A note on trapped surfaces in the Oppenheimer-Snyder solution

Masahiro Shimano[1], Naoki Tsukamoto[2], Kohji Yaijma[3] and Tomohiro Harada[4]

Department of Physics, Rikkyo University, Toshima, Tokyo 171-8501, Japan

Abstract

Although an event horizon defines a black hole, it is difficult to explain a boundary of dynamical black holes. A trapped surface is a candidate of the boundary of dynamical black holes, and is studied in various spacetimes and settings. Usually, there is no trapped surface in a Minkowski region, however, Bengtsson and Senovilla showed an interesting result as follows: in a self-similar Vaidya spacetime, they considered non-spherical trapped surfaces and showed that trapped surfaces can extend into the Minkowski region, if and only if a mass function rises fast enough [1]. In this paper we extend Bengtsson and Senovilla's way in a Oppenheimer-Snyder solution. We match two kinds of two-surfaces and construct trapped surfaces which pass through an apparent horizon.

1 Introduction

A boundary of a region in a spacetime that cannot be observed from infinity is called event horizon. The event horizon defines a boundary between an inside and an outside of a black hole. Moreover, this horizon has a teleological property: an entire future history of the spacetime must be known before a position of the event horizon can be determined. Since the event horizon is defined at future timelike infinity, a shape of the event horizon does not change. However, some quantum effect might deform the boundary of dynamical black holes. It might be impossible to explain the boundary of dynamical black holes with the event horizon.

Eardley conjectured that the boundary of the region which contains marginally outer trapped surfaces coincide with the event horizon [2]. A surface called outer trapped surface is the closed spacelike two-surface and whose outer null expansion is negative. This conjecture is very interesting, because we can translate the event horizon with outer trapped surface, i.e., we can define the event horizon constructively. In a Vaidya spacetime, Ben-Dov showed that Eardley's conjecture is true [3]. However, the outer trapped surface cannot be considered in the general spacetime, because it is defined only in asymptotically flat spacetimes [4]. Moreover, the outer trapped surface only consider an outer null ray. There remains a possibility of which we can observe an inner null ray. Since the black hole is an invisible region, to define this region we should consider a notion of which both null rays cannot be observed.

A surfaces called trapped surface defines the boundary of an invisible region. The trapped surface is a closed spacelike two-surface and whose both null expansions are negative. Since the black hole cannot be observed even from infinity, its boundary might be defined with the trapped surface. Usually, there is no trapped surface in a Minkowski spacetime. However, in the Vaidya spacetime it was showed that trapped surfaces can extend into the Minkowski region. Numerical results of Schnetter and Krishnan showed that an outer boundary of trapped surfaces can extend into the Minkowski region [5]. Moreover, Bengtsson and Senovilla considered the self-similar Vaidya spacetime, and they analytically showed that trapped surfaces can extend into the Minkowski region, if and only if a mass function rises fast enough [1].

In this paper, we extend Bengtsson and Senovilla's way into a Oppenheimer-Snyder solution, and consider the relation between the trapped surface and an apparent horizon. The surface called apparent horizon is the boundary of a trapped region. A portion of hypersurface which contains trapped surfaces is called trapped region. Therefore, usually, trapped surface does not extend outside the apparent horizon. However, by using Bengtsson and Senovilla's way we construct a non-spherical trapped surface and show that this trapped surface passes through the apparent horizon.

[1]Email address: shimano@rikkyo.ac.jp
[2]Email address: 09la008b@rikkyo.ac.jp
[3]Email address: 09la010t@rikkyo.ac.jp
[4]Email address: harada@rikkyo.ac.jp

2 Oppenheimer-Snyder solution

The Oppenheimer-Snyder solution describes a process of a gravitational collapse to the black hole. In this paper, for simplicity, we consider a collapse of a homogeneous dust from infinity. A metric inside the dust is a flat Friedmann-Robertson-Walker(FRW) solution, while the metric outside the dust is a Schwarzschild solution. These metrics are joined smoothly on a common boundary which is the surface of the collapsing dust.

The metric inside the collapsing dust is given by

$$ds_-^2 = -dt + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\right), \tag{1}$$

where t is proper time on comoving world lines, and $a(t)$ is the scale factor. Substituting this metric into the Einstein equation, we get the Friedmann equation

$$\left(\frac{a'}{a}\right)^2 = \frac{8\pi G\rho}{3}, \tag{2}$$

where ρ is a energy density of the dust, G is a gravitational constant, and a over-dash denotes a differentiation with respect to t. Because the dust has not a pressure, the dust's energy density satisfies $\rho a^3 = C$ from the energy-momentum conservation law, where C is a constant. Substituting this energy density into Eq. (2), after some calculations, we obtain

$$a(\eta) = \frac{2\pi G}{3} C(-\eta)^2, \tag{3}$$

where we have introduced comoving time η which satisfies a relation $d\eta = dt/a$, and we have assumed $a = 0$ at $\eta = 0$. The collapse begins at $\eta = -\infty$ when $a = \infty$, and it ends at $\eta = 0$ when $a = 0$. We assume that the hypersurface Σ coincides with the surface of the collapsing dust, which is located at $r = r_c$ in our comoving coordinates. The induced metric of Σ is given by

$$ds^2_{\Sigma_{\text{inside}}} = a^2(\eta)\left\{-d\eta^2 + r_c^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right\}. \tag{4}$$

On the other hand, the metric outside the dust is given by

$$ds^2 = -f d\tau^2 + f^{-1} d\chi^2 + \chi^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{5}$$

where $f = 1 - 2m/r$ and m is a gravitational mass of the collapsing dust. In this metric, the surface of the collapsing dust Σ is described by the parametric equations $\chi = \bar{R}(\eta)$ and $\tau = \bar{T}(\eta)$, where η is the same η that appears in Eq. (3). The induced metric of Σ is given by

$$ds^2_{\Sigma_{\text{outside}}} = -(F\dot{\bar{T}}^2 - F^{-1}\dot{\bar{R}}^2)d\eta^2 + \bar{R}^2(\eta)\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{6}$$

where $F = 1 - 2m/\bar{R}$, and a over-dot is the differentiation with respect to η.

Because the induced metric must be the same on both sides of the hypersurface Σ, we have

$$F\dot{\bar{T}}^2 - F^{-1}\dot{\bar{R}}^2 = a^2, \quad a^2 r_c^2 = \bar{R}^2. \tag{7}$$

Moreover, the extrinsic curvature also must be coincide with both sides of Σ. Then we get relations

$$a\dot{\beta} - \dot{a}\beta = 0, \quad a^2 = F^2\dot{\bar{T}}^2, \tag{8}$$

where $\beta = F\dot{\bar{T}} = \sqrt{\dot{\bar{R}}^2 + a^2 F}$. By using Eqs. (3), (7), (8), and $F = 1 - 2m/\bar{R}$, we obtain

$$C = \frac{3}{4\pi G}\frac{m}{r_c^3}, \quad \bar{R} = \frac{m}{2r_c^2}(-\eta)^2. \tag{9}$$

Substituting Eq. (9) into Eq. (8), after integrating with respect to η, we get

$$\bar{T} = -\frac{m}{2r_c^3}(-\eta)^3 - \frac{2m}{r_c}(-\eta) - 2m\ln\left(\frac{(-\eta) - 2r_c}{(-\eta) + 2r_c}\right) + 2m\left(20 + \ln\left(\frac{1}{3}\right)\right), \tag{10}$$

where we have considered $\bar{T} = 0$ at $-\eta = 4r_c$. Moreover, the relation of both coordinates on the hypersurface Σ is given by

$$\left(\frac{\partial}{\partial\tau}\right) = \frac{F\dot{\bar{T}}}{a}\left(\frac{\partial}{\partial\eta}\right) - \frac{\dot{\bar{R}}}{a}\left(\frac{\partial}{\partial r}\right), \quad \left(\frac{\partial}{\partial\chi}\right) = -\frac{\dot{\bar{R}}}{aF}\left(\frac{\partial}{\partial\eta}\right) + \frac{\dot{\bar{T}}}{a}\left(\frac{\partial}{\partial r}\right). \tag{11}$$

3 Trapped surface

In the Oppenheimer-Snyder spacetime, we consider two kinds of two-surfaces to obtain trapped surface which passes through the apparent horizon. One is the surface in which η and r are the function of ρ, and in which $\theta = \pi/2$. We call this surface *constant inclination surface* (CIS), and consider this surface in the inside of the collapsing dust. The other is the surface in which θ and v are the function of ρ, and in which $\chi = \chi_0$, where v is the ingoing null coordinate $v = \tau + \chi + 2m \ln|(\chi - 2m)/(2m)|$. We call this surface *constant radius surface* (CRS), and consider this surface in the outside of the collapsing dust. In order to get the trapped surface which passes through the apparent horizon, we bend these two-surfaces and match these on the hypersurface Σ.

In the inside of the collapsing dust, we bend the CIS into a quadratic curve

$$\eta = \eta_0 + \frac{E}{2}r^2, \tag{12}$$

where η_0 and E are constants. Both null expansions of this surface are

$$\vartheta_\pm = -\frac{1}{a\sqrt{1-E^2r^2}}\left[E(E^2r^2 - 2) + \frac{4}{(-\eta)}\right]. \tag{13}$$

If χ_0 satisfies

$$0 < \chi_0 < 1.6m, \tag{14}$$

both null expansions are negative.

On the other hand, in the outside of the collapsing dust, we bend the CRS into a quadrant of a circle

$$\theta^2 + (z - z_c)^2 = \frac{\pi^2}{4}, \tag{15}$$

where $z = v/\chi_0$ and $z_c = v_c/\chi_0$ is a constant. Both null expansions of this CRS are negative, if the inequality

$$\left(\frac{m}{\chi_0} - 1\right)\sqrt{\frac{2m}{\chi_0} - 1} > \frac{4}{\pi} \tag{16}$$

is satisfied. The solution of this inequality is

$$0 < \chi_0 < 0.68514m, \tag{17}$$

where we have assumed $\theta = \pi/2$ at $z = z_c$.

In order to match the CRS with the CIS on the hypersurface Σ, we match the derivative and the position of these surfaces on Σ. From Eq. (12) we obtain the derivative $d\eta/dr$ of the CIS on Σ as follows

$$\frac{d\eta}{dr} = Er. \tag{18}$$

A value of this derivative is positive. On the other hand, by using Eq. (11), we get the derivative of the CRS which is expressed by the coordinate of the inside of the collapsing dust as

$$\frac{d\eta}{dr} = \frac{(-\eta)}{2r_c}, \tag{19}$$

where we have assumed $\theta = \pi/2$ on matching point. The value of this derivative is positive too. Since both derivatives are positive, we can smoothly match the derivative. Moreover, by substituting Eq. (18) into Eq. (19) we decide the matching time

$$\eta_c = -2Er_c^2. \tag{20}$$

On this matching point, since the two-surface is $\chi = \chi_0$ and $\theta = \pi/2$, both null expansions are

$$\vartheta_{\pm} = \frac{1}{2\chi_0}\left(1 - \frac{m}{\chi_0}\right)\left(\frac{2m}{\chi_0} - 1\right)^{-1/2}. \tag{21}$$

If χ_0 satisfies

$$0 < \chi_0 < m, \tag{22}$$

both null expansions are negative.

Combining inequalities (14), (17) and (22), we can summarise that the trapped surface is constructed, if χ_0 satisfies the inequality (17).

Now, we shall check whether the trapped surface passes through the apparent horizon or not. Usually, the apparent horizon $r_{\rm AH}$ in the inside of the collapsing dust given by

$$r_{\rm AH} = \frac{a}{a'} = \frac{m}{4r_c^3}(-\eta)^3. \tag{23}$$

The apparent horizon appears $r_{\rm AH} = 2m$ when $\eta = -2r_c$, and ends $r_{\rm AH} = 0$ when $\eta = 0$. Therefore, if η_0 in Eq. (12) has a nonzero negative value, our trapped surface passes through the apparent horizon. Substituting Eq. (20) into Eq. (12) with $\eta = \eta_c$, we get

$$\eta_0 = -\frac{5}{2}Er_c. \tag{24}$$

Thus, η_0 has the nonzero negative value, and we could construct the trapped surface passes through the apparent horizon.

4 Conclusion

In the Vaidya spacetime, Bengtsson and Senovilla showed the interesting result: the trapped surface can extend into the flat region. In this paper, we have extended this result into the Oppenheimer-Snyder spacetime, and have shown that the trapped surface can pass through the apparent horizon. In order to get the trapped surface we have used two kinds of two-surfaces (we have called these surfaces CIS and CRS, respectively), and have bent the CIS and the CRS into the quadrant curve and the quadrant of the circle, respectively. We have matched the CIS and the CRS smoothly on the collapsing dust's surface Σ, and have shown that this matched surface are trapped surface, if χ_0 satisfies $0 < \chi_0 < 0.68514m$. Moreover, we have shown that this trapped surface passes through the apparent horizon.

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