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Abstract: We consider a dynamic scenario for characterizing the late Universe evolution, aiming to mitigate the Hubble tension. Specifically, we consider a metric $f(R)$ gravity in the Jordan frame which is implemented to the dynamics of a flat isotropic Universe. This cosmological model incorporates a matter creation process, due to the time variation of the cosmological gravitational field. We model particle creation by representing the isotropic Universe (specifically, a given fiducial volume) as an open thermodynamic system. The resulting dynamical model involves four unknowns: the Hubble parameter, the non-minimally coupled scalar field, its potential, and the energy density of the matter component. We impose suitable conditions to derive a closed system for these functions of the redshift. In this model, the vacuum energy density of the present Universe is determined by the scalar field potential, in line with the modified gravity scenario. Hence, we construct a viable model, determining the form of the $f(R)$ theory a posteriori and appropriately constraining the phenomenological parameters of the matter creation process to eliminate tachyon modes. Finally, by analyzing the allowed parameter space, we demonstrate that the Planck evolution of the Hubble parameter can be reconciled with the late Universe dynamics, thus alleviating the Hubble tension.

Keywords: modified gravity; Late Universe; Hubble tension; classical cosmology

1. Introduction

Over the past decade, cosmological studies of the late Universe have uncovered a significant discrepancy in data, known as the "Hubble tension". This tension arises from the differing values of the Hubble constant (H_0) measured by the SH0ES Collaboration using Type Ia Supernova (SNIa) data [1–4] and those obtained by the Planck Satellite Collaboration [5]. The discrepancy, approximately 5σ, presents a perplexing challenge, prompting new considerations regarding the dynamics of the late Universe. For a series of discussions on this topic, see [6–15]. We also underline how, nonetheless, in the presence of such a tension, the most commonly accepted cosmological model which describes the evolution of the Hubble parameter is the so-called Lambda Cold Dark Matter (ΛCDM) model [16,17]. This corresponds to including, in the Universe dynamics, a matter-like contribution and a cosmological constant. The possibility of interpreting the Hubble tension as a continuous effective variation of the Hubble constant based on redshift—where its value appears to depend on the redshift region of the astrophysical sources used for its determination—has been supported by analyses in [18–20], and is also discussed in [21,22]. Moreover, the coexistence of SNIa and baryon acoustic oscillations (BAOs) [23] (and references therein) data within the same redshift region, and their resulting tension, as BAO provides a H_0 value compatible with that from Planck, has led to the development of early dark energy (DE) models [24] and a suitable combination of early and late modified dynamics [25,26]. In particular, the analysis in [7] proposed a specific $f(R)$ model, examined within the so-called Jordan frame [27,28], to effectively address the Hubble tension in the

presence of evolutionary DE, which transits to a phantom contribution at $z > 1$. This model presents an intriguing scenario where the Hubble tension is essentially resolved at $z \gtrsim 2$, since the non-minimally coupled scalar field shows a monotonically increasing behaviour toward an asymptote. This asymptotic configuration aligns with the Planck value for the Hubble constant, occurring at a relatively low redshift.

In this work, we explore a similar approach, but rather than initially assuming the existence of evolutionary dark energy (DE), we consider a more natural physical scenario. This scenario involves the gravitational field generating weakly massive particles due to its time variation, effectively introducing a radiation component in the late Universe. Specifically, the matter creation process is treated phenomenologically, by regarding the Universe as an open thermodynamic system, as discussed in previous works like [29,30]. The particle creation rate is determined using an *ansatz* in the form of a power-law in the Hubble parameter. For a more comprehensive understanding of the matter creation process across the Universe, please see [31–34].

After constructing the dynamical system governing the late Universe dynamics and incorporating the creation of a radiation component by the gravitational background, we demonstrate how the Hubble tension could be effectively alleviated within certain favourable regions of the parameter space. This framework offers a promising depiction of the underlying mechanism potentially resolving such a cosmological issue. It is noteworthy that the current-day radiation created does not exceed a few percent, and the anticipated weakly interacting nature of these particles explains why this generated energy density remains indirectly observed in the actual Universe.

The paper is structured as follows. In Section 2, a phenomenological approach to the matter creation process is presented and the dynamics for the created energy density is fixed. In Section 3, the formulation of a metric $f(R)$ gravity in the Jordan frame is reviewed and the basic features of the proposed model are traced. In Section 4, the dynamical model corresponding to the evolution of a flat isotropic Universe in the considered modified scenario is constructed. The set of free parameters is described and the initial conditions for constructing the numerical solutions are given. In Section 5, the free parameter space is numerically investigated generating a triangular plot. A privileged set of parameters is then identified and the profile of the Hubble parameter is provided in order to show the capability of the model to alleviate the Hubble tension.

2. Matter Creation Process

In the thermodynamic framework presented in this work, the concept of matter creation from a time-varying gravitational field relies on a simple phenomenological representation. Let us start from the first principle, as follows:

$$dU = -pdV + \delta Q + \mu dN, \tag{1}$$

which has to be combined with the heat change δQ, expressed by means of the second thermodynamics principle as $\delta Q = TdS$. Here, U is the internal energy, p is the pressure, T is the temperature, S is the entropy, μ is the chemical potential, N is the particle number, and V is the volume of the considered system. Introducing now the expressions $U = \rho V$, $S = \sigma N$, and $\mu = (\rho + p)V/N - \sigma T$ (in which ρ is the total energy density of the system and σ is the entropy per particle), we can rewrite Equation (1) as follows:

$$d\rho = -(\rho + p)\left(1 - \frac{d \ln N}{d \ln V}\right)\frac{dV}{V} + T\frac{N}{V}d\sigma. \tag{2}$$

The difference from an iso-entropic system is that we only need to ensure the preservation of entropy per particle by setting $d\sigma = 0$. Hence, it is clear that, since $\sigma = S/N$, matter creation results in an increase in the entropy of the system. Therefore, Equation (2) simplifies as follows:

$$\frac{d\rho}{d\tau} = -(\rho + p)\left(1 - \frac{d \ln N}{d \ln V}\right)\frac{d \ln V}{d\tau}, \tag{3}$$

once it is applied to a homogeneous background and its evolution is tracked with respect to a clock labelled τ. Here, the time variable τ thus denotes the specific time variable associated with the considered dynamical system and, in what follows, it will be identified with the cosmological Gaussian time. The line element of a flat isotropic Universe [35] reads as follows:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right), \tag{4}$$

in which t denotes the synchronous time ($c = 1$), and $\{x, y, z\}$ are the Cartesian coordinates. Moreover, $a(t)$ stands for the cosmic scale factor responsible for the Universe expansion.

In terms of the Hubble parameter $H(t) \equiv \dot{a}/a$ (the dot denoting synchronous time differentiation), a reliable *ansatz* for the matter creation is as follows:

$$\frac{d \ln N}{d \ln V} = (H/\bar{H})^\beta, \tag{5}$$

where β and $\bar{H}$ are positive free parameters of the model and the considered fiducial volume can be set as $V = a(t)^3 V_0$, with V_0 as a generic coordinate volume which does not enter the dynamics.

Since the present-day Universe expansion rate is rather slow, it is natural to argue that the gravitational field is generating very low-mass particles [36,37]. Given that, we address this process to a radiation-like component energy density ρ_r. According to the analysis above, the dynamics of such an emerging radiation contribution is governed by the following equation (i.e., we use the equation of state $p = w\rho$ with $w = 1/3$):

$$\dot{\rho}_r = -4H\rho_r \left(1 - (H/\bar{H})^\beta\right). \tag{6}$$

This equation comes from Equation (3), once we use $\tau = t$, the *ansatz* in Equation (5), and by implementing $d \ln V/dt = 3H$. In the following sections, we embed this mechanism in the context of a metric $f(R)$ gravity in the Jordan frame.

3. Modified Cosmological Dynamics in the Jordan Frame

In the Jordan frame, the action of a metric $f(R)$ gravity can be cast as follows [27]:

$$\mathcal{S} = \frac{1}{2\chi} \int d^4x \sqrt{-g}(\phi R - V(\phi)), \tag{7}$$

where χ denotes the Einstein constant, g and R are the metric determinant and the Ricci scalar, respectively, while the non-minimally coupled scalar field ϕ is the independent degree of freedom expressing the modified gravity formulation. In particular, the potential term $V(\phi)$ is linked to the specific form of the function $f(R)$ via the following relation:

$$f(R(\phi)) = \phi \frac{dV}{d\phi} - V(\phi), \tag{8}$$

which comes from the basic definition $\phi \equiv df/dR$ and from the substitution of the field equation into the expression of V, obtained by varying the action Equation (7) with respect to ϕ, i.e., $R = dV/d\phi$. A basic viability condition for the choice of a given $f(R)$ model is that the scalar mode possesses a positively defined quadratic mass, according to the following definition [38]:

$$\mu_\phi^2 \equiv \frac{1}{3}\left(\phi \frac{d^2V}{d\phi^2} - \frac{dV}{d\phi}\right) \geq 0. \tag{9}$$

The variation in the action (7) with respect to the metric tensor yields the vacuum Einstein equations of the extended scalar–tensor theory. By taking the trace of these equations and incorporating the condition $R = dV/d\phi$, we can derive a Klein–Gordon-like equation for the scalar field. Introducing a matter source is straightforward; it involves a conserved energy–momentum tensor that represents the specific physical system under consideration.

The trace of this tensor also contributes to the Klein–Gordon-like equation for the scalar field, as discussed by [27].

We now specify the field equation for the line element of a flat isotropic Universe (see Equation (4)) in the presence of a co-moving total energy density ρ_{tot}. In particular, we consider the 00-component of the Einstein Equation [16], as follows:

$$H^2 = \frac{\chi\rho_{tot}}{3\phi} - H\frac{\dot{\phi}}{\phi} + \frac{V(\phi)}{6\phi}\,.$$ (10)

Here, $\rho_{tot} = \rho_m + \rho_r$, where ρ_m denotes the (dark and baryonic) matter component of the Universe, verifying the standard conservation law i.e., the divergenceless nature of the corresponding energy–momentum tensor, as follows:

$$\dot{\rho}_m + 3H\rho_m = 0 \;\to\; \rho_m = \rho_m^0(1+z)^3\,,$$ (11)

in which ρ_m^0 denotes the present-day value of ρ_m. We introduced the redshift variable $z(t) = 1/a(t) - 1$ (we set the present value of the cosmic scale factor equal to unity). It is worth noting that the evolution of the Universe matter component is not affected by the matter creation process. Actually, the radiation creation, as described in Equation (6), does not follow the divergenceless character of a prefect fluid with $w = 1/3$. However, it corresponds to a perfect fluid with the following time-varying equation of a state parameter:

$$w_r(t) = \frac{1}{3} - \frac{4}{3}\left(\frac{H}{\bar{H}}\right)^{\beta}\,.$$ (12)

It is the conservation law of a such an energy–momentum tensor which provides Equation (6). The second basic equation of the cosmological dynamics reads as follows:

$$\frac{dV}{d\phi} = R = 12H^2 + 6\dot{H}\,.$$ (13)

This approach is particularly suitable for determining a posteriori the form of the $f(R)$ gravity that can help alleviate the Hubble tension.

4. Reduced Dynamics

In order to transform the potential $V(\phi)$ from an ingredient which is assigned via the function $f(R)$ into a dynamical variable $V(t)$, we impose on Equation (10) the following two conditions:

$$V(\phi) = 2\chi\rho_{\Lambda} + G(\phi)\,, \qquad 6H\dot{\phi} = G\,,$$ (14)

where ρ_{Λ} is the constant value of the Universe energy density and G is a generic functional form, to be dynamically determined. Clearly, these two relations play a crucial role in giving Equation (10) a form that resembles the dynamics of a ΛCDM model for the Universe, as specified at the beginning of the next section. Introducing the critical parameters in place of the energy density according to the relations $\Omega_m^0 = \chi\rho_m/3H_0^2$, $\Omega_{\Lambda}^0 = \chi\rho_{\Lambda}/3H_0^2$, and $\Omega_r = \chi\rho_r/3H_0^2$, where H_0 is the present day value of the Hubble constant, we can rewrite Equation (10) into the following form:

$$H^2(z) = \frac{H_0^2}{\phi(z)}\left(\Omega_m^0(1+z)^3 + \Omega_{\Lambda}^0 + \Omega_r(z)\right)\,,$$ (15)

where $\Omega_r^0 = \Omega_r(z = 0)$ and the relation $\Omega_\Lambda^0 = 1 - \Omega_m^0 - \Omega_r^0$ must be satisfied, given that we set $\phi(z = 0) = 1$. This equation is coupled with the dynamical system of Equations (6), (13), and (14) recast as follows (the prime denotes z-differentiation):

$$\phi' = -\frac{G(z)}{6(1+z)H^2} , \tag{16}$$

$$G' = 3\phi'\left(4H^2 - 2(1+z)HH'\right) , \tag{17}$$

$$\Omega_r' = 4(1+z)\Omega_r\left(1 - (H/\bar{H})^\beta\right) . \tag{18}$$

The ratio of Equations (16) and (17) gives us the following relation [7]:

$$G(z) = -6A^2 H(z)(1+z)^{-2} , \tag{19}$$

where A^2 is a positive integration constant. Substituting the expression above into Equation (16), we obtain the following:

$$\phi' = \frac{A^2}{H(z)(1+z)^3} , \tag{20}$$

which describes the increasing behaviour of ϕ with the time variable z.

In this scheme, the dynamical system thus reduces to the following:

$$\phi' = A_0^2 \, (1+z)^{-3} \, \left(\phi(z)^{-1}(\Omega_m^0(1+z)^3 + 1 - \Omega_m^0 - \Omega_r^0 + \Omega_r(z))\right)^{-1/2} , \tag{21}$$

$$\Omega_r' = 4(1+z)^{-1}\Omega_r(x)\left(1 - \left(\bar{H}_0^{-2}\phi(z)^{-1}(\Omega_m^0(1+z)^3 + 1 - \Omega_m^0 - \Omega_r^0 + \Omega_r(z))\right)^{\beta/2}\right) , \tag{22}$$

where $A_0^2 \equiv A^2/H_0$ and $\bar{H}_0 \equiv \bar{H}/H_0$. Once we have calculated the function $\phi(z)$, we can also determine the potential $G(\phi)$ and, finally, the form of the resulting $f(R)$ gravity.

5. Numerical Analysis

In this section, we provide a comprehensive description of the methodology used for integrating the model. Specifically, we numerically derive the dynamical forms of $\phi(z)$ and $\Omega_r(z)$, which ultimately lead to the final expression of $H(z)$ in Equation (15). To compare our results with the standard two flat ΛCDM forms of the Hubble parameter constrained by the early Universe data [5] and by the Phanteon+ dataset [3], we introduce the following quantities:

$$H_{Pl}(z) = H_0^{Pl}(\Omega_m^{0Pl}(1+z)^3 + (1 - \Omega_m^{0Pl}))^{1/2} , \tag{23}$$

$$H_{SN}(z) = H_0^{SN}(\Omega_m^{0SN}(1+z)^3 + (1 - \Omega_m^{0SN}))^{1/2} , \tag{24}$$

with the following (H and H_0 are in units of km s^{-1} Mpc^{-1}):

$$H_0^{Pl} = 67.4 \pm 0.5 , \qquad \Omega_m^{0Pl} = 0.315 \pm 0.007 , \tag{25}$$

$$H_0^{SN} = 73.6 \pm 1.1 , \qquad \Omega_m^{0SN} = 0.334 \pm 0.018 , \tag{26}$$

In order to study the viability of the addressed scenario and also its capability in alleviating the Hubble tension, we explore the full free parameter space of the model $\{H_0, \Omega_m^0, \Omega_r^0, \bar{H}_0, A_0, \beta\}$. Equations (21) and (22) are numerically solved, spanning a grid of 15 values for each parameter, and thus collecting 15^6 sampled different models. The ranges are defined according to previously conducted tests on the integrability of the system and phenomenological considerations: we have set $72.5 \leq H_0 \leq 74.7$, $0.25 \leq \Omega_m^0 \leq 0.335$, $0.01 \leq \Omega_r^0 \leq 0.15$, $0.5 \leq \bar{H}_0 \leq 1.5$, $0.1 \leq A_0 \leq 0.5$, and $0.5 \leq \beta \leq 1.5$. We note that the selected values of H_0 are in accordance with the measurements of the Pantheon+ analysis [3], while Ω_r^0 is assumed, as already stated, to remain a small contribution to the energy density of the Universe.

The obtained models are then filtered requiring non-tachyonic modes, i.e., that Equation (9) is guaranteed, and we further impose the condition that the normalized (by H_0^2) squared mass is less or equal to unity, today. This point is relevant in order for the considered $f(R)$ model to obey the so-called "chameleon" mechanism [39]. The resulting models are thus physically viable and, to study the efficiency in order to alleviate the tension, we also require that $0.999 < H/H_{Pl}|_{z=10} < 1.001$. With this procedure, we finally obtain around 5×10^3 sampled models. We then convolve the data with a normal distribution to create a smooth density estimate. A kernel density estimate plot is a visualization method used to depict the distribution of observations in a dataset, akin to a histogram. The results are depicted in Figure 1.
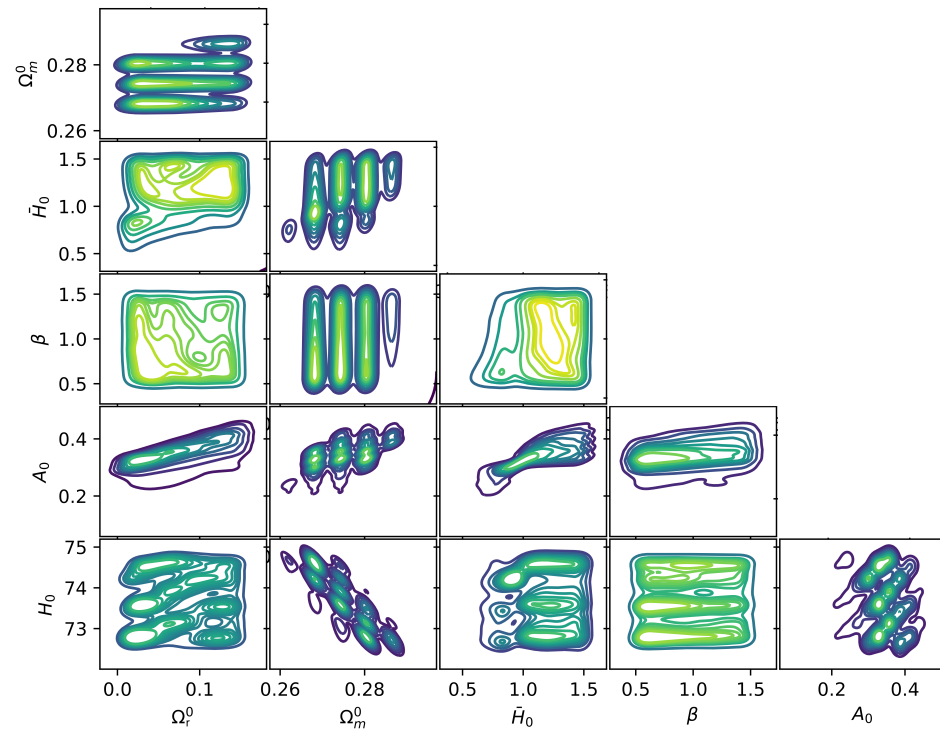

Figure 1. Density plot for all the model's parameters. The yellow region indicates the most frequent values preferred by the model.

From this analysis, preferred regions can be individualized providing the most frequent parameter sets. As an example of the model's capability in alleviating the Hubble tension, we select the following:

$$H_0 = 72.9\,, \quad \Omega_m^0 = 0.28\,, \quad \Omega_r^0 = 0.035\,, \quad \bar{H}_0 = 1.01\,, \quad A_0 = 0.33\,, \quad \beta = 0.7\,. \quad (27)$$

In Figure 2, we plot the resulting evolution of $H(z)$ from Equation (15), with this parameter setup, together with the curves in Equation (23) (blue) and Equation (24) (red), considering the corresponding errors. The tension alleviation emerges clearly, and the $H(z)$ profile overlaps $H_{Pl}(z)$ already at $z \simeq 3$, but reaches higher values of the Hubble constant in $z = 0$. In the figure, we also depict six relevant measurements for BAO sources (in the range $0.3 < z < 3$) [40–43] and the SH0ES prior for $H(z)$ today. This clearly indicates the capability of the addressed model to also alleviate the tension derived from different late Universe sources. In Figure 3, we instead plot the functions ϕ and Ω_r as functions of z. The curves are obtained by integrating Equations (21) and (22), assuming the choice of the parameters as in Equation (27). As expected, for $z > 3$, $\Omega_r(z)$ approaches zero, while $\phi(z)$ is frozen as a plateau.
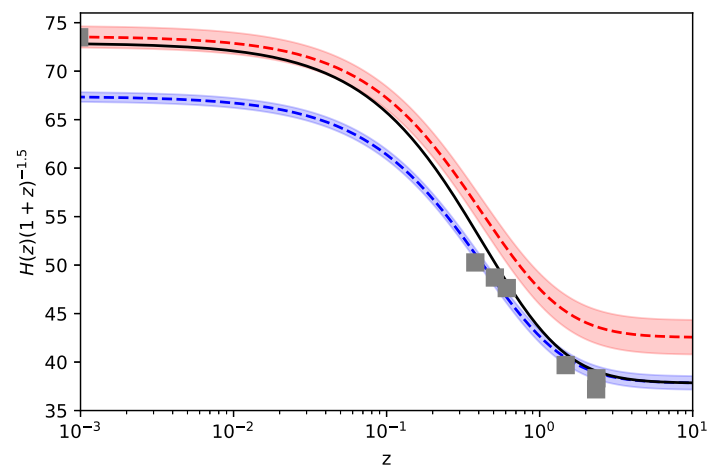
Figure 2. Plot of $H(z)$ (black) using the parameters in Equation (27) and the profiles (with the corresponding errors) of $H_{Pl}(z)$ (blue) and $H_{SN}(z)$ (red). Grey squares represent the SH0ES prior and six relevant measurements for BAO sources for $0.3 < z < 3$.
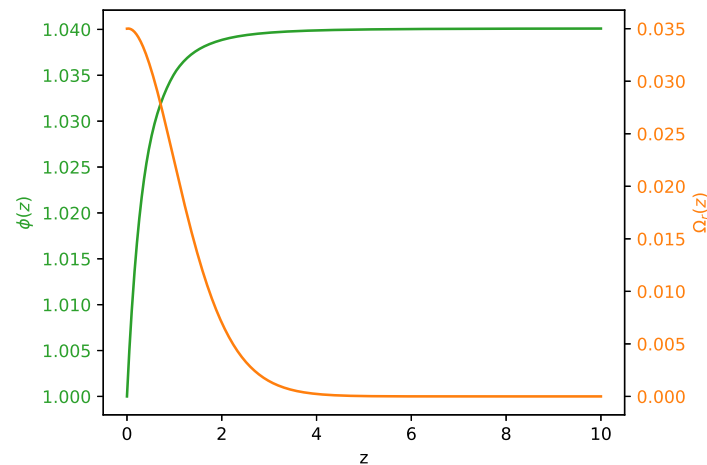
Figure 3. Plot of $\phi(z)$ (green) and $\Omega_r(z)$ (orange) integrated from Equations (21) and (22), implementing the parameters in Equation (27).

This study of the parameter space is essential for guiding the real data analysis procedure, clearly indicating the model's viability for interpreting the Hubble tension through the missing physics that must be added to the ΛCDM formulation to reconcile it with observations. The numerical study highlights how the additional components of modified metric $f(R)$ gravity and the radiation term generated by the cosmological background are crucial for mitigating the Hubble tension within the framework of a physical theory. In this context, a key aspect of the numerical filtering procedure is ensuring that the modified gravity is not associated with a tachyonic mode.

6. Conclusions

We constructed a revised dynamical model for the late Universe based on both a metric $f(R)$ gravity in the Jordan frame, similar to the approach in [7], and a phenomenological model for matter creation associated with the time variation of the cosmological gravitational field. The matter creation is described by treating the Universe as an open thermodynamic system, with the rate of particle creation phenomenologically regulated by a power-law of the Hubble parameter. The form of the $f(R)$ gravity is not assigned a priori. Indeed, by imposing suitable conditions on the generalized Friedmann equation, the potential of the non-minimally coupled scalar field $V(z) \equiv V(\phi(z))$ becomes a dynamical

variable, along with $\phi(z)$ and $\Omega_r(z)$. Consequently, it is possible to reconstruct the form of $V(\phi)$ and thus determine the function $f(R)$ governing the modified gravity.

We developed a dynamical model with six free parameters, $\{H_0, \Omega_m^0, \Omega_r^0, \bar{H}_0, A_0, \beta\}$, aimed at alleviating the Hubble tension. Our numerical analysis involved a thorough screening of all possible solutions, retaining only those parameter sets that ensured the physical consistency of the proposed scenario. These solutions also had to meet two criteria: the resulting Hubble parameter needed to exhibit the required asymptotic ΛCDM behaviour (achieved at low z values) and allow for a Hubble constant compatible with the SH0ES collaboration observations. Identifying a preferred set of parameters provided significant insights into how real data comparisons could impact the parameter space. This approach could successfully address the Hubble tension, as the Hubble parameter rapidly converges to the ΛCDM model with Planck-measured parameters as the redshift increases.

We conclude by emphasizing that the privileged values of Ω_r^0 were found to be very small, making this component reliably unobserved through direct measurement. The particles forming this radiation component, such as sterile neutrinos [44], very weakly interact, which explains their elusive nature. The key takeaway from this study is that the Hubble tension can be effectively addressed by combining metric $f(R)$ gravity with additional modifications to the standard ΛCDM model.

References

1. Scolnic, D.M.; Brout, D.; Carr, A.; Riess, A.G.; Davis, T.M.; Dwomoh, A.; Jones, D.O.; Ali, N.; Charvu, P.; Chen, R.; et al. The Pantheon+ Analysis: The Full Data Set and Light-curve Release. *Astrophys. J.* **2022**, *938*, 113. [CrossRef]
2. Scolnic, D.M.; Jones, D.O.; Rest, A.; Pan, Y.C.; Chornock, R.; Foley, R.J.; Huber, M.E.; Kessler, R.; Narayan, G.; Riess, A.G.; et al. The Complete Light-curve Sample of Spectroscopically Confirmed SNe Ia from Pan-STARRS1 and Cosmological Constraints from the Combined Pantheon Sample. *Astrophys. J.* **2018**, *859*, 101. [CrossRef]
3. Brout, D.; Scolnic, D.; Popovic, B.; Riess, A.G.; Carr, A.; Zuntz, J.; Kessler, R.; Davis, T.M.; Hinton, S.; Jones, D.; et al. The Pantheon+ Analysis: Cosmological Constraints. *Astrophys. J.* **2022**, *938*, 110. [CrossRef]
4. Riess, A.G.; Yuan, W.; Macri, L.M.; Scolnic, D.; Brout, D.; Casertano, S.; Jones, D.O.; Murakami, Y.; Anand, G.S.; Breuval, L.; et al. A Comprehensive Measurement of the Local Value of the Hubble Constant with 1 km s^{-1} Mpc^{-1} Uncertainty from the Hubble Space Telescope and the SH0ES Team. *Astrophys. J. Lett.* **2022**, *934*, L7. [CrossRef]
5. Aghanim, N.; Akrami, Y.; Ashdown, M.; Aumont, J.; Baccigalupi, C.; Ballardini, M.; Banday, A.J.; Barreiro, R.B.; Bartolo, N.; Basak, S.; et al. Planck 2018 results. VI. Cosmological parameters. *Astron. Astrophys.* **2021**, *641*, A6. Erratum in *Astron. Astrophys.* **2021**, *652*, C4. [CrossRef]
6. Schiavone, T.; Montani, G.; Bombacigno, F. f(R) gravity in the Jordan frame as a paradigm for the Hubble tension. *Mon. Not. RAS* **2023**, *522*, L72–L77. [CrossRef]
7. Montani, G.; De Angelis, M.; Bombacigno, F.; Carlevaro, N. Metric f(R) gravity with dynamical dark energy as a scenario for the Hubble tension. *Mon. Not. RAS* **2024**, *527*, L156–L161. [CrossRef]
8. Montani, G.; Carlevaro, N.; Dainotti, M.G. Slow-rolling scalar dynamics as solution for the Hubble tension. *Phys. Dark Univ.* **2024**, *44*, 101486. [CrossRef]
9. Adil, S.; Akarsu, Ö.; Di Valentino, E.; Nunes, R.; Özülker, E.; Sen, A.; Specogna, E. Omnipotent dark energy: A phenomenological answer to the Hubble tension. *Phys. Rev. D* **2024**, *109*, 023527. [CrossRef]

10. Nojiri, S.; Odintsov, S.D.; Oikonomou, V.K. Integral F(R) gravity and saddle point condition as a remedy for the H0-tension. *Nucl. Phys. B* **2022**, *980*, 115850. [CrossRef]

11. Odintsov, S.D.; Sáez-Chillón Gómez, D.; Sharov, G.S. Analyzing the H_0 tension in $F(R)$ gravity models. *Nucl. Phys. B* **2021**, *966*, 115377. [CrossRef]

12. Wang, B.; Abdalla, E.; Atrio-Barandela, F.; Pavón, D. Dark matter and dark energy interactions: Theoretical challenges, cosmological implications and observational signatures. *Rep. Prog. Phys.* **2016**, *79*, 096901. [CrossRef]

13. Naidoo, K.; Jaber, M.; Hellwing, W.A.; Bilicki, M. Dark matter solution to the H_0 and S_8 tensions, and the integrated Sachs-Wolfe void anomaly. *Phys. Rev. D* **2024**, *109*, 083511. [CrossRef]

14. Dainotti, M.; Bargiacchi, G.; Bogdan, M.; Capozziello, S.; Nagataki, S. On the statistical assumption on the distance moduli of Supernovae Ia and its impact on the determination of cosmological parameters. *J. High Energy Astrophys.* **2024**, *41*, 30–41. [CrossRef]

15. Dainotti, M.G.; Bargiacchi, G.; Bogdan, M.; Lenart, A.L.; Iwasaki, K.; Capozziello, S.; Zhang, B.; Fraija, N. Reducing the Uncertainty on the Hubble Constant up to 35% with an Improved Statistical Analysis: Different Best-fit Likelihoods for Type Ia Supernovae, Baryon Acoustic Oscillations, Quasars, and Gamma-Ray Bursts. *Astrophys. J.* **2023**, *951*, 63. [CrossRef]

16. Weinberg, S. *Cosmology*; OUP Oxford: Oxford, UK, 2008.

17. Di Valentino, E.; Mena, O.; Pan, S.; Visinelli, L.; Yang, W.; Melchiorri, A.; Mota, D.F.; Riess, A.G.; Silk, J. In the realm of the Hubble tension—a review of solutions. *Class. Quant. Grav.* **2021**, *38*, 153001. [CrossRef]

18. Dainotti, M.G.; De Simone, B.; Schiavone, T.; Montani, G.; Rinaldi, E.; Lambiase, G. On the Hubble constant tension in the SNe Ia Pantheon sample. *Astrophys. J.* **2021**, *912*, 150. [CrossRef]

19. Dainotti, M.G.; De Simone, B.; Schiavone, T.; Montani, G.; Rinaldi, E.; Lambiase, G.; Bogdan, M.; Ugale, S. On the Evolution of the Hubble Constant with the SNe Ia Pantheon Sample and Baryon Acoustic Oscillations: A Feasibility Study for GRB-Cosmology in 2030. *Galaxies* **2022**, *10*, 24. [CrossRef]

20. Xu, B.; Xu, J.; Zhang, K.; Fu, X.; Huang, Q. Model-independent test of the running Hubble constant from the Type Ia supernovae and the Hubble parameter data. *Mon. Not. RAS* **2024**, *530*, 5091–5098. [CrossRef]

21. Krishnan, C.; Colgáin, E.O.; Sheikh-Jabbari, M.M.; Yang, T. Running Hubble Tension and a H0 Diagnostic. *Phys. Rev. D* **2021**, *103*, 103509. [CrossRef]

22. Kazantzidis, L.; Perivolaropoulos, L. Hints of a Local Matter Underdensity or Modified Gravity in the Low z Pantheon data. *Phys. Rev. D* **2020**, *102*, 023520. [CrossRef]

23. Efstathiou, G. To H$_0$ or not to H$_0$? *Mon. Not. RAS* **2021**, *505*, 3866–3872. [CrossRef]

24. Giarè, W. Inflation, the Hubble Tension and Early Dark Energy: An alternative overview. *Phys. Rev. D* **2024**, *109*, 123545. [CrossRef]

25. Vagnozzi, S. Seven Hints That Early-Time New Physics Alone Is Not Sufficient to Solve the Hubble Tension. *Universe* **2023**, *9*, 393. [CrossRef]

26. Vagnozzi, S. New physics in light of the H$_0$ tension: An alternative view. *Phys. Rev. D* **2020**, *102*, 023518. [CrossRef]

27. Sotiriou, T.P.; Faraoni, V. $f(R)$ theories of gravity. *Rev. Mod. Phys.* **2010**, *82*, 451–497. [CrossRef]

28. Nojiri, S.; Odintsov, S.D. Unified cosmic history in modified gravity: From F(R) theory to Lorentz non-invariant models. *Phys. Rep.* **2011**, *505*, 59–144. [CrossRef]

29. Montani, G. Influence of particle creation on flat and negative curved FLRW universes. *Class. Quantum Gravity* **2001**, *18*, 193–203. [CrossRef]

30. Calvão, M.O.; Lima, J.A.S.; Waga, I. On the thermodynamics of matter creation in cosmology. *Phys. Lett. A* **1992**, *162*, 223–226. [CrossRef]

31. Erdem, R. Gravitational particle production and the Hubble tension. *arXiv* **2024**, arxiv:2402.16791.

32. Nunes, R.C.; Pavón, D. Phantom behavior via cosmological creation of particles. *Phys. Rev. D* **2015**, *91*, 063526. [CrossRef]

33. Nunes, R.C. Gravitationally induced particle production and its impact on structure formation. *Gen. Relativ. Gravit.* **2016**, *48*, 107. [CrossRef]

34. Lima, J.A.S.; Graef, L.L.; Pavón, D.; Basilakos, S. Cosmic acceleration without dark energy: Background tests and thermodynamic analysis. *J. Cosmol. Astropart. Phys.* **2014**, *2014*, 42. [CrossRef]

35. Efstathiou, G.; Gratton, S. The evidence for a spatially flat Universe. *Mon. Not. RAS* **2020**, *496*, L91–L95. [CrossRef]

36. Birrell, N.; Davies, P. *Quantum Fields in Curved Space*; Cambridge University Press: Cambridge, MA, USA, 1982.

37. Wald, R. *Quantum Field Theory in Curved Spacetime and Black Hole Thermodynamics*; The University of Chicago Press: Chicago, IL, USA, 1994.

38. Olmo, G.J. Post–Newtonian constraints on f(R) cosmologies in metric and Palatini formalism. *Phys. Rev. D* **2005**, *72*, 083505. [CrossRef]

39. Brax, P.; van de Bruck, C.; Davis, A.C.; Shaw, D.J. f(R) Gravity and Chameleon Theories. *Phys. Rev. D* **2008**, *78*, 104021. [CrossRef]

40. Alam, S.; Ata, M.; Bailey, S.; Beutler, F.; Bizyaev, D.; Blazek, J.A.; Bolton, A.S.; Brownstein, J.R.; Burden, A.; Chuang, C.H.; et al. The clustering of galaxies in the completed SDSS-III Baryon Oscillation Spectroscopic Survey: Cosmological analysis of the DR12 galaxy sample. *Mon. Not. RAS* **2017**, *470*, 2617–2652. [CrossRef]

41. Blomqvist, M.; du Mas des Bourboux, H.; Busca, N.G.; de Sainte Agathe, V.; Rich, J.; Balland, C.; Bautista, J.E.; Dawson, K.; Font-Ribera, A.; Guy, J.; et al. Baryon acoustic oscillations from the cross-correlation of Lyα absorption and quasars in eBOSS DR14. *Astron. Astrophys.* **2019**, *629*, A86. [CrossRef]

42. de Sainte Agathe, V.; Balland, C.; du Mas des Bourboux, H.; Busca, N.G.; Blomqvist, M.; Guy, J.; Rich, J.; Font-Ribera, A.; Pieri, M.M.; Bautista, J.E.; et al. Baryon acoustic oscillations at z = 2.34 from the correlations of Lyα absorption in eBOSS DR14. *Astron. Astrophys.* **2019**, *629*, A85. [CrossRef]

43. Hou, J.; Sánchez, A.G.; Ross, A.J.; Smith, A.; Neveux, R.; Bautista, J.; Burtin, E.; Zhao, C.; Scoccimarro, R.; Dawson, K.S.; et al. The completed SDSS-IV extended Baryon Oscillation Spectroscopic Survey: BAO and RSD measurements from anisotropic clustering analysis of the quasar sample in configuration space between redshift 0.8 and 2.2. *Mon. Not. RAS* **2021**, *500*, 1201–1221. [CrossRef]

44. Abazajian, K.N. Sterile neutrinos in cosmology. *Phys. Rept.* **2017**, *711*, 1–28. [CrossRef]