries of specific NCST models, each one with a different underlying symmetry group, thus providing a rigorous mathematical foundation for the aforementioned result.

4.2 Modern approaches

The significant advancements in pure mathematics during the 1980s contributed to a surge in interest in NCST models. This was a consequence of an enhanced receptivity of mathematics to the needs of physicists. The development of new mathematical branches and the integration of diverse, independent research areas with the needs of physics were driven by the necessity to rigorously substantiate the intuitions underlying these physical problems. It should be noted that this convergent dynamics was not absent in the 1930s and 1940s, but significantly less pronounced than in the case of NCG. The shift in attitude serves to reinforce the findings of the second phase and provides a rationale for the re-emergence of earlier ideas within the novel approaches of the third phase. In particular, as I will illustrate, two mathematical techniques emerged as particularly noteworthy and warrant further examination (see Section 4.2.2). These are the Weyl quantisation and the quantum groups. The former technique was expected to solve the issue of unboundedness, while the latter to elucidate the relationship between spacetime symmetries and the fundamental scales.[23]

For example, an important precursor to contemporary NCST approaches can be found in the semi-classical proposal known as *doubly special relativity* (DSR) [65]. The objective of this project was to treat the Planck length as a fundamental scale on a par with the speed of light c in characterising the transformation laws of inertial frames. Justified by the aforementioned localisation arguments and a measurement-oriented attitude, DSR addressed a long-standing dilemma: either break Poincaré-invariance in favour of a preferred reference frame (in which the fundamental scale is exactly λ_P), or retain it at the cost of inconsistency. The first solution implies that the fundamental length scale is associated with a background medium which selects a preferred class of inertial observers, each one measuring the value λ_P at microscopic scales. The second solution maintains SR and Poincaré-invariance but is unable to incorporate the fundamental length scale as an observer-independent property of spacetime. The "third way" of DSR is to modify the postulates of SR by introducing a new dependence on λ within the Planckian regime. In particular, the relativistic dispersion relation is deformed by the new relativistic fundamental constant and the Poincaré spacetime symmetries are replaced by their deformed quantum group.[24]

DSR suggests that it is reasonable to assume that the NCST structure would be approximately flat in its regime of applicability, despite the now pervasive nonlinearities. Building on previous work by Born and Kadyshevskii on the duality between curved momentum space and NCST [20, 21, 71], it posits that canonical and κ-Minkowski models are spatiotemporal counterparts of the coaction of different quantum groups on momentum space [72]. The objective is to ultimately construct a NCST structure for the

[23]Other examples of mathematical techniques which have proved useful in the physical applications of NCG include the cyclic and Hochschild cohomologies [4] and Kasparov's KK-theory [67].

[24]This modification addresses the incompatibility between the fundamental scale and Lorentzian boosts, which is a consequence of spatial contraction. The use of quantum groups serves to salvage the fundamental scale by introducing highly deformed boosts. Nevertheless, the proposed solution is open to question. For example, Magueijo and Smolin criticised this approach and put forth an alternative wherein the Lorentz group is maintained but its representation on momentum space is changed from linear to nonlinear [68]. DSR can therefore be described as a form of "special relativity in non-linear disguise" (the expression is from [69, p. 3]). In the limit, the two proposals are equivalent, with both recovering the standard spacetime symmetries. It is noteworthy that the momentum spaces of the two alternatives are related by nonlinear transformation maps [70]. The lack of clarity regarding the physical meaning of momentum coordinates presents two potential approaches for further investigation. One option is to select a specific theory of DSR, while the other is to identify a formulation that is independent of a particular choice of basis for momentum space.

semi-classical regime of QG.[25] However, a more comprehensive theory of QG is required to this end.

Three main NCST approaches have emerged from the enthusiastic reception of the new mathematics in the 1990s and especially in the early 2000s: the *spectral triple approach*; the *quantum group approach*; the *noncommutative approach to string theory.* Sections 4.2.1 and 4.2.2 will address the first two respectively. However, an examination of the third approach will not be undertaken here, as it would require an extensive addition of foundational and historical considerations, which would exceed the scope of this paper. In this regard, I will simply refer to a seminal paper on the subject: Seiberg and Witten's *String theory and noncommutative geometry* (1999) [76].

4.2.1 The spectral triple approach The spectral triple approach is most frequently associated with the work of Alain Connes and his collaborators. It is founded upon an algebraic-geometric duality between the C^*-algebra of smooth functions on a manifold and the underlying topology. This result, based on Gelfand duality, is employed extensively to reconstruct the manifold structure algebraically and then transform it into a noncommutative structure. As a result, the new structure provides an algebraic description of a NCST. However, upon turning on noncommutativity, the duality breaks down.[26] Consequently, the approach faces the challenge of providing new definitions for old geometric objects which are no longer well-defined. Among these are concepts such as space, localisation, and metric. Conversely, the outcomes of the old localisation problem are translated and addressed within the new algebraic framework by introducing the noncommutativity into the characterising algebra.

The metric structure is recovered by a spectral method. This method is suggested not only by the success of spectral analysis in other areas of physics, notably in QM. It is also proposed as a natural method "to adapt the basic paradigm of geometry to the new standard of length" [4, p. 179], where the latter works as an "inspiration," rather than as a full-bloaded motivation for operationalism. Connes and Marcolli describe this "new standard of length" as the expression of a unit of length in terms of a fraction of the radiation wavelength of a chosen element. It implies the identification of a unit of length as a minimum of the distance, whereas the continuity of classical (special and general relativistic) distances is the result of coarse-graining. This is based upon an analogy with the procedure used in QM, whereby the spectra of self-adjoint operators contain the possible outcomes of measurements of the corresponding quantities. The approach attempts to identify a quantum analogue of the metric. This operator, which may be regarded as a generalisation of the so-called *Dirac operator* from spinorial Riemannian geometry, has a spectrum that permits the definition of a distance in the space of noncommuting self-adjoint operators. This distance differs from the relativistic construction in that it admits inner fluctuations which, from the standpoint of particle physics, correspond to the gauge bosons that appear when the Standard Model is extended with a minimal coupling to gravity in this approach (see [4, pp. 194–196; 244–246; 248–251]). Accordingly, this algebraic methodology postulates *spectral triples* as fundamental entities. They are triples $(\mathcal{A}, \mathcal{H}, \partial)$, consisting of an algebra $\mathcal{A}$ with an operator representation on the Hilbert space $\mathcal{H}$, and a self-adjoint operator ∂ with suitable properties for deriving a metric.[27] Each spectral triple describes an associated noncommutative manifold.

In the original approach of Chamseddine and Connes, field theories at high-energy scales, which are sensible to the NCST structure, are studied from the perspective of a *spectral action principle* within an algebraic framework [78, 79]. This principle is introduced with the aim of deriving a Lagrangian for the Standard Model minimally coupled to gravity from an underlying noncommutative geometric structure, thereby providing a formally unified (albeit arguably still incomplete) derivation of the fundamental interactions. The spectral action is dependent upon both the Dirac operator of a spectral triple and a cutoff parameter. This geometric object is employed to extend the standard framework of QFT to encompass noncommutative manifolds. Indeed, the main goal of the approach, initially pursued by Connes and Lott [80] and then by Chamseddine, Connes and Marcolli [82, 4], is to provide a noncommutative extension of the Standard Model Lagrangian. The introduction of a noncommutative spectral triple

[25]This is a challenging task. DSR derives a *principle of relative locality*, whereby the position of the target differs in the spacetime reconstructed by two observers, one of whom is boosted with respect to the other. As a result, absolute localisation is not feasible in the context of DSR [73]. Furthermore, the curvature of momentum space induces a deformation of the conservation laws due to the nonlinearities [74]. Further results of DSR include spacetime fuzziness, the derivation of a threshold for photon decay into an electron-positron pair, smaller anomalies for particle reactions compared to symmetry-breaking frameworks, and the hypothesis of wavelength-dependence of the speed of light. For details, I defer to [75] and references therein.

[26]The noncommutative failure of the duality affects a number of interesting physical cases. For further insight into this issue and its implications for the geometric foundations of NCST approaches, see [77].

[27]The definition is then completed with the inclusion of chirality and charge conjugation. See [1]; also, [4] for its applications to physics.

results in the inclusion of an additional term in the Lagrangian, whereby describing background semi-classical gravity.[28] However, as suggested by Lizzi, the Chamseddine-Connes-Marcolli Lagrangian is unsatisfactory, as it leads to the incorrect prediction for the mass of the Higgs boson [83]. In contrast, the comprehensive development of renormalisation techniques for spectral triples has been pursued and promises to correct this prediction [84].

4.2.2 The quantum group approach The quantum group approach is founded upon the established link between NCST models and quantum groups, as initially proposed by Majid and Ruegg. Consequently, it attempts to reconstruct the structure of NCST in quantum gravitational regimes. Furthermore, it analyses the deformation of field theories when reconstructed on noncommutative structures. The existence of a fundamental length is typically linked to the conjecture that a theory based on quantum groups should provide an effective description of spacetime. Indeed, following an intuition that originated at the end of the second phase, the regime of applicability is constrained to the scale identified by the noncommutative parameter. In particular, star-product constructions permit a correspondence between commuting fields on NCST and noncommutative fields on commutative spacetime.[29] This duality is founded upon a mathematical relationship between Weyl and Wigner maps, Ω and Ω^{-1} respectively, which was initially introduced by [76] within the context of string theory: for two fields φ and ψ in commutative Minkowski spacetime, $\varphi \star \psi = \Omega^{-1}(\Omega(\varphi)\Omega(\psi))$.

The initial applications of the quantum groups formalism have been in the context of non-relativistic QM [85, 86] and the deformation of the Heisenberg algebra [87]. Then, the advancement of Fourier techniques to relate the quantum symmetries of NCST models with the integral expressions of field quantities prompted an investigation into field theories and a comparison with DSR and QM as sources of analogy and guidance [88, 89, 90]. This is because fields on NCST can be described by the Weyl transformation as

$$\Omega(\varphi) = \frac{1}{2\pi^{n/2}} \int \tilde{\varphi}(p) W(p) d^n p, \tag{12}$$

where $\tilde{\varphi}$ is the Fourier transform of the field φ and $W(k) = \Omega(e^{ipx})$ is a generalised Weyl system that allows expanding $\Omega(\varphi)$ in terms of plane waves on NCST [91].

Furthermore, recall that the self-duality of the underlying quantum group enables the reconstruction of a momentum space based on the representation of the translation coalgebra, endowed with a specific noncommutative composition law. As a result, the composition law of momenta modifies the product of fields on NCST due to the occurrence of the Weyl systems (with explicit dependence on momentum) in the expression (12). In other words, the relationship between composition of fields, noncommutative structure of spacetime and underlying quantum group is established thanks to the self-duality property of the latter. This property is crucial to ascertain the compatibility between the underlying NCST and the behaviour of fields defined on it. An important example is the study of scattering processes in the context of a NCST model. It can be demonstrated that the noncommutativity of spacetime affects the curvature of momentum due to the quantum Born-reciprocity (see Section 4.1), which in turn implies a modification of the energy-momentum conservation. This result calls for specific choices in the ordering of the momenta in the calculation of cross-sections as a consequence of the underlying noncommutativity [88].

Concurrently, Aschieri and collaborators initiated the application of quantum groups to the formulation of a noncommutative theory of gravity [92]. In particular, the picture they suggested extended a deformed version of the Poincaré group (called *twisted Poincaré*) to a larger group in order to obtain a deformation of the Einstein-Hilbert action. Their motivation is to identify suitable field equations to study the physics of quantum spacetimes and to illustrate how this new picture deviates from the general relativistic one, in both formalism and interpretation. It is conjectured that a deformed theory of gravity will be a theory covariant under the new deformed Hopf algebra of diffeomorphisms. The deformation parameter enters the deformed Einstein-Hilbert action as a coupling constant (with similarities to gauge

[28] Further mathematical research has included the study of exotic geometries of noncommutative space. These mathematical developments exceed the application of NCG to QG and answer to a broader interest in noncommutative geometric techniques irrespective of any spatiotemporal interpretation. See e.g. the construction of field theories on the noncommutative torus in [81].

[29] This correspondence is mathematically fruitful, but physically ambiguous. While a theory of commuting fields on NCST tracks the noncommutative effects back to the structure of spacetime at the appropriate scale, a theory of noncommutative fields on commutative spacetime assigns the cause of noncommutativity directly to the structure of high-energy fields. In other words, the second option arguably implies that noncommutativity is not a genuine feature of spacetime at high energies, but rather the result of some internal degree of freedom in the fields, which is suppressed at lower energies. This novel behaviour is demonstrated to depend on the parameters of the characteristic quantum group of the theory.

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

theories), so that deviations from the predictions of undeformed GR can be traced back to the spacetime noncommutativity.

The adoption of quantum groups and their role as symmetries of NCST models renders this approach orthogonal to Snyder's original concern regarding Lorentz-invariance. Indeed, a shared correspondence principle necessitates that the approach recovers the standard picture of special relativistic spacetime at low energy, including Lorentz-invariance. Nevertheless, this low-energy condition is also compatible with other spacetime symmetries at higher energies. The principle of Lorentz-invariance thus has two distinct effects. Firstly, it is required to obtain due to the correspondence principle. Consequently, any possible high-energy invariance principle must reduce to it in the commutative limit. Secondly, it serves as a *guiding principle* in the identification of potential symmetries. In particular, quantum groups offer suitable descriptions of these quantum symmetries, not only due to their duality with the underlying spacetime structure, but also due to the interpretation of their noncommutative parameter as a characteristic scale for the corresponding theory to apply.

It must be acknowledged that, as for the spectral triple approach, the quantum group approach also has its own limitations. Firstly, the Weyl quantisation procedure is not the sole candidate, nor is it sufficient to establish the correct order of composition of the noncommuting fields, thus creating ambiguity [91]. Secondly, the majority of theories within this approach allow the noncommutative parameter to range from the Planck length downwards. Consequently, each "theory" can be considered to account for a tower of theories with varying noncommutative parameters and, consequently, energy scales. Finally, the characterisation of the commutative limit may differ depending on the specific theory under consideration. For example, works based on DFR postulate a breakdown scale beyond which NCST undergoes symmetry-breaking, whereas other theories posit contraction as a transition mechanism.[30] The correspondence principle entails that all these differences and ambiguities be disregarded in the commutative limit, which is therefore unable to provide guidance for their resolution.

4.2.3 New frontiers I conclude this section with an overview of the recent developments in NCST research following the early 2000s. Indeed, in recent years, there has been a notable degree of cross-fertilisation between the main approaches, which has served to blur their distinctions while simultaneously delineating a number of shared driving research topics.[31]

In order to balance the vastity of topics and the length constraints to this contribution, I avoid entering into the details and simply invite the reader to see the selected references. The objective of this section is only to indicate some relevant trends within the literature on NCST approaches and the instances into which the major approaches of Sections 4.2.1 and 4.2.2 have evolved.

A significant area of research is the development of noncommutative QFT. The earliest research on this topic can be traced back to the work of DFR and Seiberg and Witten, and has been continued by both the spectral triple and the quantum group approaches. In particular, the project has concentrated on combining the results of the two approaches in order to study non-abelian gauge theories on NCST. The field has been encouraged by a seminal work by Madore, Schraml, Schupp and Wess in 2000 [97] and the introduction of the star-product to treat the problem (see e.g. [98]). It has then been developed into proper theories in algebraic QFT (see e.g. [99, 5]). More recently, the link between field theories and NCST has been investigated, e.g., in [100, 101]. In particular, [99] offers an exhaustive review of the main challenges faced by these approaches in the last twenty years, with a particular emphasis on continuations of DFR's ideas. Nevertheless, the current attempts are often inadequate due to two issues: the violation of CPT-invariance and IR/UV mixing. Furthermore, noncommutative QFT encounters significant challenges due to the failure of relativistic causality and unitarity. These issues were initially proposed by Filk [93] and subsequently elaborated upon in [102].

A more promising and currently pursued area of research concerns the definition of NCST in the field of semi-classical and QG. DSR and the quantum group approach established the basis for this investigation. These constructions are still predominantly based on coordinate systems and reference frames, in contrast to other background-independent approaches.[32] Recent results, for example, include black hole and cosmological solutions to the noncommutative Einstein field equations (see e.g. [105, 106, 107]), noncommutative geometric effects on wormholes in diverse models of gravity (see e.g. [108, 109,

[30]The calculation of the commutative limit is complicated by the fact that noncommutative QFTs predict the lack of unitarity in scattering processes and IR/UV mixing. This is defined as the dependence of high-energy, ultraviolet contributions on low-energy, infrared physics, and vice versa, upon "noncommutativisation." See [93] on noncommutative Feynman diagrams and [94] on unitarity. For the IR/UV mixing problem, classical references are [95, 96].

[31]See e.g. the contributions in *The European Physical Journal Special Topics* **232** 23–24.

[32]See e.g. [103]; see also [104] for a generalisation using tetrads.

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

Table 1: Taxonomy of the applications of NCG in modern physics

Approaches	Spectral triple approach; quantum group approach; noncommutative approach to string theory; ...
Theories	*Ordering* (symmetric, time-to-the-right, etc.); *scale* (value of q); *transition* (symmetry-breaking, contraction, etc.); *quantisation map* (Weyl, Seiberg-Witten); *invariance principles* (Lorentz-invariance, as either regulative principle or postulate; effective Lorentz-invariance)
Models	Canonical NCST; κ-Minkowski NCST; ρ-Minkowski NCST; ...

110, 111, 112]), and the emergence of noncommutative geometric effects in perturbative quantum gravity (see e.g. [113, 114]). Contact with the more recent interest in quantum reference frames was initially proposed in [115], and has since been further developed, e.g., in [116].[33]

Furthermore, the operationalist aspiration (reinvigorated by DSR) prompted the examination of potential astrophysical scenarios that could support or refute NCST models in absence of evidence from particle accelerators [117, 118]. Initial efforts in the area of *QG-phenomenology* have focused on the study of IR/UV mixing in astrophysics and the deformed dispersion relations derived in theories of QG.[34] The investigation of NCG in this context is motivated by two key reasons. Firstly, noncommutative structures emerge in numerous approaches to QG [122, 123, 26], although arguably the interpretation of noncommutativity in specific instances is unclear and not susceptible to be classified as "truly spatiotemporal." Secondly, it offers spacetime theories that may have detectable effects in cosmic high-energy messengers.[35]. The *cosmic messenger approach to phenomenology* employs a variety of mediators, including gamma rays, neutrinos, cosmic rays, gravitational waves, which are emitted by astrophysical objects and transmit information to detectors. Given that they are designed to (hopefully) reach accelerations far superior than those achievable with particle accelerators, they have been received as a promising mean of investigating the phenomena predicted by theories of QG. The contributions of NCST approaches to this area are discussed in detail in [125]. It is important to know, however, that QG-phenomenology is based on uncertain epistemological foundations and should therefore be assessed on its own merits.

5 Discussion

The history of NCG in physics is the result of a fruitful collaboration between physicists and mathematicians. Not only were physicists particularly receptive to new ideas from mathematics; mathematicians also revived old physical problems and attempted to formulate and solve them on their own initiative. The core ideas and mathematical methods were disseminated between different approaches and time periods, primarily due to the inclination of researchers engaged in one approach to contribute to others.

5.1 Driving questions and cross-contamination

The preceding brief reconstruction of the evolution of NCST approaches illustrates how the application of NCG to spacetime physics has changed since its first proposal. In particular, the research question has shifted throughout the three phases. In the first phase, NCG was introduced as a natural continuation of special attempts at solving the divergence problem inherent to QFT. It was hypothesised that modifying the structure of spacetime could prevent the derivation of infinite energy due to interactions between the electromagnetic field and a point-like source. Subsequently, in the second phase, attention was directed towards the localisation problem, which was aligned with the two general questions that remained unresolved in Snyder's work. All three versions of the localisation arguments (Bronstein's breakdown, Mead's Heisenberg-microscope and DFR's black-hole-production arguments) are based on an

[33] Intuitively, a quantum reference frame represents a modification of the relativistic notion of reference frame, taking into account the quantum nature of a particle located at its origin. This approach is motivated by the fact that dynamical operators can only be evaluated with respect to a given reference frame, while ensuring that physical laws remain invariant under transformations between frames.

[34] See also the recent application of the spectral triple approach to cosmology in [119, 120, 121, 6].

[35] See e.g. [124] and references therein

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

operationalist methodology and conclude that this is incompatible with the uncertainty relations established for regimes of strong gravitation. The localisation problem provided a renewed opportunity for the exploration of noncommutativity, which benefited from the mathematical discoveries of the 1980s. Indeed, in the third phase, the preceding results prompted a renewed investigation into the incorporation of noncommutativity into spacetime models. The primary concern was to elucidate the interrelationship between spacetime structure and field algebras.

To facilitate the organisation of these findings, the recent advancements of NCG within physics suggest a possible classification in three levels (see Table 1). The first level concerns the various *approaches* to NCST and the specific methodological assumptions that underpin them. As previously stated, the distinction between approaches is often blurred and conventional. This is because techniques derived from one approach can be readily translated into another.

Modern approaches exhibit notable similarities. The majority of mathematical techniques, goals and strategies are shared by all. This cross-contamination is evidenced by the recurrence of methodologies and results, such as Born-reciprocity or the privileged use of an algebraic framework. Notwithstanding, there are important differences with regard to the initial physical assumptions. For example, there is a lack of consensus regarding the fundamental symmetry group (whether it should be the Poincaré group, as for DFR, or a deformation thereof), as well as the transition mechanism to lower energies.

Furthermore, these approaches exemplify the use of the analogy with QM in theory construction, as discussed in Section 2. The aforementioned analogy guides the mathematical construction and physical interpretation of NCST theories. However, it is also important to acknowledge its limitations. The formal analogy leaves a number of physical and conceptual issues open, thus encouraging further research and cross-fertilisation. One example is the disappearance of spacetime points, which builds on the analogy with points in quantum phase space but also imposes further restrictions in the form of generalised uncertainty relations [126].

The second level of classification concerns the various combinations of ordering prescription, scale-value, transition mechanism from the noncommutative to the commutative regime, and quantisation map. Each combination of choices defines a *theory* within an approach. Roughly speaking, each theory is tasked with identifying possible configurations of fields and spacetime at that scale and making predictions about their dynamics. As a methodological premise, it is assumed that one theory will be correct for each value of the noncommutative parameter q. However, it is currently *impossible* to determine which theory will prove to be correct, and this may remain the case in the future.

Finally, a third level concerns the selection of a specific spacetime *model*, i.e. a particular realisation of the algebra of spacetime coordinate operators. For each value of the noncommutative parameter q, the selection of different NCST models will be provably inequivalent. While the expressions of field quantities and composition laws inevitably vary between models, the logical construction of the theories remains largely independent of the specific choice of NCST. As previously stated, it is inevitable that one spacetime model will be proven to be correct. A correspondence principle stipulates that every possible model must reduce to Minkowski spacetime as q is sent to zero. However, investigations into the general relativistic, commutative limit have only recently commenced due to the intricate interplay between the noncommutative scale and the spacetime structure. Moreover, these studies are mainly related to those approaches that aim to develop a comprehensive theory of QG, as exemplified by the work of Aschieri and collaborators.

5.2 Noncommutativity and divergences

The link between the research in NCG and the appearance of the divergences in QFT is substantiated by the initial NCST models. As previously mentioned, Snyder identifies the necessity for a noncommutative algebra of spacetime coordinate operators as a direct consequence of dissatisfaction with the regularisation methods of his time. A proper characterisation of this link is crucial to analyse the precise interplay between noncommutativity and divergences during the first phase.

It is noteworthy that the utilisation of NCG was prompted by a very specific attempt to find a solution to the divergence problem. This attempt entailed the introduction of a new fundamental length scale to screen off smaller-scale contributions to the field's energy. The primary motivation was Heisenberg's conjecture that the transformation of the Minkowski spacetime structure into a lattice might provide a natural, observer-independent justification for the elimination of the divergences.

Heisenberg's solution was to replace the partial derivatives of the wave function with difference quotients over the fundamental scale $\lambda = \frac{\hbar}{m_p c}$, where m_p represents the mass of the proton. On the one hand, from a geometric standpoint, the introduction of a minimal spacing of λ rendered the notion of a point-like source impossible, thus precluding the source-field interaction, which was believed to be the origin of the divergence of the self-energy of the electron. On the other hand, from a field-theoretic

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

standpoint, it implied that elementary particles cannot be localised within point-like spatial regions at any given moment in time. Instead, field quantities must be smeared out over spatial regions of finite extension.

This modification introduced nonlocality into the theory. Furthermore, the solution of the modified Klein-Gordon equation on the lattice revealed that both the electron and proton appeared in the energy spectrum as solutions to the same eigenvalue equation. Despite these results being subsequently refuted, Heisenberg's lattice served as a template or source of inspiration for analogous solutions that sought to introduce a fundamental length scale in order to counteract the divergence of the self-energy in QED.

In this context, the introduction of noncommutativity did not arise as an immediate consequence of Heisenberg's solution. In other words, it was not put forth as a direct *solution* to the divergence problem. On the contrary, it was proposed as a natural *suggestion* to implement solutions in line with the Heisenberg lattice. The analogy with the quantum phase space provided an appropriate mathematical framework for incorporating discreteness into the description of spacetime, thereby rendering the fundamental scale a property of spacetime. This implied that the new quantity was *universal*, in the sense of being independent of the specifics of the fields, which may or may not diverge in the high-energy regime. The *instrumental role of noncommutativity* with respect to the problem of providing a solution to the divergence problem is evidenced by the following fact: NCG was never developed into a proper solution to the latter.

As previously illustrated in Section 2, the Heisenberg lattice resulted in the violation of Lorentz-invariance. The special metatheoretical role assigned to Lorentz-invariance and its preservation implied that such solution was not viable. In other words, discrete solutions in line with the Heisenberg lattice were regarded as "distasteful" because, had they been viable, they would have contravened a physical principle of higher nomological status. Consequently, Snyder's objective was to identify a novel methodology that can simultaneously preserve Lorentz-invariance and reconcile it with the derivation of a discrete structure. This model indeed promised to reconcile the spacetime symmetries with those solutions to the divergences that are analogous to the Heisenberg lattice.

Admittedly, in his 1947 paper, Snyder merely *hints* at the fact that the discreteness of the spectra of spacetime coordinate operators may contribute to solving the divergence problem, provided that the discreteness proposal turns out to be successful. Nevertheless, he never elaborates on this idea further, for instance by presenting a compelling argument for how discreteness (as expressed by noncommutativity) can solve the divergence problem.

In light of this reconstruction, it would be erroneous to understand the link between NCG and divergences as an attempt to solve the latter by means of the former. Rather, the concept of noncommutativity and its mathematical methods were originally *motivated* by the divergence problem. It was born as an attempt to reconcile certain types of solutions to the divergence problem with the requisite of Lorentz invariance. From a historical perspective, it was ultimately proven that these discrete attempts were unsuccessful. The divergence problem was resolved only decades after Snyder's proposal thanks to the introduction of renormalisation techniques. These "relieved theoretical physics [...] from the attempts (and the consequences thereof) at introducing fundamental length into the formalism" [12, p. 64]. However, the idea of a fundamental length was further explored and motivated by a *different* set of arguments in the second phase.

5.3 Spacetime, fields, and the fate of Lorentz-invariance in NCG

The emphasis on the conditions governing the parameter q indicates that NCST approaches are particularly prone to accept both the idea of scale-dependent spacetime and fields, as well as the identification of a precise regime of applicability for the theory. It is essential to specify this regime at the outset, or at the very least, to make it explicit in the form of a variable noncommutative parameter. Indeed, the fundamental scale (whether of dimensions of length, energy or mass) and the type of dependence of the physical quantities on it (whether, e.g., linear or quadratic) *are* the defining characteristics of the noncommutative theory, regardless of the details of its construction, e.g. which approach was used to develop it.

It is a fundamental assumption of NCST that every theory is an *effective theory*. This implies that NCG is employed to describe an effective spacetime structure that holds appropriate at a specific energy scale, wherein the energy scale is inherently linked to, or encapsulated by, the deformation or noncommutative parameter associated with the corresponding algebra. For example, this is one of the reasons why Seiberg and Witten derive NCG from string theory under specific conditions, or noncommutative structures emerge from alternative approaches to QG. In more mathematically oriented works, the spectral triple approach may obscure this characteristic scale (see e.g. [1, 3]), whereas in the development of noncommutative extensions of QFT, it constitutes a central point of interest. Conversely, the quantum

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

group approach permits the deformation parameter to vary freely, subject to additional constraints on its limit.[36]

Furthermore, this reconstruction reveals a profound interest in the relationship between field theories and spacetime at the fundamental level. For example, Snyder makes reference to Born-reciprocity in an attempt to reformulate the differential calculus for his NCST model. Indeed, the modified commutation relations give rise to a novel differential structure, as arbitrary displacements (associated with the tangent structure of NCST) are no longer defined for distances shorter than the fundamental length. This connection is further elucidated by the recognition that a deformation of the spacetime structure induces a compatible deformation of its group of symmetries. In particular, the work of Majid and Ruegg on Hopf algebras has provided the formalism to explicitly understand the connection between deformed translations and the noncommutative parameter, thus ensuring the consistency between the differential calculus (induced by a certain representation of said translations) and the underlying NCST.

Ultimately, the transition from the previous approaches to the modern ones resulted in a rebalancing of the relative importance assigned to the various aspects of the theory. Snyder's work exemplifies a pervasive tendency to prioritise Lorentz-invariance above all else. He derives noncommutativity from the joint assumption of discreteness of the spectra of coordinate operators (due to the introduction of a fundamental length) and invariance of the squared spacetime distance under the action of the Lorentz group. In contrast, modern approaches are more confident in abandoning Lorentz-invariance, which is only recovered at low energies.[37] Indeed, high-energy regimes leave some room for speculation about alternative symmetry groups at the more fundamental level, provided that compatibility with established physical principles is maintained. This is particularly evident in the quantum group approach, which is founded upon deformations of the standard spacetime symmetries at high energies. Furthermore, this modification eliminates the incompatibility between a fundamental length scale and Fitzgerald-Lorentz boosts. Indeed, if the existence of a fundamental length is postulated as a higher-rank assumption in the construction of the theory, then the group of transformations between inertial frames must accommodate this choice, rather than the other way around.[38]

This suggests that modern approaches take Lorentz-invariance as a regulative principle, rather than a postulate of the fundamental theory. In other words, when constructing such a theory, it is probable that Lorentz-invariance will be violated at the energy scales that are to be investigated with NCG. In contrast, invariance is imposed as a constraint on the form of the equations *in the commutative limit*. In order to supplement the new invariance principles, it is necessary to specify a new quantum group of spacetime symmetries that corresponds to the chosen model. The field theory described on this spacetime model thus conforms to these symmetries, as indicated in Section 4.2.2, and therefore satisfies this new invariance principle for the high-energy regime.[39]

The recovery of Lorentz-invariance only in the commutative limit, which we refer to as *effective Lorentz-invariance* for brevity, is a particular instance of a more general correspondence principle that characterises theories of QG: the new theory and the old, low-energy theory must make compatible predictions in overlapping domains of applicability.[40] All different realisations of noncommutativity in spacetime obey this correspondence principle. Consequently, the latter works on the theory as both: (i) an *external constraint for its acceptance*, meaning that the NCST should be rejected in the event of its failure to satisfy the aforementioned principle; and (ii) an *expected means of confirmation*, for example, in the case of QG-phenomenology, by decreasing the trust in the theory, were the latter to predict a privileged reference frame at low energy. Moreover, effective Lorentz-invariance can interact with the deformation

[36]One might inquire as to the extent to which these latter approaches adopt the so-called "effective field theory approach" from QFT and related research in QG. Admittedly, a comprehensive renormalisation group analysis has yet to be conducted for the majority of these proposals, with just the few exceptions mentioned above. This absence is ultimately due to the difficulties in specifying the exact nature of the commutative limit. Despite its apparent mathematical simplicity (just compute the limit and cancel all negligible terms at low energy), the physical process described by this calculation is unclear. For example, does it constitute a dynamic or a static case of reduction? Does the theory's trajectory in theory space reach a singularity?

[37]On this tendency, see e.g. [127, pp. 202–203].

[38]A third approach is that of Magueijo and Smolin in DSR (see note 23). They preserve both Lorentz-invariance and the fundamental length, but modify the representation of the symmetries in momentum space. This alteration is not merely a mathematical one, as the resulting theory appears to be inequivalent to Amelino-Camelia's version of DSR despite the nonlinear maps. Nevertheless, to my knowledge, the viability of this option for a proper theory of noncommutative gravity has yet to be explored further.

[39]This is roughly the strategy used in [128], which provides an explicit example of construction of a quantum group for NCST.

[40]For further insight into the relationship between correspondence and fundamentality, see e.g. [129]. In the following, I use Crowther's [11] classification to specify the various roles played by this principle in NCST approaches.

strategy, functioning as a *guiding principle* for the latter. In other words, it provides a heuristic basis for the definition of new quantum symmetries and informs the new, highly deformed objects with a physical interpretation.[41]

6 Conclusion

NCST approaches represent a fascinating case study in the construction of a theory of QG. Nevertheless, they have received much less attention in the past than other approaches, such as loop quantum gravity or string theory. Noncommutative structures are a recurrent theme in the latter, yet physicists frequently employ the tools of NCG without any spatiotemporal interpretation along the lines of the spectral triple or quantum group approaches. For instance, the work of Seiberg and Witten has significantly encouraged string theorists to employ the techniques of NCG as purely mathematical instruments. In contrast, NCST approaches strive to be "fully physical" theories. NCG certainly offers a clear picture as a mathematical theory; however, its application to physical situations and subsequent interpretation is currently unclear or heretogeneous. The epistemological status of NCST theories as *physical* theories is controversial due to the lack of confirming evidence, as is the case with other theories of QG. NCST theories also exemplify the peculiarities of a research programme born out of both physical and purely mathematical necessities. The broad scope of applications of NCG in physics is a sign of its usefulness, but also presents an obstacle to its clear physical interpretation across diverse investigative contexts. Despite the differences in applications, core ideas have been shared by physicists working in this area, from its beginnings to the present day, including: operationalism, effective Lorentz-invariance, correspondence in the commutative limit, analogy to QM, and an algebraic treatment of the spacetime structure.

Notwithstanding the possibility of identifying a common core, the application of NCG to spacetime physics remains highly diversified. As previously demonstrated, the initial objective was to address the divergence problem in QFT. Snyder (building on Heisenberg) proposed a radical shift in our understanding of the structure of spacetime, from continuous to discrete. The noncommutativity of spacetime structure was the first notable conjecture to bring the different approaches closer together in the 2000s. Operationalist interpretations of Snyder's results were well aligned with the broader concerns of proving the existence of a fundamental scale and determining its value. Noncommutativity provided a natural means of integrating constraints on localisability into the theoretical framework. This was achieved by deriving the relevant methods from the construction of phase space in orthodox QM.

Furthermore, the advent of new mathematical techniques furnished physicists engaged in this field with a shared vocabulary during the early 1990s. The application of these results from independent mathematical research provides a second, converging force for NCST approaches. It also elucidates the resurgence of NCST models and the divergence from Snyder's concerns and assumptions. It is noteworthy that fundamental Lorentz-invariance is abandoned by physicists working within the quantum group approach. Instead, it is required that this principle be recovered in the commutative limit. The proliferation of methods, interpretations and conjectures has resulted in a lack of clarity regarding the distinction between the main approaches currently in use. Exceptions concern the overall direction of research: particle physics for the spectral triple approach; QG for the quantum group approach. Despite the differences between the two approaches, the common intuition is that a solution to the respective research questions necessitates a comprehensive examination of the noncommutative structure of spacetime at high energies.

In conclusion, the history of NCG in physics exhibits a process of "interlaced convergence." First, a number of conjectures, methodologies, strategies, goals and interpretations were developed independently. Subsequently, a series of driving problems brought these together in pairs at different times. The reconstruction of this proliferation of instances reveals a complex web, or "fabric," in which earlier attempts are recovered and modified to meet new understandings and research questions. The various combinations can be explained by a general movement of convergence that continues to this day.

Acknowledgements

Many thanks to the participants of the Helsinki 2024 conference "Physics and Reality" for the helpful comments. I am particularly grateful to Elena Castellani and Fedele Lizzi for carefully reading a longer draft of this paper and for their invaluable comments. I am also grateful to two anonymous referees for

[41] One illustrative example is that of deformed boosts and the associated dispersion relations for the κ-Poincaré group. The two mathematical expressions differ significantly from their contracted counterparts due to the presence of additional terms with fundamental scale dependence (see e.g. [128]). Their interpretation as deformed Lorentzian boosts and deformed dispersion relations stems from the fact that they reduce to these counterparts in the commutative limit. In other words, their interpretations are inherited from their respective limits.

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

pointing out some criticalities in a previous draft and suggesting improvements. Special thanks to Clara Costa for reading and commenting on the final draft of this paper.

References

[1] Connes A 1995 *Noncommutative Geometry* (Cambridge: Academic Press)

[2] Madore J 1999 *An Introduction to Noncommutative Differential Geometry and its Physical Applications* 2nd ed (Cambridge: Cambridge University Press)

[3] Várilly J C 2006 *An Introduction to Noncommutative Geometry* (Zürich: European Mathematical Society)

[4] Connes A and Marcolli M 2007 *Noncommutative Geometry, Quantum Fields and Motives* (Providence: American Mathematical Society)

[5] van Suijlekom W D 2015 *Noncommutative Geometry and Particle Physics* (Dordrecht: Springer)

[6] Marcolli M 2018 *Noncommutative Cosmology* (Singapore: World Scientific)

[7] Heller M, Lambert D and Madore M 1998 A brief history of noncommutative space-time *Acta Cosmologica* **24** 1 51–69

[8] Blum A S and Rickles D 2018 *Quantum gravity in the first half of the twentieth century: A sourcebook* (Berlin: Edition Open Access)

[9] Carlip S 2001 Quantum gravity: A progress report *Reports on Progress in Physics* **64** 8 855

[10] Castellani E 2012 Early string theory as a challenging case study for philosophers *The Birth of String Theory* ed Cappelli A, Castellani E, Colombo F and Di Vecchia P (Cambridge: Cambridge University Press) pp 63–78

[11] Crowther K 2021 Defining a crisis: The role of principles in the search for a theory of quantum gravity *Synthese* **198** Suppl 14 S3489–S3516

[12] Hagar A 2024 *Discrete or Continuous? The Quest for Fundamental Length in Modern Physics* (Cambridge: Cambridge University Press)

[13] Heisenberg W and Pauli W 1929 Zur Quantendynamik der Wellenfelder *Zeitschrift für Physik* **56** 1–61

[14] Landau L and Peierls R 1931 Extensions of the uncertainty principle to relativistic quantum theory *Selected Scientific Papers of Sir Rudolf Peierls, with Commentary by the Author* ed Dalitz R H and Peierls R 1997 (Singapore World Scientific) pp 83–94 reprinted

[15] Born M and Infeld L 1934 Foundations of the new field theory *Proceedings of the Royal Society of London A* **144** 852 425–451

[16] Dirac P 1938 Classical theory of radiating electrons *Proceedings of the Royal Society of London A* **167** 148–169

[17] Heisenberg W 1930 The self-energy of the electron *Early Quantum Electrodynamics* ed Miller A 1994 (Cambridge: Cambridge University Press) pp 121–128

[18] Snyder H S 1947 Quantized space-time *Physical Review* **71** 1 38–41

[19] Snyder H S 1947 The electromagnetic field in quantized space-time *Physical Review* **72** 1 68–71

[20] Born M 1938 A suggestion for unifying quantum theory and relativity *Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences* **165** 921 291–303

[21] Born M 1949 Reciprocity theory and elementary particles *Reviews of Modern Physics* **21** 3 463–473

[22] Flint H T 1948 The quantization of space and time *Physical Review* **74** 2 209–210

[23] Bohr N and Rosenfeld L 1950 Field and Charge Measurements in Quantum Electrodynamics *Physical Review* **78** 6 794–798

[24] Amelino-Camelia G 2003 The three perspectives on the quantum-gravity problem and their implications for the fate of Lorentz symmetry (arXiv:gr-qc/0309054)

[25] Bronstein M 2012 Republication of: Quantum theory of weak gravitational fields *General Relativity and Gravitation* **44** 267–283

[26] Hossenfelder S 2013 Minimal length scale scenarios for quantum gravity *Living Reviews in Relativity* **16** 2

[27] Mead C A 1964 Possible connection between gravitation and fundamental length *Physical Review* **135** 3B B849–B862

[28] Wheeler J A 1955 Geons *Physical Review* **97** 2 511–536

[29] Wheeler J A and Ford K *Geons, Black Holes, and Quantum Foam. A Life in Physics* (New York and London: W. W. Norton & Company)

[30] Schwarz J H 2012 Gravity, unification, and the supersting *The Birth of String Theory* ed Cappelli A, Castellai E, Colombo F and Di Vecchia P (Cambridge: Cambridge University Press) pp 37–62

[31] Amati D, Ciafaloni M and Veneziano G 1989 Can spacetime be probed below the string size? *Physics Letters B* **216** 1–2 41–47

[32] Doplicher S, Fredenhagen K and Robert J E 1994 Spacetime quantization induced by classical gravity *Physics Letters B* **331** 1–2 39–44

[33] Doplicher S, Fredenhagen K and Robert J E 1995 The quantum structure of spacetime at the Planck scale and quantum fields *Communications in Mathematical Physics* **172** 187–220

[34] Kosiński P, Lukierski J and Maślanja P 2001 κ-deformed Wigner construction of relativistic wave functions and free fields on κ-Minkowski space *Nuclear Physics B – Proceedings Supplements* **102–103** 161–168

[35] Hellund E J and Tanaka K 1954 Quantized space-time *Physical Review* **94** 1 192–195

[36] Lizzi F, Scala L and Vitale P 2022 Localization and observers in ρ-Minkowski spacetime *Physical Review D* **106** 025023

[37] Sternheimer D 1998 Deformation quantization: Twenty years after *Particles, Fields and Gravitation* ed Rembieliński J (Melville NY: AIP Press) pp. 107–145

[38] Dito G and Sternheimer D 2002 Deformation quantization: genesis, developments and metamorphoses *Deformation quantization* ed Halbout G (Berlin: de Gruyter) pp. 9–54

[39] Faddeev L 2006 History and perspectives of quantum groups *Milan Journal of Mathematics* **74** 279–294

[40] Weyl H 1931 *The Theory of Groups and Quantum Mechanics* (New York: Dover Publications, Inc.)

[41] Wigner E 1932 On the quantum correction for thermodynamic equilibrium *Physical Review* **40** 749–759

[42] Moyal J E 1947 Quantum mechanics as a statistical theory *Mathematical Proceedings of the Cambridge Philosophical Society 45* 1 99–124

[43] Groenewold H J 1946 On the principles of elementary quantum mechanics *Physica* **12** 7 405–460

[44] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 Deformation theory and quantization. I. Deformations of symplectic structures *Annals of Physics* **111** 61–110

[45] Bayen F, Flato M, Fronsdal C, Lichnerowicz A and Sternheimer D 1978 Deformation theory and quantization. II. Physical applications *Annals of Physics* **111** 111–151

[46] Celeghini E, Giachetti R, Sorace E and Tarlini M 1992 Contractions of quantum groups *Quantum Groups: Proceedings of Workshops held in the Euler International Mathematical Institute, Leningrad, Fall 1990* ed. Kulish P. P. (Berlin, Heidelberg: Springer-Verlag) pp 221–244

[47] Gerstenhaber M 1974 On the deformation of rings and algebras *Annals of Mathematics* **79** 59–103

[48] Majid S 1995 *Foundations of Quantum Group Theory* (Cambridge: Cambridge University Press)

[49] Kontsevich M 1997 Deformation quantization of Poisson manifolds *Letters in Mathematical Physics* **66** 157–216

[50] Faddeev L 1982 Integrable models in 1+1 dimensional quantum field theory *Technical report: Centre d'Etudes Nucleaires de Saclay 91*

[51] Drinfel'd V 1985 Hopf algebras and the quantum Yang-Baxter equation *Yang-Baxter Equation in Integrable Systems* ed Jimbo M (Singapore: World Scientific) pp. 264–268 reprinted

[52] Drinfel'd V 1988 Quantum groups *Journal of Soviet Mathematics* **41** 898–915

[53] Jimbo M 1985 A q-difference analogue of $u(\mathfrak{g})$ and the Yang-Baxter equation *Letters in Mathematical Physics* **10** 63–69

[54] Woronowicz S L 1987 Twisted $su(2)$ group. An example of a non-commutative differential calculus *Publications of the Research Institute for Mathematical Sciences* **23** 1 117–181

[55] Woronowicz S L 1989 Differential calculus on compact matrix pseudogroups (quantum groups) *Communications in Mathematical Physics* **122** 125–170

[56] Manin Y I 1989 Multiparametric quantum deformation of the general linear supergroup *Communications in Mathematical Physics* **123** 163–175

[57] Majid S 1988 Hopf algebras for physics at the Planck scale *Classical and Quantum Gravity* **5** 1587–1606

[58] Majid S 1990 Physics for algebraists: Non-commutative and non-cocommutative Hopf algebras by a bicrossproduct construction *Journal of Algebra* **130** 17–64

[59] Lukierski J, Ruegg H, Nowicki A and Tolstoy V N 1991 q-deformation of Poincaré algebra *Physics Letters B* **264** 3/4 331–338

[60] Lukierski J, Nowicki A and Ruegg H 1992 New quantum Poincaré algebra and κ-deformed field theory *Physics Letters B* **293** 344–352

[61] Nowicki A, Sorace E and Tarlini M 1993 The quantum Dirac equation associated to κ-Poincaré *Italian workshop on quantum groups. Florence, February 3 to 6, 1993* ed Celeghini E and Tarlini M (arXiv:hep-th/9304160) 33–34

[62] Biedenharn L, Mueller B and Tarlini M 1993 The Dirac-Coulomb problem for the κ-Poincaré quantum group *Physics Letters B* **318** 613–616

[63] Lukierski J, Ruegg H and Rühl W 1993 From κ-Poincaré algebra to κ-Lorentz quasigroup. A deformation of relativistic symmetry *Physics Letters B* **313** 357–366

[64] Ruegg H 1993 Q-deformation of semisimple and nonsemisimple Lie algebras *NATO Science Series C* **409** 45–82

[65] Amelino-Camelia G 2002 Relativity in spacetimes with short-distance structure governed by an observer-independent (Planckian) length scale *International Journal of Modern Physics D* **11** 1 35–59

[66] Majid S and Ruegg H 1994 Bicrossproduct structure of κ-Poincaré group and non-commutative geometry *Physics Letters B* **334** 3/4 348–354

[67] Kasparov G G 1981 The operator K-functor and extensions of C^*-algebras *Mathematics of the USSR-Izvestiya* **16** 3 513

[68] Magueijo J and Smolin L 2001 Lorentz invariance with an invariant energy scale *Physical Review Letters* **88** 19 190403

[69] Kowalski-Glikman J 2003 Doubly special relativity and quantum gravity phenomenology (arXiv:hep-th/0312140v1)

[70] Kowalski-Glikman J and Nowak S 2002 Doubly special relativity and de Sitter space *Classical and Quantum Gravity* **20** 4799

[71] Kadyshevskii V G 1962 On the theory of quantization of spacetime *Soviet Physics JETP* **14** 6 1340–1346

[72] Cortés J L and Gamboa J 2005 Quantum uncertainty in doubly special relativity *Physical Review D* **71** 065015

[73] Amelino-Camelia G, Freidel L, Kowalski-Glikman J and Smolin L 2011 The principle of relative locality *Physical Review D* **84** 084010

[74] Judes S and Visser M 2003 Conservation laws in "doubly special relativity" *Physical Review D* **68** 045001

[75] Amelino-Camelia G 2009 Doubly-Special Relativity: Facts, Myths and Some Key Open Issues *Recent Developments in Theoretical Physics* ed Ghosh S and Kar Guruprasad (Singapore: World Scientific) pp 123–170

[76] Seiberg N and Witten E 1999 String theory and noncommutative geometry *Journal of High Energy Physics* **09** 032

[77] Lizzi F 2009 Noncommutative spaces *Noncommutative Spacetimes. Symmetries in Noncommutative Geometry and Field Theory* ed Aschieri P, Dimitrijević M, Kulish P, Lizzi F and Wess J (Berlin Heidelberg: Springer)

[78] Chamseddine A H and Connes A 1996 Universal formula for noncommutative geometry actions: Unification of gravity and the Standard Model *Physical Review Letters* **77** 4868

[79] Chamseddine A H and Connes A 1997 The spectral action principle *Communications in Mathematical Physics* **186** 731–750

[80] Connes A and Lott J 1990 Particle models and noncommutative geometry *Nuclear Physics B (Proc. Suppl.)* **18** B 29–47

[81] Connes A, Douglas M R and Schwarz A 1998 Noncommutative geometry and matrix theory: Compactification on tori *Journal of High Energy Physics* **02** 003

[82] Chamseddine A H, Connes A and Marcolli M 2007 Gravity and the Standard Model with neutrino mixing *Advances in Theoretical and Mathematical Physics* **11** 6 991–1089

[83] Lizzi F 2005 Spectral geometry for quantum spacetime *Il Nuovo Cimento C* **38** 165

[84] Grosse H and Wulkenhaar R 2004 Renormalisation of noncommutative scalar field theories *Lie Theory and Its Applications in Physics V* ed Doebner H D and Dobrev V K (Singapore: World Scientific)

[85] Kempf A 1992 Quantum group-symmetric Fock spaces with Bergmann-Fock representation *Letters in Mathematical Physics* **26** 1–12

[86] Kempf A 1993 Uncertainty relation in quantum mechanics with quantum group symmetry *Journal of Mathematical Physics* **35** 4483–4496

[87] Lukierski J, Ruegg H and Zakrzewski W J 1995 Classical and quantum mechanics of free κ-relativistic systems *Annals of Physics* **243** 90–116

[88] Amelino-Camelia G and Majid S 2000 Waves on noncommutative space-time and gamma-ray bursts *International Journal of Modern Physics A* **15** 27 4301–4323

[89] Chaichian M, Kulish P P, Nishijima K and Tureanu A 2004 On a Lorentz-invariant interpretation of noncommutative space-time and its implications on noncommutative QFT *Physics Letters B* **604** 1/2 98–102

[90] Chaichian M, Prešajder P and Tureanu A 2005 New concept of relativistic invariance in noncommutative space-time: Twisted Poincaré symmetry and its implications *Physical Review Letters* **94** 151602

[91] Agostini A, Lizzi F and Zampini A 2002 Generalized Weyl systems and κ-Minkowski space *Modern Physics Letters A* **17** 32 2105–2126

[92] Aschieri P, Blohmann C, Dimitrijević M, Meyer F, Schupp P and Wess J 2005 A gravity theory on noncommutative spaces *Classical and Quantum Gravity* **22** 3511

[93] Filk T 1996 Divergencies in a field theory on quantum spacetime *Physics Letters B* **376** 53–58

[94] Gomis J and Mehen T 2000 Space-time noncommutative field theories and unitarity *Nuclear Physics B* **591** 1/2 265–276

[95] Matusis A, Susskind L and Toumbas N 2000 The IR/UV connection in the noncommutative gauge theories *Journal of High Energy Physics* **12** 002

[96] Hayakawa M 2000 Perturbative analysis on infrared aspects of noncommutative QED on r^4 *Physics Letters B* **478** 1–3 394–400

[97] Madore J, Schraml S, Schupp P and Wess J 2000 Gauge theory on noncommutative spaces *The European Physical Journal C* **16** 161–167

[98] Jurco B, Möller L, Schraml S, Schupp P and Wess J 2001 Construction of non-abelian gauge theories on noncommutative spaces *The European Physical Journal C* **21** 383–388

[99] Bahns D, Doplicher S, Morsella G and Piacitelli G 2015 Quantum Spacetime and Algebraic Quantum Field Theory *Advances in Algebraic Quantum Field Theory* ed Brunetti R, Doppiaggi C, Fredenhagen K and Yngvason J (Cham: Springer) pp 289–330

[100] Fiore G 2010 On second quantization on noncommutative spaces with twisted symmetries *Journal of Physics A* **43** 155401

[101] Gubitosi G, Lizzi F, Relancio J J and Vitale P 2022 Double quantization *Physical Review D* **105** 126013

[102] Alvarez-Gaumé L, Barbón J L F and Zwicky R 2001 Remarks on time-space non-commutative field theories *Journal of High Energy Physics* **05** 057

[103] Wess J 2004 Deformed coordinate spaces; Derivatives *Mathematical, Theoretical and Phenomenological Challenges Beyond the Standard Model* ed Djordjević G, Nešić and Wess J (Singapore: World Scientific) pp 122–128

[104] Kober 2011 Canonical noncommutativity algebra for the tetrad field in general relativity *Classical and Quantum Gravity* **28** 225021

[105] Aschieri P and Castellani L 2010 Noncommutative gravity solutions *Journal of Geometry and Physics* **60** 375–393

[106] Dimitrijević Ćirić M, Dordević D, Gočanin D, Nikolić B and Radovanović V 2023 Noncommutative $SO(2,3)_\star$ gauge theory of gravity *The European Physical Journal Special Topics* **232** 3747–3770

[107] Herceg N, Jurić T, Samsarov A and Smolić I 2024 Metric perturbations in noncommutative gravity *Journal of High Energy Physics* **06** 130

[108] Kavya N S, Venkatesha V, Mustafa G and Sahoo P K 2023 On possible wormhole solutions supported by non-commutative geometry within $f(\mathcal{R},\mathcal{L}_m)$ gravity *Annals of Physics* **455** 169383

[109] Kavya N S, Venkatesha V, Mustafa G and Sahoo P K 2024 Exploring wormhole solutions in curvature-matter coupling gravity supported by noncommutative geometry and conformal symmetry *Chinese Journal of Physics* **87** 751–765

[110] Baruah A, Goswami P and Deshamukhya A 2023 Non-commutative wormholes in $f(R)$ gravity satisfying the energy conditions *New Astronomy* **99** 101956

Journal of Physics: Conference Series **2948** (2025) 012011 doi:10.1088/1742-6596/2948/1/012011

[111] Song Q and Güdekli E 2023 Traversable wormholes in Rastall teleparallel gravity with non-commutative geometry *New Astronomy* **100** 101993

[112] Rahman N, Kalam M, Das A, Islam S, Rahaman F and Murshid M 2023 Thin-shell wormhole under non-commutative geometry inspired Einstein-Gauss-Bonnet gravity *The European Physical Journal Plus* **138** 146

[113] Fröb M B, Much A and Papadopoulos K 2023 Noncommutative geometry from perturbative quantum gravity *Physical Review D* **107** 064041

[114] Fröb M B, Lima W C C, Much A and Papadopoulos K 2023 Noncommutative geometry from perturbative quantum gravity in de Sitter spacetime *Physical Review D* **108** 086003

[115] Girelli F and Poulin D 2008 Quantum reference frames and deformed symmetries *Physical Review D* **77** 104012

[116] Ballesteros A, Giacomini F and Gubitosi G 2021 The group structure of dynamical transformations between quantum reference frames *Quantum* **5** 470

[117] Amelino-Camelia G 2001 Testable scenario for relativity with minimum length *Physics Letters B* **510** 255–263

[118] Amelino-Camelia G, Smolin and Starodubtsev A 2004 Quantum symmetry, the cosmological constant and Planck scale phenomenology *Classical and Quantum Gravity* **21** 3095

[119] Nelson W and Sakellariadou M 2010 Cosmology and the noncommutative approach to the Standard Model *Physical Review D* **81** 085038

[120] Nelson W, Ochoa J and Sakellariadou M 2010 Constraining the Noncommutative Spectral Action via Astrophysical Observations *Physical Review Letters* **105** 101602

[121] Buck M, Fairbairn M and Sakellariadou M 2010 Inflation in models with Conformally Coupled Scalar Fields: An application to the Noncommutative Spectral Action *Physical Review D* **82** 043509

[122] Garay L J 1995 Quantum gravity and minimum length *International Journal of Modern Physics A* **10** 145–166

[123] Matschull H-J and Welling M 1998 Quantum mechanics of a point particle in (2+1)-dimensional gravity *Classical and Quantum Gravity* **15** 2981–3030

[124] Kobakhidze A, Lagger C and Manning A 2016 Constraining noncommutative spacetime from GW150914 *Physical Review D* **94** 6 064033

[125] Addazi A et al. 2022 Quantum gravity phenomenology at the dawn of the multi-messenger era–A review *Progress in Particle and Nuclear Physics 125* 103948

[126] Huggett N, Lizzi F and Menon T 2021 Missing the point in noncommutative geometry *Synthese* **199** 4695–4728

[127] Carlip S 2014 Challenges for emergent gravity *Studies in History and Philosophy of Modern Physics* **46** 200–208

[128] Agostini A, Amelino-Camelia G, Arzano M, Marciano A and Tacchi R A 2007 Generalizing the Noether theorem for Hopf-algebra spacetime symmetries *Modern Physics Letters A* **22** 24 1779–1786

[129] Crowther K 2018 Inter-theory relations in quantum gravity: Correspondence, reduction, and emergence *Studies in History and Philosophy of Modern Physics* **63** 74–85