Supersymmetric higher derivative couplings and their applications

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Supersymmetric higher derivative couplings and their applications

Supersymmetrische koppelingen met hogere afgeleiden en hun toepassingen

(met een samenvatting in het Nederlands)

Accoppiamenti a piú alte derivate e loro applicazioni

(con un riassunto in Italiano)

Proefschrift

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Ivano Lodato

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Promotor: Prof. dr. B.Q.P.J. de Wit

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Publications

This thesis is based on the following publications

- *Non-renormalization theorems and $\mathcal{N} = 2$ supersymmetric backgrounds*,
  D. Butter, B. de Wit and I. Lodato,
  JHEP **03** (2014) 131, arXiv:1401.6591 [hep-th]

- *New higher-derivative invariants in N=2 supergravity and the Gauss-Bonnet term*,
  D. Butter, B. de Wit, S. M. Kuzenko and I. Lodato,
  JHEP **12** (2013) 062, arXiv:1307.6546 [hep-th]

- *The fate of flat directions in higher derivative gravity*,
  N. Banerjee, S. Dutta and I. Lodato,
  JHEP **05** (2013) 027, arXiv:1301.6773 [hep-th]

and on the review paper, in progress,

- *On Noether's theorems and their modern applications*,
  B. de Wit, N.D. Hari Dass, S. Katmadas, I. Lodato.

"The skull's right eye is where I saw the gold."

OP-149,165

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Chapter 1

Introduction

Black holes are perhaps the most intriguing objects in Nature. They are formed when a massive star at the end of its life cycle collapses under the weight of its own gravity and their gravitational pull is so strong that prevents anything, including light, from escaping. The point of no return for an infalling body or radiation is marked by a theoretical surface, called *event horizon*, which encloses the black hole: anything crossing the horizon will be absorbed by the strong gravitational field and lose any causal connection with the region outside, hence the name event horizon. Despite the fact that even light hitting the horizon will be absorbed and not emitted back, the existence of black holes in the Universe can still be inferred by studying the rotation of stars in a galaxy or observing bright accretion disk composed by heated matter falling inside them. These experimental aspects are surely interesting and relevant but, in this work, we will focus instead on studying theoretical features of black holes that play a crucial role towards the formulation of a consistent theory unifying quantum mechanics and gravity.

Black holes were first discovered in 1916 by Schwarzschild and independently by Droste [1, 2], as classical dynamical solutions of Einstein's equations. It was only 40 years later that Finkelstein interpreted them as perfect black bodies, absorbing anything that crosses the event horizon. Contrary to what one might believe, classic black holes are very simple objects, completely described in terms of very few parameters: mass, charges and angular momenta. This led the famous physicist Chandrasekhar to write that *"Black holes are, almost by definition, the most perfect macroscopic objects there are in the universe.. they are the simplest object as well"*. Based on Einstein's theory of gravity, these few parameters are also related among each other by a series of laws, the four laws of black hole mechanics [3], describing physical properties classic black holes should satisfy (see section 1.1.2). For instance, the 2^{nd} law states that if two black holes merge, the total horizon area is not smaller than the sum of the horizon areas of its constituents. It is easy to realize the resemblance to the 2^{nd} law of thermodynamics, where the area of the horizon would be identified with the thermodynamic entropy of an isolated system. Similar abstract identifications can be derived by comparison with the other laws of thermodynamics but, when the laws of black hole mechanics were discovered, it was largely believed that black holes were perfectly absorbing black bodies at zero temperature. Very few scientists, such as Bekenstein, were instead convinced [4] that black holes do possess a thermodynamic entropy proportional to their horizon area. This matter was settled soon after by Hawking [5] who, in the attempt to disprove Bekenstein's conjecture, actually proved him right. In fact, with a semi-classical calculation, Hawking showed that black holes are almost perfect black bodies, absorbing and *emitting* radiation of a certain non-zero temperature T, inversely proportional to their mass. He also derived the precise relationship between entropy and horizon area A of a black hole

$$S = \frac{k_{\rm B}\, c^3}{G\,\hbar}\frac{A}{4}\,. \tag{1.0.1}$$

The discovery of the so-called *Hawking radiation* established then the physical relevance of the laws of black hole mechanics which, initially thought of as a curious mathematical coincidence, actually describe the properties of thermodynamic objects.

To fully appreciate the implications of this discovery, we remind the reader that thermodynamics studies the transformation of heat into energy (or work) through measurable macroscopic quantities describing a physical system, e.g. the temperature T, the pressure p, the volume V, related by the *equations of state*. The concept of entropy arises also in thermodynamics, to formally explain the observation that, even though work, or energy, can always be *fully* transformed in heat, the vice-versa is not true.

It is nowadays known that the laws of thermodynamics, postulated based on empirical evidence, can be explained by the microscopic kinetic theory describing the mechanical motion of a huge number N of atoms or molecules via a statistical approach. Thermodynamic quantities are then *averaged* properties of a complicated mechanical system, specified by $6N$ variables (position and velocity of the N components), and they describe accurately the thermodynamics of a macroscopic system only in the *thermodynamic limit* of $V \to \infty$ and $N \to \infty$, with the ratio V/N finite. For instance, the microscopic (or statistical) entropy of a gas, given by the Boltzmann's formula

$$S_{\rm micro} = k_{\rm B}\,\ln\Omega\,, \tag{1.0.2}$$

numerically equals the macroscopic entropy $S = \int dQ/T$ only in such limit. Here Ω is the number of microstates accessible by the ensemble of N atoms or molecule composing the gas which are consistent with its macroscopic thermodynamic properties (e.g. p, V, T). In the thermodynamic limit $V \to \infty$, $N \to \infty$ the number of microstates available to the system diverges and, at the same time, the statistical fluctuations become negligible, leading to an accurate macroscopic description.
The discovery of Hawking radiation was suggestive of the fact that general relativity could explain the thermodynamics of black holes and be considered a macroscopic averaged description of an underlying microscopic theory describing a large number of degrees of freedom composing a black hole. For 20 years it remained unclear whether such microscopic theory, yielding the same results of general relativity in a certain thermodynamic limit existed, until Strominger and Vafa [6] considered a microscopic realization of a five-dimensional black hole as a bound state of D-branes, in the context of string theory, and showed that its statistical entropy (1.0.2) matches the thermodynamic entropy given by the area-law (1.0.1). In particular, the two results match when the thermodynamic limit of *large black hole charges* is considered. It is important to stress that this microscopic entropy results was achieved by exploiting a special symmetry, *supersymmetry*, which plays a crucial role in string theory. Despite the fact that no experimental results so far confirmed the existence of such symmetry in Nature, and hence, ideally, the microscopic results should be derived without its use, calculations worked out in a general, non-supersymmetric setting are typically prohibitive. Supersymmetry instead offers an elegant framework where explicit calculations are actually possible. This motivates the choice made in this work to consider theories of gravity invariant under supersymmetry.
Shortly after the classical results of [6], quantum corrections to the entropy of black holes were considered. In [7], subleading (in the limit of large charges) corrections to the microscopic entropy of four-dimensional supersymmetric black holes were calculated. [1] This result was found again to be in agreement with the correspondent quantum corrected macroscopic black hole entropy [8–11], which was calculated in the context of the low energy effective description of string theory, *supergravity*. Supergravity theories were in fact known since long as locally supersymmetric generalizations of Einstein's theory and, being effective theories, they encode quantum corrections into higher (than two) derivative couplings, suppressed in the classical limit[2] The interesting point to raise is that, while the

[1] As it turns out, subleading quantum corrections to the entropy of black hole are connected to finite-size effects. This is because of the attractor mechanism which fixes spacetime quantities like curvature and expansion parameters in terms of the charges. We will discuss this later on in this chapter and more in detail, with practical applications, in Chapter 6.

[2] In section 1.3 we will show how these couplings arise when considering an effective Wilsonian descriptions of quantum processes on which this work is focused.

microscopic result was obtained in a very general context, the macroscopic result was derived by considering only a restricted set of quantum corrections to the entropy of the correspondent black hole configuration, which could be extracted from the only higher derivative invariant known at the time. Nevertheless the matching was proven to be exact.

This will be the starting point of this thesis and its main motivation: in Chapter 4 we will explicitly construct a large class of higher derivative invariants in supergravity, which encodes new one-loop quantum corrections and in Chapter 5 we will show that it does not contribute to the entropy or charges of any of its dynamical macroscopic black hole configurations. This result is likely to be the final confirmation of the results obtained in [8–10] for the thermodynamic entropy of four-dimensional supersymmetric (BPS) black hole configurations at the full one-loop quantum level, which was already found to be in agreement with the calculations of [7]. Of course, before explicitly showing these results, we will introduce all the relevant concepts and topics on which our analysis hinges on. In this first chapter, we will present a pedagogical treatment of general relativity, black hole physics and the laws of black hole mechanics, followed by an introductory discussion on supersymmetric theories and the supersymmetry algebra structure. In view of our treatment of quantum corrections to supersymmetric theories of gravity, which are encoded in higher derivative couplings, we will also explain the salient features of effective theories. The covariant phase space approach, presented in Chapter 2, is perfectly suited to study such theories and also allows for an elegant description of symmetries and conserved quantities. The main result presented is the general procedure, based on the first law of black hole mechanics [12], to calculate quantum corrections to the macroscopic entropy of black holes due to higher derivative couplings. In Chapter 3 we will introduce supergravity theories in four dimensions in the superconformal context. The formalism explained in that chapter will then be adopted to work out the analysis of Chapter 4 and 5. The general treatment of supersymmetric higher derivative couplings in four dimensions will be put on more concrete grounds in Chapter 6 where we will analyze explicit black hole solutions in five- and ten-dimensional supergravity theories, including supersymmetric higher derivative corrections. The goal is to study the restrictions imposed by supersymmetry on the scalar sector of such theories, and how such restrictions can be modified by higher derivative (quantum) corrections. The general formulas presented in Chapter 2 will also be used explicitly to calculate the quantum corrections to all the conserved charges and the entropy of the black hole configurations under examination.

1.1 Black hole thermodynamics

As we already discussed in the introduction, the study of black holes has a central role in the quest for a consistent theory of quantum gravity, because it might give some insight on the quantum nature of the gravitational force. In this respect, a crucial role was played by the discovery of the Hawking radiation [5], a thermodynamic emission of particles from the black hole to spatial infinity, that arises from considering quantum field on a classical background. Thanks to this discovery, the laws of black hole mechanics [3] assume nowadays a physical significance that goes beyond any initial expectation. The possibility of evaluating the macroscopic black hole entropy from a classical theory of gravity is, in fact, complemented by the microscopic description of black holes and their entropy, in the context of string theory. Since the string theory description relies heavily on supersymmetry, the classical effective theory to consider for the macroscopic calculation is not general relativity, but its supersymmetric extension, supergravity, which effectively encodes quantum corrections in higher derivative couplings.
In this section we want to pave the way for the following analysis of supergravity theories, by considering Einstein's general relativity and few simple examples of black hole solutions. This will allow us to introduce a number of concepts, relevant for the following discussions, in a simpler non-supersymmetric context, such as black hole *horizons* and *extremality*. We will also present the laws of black hole mechanics, without an explicit derivation which goes beyond the scope of this work. The material presented here will be complemented in the next chapter, where a procedure [12] to calculate sub-leading corrections to the area of black holes due to higher derivative couplings in general relativity and, more importantly for this work, in supergravity will be presented.

1.1.1 Black holes in general relativity

General relativity is a theory invariant under local coordinate transformation, or diffeomorphism, of the form $x^\mu \to x^\mu + \xi^\mu(x)$. The only necessary dynamical field of the theory is the metric, $g_{\mu\nu}$, a symmetric two-index tensor which gauges diffeomorphism invariance and describes the space-time geometry. We restrict the following analysis to 4 space-time dimensions ($\mu, \nu = 0, \ldots, 3$), although extensions to different space-time dimensions are trivial.
The starting point is the Einstein-Hilbert action including a coupling between

gravity and matter and a cosmological constant term Λ

$$S = \int \mathrm{d}^4x\sqrt{-g}\Big[-\frac{1}{2\kappa^2}(\mathcal{R}-\Lambda)+\mathcal{L}_M\Big] \ . \qquad (1.1.1)$$

In the equation above, $\kappa^2 = 8\,\pi\,G$ and G is the Newton's constatnt, g is the determinant of the metric and the factor $\sqrt{-g}$, called *measure* always appears in diffeomorphism invariant theories, as otherwise the Lagrangian would not transform into a total derivative under diffeomorphism transformation. Furthermore the quantity $\sqrt{-g}\,\mathrm{d}^4x$ is the invariant volume in a covariant theory of gravity and the scalar $\mathcal{R}$, the Ricci scalar, is obtained from contractions of the Riemann curvature tensor (see section 2.5). Finally the term $\mathcal{L}_M$ describes the matter content of the theory. From the Lagrangian one immediately obtains the equation of motion for the metric,

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2}(\mathcal{R}-\Lambda)g_{\mu\nu} = 2\,\kappa^2\,T_{\mu\nu} \ , \qquad (1.1.2)$$

where $T_{\mu\nu} \equiv -\frac{\delta\mathcal{L}_M}{\delta g^{\mu\nu}} + \frac{1}{2}\mathcal{L}_M g_{\mu\nu}$. Mathematically, Einstein's equations are a system of coupled partial differential equations. Physically, this reflects the fact that that gravity is a source of (potential) gravitational energy and, given the famous relativistic relation $E = Mc^2$, gravitational energy becomes source of mass, and then gravity again.

Another source of gravitational energy is the cosmological term Λ, associated with the vacuum energy density of space-time. Its value can be positive, negative or vanishing and each choice leads to a different maximally symmetric vacuum solution of the dynamical Einstein's equations: de Sitter (dS), Anti de Sitter (AdS) and Minkowski spaces possessing positive, negative and zero curvatures respectively. For simplicity we will consider in this chapter $\Lambda = 0$, whereas in Chapter 6 we will consider the case of $\Lambda < 0$ [3].

When Einstein first found the dynamical equations (1.1.2), he was convinced it would probably take many decades before someone would find a non-trivial analytical solution. However, just few months later, Schwarzschild (and independently Droste) [1, 2] found a solution in the vacuum ($\mathcal{L}_M = 0$). The starting point was to impose spherical and time-reversal symmetry, and time-independence of the metric ansatz. Furthermore, he also required the solution to degenerate asymptotically (in the limit $r \to \infty$) into the flat Minkowski metric $\eta_{\mu\nu} = (-1, 1, 1, 1)$ The line element version of the solution for the metric $g_{\mu\nu}$ reads:

$$\mathrm{d}s^2 = \Big(1-\frac{2M}{r}\Big)\mathrm{d}t^2 + \Big(1-\frac{2M}{r}\Big)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2+\sin\theta^2\mathrm{d}\phi^2) \ , \qquad (1.1.3)$$

[3] We refer to [13] for a pedagogical treatment of all these solutions and their importance in cosmology and many other branches of physics, such as the AdS/CFT correspondence.

where (t, r, θ, ϕ) are the coordinates parametrising the space-time. This is the celebrated *Schwarzschild solution* describing a black hole of mass M situated at the origin $r = 0$ of the space-time. At a first look, it seems this metric is singular at two different radial coordinates, $r = 0$ and $r = 2M$. However, a deeper analysis confirms that, while $r = 0$ is a real singularity of the space-time (the Ricci scalar curvature diverges there and so does the gravitational force), $r = 2M$ is a coordinate singularity, i.e. can be eliminated by choosing a different system of coordinates. This means that, locally, the horizon is not a problematic surface. It is nevertheless a defining global characteristic of the black hole solution: it corresponds to the radial coordinate of the *event-horizon*, the special surface from which no physical interior orbit can escape. It is indeed an horizon as anything that falls beyond it is not causally connected to the exterior region anymore and anything that approaches it from the exterior region will become infinitely red-shifted (being that light or a space-ship).
Other solutions of (1.1.2) can be obtained by considering slightly different theories or by imposing a different ansatz for the metric. For instance, consider a black hole of mass M containing electric and magnetic charges (Q, P) at the coordinates-origin $(r = 0)$. The presence of charges can be mathematically implemented by considering Maxwell theory as a non-trivial matter Lagrangian, $\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$. The resulting solution is the Reissner-Nordström metric [14, 15], describing a static, spherically symmetric black hole,

$$\mathrm{d}s^2 = \Big(1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}\Big)\mathrm{d}t^2 + \Big(1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}\Big)^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2) \ ,$$
$$F_{rt} = \frac{Q}{r^2} \ , \quad F_{\theta\phi} = P \sin\theta \ . \tag{1.1.4}$$

The solution is asymptotically flat as before, but now there are three values of r for which the metric diverges. Again, $r = 0$ is the only real singularity of the space-time, while $r = r_\pm$ with $r_\pm = M \pm \sqrt{M^2 - Q^2 - P^2}$ are coordinate singularities and physically represents the radial coordinates of the inner and outer horizons. Obviously, for the equality to make sense the mathematical constraint $\sqrt{Q^2 + P^2} \leq M$ is intended.[4] Observe that, in the limit case $M = \sqrt{Q^2 + P^2}$ the two horizons coalesce, $r_+ = r_-$ and the black hole is called *extremal.*
Extremal black holes are special in that their near-horizon geometry has enhanced symmetries, when compared to the full solution.[5] This is relatively easy to show for the extremal limit of the Reissner-Nordström solution. Define first the new

[4]As it turns out, the mathematical constraint $\sqrt{Q^2 + P^2} \leq M$ has deep physical implications connected to the cosmic censorship hypothesis. See [16] for a clear exposition.

[5]Supersymmetric black holes are all extremal and their near-horizon geometry will show an enhanced supersymmetry invariance.

coordinates

$$\tau = \lambda t / r_+^2 \ , \qquad \rho = \lambda^{-1}(r - r_+) \ ,$$

with λ arbitrary constant. In the new coordinate system and in the near-horizon limit $\lambda \to 0$ (keeping ρ fixed, the original coordinate r approaches r_+) the solution takes the form

$$\begin{aligned} \mathrm{d}s^2 &= r_+^2\big(-\rho^2 \mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{\rho^2}\big) + r_+^2(\mathrm{d}\theta^2 + \sin\theta^2 \mathrm{d}\phi^2) \ , \\ F_{\rho\tau} &= Q \ , \quad F_{\theta\phi} = P \sin\theta \ , \end{aligned} \tag{1.1.5}$$

which is also a proper solution of the Einstein-Maxwell equations. The resulting metric is the direct product of two 2-dimensional spaces. The second space remains unmodified after the change of coordinates: it is a 2-sphere S^2 labelled by (θ, ϕ), invariant under the symmetry group SO(3). The first space, labelled by (ρ, τ) is known as Anti de Sitter space AdS_2, and it possesses an $\mathrm{SO}(2,1)$ symmetry which was not present in the original (r, t) space in (1.1.4). This phenomenon of symmetry enhancement at the horizon of a black hole is related to the so-called *attractor mechanism* which restricts the field configurations to specific constant values proportional to the black hole charges. We will come back to this issue in Chapter 6.

The solutions presented so far were found by imposing a static, spherically symmetric, metric ansatz. More generally, a stationary and only axially-symmetric space-time ansatz for the metric can be considered: the solutions obtained will describe rotating black holes, referred to as Newmann-Kerr or just Kerr, depending on whether they possess electric (magnetic) charges or not. We will encounter solutions of this kind later in this thesis, so we will not go into more detail for now.

It is worth mentioning that, within Einstein's theory of gravity, one can prove the so-called uniqueness theorems [17–22] for stationary black hole solutions in absence of matter in their exterior region. The first theorem states that if a black hole is static, then it *must* be spherically symmetric and is described by the Schwarzschild (or Reissner-Nordström) solution. The second theorem states that if a black hole is stationary and axially symmetric, it will be described by the Kerr (Newmann-Kerr) solutions. These uniqueness theorems allow for a very simple description of all classical black holes and justify the famous statement by Wheeler, "black holes have no hair". At the same time, black holes are thermodynamic objects which possess an enormous entropy. In reality, then, the full description of a black hole is quite complicated and leads to many yet unsolved problems, e.g. the information paradox. We will not dwell on these issues here.

To conclude this section we want to introduce a quantity relevant in the study of black holes, called *surface gravity* κ_S, defined as the force necessary for an observer at infinity to hold in a fixed position on the event horizon a (unit) mass by means of an infinitely long, massless string. To calculate such force one equates the work done by the observer at infinity to slightly move the particle of a small proper distance δs to the (red-shifted) work done at the particle's position. For Schwarzschild and Reissner-Nordström solutions the surface gravity is given respectively by:

$$\kappa_S = \frac{1}{4M} \, , \qquad \kappa_S = \frac{r_+ - r_-}{2r_+^2} \, . \tag{1.1.6}$$

It is easy to note that for extremal (and, as a consequence, for all supersymmetric) black holes the surface gravity is identically zero. It will be immediately clear in the following the role this quantity plays in the thermodynamic aspects of a black hole.

1.1.2 Black hole mechanics *vs* the laws of thermodynamics

In this section we want to present the law of black hole mechanics [3]. Note that these laws were derived in the context of Einstein's general relativity coupled to classical matter but extension thereof for theories of gravity containing higher derivative effective couplings will be treated in the next chapter. We refrain from giving any details on the derivations of the laws, referring to [23] for a clear and exhaustive treatment. The laws of black hole mechanics can be stated as follows:

- 0^{th} law. The surface gravity κ_S of a stationary black hole is uniform over the entire event-horizon.

- 1^{st} law. It encodes the principle of energy conservation. Given a quasi-static process connecting two infinitesimally close black hole solutions, and indicating the change δM of the mass as a function of the changes in the angular momentum δJ, the area δA and the charges δQ, it is possible to prove the following identity:

$$\delta M = \frac{\kappa_S}{8\pi}\delta A + \omega \delta J + \mu \delta Q \, , \tag{1.1.7}$$

  where ω is the angular velocity of the rotating black hole and $\mu = Q/r_+$.

- 2^{nd} (area) law. Consider non-stationary processes in a space-time containing black holes which can collide and fuse together. Under some "reasonable" assumption on the time evolution of the system and the matter content of the theory, the sum of the horizon areas of all black holes never decreases, i.e. $\delta A_{\rm tot} \geq 0$,

- $3^{\rm rd}$ law. There are no physical processes that can reduce the surface gravity of a black hole to zero. This means that a non-extremal black hole can never become extremal in a finite amount of time.

A few relevant comments are in order. In the second law the area of the event horizon A can be identified with the thermodynamic entropy. Even though the semi-classical process by which black hole emit particles will decrease their mass and, as a consequence, their area, seemingly violating the area law, it is important to realize that such violations can happen only locally and the total entropy of an isolated system (in this case the full system black hole plus radiation emitted by it) never decreases. The identification area-entropy leads to the identification between the surface gravity and the temperature of a black hole, as can be easily seen from, say, the $0^{\rm th}$ law. The first law can formally be compared to the first law of thermodynamics for a grand-canonical ensemble where, for instance, μ and Q represent the black hole equivalent of chemical potential and particle number. This shows explicitly the connection between number of particles in a grand-canonical ensemble and black hole charges, which assumes a crucial importance when microscopic and macroscopic entropies are compared. In fact the thermodynamic limit $N \to \infty$ in statistical thermodynamics coincides with the limit of large charges in string theory, the theory explaining the microscopic structure of black holes. Finally, we want to comment shortly on the third law. Notice from (1.1.6) that the surface gravity, hence the temperature of a Schwarzschild black hole increases as its mass M becomes smaller through evaporation. The higher the temperature the bigger the amount of energy and particles radiated, leading to an even smaller mass, and so on. The final result of this process would presumably be complete evaporation from which one can conclude that Schwarzschild black holes are not quantum mechanically stable. An analogous phenomenon can not happen instead for charged (or rotating) black holes, which possess two horizons. In the extremal limit, the horizons coalesce and the black hole has a minimum non-zero mass corresponding to its charges (or angular momenta). For example, from (1.1.6) it is immediate to realize that the extremal limit of a Reissner-Nordström black hole corresponds to a zero temperature condition. Then, a charged (or rotating) black hole configuration possesses a stable ground state ($T = 0$) given by its extremal limit which, according to the $3^{\rm rd}$ law, can never be reached through any physical process in a finite amount of time.

1.2 Supersymmetry: a gauge theory example

Quantum particles are divided in two main classes, bosons and fermions, possessing respectively integer and half-integer spin. The spin-statistics theorem explains how their quantum behaviors differ, as the wave functions of a system of identical bosons or fermions are symmetric or anti-symmetric, respectively, under position exchange of two particles. As a consequence, an infinite amount of identical bosons can share a single quantum state whereas only one fermion can occupy each quantum state. Despite these radical differences, fermions and bosons can be intrinsically and dynamically connected by a special symmetry, *supersymmetry*, theoretically discovered in the early seventies [24–26], which transforms bosonic states $|B\rangle$ in fermionic states $|F\rangle$, and viceversa. If Q is the generator of supersymmetry transformation then schematically

$$Q\,|B\rangle = |F\rangle\ , \qquad Q\,|F\rangle = |B\rangle\ . \tag{1.2.1}$$

Supersymmetry then transforms the physical states of a theory by mixing their spin content. This imposes strong restrictions on the theory since the invariance under (1.2.1) forces the number of bosonic and fermionic degrees of freedom need to be equal. This leads to elegant structures such as supermultiplets and superfields which will be described in more detail shortly. Furthermore, by balancing the spin content of both equations (1.2.1), it is easy to realize that supersymmetry generators Q must be anti-commuting spinors: hence supersymmetry is a *fermionic* symmetry. A theory can also be invariant under many "copies" $\mathcal{N}$ of SUSY transformations. For $\mathcal{N} \geq 2$, we generically referred to as *extended* supersymmetry. As $\mathcal{N}$ grows, the field content must be widened to accommodate particles of increasing spins and, at the same time, the resulting theories are more constrained.

This general discussion can be put on more concrete grounds by illustrating a pedagogical example: the $\mathcal{N} = 2$ supersymmetric non-interacting abelian gauge theory in four dimensions, which is an extension of Maxwell theory invariant under $\mathcal{N} = 2$ global supersymmetries.

The field content of the theory comprises a complex scalar X, an abelian gauge field A_μ, a pair of Majorana fermions decomposed in chiral and anti-chiral components denoted respectively by Ω^i and Ω_i, and a triplet of scalars $Y_{ij} = Y_{ji}$ $(i, j = 1, 2)$ satisfying the reality constraint

$$Y^{ij} \equiv (Y_{ij})^* = \varepsilon^{ik}\varepsilon^{jl}\,Y_{kl}\ .$$

The Majorana constraint also implies that $\bar{\Omega}^i = \Omega^{i\,\mathrm{T}}\, C$, and likewise $\bar{\Omega}_i = {\Omega_i}^{\mathrm{T}}\, C$, where C is the charge conjugation matrix and $\bar{\Omega}^i$ and $\bar{\Omega}_i$ are the standard Dirac conjugates of Ω_i and Ω^i, respectively[6].
The invariant Lagrangian for this theory takes the following form,

$$\mathcal{L} = -\,2\,\partial_\mu X\,\partial^\mu \bar{X} - \tfrac{1}{2}\bar{\Omega}^i\,\overset{\leftrightarrow}{\not{\partial}}\,\Omega_i - \tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{4}\,Y_{ij}\,Y^{ij}\,, \tag{1.2.2}$$

which is simply a sum of kinetic terms for all fields except for the triplet Y_{ij}. Therefore, the latter fields are not describing any dynamical degrees of freedom, since their equations of motion are algebraic and imply that these scalars must vanish for any solution. For this reason the Y_{ij} are often called *auxiliary fields.*
The supersymmetry transformations rules which leave the above Lagrangian invariant, up to a total derivative term, read [7]

$$\begin{aligned}
\delta X =&\, \bar{\epsilon}^i\Omega_i\,,\\
\delta\Omega_i =&\, 2\not{\partial}X\epsilon_i + \tfrac{1}{2}\varepsilon_{ij}F_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j\,,\\
\delta W_\mu =&\, \varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j + \varepsilon_{ij}\bar{\epsilon}^i\gamma_\mu\Omega^j\,,\\
\delta F_{\mu\nu} =&\, -\,2\,\varepsilon^{ij}\bar{\epsilon}_i\gamma_{[\mu}\partial_{\nu]}\Omega_j - 2\,\varepsilon_{ij}\bar{\epsilon}^i\gamma_{[\mu}\partial_{\nu]}\Omega^j\,,\\
\delta Y_{ij} =&\, 2\,\bar{\epsilon}_{(i}\not{\partial}\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\not{\partial}\Omega^{l)}\,,
\end{aligned} \tag{1.2.3}$$

where ϵ^i and ϵ_i are the chiral components of a Majorana doublet of constant spinor parameters. These parameters satisfy the chirality conditions $\gamma_5\epsilon^i = \epsilon^i$ and $\gamma_5\epsilon_i = -\epsilon_i$. We may therefore conclude that the transformations (1.2.3) define a symmetry of the theory for any constant (anti-commuting) parameters ϵ^i. Since this symmetry is parametrised by two independent Majorana spinors, it is known as $\mathcal{N} = 2$ supersymmetry.
Note that the transformation rule for the auxiliary fields Y_{ij} is proportional to the equations of motion for the fermion field Ω_i. This simple observation has very deep consequences, especially in view of our treatment of higher derivative couplings: in fact, by using the equation of motion for the auxiliary field, i.e. $Y_{ij} = 0$, the Lagrangian (1.2.2) is invariant under a modified version of the transformation rules (1.2.3), in which more terms, proportional to the equation of motion of the fermion fields, have to be included only in the SUSY variation $\delta F_{\mu\nu}$. As a consequence, the modified transformation rules do not possess the same symmetric structure of (1.2.3) and many complications arise. We will come back to this issue later on in this chapter.

[6] We refer to Appendix A for notation and conventions on gamma matrices and fermion fields.
[7] The Lagrangian (1.2.2) is also invariant under a rigid $\mathrm{U}(2) \simeq \mathrm{SU}(2)\times\mathrm{U}(1)$ group, which is this case, corresponds to the R-symmetry group. See section 1.2.2 for a rigorous definition.

For now, we want to exploit the above example of $\mathcal{N} = 2$ Maxwell theory to analyze the general field structure of supersymmetric theories, and introduce the notion of supermultiplet and superfield. These structures arise as representations of the supersymmetry algebra, presented in section 1.2.2.

1.2.1 Supermultiplets, superspace and superfields

The pure Maxwell theory depends on one dynamical field, the gauge connection A_μ. In the example above, supersymmetry invariance required the inclusion of other fermionic and bosonic fields, connected through the SUSY transformations. More generally, supersymmetry strongly constrains a theory in that it fixes all the relative coefficients and couplings. Also, as we already noted below equation (1.2.1), the number of bosonic and fermionic degrees of freedom of a supersymmetric theory must be balanced, since each bosonic (fermionic) field is transformed in its fermionic (bosonic) superpartner. It is then natural to group all the fields connected among each other by supersymmetry within a *supermultiplet*. Supersymmetric Lagrangians can always be written as functions of supermultiplets: for instance, the $\mathcal{N} = 2$ Maxwell theory analyzed in the previous section depends only on the *vector multiplet* with components $(X, \Omega_i, A_\mu, Y_{ij})$.
By definition then, all supermultiplets describe an equal number of bosonic and fermionic degrees of freedom and this property straightforwardly extends to all supersymmetric Lagrangians, which are functions of supermultiplets. Moreover, the equality holds both on-shell, where only the physical degrees of freedom obtained after the use of the dynamical equations are used, and off-shell, where no use of the dynamical equations is made. For instance, a simple off-shell counting for the $\mathcal{N} = 2$ vector multiplet in four space-time dimensions used above gives in fact eight fermionic degrees of freedom (2 degrees of freedom for each value of i in Ω_i and Ω^i), three degrees of freedom from the gauge field W_μ (not four, as one is absent because of gauge invariance), two degrees of freedom from the complex scalar X, and finally three degrees of freedom from the triplet Y_{ij}. The on-shell counting gives instead four fermionic degrees of freedom (four are constrained by the Dirac equation), four bosonic ones associated with the (massless) gauge field and the complex scalar field X. The auxiliary fields Y_{ij} are omitted from the on-shell counting since their dynamical equations are $Y_{ij} = 0$.
Supermultiplets in space-time can be expressed in a more elegant and compact way in terms of *superfields* in *superspace* ([27–29], see also [30, 31] for an exhaustive treatment of the $\mathcal{N} = 1$ case). Superspace is an extension of space-time. Each point in superspace is labelled by the usual (bosonic) coordinates x^μ and some

additional fermionic coordinates $\theta, \bar{\theta}$, which are linked by the following schematic supersymmetry transformations:

$$\begin{aligned} \delta x^\mu &= \bar{\theta}\gamma^\mu \epsilon \ , \\ \delta\theta &= \epsilon \ , \\ \delta\bar{\theta} &= \bar{\epsilon} \ . \end{aligned} \tag{1.2.4}$$

A supersymmetry transformations hence acts as a rigid translation operator for the fermionic coordinates θ. We will see shortly how this feature is generalized to fields in space-time.

In the case of the $\mathcal{N}=2$ SUSY Maxwell theory in $D=4$ illustrated before, each point of flat superspace is labelled by its space-time coordinates and eight total Grassmann variables θ^i_α and $\bar{\theta}_{i\dot{\alpha}}$, which are the chiral and anti-chiral components of constant Majorana spinors (we refer to Appendix A for the details on the chiral and spinor notation used in this work). The index $i=1,2$ is connected to the number of supersymmetries $\mathcal{N}$ while $\alpha=1,2$ and $\dot{\alpha}=\dot{1},\dot{2}$ count the chiral and anti-chiral fermionic components, which satisfy the chirality conditions $(1-\gamma_5)\theta^i_\alpha = (1+\gamma_5)\bar{\theta}_{i\,\dot{\alpha}} = 0$ It is important to remark that the concept of functions in superspace and the linked notions of differentiation and integration can be rigorously defined (see [31, 32] for a simple treatment). This allows to defined a superfield $\Phi(x^\mu, \theta, \bar{\theta})$ as a function of the superspace coordinates which can be Taylor expanded in terms of the θ coordinates. Such expansion is always finite, given the anti-commuting properties of the Grassmann variables, and is expressed in terms of x^μ-dependent functions corresponding to the fields $\phi^i(x)$ contained within a supermultiplet. This means that there always exists a $1:1$ correspondence between supermultiplets and superfields. To give an example, a generic complex scalar superfield in $\mathcal{N}=2$ superspace contains $2\cdot 2^{4\mathcal{N}} = 2^8+2^8$ field components, as it is easy to realize by considering all the possible non-zero combinations of Grassmann variables in the expansion. From a scalar superfield it is possible to define a (anti-)chiral superfield, by requiring no θ coordinates of (positive) negative chirality to appear in the expansion. This constraint reduces the number of fermionic plus bosonic degrees of freedom to $16+16$. The resulting chiral superfield reads [8]

$$\begin{aligned} \Phi(x^\mu, \theta^i, \bar{\theta}^i) =& A(y) + \bar{\theta}^{i\,\alpha}\psi_{i\,\alpha}(y) + \tfrac{1}{2}\bar{\theta}^{i\,\alpha}\theta^j_\alpha B_{ij}(y) + \tfrac{1}{2}\varepsilon_{ij}\bar{\theta}^{i\,\alpha}(\gamma^{ab})_\alpha{}^\beta\theta^j_\beta F^-_{ab}(y) \\ &+ \tfrac{1}{3}\left[\varepsilon_{ij}\bar{\theta}^{i\,\alpha}(\gamma^{ab})_\alpha{}^\beta\,\theta^j_\beta\right]\bar{\theta}^{k\,\delta}(\gamma^{ab})_\delta{}^\epsilon\Lambda_{k\,\epsilon}(y) + \tfrac{1}{12}\left[\varepsilon_{ij}\bar{\theta}^{i\,\alpha}(\gamma^{ab})_\alpha{}^\beta\,\theta^j_\beta\right]^2 C(y) \ . \end{aligned} \tag{1.2.5}$$

[8]The fields contained in the superfield depend on a new complex coordinate, $y^\mu = x^\mu + \bar{\theta}^i\gamma^\mu\theta_i$, whose presence will be justified in chapter 3. For now, we want to observe that, although the chiral superfield depends explicitly only on positive chirality Grassmann coordinate, the fields that compose it still depend implicitly on θ_i.

The fields A and C are complex scalars, while B_{ij} and F^-_{ab} are a complex triplet under SU(2) and a complex anti-selfdual tensor respectively. The fermionic fields ψ_i and Λ_i are instead positive chirality components of Majorana spinors. A chiral multiplet, then, can be schematically written as $(A, \psi_i, B_{ij}, F^-_{ab}, \Lambda_i, C)$. Note now that for a $\mathcal{N} = 2$ SUSY theory in four dimensions, a scalar chiral superfield cannot have more components, because any product of more than four positive (or negative, for an anti-chiral multiplet) chirality θ's will vanish. On the other hand, a chiral multiplet can be further reduced by imposing more restrictive conditions on the field components. For instance, a vector multiplet is obtained from a chiral multiplet via the following constraints:

$$\begin{aligned} B_{ij} &= \varepsilon_{ik}\varepsilon_{jl}B^{kl} \ , \\ \Lambda_{i\,\alpha} &= -\varepsilon_{ij}\partial\!\!\!/\psi^j_\alpha \ , \\ C &= -2\partial^2\bar{A} \ , \\ \partial_a F^{-\,ab} &= \partial_a F^{+\,ab} \ , \end{aligned} \tag{1.2.6}$$

and using the identifications $A \to X$, $\psi_i \to \Omega_i$ $B_{ij} \to Y_{ij}$. The first equation is just a reality constraint on B_{ij} while the last represents a Bianchi identity so that F^-_{ab} is identified with the anti-selfdual component of a physical field strength $F_{ab} = F^+_{ab} + F^-_{ab} = 2\,\partial_{[a}W_{b]}$. The constraints (1.2.6) reduce the number of off-shell degrees of freedom to $8 + 8$.
This concludes, for the time being, our treatment of elementary multiplet in flat space-time (for generalizations to curved space-time, we refer to Chapter 3). In the next section we will explain how supermultiplets arise as representations of the supersymmetry algebra.

1.2.2 The supersymmetry algebra

In physics the concept of symmetry plays a crucial role. A symmetry is an invariance of a certain system under a group of transformations. In many practical applications, and throughout this whole thesis, one is interested in continuous group of transformations, *Lie groups*, which can be defined *locally* in terms of its generators T^i. The number of generators corresponds to the dimension of the group G itself. An infinitesimal transformation g around the identity is given by $g \simeq 1 + \xi_i T^i_R$, for some continuous infinitesimal parameters ξ_i. Call now $g_{1,2}$ two distinct infinitesimal transformations. The composite transformation $\delta_1\delta_2 - \delta_2\delta_1$

will still be an infinitesimal transformation belonging to the group G. This statement can be expressed formally through the following generators relation,

$$[T^i, T^j] = f^{ij}{}_k T^k \ , \tag{1.2.7}$$

where $f^{ij}{}_k$ are called the *structure constants* of the group and define completely its local properties. The equation above represents the Lie algebra of the group G, and holds for all group generators [9].

Relativistic field theories in four space-time dimensions must be invariant under the Poincaré group, composed by a four-dimensional rigid translation group, whose generators are the 4-momentum operators P_μ, and the six-dimensional Lorentz group, whose generators are $M_{\mu\nu} = -M_{\nu\mu}$. Each of these generators is associated to a conserved quantity: for instance P_0 is the generator of time translation and it is related to a conserved quantity, the Hamiltonian of the system. Analogously P_i generate spatial translations and are related to the conserved momentum (for further details on the connection between generators of a symmetry and conserved quantities we refer to Chapter 2). The Poincaré algebra, on a flat Minkowski background, reads

$$[P_\mu, P_\nu] = 0 \ ; \qquad [P_\rho, M_{\mu\nu}] = 2\, P_{[\mu}\, \eta_{\nu]\rho} \ ;$$
$$[M_{\mu\nu}, M_{\rho\sigma}] = 2\, \eta_{\rho[\nu} M_{\mu]\sigma} - 2\, \eta_{\sigma[\nu} M_{\mu]\rho} \ . \tag{1.2.8}$$

Typically, relativistic field theories are also invariant under continuous groups of internal symmetries, called gauge symmetries, which mix fields into each other and express redundancies in the mathematical formulation of the physical system. As it turns out, by studying the (bosonic) symmetries of the S-matrix in a relativistic field theory with a non-zero mass-gap (i.e. the mass of the lightest field excitation, for at least one field, is non-zero), Coleman and Mandula showed [34] that it is impossible to mix gauge ($\mathcal{I}$) and Poincaré ($\mathcal{P}$) symmetries and, as a consequence, the maximal group of (bosonic) symmetries of an S-matrix is $\mathcal{I} \otimes \mathcal{P}$. Shortly after, a new kind of symmetry, supersymmetry, generated by fermionic operators satisfying anti-commuting relations was discovered [25, 26, 35]. This led to an extension of the Coleman-Mandula theorem, proved by Haag-Lopuszański-Sohnius (HLS) [36]: the maximal group of symmetries of an S-matrix can include also different copies of supersymmetry transformations generated only by spin 1/2 fermionic operators Q [10]. The resulting algebra is called the *super-Poincaré algebra* and for the $\mathcal{N} = 2$

[9] We refer to [33] for further details on Lie groups, Lie algebras and representation theory.

[10] In general, supersymmetry transformations can be parametrized by spinors of higher half-integer spin but, when projected onto the physical Hilbert space, only the generators of spin 1/2 will survive.

case it reads [11]

$$\{Q_{i\,\alpha}, \bar{Q}^j_{\dot\beta}\} = -2\delta^j_i(\gamma^a)_{\alpha\dot\beta}P_a \ , \qquad \{Q_{i\,\alpha}, Q_{j\,\beta}\} = 2\,\epsilon_{\alpha\beta}\,\varepsilon_{ij}\,g(\Lambda) \ ,$$
$$[M_{ab}, \bar{Q}^i_{\dot\alpha}] = -\tfrac{1}{2}\,\bar{Q}^i_{\dot\beta}\,(\gamma_{ab})^{\dot\beta}{}_{\dot\alpha} \ , \qquad [P_a, \bar{Q}^i_{\dot\alpha}] = -\tfrac{1}{2}\,\bar{Q}^i_{\dot\beta}\,(\gamma_a)^{\dot\beta}{}_{\dot\alpha} \ . \qquad (1.2.9)$$

The supersymmetry generators Q, also called *supercharges*, are the quantities conserved under a supersymmetry transformation and their number depends on the space-time dimension D of the theory and $\mathcal{N}$. In fact, a theory preserving $\mathcal{N}$ supersymmetries contains $\mathcal{N}$ irreducible Lorentz spinors whose number of independent components depends on the spinorial representations of the Lorentz algebra available in D dimensions. The number of conserved supercharges is then given by the number of independent spinor components times the number $\mathcal{N}$ of preserved supersymmetries. For the $\mathcal{N} = 2$ $D = 4$ case at hand the two supersymmetries are parametrized by two Majorana spinors, irreducible representation of the Lorentz algebra, which have four independent components each, so that the number of conserved supercharges is 8. Also, once the spinorial representation for the supercharges is chosen, there exists a group of transformations which commutes with the Lorentz group and rotates the supercharges, leaving the supersymmetry algebra invariant. The largest such group is referred to as the *R-symmetry group*. This group can also be realized as a manifest invariance of the theory, as it was the case for the supersymmetric gauge theory analyzed in section 1.2, and when that happens, it becomes part of the supersymmetry algebra.
Note that the anti-commutator relations (1.2.9) between supercharges of opposite chirality closes on a space-time translation P_a, just as a supersymmetry transformation of the superspace Grassmann coordinate causes a fermionic shift. On the other hand, two supersymmetry transformations of the same chirality close on a (field dependent) gauge transformation $g(\Lambda)$ of fixed parameter Λ.
To obtain important insights on the physical interpretations of both operators P_a and $g(\Lambda)$, consider the action of the algebra of infinitesimal supersymmetry variations δ_Q on the vector multiplet fields $(X, \Omega_i, A_\mu, Y_{ij})$. This choice simplifies the explicit calculations because the variations (1.2.3) include supersymmetry transformations of both chiralities, so that the anti-commutators in (1.2.9) are described by one unique *commutator* (the SUSY parameters ϵ_i, ϵ^i are also anti-commuting). Two supersymmetry transformations with rigid parameters ϵ_{1i} and ϵ_{2i}, acting on the scalar field X, yield a rigid translation,

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]X = -2\,\xi^\mu\partial_\mu X \ . \qquad (1.2.10)$$

[11] We are voluntarily neglecting the *central charge* operator $\mathcal{Z}_{ij}$ in the algebra since throughout this work it can effectively be considered vanishing. For further details on the central charges topic, we refer to [37]

with constant parameter

$$\xi^\mu = \bar\epsilon_1^i \gamma^\mu \epsilon_{2i} - \bar\epsilon_2^i \gamma^\mu \epsilon_{1i} \ . \tag{1.2.11}$$

An analogous relation can be proven for the auxiliary field Y_{ij}. Particular attention must instead be paid when considering the action of the algebra on the fields Ω_i and W_μ. Let us start with the gauge field W_μ, for which the application of two consecutive SUSY transformations gives (use (1.2.3) and (A.4))

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]W_\mu = -2\,\xi^\nu F_{\nu\mu} + 4\,\varepsilon^{ij}\bar\epsilon_{1i}\epsilon_{2j}\partial_\mu X \ . \tag{1.2.12}$$

Note the final expression contains two field dependent gauge transformations, which can be justified on the basis of the invariance under a local U(1) transformation we tacitly assumed when we considered the physical gauge connection W_μ. The first term, with gauge parameter $\lambda = \xi^\nu W_\nu$, adds to the rigid translation to give a *gauge covariant translation.* [12] This is the correct interpretation for the operator P_a, which acts in a gauge covariant way on the fields. The second term is a gauge transformation with field dependent parameter $\Lambda = 4\,\varepsilon^{ij}\bar\epsilon_{1i}\epsilon_{2j}X$ which derives from the second relation of (1.2.9).

Since the other fields of the theory, including Ω_i, are neutral under this local U(1) transformation, the algebra closes on them yielding solely a rigid translations. However, the situation for the fermion fields changes radically when the auxiliary fields Y_{ij} are eliminated from the action and the transformation rules, through their equations of motion $Y_{ij} = 0$. In this case, the algebra on Ω_i reads,

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)]\Omega_i = -2\xi^\mu\partial_\mu\Omega_i + \left(\tfrac{1}{2}\,\bar\epsilon_1^j\gamma^a\epsilon_{2i}\,\gamma_a\partial\!\!\!/\Omega_j + \tfrac{1}{2}\,\bar\epsilon_{1j}\gamma^a\epsilon_2^j\,\gamma_a\partial\!\!\!/\Omega_i + \cdots - (1 \leftrightarrow 2)\right) , \tag{1.2.13}$$

where all the new terms, even the ones that for simplicity were left out and indicated by ... are proportional to the field equations for Ω_i. This means that, without the auxiliary fields, the supersymmetry algebra closes on translations only on-shell. Of course, for the simple example of free theory at hand, the presence of terms proportional to the equation of motion in the algebra does not constitute a tremendous obstacle but, when dealing with interactive theories containing higher derivative interactions, it is of utmost importance to consider, whenever possible, an off-shell formulation. Without auxiliary fields, the algebra and the transformation rules will depend on the dynamical equations for the fields, hence on the specific theory under consideration. The addition of auxiliary fields instead assures an off-shell description in which the algebra and the transformation rules

[12] The standard Lie algebra is generalized to a so-called *soft algebra*. The main feature of such algebras is that the structure constants are not constant anymore but are field dependent functions. For a complete exposition on soft algebras, we refer the reader to [37].

are fixed independently from the specific theory considered.
We want to close this section by making an important remark: it is evident from (1.2.9) that, if the supersymmetry transformation is made local, or gauged, then so will be the resulting translations. This means that a theory of gauged supersymmetry gives rise to a theory invariant under general coordinate transformations, i.e. a theory of gravity called *supergravity*. From Chapter 3 on, we will only consider theories of this kind.

1.3 Effective actions and higher derivative couplings

In this section we want to explain some general feature of effective field theories to give some insight on the connection between string theory and supergravity theories, which is at the center of this work.
The idea of effective theory is implicitly behind any physical theory. It is a known fact that natural phenomena happen at very different distance scales and it is often convenient to select just the sectors of a theory relevant to predict physical results up to a certain distance or energy scale (cutoff). To make this statement clearer, it is sufficient to realize that the description, say, of the classic kinematic motion of a macroscopic body does not depend on the details of its atomic structure. Classical mechanics then effectively describes the laws of quantum mechanics, for distance scales much larger than the atomic scale. A similar reasoning was already used in the introduction where we presented thermodynamics as the averaged description of a statistical microscopic theory.
More generally, the low energy, long distance dynamics decouples from the high energy, small distance dynamics. Consequently, one can describe low energy physics by considering an effective theory that depends explicitly only on the low energy degrees of freedom, without specifying the details of the high energy sector. This will of course render the calculations simpler, since only the dynamics relevant below a certain cutoff will be taken into account. At the same time, probing the high energy sector is still possible through the analysis of effective interaction terms in the action. To explicitly show how these interactions arise, we treat in the following the simple example of an effective theory that goes back to Euler and Heisenberg. Then we will present the general setting, based on operator expansions, used to describe effective theories.
For detailed introductions to effective theories we refer to the review articles [38–41].

1.3.1 A toy model: the Euler-Heisenberg Lagrangian

Quantum Electro-Dynamics (QED) is a theory describing the interaction between fermion fields, such as the electron, and a bosonic gauge field, the photon. The physical predictions of this theory are, in principle, valid at any energy scale because the theory is renormalizable [13]. But for now, we are interested in deriving the scattering amplitude for a light-light scattering process in QED, at energy scales E much lower than the electron mass m_e. Practically speaking, this means that the relevant ultraviolet (UV) cut-off of the theory is $\Lambda = m_e$. The process $\gamma\gamma \to \gamma\gamma$ receives contributions at leading order from the box diagram (a) in Figure 1.1. For $E << \Lambda$, one can effectively substitute each electron propagator with a factor of $1/m_e$, and obtain the four-photon vertex (b) in Figure 1.1. The effective description of this process was first derived in [42]. The resulting Euler-Heisenberg Lagrangian contains all the terms allowed by gauge, Lorentz, Charge conjugation and parity symmetries [14] and reads,

$$\mathcal{L}_{\text{eff}} = -\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + a(F_{\mu\nu}F^{\mu\nu})^2 + bF_{\mu\nu}F^{\nu\rho}F_{\rho\sigma}F^{\sigma\mu} + \mathcal{O}(F^6)\ . \tag{1.3.1}$$

The theory contains now only degrees of freedom with energy (momentum) below the UV cut-off scale Λ and the effects of the electron fields are encoded in the higher derivative photon self-interactions. The final result for the scattering process can be easily calculated: each field strength will give a factor of ω, the energy of the photon field, so the amplitude of the process $\gamma\gamma \to \gamma\gamma$ will have a quartic dependence on ω, as expected. Furthermore, from dimensional analysis, we can also conclude that $a \sim b \sim 1/m_e^4$ while the terms containing six field strengths are suppressed by higher powers of the cutoff m_e. The exact value for the coefficients can be obtained by explicitly evaluating the loop diagram (a) in Figure 1.1.

This example, although quite simple, clarifies the procedure used to practically

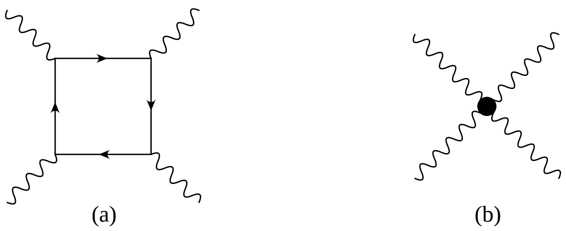


Figure 1.1: (a) four-photon scattering in QED; (b) effective four-photon scattering in Euler-Heisenberg theory (1.3.1).

[13] We know now that the QED is an effective description of the Standard Model valid for energy scales below the mass of the $W^{\pm}$ and Z bosons.

[14] It is assumed that the electromagnetic field is uniform, so terms of the form $\partial_\mu F_{\nu\rho}\partial^\mu F^{\nu\rho}$ do not appear in the expression (1.3.1).

incorporate quantum corrections into effective theories: first consider only the one-particle-irreducible quantum diagrams with the external particles having energy (momentum) below the UV cut-off of the effective theory. Then write down the correspondent effective couplings, which would typically contain higher derivative terms. Finally, since the two theories are characterized by the same infrared (low-energy) behavior, via a matching procedure one can give a specific value to the couplings which mock up the effects of the high energy modes in the low-energy theory.

1.3.2 Wilsonian action: top-down and bottom-up approaches

In this section we want to make formal the intuitive procedure used to construct the Euler-Heisenberg Lagrangian. Let us start from a generic quantum field theory with energy cutoff Λ. We are interested in deriving the effective theory for energy $E << \Lambda$, which has the same infrared properties of the fundamental theory. First one has to divide the field content, among heavy and light (compared to the energy scale E) fields, ϕ_H and ϕ_L respectively. As it is customary, consider then the path integral formulation of the theory, from which information about the quantum nature of interactions and scattering processes can be derived. If the theory is described by the action $S(\phi_L, \phi_H)$, then the path integral reads,

$$\int \mathcal{D}\phi_L \mathcal{D}\phi_H e^{i\,S(\phi_L,\phi_H)} = \int \mathcal{D}\phi_L e^{i\,S_\Lambda(\phi_L)} \, . \qquad (1.3.2)$$

In the right hand side of the equality, the degrees of freedom of the heavy particles of the theory have been integrated out and the information about them is now contained inside the *Wilsonian* low energy local [15] effective action S_Λ in the form of couplings and symmetries. This procedure, although formally solid, is however very cumbersome in practice. In the example of the previous section, for instance, we considered only two couplings in the Lagrangian for the photon field to effectively encode the diagram (a) in Figure 1.1 but, in general, an infinite number of these diagrams can be considered to which it will correspond an infinite number of couplings in the effective action. Furthermore, up until now we have implicitly assumed that the underlying high energy theory is known and understood. The use of an effective theory below a certain UV cut-off is then convenient because

[15] Note that locality is embedded in the theory since we are only integrating out the high energy degrees of freedom and high momenta. In the QED example of the previous section, for instance, tha action (1.3.1) is local because $m_e > 0$, but taking the limit $m_e \to 0$ would lead to a non-local effective action. The appearance of non-local terms is often an issue when dealing with integration over massless modes.

it singles out a specific low energy sector of the theory and, through a matching procedure, fixes the explicit values for the couplings. This approach is called *top-down*.
On the other hand, in many practical applications one does not know the specifics of the fundamental theory or the matching procedure can be too hard to be explicitly worked out. In this case, the effective theory can be constructed by simply writing down all the interactions allowed on the base of symmetries, which are in principle again an infinite number. This approach is called *bottom-up* and it is used, among many other examples, in theories of supergravity of interest in this thesis. For a bottom-up approach to effective theories, the identity (1.3.2) should be read from right to left: based on symmetry arguments one proposes a consistent ansatz for the effective action to extrapolate information about the unknown, or not well-understood, high energy theory.
Obviously, in both approaches to effective theories it is important to distinguish, among the infinite number of possible interactions, the few that are relevant to describe a physical processes below a certain scale and up to a certain level of accuracy. This can be done by writing the effective action as a power expansion in terms of local operators of the form (in natural units $[S] = [\hbar] = 1$),

$$S_\Lambda = \int d^D x \sum_k c_k \mathcal{O}_k \ .$$

If an operator $\mathcal{O}_k$ has (energy) dimension α_k then $[c_k] = [E^{D-\alpha_k}] = D - \alpha_k$. A simple example of one such operator could be the mass term in a $D = 4$ scalar quantum field theory, $m^2\phi^2$, where $[m] = 1$. The important message is that the dimension of an operator determines univocally its behavior. Specifically, an operator $\mathcal{O}_k$ for which $\alpha_k > D$ is called *irrelevant*, because it will be suppressed by powers of $1/\Lambda^{\alpha_k - D}$. Contrary, if $\alpha_k < D$, then the operator is called *relevant*, because it becomes more important at low energies. Finally, when $\alpha_k = D$, then $\mathcal{O}_k$ is a *marginal* operator: they have equal importance at all energy scales. Often, there is only a finite number of relevant or marginal operators, so that the effective theory can be expressed in terms of a finite number of parameters. To be more precise, the low energy effective theory describes the high energy physics through relevant and marginal couplings, while irrelevant couplings give information about small corrections, the higher the dimension the smaller the correction.

In the context of interest for this work, supergravity theories describe the same infra-red physical phenomena of superstring theories since they contain the same low energy degrees of freedom. In order to obtain results that are valid at higher (stringy) energies and include quantum corrections, the contributions coming from irrelevant, non-renormalizable, higher derivative couplings in supergravity must be included and analyzed. This is the main topic of this work.

Chapter 2

The covariant phase space and the Noether potential

Effective actions describe the full quantum-mechanical dynamics below some given mass scale, and are obtained by integrating over all the modes of the quantum theory above this mass scale. This explains why one must effectively deal with higher derivative couplings, as it was shown in section 1.3. This is the point of view taken in this thesis which applies also to general relativity where attention cannot be restricted to just the Einstein-Hilbert action. Of course, many subtleties arise when defining the Wilsonian action, especially when dealing with gauge theories and general relativity, but we will not dwell on those issues. The crucial point is that quantum corrections induce new couplings in the effective action, which can nevertheless treated "classically". No quantization is needed. Hence an efficient procedure to deal with all the new couplings and maintain explicit covariance becomes necessary. The covariant phase space approach offers all these advantages and leads to a structured and elegant description of symmetries and conserved quantities which are central to the study of higher derivative effective theories[1]. It is worth mentioning at this point that the treatment of higher derivatives has a long history. For classical mechanics, this goes back to the work of Ostrogradski (see, e.g. [44]), who derived the Hamiltonian formulation for Lagrangians with higher time derivatives. In the context of field theory an early discussion of higher derivative interactions was given by Pais and Uhlenbeck [45]. The realization that Lagrangians with higher derivatives have a Hamiltonian formulation with a corresponding phase space description is a relevant one, and it plays an important role in works that were carried out much later. We will return to this aspect, and

[1]The material treated in this chapter is part of a forthcoming review article [43].

to the connection between covariant and canonical (Hamiltonian) description in section 2.4.

Let us start the analysis from the Euler-Lagrange equations to be satisfied by functions in order that certain functionals of these functions are stationary. These equations hold for functions $\phi(x_1, x_2, \ldots, x_n)$ of some independent variables $(x_1, x_2, \ldots, x_n)$, and a functional of this function, where the generalization to cases with many functions is straightforward. In the context of this chapter the functions are *fields* generically denoted by $\phi^i(x)$, where x^μ are coordinates of a d-dimensional space-time and the functional is the action integral,

$$S[\phi(x)] = \int \mathrm{d}^d x\ \mathcal{L}(\phi, \partial_\mu \phi, \partial_\mu \partial_\nu \phi, \ldots)\,. \tag{2.0.1}$$

Here $\mathcal{L}$ is the Lagrangian density, which may depend on the fields and arbitrary (multiple) space-time derivatives thereof. Obviously point-particle dynamics can be handled similarly. The following general identity plays an important role in the derivation of Euler-Lagrange equations as well as of Noether's theorems. It decomposes the change of the Lagrangian induced by arbitrary variations of the fields, $\phi^i \to \phi^i + \delta\phi^i$, into two terms (we always sum over repeated indices unless specified differently),

$$\delta\mathcal{L} = E_i(\phi)\,\delta\phi^i + \partial_\mu \Theta^\mu(\phi, \delta\phi)\,, \tag{2.0.2}$$

where we stress that E_i and Θ^μ both depend on the fields and derivatives thereof, whereas Θ^μ is also linearly proportional to the variations $\delta\phi^i(x)$ and their derivatives. This is an identity which always holds; terms involving derivatives of $\phi^i(x)$ in $\mathcal{L}$ will give terms proportional to the corresponding derivatives of $\delta\phi^i(x)$, which can be written as terms proportional to $\delta\phi^i(x)$ plus total derivatives. We further stress that at this stage the variations $\delta\phi^i$ are not subject to any restriction, and they may depend on the fields ϕ^i or derivatives thereof, as well as on explicit space-time coordinates. We will be dealing with local Lagrangians, so that E_i and Θ^μ will be local functions of ϕ^i and their derivatives, although in principle this is not essential.

It is straightforward to find explicit expressions for the quantities E_i and Θ^μ that appear in this equation,

$$
\begin{aligned}
E_i(\phi) &= \frac{\partial \mathcal{L}}{\partial \phi^i} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} \right) + \partial_\mu \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi^i)} \right) - \cdots \,, \\
\Theta^\mu(\phi, \delta\phi) &= \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^i)} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu \phi^i)} \right) + \cdots \right] \delta\phi^i \\
&\quad + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\nu \partial_\mu \phi^i)} - \partial_\rho \left(\frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu \partial_\mu \phi^i)} \right) + \cdots \right] \partial_\nu \delta\phi^i \\
&\quad + \left[\frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\nu \partial_\mu \phi^i)} - \partial_\sigma \left(\frac{\partial \mathcal{L}}{\partial(\partial_\sigma \partial_\rho \partial_\nu \partial_\mu \phi^i)} \right) + \cdots \right] \partial_\rho \partial_\nu \delta\phi^i \\
&\quad + \cdots \,.
\end{aligned} \tag{2.0.3}
$$

Here the ‘current’ Θ^μ is linearly proportional to $\delta\phi^i$ and derivatives thereof. As we shall see, it is conceptually different from the *Noether current*, to be introduced in section 2.2, because it is based on arbitrary field variations, rather than on variations associated with invariances of the action and the equations of motion.

As it is well-known, the significance of $E_i(\phi)$ to the Euler-Lagrange variational problem is that $E_i(\phi) = 0$ are precisely the Euler-Lagrange equations to be satisfied by the fields $\phi^i(x)$ in order that the action (2.0.1) is stationary. To see this let us restrict the $\delta\phi^i$ to be such that they vanish at the boundary of integration in (2.0.1), but are otherwise arbitrary. Then the $\partial_\mu \Theta^\mu$ term does not contribute to $\delta S[\phi(x)]$ and one has,

$$
0 = \delta S[\phi(x)] = \int \mathrm{d}^d x \, E_i(\phi) \, \delta\phi^i \,. \tag{2.0.4}
$$

Since $\delta\phi^i$ is arbitrary in the bulk, this is possible only if the field equations $E_i(\phi) = 0$. The variational principle is known as Hamilton’s principle. It is easy to verify that the field equations will not change when one adds a total derivative $\partial_\mu v^\mu$ to the Lagrangian. Such an addition changes the action by a boundary term, so that only Θ^μ will be modified. More generally, Θ^μ has ambiguities of the following form,

$$
\Theta^\mu(\phi, \delta\phi) \to \Theta^\mu(\phi, \delta\phi) + \delta\big[v^\mu(\phi)\big] + \partial_\nu v^{\mu\nu} \,, \tag{2.0.5}
$$

where $v^{\mu\nu}$ is any anti-symmetric tensor. This last ambiguity is known as an improvement term: a term that is conserved owing to its special form without requiring any restrictions on the fields. Both these ambiguities will receive due attention in what follows.

2.1 The covariant phase space

The framework presented in the previous section is well suited for setting up the the so-called covariant phase space description. Here the quantity $\Theta^\mu(\phi, \delta\phi)$ defined in (2.0.2) plays a key role, as it leads to a new vector density ω^μ on the space of field configurations, which eventually will lead to the definition of a closed symplectic form on the covariant phase space. The definition of ω^μ is based on considering two independent variations in the field configuration space, $\delta_1\phi^i$ and $\delta_2\phi^i$, and reads,

$$\omega^\mu(\delta_1\phi, \delta_2\phi; \phi) = \delta_1\, \Theta^\mu(\phi, \delta_2\phi) - \delta_2\, \Theta^\mu(\phi, \delta_1\phi)\,. \tag{2.1.1}$$

The vector density $\Theta^\mu(\phi, \delta\phi)$ is known as the *symplectic potential*; the vector density $\omega^\mu(\delta_1\phi, \delta_2\phi; \phi)$ that follows from it is closely related to the so-called *symplectic current density*, to be introduced shortly.

The above definition (2.1.1) involves also double variations $\delta_1\,\delta_2\phi^i$ and is only an anti-symmetric bilinear in $\delta_1\phi$ and $\delta_2\phi$ when $[\delta_1, \delta_2]\phi^i = 0$. For more general variations that do not necessarily commute one therefore uses the following, modified, expression for ω^μ,

$$\omega^\mu(\delta_1\phi, \delta_2\phi; \phi) = \delta_1\, \Theta^\mu(\phi, \delta_2\phi) - \delta_2\, \Theta^\mu(\phi, \delta_1\phi) - \Theta^\mu(\phi, [\delta_1, \delta_2]\phi)\,. \tag{2.1.2}$$

Besides ensuring that ω^μ continues to be an anti-symmetric bilinear for arbitrary variations $\delta\phi$, this definition has the virtue of being insensitive to the ambiguity in Θ^μ proportional to $\delta[\upsilon^\mu(\phi)]$ that was noted in (2.0.5). This particular ambiguity is associated with the presence of possible boundary terms in the Lagrangian. Note that the ambiguitities of the form $\partial_\nu\, \upsilon^{\mu\nu}$ (the so-called improvement terms) will still remain at this stage.

The definition (2.1.2) will also ensure that ω^μ is a conserved current on the space of solutions. The latter follows by first considering $(\delta_1\,\delta_2 - \delta_2\,\delta_1 - \delta_3)\mathcal{L}$, which vanishes identically for $\delta_3\phi^i = [\delta_1, \delta_2]\phi^i$, and subsequently invoking the basic relation (2.0.2). In this way one directly derives the following equation for the divergence of ω^μ,

$$\partial_\mu\omega^\mu(\delta_1\phi, \delta_2\phi; \phi) = \delta_2 E_i\, \delta_1\phi^i - \delta_1 E_i\, \delta_2\phi^i\,, \tag{2.1.3}$$

for general ϕ and $\delta\phi$. The current (2.1.2) is not generally conserved, but it is conserved when projected to the subspace $\bar{\mathcal{M}}$ that comprises all the solutions of the field equations, $E_i(\phi) = 0$. The latter follows on noting that, by definition, every displacement in $\bar{\mathcal{M}}$ will connect two different solutions. Since we assume that we are dealing with a continuous variety of solutions, so that the space $\bar{\mathcal{M}}$ is continuous, it is clear that one can consider infinitesimal displacements $\bar{\delta}\bar{\phi}^i$ of a solution $\bar{\phi}^i$ leading to a neighboring point on $\bar{\mathcal{M}}$, so that both $E_i(\bar{\phi}) = 0$ and $E_i(\bar{\phi} + \bar{\delta}\bar{\phi}) = 0$. Therefore, the first-order variation $\bar{\delta}E_i(\bar{\phi}) \approx E_i(\bar{\phi} + \bar{\delta}\bar{\phi}) - E_i(\bar{\phi})$ also vanishes. Consequently, when projected to $\bar{\mathcal{M}}$, the symplectic current (2.1.2), defines the conserved *symplectic current density* [46–50],

$$\bar{\omega}^\mu(\bar{\delta}_1\bar{\phi}, \bar{\delta}_2\bar{\phi}; \bar{\phi}) = \bar{\delta}_1\, \bar{\Theta}^\mu(\bar{\phi}, \bar{\delta}_2\bar{\phi}) - \bar{\delta}_2\, \bar{\Theta}^\mu(\bar{\phi}, \bar{\delta}_1\bar{\phi}) - \bar{\Theta}^\mu(\bar{\phi}, [\bar{\delta}_1, \bar{\delta}_2]\bar{\phi})\,, \tag{2.1.4}$$

as follows straightforwardly from (2.1.3). Provided the fields are subject to suitable boundary conditions at spatial infinity, one can define a time-independent symplectic form by integrating (2.1.4) over a Cauchy hypersurface Σ,

$$\bar{\Omega}(\bar{\delta}_1\bar{\phi}, \bar{\delta}_2\bar{\phi}; \bar{\phi}) = \int_\Sigma \bar{\omega}^\mu(\bar{\delta}_1\bar{\phi}, \bar{\delta}_2\bar{\phi}; \bar{\phi})\, \mathrm{d}\Sigma_\mu\,, \tag{2.1.5}$$

The space of solutions $\bar{\mathcal{M}}$ defines the (on-shell) *covariant phase space* (these concepts are lucidly elaborated in [46, 47, 50]). The reason why it is called *covariant*, is that the introduction of a Cauchy surface is postponed until the end and the use of any auxiliary variables (as the momenta in the *canonical* phase space) is not necessary. The covariant phase space can then be regarded as being based on entire histories, without referring to specific initial conditions.

Motivated by the existence of this symplectic form, we will now present a more detailed introduction of the covariant phase space, where it is convenient to first discuss the theory on a given, but arbitrary, time slice, without imposing the field equations. On such a slice the independent variables are the fields ϕ defined over the time-slice as well as the (single or possibly multiple) time derivatives of the fields. Obviously spatial derivatives do not constitute separate variables, as the fields have been specified already over the whole time slice. In this slice, the vector density $\omega^\mu(\delta_1\phi, \delta_2\phi; \phi)$ depends on the variations of the fields as well as variations of the space-time derivatives of the fields. More precisely, if the order of the highest time derivative in the Lagrangian equals M, then variations of the time derivatives will generically occur in $\omega^\mu(\delta_1\phi, \delta_2\phi; \phi)$ upto order $2M - 1$. Consequently $\omega^\mu(\delta_1\phi, \delta_2\phi; \phi)$ can be viewed as an anti-symmetric quadratic form of the differentials of the fields and of their (possibly multiple) time derivatives which span a $2M$-dimensional manifold (at each point on the time slice, and for every type of field separately). This manifold is the *covariant phase space* $\mathcal{M}$. Note that it is a bit ambiguous to determine the highest time derivative in a Lagrangian in view of the fact that the Lagrangian is defined up to total derivatives. However, neither the equations of motion nor the vector density $\omega^\mu(\delta_1\phi, \delta_2\phi; \phi)$ are affected by the presence of total derivatives, so that they will always ensure a reliable count of the dimensionality of the covariant phase space.[2]

Provided the fields are subject to suitable boundary conditions at spatial infinity, one can define the quantity $\Omega(\delta_1\phi, \delta_2\phi; \phi)$, without the use of any field equations, by integrating (2.1.2) over a Cauchy hypersurface Σ,

$$\Omega(\delta_1\phi, \delta_2\phi; \phi) = \int_\Sigma \omega^\mu(\delta_1\phi, \delta_2\phi; \phi)\, \mathrm{d}\Sigma_\mu\,. \tag{2.1.6}$$

It should be noted that $\Omega\big(\delta_1\phi, \delta_2\phi; \phi\big)$ is time-dependent, because we refrained from imposing the field equations, but it nevertheless provides a useful structure on the covariant phase space. By taking the fields and their $(2M-1)$-th time-differentials as the 'coordinates' ζ^α, at least locally, of $\mathcal{M}$, $\Omega(\delta_1\phi, \delta_2\phi; \phi)$ can be locally expressed in these coordinates as

$$\Omega(\delta_1\phi(\zeta), \delta_2\phi(\zeta); \phi(\zeta)) = \Omega_{\alpha\beta}(\zeta)\, \delta_1\zeta^\alpha\, \delta_2\zeta^\beta\,, \tag{2.1.7}$$

where we made use of the fact that $\Omega(\delta_1\phi(\zeta), \delta_2\phi(\zeta); \phi(\zeta))$ is an anti-symmetric bilinear in $\delta_1\phi$ and $\delta_1\phi$. Of course, one may also use modified coordinates, as the tensor $\Omega_{\alpha\beta}$ is a covariant tensor in the space $\mathcal{M}$. However, some care should be exercised here so that the time slice description remains consistent (implying that the reparametrization should be non-singular).

For a systematic treatment, one may use the anti-symmetric bilinearity and introduce differential forms of rank one, denoted by $\mathrm{d}\zeta^\alpha$. Under diffeomorphisms these transform the same way as the coordinate differentials $\delta\zeta^\alpha$, but otherwise the two objects are very different. For one thing, the coordinate differentials $\delta\zeta^\alpha$ are commuting, whereas the one-forms $\mathrm{d}\zeta^\alpha$ are anti-commuting. Furthermore, $\delta\zeta^\alpha$, being a coordinate differential is a *contravariant vector*, to be associated with the *tangent space* $T(\mathcal{M})$, while $\mathrm{d}\zeta^\alpha$ is actually associated with $T^*(\mathcal{M})$, the space *dual* to the tangent space [51]. The anti-symmetric bilinearity of $\Omega_{\alpha\beta}$ allows one to construct the two-form

$$\Omega = \tfrac{1}{2}\Omega_{\alpha\beta}\, \mathrm{d}\zeta^\alpha \wedge \mathrm{d}\zeta^\beta\,. \tag{2.1.8}$$

A careful distinction has to be made between Ω of (2.1.8), and $\Omega(\delta_1\phi(\zeta), \delta_2\phi(\zeta); \phi(\zeta))$ of (2.1.7); the former is a two-form, which is an element of the *exterior algebra*

[2] Note that the highest time derivative in the field equations will generically be of order $2M$, so that the solutions can be given upon specifying initial conditions for ϕ and all its multiple time derivatives up to degree $2M-1$.

$\bigwedge T^*(\mathcal{M})$ defined independently of any δ,[3] while the latter is a function on $\mathcal{M}$ that explicitly depends on a pair of $\delta\phi(\zeta)$. The latter is to be understood as the value of the two-form Ω evaluated on the pair of vector fields $\delta_1\phi$ and $\delta_2\phi$.[4]

Nevertheless, one can formally think of regarding the two-form Ω, obtained from $\omega^\mu(\delta_1\phi(\zeta), \delta_2\phi(\zeta); \phi(\zeta))$, by replacing the commuting $\delta_1\zeta^\alpha$ and $\delta_2\zeta^\beta$ with the anti-commuting $\mathrm{d}\zeta^\alpha$ and $\mathrm{d}\zeta^\beta$, dropping the indices $1, 2$ on δ, as advocated in [46, 47]. By similar reasoning, it follows that $\Theta^\mu(\phi(\zeta), \delta\phi(\zeta))$ and $\omega^\mu(\delta_1\phi(\zeta), \delta_2\phi(\zeta); \phi(\zeta))$ can be regarded as forms. Hence we write,

$$\begin{aligned} \Theta^\mu(\phi(\zeta), \delta\phi(\zeta)) &= \Theta^\mu{}_\alpha(\zeta)\,\delta\zeta^\alpha\,, \\ \omega^\mu(\delta_1\phi(\zeta), \delta_2\phi(\zeta); \phi(\zeta)) &= \omega^\mu{}_{\alpha\beta}(\zeta)\,\delta_1\zeta^\alpha\,\delta_2\zeta^\beta\,. \end{aligned} \tag{2.1.9}$$

The implication of (2.1.4) for these components is then

$$\omega^\mu{}_{\alpha\beta} = \partial_\alpha\,\Theta^\mu{}_\beta - \partial_\beta\,\Theta^\mu{}_\alpha\,. \tag{2.1.10}$$

Therefore, the $\Theta^\mu{}_\alpha$ behave like gauge potentials (hence the name *symplectic potential*), and the $\omega^\mu{}_{\alpha\beta}$ like the corresponding abelian field strengths. Obviously $\omega^\mu{}_{\alpha\beta}$ obeys a Bianchi identity,

$$\partial_\alpha\omega^\mu{}_{\beta\gamma} + \partial_\beta\omega^\mu{}_{\gamma\alpha} + \partial_\gamma\omega^\mu{}_{\alpha\beta} = 0\,. \tag{2.1.11}$$

Upon defining the one-form $\Theta^\mu = \Theta^\mu{}_\alpha\,\mathrm{d}\zeta^\alpha$ and the two-form $\omega^\mu = \frac{1}{2}\omega^\mu{}_{\alpha\beta}\,\mathrm{d}\zeta^\alpha \wedge \mathrm{d}\zeta^\beta$ (in line with (2.1.8)), our above conclusions can be elegantly summarized in the language of differential forms,

$$\omega^\mu = \mathrm{d}\,\Theta^\mu\,, \qquad \mathrm{d}\,\omega^\mu = 0\,, \qquad \mathrm{d}\,\Omega = 0\,, \tag{2.1.12}$$

where d is the so-called *exterior derivative* which satisfies $\mathrm{d}^2 = 0$. Obviously ω^μ and Ω are closed forms. Returning to our earlier remark that the ambiguity in Θ^μ arising out of the $\delta[\upsilon^\mu(\phi)]$ term induced by adding a boundary term to the action (c.f. 2.0.5), does not contribute to ω^μ, this can now be understood as the gauge freedom associated with the symplectic potential. In the form language, this particular ambiguity in Θ^μ is representable as $\mathrm{d}\upsilon^\mu$ and will therefore not contribute to ω^μ. This is in agreement with our conclusion earlier in this section.

[3]Unlike tensors whose direct products again yield tensors, anti-symmetric tensors upon direct product do not yield anti-symmetric tensors; a further anti-symmetrization has to be performed. This is the essence of the exterior algebra.

[4]If ω is an r-form $\omega = \frac{1}{r!}\omega_{\alpha_1\ldots\alpha_r}\,\mathrm{d}\zeta^{\alpha_1} \wedge \ldots \wedge \mathrm{d}\zeta^{\alpha_r}$, and X_i are r vector fields, then $\omega(X_1, \ldots X_r) = \omega_{\alpha_1\ldots\alpha_r}\,X_1^{\alpha_1} \ldots X_r^{\alpha_r}$.

Now we return to the *on-shell* covariant phase space $\bar{\mathcal{M}}$ by projecting upon the space of solutions of the field equations, denoted by $\bar{\phi}$. In this case the *independent* initial values Φ^α can be taken, at least locally, to be the coordinates of $\bar{\mathcal{M}}$, which in fact correspond to entire histories consistent with the equations of motion. All the structures defined above for $\mathcal{M}$ can now be repeated verbatum. In particular, we define

$$\bar{\Omega}(\bar{\delta}_1\bar{\phi}, \bar{\delta}_2\bar{\phi}; \bar{\phi}) = \int_\Sigma \bar{\omega}^\mu(\bar{\delta}_1\bar{\phi}, \bar{\delta}_2\bar{\phi}; \bar{\phi})\, \mathrm{d}\Sigma_\mu\,, \tag{2.1.13}$$

where the crucial difference with (2.1.6) is that the above quantity is time-independent because $\bar{\omega}^\mu$ is conserved by virtue of (2.1.3). Here we recall that the off-shell quantities ζ^α and the initial conditions Φ^α have a one-to-one relation so that the dimensionality of the off-shell and the on-shell covariant phase space is the same. We can then enumerate the independent solutions $\bar{\phi}$ as functions of the initial conditions so that we may write $\bar{\phi}(x;\Phi)$. Then, $\bar{\Omega}$ can be locally expressed in these coordinates Φ^α as $\bar{\delta}\bar{\phi}$ can be decomposed into the derivatives $\partial_\alpha\bar{\phi}(x,\Phi)$ with respect to the Φ^α, so that we may write

$$\bar{\Omega}(\bar{\delta}_1\bar{\phi}(\Phi), \bar{\delta}_2\bar{\phi}(\Phi); \bar{\phi}(\Phi)) = \bar{\Omega}_{\alpha\beta}(\Phi)\, \delta_1\Phi^\alpha\, \delta_2\Phi^\beta\,, \tag{2.1.14}$$

which takes the same form as (2.1.7). For the same reasons as discussed previously $\bar{\Omega}_{\alpha\beta}$ is an anti-symmetric covariant tensor of rank two on $\bar{\mathcal{M}}$, which can be associated with a closed two-form. For the on-shell phase space this two-form is constant in time.

Clearly it is suggestive to directly identify $\bar{\Omega}$ with the symplectic form on $\bar{\mathcal{M}}$. To do so, $\bar{\Omega}$ must satisfy an additional and crucial criterion, namely it has to be *nondegenerate*. In fact, in the case of gauge theories, $\bar{\Omega}$ as constructed above is degenerate as it vanishes when one of the variations is a gauge variation. Therefore special care has to be exercised in this case to ensure that the symplectic form is not degenerate. In that context (2.1.14) is sometimes referred to as the *presymplectic form* to indicate that it is not yet referring to the *physical* phase space. The physical phase space is obtained from $\bar{\mathcal{M}}$ upon division by the gauge group. This division precisely removes the sources of degeneracy of $\bar{\Omega}$. The covariant phase space approach to gauge theories and general relativity is by now well developed [50, 52], although not usually directed to Lagrangians with higher derivative couplings.

When the covariant phase space is equipped with a closed, non-degenerate symplectic form, one can introduce the following bracket structure on the covariant phase space [46, 47],

$$\big\{A(\zeta), B(\zeta)\big\}_{\text{cov}} = \Omega^{\alpha\beta}\,\frac{\partial A}{\partial\zeta^\alpha}\,\frac{\partial B}{\partial\zeta^\beta}\,, \tag{2.1.15}$$

in direct analogy with the Poisson bracket known from the canonical phase space. Here the two-rank tensor $\bar{\Omega}^{\alpha\beta}$ is the inverse of the tensor defined in (2.1.8) (or (2.1.14) when we put the theory on-shell). As we shall see shortly, comparison with the canonical Poisson bracket may require an overall minus sign in (2.1.15), depending on how we have ordered the ζ^α. Because the symplectic form $\Omega_{\alpha\beta}$ is closed (irrespective of whether we have imposed the equations of motion), it readily follows that the Jacobi identity is valid for the bracket (2.1.15),

$$\big\{\{A,B\}_{\rm cov},C\big\}_{\rm cov}+\big\{\{B,C\}_{\rm cov},A\big\}_{\rm cov}+\big\{\{C,A\}_{\rm cov},B\big\}_{\rm cov}=0\,. \qquad (2.1.16)$$

Obviously the bracket can also be defined when the theory is put on-shell.

To put our discussions of the covariant phase space in a broader perspective we make a comparison with the salient features of the so-called *canonical* phase space. Illustrating this with the simplest mechanical example described by the Lagrangian $L=\frac{1}{2}\dot q^2-V(q)$, one easily derives $\Theta^t=\dot q\,\mathrm{d}q$, so that $\omega^t=\mathrm{d}\dot q\wedge\mathrm{d}q$. Provided we choose off-shell phase space differentials, $\zeta^1=q(t)$, and $\zeta^2=\dot q(t)$ at a given time slice, the matrix ${\omega^t}_{\alpha\beta}$ satisfies ${\omega^t}_{12}=-{\omega^t}_{21}=-1$ with the diagonal components vanishing. For the canonical phase space, one first constructs the canonical momentum $p=\partial L/\partial\dot q$, which in this case equals $\dot q$. Subsequently $\dot q$ is eliminated in favour of p, and a Hamiltonian $H=\dot q\,p-L=\frac{1}{2}p^2+V(q)$ is introduced, which is a function of p and q. In analogy with the more general discussion earlier, we can impose the equations of motion, which in this case are represented by Hamilton's equations,

$$\dot q=\frac{\partial H}{\partial p}\,,\qquad \dot p=-\frac{\partial H}{\partial q}\,, \qquad (2.1.17)$$

and adopt coordinates for the on-shell canonical phase space. To obtain the conventional formula, these phase space coordinates are labeled according to $\zeta^1=p$ and $\zeta^2=q$, which act as initial data for the time evolution of the system. In terms of them the non-degenerate symplectic two-form equals[5]

$$\omega=\mathrm{d}p\wedge\,\mathrm{d}q=\tfrac{1}{2}\omega_{\alpha\beta}\,\mathrm{d}\zeta^\alpha\,\mathrm{d}\zeta^\beta\,, \qquad (2.1.18)$$

which coincides with the previous choice for the covariant phase space upon replacing p by $\dot q$. The corresponding matrix $\bar\omega_{\alpha\beta}$ has a different overall sign than in the covariant phase space description, which is related to the different ordering of $(q,\dot q)$ and (p,q) used in the two approaches. Obviously, ω is non-degenerate, and the inverse of the matrix $\omega_{\alpha\beta}$, denoted by $\omega^{\alpha\beta}$, equals $\omega^{12}=-\omega^{21}=-1$. The symplectic form is also closed as can be seen from $\omega=\mathrm{d}(q\,\mathrm{d}p)$. Indeed, the

[5]We use ω, which is standard notation for the symplectic form in mechanics. This is precisely the quantity that one obtains from (2.1.13) in the zero-volume limit.

one-form $q\,\mathrm{d}p$ is the precise analog of the Θ^t one-form based on the covariant approach. The freedom to alter $q\,\mathrm{d}p$ by the addition of the one-form $\mathrm{d}A$, where A is any function on the phase space, is the analog of the one-form ambiguity of $\bar{\Theta}^\mu$ already discussed below (2.1.12).

To reiterate the differences and commonalities between the canonical and covariant phase spaces, both are governed by the existence of non-degenerate closed symplectic two-forms, and both have bracket structures that are very similar. The differences arise in the fact that, for the canonical phase space, coordinates are provided by coordinates and canonical momenta, while for the covariant phase space they are provided by coordinates and their various time-differentials. The introduction of canonical momenta is rather straightforward for theories that depend only on first-order time derivatives, but in the general case with higher order time derivatives, more auxiliary variables are required in order to derive an appropriate phase space, following Ostrogradski and Dirac [44, 53]. We will sketch the procedure to be followed in presence of higher time derivative terms in the canonical formalism in section 2.4 for a simple point particle example, giving instead a detailed treatment based on the covariant approach.

2.2 The Noether current and the Noether potential

When the action is invariant under a certain continuous group of transformations, possibly up to boundary terms, then we expect the set of field equations to be invariant (meaning that the field equations will transform into field equations). The reason is that the field equations follow from Hamilton's principle which requires the action to be stationary under general variations that vanish at the boundary (see the discussion in the introduction to this chapter). Therefore the emergence of boundary terms under the symmetry transformations will have no impact on the field equations.

We will bypass the details in the formulation of Noether's theorems. The crucial implication of the theorems, both in the case of rigid and of local invariance, is that when the fields are *on-shell*, that is, when they satisfy their respective Euler-Lagrange equations, there exist corresponding *conserved currents*. Before presenting the derivation of this result, which makes central use of the mathematical identity (2.0.2), let us first return to the invariance statement. When the action is invariant up to boundary terms, then the Lagrangian must be invariant up to a total derivative. Hence we may generally assume,

$$\delta_\xi \mathcal{L} = \partial_\mu N^\mu(\phi, \xi)\,, \tag{2.2.1}$$

where N^μ depends both on the fields and on the transformation parameters ξ (and possibly on their derivatives). Here it makes no difference whether the variation belongs to a rigid or to a local invariance. General relativity and locally supersymmetric theories of interest in this work are examples where N^μ is always non-vanishing, while in gauge theories N^μ usually vanishes, unless the theory contains Chern-Simons terms (see Appendix C). Observe that N^μ also has an inherent ambiguity depending on possible boundary terms in the Lagrangian and is only defined modulo a form-conserved improvement term, just as was already noted for the current Θ^μ in (2.0.5). Therefore there are ambiguities

$$N^\mu(\phi,\xi) \to N^\mu(\phi,\xi) + \delta_\xi\big[\upsilon^\mu(\phi)\big] + \partial_\nu w^{\mu\nu}\,, \tag{2.2.2}$$

where $w^{\mu\nu}$ is any anti-symmetric tensor.

Subsequently one combines the expression for $\delta_\xi\mathcal{L}$ coming from the general identity (2.0.2) with its explicit variation (2.2.1),

$$\delta_\xi\mathcal{L} = E_i\,\delta_\xi\phi^i + \partial_\mu\,\Theta^\mu(\phi,\delta_\xi\phi) = \partial_\mu\,N^\mu(\phi,\xi)\,. \tag{2.2.3}$$

It then follows straightforwardly that, when all the fields satisfy their respective Euler-Lagrange equations, the *Noether current*,

$$J^\mu(\phi,\xi) = \Theta^\mu(\phi,\delta_\xi\phi) - N^\mu(\phi,\xi)\,, \tag{2.2.4}$$

is conserved. As before, we are using a shorthand notation by indicating all the fields under considerations by ϕ. When dealing with gauge theories, ϕ would thus collectively stand for both matter and gauge fields.

We already stated that field equations transform under the invariance into field equations. Therefore, solutions of the field equations will transform into solutions, so that the invariance transformations consistently induce transformations in the space of solutions $\bar{\mathcal{M}}$ that was introduced in the previous section.

There are several additional issues that should be emphasized here. First, while $\Theta^\mu(\phi,\delta_\xi\phi)$ depends by definition on the fields and on the variations $\delta_\xi\phi$, this is not the case for N^μ, simply because the manipulations that are required to bring the variation of the Lagrangian into the form (2.2.1) will in general only yield an expression for N^μ that depends on the parameters ξ and no longer on the variations $\delta_\xi\phi$. The examples that we discuss will demonstrate this fact. The same comment will thus obviously apply to J^μ. Secondly, the Noether current will not be affected by changes of the Lagrangian by boundary terms, as follows from (2.0.5) and (2.2.2). As we saw before, the same property holds for the symplectic current density $\omega^\mu(\delta_1\phi, \delta_2\phi; \phi)$. For this reason we will no longer need to refer to ambiguities induced by boundary terms in the Lagrangian. The ambiguities associated with improvement terms, however, remain and we will discuss them in due course. Thirdly, it is important to emphasize that the current (2.2.4) satisfies an *ordinary* conservation law, i.e. $\partial_\mu J^\mu(\phi,\xi) = 0$, even in the case of non-abelian gauge fields and diffeomorphism invariant theories. This is related to the fact that the currents contain the (local) parameters ξ. Observe also that the current transforms as a vector density, as both terms in (2.2.4) are generated by variations of the Lagrangian (which transforms as a density).

To study the Noether current associated with a local abelian gauge invariance, we consider in the following the theory of electrodynamics described by the Lagrangian

$$\mathcal{L} = -\bar\Psi\gamma^\mu(\partial_\mu - \mathrm{i}eA_\mu)\Psi - m\bar\Psi\Psi - \tfrac{1}{4}F_{\mu\nu}{}^2\,, \tag{2.2.5}$$

where the field strength tensor is defined by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This Lagrangian is invariant under local gauge transformations, whose infinitesimal form is $\delta_\xi\Psi(x) = \mathrm{i}e\xi(x)\,\Psi(x)$, $\delta_\xi A_\mu(x) = \partial_\mu\xi(x)$. In this case, the quantities introduced in (2.0.2) and (2.2.1) take the form,

$$\begin{aligned}
E_{\bar\Psi} &= -\big(\partial\!\!\!/ - \mathrm{i}eA\!\!\!/ + m\big)\Psi\,, \quad E_\Psi = -\bar\Psi\big(-\overleftarrow{\partial\!\!\!/} - \mathrm{i}eA\!\!\!/ + m\big)\,,\\
E_A{}^\mu &= -\partial_\nu F^{\mu\nu} + \mathrm{i}e\bar\Psi\gamma^\mu\Psi\,,\\
\Theta^\mu &= -\bar\Psi\gamma^\mu\,\delta\Psi - F^{\mu\nu}\,\delta A_\nu\,,\\
N^\mu &= 0\,.
\end{aligned} \tag{2.2.6}$$

From this it follows that the Noether current associated with local gauge invariance equals,

$$J^\mu(A,\Psi,\bar\Psi,\xi) = -\mathrm{i}e\,\xi\,\bar\Psi\gamma^\mu\Psi - F^{\mu\nu}\,\partial_\nu\xi\,, \tag{2.2.7}$$

which is conserved for *arbitrary* $\xi(x)$ when $E_\Psi = E_{\bar\Psi} = E_A{}^\mu = 0$.

Many treatments of electrodynamics refer only to the first term in (2.2.7) as the conserved current without including $\xi(x)$. However, this does not represent the Noether current associated with the gauge invariance, but the current associated with the constant phase transformations of the fermion field. A related aspect is that the Noether current (2.2.7) is conserved only when *all* the fields satisfy their equations of motion. This again contrasts with the usual description. For example, in electrodynamics the current $i\bar{\Psi}\gamma^\mu\Psi$ is conserved as long as the fields $\bar{\Psi}$ and Ψ satisfy their equations of motion. It is not necessary for the gauge fields to be on-shell for this. On the other hand, consistency already requires that the photon field must couple to a conserved current without using the field equations for the matter fields, as can be inferred from considering an on-shell photon field and, consequently, the equation $\partial_\mu E_A{}^\mu = 0$. So clearly these matters are intertwined.

A noteworthy feature of the Noether current (2.2.7), is revealed by considering the combination,

$$J^\mu(A,\Psi,\bar{\Psi},\xi) + \xi\, E_A{}^\mu = \partial_\nu\big(-\xi\, F^{\mu\nu}\big)\,. \tag{2.2.8}$$

This equation states that when the gauge fields are on-shell (even without the matter fields being explicitly on-shell) the Noether current can be expressed as

$$J^\mu(A,\Psi,\bar{\Psi},\xi) = \partial_\nu\big(-\xi\, F^{\mu\nu}\big)\,. \tag{2.2.9}$$

The right-hand side of this equation does not depend on the fermion fields, but this is only an artifact of this particular model.[6] The observation that the conserved Noether current can generally be written as,

$$J^\mu(\phi,\xi) = \partial_\nu\, \mathcal{Q}^{\mu\nu}(\phi,\xi)\,, \tag{2.2.10}$$

in the space of solutions $\bar{\mathcal{M}}$, is important in the general context of this chapter. The tensor $\mathcal{Q}^{\mu\nu}$ is called the *Noether potential*. For the Lagrangian (2.2.5) it is equal to $\mathcal{Q}^{\mu\nu} = -\xi\, F^{\mu\nu}$, which is a local expression in terms of the fields. There is a subtlety regarding (2.2.10) that is important to clarify. Given a current conservation of the form,

$$\partial_\mu\, J^\mu(\bar{\phi},\xi) = 0 \tag{2.2.11}$$

for fields $\bar{\phi}$ satisfying the equation of motion, there is a significant difference between the cases when the invariances in question are rigid or local. For rigid invariances one can trivially find a non-local expression for $\mathcal{Q}^{\mu\nu}$,

$$\mathcal{Q}^{\mu\nu} = -\Box^{-1}\big[\partial^\mu J^\nu - \partial^\nu J^\mu\big]\,, \tag{2.2.12}$$

satisfying (2.2.10). One may wonder whether there are circumstances under which such a Noether potential becomes *local*. For rigid invariances there are only finitely many independent parameters involved in ξ and consequently only finitely many

[6] The reader may for instance verify that, when adding a moment coupling to the Lagrangian (2.2.5) proportional to $\bar{\Psi}\gamma^\mu\gamma^\nu\Psi\, F_{\mu\nu}$, the right-hand side of (2.2.8) will contain also terms that depend on the fermion fields.

independent conditions are implied by (2.2.11). This is in general not sufficient to ensure locality of the Noether potential.

On the other hand, when the invariance in question is local, then there are *infinitely* many conditions implicit in the conservation equation (2.2.11). This makes the situation highly restrictive, and forces the resulting Noether potential to be local (the reader may consult [54] for a discussion of such issues). Therefore in practical applications the Noether potential is a useful concept mainly in the context of local invariances.

Now, since the Noether current (2.2.9) is conserved for arbitrary $\xi(x)$, one could get the (incorrect) view that in locally invariant theories there are infinitely many conserved charges [7]. Indeed, using (2.2.10) it is straightforward to write down the corresponding charge, enclosed by a $(d-2)$-dimensional surface S. [8]

$$\begin{aligned} q_\xi &= \oint_S \mathcal{Q}^{\mu\nu}\, \mathrm{d}S_{\mu\nu} \\ &= \oint_S \xi(x)\, \boldsymbol{E}(x)\cdot \mathrm{d}\boldsymbol{S}\,, \end{aligned} \tag{2.2.13}$$

where $\mathrm{d}S_{\mu\nu}$ denotes a surface element on the boundary. Obviously S has the topology of S^{d-2} with the normal vector denoted by $\boldsymbol{S}$ and the Noether potential as given in (2.2.9) with $\boldsymbol{E}$ the electric field. Except for the presence of the arbitrary gauge function $\xi(x)$, this is the standard flux integral based on Gauss' law. To understand this issue and appreciate that there exists nevertheless just one conserved charge, it is important to realize that we are considering this formula in the presence of charged fields which can transfer the charge through S. To measure a conserved charge one must assume that S is chosen in a region where the field configuration is gauge invariant. This implies that ξ be constant and $\Psi = 0$, so that up to a normalization factor one is thus dealing with a single conserved charge. In the non-abelian case the condition on ξ will take the form $D_\mu \xi = 0$.

One can formulate this more mathematically by returning to (2.2.8) and observing that, when the fields satisfy their field equations and the current is proportional to the gauge variations of the fields, as is the case in (2.2.7), then the dual of the Noether potential, ${}^*\mathcal{Q}$, must be a closed $(d-2)$-form. If there are charges enclosed by S, then this $(d-2)$-form will be closed but not exact, so that S is to be regarded as a non-trivial cycle. In that case, the charge is measured by the integral of ${}^*\mathcal{Q}$

[7] This point was first raised in [55].

[8] We will pay due attention to the definition of the surface integral in chapter 6, where an explicit calculation of black hole charges involving the Noether potential will be worked out. For a rigorous treatment, we refer to [23].

over the $(d-2)$-dimensional cycle S. This argument leads to a partial restriction of the ambiguities in the Noether potential.

Perhaps this is a point to refer again to the ambiguities in the definition of the Noether current. In the standard treatment of conserved currents it is well known that the conserved currents are defined upto the addition of the so-called *improvement terms*. We have mentioned those ambiguities already at several points, most particularly below (2.0.5) and (2.2.2). Obviously the current expressed by (2.2.10) seems to represent just an improvement term, so one might be tempted to consider the possibility of absorbing this term into the original Noether current. In that case the current would vanish identically upon using the equations of motion! Indeed, it is often stated that the inclusion of improvement terms in the currents does not alter the value of charges. Care must be exercised in this. If arbitrary improvement terms could have been added, such as terms proportional to $\partial_\nu F^{\mu\nu}$, the total current and hence the charge, would vanish, something that is obviously absurd. The flaw here is, of course, in the fact that this particular improvement term would then not fall off sufficiently rapidly. Indeed, one normally argues that improvement terms do not affect the charge because they give contributions to the charge density that take the form of divergences $\vec{\nabla}\cdot\vec{V}$. After Fourier transforming, the charge becomes proportional to the photon momentum, so that the form factor at zero momentum is not affected. Hence this argument can equivalently be seen as an assumption about the analytic behavior of the improvement term at small photon momentum.

Clearly, if the Noether potential is to play a fundamental role, then one must be able to formulate specific criteria to sufficiently control the ambiguities caused by possible improvement terms. As will it turn out, these criteria will depend on the particular context in which the Noether potential is used (the interested reader can find some important observation concerning this point is Appendix C. We refer to [43] for a thorough discussion).

2.3 Conserved quantities and symmetry generators in phase space

In the previous sections we have identified a symplectic potential Θ^μ that plays a central role in the phase space description of the theory and we have derived the existence of conserved Noether currents associated with continuous invariances of the action. In point-particle mechanics the situation is similar, but the Noether *current* is now just a conserved *charge*. In this context it is known that that this charge also plays the role of the *generator* of the original symmetry transformations. Consequently, there is a tendency to equate the two concepts. The purpose of this section is to investigate the precise relationship between them, in particular for field theories, in the more general setting of the covariant phase space approach.

As a prelude let us first briefly recall the situation in point particle mechanics in the context of the canonical phase space description, with phase space coordinates p_i, q^i and a symplectic form $\omega = \mathrm{d}p_i \wedge \mathrm{d}q^i$. Consider the infinitesimal transformations in phase space that leave the Poisson bracket invariant. As it turns out these transformations take the form,

$$\delta_\xi\, p_i = \{p_i, G_\xi\}_{\text{PB}} = -\frac{\partial G_\xi}{\partial q^i}\,, \qquad \delta_\xi\, q^i = \left\{q^i, G_\xi\right\}_{\text{PB}} = \frac{\partial G_\xi}{\partial p_i}\,, \tag{2.3.1}$$

where the quantity $G_\xi(q,p)$ is known as the *generator* and ξ denotes the infinitesimal parameter associated with these transformations.[9] Transformations of this type are called *canonical transformations*. The invariance of the Poisson brackets under these transformations can be verified by evaluating the Poisson bracket of the the new phase space coordinates $p_i + \delta_\xi p_i$ and $q^j + \delta_\xi q^j$, which, up to terms quadratic in ξ, are equal to the Poisson brackets of the original p_i and q^i.

Let us now consider the variation of the generator $G_\xi(q,p)$ under arbitrary variations of p_i and q^i, denoted by δp_i and δq^i,

$$\delta G_\xi = \frac{\partial G_\xi}{\partial p_i}\,\delta p_i + \frac{\partial G_\xi}{\partial q^i}\,\delta q^i = \delta p_i\,\delta_\xi q^i - \delta q^i\,\delta_\xi p_i\,. \tag{2.3.2}$$

Note that the right-hand side of this equation involves the same anti-symmetric matrix that appears in the symplectic form ω of the canonical phase space. In fact, this result is very general and applies to any generator of a canonical transformation. Therefore it not surprising that an analogous result follows immediately for the covariant phase space. To demonstrate this we work out everything in terms of an arbitrary choice of phase space coordinates denoted by ζ^α. From (2.1.15) it follows that, for any function $A(\zeta^\alpha)$ on the covariant phase space, $\left\{A, \zeta^\alpha\right\}_{\text{cov}} = -\Omega^{\alpha\beta}\,\partial_\beta A$, leading to $\partial_\alpha A = \Omega_{\alpha\beta}\left\{\zeta^\beta, A\right\}$. The result for the variation δG_ξ follows,

$$\delta G_\xi = \partial_\alpha G_\xi\,\delta\zeta^\alpha = \Omega_{\alpha\beta}\left\{\zeta^\beta, G_\xi\right\}_{\text{cov}}\delta\zeta^\alpha\,, \tag{2.3.3}$$

[9]We assume that G_ξ does not explicitly depend on time, although this is not quite necessary at this stage.

which, upon using $\delta_\xi\,\zeta^\alpha = \big\{\zeta^\alpha, G_\xi\big\}_{\rm cov}$, can be rewritten as,

$$\delta G_\xi = \Omega_{\alpha\beta}\,\delta\zeta^\alpha\,\delta_\xi\zeta^\beta = \Omega\big(\delta\phi(\zeta), \delta_\xi\phi(\zeta); \phi(\zeta)\big)\,, \tag{2.3.4}$$

which is the generalization of (2.3.2). A careful examination of this result reveals that the non-degeneracy of the symplectic form was crucial in its derivation. Yet, the final result $\delta G_\xi = \Omega(\delta\phi, \delta_\xi\phi; \phi)$ makes sense even in cases where the symplectic form Ω is degenerate. At the end of section 2.4 we will present an explicit example for point particle mechanics with higher time derivatives which exhibits these characteristic features of the covariant phase space description.

Up to this point we were referring to generic canonical transformations, but let us now explore the consequences for a transformation that leaves the Hamiltonian invariant. It is easy to see from (2.3.1) that this implies

$$\big\{H, G_\xi\big\} = 0\,. \tag{2.3.5}$$

Using Hamilton's equations this proves that G_ξ is a conserved quantity,

$$\frac{\mathrm{d}G_\xi}{\mathrm{d}t} = 0\,. \tag{2.3.6}$$

Hence we conclude that conserved quantities generate the symmetry transformations that leave the Hamiltonian invariant. The purpose of this chapter is to analyze this relationship in the context of field theory. We will do so by investigating the relation (2.3.4) for conserved currents, but usually without projecting the fields and their variations to the space $\bar{\mathcal{M}}$ of solutions to the field equations.

Before returning to the field theory context, this is a good place to explain some subtle features about the variations δ (or $\bar\delta$) and δ_ξ. First of all, the variations $\delta_\xi\phi$ are specific variations that correspond to an invariance property. As a result these variations may themselves depend on the fields. This must in general be the case when one considers several independent, mutually non-commuting, symmetries, parametrized by different parameters $\xi = \alpha, \beta, \ldots$, so that two of these variation must lead to a third one according to $[\delta_\alpha,\, \delta_\beta] = \delta_\gamma$. This can only be realized on the fields when the variations $\delta_\xi\phi$ will depend on ϕ. Perhaps it is helpful to present an illustrative example at this point. Consider a triplet of fields $\boldsymbol{\phi}$ transforming under three-dimensional rotations. The infinitesimal rotations then read $\delta_\xi\boldsymbol{\phi} = \boldsymbol{\xi}\times\boldsymbol{\phi}$. Consider now the product of two such variations, with parameters $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$, and form the commutator. In this case the commutator is non-vanishing in general, as is shown by the relation $[\delta_1, \delta_2]\boldsymbol{\phi} = (\boldsymbol{\xi}_1\times\boldsymbol{\xi}_2)\times\boldsymbol{\phi}$.

The variations $\delta\phi$ can be either generic field variations not subject to any restriction, or a special subset that do not depend on the fields themselves, so that δ acts as a derivative in the field configuration space (the $\bar\delta\bar\phi$ are similar as they are induced by generic variations $\delta\Phi$ in the space of solutions $\bar{\mathcal{M}}$). In the latter case they can therefore be generated by the functional derivative,

$$\delta = \int \mathrm{d}^4x\, \delta\phi^i(x)\, \frac{\partial}{\partial\phi^i(x)}\,, \tag{2.3.7}$$

where the functions $\delta\phi^i(x)$ may be subject to certain restrictions, such as the condition that they will vanish at the boundary of the integration domain. Hence it is clear what is meant by $\delta\,\delta_\xi\phi$, namely that we vary the actual expression $\delta_\xi\phi$ as a function of ϕ by changing $\phi \to \phi + \delta\phi$. The question then remains how to define the expression $\delta_\xi\,\delta\phi$, because the symmetry transformations have originally not been defined as acting on $\delta\phi$. The proper result follows from applying δ_ξ on $\phi + \delta\phi$ and subtract from it $\delta_\xi\phi$. Hence

$$\delta_\xi\,\delta\phi \equiv \delta_\xi[\phi + \delta\phi] - \delta_\xi\phi\,. \tag{2.3.8}$$

Expanding this expression to first order in $\delta\phi$, one obtains the same result as for $\delta\,\delta_\xi\phi$. Therefore we conclude that[10],

$$\delta_\xi\,\delta\phi = \delta\,\delta_\xi\phi\,. \tag{2.3.9}$$

The reader may verify that this result also applies when the transformation $\delta_\xi\phi$ contains space-time derivatives. They reader may want to return to the example presented at the end of the previous paragraph and show that the above arguments imply that $\delta\boldsymbol{\phi}$ will transform as a three-vector, just as the original field $\boldsymbol{\phi}$.

Now let us return to the main topic of this section, which is to examine the relation between conserved Noether currents and the corresponding generators in the covariant phase space. Obviously, in the context of field theory one is dealing with conserved Noether currents which have an ambiguity in the form of improvement terms that leave the conservation law unaffected. The change of these currents under variations of the fields (possibly projected to the space of solutions $\bar{\mathcal{M}}$) is therefore expected to be consistent with (2.3.4), up to improvement terms and possibly field equations. Before presenting a few explicit examples we will first present some general arguments to show that this expectation is indeed correct.

[10]This result applies also to $\bar\delta\bar\phi$. To exhibit the latter, observe that $\bar\delta$ is generated by shifts of the Φ^α, so that $\bar\delta\,\delta_\xi\bar\phi = \delta\Phi^\alpha\,\partial_\alpha\bar\phi^j\,\partial_j(\delta_\xi\bar\phi)$. On the other hand, $\delta_\xi\,\bar\delta\bar\phi = \delta_\xi\big(\bar\phi(\Phi+\delta\Phi) - \bar\phi(\Phi)\big)$ leads to exactly the same result, so that the two variations δ_ξ and $\bar\delta$ are commuting.

Consider the following identity which applies to any variation $\delta\phi^i$,

$$\delta J^\mu(\phi,\xi) - \omega^\mu(\delta\phi,\delta_\xi\phi;\phi) = \delta_\xi\Theta^\mu(\phi,\delta\phi) + \Theta^\mu(\phi,[\delta,\delta_\xi]\phi) - \delta N^\mu(\phi,\xi)\,, \quad (2.3.10)$$

using the definitions (2.2.4) and (2.1.2). In the case that the Lagrangian is strictly invariant (i.e. $N^\mu(\phi,\xi)=0$) and that the variations $\delta\phi^i$ satisfy (2.3.9), we derive the expected result provided that $\Theta^\mu(\phi,\delta\phi)$ is invariant under the infinitesimal variations $\delta_\xi\phi$. This particular case is very common. Returning to the more general case we make use of two known equations,

$$\begin{aligned}\partial_\mu\omega^\mu(\delta\phi,\delta_\xi\phi;\phi) &= \delta_\xi E_i\,\delta\phi^i - \delta E_i\,\delta_\xi\phi^i\,,\\ \partial_\mu J^\mu(\phi,\xi) &= -\,E_i\,\delta_\xi\phi^i\,.\end{aligned} \quad (2.3.11)$$

The first equation is identical to (2.1.3) and holds for arbitrary $\delta\phi^i$. The second equation expresses current conservation and follows directly from (2.2.3). Adopting from now on variations $\delta\phi^i$ that are subject to (2.3.9), it follows that

$$\partial_\mu\big[\delta J^\mu(\phi,\xi) - \omega^\mu(\delta\phi,\delta_\xi\phi;\phi)\big] = -\delta_\xi(E_i\,\delta\phi^i)\,, \quad (2.3.12)$$

where we made use of (2.0.2) and (2.2.1). Observe that this result holds for general field configurations and arbitrary functions $\delta\phi^i(x)$, which makes it very restrictive. For instance, when the $\delta_\xi\phi^i$ (and therefore the $\delta_\xi\,\delta\phi^i$) do not involve derivatives of the fields, then both sides of the equation (2.3.12) should vanish *separately*, because the right-hand side will not contain terms proportional to space-time derivatives of $\delta\phi$ so that it will never constitute a total derivative for *arbitrary* $\delta\phi$. Although gauge transformations involve a derivative of the gauge *parameter*, they usually do not contain space-time derivatives on the fields, so that the both sides of (2.3.12) must vanish. The obvious examples where the right-hand side of (2.3.12) is nontrivial concern space-time symmetries and supersymmetry.

We can summarize the above discussion by defining a vector density $\mathcal{C}^\mu(\delta\phi,\xi;\phi)$, (c.f. 2.3.10),

$$\delta J^\mu(\phi,\xi) + \mathcal{C}^\mu(\delta\phi,\xi;\phi) = \omega^\mu(\delta\phi,\delta_\xi\phi;\phi)\,, \quad (2.3.13)$$

which, according to (2.3.12), satisfies the divergence equation

$$\partial_\mu\mathcal{C}^\mu(\delta\phi,\xi;\phi) = \delta_\xi(E_i\delta\phi^i)\,. \quad (2.3.14)$$

It is possible to construct a solution of the latter equation without imposing the field equations. First consider the right-hand side of the equation and note that it is proportional to $\delta\phi^i(x)$ and, depending on the nature of the variations δ_ξ, on space-time derivatives of $\delta\phi^i(x)$. Both these terms will be multiplied by terms proportional to the field equations and their derivatives. Assuming that the highest derivative in $\delta_\xi\phi^i$ is of n-th order, one can construct a solution of (2.3.14) by expanding $\mathcal{C}^\mu(\delta\phi,\xi;\phi)$ in terms of $\delta\phi^i(x)$ and their multiple space-time derivatives up to order $n-1$. It is important to realize that the right-hand side of (2.3.14) will have to satisfy a number of subtle consistency requirements in order that there exists a solution at all. The underlying symmetry will of course ensure that these requirements are met and we will explicitly demonstrate this in examples at the end of this section where we will consider a model with rigid supersymmetry. Obviously the solution is unique up to improvement terms. When $n=0$, when the transformation rules do not contain derivatives of the fields, the two sides of the equation cannot match so that both terms must vanish separately, as we have already observed before. Note that our arguments share some features with the proof of a related result presented in [54].

Hence we have now generally established that $\delta J^\mu(\phi,\xi)$ and $\omega^\mu(\delta\phi,\delta_\xi\phi;\phi)$ are equal up to improvement terms and terms proportional to equations of motion and their derivatives. The above results can be understood by considering an example of symmetries with non-vanishing $\mathcal{C}^\mu(\delta\phi,\xi;\phi)$: the supersymmetric $\mathcal{N}=2$ Maxwell theory already treated in section 1.2. As we pointed out, under the transformations (1.2.3) the Lagrangian (1.2.2) changes by a total derivative,

$$\begin{aligned}\delta_\epsilon\mathcal{L} =\partial_\mu\Big[&\bar\epsilon_i\gamma^\mu\big(-\partial\!\!\!/ X\,\Omega^i+\tfrac12\,Y^{ij}\,\Omega_j-\tfrac14\varepsilon^{ij}F_{ab}\gamma^{ab}\Omega_j\big)\\ &+\bar\epsilon^i\gamma^\mu\big(-\partial\!\!\!/\bar X\,\Omega_i+\tfrac12\,Y_{ij}\,\Omega^j-\tfrac14\varepsilon_{ij}F_{ab}\gamma^{ab}\Omega^j\big)\Big]\,. \end{aligned}\qquad(2.3.15)$$

We may therefore conclude that the transformations (1.2.3) define a symmetry of the theory for any constant (anti-commuting) parameters ϵ^i. Since this symmetry is parameterised by two independent Majorana spinors, it is known as $N=2$ supersymmetry.

One can now proceed with the generic variation of the Lagrangian to obtain the equations of motion and the associated symplectic potential Θ^μ, according to (2.0.2),

$$\begin{aligned}E_{\bar X} &=2\,\partial^\mu\partial_\mu X\,, \qquad & E_{\bar\Omega i} &= -\partial\!\!\!/\Omega_i\,,\\ E_A{}^\mu &= -\,\partial_\nu F^{\mu\nu}\,, \qquad & E_{Y^{ij}} &= \tfrac12\,Y_{ij}\,,\\ \Theta^\mu &= -\,2\,\delta X\,\partial^\mu\bar X-2\,\delta\bar X\,\partial^\mu X-\tfrac12\,\bar\Omega^i\gamma^\mu\,\delta\Omega_i+\tfrac12\,\delta\bar\Omega^i\gamma^\mu\,\Omega_i-F^{\mu\nu}\,\delta A_\nu\,. \end{aligned}\qquad(2.3.16)$$

Combining the result of (2.3.15) with the symplectic potential according to (2.2.4), we obtain the Noether current associated with supersymmetry, often called the supercurrent,

$$J^\mu(\phi,\epsilon) = -\big(2\,\bar\epsilon^i \partial\!\!\!/ \bar X + \tfrac12 \varepsilon^{ij}\,\bar\epsilon_j F_{\rho\sigma}\gamma^{\rho\sigma}\big)\,\gamma^\mu\,\Omega_i - \big(2\,\bar\epsilon_i \partial\!\!\!/ X + \tfrac12\varepsilon_{ij}\,\bar\epsilon^j F_{\rho\sigma}\gamma^{\rho\sigma}\big)\,\gamma^\mu\,\Omega^i\,. \tag{2.3.17}$$

It easily follows that it is indeed conserved upon use of the equations of motion. Note that it does not contain the auxiliary fields Y_{ij}.

One can directly compute the variation of the Noether current and compare it with the symplectic current density. After some effort which involves rearrangement of gamma matrices one obtains the result

$$\delta J^\mu(\phi,\epsilon) = \,\omega^\mu(\delta\phi,\delta_\epsilon\phi;\phi) - \mathcal{C}^\mu(\delta\phi,\epsilon;\phi)\,, \tag{2.3.18}$$

where $\mathcal{C}^\mu(\delta\phi,\epsilon;\phi)$ equals

$$\begin{aligned}\mathcal{C}^\mu(\delta\phi,\epsilon;\phi) =\; & -\partial_\nu\big[2\,\delta\bar X\,\bar\epsilon^i\,\gamma^{\mu\nu}\Omega_i + 2\,\delta X\,\bar\epsilon_i\,\gamma^{\mu\nu}\Omega^i + \big(\varepsilon^{ij}\,\bar\epsilon_i\gamma^{\mu\nu\rho}\Omega_j + \varepsilon_{ij}\,\bar\epsilon^i\gamma^{\mu\nu\rho}\Omega^j\big)\,\delta A_\rho\big]\\ & + 2\,\big[\delta X\,\bar\epsilon_i\gamma^\mu + \tfrac12\varepsilon_{ij}\,\bar\epsilon^j\gamma^{\mu\nu}\,\delta A_\nu\big]\partial\!\!\!/\Omega^i + 2\,\big[\delta\bar X\,\bar\epsilon^i\gamma^\mu + \tfrac12\varepsilon^{ij}\,\bar\epsilon_j\gamma^{\mu\nu}\,\delta A_\nu\big]\partial\!\!\!/\Omega_i\\ & + \bar\epsilon^j\gamma^\mu\delta\Omega^i\,Y_{ij} + \bar\epsilon_i\,\gamma^\mu\,\delta\Omega_j\,Y^{ij}\,.\end{aligned} \tag{2.3.19}$$

This expression decomposes into an improvement term and a number of terms proportional to the equations of motion. Hence its divergence will also be proportional to the equations of motion. This is consistent with the more general conclusions presented below (2.3.13) and (2.3.14). To verify the validity of (2.3.14), consider the following expression,

$$E_i\,\delta\phi^i = 2\,\partial^\mu\partial_\mu X\,\delta\bar X + 2\,\partial^\mu\partial_\mu\bar X\,\delta X - \delta\bar\Omega^i\partial\!\!\!/\Omega_i + \partial_\mu\bar\Omega^i\gamma^\mu\delta\Omega_i - \partial_\nu F^{\mu\nu}\,\delta A_\mu + \tfrac12 Y^{ij}\,\delta Y_{ij}\,. \tag{2.3.20}$$

To determine its behavior under supersymmetry we use (2.3.9), so that the variations $\delta\phi$ transform in the same way as the original fields as $\delta_\epsilon(\delta\phi^i) = \delta(\delta_\epsilon\phi^i)$. Furthermore we need the supersymmetry transformation of the field equations given in (2.3.16),

$$\begin{aligned}\delta_\epsilon E_{\bar X} =\; & -2\bar\epsilon^i\partial\!\!\!/ E_{\bar\Omega^i}\,,\\ \delta_\epsilon E_{\bar\Omega^i} =\; & -E_{\bar X}\epsilon_i + \varepsilon_{ij}E_A{}^\mu\,\gamma_\mu\epsilon^j - 2\,\partial\!\!\!/ E_{Y^{ij}}\epsilon^j\,,\\ \delta_\epsilon E_A{}^\mu =\; & -\tfrac12\varepsilon^{ij}\bar\epsilon_i\gamma^{\mu\rho}\partial_\rho E_{\bar\Omega^j} - \tfrac12\varepsilon_{ij}\bar\epsilon^i\gamma^{\mu\rho}\partial_\rho E_{\bar\Omega_j}\,,\\ \delta_\epsilon E_{Y^{ij}} =\; & -\bar\epsilon_{(i}E_{\bar\Omega^{j)}} - \varepsilon_{ik}\,\varepsilon_{jl}\,\bar\epsilon^{(k}E_{\bar\Omega_{l)}}\,.\end{aligned} \tag{2.3.21}$$

By direct computation one then verifies that

$$\partial_\mu\mathcal{C}^\mu(\delta\phi,\epsilon;\phi) = \delta_\epsilon\big(E_i\,\delta\phi^i\big)\,. \tag{2.3.22}$$

The results (2.3.18) and (2.3.22) are therefore in full agreement with (2.3.13) and (2.3.14) and the general conclusions drawn from them.

2.4 A classic higher derivative theory and the phase space description

As already mentioned in the introduction, Lagrangian theories with higher derivative couplings can be cast in an Hamiltonian form. To explain this in some detail we use an example of a point particle in one dimension described by a Lagrangian that depends on the particle coordinate, and its associated velocity and acceleration. This Lagrangian takes the form,

$$L(q,\dot q,\ddot q) = \tfrac{1}{2}\dot q^2 + \tfrac{1}{2}\mu\,\ddot q^2 - V(q)\,, \tag{2.4.1}$$

with μ an arbitrary parameter, and leads to the following Euler-Lagrange equation of motion,

$$E(q) = -\mathrm{d}_t{}^2 q + \mu\,\mathrm{d}_t{}^4 q - V'(q) = 0\,, \tag{2.4.2}$$

where here and henceforth in this section we use the notation $\mathrm{d}_t = \mathrm{d}/\mathrm{d}t$. To solve (2.4.2), one needs *four* initial conditions. Hence the solution space is four-dimensional. We also observe that the symplectic potential that one encounters in the variation of the Lagrangian, equals

$$\Theta^t(q,\delta q) = \mathrm{d}_t q\,\delta q + \mu\,\mathrm{d}_t{}^2 q\,\overset{\leftrightarrow}{\mathrm{d}_t}\,\delta q\,, \tag{2.4.3}$$

as follows directly from (2.0.3). We note that the addition of terms to the Lagrangian that can be written as an overall time derivative may induce even higher derivatives in the Lagrangian. However, while those terms do affect the symplectic potential Θ^t, neither the equations of motion nor the symplectic density ω^t, will change under this modification of the Lagrangian, as we have already stressed at earlier occasions.

In the presence of higher time derivatives, the standard canonical structure does not exist. To see this, note that the standard definition of the canonical momentum yields,

$$p \equiv \frac{\partial L}{\partial \dot q} = \dot q\,, \tag{2.4.4}$$

which can, as usual, be inverted to express $\dot q$ in terms of p. But the Hamiltonian, constructed according to the standard expression $H = \dot q\,p - L$, will still contain time derivatives of p and is no longer a function of q and p only. On the other hand, the *energy* is a conserved quantity associated with constant shifts in the time variable, $t \to t + \xi$, so one can alternatively use the definition (2.2.4) based on Noether's theorem to determine the energy as a function of $q(t)$ and its time derivatives. Using $\delta_\xi q = -\xi \mathrm{d}_t q$ and choosing $\xi = -1$ the result reads

$$\begin{aligned} \mathcal{E} &= \Theta(q, \mathrm{d}_t q) - L(q, \mathrm{d}_t q, {\mathrm{d}_t}^2 q) \\ &= \tfrac{1}{2}(\mathrm{d}_t q)^2 + \tfrac{1}{2}\mu\, ({\mathrm{d}_t}^2 q)^2 - \mu\, {\mathrm{d}_t}^3 q\, \mathrm{d}_t q + V(q)\,, \end{aligned} \tag{2.4.5}$$

Indeed, one can easily verify that $\mathcal{E}$ is time-independent by virtue of the equation of motion (2.4.2).

The difficulties in setting up a canonical phase space are tied up with the fact that the dimensionality of phase space must match the dimensionality of the solution space. To circumvent these problems, Ostrogradski suggested introducing enough auxiliary variables so that the Lagrangian can be expressed in terms of the (extended) coordinates and their velocities only [44]. For the example at hand, this can be achieved by using an auxiliary variable $u = \dot{q}$. Then $\ddot{q} = \dot{u}$, so that an alternative, but equivalent, Lagrangian can be obtained with a Lagrange multiplier λ to impose the equality between u and $\dot{q}$,

$$L'(q, u, \lambda, \dot{q}, \dot{u}) = \tfrac{1}{2}\dot{q}^2 + \tfrac{1}{2}\mu\, \dot{u}^2 - V(q) - \lambda(\dot{q} - u)\,. \tag{2.4.6}$$

It is to be noted that λ also has to be treated as a dynamical variable (a 'coordinate'), although its corresponding velocity $\dot{\lambda}$ is not present. Thus the configuration space is now labelled by (q, u, λ) and the extended phase space is six-dimensional.

Starting from the Lagrangian L', the conversion to a Hamiltonian description is straightforward, except that one will be dealing with a *constrained* Hamiltonian system. We refer to the original paper by Dirac [53] for an extensive treatment and just sketch, for completeness, the procedure to be used.
From the Lagrangian L' one can evaluate the canonical momenta and the Hamiltonian H_0. Note that the variable λ is not dynamical, so its conjugate momentum must vanish, at least in the subspace of the phase space where physical solutions lie. This is a *primary* constraint to be imposed on the Hamiltonian through a Lagrange multiplier. Furthermore, by ensuring that this constraint will hold at all times, a *secondary* constraint is found. Both constraints must be imposed on the canonical phase space, whose dimension is then reduced from 6 to 4, as expected (see [43] for a detailed discussion on the constrained Hamiltonian systems).

In the covariant approach, one can study the symplectic structure directly on the basis of the higher derivative Lagrangian (2.4.1) and compare it to the Lagrangian (2.4.6) containing at most first-order time derivatives. From the expression for Θ^t given in (2.4.3), we can derive the form for the symplectic density,

$$\omega^t(\delta_1 q, \delta_2 q; q) = -\delta_1 q \overset{\leftrightarrow}{\mathrm{d}_t} \delta_2 q + (\mathrm{d}_t{}^2\, \delta_1 q) \overset{\leftrightarrow}{\mathrm{d}_t} \delta_2 q - (\mathrm{d}_t{}^2\, \delta_2 q) \overset{\leftrightarrow}{\mathrm{d}_t} \delta_1 q \,, \tag{2.4.7}$$

using the definition (2.1.2). As explained in section 2.1, this quantity should be time independent when projected to the space of solutions $\bar{\mathcal{M}}$ and equal to $\bar\Omega(\bar\delta_1\bar q, \bar\delta_2\bar q; \bar q)$. In the model at hand, this implies that $\bar q$ satisfies the Euler-Lagrange equation (2.4.2), and the variations $\bar{\delta q}$ satisfy the fourth-order differential equation,

$$\big[\mathrm{d}_t{}^4 - \mathrm{d}_t{}^2 - V''(\bar q)\big]\ \bar\delta\bar q = 0\,. \tag{2.4.8}$$

The reader can verify explicitly that (2.4.7) is indeed conserved under the conditions given above.

Likewise we can consider the expression for the symplectic current that one obtains from the corresponding theory with only first derivatives that is described by the Lagrangian L' specified in (2.4.6). The steps are familiar by now and we directly write down the results for the equations of motion, E_q, E_u and E_λ and for Θ^t,

$$\begin{aligned} E_q &= -\,\ddot q + \dot\lambda - V'(q) = 0\,,\\ E_u &= -\,\mu\,\ddot u + \lambda = 0\,,\\ E_\lambda &= -\,\dot q + u = 0\,,\\ \Theta^t &= (\dot q - \lambda)\,\delta q + \dot u\,\delta u\,. \end{aligned} \tag{2.4.9}$$

It is easy to see that on the solution space, obtained by eliminating both u and λ by using their equations of motion, one has $u = \dot q$ and $\lambda = \mu\,\ddot u = \mu\,\dddot q$, which leads to the same expression for Θ^t as in (2.4.3).[11]

Let us now return to the expression for the conserved energy (2.4.5) based on the original Lagrangian (2.4.1) and consider its variation, $q(t) \to q(t) + \delta q(t)$, for arbitrary functions $\delta q(t)$. The result takes the form,

$$\begin{aligned} \delta\mathcal{E} &= \mathrm{d}_t\delta q\,\mathrm{d}_t q + \mu\big[\mathrm{d}_t{}^2\delta q\,\mathrm{d}_t{}^2 q - \mathrm{d}_t{}^3\delta q\,\mathrm{d}_t q - \mathrm{d}_t\delta q\,\mathrm{d}_t{}^3 q\big] + V'(q)\,\delta q\\ &= -\,\delta q\,\mathrm{d}_t{}^2 q + \mathrm{d}_t\delta\,\mathrm{d}_t q + \mu\big[\delta q\,\mathrm{d}_t{}^4 q - \mathrm{d}_t\delta q\,\mathrm{d}_t^3 q + \mathrm{d}_t{}^2\delta q\,\mathrm{d}_t{}^2 q - \mathrm{d}_t{}^3\delta q\,\mathrm{d}_t q\big] - E(q)\,\delta q\,. \end{aligned} \tag{2.4.10}$$

[11]This is not a surprise as the elimination of u and λ follows from the same variational principle that has been used to derive the two expressions for Θ^t, and we only imposed the equations of motion $E_u = 0$ and $E_\lambda = 0$, whereas the δu variation is rewritten as $\delta\dot q$ without restricting δq.

Recall that, in the conventions of this chapter, the conserved quantity is written as $-\xi\,\mathcal{E}$, while $\delta_\xi q = -\xi \mathrm{d}_t q$. Therefore this result can be expressed as,

$$\delta\big(-\xi\mathcal{E}\big) - \xi E(q)\,\delta q = \omega(\delta q, \delta_\xi q; q)\,. \tag{2.4.11}$$

The presence of the equation of motion term is in agreement with the arguments that led to to equations (2.3.13) and (2.3.14). In particular the term $\mathcal{C}^t = \xi\,E(q)\,\delta q$ satisfies precisely $\partial_t \mathcal{C}^t = -\delta_\xi(E(q)\,\delta q)$.

Subsequently, we identify q and its first, second and third derivative as the coordinates ζ^α of the four-dimensional *covariant phase space*,

$$\zeta^1 \equiv q\,, \quad \zeta^2 \equiv \mathrm{d}_t q\,, \quad \zeta^3 \equiv \mathrm{d}_t{}^2 q\,, \quad \zeta^4 \equiv \mathrm{d}_t{}^3 q\,. \tag{2.4.12}$$

In terms of these variables we can write the symplectic density (2.4.7) as

$$\begin{aligned}\omega^t(\delta_1 q, \delta_2 q) &= -\,\delta_1\zeta^1\,\delta_2\zeta^2 + \mu\,\delta_1\zeta^3\,\delta_2\zeta^2 - \mu\,\delta_1\zeta^4\,\delta_2\zeta^1 - \big(1 \leftrightarrow 2\big) \\ &= \omega_{\alpha\beta}\,\delta_1\zeta^\alpha\,\delta_2\zeta^\beta\,,\end{aligned} \tag{2.4.13}$$

where the skew-symmetric matrix $\omega_{\alpha\beta}$ and its inverse $\omega^{\alpha\beta}$ equal,

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & -1 & 0 & \mu \\ 1 & 0 & -\mu & 0 \\ 0 & \mu & 0 & 0 \\ -\mu & 0 & 0 & 0 \end{pmatrix}\,, \qquad \omega^{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & -\mu^{-1} \\ 0 & 0 & \mu^{-1} & 0 \\ 0 & -\mu^{-1} & 0 & -\mu^{-2} \\ \mu^{-1} & 0 & \mu^{-2} & 0 \end{pmatrix}\,. \tag{2.4.14}$$

With this result we can determine the covariant brackets, using the definition (2.1.15).

Furthermore, the energy $\mathcal{E}$ can be expressed in terms of the same coordinates ζ^α,

$$\mathcal{E} = \tfrac{1}{2}(\zeta^2)^2 + \tfrac{1}{2}\mu\,(\zeta^3)^2 - \mu\,\zeta^4\,\zeta^2 + V(\zeta^1)\,, \tag{2.4.15}$$

from which one can then straightforwardly derive the following expression for the covariant brackets,

$$\big\{\mathcal{E}, \zeta^\alpha\big\}_{\mathrm{cov}} - \delta_4^\alpha\,\mu^{-1} E(\zeta) = -\mathrm{d}_t \zeta^\alpha\,, \tag{2.4.16}$$

where $E(\zeta)$ is just the field equation (2.4.2). Note that this result is complementary to (2.4.11), in line with the arguments presented at the beginning of section 2.3. It reflects the well-known result that the Hamiltonian only generates time translations provided the equations of motion have been imposed.

The above discussion can be easily extended to other conserved quantities. To briefly illustrate this let us replace the coordinate q in the example above by two coordinates $\boldsymbol{q} = (q_1, q_2)$, so that we will be discussing a particle in two dimensions. For calculational convenience, we write $\boldsymbol{q}$ as a complex coordinate $q = \frac{1}{2}\sqrt{2}\big(q_1 + \mathrm{i}q_2\big)$. The Lagrangian then takes the form,

$$L(q,\dot q,\ddot q) = |\dot q|^2 + \mu\,|\ddot q|^2 - V(|q|)\,. \tag{2.4.17}$$

The equation of motion and the symplectic potential read,

$$\begin{aligned} E(q,q^*) =&\, -\mathrm{d}_t{}^2 q^* + \mu\,\mathrm{d}_t{}^4 q^* - \partial_q V(|q|)\,,\\ \Theta^t(q,q^*,\delta q,\delta q^*) =&\, \mathrm{d}_t q^*\,\delta q + \mu\,\mathrm{d}_t{}^2 q^* \overset{\leftrightarrow}{\mathrm{d}}_t\,\delta q + \text{h.c.}\,, \end{aligned} \tag{2.4.18}$$

Infinitesimal rotations now take the form $\delta_\xi q = \mathrm{i}\xi\, q$, $\delta_\xi \bar q = -\mathrm{i}\xi\, q^*$, where ξ denotes a infinitesimal rotation angle, and obviously leave the Lagrangian invariant. The corresponding constant of the motion $\mathcal{J}$ is proportional to the angular momentum and equal to

$$\begin{aligned} \mathcal{J} =&\,\Theta(q,q^*,\delta_\xi q,\delta_\xi q^*)\\ =&\,\mathrm{i}\xi\big[-q^*\overset{\leftrightarrow}{\mathrm{d}}_t\, q + \mu\,\mathrm{d}_t{}^2 q^*\overset{\leftrightarrow}{\mathrm{d}}_t\, q + \mu\, q^*\overset{\leftrightarrow}{\mathrm{d}}_t\,\mathrm{d}_t{}^2 q\big]\,. \end{aligned} \tag{2.4.19}$$

One can easily verify that $\mathcal{J}$ is a constant of the motion by virtue of the field equation given in (2.4.18).

Let us now consider the change of the angular momentum $\mathcal{J}$ induced by arbitrary changes of q and q^*, which takes the following form,

$$\delta\mathcal{J} = \omega(\delta q,\delta q^*,\delta_\xi q,\delta_\xi q^*;q,q^*)\,. \tag{2.4.20}$$

Clearly $\mathcal{J}$ acts as the generator for the rotations. Note the difference with the conserved quantity $\mathcal{E}$, which according to (2.4.11) only generates time translations provided the equations of motion are satisfied.

In this case the covariant phase space coordinates are now complex, so that the conserved quantities are functions of $(\zeta^\alpha,\zeta^{\bar\alpha})$. The expressions for the energy and angular momentum are as follows,

$$\begin{aligned} \mathcal{E}(\zeta^\alpha,\zeta^{\bar\alpha}) =&\,\zeta^2\zeta^{\bar 2} + \mu\big(\zeta^{\bar 3}\zeta^3 - \zeta^{\bar 2}\zeta^4 - \zeta^{\bar 4}\zeta^2\big) + V(\zeta^{\bar 1},\zeta^1)\,,\\ \mathcal{J}(\zeta^\alpha,\zeta^{\bar\alpha}) =&\,\mathrm{i}\xi\big(-\zeta^{\bar 1}\zeta^2 + \zeta^{\bar 2}\zeta^1 + \mu\,\zeta^{\bar 1}\zeta^4 - \mu\,\zeta^{\bar 4}\zeta^1 - \mu\,\zeta^{\bar 2}\zeta^3 + \mu\,\zeta^{\bar 3}\zeta^2\big)\,. \end{aligned} \tag{2.4.21}$$

The bracket will now involve an 8×8 matrix ω which decomposes into two off-diagonal blocks according to

$$\omega = \begin{pmatrix} 0 & \omega^{\alpha\bar{\beta}} \\ \omega^{\bar{\alpha}\beta} & 0 \end{pmatrix} , \tag{2.4.22}$$

where the 4×4 sub-matrices $\omega^{\bar{\alpha}\beta}$ and $\omega^{\alpha\bar{\beta}}$ are both equal to the matrix $\omega^{\alpha\beta}$ given in (2.4.14). With this result the bracket takes the form

$$\{A, B\}_{\text{cov}} = \omega^{\alpha\beta} \left[\frac{\partial A}{\partial \zeta^{\alpha}} \frac{\partial B}{\partial \zeta^{\bar{\beta}}} + \frac{\partial A}{\partial \zeta^{\bar{\alpha}}} \frac{\partial B}{\partial \zeta^{\beta}} \right] . \tag{2.4.23}$$

Using this result one easily verifies the relation

$$\left\{ \mathcal{J}, \zeta^{\alpha} \right\}_{\text{cov}} = \mathrm{i}\xi\, \zeta^{\alpha} , \qquad \left\{ \mathcal{J}, \zeta^{\bar{\alpha}} \right\}_{\text{cov}} = -\mathrm{i}\xi\, \zeta^{\bar{\alpha}} , \tag{2.4.24}$$

which shows that the angular momentum generates the rotations. This result is analogous to (2.4.16) except that there is no term proportional to the equation of motion, as is expected based on the observations made at the beginning of section 2.3. Furthermore one easily establishes that $\left\{ \mathcal{J}, \mathcal{E} \right\}_{\text{cov}} = 0$, which once more shows that angular momentum is a conserved quantity.

The model of this section demonstrates the virtues of the covariant phase space approach. Without introducing new variables and dealing with constraints, thus retaining the theory in its original form, one calculates conserved quantities and defines the appropriate covariant brackets. The brackets play a role that is very similar to the role that the Poisson brackets play in the canonical phase space. Their consequences are exactly the same. Quantities whose covariant brackets with the 'Hamiltonian' (the conserved quantity associated with time shifts) vanish are constants of the motion. Furthermore the conserved quantities generate the corresponding infinitesimal symmetry transformation on the phase space variables through the covariant brackets.

2.5 The first law of black hole mechanics

In this section we will explain a procedure, due to Wald and based on the covariant phase space approach [12, 56], to obtain the conserved charges and entropy of a black hole solution in any theory of gravity, even including higher derivative couplings. Before doing so, we want to introduce a number of important objects which are indispensable to any treatment of general relativity and complement the discussion in section 1.1.1. There we defined general relativity as a theory invariant under diffeomorphism transformations. This means that the invariant action must be of the form

$$S = \int \mathrm{d}^4x \sqrt{-g}\mathcal{L}$$

where $\mathcal{L}$ is a scalar and $\mathrm{d}V_{\rm inv} = \mathrm{d}^4x \sqrt{-g}$ is the invariant volume element in theories of gravity with a dynamical metric. To prove this statement, it is sufficient to consider the transformation of $\mathrm{d}V_{\rm inv}$ under diffeomorphism $x^\mu \to x'^\mu = x^\mu + \xi^\mu(x)$, i.e.

$$\begin{aligned} \mathrm{d}^4\, x \to \mathrm{d}^4\, x' &= \left|\frac{\partial x'^\mu}{\partial x^\mu}\right| \mathrm{d}^4\, x = \mathcal{J}\, \mathrm{d}^4\, x\, ; \\ g_{\mu\nu}(x) \to g'_{\rho\sigma}(x') &= \frac{\partial x^\mu}{\partial x'^\rho}\frac{\partial x^\nu}{\partial x'^\sigma} g_{\mu\nu}(x)\, , \\ g(x) \to g'(x') &= \mathcal{J}^{-2} g(x)\ , \end{aligned} \tag{2.5.1}$$

where $\mathcal{J}$ is the Jacobian of the transformation, and g is the determinant of the metric tensor. Quantities like g or d^4x that transform with powers $|\mathcal{J}|^W$ of the Jacobian are also called *densities* of weight W [12]. The covariant derivative of a tensor density $\mathcal{I}^{\alpha,\cdots}{}_{\gamma,\ldots}$ of weight W reads

$$\nabla_\mu \mathcal{I}^{\alpha,\cdots}{}_{\gamma,\ldots} = \partial_\mu \mathcal{I}^{\alpha,\cdots}{}_{\gamma,\ldots} + \Gamma^\alpha_{\mu\lambda}\, \mathcal{I}^{\lambda,\cdots}{}_{\gamma,\ldots} + \cdots - \Gamma^\lambda_{\mu\gamma}\, \mathcal{I}^{\alpha,\cdots}{}_{\lambda,\ldots} - \cdots - W\, \Gamma^\rho_{\rho\mu}\, \mathcal{I}^{\alpha,\cdots}{}_{\gamma,\ldots}\ . \tag{2.5.2}$$

The covariant derivative of tensorial quantities can then be obtained from this formula by setting $W = 0$, which is equivalent to saying that each tensorial quantity is a density of weight zero. The symbols $\Gamma^\alpha_{\beta\gamma}$ used above are the Christoffel connections. They can be written in terms of the metric as,

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2} g^{\alpha\rho}\big(\partial_\beta g_{\rho\gamma} + \partial_\gamma g_{\beta\rho} - \partial_\rho g_{\beta\gamma}\big)\ , \tag{2.5.3}$$

which show that $\Gamma^\alpha_{\beta\gamma}$ is symmetric in its lowest indices [13]. This connection has a natural curvature tensor associated, the Riemann tensor $\mathcal{R}^\rho{}_{\sigma\mu\nu}$ defined as,

$$\mathcal{R}^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}{}^\rho - \partial_\nu \Gamma_{\mu\sigma}{}^\rho + \Gamma_{\mu\lambda}{}^\rho \Gamma_{\nu\sigma}{}^\lambda - \Gamma_{\nu\lambda}{}^\rho \Gamma_{\mu\sigma}{}^\lambda\ . \tag{2.5.4}$$

[12] This definition finally explains why, insofar, we have called the symplectic vector θ^μ a vector density. While it is true that any theory implicitly contains a metric, only theories invariant under diffeomorphisms allow for a dynamical one, which in turns leads to the transformations (2.5.1).

[13] For our purposes in this work, we will always consider torsion free theories, satisfying the metric postulate $\nabla_\mu g_{\nu\rho} = 0$, so that the symbols Γ are defined by (2.5.3).

The Riemann curvature arises also from the commutators of covariant derivatives, applied on a generic tensor $A^{\alpha}{}_{\beta}$ as follows

$$[\nabla_\mu, \nabla_\nu] A^{\alpha}{}_{\beta} = \mathcal{R}^{\alpha}{}_{\lambda\mu\nu} A^{\lambda}{}_{\beta} - \mathcal{R}^{\lambda}{}_{\beta\mu\nu} A^{\alpha}{}_{\lambda} \,. \tag{2.5.5}$$

Contractions of the Riemann tensor lead to the Ricci tensor and the Ricci scalar (already used in section 1.1.1)

$$\mathcal{R}_{\mu\nu} = \mathcal{R}^{\rho}{}_{\mu\rho\nu} \,, \quad \mathcal{R} = g^{\mu\nu} \mathcal{R}_{\mu\nu} \,. \tag{2.5.6}$$

Although the Riemann tensor measures the curvature of the space-time, the coordinate dependence of its components makes it tricky to assert when it becomes infinite. On the other hand, the Ricci scalar is coordinate independent, so if becomes singular at a certain point, then so does the curvature of the space-time. Perhaps this is the best moment to introduce a very important concept that will be used extensively in the following chapters, the *tangent space*. If Einstein's equations admit a regular (up to a finite number of singularity points) solution for the metric, one can always find a specific reparametrization that brings the second equation of (2.5.1) in the form

$$\eta_{\rho\sigma}(P') = \frac{\partial x^\mu}{\partial x'^{\rho}} \frac{\partial x^\nu}{\partial x'^{\sigma}} g_{\mu\nu}(P) \,.$$

This means that, at least locally in a (regular) neighbourhood of a point P of the space-time, the metric can always be written as Minkowski flat space, called in this case *local Lorentz frame* or *tangent space* at P. This observation turns out to be crucial in supersymmetric theories of gravity (supergravity) since no consistent description of fermion fields in curved space-time exists, and so one always makes use of the tangent space description, which allows to rigorously define fermion fields and their transformations. As we shall see, in that case the natural connections to covariantize ordinary derivatives on a generic curved manifold are not the Christoffel connections, but the *spin connection* field $\omega_\mu{}^{ab}$ which can be considered the gauge field of local Lorentz transformations in the tangent space (see for instance [57] for an extensive treatment of this issue).

We are now ready to derive the first law of black hole mechanics. Consider a gravity theory described by a Lagrangian density of the form $\sqrt{-g}\, L(g_{\mu\nu}, \mathcal{R}_{\mu\nu\sigma\tau}, \dots)$. Here L is just a scalar quantity which depends on arbitrary powers of the Riemann tensor and contractions thereof. For the two derivative Einstein-Hilbert theory $L \sim \mathcal{R}$, but we want to consider here any generic covariant higher derivative

Lagrangian. Under diffeomorphism with parameter $\xi^\mu(x)$[14], the scalar L is simply translated,

$$\delta_\xi L = \xi^\alpha \partial_\alpha L \ ,$$

while, from the symmetry variation of the metric tensor,

$$\delta_\xi g_{\mu\nu} = 2\nabla_{(\mu}\xi_{\nu)} \ , \tag{2.5.7}$$

and using the identity $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\, g^{\alpha\beta}\delta g_{\alpha\beta}$, one obtains

$$\delta_\xi \sqrt{-g} = \nabla_\mu(\xi^\mu \sqrt{-g}) \ .$$

Then, under a symmetry variation, *every* Lagrangian density of weight 1 transforms as

$$\delta_\xi(\sqrt{-g}L) = \partial_\mu N^\mu = \partial_\mu(\xi^\mu\sqrt{-g}L) = \nabla_\mu(\xi^\mu\sqrt{-g}L) \ , \tag{2.5.8}$$

where the last identity holds because the quantity inside the parenthesis is a vector density. In section 2.3 we illustrated the relationship between the generic variation of a current and the symplectic form ω, that takes the form (2.3.13). For the case at hand we get, after plugging in the value of N^μ:

$$\delta J^\mu = \omega^\mu(\delta\phi, \delta_\xi\phi; \phi) + \partial_\nu\big(2\,\xi^{[\nu}\Theta^{\mu]}\big) \ , \tag{2.5.9}$$

where we used the result for the Lie derivative of a vector I^μ, $\delta_\xi I^\mu = \xi^\nu\partial_\nu I^\mu - I^\nu\partial_\nu\xi^\mu$. Now the crucial observation is that if (2.5.7) vanishes, namely if a *Killing vector* ξ^μ exists and is time-like ($\xi^\mu\xi_\mu = -1$ after normalization), then one can define an Hamiltonian H generating the evolution along the integral curves [15] of ξ^μ. The generic variation of the Hamiltonian δH is related to the time-independent symplectic form Ω by (2.3.4). Combining with (2.5.9) we get,

$$\delta H = \delta\int_{\mathcal{C}} \mathrm{d}\Sigma_\mu J^\mu - 2\int_{\mathcal{C}} \mathrm{d}\Sigma_\mu\, \partial_\nu(\xi^{[\nu}\Theta^{\mu]}) \ . \tag{2.5.10}$$

This shows that the current is an Hamiltonian density (up to total derivatives). Now, if the linearized equation of motion are satisfied, so given a certain solution $\bar\phi$, we have $\big[\partial E_i(\phi)/\partial\phi\big]\Big|_{\bar\phi} = 0$, then J^μ and its generic variation in the space of solutions, δJ^μ, are conserved quantities. We can then write the equality, valid on the on-shell phase space, $\delta J^\mu = \delta\partial_\nu Q^{\mu\nu} = \partial_\nu\delta Q^{\mu\nu}$. Upon substituting this identity into (2.5.10), it is immediate to realize that the Hamiltonian, which generates time

[14] The diffeomorphism transformations δ_ξ are also called Lie derivative, see for instance [16, 58]. We will use these two names interchangeably.

[15] Generally speaking, a Killing vector is an infinitesimal generator of isometries of the spacetime manifold. We refer for instance to [16] for an extensive treatment of the subject.

translations, is just a surface term [16] [12, 56],

$$\delta H = \int_{\partial\mathcal{C}} \mathrm{d}\Sigma_{\mu\nu} \left[\delta Q^{\mu\nu} - 2\,\xi^{[\nu}\Theta^{\mu]}\right] . \tag{2.5.11}$$

Since the variations in equation (2.5.10) are taken in the space of solutions, δH will not depend on the choice of the integration Cauchy surface or its boundary $\partial\mathcal{C}$, as we already hinted at in section 2.1. Take also $\xi(x) = t^\alpha$ to be the Killing vector parametrizing infinitesimal time translations at infinity. The associated conserved quantity, given as a surface integral at infinity, is the energy of, or equivalently the mass enclosed by the space-time. The above expression can be readily generalized to give the generic variation of a symmetry generator as long as ξ^μ is the Killing vector parametrizing that symmetry. For instance, if $\xi(x) = \varphi^\alpha$ parametrizes infinitesimal rotations at infinity, the expression above will give the generic variation of the angular momentum. The results read [12, 56],

$$\begin{aligned} \delta M &= \tfrac{1}{2}\int_{\Sigma_\infty} \delta Q^{\mu\nu}(t) - 2\,t^{[\nu}\Theta^{\mu]} \ , \\ \delta J &= -\tfrac{1}{2}\int_{\Sigma_\infty} \delta Q^{\mu\nu}(\varphi) \ , \end{aligned} \tag{2.5.12}$$

where we called Σ_∞ the boundary of the Cauchy surface $\mathcal{C}$ at infinity. Notice that in the second equality the term proportional to Θ^μ vanishes because the Killing vector φ is tangent to the integration surface.

These expression can be manipulated to give the value of the conserved charges, whenever the symplectic potential Θ can be written as the arbitrary variation of another tensor ϑ, as follows

$$\begin{aligned} M &= \tfrac{1}{2}\int_{\Sigma_\infty} Q^{\mu\nu}(t) - 2\,t^{[\nu}\vartheta^{\mu]} \ , \\ J &= -\tfrac{1}{2}\int_{\Sigma_\infty} Q^{\mu\nu}(\varphi) \ . \end{aligned} \tag{2.5.13}$$

It is straightforward to check that, starting from the Einstein-Hilbert Lagrangian (1.1.1), the equations (2.5.13) give the well-known ADM mass and Komar integral for the angular momentum [59, 60].

Consider a stationary black hole solution, such as the Kerr solution, which admits a Killing vector field $\xi^\mu = t^\mu + \omega^{(i)}\varphi^\mu_{(i)}$, with $\omega^{(i)}$ angular velocities and the summation over i is intended for solutions in dimensions higher than four, where the conservation of more than one angular momentum is possible. Suppose also that the Killing vector ξ^μ is such that, for every field ϕ of the theory, $\delta_\xi\phi = 0$. Then

[16] For further details on the surface element $\mathrm{d}\Sigma_{\mu\nu}$ we refer to Chapter 6 where calculations of such integrals are explicitly worked out.

the symplectic form, being a bilinear in $\delta\phi$ and $\delta_\xi\phi$, will vanish identically and so will the arbitrary variation of the Hamiltonian. This means that the integrals (2.5.10) and (2.5.11) vanish. Take now the boundary of $\mathcal{C}$ to be the union of two sub-spaces, one at infinity and the other one surrounding the black hole horizon, i.e. $\partial\mathcal{C} = \Sigma_\infty \cup \Sigma_{\rm hor}$. Then by using the same manipulations as above, one derives the first law of thermodynamics (see (1.1.7)),

$$\frac{\kappa_s}{2\pi}\delta S = \delta M - \omega^{(i)}\delta J_{(i)} \tag{2.5.14}$$

with δM and δJ as in (2.5.12) and the variation of the entropy given by,

$$\frac{\kappa_s}{2\pi}\delta S = \tfrac{1}{2}\int_{\Sigma_{\rm hor}} \delta Q^{\mu\nu}(\xi) - 2\,\xi^{[\nu}\Theta^{\mu]}\ . \tag{2.5.15}$$

For Einstein-Hilbert theory, the Noether potential reads

$$Q^{\mu\nu}_{\rm EH} = 2\,g^{\rho[\mu}g^{\nu]\sigma}\nabla_{[\rho}\xi_{\sigma]}\ ,$$

from which one obtains [12, 56] the celebrated area law, $S = A/4$. If the gravitational Lagrangian is also invariant under an abelian gauge symmetry and admits as a solution of the dynamical equations a charged black hole, such as Kerr-Newmann solution, then the electric (magnetic) charge can be immediately calculated by using the results presented in the previous section.

The important point is that the procedure used to obtain a conserved current and the Noether potential that stems from it on-shell can be straightforwardly extended, for every conserved charge, also to effective theories of gravity that include higher derivative corrections. In later chapter we will encounter effective corrections of the form $\mathcal{R}^2$, $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$, etc., in the context of supergravity and we will make explicit use of the formulas derived in this section and, more generally, in this chapter.

Chapter 3

4D $\mathcal{N} = 2$ conformal supergravity

In this chapter we will introduce the basic aspects of four-dimensional $\mathcal{N} = 2$ supergravity and its coupling with matter multiplets, such as the vector and chiral multiplets, already introduced in Chapter 1 in the context of global supersymmetry. As explained in section 1.2.2, it is convenient to formulate (locally) supersymmetric theories off-shell, especially when dealing with effective Lagrangians containing higher derivative couplings, since no use of the dynamical equation is needed to close the algebra on its multiplet representations. Nevertheless, local Poincaré supergravity and its matter couplings are expressed in terms of large irreducible multiplet representations which transform, under local supersymmetry, in a complicated non-linear fashion. To deal with these issues, the superconformal formalism is used, which allows to obtain a simpler off-shell supergravity theory described in terms of the smallest possible representations on which the algebra is linearly realized [57]. Such formalism relies on the conformal group, the largest group of space-time symmetries of a field theory, which includes Poincaré symmetry together with other gauge invariances, such as scale transformations $\mathbb{D}$. It is easy to realize that the invariance under a larger (than Poincaré) symmetry group imposes more restrictive conditions on the field content and the possible interactions of a theory, which can also be studied in a more structured and systematic way. On the other hand, the requirement of scale invariance is unphysical, since it does not allow for a privileged energy or mass scale. This is why in many practical applications, e.g. the evaluation of quantum corrections in theories of supergravity of interest for this work, conformal invariance must be broken for the theory to yield physical results.
The concept of *gauge equivalence* reconciles these intertwined matters and is at the center of the superconformal formalism: the goal is to construct a theory of supergravity invariant under the gauge symmetries of the conformal group [61–63].

These gauge symmetries are then identified with space-time symmetries, by imposing conventional constraints (we will come back to this issue shortly). Finally, the physical Poincaré theory is obtained from the conformal supergravity theory via gauge-fixing, through the use of the so-called *compensating multiplets.*
The procedure is best explained and shown by exhibiting some simple examples, in the context of gauge theories and (non-supersymmetric) gravity in d space-time dimensions.
The first example we want to consider is the Proca Lagrangian describing a non-interacting massive vector field V_μ,

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 - \frac{1}{2}\, m^2 V_\mu^2 \ . \tag{3.0.1}$$

It is immediate to realize that the presence of the mass term breaks the explicit gauge invariance of the kinetic term, $\delta V_\mu = \partial_\mu \lambda$. It is however possible to obtain a locally gauge invariant Lagrangian via the field redefinition

$$V_\mu = W_\mu - m^{-1}\partial_\mu \phi \ , \tag{3.0.2}$$

which is invariant under the combined local gauge transformations $\delta W_\mu = \partial_\mu \lambda(x)$ and $\delta\phi = m\,\lambda(x)$. These transformations are an invariance of the Lagrangian re-written in terms of the new fields [64], i.e.

$$\mathcal{L} = -\frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2 - \frac{1}{2}(D_\mu \phi)^2 \ , \tag{3.0.3}$$

with $D_\mu\phi = \partial_\mu\phi - m\,W_\mu$. The first remark to make is that the inclusion of the scalar field ϕ has not modified the number of degrees of freedom of the theory but is, instead, perfectly balanced by the new local gauge invariance. We simply rearranged the d off-shell degrees of freedom initially associated to a massive vector field into a massless gauge field A_μ $(d-1)$ and a real scalar field ϕ. The field redefinition (3.0.2) makes explicit the decomposition of transversal and longitudinal degrees of freedom of the vector field V_μ now associated to the vector field A_μ and the scalar ϕ and maintains the manifest Lorentz-covariance of the theory. The same decomposition could have been achieved through *non-local* projection operators which commute with the Lorentz transformations, making the massive vector field a reducible representation of the Lorentz algebra. But field redefinitions, such as (3.0.2), have the advantage of being local and, for the case of supergravity theories, they will be expressed from the start in terms of the smallest irreducible multiplets, representations of the superconformal algebra.
The crucial role in the procedure is played by the extra degrees of freedom, the

compensating scalar field ϕ which couples in a gauge invariant fashion to the massless connection A_μ and constitutes the bridge between the two formulations. In fact, it is sufficient to gauge-fix its value via a gauge transformation with parameter $\lambda = -m^{-1}\phi$ to obtain the Proca Lagrangian we started with. The two theories are then *gauge equivalent*.
With the next example, quite relevant to this thesis, we want to show how gauge equivalence can be exploited to obtain a scale invariant version of the Einstein-Hilbert Lagrangian,

$$-2\kappa^2\,\mathcal{L} = \sqrt{-g}\mathcal{R}\,. \tag{3.0.4}$$

Under dilatations, the metric $g_{\mu\nu}$ transforms as

$$\delta_D\, g_{\mu\nu} = -2\Lambda_D\, g_{\mu\nu}\,. \tag{3.0.5}$$

From this, the transformations of the Ricci scalar $\mathcal{R}$ and the density $\sqrt{-g}$ are easily derived,

$$\delta_D\sqrt{-g} = -d\,\Lambda_D\sqrt{-g}\,, \qquad \delta_D\mathcal{R} = 2\,\Lambda_D\mathcal{R} - 2(d-1)\Box\Lambda_D\,, \tag{3.0.6}$$

where the operator $\Box = D^\mu D_\mu$ is the d'Alambertian, and the derivatives D_μ are covariantized with respect to diffeomorphism through the Christoffel connections Γ in (2.5.3). It is an easy exercise to show that the Einstein-Hilbert Lagrangian varies, under a scale transformation, into

$$-2\kappa^2\,\delta_D\mathcal{L} = (2-d)\,\Lambda_D\mathcal{R} - 2(d-1)\Box\Lambda_D\,.$$

To obtain a scale invariant version of (3.0.4), one adds a compensating scalar field ϕ to the theory, transforming under the local gauge dilatations as

$$\delta_D\phi = \frac{1}{2}(d-2)\Lambda_D\phi\,,$$

that can be used to define a scale invariant metric $\tilde{g}_{\mu\nu}$ through the field redefinition

$$\tilde{g}_{\mu\nu} = \phi^{\frac{4}{d-2}}g_{\mu\nu}\,. \tag{3.0.7}$$

Then the scale invariant version of the Einstein-Hilbert Lagrangian reads

$$-2\kappa^2\,\mathcal{L}_S = \sqrt{-\tilde{g}}\tilde{\mathcal{R}}\,,$$

which, in terms of the original fields $g_{\mu\nu}$ and ϕ, it takes the form

$$-2\kappa^2\,\mathcal{L}_S = \sqrt{-g}\big(\phi^2\mathcal{R} - \frac{4(d-1)}{d-2}\partial^\mu\phi\,\partial_\mu\phi\big)\ . \tag{3.0.8}$$

Now this Lagrangian is locally scale invariant only if one forces a negative sign for the kinetic term of the compensating scalar, which is then unphysical. Nevertheless, the field redefinition (3.0.7) allowed for a decomposition of the $\frac{1}{2}d(d-1)$ independent field components of the original metric $g_{\mu\nu}$[1] in the $\frac{1}{2}(d-2)(d+1)$ off-shell degrees of freedom of its massive spin-2 part, $\tilde{g}_{\mu\nu}$, and the scaling part, ϕ and then solves a general mismatch between the off-shell degrees of freedom of the theory and the independent field components (see [57] for an extensive treatment of this issue in (super)gravity).[2]

As before, it is easy to obtain the original Poincaré Lagrangian, by imposing the gauge-fixing condition $\phi = 1$ or, equivalently, by rescaling the scalar through a finite scale transformation with parameter $\exp\Lambda_D = \phi^{-1}$. Again, the two Lagrangian are equivalent up to a gauge transformation.
These examples of gauge equivalence exhibited the main feature of the procedure used to obtain a Lagrangian invariant an extra gauge symmetry through the use of compensating fields, which ensures the number of degrees of freedom to remain unvaried. In the following we will extend the gauge equivalence procedure to the full conformal group of symmetries and explicitly construct two- and four-derivatives conformally invariant actions. We will then present their $\mathcal{N} = 2$ supersymmetric generalizations in curved space (conformal supergravity), obtained through the use of *compensating multiplets*. The gauge-fixing procedure to obtain super-Poincaré actions will not be treated in detail, but we will comment on it in various instances.

3.1 Conformal gravity

We want to exploit now the concept of gauge equivalence to construct theories of gravity invariant under the full conformal group of symmetries. The procedure use to obtain a conformal gauge equivalent version of the Einstein-Hilbert action (3.0.4) will also be extended to construct conformally invariant higher derivative

[1] In fact, a symmetric rank 2 tensor has $\frac{d(d+1)}{2}$ field components but only $\frac{d(d-1)}{2}$ are independent, because of the invariance under d diffeomorphisms.

[2] Alternatively one can make use of the Einstein's equation to fix the scale of the metric, similarly to what happens when considering massive vector fields. However, the resulting theory will be on-shell.

Lagrangians. To keep a uniform exposition henceforth, we will work in four space-time dimensions. The conformal group is the group of transformations SO(4, 2) that leaves the light-cone invariant. It is generated by translations P_a and Lorentz transformations M_{ab} of the Poincaré group, dilatations $\mathbb{D}$ and the conformal boosts, or special conformal transformations K_a. The connections and parameters associated to this group of transformations are listed in the table below. The indices $a, b, \dots$ run from $0, \dots, 3$ and they parametrize a flat Minkowski internal space. The Greek indices $\mu, \nu, \dots = 0, \dots, 3$ are curved indices in space-time.

generator	P^a	M^{ab}	$\mathbb{D}$	K^a
gauge fields	$e_\mu{}^a$	$\omega_\mu{}^{ab}$	b_μ	$f_\mu{}^a$
parameters	ξ^a	ε^{ab}	Λ_D	Λ_K^a

Note that at this point, despite the nomenclature used, these operators act as an internal symmetry group and not as space-time transformations. Hence, the infinitesimal transformations of the gauge fields can be obtained from the algebra of the group SO(4, 2)[3] and read,

$$\begin{aligned} \delta e_\mu{}^a &= \mathcal{D}_\mu \xi^a - \Lambda_D\, e_\mu{}^a + \varepsilon^{ab} e_{\mu b} \ , \\ \delta \omega_\mu{}^{ab} &= \mathcal{D}_\mu \varepsilon^{ab} + 2\Lambda_K^{[a}\, e_\mu^{b]} + 2\xi^{[a} f_\mu{}^{b]} \ , \\ \delta b_\mu &= \partial_\mu \Lambda_D + \Lambda_K^a e_{\mu a} - \xi^a\, f_{\mu a} \ , \\ \delta f_\mu{}^a &= \mathcal{D}_\mu \Lambda_K^a + \Lambda_D\, f_\mu{}^a + \varepsilon^{ab} f_{\mu b} \ , \end{aligned} \tag{3.1.1}$$

where the derivatives $\mathcal{D}$ are covariantized with respect to dilatations and Lorentz transformations, i.e.

$$\mathcal{D}_\mu \xi^a = \partial_\mu \xi^a + b_\mu\, \xi^a - \omega_\mu{}^{ab} \xi_b \ . \tag{3.1.2}$$

The full covariant derivative, containing also the conformal boosts connection $f_\mu{}^a$, will be indicated with D_μ. In complete analogy to the case of non-abelian gauge theories, to define the curvature tensors of all the conformal gauge fields, which are gauge covariant quantities, one can use the commutator of covariant derivatives D_μ. Its action on a general covariant field (indicated with $\cdot$ in the following) is given by,

$$[D_\mu, D_\nu](\cdot) = \big(-\tfrac{1}{2} R_{\mu\nu}{}^{ab}(M)\mathbb{M}_{ab} + R_{\mu\nu}{}^a(P)\mathbb{P}_a + R_{\mu\nu}(D)\mathbb{D} + R_{\mu\nu}{}^a(K)\mathbb{K}_a \big)(\cdot) \ , \tag{3.1.3}$$

[3]This group is double-covered by the group SU(2, 2), just as SO(3) is double-covered by SU(2).

where we defined,

$$
\begin{aligned}
R(M)_{\mu\nu}{}^{ab} &= 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\,\omega_{\nu]c}{}^{b} - 4\,f_{[\mu}{}^{[a}\,e_{\nu]}{}^{b]}\ ,\\
R(P)_{\mu\nu}{}^{a} &= 2\,\mathcal{D}_{[\mu}\,e_{\nu]}{}^{a}\ ,\\
R(D)_{\mu\nu} &= 2\,\partial_{[\mu}\,b_{\nu]} - 2\,f_{[\mu}{}^{a}\,e_{\nu]a}\ ,\\
R(K)_{\mu\nu}{}^{a} &= 2\,\mathcal{D}_{[\mu}\,f_{\nu]}{}^{a}\ .
\end{aligned}
\tag{3.1.4}
$$

We emphasize again that, at this point, these curvatures represent only internal field strengths. But in a theory of gravity the local translation operator should be identified with general coordinate transformations. Furthermore, the theory should be formulated in a translationally invariant way. To achieve these goal, one first identifies the space where the P operator acts with the Minkowski frame for the tangent space of the curved space-time manifold. The gauge field $e_\mu{}^a$ is then an invertible map and interpreted as the vielbein of the theory [4]. Then the so-called *conventional constraints* must be imposed

$$
R(P)_{\mu\nu}{}^{a} = 0\,, \qquad R(M)_{\mu\nu}{}^{ab}\,e_b{}^{\nu} = 0\,,
\tag{3.1.5}
$$

which allow the necessary symmetry identifications and reduce the number of independent degrees of freedom. For instance, the first constraint on $R(P)$ can be derived by imposing the vielbein P transformation

$$
\begin{aligned}
\delta_P e_\mu^a = \mathcal{D}_\mu \xi^a &= \xi^\nu \partial_\nu e_\mu^a + \partial_\mu \xi^\nu\, e_\nu^a + \xi^\nu b_\nu e_\mu{}^a - \xi^\nu\,\omega_\nu{}^{ab} e_{\mu b} + \xi^\nu\, R(P)_{\mu\nu}{}^a\\
&= \delta_{\rm cov} e_\mu^a + \xi^\nu\, R(P)_{\mu\nu}{}^a\ ,
\end{aligned}
\tag{3.1.6}
$$

to close only on the covariant general coordinate transformations $\delta_{\rm cov}$. The constraint can be solved to yield the spin-connection

$$
\omega_\mu{}^{ab} = -2\,e^{\nu[a}\partial_{[\mu}e_{\nu]}^{b]} - e^{\nu[a}e^{b]\rho}e_{\mu c}\partial_\rho e_\nu{}^c - 2\,e_\mu{}^{[a}e^{b]\nu}b_\nu\ .
\tag{3.1.7}
$$

Analogously, the explicit expression for the K-connection field $f_\mu{}^a$ can be found by solving the second constraint in terms of $e_\mu{}^a$ and b_μ,

$$
f_\mu{}^a = \tfrac{1}{2}\mathcal{R}(e,b)_\mu{}^a - \tfrac{1}{12}e_\mu{}^a\,\mathcal{R}(e,b)\,, \qquad f_\mu{}^\mu \equiv f = \tfrac{1}{6}\,\mathcal{R}(e,b)\,,
\tag{3.1.8}
$$

where $\mathcal{R}(e,b)_{\mu\nu}{}^{ab}$ denotes the curvature associated with the spin connection,

$$
\mathcal{R}(e,b)_{\mu\nu}{}^{ab} = 2\,\partial_{[\mu}\omega(e,b)_{\nu]}{}^{ab} - 2\,\omega(e,b)_{[\mu}{}^{ac}\,\omega(e,b)_{\nu]c}{}^{b}\,.
\tag{3.1.9}
$$

[4] For an extensive review on the vielbein formulation of general relativity, we refer to [57].

Because the constraints (3.1.5) are invariant under Lorentz transformations, dilatations and conformal boosts, the transformation rules (3.1.1) remain unaffected. For the supersymmetric extension this will no longer be the case and additional terms will emerge.

We want to use now the above formalism to construct two conformally invariant Lagrangians: the first one reduces to the two-derivative Einstein-Hilbert Lagrangian after the usual gauge-fixing procedure; the second Lagrangian instead contains four-derivative couplings, relevant to this work, but contrary to the naive expectations no curvature squared terms appear. We will further comment on these points at the end of this section.
Let us start by considering a scalar field ϕ transforming under dilatations as

$$\delta_{\rm D}\phi = w\,\Lambda_{\rm D}\phi\,, \tag{3.1.10}$$

where the constant w is known as the Weyl weight. It is now straightforward (but more and more tedious) to determine explicit expressions for multiple conformally covariant derivatives of ϕ and their transformation behavior under K-transformations (c.f. appendix B of [65]),

$$\begin{aligned} D_\mu\phi =& \mathcal{D}_\mu\phi = \partial_\mu\phi - wb_\mu\phi\,,\\ D_\mu D_a\phi =& \mathcal{D}_\mu D_a\phi + w\,f_{\mu a}\,\phi\,,\\ D_\mu\Box_{\rm c}\,\phi =& \mathcal{D}_\mu\Box_{\rm c}\phi + 2(w-1)f_\mu{}^a D_a\phi\,,\\ \Box_{\rm c}\Box_{\rm c}\,\phi =& \mathcal{D}_a D^a\Box_{\rm c}\phi + (w+2)f\,\Box_{\rm c}\phi + 2(w-1)f_{\mu a}\,D^\mu D^a\phi\,, \end{aligned} \tag{3.1.11}$$

whose variations under K-transformations read,

$$\begin{aligned} \delta_{\rm K}D_a\phi =& -w\,\Lambda_{{\rm K}a}\,\phi\,,\\ \delta_{\rm K}D_\mu D_a\phi =& -(w+1)\big[\Lambda_{{\rm K}\mu}\,D_a + \Lambda_{{\rm K}a}\,D_\mu\big]\phi + e_{\mu a}\Lambda_{\rm K}{}^b\,D_b\phi\,,\\ \delta_{\rm K}\Box_{\rm c}\phi =& -2(w-1)\Lambda_{\rm K}{}^a\,D_a\phi\,,\\ \delta_{\rm K}D_\mu\Box_{\rm c}\phi =& -(w+2)\Lambda_{{\rm K}\mu}\,\Box_{\rm c}\phi - 2(w-1)\Lambda_{\rm K}{}^a\,D_\mu D_a\phi\,,\\ \delta_{\rm K}\Box_{\rm c}\Box_{\rm c}\phi =& -2(w-1)\Lambda_{\rm K}{}^a\,\Box_{\rm c}\,D_a\phi - 2(w+1)\Lambda_{\rm K}{}^a D_a\Box_{\rm c}\phi\,. \end{aligned} \tag{3.1.12}$$

where we used the symbol $\Box_{\rm c} = D_\mu D^\mu$. It turns out that, for specific Weyl weights, $\Box_{\rm c}\phi$ and $\Box_{\rm c}\Box_{\rm c}\phi$ are K-invariant [5],

$$\begin{aligned} \delta_{\rm K}\Box_{\rm c}\phi =&\ 0\,, && (\text{for } w=1)\,,\\ \delta_{\rm K}\Box_{\rm c}\Box_{\rm c}\phi =&\ 2\,\Lambda_{\rm K}{}^a\big(\Box_{\rm c}\,D_a - D_a\Box_{\rm c}\big)\phi = 0\,, && (\text{for } w=0)\,, \end{aligned} \tag{3.1.13}$$

where, to prove the last part of the second equation, we rewrote $\Box_{\rm c}\,D_a\phi - D_a\Box_{\rm c}\phi = D^b\big[D_b, D_a\big]\phi + \big[D_b, D_a\big]D^b\phi$ and made use of the Ricci identity and the curvature constraints. From (3.1.13) one derives two conformally invariant Lagrangians by multiplying with a similar scalar field ϕ' of the same Weyl weight as ϕ,

$$\begin{aligned} e^{-1}\mathcal{L} \propto \phi'\,\Box_{\rm c}\phi =& -\mathcal{D}^\mu\phi'\,\mathcal{D}_\mu\phi + f\,\phi'\phi\,, && (\text{for } w=1)\\ e^{-1}\mathcal{L} \propto \phi'\,\Box_{\rm c}\Box_{\rm c}\phi =&\ \mathcal{D}^2\phi'\,\mathcal{D}^2\phi + 2\,\mathcal{D}^\mu\phi'\big[2\,f_{(\mu}{}^a e_{\nu)a} - f\,g_{\mu\nu}\big]\mathcal{D}^\nu\phi\,, && (\text{for } w=0) \end{aligned} \tag{3.1.14}$$

up to total derivatives. Both the above expressions are symmetric in ϕ and ϕ'.

Let us comment on the two Lagrangians (3.1.14). In both Lagrangians the dependence on b_μ will cancel as a result of the invariance under conformal boosts, since that is the only independent field which transforms non-trivially under K-transformations. In the first Lagrangian one may then adjust the product $\phi'\phi$ to a constant by means of a local dilatation. In that case the second term of the Lagrangian is just proportional to the Ricci scalar, so that one obtains the Einstein-Hilbert term. The kinetic term for the scalars depends on the choice made for ϕ' and ϕ. For instance, when the two fields are the same, then ϕ equals a constant; when they are not the same (elementary or composite) fields, the kinetic term can be exclusively written in terms of ϕ and will be proportional to $\phi^{-2}\big(\partial_\mu\phi\big)^2$. In that case the first Lagrangian describes an elementary or a composite scalar field coupled to Einstein gravity.

The situation regarding the second higher derivative Lagrangian is fundamentally different, because one cannot adjust the scalar fields to any particular value by local dilatations in view of the vanishing Weyl weight. The scalar fields may be equal to constants (in which case the Lagrangian vanishes) or to homogeneous functions of other fields such that the combined Weyl weight remains zero, without affecting the invariance under local dilatations. Either case, no $f_\mu{}^a$ (hence curvature) squared terms can arise, as it is easy to see by imposing $w=0$ in (3.1.11). Nevertheless, this example uncovers the structure of the four-derivative

[5] Poincaré and Lorentz invariance are manifest. Furthermore, the K-invariants presented have the correct Weyl weight for scale invariant actions. We refer to section 3.4 for a detailed discussion.

conformal Lagrangian (3.1.14) and the restrictions imposed on the scalar fields by K-invariance. These insights will be used in the next chapter to construct, from a non-linear scalar field, a similar four-derivative conformal Lagrangian which instead does contain curvature squared terms.

3.2 $\mathcal{N} = 2$ conformal supergravity: the Weyl multiplet

In order to construct conformal theories of supergravity [66–69] it is necessary to illustrate their gauge and matter structure. The aim is again to describe gravity as a gauge theory of the superconformal group, through the aid of compensating multiplets (such as the vector multiplet, see section 3.3). This will provide an off-shell formulation of the theory in which the extra gauge invariances allow for a correct matching between off-shell degrees of freedom and number of independent field components, within supermultiplets. A related aspect is that representations of the off-shell superconformal algebra are easier to find than their Poincaré counterparts. More importantly, the large group of invariances will select minimal representations of the algebra which are naturally suited to write down the supersymmetric extension of particular classes of higher curvature terms relevant to this thesis.

In this section we will focus our attention on the supermultiplet containing all the gauge fields of the $\mathcal{N} = 2$ superconformal algebra, the *Weyl multiplet*, which will be presented in its full glory shortly. We start from the $\mathcal{N} = 2$ superconformal group $\mathrm{SU}(2,2|2)$ that can be fully derived by adding 4 Majorana supercharges to the conformal algebra. Half of those are the Q-supersymmetry generators $Q^i(Q_i)$ of the super Poincaré algebra and satisfy the anti-commutation relationships (1.2.9). This algebra is, as was already noted, invariant under the R-symmetry group $\mathrm{U}(2) \simeq \mathrm{SU}(2) \times \mathrm{U}(1)$, which acts chirally on the fermions of the theory. The Majorana spinors can then be taken in the (anti-)fundamental representation of this group, and their chirality is indicated by an upper or lower SU(2) index, as before. Complex (and hermitian) conjugation accompanies the raising or lowering of the chiral indices (the assignments of chirality for the fermions, together with the two abelian charges, the Weyl and U(1) weights of all the fields of the Weyl multiplet, are given in Table B.1 of Appendix B).

Now, the commutator between the Q-supersymmetry generators $Q^i(Q_i)$ and the conformal boosts generator K_a closes on new supercharges $S^i(S_i)$, which generate an auxiliary fermionic symmetry, S-supersymmetry,

$$[K_a, Q^i] = -\gamma_a S^i \ . \tag{3.2.1}$$

The S-supercharges transform, analogously to the Q-supercharges (see (1.2.9)), as Lorentz spinors and they close on conformal boosts, i.e.

$$\{S^i, S_j\} = \gamma^a \, K_a \, \delta^i_j \ . \tag{3.2.2}$$

It is important to notice that the closure of the superconformal algebra, particularly the anti-commutator between Q^i and S_j, requires the inclusion of the U(2) R-symmetry algebra as well[6].
The local transformations of the superconformal group, initially considered as internal, are then translations P_a, Lorentz transformations M_{ab}, dilatations $\mathbb{D}$, special conformal transformations K_a, Q- and S-supersymmetries and the R-symmetry U(1) and SU(2) transformations, $\mathbb{A}$ and $V^i{}_j$. The corresponding gauge fields are contained within the Weyl multiplet and are summarized below.

generator	P_a	M_{ab}	$\mathbb{D}$	K_a	Q_i	S_i	V_i^j	$\mathbb{A}$
gauge fields	$e_\mu{}^a$	$\omega_\mu{}^{ab}$	b_μ	$f_\mu{}^a$	$\psi_\mu{}^i$	$\phi_\mu{}^i$	$\mathcal{V}_\mu{}^i{}_j$	A_μ
parameters	ξ^a	ε^{ab}	Λ_D	Λ_K^a	ϵ^i	η^i	$\Lambda_{\mathrm{SU(2)}}{}^i{}_j$	$\Lambda_{\mathrm{U(1)}}$

where the V_i^j gauge field is anti-hermitean and traceless, as appropriate for the generators of the SU(2) group, i.e. $\mathcal{V}_\mu{}^i{}_j = -\mathcal{V}_{\mu j}{}^i$ and $\mathcal{V}_\mu{}^i{}_i = 0$.
Note that the transformation rules for the gauge fields of the conformal group presented in (3.1.1) and, as a consequence, the associated curvatures (3.1.4) need to be complemented by additional terms due to supersymmetry. We will keep the same notation used in the previous section for the conformal gauge field curvatures, keeping in mind that the expressions will now have extra terms due to supersymmetry (see Appendix B for details). Also, the covariant derivative D_μ needs to be covariantized under all the superconformal gauge symmetries (except P-transformations for reasons that are clearly explained in chapter 11 of [37]), whilst the bosonic covariant derivative $\mathcal{D}_\mu$ contains all the bosonic gauge fields except the conformal boosts' f_μ^a which are non-linearly realized.
To obtain a theory of supergravity from the superconformal gauge theory we again need to impose conventional constraints, invariant under the superconformal group of symmetries. This will bring modifications of the transformation rules of the

[6] For further details about the (anti-)commutator structure of the superconformal group we refer to Appendix C of [70].

dependent fields. For instance, the constraint $R(P)_{\mu\nu}{}^a$ is not supersymmetry-invariant, which means that the dependent supersymmetry covariant spin connection $\omega_\mu{}^{ab}(e,b,\psi)$ will transform under supersymmetry with some additional terms compared to the the corresponding independent field (for a clear exposition of this issue, we refer to Chapter 16 of [37]). As a consequence, the Lorentz curvature will be covariant under the new transformation rules of the spin connection only after the addition of new terms proportional to the gravitinos $\psi_\mu{}^i$ and the S-supersymmetry connection $\phi_\mu{}^i$. More generally, the conventional constraints one can impose to obtain a conformal supergravity theory, where the superconformal group acts as space-time symmetry, assume a superconformally invariant form only if new fields are introduced. This is a welcome addition, because new fields are necessary for the construction of the Weyl supermultiplet, based on a simple off-shell counting of degrees of freedom. In fact, the gauge fields considered so far include two fermionic gauge fields, which describe 8 degrees of freedom each (16 total minus 8 because of supersymmetry invariance), and 4 bosonic fields $e_\mu{}^a$, b_μ, $\mathcal{V}_\mu{}^i{}_j$ and A_μ, which encompass a total of 17 degrees of freedom, arranged in $(16-4-6-1)$, $(4-4)$, $(12-3)$ and $(4-1)$ off-shell degrees of freedom respectively. To balance the counting, we must add three auxiliary fields, the (anti-)selfdual tensor $T_{ab}{}^{ij}$, anti-symmetric in both the ab and ij indices (which means this tensor is a singlet under SU(2) transformations) (6), the real scalar D (1) and the chiral spinor χ^i (8). By adding these fields, the fermionic and bosonic degrees of freedom of the Weyl multiplet are balanced and total $24+24$. Accordingly, the conventional constraints express the bosonic fields $\omega_\mu{}^{ab}$ and $f_\mu{}^a$ and the fermionic gauge field $\phi_\mu{}^i$ as a function of the other independent gauge fields (see Appendix B).

Independent fields	$e_\mu{}^a$	b_μ	$\psi_\mu{}^i$	$\mathcal{V}_\mu{}^i{}_j$	A_μ	$T_{ab}{}^{ij}$	χ^i	D

The Q-supersymmetry algebra then closes on covariant general coordinate transformations on all field of the Weyl multiplet, i.e.

$$[\delta(\epsilon_1),\delta(\epsilon_2)](\cdot) = \big(\delta_{\rm cov}(\xi) + \delta_M(\varepsilon) + \delta_K(\Lambda_K) + \delta_S(\eta) + \delta_{\rm gauge}\big)(\cdot)\ , \tag{3.2.3}$$

with the additional covariantization under supersymmetry transformations that depend on the auxiliary fields of the Weyl multiplet as follows

$$\begin{aligned}
\varepsilon_{ab} &= \bar\epsilon_1^i\,\epsilon_2^j\,T_{ab\,ij} + \text{h.c.}\ ,\\
\Lambda_K^a &= \bar\epsilon_1^i\,\epsilon_2^j\,D_b T^{ba} ij - \tfrac{3}{2}\,\bar\epsilon_2^i\,\gamma^a\,\epsilon_{1i}\,D + \text{h.c.}\ ,\\
\eta_i &= 6\,\bar\epsilon_{[1i}\,\epsilon_{2]j}\,\chi^j\ ,
\end{aligned} \tag{3.2.4}$$

Hence the local translations P are traded for super-covariant general coordinate transformations. The parameter of the covariant general coordinate transformation is given in (1.2.11) and it will combine with the gauge fields of the superconformal algebra to give field dependent gauge transformations (see (1.2.12) or (3.1.6)). Finally the gauge transformation δ_{gauge} takes into account all the possible additional, internal gauge symmetries of the theory which commute with the superconformal algebra (the R-symmetry generators are NOT one such example). Once the conventional constraints are imposed and the gauge fields of the Weyl multiplet are recognized to be space-time symmetries, all that is left to introduce are the matter multiplets, which act as compensators. By considering their coupling to gravity, it is possible to obtain different versions of Poincaré supergravity [67, 71]. It is important to stress that few special (super)conformal Lagrangians can also be obtained without the aid of compensating multiplets. We will come back to this issue shortly when we will introduce a covariant version of the Weyl multiplet.

3.3 The chiral and vector multiplet

In this section we want to introduce the basic matter multiplets which act as compensators in the conformal formulation of supergravity. To prepare the reader for the analysis of the following two chapters, superspace and components notations are used in parallel and, for clarity of exposition, the spinor index structure is reinstated.

The first matter multiplet we will treat is the chiral multiplet, already introduced in section 1.2.1. As we mentioned before, a general $\mathcal{N}=2$ complex scalar superfield Φ_S would encompass 256+256 degrees of freedom. One can require such superfield to depend only on chiral Grassmann coordinates via the superfield constraint

$$\bar{D}^{\dot{\alpha}i}\Phi_S = 0 \; , \tag{3.3.1}$$

with the tangent space derivatives defined by (we refer to Appendix A for further details on the notation)

$$\partial_a = \frac{\partial}{\partial x^a} \; , \qquad D_{\alpha i} = \frac{\partial}{\partial \theta^{\alpha i}} + \mathrm{i}(\sigma^a)_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}{}_i \frac{\partial}{\partial x^a} \; , \quad \bar{D}^{\dot{\alpha}i} = \frac{\partial}{\partial \bar{\theta}_{\dot{\alpha}i}} + \mathrm{i}(\bar{\sigma}^a)^{\dot{\alpha}\alpha}\,\theta_\alpha{}^i \frac{\partial}{\partial x^a} \; . \tag{3.3.2}$$

Here our superspace notation reflects the fact that in flat superspace world and tangent-space indices can be identified. In the context of curved superspace a

	A	ψ_i	B_{ij}	F^-_{ab}	Λ_i	C
w	w	$w+\frac{1}{2}$	$w+1$	$w+1$	$w+\frac{3}{2}$	$w+2$
c	$-w$	$-w+\frac{1}{2}$	$-w+1$	$-w+1$	$-w+\frac{3}{2}$	$-w+2$
γ_5		$+$			$+$	

Table 3.1: Weyl (w) and chiral c weights of the chiral multiplet components. The chirality (γ_5) of the fermion fields is also indicated.

vector tangent-space derivative ∇_a and spinor tangent-space derivatives $\nabla_{\alpha i}$ and $\bar{\nabla}^{\dot{\alpha} i}$, direct extensions of the derivatives in (3.3.2), are employed.[7]. The constraint (3.3.1) reduces the off-shell degrees of freedom of Φ_S to 16+16 and yields the scalar chiral superfield Φ [8]. The chiral supermultiplet is composed of a complex scalar field A, a pair of SU(2) doublets of chiral fermions ψ_i and Λ_i, a complex SU(2) triplet of scalars B_{ij} and an anti-selfdual Lorentz tensor F^-_{ab}. These components are given, in terms of the corresponding superfield Φ, by [67, 72],

$$\begin{aligned}
&A := \Phi|_{\theta=0}\ , &&\Psi_{\alpha i} := D_{\alpha i}\Phi|_{\theta=0}\ , &&B_{ij} := -\tfrac{1}{2}D_{ij}\Phi|_{\theta=0}\ ,\\
&F^-_{ab} := -\tfrac{1}{4}(\sigma_{ab})_\alpha{}^\beta D_\beta{}^\alpha\Phi|_{\theta=0}\ , &&\Lambda_{\alpha i} := \tfrac{1}{6}\varepsilon^{jk}D_{\alpha k}D_{ji}\Phi|_{\theta=0}\ , &&C := -2D^4\Phi|_{\theta=0}\ ,
\end{aligned} \tag{3.3.3}$$

where

$$D_{ij} := -D_{\alpha(i}D^\alpha{}_{j)}\,, \qquad D_{\alpha\beta} := -\varepsilon^{ij}D_{(\alpha i}D_{\beta)j}\,. \tag{3.3.4}$$

Under dilatations and chiral U(1) transformations (with constant parameters $\Lambda_{\rm D}$ and $\Lambda_{\rm A}$ in flat superspace) the superspace coordinates change according to

$$x' = \exp\big[-\Lambda_{\rm D}\big]\,x\,, \qquad \theta' = \exp\big[-\tfrac{1}{2}(\Lambda_{\rm D}+{\rm i}\Lambda_{\rm A})\big]\,\theta\,, \qquad \bar\theta' = \exp\big[-\tfrac{1}{2}(\Lambda_{\rm D}-{\rm i}\Lambda_{\rm A})\big]\,\bar\theta\,, \tag{3.3.5}$$

and superfields $\Psi(x,\theta,\bar\theta)$ are usually assigned to transform as

$$\Psi'(x',\theta',\bar\theta') = \exp\big[w\,\Lambda_{\rm D} + {\rm i}c\,\Lambda_{\rm A}\big]\,\Psi(x,\theta,\bar\theta)\,, \tag{3.3.6}$$

where w and c are called the Weyl and the chiral weight. For chiral multiplets these weights are related by $c=-w$. In that case the Weyl weight of A equals w and the highest-θ component C has weight $w+2$ (the list of all the weights assignments is given in Table 3.1). All the components scale homogeneously and since there are no chiral superfield components with Weyl weight less than w it follows that

[7] For simplicity, we will use in this section only the flat superspace notation, which can be easily generalized to curved superspace.

[8] The presence of the second term in the superspace derivatives (3.3.2) justifies the use of the complex space-time coordinate $y^\mu = x^\mu + \epsilon_{ij}\bar{\theta}^i\gamma^\mu\theta^j$ in (1.2.5).

A must be invariant under S-supersymmetry. This implies that it is also invariant under K transformations. Such a chiral superfield is called a *conformal primary* field. All these properties can be derived systematically on the basis of the rigid superconformal algebra using the chiral constraint (3.3.1).

The curved superspace generalization thereof leads to the following Q- and S-supersymmetry transformations rules of a chiral multiplet of generic Weyl weight w in a superconformal background [67, 72]

$$\begin{aligned}
\delta A =& \bar{\epsilon}^i \Psi_i \,, \\
\delta \Psi_i =& 2\, \not{D} A \epsilon_i + B_{ij}\, \epsilon^j + \tfrac{1}{2}\gamma^{ab} F^-_{ab}\, \varepsilon_{ij}\epsilon^j + 2\, w A\, \eta_i \,, \\
\delta B_{ij} =& 2\, \bar{\epsilon}_{(i} \not{D} \Psi_{j)} - 2\, \bar{\epsilon}^k \Lambda_{(i}\, \varepsilon_{j)k} + 2(1-w)\, \bar{\eta}_{(i} \Psi_{j)} \,, \\
\delta F^-_{ab} =& \tfrac{1}{2}\varepsilon^{ij}\, \bar{\epsilon}_i \not{D}\gamma_{ab}\Psi_j + \tfrac{1}{2}\bar{\epsilon}^i \gamma_{ab}\Lambda_i - \tfrac{1}{2}(1+w)\, \varepsilon^{ij}\bar{\eta}_i \gamma_{ab}\Psi_j \,, \\
\delta \Lambda_i =& -\tfrac{1}{2}\gamma^{ab}\not{D} F^-_{ab}\epsilon_i - \not{D} B_{ij}\varepsilon^{jk}\epsilon_k + C\varepsilon_{ij}\, \epsilon^j + \tfrac{1}{4}\big(\not{D} A\, \gamma^{ab} T_{abij} + w\, A\, \not{D}\gamma^{ab} T_{abij}\big)\varepsilon^{jk}\epsilon_k \\
&- 3\, \gamma_a \varepsilon^{jk}\epsilon_k\, \bar{\chi}_{[i}\gamma^a \Psi_{j]} - (1+w)\, B_{ij}\varepsilon^{jk}\, \eta_k + \tfrac{1}{2}(1-w)\, \gamma^{ab}\, F^-_{ab}\eta_i \,, \\
\delta C =& -2\, \varepsilon^{ij}\bar{\epsilon}_i \not{D}\Lambda_j - 6\, \bar{\epsilon}_i \chi_j\, \varepsilon^{ik}\varepsilon^{jl} B_{kl} \\
&- \tfrac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big((w-1)\, \bar{\epsilon}_i \gamma^{ab}\not{D} T_{abjk}\Psi_l + \bar{\epsilon}_i\gamma^{ab} T_{abjk}\not{D}\Psi_l\big) + 2\, w\varepsilon^{ij}\bar{\eta}_i\Lambda_j \,. \qquad (3.3.7)
\end{aligned}$$

In the above equations, the derivatives D are covariantized with respect to the gauge transformations of the superconformal algebra appropriate for each field, as shown in (B.2). Note that the transformation rules are linear in the multiplet field. This feature of the off-shell superconformal formulation is however lacking in the corresponding Poincaré formulation, as we will see in Chapter 5.

A number of remarks are in order at this point. The constraint (3.3.1) manifests itself in the transformation rules of the lowest component of the multiplet, which transforms under Q-supersymmetry only through local parameters of one chirality, ϵ^i. As long as this property is satisfied, a chiral multiplet can be defined. This means that the lowest component A needs not be elementary, but can instead be given as a function of different fields. By making use of techniques known as *multiplet calculus* [68] (some explicit expressions are shown in Appendix B), one can then obtain a *composite* chiral multiplet, whose properties are dictated by the separate multiplets that compose it. For instance, the multiplication of two chiral multiplets of Weyl weight w_1 and w_2 gives rise to a chiral multiplet of weight $w = w_1 + w_2$. Non-trivial functions of chiral superfields can also be defined via the multiplet calculus rules, as long as a proper Weyl weight can be assigned to the resulting multiplet.

It is also possible to reduce chiral multiplet by imposing specific constraints, if their Weyl weight is 1 [72, 73]. One such supermultiplets is the vector multiplet, already introduced in the context of rigid supersymmetry in section 1.2.1, which is also an off-shell representation of the full superconformal algebra. Besides the chiral constraint (3.3.1), vector superfields in flat superspace obey the additional constraint [67, 72],

$$D_{ij}\mathcal{X} = \varepsilon_{ik}\varepsilon_{jl}\,\bar{D}^{kl}\bar{\mathcal{X}}\ , \tag{3.3.8}$$

that halves the number of independent field components by expressing the higher θ components in terms of space-time derivatives of the lower-θ components. Namely, the highest component of the chiral multiplet, Λ_i and C are not independent anymore. Even more importantly, a reality condition is imposed on the SU(2) triplet B_{ij} [9] and a Bianchi identity for the Lorentz tensor F^-_{ab}, which now can be interpreted as a physical field strength. The independent field of the resulting multiplet are denoted by $(X, \Omega_i, Y_{ij}, \hat{F}^-_{ab})$. In the local superconformal context they are connected to the components of the original chiral multiplet through the identifications (generalizing the set of equations (1.2.6)),

$$\begin{aligned}
A|_{\rm vec} &= X\ ,\\
\psi_i|_{\rm vec} &= \Omega_i\ ,\\
B_{ij}|_{\rm vec} &= Y_{ij} = \varepsilon_{ik}\varepsilon_{jl}\,Y^{kl}\ ,\\
F^-_{ab}|_{\rm vec} &= \hat{F}^-_{ab} = F^-_{ab} + \tfrac{1}{4}\,(\bar{\psi}_\rho{}^i\gamma_{ab}\gamma^\rho\,\Omega^j + \bar{X}\,\bar{\psi}_\rho{}^i\gamma^{\rho\sigma}\gamma_{ab}\psi_\sigma{}^j + \text{c.c} - \bar{X}\,T_{ab}{}^{ij})\epsilon_{ij}\ ,\\
\Lambda_i|_{\rm vec} &= -\varepsilon_{ij}\,\not{D}\Omega^j\ ,\\
C|_{\rm vec} &= -2\Box_{\rm c}\bar{X} - \tfrac{1}{4}\hat{F}^+_{ab}\,T^{ab}{}_{ij}\epsilon^{ij} - 3\bar{\chi}_i\Omega^i\ ,
\end{aligned} \tag{3.3.9}$$

where the symbol $\Box_{\rm c} = D^\mu D_\mu$ is the superconformal d'Alambertian. Furthermore we used $\hat{F}_{ab}$ to indicate the supercovariantization of the abelian field strength $F_{ab} = 2\,e_a{}^{[\mu}\,e_b{}^{\nu]}\,\partial_\mu W_\nu$, which is obtained from (3.3.9),

$$\begin{aligned}
\hat{F}_{\mu\nu} =& F^+_{\mu\nu} + F^-_{\mu\nu} - \epsilon^{ij}\bar{\psi}_{[\mu i}(\gamma_{\nu]}\Omega_j + \psi_{\nu]j}\,X) - \epsilon_{ij}\bar{\psi}_{[\mu}{}^i(\gamma_{\nu]}\Omega^j + \psi_{\nu]}{}^j\,\bar{X})\\
&- \tfrac{1}{4}\,(X\,T_{\mu\nu ij}\,\epsilon^{ij} + \bar{X}\,T_{\mu\nu}{}^{ij}\,\epsilon_{ij})\ ,
\end{aligned} \tag{3.3.10}$$

and satisfies the Bianchi identity,

$$D^b(\hat{F}^+_{ab} - \hat{F}^-_{ab} + \tfrac{1}{4}\,X\,T_{ab\,ij}\epsilon^{ij} - \tfrac{1}{4}\,\bar{X}\,T_{ab}{}^{ij}\epsilon_{ij}) + \tfrac{3}{4}\,(\bar{\chi}_i\,\gamma_a\,\Omega_j\,\varepsilon^{ij} - \bar{\chi}^i\,\gamma_a\,\Omega^j\,\varepsilon_{ij}) = 0\ . \tag{3.3.11}$$

In Table 3.2 we show the Weyl and chiral weight assignment for the vector multiplet components. The Q- and S-supersymmetry transformation rules for the vector

[9] This constraint can be immediately derived from the superspace identity (3.3.8) by considering the lowest order terms in the θ^i expansion, which are indeed proportional to $B_{ij}(x)$.

	X	Ω_i	W_μ	Y_{ij}
w	1	$\frac{3}{2}$	0	2
c	-1	$-\frac{1}{2}$	0	0
γ_5		$+$		

Table 3.2: Weyl (w) and chiral c weights of the vector multiplet components. The chirality (γ_5) of the fermion field Ω_i is also indicated.

multiplet in a conformal background take the form,

$$\begin{aligned}
\delta X &= \bar{\epsilon}^i \Omega_i \,, \\
\delta\Omega_i &= 2\not{D} X \epsilon_i + \tfrac{1}{2}\varepsilon_{ij} \hat{F}_{\mu\nu}\gamma^{\mu\nu}\epsilon^j + Y_{ij}\epsilon^j + 2\, X\, \eta_i \,, \\
\delta W_\mu &= \varepsilon^{ij}\bar{\epsilon}_i\big(\gamma_\mu \Omega_j + 2\,\psi_{\mu j}\, X\big) + \varepsilon_{ij}\bar{\epsilon}^i\big(\gamma_\mu\Omega^j + 2\,\psi_\mu{}^j\,\bar{X}\big)\,, \\
\delta Y_{ij} &= 2\,\bar{\epsilon}_{(i}\not{D}\Omega_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{\epsilon}^{(k}\not{D}\Omega^{l)}\,.
\end{aligned} \tag{3.3.12}$$

It is possible although cumbersome to check that the transformation rules (3.3.7) and (3.3.12) satisfy the supercovariant form of the supersymmetry algebra (3.2.3). For instance, since W_μ transforms non-trivially under the U(1) transformations, the algebra of two supersymmetry transformations will close on it as follows, [10]

$$\begin{aligned}
[\delta(\epsilon_1),\delta(\epsilon_2)]W_\mu = &\, \partial_\mu\Lambda + \xi^\rho\partial_\rho W_\mu + \partial_\mu\xi^\rho W_\rho - \partial_\mu(\xi^\rho\, W_\rho) \\
& - \xi^\rho\big(\tfrac{1}{2}\,\varepsilon_{ij}\,\bar{\psi}_\rho{}^i\gamma_\mu\,\Omega^j + \varepsilon_{ij}\,\bar{X}\,\bar{\psi}_\rho{}^i\,\psi_\mu{}^j + \text{h.c.}\big)\,.
\end{aligned} \tag{3.3.13}$$

where the first term is just field dependent abelian gauge transformation with parameter $\Lambda = 2\bar{X}\bar{\epsilon}_2^i\,\epsilon_1^j\,\varepsilon_{ij} + \text{h.c.}$. We also obtain a general coordinate transformation with parameter ξ^μ, given in (1.2.11), covariantized with respect to gauge transformations with parameter $(-\xi^\rho\, W_\rho)$ and supersymmetry transformations with parameter $(-\frac{1}{2}\,\xi^\rho\,\psi_{\rho i})$ (and its hermitian conjugate). This result is general: the algebra of supersymmetry transformations closes into supercovariant general coordinate transformations and, for the gauge connection W_μ one must add an explicit gauge transformations (similar features were already noted in the simplest context of the rigid super-Poincaré algebra in section 1.2.2).

Another example of reduced chiral multiplet is the covariant Weyl multiplet **W**, which can be used, as we will show in the next chapter, for the construction of an higher derivative superconformal invariant without the need of compensating multiplets. The covariant Weyl multiplet stems from a chiral anti-selfdual tensor superfield $W_{\alpha\beta}$, symmetric in $(\alpha\beta)$, subject to the flat superspace constraint

[10] Explicit use is made of the identity $\xi^\rho\, F_{\rho\mu} = \xi^\rho\partial_\rho W_\mu + \partial_\mu\xi^\rho W_\rho - \partial_\mu(\xi^\rho\, W_\rho)$.

$D^{\alpha\beta}W_{\alpha\beta} = \bar{D}_{\dot{\alpha}\dot{\beta}}\bar{W}^{\dot{\alpha}\dot{\beta}}$. Of course, this flat superspace definition can be lifted to curved superspace. We refer to Appendix B for further details on the field content and transformation rules of this reduced supermultiplet on a superconformal background.

Now that we have introduced the gauge and (part of) the matter representations of the superconformal algebra, we are ready to construct conformally invariant supergravity theories. This will be the topic of the next section.

3.4 Supersymmetric density formulas

The interplay between the Weyl multiplet and the compensating chiral/vector multiplets is made explicit in this section, where we will present the $\mathcal{N} = 2$ supersymmetric extensions of the Lagrangians (3.1.14). To keep the treatment as linear as possible, we will first explain how rigidly superconformal invariants are constructed in flat (super)space. This will allow to introduce, in a simple context, a composite multiplet, the kinetic multiplet, which is a central object in the treatment of the next chapters. The analysis will then be extended to curved superspace: we present the supersymmetric density formula for a chiral multiplet coupled to conformal supergravity and define from it a class of superconformal Lagrangians which describe the interaction between gravity and $n_V + 1$ vector multiplets, one of which acts as a compensator.

3.4.1 Supersymmetric Lagrangian in flat superspace

The procedure used to construct superconformally invariant Lagrangians in both flat and curved superspace can be summarized in a few elementary steps. The starting point is to identify a supersymmetric invariant quantity, i.e. a quantity that transforms under supersymmetry transformations into a total derivative (see (2.2.1)), which vanishes when integrated over the whole space-time. Secondly, the choice of this Lagrangian should be physical, i.e. the Lagrangian should describe some dynamical degrees of freedom. Lastly, as we did in the previous examples of this chapter, we impose the invariance under the (rigid) conformal group by choosing compensating fields of the correct Weyl weight. Indeed, as we have already seen before, in both the global and local case, to obtain a conformally invariant action, the Lagrangian density (including compensating fields) must have weights $w = 4$, $c = 0$. Since we already discussed at length the procedure used to make conformal invariance explicit, we will not comment any further on it, but instead present only the final results.

Let us start from the first step, namely identifying a quantity that transforms, under rigid supersymmetry, into a total derivative. The simplest example available, which lies at the base of the procedure for constructing superconformal invariants also in curved superspace, is the highest component of a chiral multiplet which transforms as a total derivative under supersymmetry, $\delta C = \partial_\mu(-2\,\varepsilon^{ij}\bar{\epsilon}_i\gamma^\mu\Lambda_j)$ and is then a supersymmetric invariant. This is still not a physical Lagrangian since it does not contain any kinetic terms. To solve this issue, the general procedure is to make use of the multiplet calculus techniques (see Appendix B) and consider a composite chiral multiplet Φ^2. Subsequently we apply the vector constraint (3.3.8) on the chiral multiplet Φ which automatically implies the identifications (1.2.6) (or (3.3.9) in curved superspace). It is an easy exercise to show that the final Lagrangian reads,

$$\mathcal{L} = -\tfrac{1}{2}\,C|_{\mathcal{X}^2} = 2\,X\Box\bar{X} - \tfrac{1}{2}\bar{\Omega}^i\,\overleftrightarrow{\partial\!\!\!/}\,\Omega_i - \tfrac{1}{4}\,F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{4}\,Y_{ij}\,Y^{ij}\,. \tag{3.4.1}$$

The result obtained is a physical rigidly supersymmetric Lagrangian, quadratic in space-time derivatives (like the first Lagrangian in (3.1.14)), invariant also under the rigid conformal group. Note that it corresponds, up to a total derivatives, to the Lagrangian (1.2.2).

This simple example can of course be worked out also in superspace. In fact, more generally, the construction of supersymmetric invariants in superspace is greatly simplified by the presence of compact structure such as superfields. To understand the procedure used, we remind the reader that translation invariant actions in space-time are constructed by considering Lagrangian densities, which transform as total derivatives and whose variation under the symmetry is hence zero when integrated over the whole space-time. Analogously, to obtain a supersymmetric invariant one must choose a quantity that transforms as a total derivative under supersymmetry. But since the generators of supersymmetry transformations $Q_{\alpha i} = \mathrm{i}\,\partial/\partial\theta^{\alpha i} + (\sigma^a)_{\alpha\dot{\alpha}}\,\bar{\theta}^{\dot{\alpha}}{}_i\,\partial/\partial x^a$ act as rigid translations in superspace [11] (see (1.2.4)), and they are expressed in terms of Grassmann and bosonic derivatives, any generic superfield integrated over the whole superspace is a supersymmetric invariant. Its supersymmetric variation in fact will yield Grassmann total derivatives terms, which vanish by construction when integrated over the full set of Grassmann variables. The remaining terms in the variation are, similarly, bosonic

[11] Note that the supersymmetry generators Q resembles the derivative operators in superspace D in (3.3.2). The former generate isometries while the latter generate covariant translations in flat superspace.

total derivatives which also vanish once integrated over the full space-time.
So the *full superspace* integral of any generic superfield $\Sigma(x^\mu, \theta, \bar{\theta})$

$$\int \mathrm{d}^4\theta \, \mathrm{d}^4\bar{\theta} \, \Sigma \, , \tag{3.4.2}$$

is a supersymmetric invariant. It is important to note now that the above superspace integral will select only the highest component of the superfield Σ. To understand why, remember that Grassmann integration corresponds to Grassmann derivation, an important fact which will be used shortly to define a new type of multiplet.

For now, we want to present a simple application of this abstract superspace discussion, by constructing the Lagrangian (3.4.1), this time starting from a superspace integral. First of all, we notice that the chiral multiplet Φ^2 depends explicitly only on the positive chirality Grassmann coordinates (see (1.2.5)). This means that only supersymmetry transformations of positive chirality will act as a rigid translation in the chiral subspace of the full superspace. Nevertheless the previous considerations still apply from which it follows that supersymmetric invariant can also be obtained by integrating a chiral superfield over a chiral subspace of the full superspace. Specifically, an integral of this kind will again single out the highest component, C, of the chiral multiplet (an integration over the full superspace will obviously yield a vanishing integral). The expression reads

$$\mathcal{L} = \int \mathrm{d}^4\theta \, \Phi^2 \, . \tag{3.4.3}$$

Integrals of the kind of (3.4.3) are referred to as *chiral superspace* integrals. For this invariant to describe dynamical degrees of freedom, we again impose the constraint (3.3.8) on the chiral superfield Φ and obtain a Lagrangian density in terms of the vector multiplet fields.
Finally, we want to note again that the choice of vector multiplets makes the final Lagrangian (3.4.1) also invariant under rigid conformal transformations, since $\mathcal{L}$ has weights $w = 4$ and $c = 0$ [12].

The procedure explained so far to construct rigidly supersymmetric (and conformal) invariants by using full or chiral superspace integrals is very general and can also be used to obtain higher derivative Lagrangians (such as the second Lagrangian in (3.1.14)). To show how, let us introduce the so-called *kinetic multiplet.* The kinetic multiplet appeared first in the context of $\mathcal{N} = 1$ tensor calculus [74]

[12] To prove this statement we note that $\int d\theta$ transforms under dilatations and U(1) transformations as an inverse Grassmann coordinate (see (3.3.5)).

where it was used to construct a Lagrangian for the kinetic terms. To maintain a uniform treatment throughout this work, in here we will only consider the $\mathcal{N}=2$ case, treated in [67]. For $\mathcal{N}=2$, the superspace depends on four θ and four $\bar\theta$ Grassmann coordinates, so that we have the following identification, up to total derivative terms,

$$\int \mathrm{d}^4\theta\, \mathrm{d}^4\bar\theta\, \Phi'\, \bar\Phi = \int \mathrm{d}^4\theta\, \Phi' \left(\bar D^4 \bar\Phi\right) = A' \,\Box\Box \bar A + \cdots \,, \tag{3.4.4}$$

where $\bar D^4 = \frac{1}{48}\varepsilon_{ik}\varepsilon_{jl}\, \bar D^{ij}\, \bar D^{kl}$ and A and A' are the lowest-θ components of the chiral superfields Φ and Φ', respectively. Obviously this class of Lagrangians defines a rigidly superconformal version (in flat space) of the second Lagrangian in (3.1.14)[13]. Again, we note that in order for the action to be superconformally invariant, the chiral superfields Φ and Φ' must both have vanishing Weyl weights, implying that A and A' are scale invariant.

The intermediate equality in (3.4.4) involves the so-called $\mathcal{N}=2$ *kinetic multiplet* $\mathbb{T}(\bar\Phi)$ [67], conventionally normalized as $\mathbb{T}(\bar\Phi) := -2\, \bar D^4 \bar\Phi$. When Φ has zero Weyl weight the highest-θ component of the chiral superfield Φ, denoted by C, is S-supersymmetric. Since $\bar C$ equals the lowest-θ component of $\mathbb{T}(\bar\Phi)$, the kinetic multiplet is therefore a conformal primary chiral superfield. The kinetic multiplet itself thus has Weyl weight $w=2$. Its flat-space components are

$$\begin{aligned} A|_{\mathbb{T}(\bar\Phi)} &= \bar C \,, & \Psi_i|_{\mathbb{T}(\bar\Phi)} &= -2\, \varepsilon_{ij}\, \partial\!\!\!/ \Lambda^j \,, \\ B_{ij}|_{\mathbb{T}(\bar\Phi)} &= -2\, \varepsilon_{ik}\varepsilon_{jl}\, \Box B^{kl} \,, & F^-_{ab}|_{\mathbb{T}(\bar\Phi)} &= -4\big(\delta_a{}^{[c}\delta_b{}^{d]} - \tfrac12 \varepsilon_{ab}{}^{cd}\big)\partial_c\partial^e F^+_{ed} \,, \\ \Lambda_i|_{\mathbb{T}(\bar\Phi)} &= 2\, \Box\, \partial\!\!\!/ \Psi^j \varepsilon_{ij} \,, & C|_{\mathbb{T}(\bar\Phi)} &= 4\, \Box\Box \bar A \,. \end{aligned} \tag{3.4.5}$$

They transform as a chiral multiplet, while depending on the components of the anti-chiral multiplet $\bar\Phi$.

3.4.2 Superconformal Lagrangian in curved superspace

The two types of actions discussed so far in flat superspace have a straightforward extension to curved superspace. They require general chiral multiplets and vector multiplets, which are contained respectively in chiral and reduced chiral superfields. To couple these to conformal supergravity in superspace requires merely the covariantization of the chiral constraint and the reducibility constraint, respectively. Practically speaking, the flat superspace derivatives $(\partial_a, D_{\alpha i}, D^{\dot\alpha i})$ are

[13] It is easy to see that, in the $\mathcal{N}=1$ case, the kinetic multiplet is given by $\mathbb{T}(\bar\Phi) \sim \bar D^2 \bar\Phi$. This explains why it was originally used to obtain the kinetic terms.

substituted by $(\nabla_a, \nabla_{\alpha i}, \nabla^{\dot{\alpha} i})$, which include all the proper superconformal covariantizations (we refer to [70] and references therein for further details).
Just as in flat superspace, invariant actions in curved superspace are constructed in two ways. A full superspace integral involves an integral over the eight Grassmann coordinates of some superspace Lagrangian, which we denote using the symbol $\mathscr{L}$ (to distinguish it from a component Lagrangian $\mathcal{L}$),

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\, E\, \mathscr{L} \ . \tag{3.4.6}$$

The measure factor $E = \mathrm{Ber}(E_M{}^A)$ is the Berezinian (or superdeterminant) of the superspace vielbein and plays the same role as the vierbein determinant e on a bosonic manifold. In order for the action to be invariant under the supergravity gauge group, the superspace Lagrangian $\mathscr{L}$ must be a conformal primary scalar with Weyl and chiral weight zero. A chiral superspace integral can instead be written as

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \mathscr{L}_{\mathrm{ch}} \,, \tag{3.4.7}$$

where $\mathcal{E}$ is the appropriate chiral measure and the Lagrangian $\mathscr{L}_{\mathrm{ch}}$ must be covariantly chiral (i.e. subject to $\bar{\nabla}^{\dot{\alpha} i}\mathscr{L}_{\mathrm{ch}} = 0$) and a conformal primary with Weyl weight 2 and chiral weight -2.[14] Hence, $\mathscr{L}_{\mathrm{ch}}$ must be a chiral superfield with $w = -c = 2$. Generally, any integral over the full superspace can be rewritten (up to a total derivative) as an integral over chiral superspace,

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathrm{d}^4\bar{\theta}\, E\, \mathscr{L} = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \bar{\nabla}^4 \mathscr{L} \tag{3.4.8}$$

using the chiral projection operator $\bar{\nabla}^4$,

$$\bar{\nabla}^4 = \tfrac{1}{48}\varepsilon_{ik}\varepsilon_{jl}\bar{\nabla}^{kl}\bar{\nabla}^{ij} \ , \quad \bar{\nabla}^{ij} := \bar{\nabla}_{\dot{\alpha}}{}^{(i}\bar{\nabla}^{\dot{\alpha} j)} \ . \tag{3.4.9}$$

This is a non-trivial statement in curved superspace: one must check that $\bar{\nabla}^4\mathscr{L}$ is indeed chiral and annihilated by S-supersymmetry. Of course, superspace integrals can be related to the usual integrals over the bosonic manifold by performing the θ integrals. For instance, from (3.4.7) one obtains [75]

$$\int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\, \mathscr{L}_{\mathrm{ch}} = \int \mathrm{d}^4x\, \mathcal{L}_{\mathrm{ch}} \tag{3.4.10}$$

[14] To understand why, remember that the Weyl and chiral weights of the measure $\mathcal{E}$ can be easily derived on the tangent space, where they correspond to the weights of the flat superspace measure $\mathrm{d}^4x\, \mathrm{d}^4\theta$. From (3.3.5), one obtains $w = -2$ and $c = 2$ for the chiral measure $\mathcal{E}$.

where $\mathcal{L}_{\text{ch}}$ is given, in four-component notation by [72],

$$\begin{aligned} e^{-1}\mathcal{L}_{\text{ch}} = & C - \varepsilon^{ij}\,\bar{\psi}_{\mu i}\gamma^{\mu}\Lambda_j - \tfrac{1}{8}\bar{\psi}_{\mu i}T_{ab\,jk}\gamma^{ab}\gamma^{\mu}\Psi_l\,\varepsilon^{ij}\varepsilon^{kl} - \tfrac{1}{16}A(T_{ab\,ij}\varepsilon^{ij})^2 \\ & - \tfrac{1}{2}\bar{\psi}_{\mu i}\gamma^{\mu\nu}\psi_{\nu j}\,B_{kl}\,\varepsilon^{ik}\varepsilon^{jl} + \varepsilon^{ij}\bar{\psi}_{\mu i}\psi_{\nu j}(F^{-\mu\nu} - \tfrac{1}{2}A\,T^{\mu\nu}{}_{kl}\,\varepsilon^{kl}) \\ & - \tfrac{1}{2}\varepsilon^{ij}\varepsilon^{kl}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar{\psi}_{\mu i}\psi_{\nu j}(\bar{\psi}_{\rho k}\gamma_{\sigma}\Psi_l + \bar{\psi}_{\rho k}\psi_{\sigma j}\,A) + \text{h.c.}\,. \end{aligned} \tag{3.4.11}$$

This superconformally invariant Lagrangian describes the couplings of a $w = 2$ chiral multiplet with the superconformal background fields. However, it does not contain any kinetic terms for the fields, so it does not describe any physical degrees of freedom. To solve this issue, one could use the techniques of multiplet calculus and obtain, from a composite chiral multiplet, a physical Lagrangian for the component multiplets, as was done for the rigid supersymmetric Lagrangian in the previous section, equations (3.4.1)-(3.4.3). The curved space generalization of that Lagrangian reads simply

$$\tfrac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\mathcal{X}^2\ . \tag{3.4.12}$$

Its component expression was given in [72]. A generalization thereof to include an arbitrary number $n_V + 1$ of vector multiplets was given in [76]. There, the starting point was to consider the lowest component of a composite chiral multiplet $A|_{\text{comp}}$, whose most general form is a function of the vector multiplets scalars X^I, $I = 0, \ldots, n_V$, i.e.

$$A|_{\text{comp}} = F(X^I)\ .$$

Some restrictions must be imposed on this function. First of all, for $A|_{\text{comp}}$ to transform (anti-)chirally under supersymmetry transformation, the function F must be (anti-)holomorphic, namely it should depend only on X^I ($\bar{X}^I$). Secondly, to obtain an invariant Lagrangian from (3.4.11), $A|_{\text{comp}}$ must have weight $w = 2$ so the function F must be homogeneous of degrees 2,

$$F(\lambda\,X^I) = \lambda^2\,F(X^I)\ ,$$

for any complex $\lambda \neq 0$, since the lowest components of the vector multiplets $\mathcal{X}^I$ have Weyl weight 1. The function F is then referred to as the *prepotential*. At this point, one can use multiplet calculus results (see (B.16)) to obtain the values for each component of the composite chiral multiplets, which can then be plugged in the chiral density formula (3.4.11) to yield the following superconformal

Lagrangian (for convenience we show only the bosonic terms)

$$\begin{aligned} e^{-1}\,\mathcal{L} \;=\;& \mathrm{i}\mathcal{D}^\mu F_I\,\mathcal{D}_\mu \bar{X}^I - \mathrm{i}F_I\,\bar{X}^I(\tfrac{1}{6}\mathcal{R} - D) - \tfrac{1}{8}\mathrm{i}F_{IJ}\,Y^I_{ij}Y^{Jij} \\ & +\tfrac{1}{4}\mathrm{i}F_{IJ}(F^{-I}_{ab} - \tfrac{1}{4}\bar{X}^I T^{ij}_{ab}\varepsilon_{ij})(F^{-Jab} - \tfrac{1}{4}\bar{X}^J T^{ijab}\varepsilon_{ij}) \\ & -\tfrac{1}{8}\mathrm{i}F_I(F^{+I}_{ab} - \tfrac{1}{4}X^I T_{abij}\varepsilon^{ij})T^{ab}_{ij}\varepsilon^{ij} - \tfrac{1}{32}\mathrm{i}F(T_{abij}\varepsilon^{ij})^2 + \text{h.c.}\,, \end{aligned} \tag{3.4.13}$$

where F_I and F_{IJ} are the first and second derivative of the prepotential with respect to the scalar fields X^I. Note that this theory is, *per se*, inconsistent as it is easy to realize by considering the on-shell theory where the auxiliary field D is substituted by its dynamical equation $F_I\,\bar{X}^I = 0$, which would make also the Einstein-Hilbert term disappear. To consistently gauge fix $\mathcal{N}=2$ superconformal gravity to Poincaré supergravity one then needs two compensating multiplets. One of those is typically a vector multiplet arbitrary chosen among the $n_V + 1$, while for the second multiplet different choices are available[15]. These choices are presented in the chapter 5, where the conditions imposed by full supersymmetry on the superconformal fields will be analyzed (for a pedagogical discussion on the gauge-fixing procedure we refer instead to [37] and reference therein).
We note at this point that the Lagrangian (3.4.13) can be easily generalized to include higher order curvature terms simply by considering a prepotential of the form $F(X^I, \mathbf{W}^2)$, with $\mathbf{W}$ covariant Weyl multiplet (see Appendix B). The corresponding class of supersymmetric invariants was analyzed in [9, 11] to obtain the first corrections to entropy of supersymmetric black holes due to higher derivative couplings. We will come back to the result of this calculation in the next chapter.

For completeness we note that also the higher derivative Lagrangian (3.4.4) generalizes to curved conformal superspace in a completely straightforward manner:

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}\,E\,\Phi'\bar{\Phi} = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Phi'\bar{\nabla}^4\bar{\Phi}\;. \tag{3.4.14}$$

We have emphasized that the same action can be written using (3.4.8) as a chiral integral of the product of Φ' and the kinetic multiplet $\mathbb{T}(\bar{\Phi})$. At the component level, the Lagrangian is the supersymmetrization of $A'\Box_{\mathrm{c}}\Box_{\mathrm{c}}\bar{A}$ and was analyzed in [65]. This class of higher derivative action admits an obvious generalization in the presence of several chiral multiplets Φ^I with weights w_I. Introducing a

[15]This can be understood on the base of degrees of freedom counting. The minimal off-shell representation of Poincaré supergravity contains in fact $40+40$ degrees of freedom [66, 77], whereas the Weyl multiplet and a vector multiplet add up to $32+32$ degrees of freedom. Another compensating multiplet is then needed, which contains exactly $8+8$ degrees of freedom. The three viable choices are the non-linear multiplet, the tensor multiplet and the hypermultiplet [67]

homogeneous function $\mathcal{H}(\Phi, \bar{\Phi})$ of weight zero,

$$\sum_I w_I \Phi^I \mathcal{H}_I = 0 \ , \tag{3.4.15}$$

where $\mathcal{H}_I := \partial \mathcal{H} / \partial \Phi^I$, one can construct a higher derivative action by integrating $\mathcal{H}$ over the full superspace,[16]

$$\int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \mathrm{d}^4\bar{\theta} \, E \, \mathcal{H} \ . \tag{3.4.16}$$

By virtue of the formula (3.4.8) and its complex conjugate, one can show that the action is invariant under the Kähler-like transformations

$$\mathcal{H} \to \mathcal{H} + \Lambda(\Phi) + \bar{\Lambda}(\bar{\Phi}) \tag{3.4.17}$$

where the holomorphic function $\Lambda(\Phi)$ must similarly be homogeneous. It follows that the component action will depend only on the Kähler metric $\mathcal{H}_{I\bar{J}}$, which is subject to the homogeneity condition

$$\sum_I w_I \Phi^I \mathcal{H}_{I\bar{J}} = 0 \ . \tag{3.4.18}$$

The locally supersymmetric version was analyzed in [65], with particular attention paid to the special case where the chiral multiplets were vector multiplets $\mathcal{X}^I$ with $w = 1$ or the Weyl-squared chiral multiplet $W^{\alpha\beta} W_{\alpha\beta}$ with $w = 2$. This class can be broadened further while maintaining the Kähler structure by considering the chiral multiplets Φ^I to be themselves composite in various ways.
The situation will be quite different for the class of supersymmetric invariants analyzed in the next chapter, for which, as we shall show, the homogeneity conditions will not be satisfied.

[16] Similar structures were considered in the context of low-energy effective actions in flat space [78–82].

Chapter 4

A new class of higher derivative couplings in $\mathcal{N} = 2$ supergravity

This chapter is directed to an extension of certain classes of higher derivative invariants in $\mathcal{N} = 2$ supergravity. From the technical point of view, such a study is facilitated by the fact that there exist formulations of $\mathcal{N} = 2$ supergravity where supersymmetry is realized off-shell, i.e. without involving the equations of motion associated with specific Lagrangians.[1] In that case there exist well-established methods such as superspace and component calculus, treated in the previous chapter, that enable a systematic study. There exists a healthy variety of approaches: in this chapter we will make use of conformal superspace [75] which is closely related to the superconformal multiplet calculus [68, 69] that is carried out in component form.[2] We will be using these methods in parallel.

Some higher derivative invariants in $\mathcal{N} = 2$ supersymmetry and supergravity have been known for some time, such as those involving functions of the field strengths for supersymmetric gauge theories [78–82], the chiral invariant containing the square of the Weyl tensor, possibly coupled to matter chiral multiplets [86], which we hinted at at the end of the previous chapter (see also Appendix B), and invariants for tensor multiplets [87]. A full superspace integral has also been used to generate an $\mathcal{R}^4$ term in the context of "minimal" Poincaré supergravity [88]. More recently, a large class of higher derivative supersymmetric invariants

[1] For higher extended supersymmetry the application of methods such as these becomes problematic for the simple reason that off-shellness is not realized, up to a few notable exceptions such as the Weyl multiplet in $\mathcal{N} = 4$ supergravity.

[2] Other off-shell methods include the $\mathcal{N} = 2$ harmonic [83] and projective [84, 85] superspace approaches, which make it possible to realize the most general off-shell supergravity-matter couplings.

was constructed using the superconformal multiplet calculus, corresponding to integrals over the full $\mathcal{N} = 2$ superspace [65].[3] This action involved arbitrary chiral multiplets, which could play the role of composite fields consisting of homogeneous functions of vector multiplets. This entire class had the remarkable property that the corresponding invariants and their first derivatives (with respect to the fields or to coupling constants) vanish in a fully supersymmetric background. This result ensures that these invariants do not contribute to either the entropy or the electric charge of BPS black holes. Actions of this class have also been used recently to study supergravity counterterms and the relation between off-shell and on-shell results [89]. Furthermore, in [90], higher derivative actions were constructed in projective superspace by allowing vector multiplets and/or tensor multiplets to be contained in similar homogeneous functions of other multiplets. Because the invariants derived in [65, 87, 90] can involve several independent homogeneous functions at the same time, they cannot be classified concisely, although this forms no obstacle when considering applications.

Nevertheless, these broad classes do not exhaust the possibilities for higher derivative invariants. A previously unknown 4D higher derivative term was identified recently in [91] when applying off-shell dimensional reduction to the 5D mixed gauge-gravitational Chern-Simons term [92]. It turned out to involve a Ricci-squared term $\mathcal{R}^{ab}\mathcal{R}_{ab}$ multiplied by the ratio of vector multiplets. This curvature combination does not appear in the previous known invariants and is suggestive of the Gauss-Bonnet term, whose $\mathcal{N} = 2$ extension has, remarkably, never been constructed before.

A related issue, also involving the Gauss-Bonnet term, arose several years ago in a different context: the calculation of black hole entropy from higher derivative couplings in an effective supergravity action. It was observed in a certain model [93] that one could calculate the entropy of a BPS black hole by considering the effective action involving the product of a dilaton field with the Gauss-Bonnet term *without* supersymmetrization. This result agreed with the original calculation based on the square of the Weyl tensor, which depended critically on its full supersymmetrization [9, 11], but it remained unclear why the non-supersymmetric approach of [93] would yield the same answer and whether the outcome was indicative of some deeper result.

[3] The action considered in [88] can be interpreted within the conformal framework of [65] as the full superspace integral of $\mathcal{H} = (T_{ab\,ij})^2 (T^{cd\,kl})^2 / (X_0 \bar{X}_0)^2$ where X_0 is a compensating vector multiplet, in the presence of an additional non-linear multiplet.

Both of these issues would be resolved by a full knowledge of the $\mathcal{N}=2$ Gauss-Bonnet invariant and the broader class of higher derivative supersymmetric invariants to which it belongs. The goal of this chapter is to present this class and to discuss whether it shares the same properties with the previously explored classes of invariants. Furthermore, a detailed analysis of the dimensional reduction of the 5D (supersymmetrization of the) mixed gauge-gravitational Chern-Simons term, from which the Gauss-Bonnet term in $\mathcal{N}=2$ 4D supergravity will be given.

4.1 A novel class of higher derivative couplings

In this section we want to first introduce the non-supersymmetric conformal expressions for the Gauss-Bonnet invariant. Subsequently, in analogy to the previous chapter, we will give the supersymmetric generalizations using the formal superspace notation, leaving the explicit details for the following sections.

Let us first briefly recall some features of the Gauss-Bonnet invariant as well as other invariants quadratic in the Riemann tensor. In this introductory section we restrict ourselves to bosonic fields; the supersymmetric extension will be discussed in the subsequent sections. In four space-time dimensions there are two terms quadratic in the Riemann tensor whose space-time integral defines topological invariants: these are the Pontryagin density,

$$\mathcal{L}_{\mathrm{P}} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{R}_{\mu\nu}{}^{\lambda\tau}\,\mathcal{R}_{\rho\sigma\lambda\tau}\,, \tag{4.1.1}$$

and the Euler density,

$$e^{-1}\mathcal{L}_{\chi} = \tfrac{1}{4}\,\varepsilon^{\mu\nu\rho\sigma}\,\mathcal{R}_{\mu\nu}{}^{\lambda\tau}\,\mathcal{R}_{\rho\sigma}{}^{\delta\epsilon}\,\varepsilon_{\lambda\tau\delta\epsilon} = \mathcal{R}^{\mu\nu\rho\sigma}\mathcal{R}_{\mu\nu\rho\sigma} - 4\mathcal{R}^{\mu\nu}\mathcal{R}_{\mu\nu} + \mathcal{R}^2\,. \tag{4.1.2}$$

The integral of the Euler density is the Gauss-Bonnet invariant. Their difference can be made more apparent by trading the Riemann tensor for the Weyl tensor, $C_{\mu\nu}{}^{\rho\sigma} = \mathcal{R}_{\mu\nu}{}^{\rho\sigma} - 2\,\delta_{[\mu}{}^{[\rho}\,\mathcal{R}_{\nu]}{}^{\sigma]} + \tfrac{1}{3}\delta_{\mu}{}^{[\rho}\delta_{\nu}{}^{\sigma]}\,\mathcal{R}$,

$$\mathcal{L}_{\mathrm{P}} = \tfrac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\,C_{\mu\nu}{}^{\lambda\tau}\,C_{\rho\sigma\lambda\tau}, \qquad e^{-1}\mathcal{L}_{\chi} = C^{\mu\nu\rho\sigma}\,C_{\mu\nu\rho\sigma} - 2\,\mathcal{R}^{\mu\nu}\,\mathcal{R}_{\mu\nu} + \tfrac{2}{3}\mathcal{R}^2\,. \tag{4.1.3}$$

These equations are not a good basis for discussing supersymmetric extensions. Rather, it turns out that the following combinations are more natural, from a supersymmetric perspective,

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\mathrm{W}}{}^{\pm} &= \tfrac{1}{2}C_{\mu\nu}{}^{ab}\,C^{\mu\nu cd}\big[\eta_{ac}\eta_{bd} \pm \tfrac{1}{2}\varepsilon_{abcd}\big] = C_{\mu\nu}{}^{ab\pm}C^{\mu\nu\pm}_{\ \ \ \ ab}\,,\\
e^{-1}\mathcal{L}_{\mathrm{NL}} &= -\,\mathcal{R}^{\mu\nu}\,\mathcal{R}_{\mu\nu} + \tfrac{1}{3}\mathcal{R}^2\,.
\end{aligned}
\tag{4.1.4}
$$

The first expression is the square of the anti-selfdual (selfdual) Weyl tensor, which belongs to a chiral (anti-chiral) multiplet (see (B.19)), and whose superextension has been known for a long time [86]. The supersymmetric extension of the second term will be one of the results of this chapter.

To explain the strategy we will follow, let us first restrict ourselves for simplicity to bosonic terms only. It is possible to construct invariants which involve the Weyl tensor as any scalar field of Weyl weight zero times the square of the Weyl tensor defines a conformally invariant Lagrangian. But how to include invariants such as the four-dimensional Gauss-Bonnet term is less obvious. As it turns out, the crucial assumption made so far in the construction of (super)conformal invariants is that the scalar fields transform linearly under dilatations. To demonstrate how the situation changes when this is not the case, let us repeat the previous construction (3.1.11) for $\ln\phi$, which transforms inhomogeneously under dilatations, $\delta_{\mathrm{D}}\ln\phi = w\,\Lambda_{\mathrm{D}}$. We derive the following definitions,

$$
\begin{aligned}
D_\mu \ln\phi &= \mathcal{D}_\mu \ln\phi = \partial_\mu \ln\phi - w\,b_\mu\,,\\
D_\mu D_a \ln\phi &= \mathcal{D}_\mu D_a \ln\phi + w\,f_{\mu a}\,,\\
D_\mu \Box_{\mathrm{c}}\,\ln\phi &= \mathcal{D}_\mu \Box_{\mathrm{c}} \ln\phi - 2\,f_\mu{}^a D_a \ln\phi\,,\\
\Box_{\mathrm{c}}\Box_{\mathrm{c}}\,\ln\phi &= \mathcal{D}_a D^a \Box_{\mathrm{c}} \ln\phi + 2f\,\Box_{\mathrm{c}} \ln\phi - 2\,f_{\mu a}\,D^\mu D^a \ln\phi\,.
\end{aligned}
\tag{4.1.5}
$$

The equations above show an interesting systematics, namely that, after applying a certain number of covariant derivatives on $\ln\phi$, these expressions take the same form as in (3.1.12) with $w = 0$. However, it is important to realize that the details implicit in the multiple covariant derivatives will still depend on the characteristic features associated with the logarithm. The same observation can be made for the K-transformations of multiple derivatives which also transform as if one were dealing with a $w = 0$ scalar field,

$$
\begin{aligned}
\delta_{\mathrm{K}} D_a \ln\phi &= -\,w\,\Lambda_{\mathrm{K}a}\,, \qquad \delta_{\mathrm{D}} D_a \ln\phi = \Lambda_{\mathrm{D}}\,D_a \ln\phi\,,\\
\delta_{\mathrm{K}} D_\mu D_a \ln\phi &= -\,\big[\Lambda_{\mathrm{K}\mu}\,D_a + \Lambda_{\mathrm{K}a}\,D_\mu\big]\ln\phi + e_{\mu a}\Lambda_{\mathrm{K}}{}^b\,D_b \ln\phi\,,\\
\delta_{\mathrm{K}} \Box_{\mathrm{c}} \ln\phi &= 2\,\Lambda_{\mathrm{K}}{}^a\,D_a \ln\phi\,,\\
\delta_{\mathrm{K}} D_\mu \Box_{\mathrm{c}} \ln\phi &= -\,2\,\Lambda_{\mathrm{K}\mu}\,\Box_{\mathrm{c}} \ln\phi + 2\,\Lambda_{\mathrm{K}}{}^a\,D_\mu D_a \ln\phi\,,\\
\delta_{\mathrm{K}} \Box_{\mathrm{c}}\Box_{\mathrm{c}} \ln\phi &= 2\Lambda_{\mathrm{K}}{}^a\,\Box_{\mathrm{c}}\,D_a \ln\phi - 2\,\Lambda_{\mathrm{K}}{}^a D_a \Box_{\mathrm{c}} \ln\phi = 0\,.
\end{aligned}
\tag{4.1.6}
$$

In four space-time dimensions the only conformally invariant Lagrangian based on the above expression must be equal to $\Box_{\rm c}\Box_{\rm c}\ln\phi$, possibly multiplied with a scalar field of zero Weyl weight. This constitutes the non-linear version of the second Lagrangian in (3.1.14), namely $\sqrt{g}\,\phi'\Box_{\rm c}\Box_{\rm c}\ln\phi$, where ϕ has a non-vanishing, but arbitrary Weyl weight w and ϕ' has zero Weyl weight. Taking the explicit form of $\Box_{\rm c}\Box_{\rm c}\ln\phi$ this Lagrangian is given by

$$\sqrt{g}\,\phi'\Box_{\rm c}\Box_{\rm c}\ln\phi = \sqrt{g}\,\phi'\Big\{\big(\mathcal{D}^2\big)^2\ln\phi - 2\,\mathcal{D}^\mu\big[\big(2\,f_{(\mu}{}^a e_{\nu)a} - f\,g_{\mu\nu}\big)\mathcal{D}^\nu\ln\phi\big] \\ + w\big[\mathcal{D}^2 f + 2\,f^2 - 2\,(f_\mu{}^a)^2\big]\Big\}. \qquad (4.1.7)$$

There are two features to note about this Lagrangian. The first is that its dependence on $\ln\phi$ is isolated in the first line on the right-hand side, which is a total derivative when ϕ' is constant. In other words, the action is independent of the choice of $\ln\phi$ when ϕ' is constant. The second feature is that the Lagrangian is K-invariant, so all the b_μ terms must drop out. Equivalently, one can adopt a K-gauge where $b_\mu = 0$. Using (3.1.8), one finds

$$\mathcal{D}^2 f + 2\,f^2 - 2\,(f_\mu{}^a)^2 = \tfrac{1}{6}\mathcal{D}^2\mathcal{R} - \tfrac{1}{2}\mathcal{R}^{ab}\,\mathcal{R}_{ab} + \tfrac{1}{6}\mathcal{R}^2\ , \qquad (4.1.8)$$

which is proportional to $\mathcal{L}_{\rm NL}$ (c.f. 4.1.4) up to a total covariant derivative. When combined with the square of the Weyl tensor with an appropriate relative normalization one obtains the Gauss-Bonnet invariant up to a total covariant derivative

$$\begin{aligned} e^{-1}\mathcal{L}_\chi =&\, C^{abcd}C_{abcd} + 4\,w^{-1}\,\Box_{\rm c}\Box_{\rm c}\ln\phi \\ =&\, C^{abcd}C_{abcd} - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{2}{3}\mathcal{R}^2 + \tfrac{2}{3}\mathcal{D}^2\mathcal{R} \\ &+ 4w^{-1}\Big\{\big(\mathcal{D}^2\big)^2\ln\phi + \mathcal{D}^a\Big(\tfrac{2}{3}\mathcal{R}\,\mathcal{D}_a\ln\phi - 2\,\mathcal{R}_{ab}\,\mathcal{D}^b\ln\phi\Big)\Big\}\ , \end{aligned} \qquad (4.1.9)$$

where we have taken the gauge $b_\mu = 0$ in the second equality. Discarding the (explicit) total derivatives, this result reduces to the Euler density. Alternatively the dilatation gauge $\phi = 1$ reduces it to

$$e^{-1}\mathcal{L}_\chi = C^{abcd}C_{abcd} - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{2}{3}\mathcal{R}^2 + \tfrac{2}{3}\mathcal{D}^2\mathcal{R}\,. \qquad (4.1.10)$$

This differs from the usual Euler density (4.1.2) by an explicit total derivative. Obviously additional invariants are obtained by multiplying this result with a $w = 0$ independent (composite or elementary) scalar field ϕ'.

The above relatively simple bosonic Lagrangians indicate how higher derivative couplings will be characterized in this chapter. As we shall argue in the next section, all these Lagrangians have an $\mathcal{N} = 2$ supersymmetric counterpart based on chiral superfields. These include the well-known Lagrangians quadratic in derivatives, the class of higher derivative Lagrangians discussed in [65], and a new class of Lagrangians based on $\sqrt{g}\,\phi'\,\Box_{\rm c}\Box_{\rm c}\ln\phi$, where ϕ' and ϕ are the lowest components of chiral multiplets with $w' = 0$ and $w \neq 0$. This last class must contain the $\mathcal{N} = 2$ supersymmetric higher derivative invariant whose traces were found upon reducing the 5D higher derivative invariant coupling to four dimensions [91]. We will give a detailed overview in the last section of this chapter.

4.1.1 logΦ chiral multiplet and Gauss-Bonnet invariant in flat $\mathcal{N} = 2$ superspace

Following the same procedure outlined in the previous chapter, incorporating the Lagrangian (4.1.7) in the context of chiral multiplets seems rather obvious. Taking $\bar\Phi$ to be an anti-chiral multiplet of weight w, we consider the chiral integral

$$\int \mathrm{d}^4\theta\, \Phi' \left(\bar D^4 \ln\bar\Phi\right) = A'\,\Box\Box\ln\bar A + \cdots \ , \tag{4.1.11}$$

where Φ' is a $w = 0$ chiral superfield and A' denotes its lowest component. Naively, this resembles the previous action (3.4.4), but there is a crucial difference: the anti-chiral multiplet $\bar\Phi$ has arbitrary Weyl weight w and so $\ln\bar\Phi$ transforms non-linearly under dilatations. Remarkably, the corresponding kinetic multiplet $\mathbb{T}(\ln\bar\Phi) := -2\,\bar D^4\ln\bar\Phi$ is nevertheless a conformal primary chiral multiplet in flat superspace.[4] In other words, it transforms linearly under dilatations with $w = 2$ and its lowest component is invariant under S-supersymmetry.

We should stress that the non-linearities in $\mathbb{T}(\ln\bar\Phi)$ are of two different types. First of all, the logarithm leads to an anti-chiral superfield that will depend non-linearly on the components of $\bar\Phi$. Because of this behavior, the superconformal transformations will also be realized in a non-linear fashion, and as a result the covariantizations that are required in curved superspace will involve non-linearities depending on the Weyl weight w. In spite of all these complications, there is a rather systematic way of writing the various components of $\mathbb{T}(\ln\bar\Phi)$, although the various explicit expressions tend to become rather complicated, especially because they involve higher space-time derivatives. These non-linearities are the reason why the kinetic multiplet $\mathbb{T}(\ln\bar\Phi)$ differs in a crucial way from the original one $\mathbb{T}(\bar\Phi)$.

[4] The multiplet $\mathbb{T}(\ln\bar{\mathcal{X}})/\mathcal{X}^2$ was considered in [80] with $\mathcal{X}$ a reduced chiral superfield, and shown to be a $w = 0$ conformal primary. The extension of that analysis to $\mathbb{T}(\ln\bar\Phi)$ for an arbitrary anti-chiral multiplet $\bar\Phi$ is completely straightforward.

As a first step in constructing the components of $\mathbb{T}(\ln\bar\Phi)$, we must replace the components of $\bar\Phi$ in (3.4.5) with those of $\ln\bar\Phi$. This will simply involve replacing $\bar A \to \bar A|_{\ln\bar\Phi}$, …, $\bar C \to \bar C|_{\ln\bar\Phi}$, where the components of the multiplet $\ln\Phi$ are identified as (see (B.16))

$$\begin{aligned}
A|_{\ln\Phi} &= \ln A\,, & \Psi_i|_{\ln\Phi} &= \frac{\Psi_i}{A}\,,\\
B_{ij}|_{\ln\Phi} &= \frac{B_{ij}}{A} + \frac{1}{2A^2}\bar\Psi_{(i}\Psi_{j)}\,, & F^-_{ab}|_{\ln\Phi} &= \frac{F^-_{ab}}{A} + \frac{1}{8A^2}\,\varepsilon^{ij}\bar\Psi_i\gamma_{ab}\Psi_j\,,\\
\Lambda_i|_{\ln\Phi} &= \frac{\Lambda_i}{A} + \frac{1}{2A^2}\big(B_{ij}\varepsilon^{jk}\Psi_k + \tfrac12 F^-_{ab}\gamma^{ab}\Psi_i\big) + \frac{1}{24A^3}\gamma^{ab}\Psi_i\varepsilon^{jk}\bar\Psi_j\gamma_{ab}\Psi_k\,, \\
C|_{\ln\Phi} &= \frac{C}{A} + \frac{1}{4A^2}\big(\varepsilon^{ik}\varepsilon^{jl}B_{ij}B_{kl} - 2F^{-ab}F^-_{ab} + 4\varepsilon^{ij}\bar\Lambda_i\Psi_j\big)\\
&\quad + \frac{1}{2A^3}\big(\varepsilon^{ik}\varepsilon^{jl}B_{ij}\bar\Psi_k\Psi_l - \tfrac12\varepsilon^{kl}F^-_{ab}\bar\Psi_k\,\gamma^{ab}\Psi_l\big) - \frac{1}{32A^4}\varepsilon^{ij}\bar\Psi_i\,\gamma_{ab}\Psi_j\varepsilon^{kl}\bar\Psi_k\gamma^{ab}\Psi_l\,.
\end{aligned} \tag{4.1.12}$$

When the chiral superfield Φ has zero Weyl weight, the logarithm is merely a field redefinition in superspace, which has no direct consequences. However, in the superconformal setting that we are considering, this is no longer the case for non-zero Weyl weight and the two chiral multiplets Φ and $\ln\Phi$ are very different. In particular $\ln\Phi$ does not satisfy the assignment (3.3.6) as it transforms *inhomogeneously* under (constant) dilatations and chiral U(1) transformations,

$$\delta A|_{\ln\Phi} = w\left(\Lambda_{\rm D} - {\rm i}\Lambda_{\rm A}\right)\,. \tag{4.1.13}$$

There are further inhomogeneous transformations, such as S-supersymmetry that acts inhomogeneously on $\Psi_i|_{\ln\Phi}$. However, the higher θ components all scale consistently as if they belong to a $w = 0$ chiral multiplet. In flat superspace this phenomenon also extends to the Q- and S-supersymmetry transformations, although, as we shall see later, there are some minor exceptions in curved superspace. The explicit components in $\mathbb{T}(\ln\bar\Phi)$ will take a rather different form than in $\mathbb{T}(\bar\Phi)$, but much of the global structure of $\mathbb{T}(\ln\bar\Phi)$ will still match that of $\mathbb{T}(\bar\Phi)$. In particular, the highest θ-component, $C|_{\ln\Phi}$ will remain *invariant* under S-supersymmetry, irrespective of the value of the Weyl weight of Φ. As explained earlier, the latter implies that the kinetic multiplet $\mathbb{T}(\ln\bar\Phi)$, defined from a generic chiral multiplet Φ of arbitrary Weyl weight w, will constitute a conformal primary $w = 2$ chiral multiplet. This observation is essential as it forms the basis for the approach followed in this chapter. We will be more explicit in section 4.2.

Finally, to obtain the supersymmetric generalization of $\mathcal{L}^-_{\rm W}$, one needs the Weyl tensor, which turns out to be one of the components of the Weyl multiplet $\mathbf{W}$

(introduced in section 3.3, see Appendix B for further details). At the linearized level, we can work with flat superspace, and we find

$$\mathcal{L}^-_{\rm W} = -\int \mathrm{d}^4\theta\, W_{\alpha\beta}W^{\alpha\beta} = C^{abcd-}C^-_{abcd} + \cdots\,. \tag{4.1.14}$$

From these results we can now define characteristic terms of the (linearized and complex) expression for the Gauss-Bonnet density in flat superspace,

$$\begin{aligned}\mathcal{L}^-_\chi &= -\int \mathrm{d}^4\theta\,\left\{W_{\alpha\beta}\,W^{\alpha\beta} + w^{-1}\,\mathbb{T}(\ln\bar\Phi)\right\}\\ &= \tfrac{1}{2}C^{abcd}C_{abcd} - \tfrac{1}{2}C^{abcd}\tilde{C}_{abcd} + 2w^{-1}\Box\Box\ln\bar{A} + \cdots\,,\end{aligned} \tag{4.1.15}$$

where the additional terms depend on the remaining components of the linearized Weyl multiplet. We already have seen how to use the density formula (3.4.11) to construct locally supersymmetric invariants from a generic weight-two chiral multiplet, analogous to chiral superspace integrals. The full Lagrangian corresponding to the Weyl multiplet action (4.1.14), given long ago in [86], falls into this class, as does the action (3.4.4) built upon the kinetic multiplet $\mathbb{T}(\bar\Phi)$, whose locally supersymmetric version was shown to be a conformal primary chiral multiplet in [65]. For the more complicated Lagrangian (4.1.11), the key property to determine is similarly whether $\mathbb{T}(\ln\bar\Phi)$ similarly exists as a proper chiral multiplet; once that is established, the locally supersymmetric extension follows. One can then, as a simple application, construct the $\mathcal{N}=2$ Gauss-Bonnet invariant using the non-linear version of (4.1.15), which we can immediately deduce must look like

$$\begin{aligned}e^{-1}\mathcal{L}^-_\chi &= \tfrac{1}{2}C^{abcd}C_{abcd} - \tfrac{1}{2}C^{abcd}\tilde{C}_{abcd} + 2\,w^{-1}\,\Box_{\rm c}\Box_{\rm c}\ln\bar{A} + \cdots\\ &= \tfrac{1}{2}C^{abcd}C_{abcd} - \tfrac{1}{2}C^{abcd}\tilde{C}_{abcd} - \mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{1}{3}\mathcal{R}^2 + \tfrac{1}{3}\mathcal{D}^2\mathcal{R}\\ &\quad + 2w^{-1}\Big\{(\mathcal{D}^2)^2\ln\bar{A} + \mathcal{D}^a\big(\tfrac{2}{3}\mathcal{R}\,\mathcal{D}_a\ln\bar{A} - 2\mathcal{R}_{ab}\,\mathcal{D}^b\ln\bar{A}\big)\Big\} + \cdots\end{aligned} \tag{4.1.16}$$

where the missing terms depend on the rest of the Weyl and chiral multiplets.

4.1.2 The $\mathbb{T}(\ln\bar\Phi)$ multiplet in curved superspace

In this short section we want to comment on some general aspects of the curved superspace generalization of the supersymmetric action (4.1.11).

We already noticed in section 3.4.2 that the class of higher derivative chiral superspace integrals constructed in [65] lift naturally to full superspace integrals by stripping away an operator $\bar\nabla^4$ as in (3.4.14). However, it turns out that the curved version of the action (4.1.11), namely

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Phi'\bar{\nabla}^4 \ln\bar{\Phi}\ , \tag{4.1.17}$$

where Φ' has weight $w' = 0$ and Φ has nonzero weight w, does *not* belong to this class. At first glance, a naive application of (3.4.8) would seem to indicate

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Phi'\bar{\nabla}^4 \ln\bar{\Phi} \stackrel{?}{=} \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}\,E\,\Phi' \ln\bar{\Phi} \tag{4.1.18}$$

with the full superspace Lagrangian falling into the class of generic function $\mathcal{H}$ considered in section 3.4.2. However, the proposed Lagrangian $\mathcal{H} = \Phi' \ln\bar{\Phi}$ transforms inhomogeneously under dilatations and so is *not* permissible; in other words, $\mathcal{H}$ does not obey the homogeneity conditions (3.4.15) or (3.4.18).[5] Nevertheless, the left-hand side of (4.1.18) *does* transform appropriately. This is because the kinetic multiplet $\mathbb{T}(\ln\bar{\Phi})$ is a conformal primary chiral multiplet of weight $w = 2$. Both conditions are straightforward enough to check (we refer to [70] for further details). Now by comparing to the flat space limit, it is obvious that

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Phi'\,\bar{\nabla}^4 \ln\bar{\Phi} = \int \mathrm{d}^4x\,e\,A'\,\Box_{\mathrm{c}}\Box_{\mathrm{c}} \ln\bar{A} + \text{additional terms}\,. \tag{4.1.19}$$

The complete expression, which we will present in this chapter, corresponds to a *new chiral supersymmetric invariant.*

This invariant has already appeared in physical applications. In [91], the 5D mixed gauge-gravitational Chern-Simons invariant [92] was dimensionally reduced, and a characteristic subset of 4D terms was obtained which broke down into three classes. The first class was easily identified as the usual chiral superspace integral of a holomorphic function. Another class seemed to coincide with the full superspace integral of a real function $\mathcal{H} \sim \Phi' \ln\bar{\Phi} + \text{h.c.}$, while the remainder, involving terms of the Gauss-Bonnet variety, could not be identified with any currently known invariant. It is clear to us now that these latter two classes of terms are actually contained within the single invariant (4.1.19), which is intrinsically chiral and cannot be decomposed further in a manifestly superconformal way.

Before setting out to calculate the expression (4.1.19) explicitly, we should make an important observation. In the introduction, we noted that the non-linear Lagrangian (4.1.7) with ϕ' constant, must depend on the field $\ln\bar{\phi}$ only via total

[5] This obstruction is specific for curved superspace. For flat superspace, $\mathcal{H}$ must be homogeneous only up to Kähler transformations; see e.g. [80] where such actions were considered.

derivative terms. We expect the same should hold for its supersymmetrized version, namely that when Φ' is constant in (4.1.19) the dependence on $\ln\Phi$ is only in the form of total-derivative terms. To see this, suppose we have two such kinetic multiplets built out of the logarithm of two different anti-chiral superfields $\bar\Phi_1$ and $\bar\Phi_2$, taken to have the same weight w for simplicity. The difference is obviously

$$\bar\nabla^4 \ln\bar\Phi_1 - \bar\nabla^4 \ln\bar\Phi_2 = \bar\nabla^4 \ln(\bar\Phi_1/\bar\Phi_2)\,, \tag{4.1.20}$$

and the quantity under the spinor derivatives on the right-hand side is actually a proper weight-zero multiplet. It follows that any chiral integrand involving such a difference can be written as a full superspace integral and then as an anti-chiral superspace integral, discarding total derivatives in the equalities. Hence,

$$\begin{aligned}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Phi'\,\bar\nabla^4\ln(\bar\Phi_1/\bar\Phi_2) &= \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar\theta\,E\,\Phi'\ln(\bar\Phi_1/\bar\Phi_2)\\ &= \int \mathrm{d}^4x\,\mathrm{d}^4\bar\theta\,\bar{\mathcal{E}}\,(\nabla^4\Phi')\ln(\bar\Phi_1/\bar\Phi_2)\;.\end{aligned} \tag{4.1.21}$$

Taking the weight-zero chiral superfield Φ' to be actually constant, it follows that the right-hand side of (4.1.21) vanishes and therefore

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\bar\nabla^4\ln\bar\Phi_1 = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\bar\nabla^4\ln\bar\Phi_2\;. \tag{4.1.22}$$

In other words, the integral $\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\bar\nabla^4\ln\bar\Phi$ is *independent* of the components of $\ln\bar\Phi$ up to total derivatives. This observation will be an important check that we have correctly calculated the additional terms in (4.1.19). It is to this task which we now turn.

4.2 The component structure of the kinetic multiplet

In this section, we proceed to construct the kinetic multiplet $\mathbb{T}(\ln\bar\Phi)$ in supergravity along with the corresponding Lagrangian (4.1.11). The starting point is the formula (3.3.7) for the Q- and S-supersymmetry transformations of a general $\mathcal{N}=2$ chiral multiplet Φ with Weyl weight w in four-component notation [65, 67, 72]. Now we wish to construct the non-linear version of the kinetic multiplet, $\mathbb{T}(\ln\bar\Phi)$. As we have already alluded to in section 4.1.1, we will choose to define the components of $\ln\Phi$ using (4.1.12), which coincides with using the curved superspace version of (3.3.3) with Φ replaced by $\ln\Phi$. It is straightforward to determine the Q- and S-supersymmetry transformation rules of these components,

$$
\begin{aligned}
\delta\hat{A} &= \bar{\epsilon}^i\hat{\Psi}_i\,,\\
\delta\hat{\Psi}_i &= 2\,\not{D}\hat{A}\epsilon_i + \hat{B}_{ij}\,\epsilon^j + \tfrac{1}{2}\gamma^{ab}\hat{F}^-_{ab}\,\varepsilon_{ij}\epsilon^j + 2\,w\,\eta_i\,,\\
\delta\hat{B}_{ij} &= 2\,\bar{\epsilon}_{(i}\not{D}\hat{\Psi}_{j)} - 2\,\bar{\epsilon}^k\hat{\Lambda}_{(i}\,\varepsilon_{j)k} + 2\,\bar{\eta}_{(i}\hat{\Psi}_{j)}\,,\\
\delta\hat{F}^-_{ab} &= \tfrac{1}{2}\varepsilon^{ij}\,\bar{\epsilon}_i\not{D}\gamma_{ab}\hat{\Psi}_j + \tfrac{1}{2}\bar{\epsilon}^i\gamma_{ab}\hat{\Lambda}_i - \tfrac{1}{2}\,\varepsilon^{ij}\bar{\eta}_i\gamma_{ab}\hat{\Psi}_j\,,\\
\delta\hat{\Lambda}_i &= -\,\tfrac{1}{2}\gamma^{ab}\not{D}\hat{F}^-_{ab}\epsilon_i - \not{D}\hat{B}_{ij}\varepsilon^{jk}\epsilon_k + \hat{C}\varepsilon_{ij}\,\epsilon^j + \tfrac{1}{4}\big(\not{D}\hat{A}\,\gamma^{ab}T_{abij} + w\,\not{D}\gamma^{ab}T_{abij}\big)\varepsilon^{jk}\epsilon_k\\
&\quad -3\,\gamma_a\varepsilon^{jk}\epsilon_k\,\bar{\chi}_{[i}\gamma^a\hat{\Psi}_{j]} - \hat{B}_{ij}\varepsilon^{jk}\,\eta_k + \tfrac{1}{2}\,\gamma^{ab}\,\hat{F}^-_{ab}\eta_i\,,\\
\delta\hat{C} &= -\,2\,\varepsilon^{ij}\bar{\epsilon}_i\not{D}\hat{\Lambda}_j - 6\,\bar{\epsilon}_i\chi_j\,\varepsilon^{ik}\varepsilon^{jl}\hat{B}_{kl} + \tfrac{1}{4}\varepsilon^{ij}\varepsilon^{kl}\big(\bar{\epsilon}_i\gamma^{ab}\not{D}T_{abjk}\hat{\Psi}_l - \bar{\epsilon}_i\gamma^{ab}T_{abjk}\not{D}\hat{\Psi}_l\big)\,.
\end{aligned}
\tag{4.2.1}
$$

Comparing these transformation laws to those in (3.3.7), one notes the appearance of non-linearities involving the weight w. Every term is linear in the components of $\hat{\Phi} = \ln\Phi$ except for the terms proportional to w, which are independent of $\ln\Phi$. As discussed earlier, this arises ultimately from the inhomogeneous transformation of $\ln\Phi$ under dilatations. Note, however, that the covariant derivatives in (4.2.1) do also depend on the Weyl weight and therefore contain similar terms. For instance, consider the transformation (4.1.13): it obviously requires a term $-w\big(b_\mu - \mathrm{i}A_\mu\big)$ in the covariant derivative $D_\mu\hat{A}$ which then no longer depends on $\ln\Phi$.

As mentioned in section 4.1.1, the highest component $\hat{C}$ of $\ln\Phi$ is a weight 2 conformal primary and (anti-)chiral under Q-supersymmetry. This means we may use $\bar{\hat{C}}$ as the lowest component of a chiral multiplet, which will be the kinetic multiplet $\mathbb{T}(\ln\bar{\Phi})$. Within superspace, we can define its components exactly as in (3.3.7), with $\bar{\Phi}$ replaced by $\ln\bar{\Phi}$, and the subsequent computational steps are as outlined before, except for the generation of terms involving w.

An alternative procedure is to begin with the condition $A|_{\mathbb{T}(\ln\bar{\Phi})} = \bar{\hat{C}}$ and derive $\Psi_i|_{\mathbb{T}(\ln\bar{\Phi})}$ by applying a Q-supersymmetry transformation to both sides. Continuing in this way, one can build up the entire multiplet. This was the procedure that was originally applied to the linear kinetic multiplet $\mathbb{T}(\bar{\Phi})$ in [65], but which is now considerably more involved. A convenient way of applying the same strategy is to focus only on the w-dependent terms by unpackaging the full covariant derivatives. Although this sacrifices manifest covariance, it exploits the high degree of overlap between $\mathbb{T}(\ln\bar{\Phi})$ and the kinetic multiplet $\mathbb{T}(\bar{\Phi})$ studied in [65].

We have followed both lines of approach and confirmed agreement between them, up to the fermionic terms in $C|_{\mathbb{T}(\ln\bar{\Phi})}$; these have passed other non-trivial checks using S-supersymmetry. The result is (in four component notation),

$$
\begin{aligned}
A|_{\mathbb{T}(\ln\bar\Phi)} =& \hat{\bar{C}}\,,\\
\Psi_i|_{\mathbb{T}(\ln\bar\Phi)} =& -2\,\varepsilon_{ij}\not{D}\hat\Lambda^j - 6\ \varepsilon_{ik}\varepsilon_{jl}\chi^j\hat B^{kl} - \tfrac14\varepsilon_{ij}\varepsilon_{kl}\,\gamma^{ab}T_{ab}{}^{jk}\ \overset{\leftrightarrow}{\not{D}}\ \hat\Psi^l\,,\\
B_{ij}|_{\mathbb{T}(\ln\bar\Phi)} =& -2\,\varepsilon_{ik}\varepsilon_{jl}\big(\Box_{\rm c}+3\,D\big)\hat B^{kl} - 2\,\hat F^+_{ab}\,R(\mathcal{V})^{ab\,k}{}_i\,\varepsilon_{jk}\\
&-6\,\varepsilon_{k(i}\,\bar\chi_{j)}\hat\Lambda^k + 3\,\varepsilon_{ik}\varepsilon_{jl}\hat{\bar\Psi}{}^{(k}\not{D}\chi^{l)}\,,\\
F^-_{ab}|_{\mathbb{T}(\ln\bar\Phi)} =& -\big(\delta_a{}^{[c}\delta_b{}^{d]} - \tfrac12\varepsilon_{ab}{}^{cd}\big)\big[4\,D_cD^e\hat F^+_{ed} + (D^e\hat{\bar A}\,D_cT_{de}{}^{ij} + D_c\hat{\bar A}\,D^eT_{ed}{}^{ij})\varepsilon_{ij} - wD_cD^eT_{ed}{}^{ij}\varepsilon_{ij}\big]\\
&+\Box_{\rm c}\hat{\bar A}\,T_{ab}{}^{ij}\varepsilon_{ij} - R(\mathcal{V})^-{}_{ab}{}^i{}_k\,\hat B^{jk}\,\varepsilon_{ij} + \tfrac18 T_{ab}{}^{ij}\,T_{cdij}\hat F^{+cd} - \varepsilon_{kl}\,\hat{\bar\Psi}{}^k\ \overset{\leftrightarrow}{\not{D}}\ R(Q)_{ab}{}^l\\
&-\tfrac94\varepsilon_{ij}\,\hat{\bar\Psi}{}^i\gamma^c\gamma_{ab}D_c\chi^j + 3\,\varepsilon_{ij}\bar\chi^i\gamma_{ab}\not{D}\hat\Psi^j + \tfrac38 T_{ab}{}^{ij}\varepsilon_{ij}\bar\chi_k\hat\Psi^k\,,\\
\Lambda_i|_{\mathbb{T}(\ln\bar\Phi)} =& 2\,\Box_{\rm c}\not{D}\hat\Psi^j\varepsilon_{ij} + \tfrac14\gamma^c\gamma_{ab}(2\,D_cT^{ab}{}_{ij}\,\hat\Lambda^j + T^{ab}{}_{ij}\,D_c\hat\Lambda^j)\\
&-\tfrac12\varepsilon_{ij}\big(R(\mathcal{V})_{ab}{}^j{}_k + 2{\rm i}\,R(A)_{ab}\delta^j{}_k\big)\,\gamma^c\gamma^{ab}D_c\hat\Psi^k\\
&+\tfrac12\,\varepsilon_{ij}\big(3\,D_bD - 4{\rm i}D^aR(A)_{ab} + \tfrac14 T_{bc}{}^{ij}\ \overset{\leftrightarrow}{D}_a\ T^{ac}{}_{ij}\big)\,\gamma^b\hat\Psi^j\\
&-2\,\hat F^{+ab}\,\not{D}R(Q)_{abi} + 6\,\varepsilon_{ij}D\,\not{D}\hat\Psi^j\\
&+3\,\varepsilon_{ij}\,\big(\not{D}\chi_k\,\hat B^{kj} + \not{D}\hat{\bar A}\,\not{D}\chi^j\big)\\
&+\tfrac32\big(2\,\not{D}\hat B^{kj}\varepsilon_{ik} + \not{D}\hat F^+_{ab}\gamma^{ab}\,\delta^j_i + \tfrac14\varepsilon_{kl}T_{ab}{}^{kl}\,\gamma^{ab}\,\not{D}\hat{\bar A}\,\delta_i{}^j\big)\chi_j\\
&+\tfrac94\,(\bar\chi^l\gamma_a\chi_l)\,\varepsilon_{ij}\gamma^a\hat\Psi^j - \tfrac92\,(\bar\chi_i\gamma_a\chi^k)\,\varepsilon_{kl}\gamma^a\hat\Psi^l\\
&-\tfrac32 w\,\varepsilon_{jk}\,D^aT_{ab}{}^{jk}\gamma^b\chi_i\,,\\
C|_{\mathbb{T}(\ln\bar\Phi)} =& 4(\Box_{\rm c}+3D)\Box_{\rm c}\hat{\bar A} + 6(D_aD)\,D^a\hat{\bar A} - 16\,D^a\Big(R(D)^+_{ab}D^b\hat{\bar A}\Big)\\
&-D^a(T_{abij}T^{cbij}D_c\hat{\bar A}) - \tfrac12 D^a(T_{abij}T^{cbij})D_c\hat{\bar A} - 9\,\bar\chi_j\gamma^a\chi^j\,D_a\hat{\bar A}\\
&+\tfrac12 D_aD^a(T_{bcij}\hat F^{bc+})\varepsilon^{ij} + 4\varepsilon^{ij}D_a\Big(D^bT_{bcij}\hat F^{ac+} + D^b\hat F^+_{bc}T^{ac}{}_{ij}\Big)\\
&-\tfrac92\varepsilon^{jk}\bar\chi_j\gamma^{ab}\chi_k\hat F^+_{ab} + 9\bar\chi_j\chi_k\hat B^{jk} + \tfrac1{16}(T_{ab}{}^{ij}\varepsilon_{ij})^2\hat{\bar C}\\
&+6D^aD_a\bar\chi_j\hat\Psi^j + 3\bar\chi_j\not{D}\not{D}\hat\Psi^j + 3D_a(\bar\chi_j\gamma^a\not{D}\hat\Psi^j) + 9D\bar\chi_j\hat\Psi^j\\
&-8D^a\bar R(Q)_{abj}D^b\hat\Psi^j + 6D_b\bar\chi_j\gamma^b\not{D}\hat\Psi^j\\
&+\tfrac32 D^aT_{abij}\bar\chi^i\gamma^b\hat\Psi^j + 3D^a(T_{abij}\bar\chi^i\gamma^b\hat\Psi^j) + \tfrac32 D^a(T_{abij}\bar\chi^i)\gamma^b\hat\Psi^j\\
&+3\Big(\tfrac12 R(\mathcal{V})^+_{ab}{}^i{}_j - R(D)^+_{ab}\delta^i{}_j\Big)\bar\chi_i\gamma^{ab}\hat\Psi^j - 2R(\mathcal{V})^+_{ab}{}^i{}_j\bar R(Q)^{ab}{}_i\hat\Psi^j - \tfrac12 T^{ab}{}_{ij}\bar R(S)^+_{ab}{}^i\hat\Psi^j\\
&+\tfrac18\varepsilon^{ij}T_{abij}\Big(3\bar\chi_k\gamma^{ab}\hat\Lambda^k + 2\bar R(Q)^{ab}_k\hat\Lambda^k\Big)\\
&+w\Big\{9\bar\chi_j\not{D}\chi^j - R(\mathcal{V})^+_{ab}{}^i{}_jR(\mathcal{V})^{ab+j}{}_i - 8R(D)^+_{ab}R(D)^{ab+}\\
&\qquad - D^aT_{abij}D_cT^{cbij} - D^a(T_{abij}D_cT^{cbij})\Big\}\,.
\end{aligned}
\tag{4.2.2}
$$

The result agrees with the corresponding expressions for the usual kinetic multiplet discussed in [65] by taking $w=0$. In this limit, the superfield $\ln\bar\Phi$ becomes a normal $w = 0$ anti-chiral multiplet with $\mathbb{T}(\ln \bar{\Phi})$ its associated kinetic multiplet.

Now we can calculate the component Lagrangian $\mathcal{L}$ corresponding to the action

$$-2 \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \mathcal{E} \, \Phi' \, \mathbb{T}(\ln \bar{\Phi}) \; . \tag{4.2.3}$$

This is a straightforward application of the product rule (B.15) and (3.4.11). We will ignore all fermions, which significantly simplifies the resulting expression. Expanding out the covariant d'Alembertians using, for example, the expression for $f_\mu{}^a$ given in (B.6) leads to,

$$\begin{aligned} C|_{\mathbb{T}(\ln \bar{\Phi})} = {} & \mathcal{D}_a V^a + \tfrac{1}{16} (T_{ab\,ij} \varepsilon^{ij})^2 \, \hat{\bar{C}} \\ & + w\big\{ -2\,\mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{2}{3}\mathcal{R}^2 - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+\,i}_{ab\;\;j} R(\mathcal{V})^{ab+j}{}_i \\ & \quad + \tfrac{1}{128} T^{abij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cdkl} + T^{acij} D_a D^b T_{bcij} \big\} \, , \end{aligned} \tag{4.2.4}$$

where V^a is given by

$$\begin{aligned} V^a = {} & 4\mathcal{D}^a \mathcal{D}^2 \hat{\bar{A}} - 8\mathcal{R}^{ab}\mathcal{D}_b \hat{\bar{A}} + \tfrac{8}{3}\mathcal{R}\mathcal{D}^a \hat{\bar{A}} + 8D\,\mathcal{D}^a \hat{\bar{A}} - 8\mathrm{i}\,R(A)^{ab}\mathcal{D}_b \hat{\bar{A}} \\ & - 2T^{acij}T_{bcij}\mathcal{D}^b \hat{\bar{A}} + \tfrac{1}{2}\varepsilon^{ij}\mathcal{D}^a T_{bcij} \hat{F}^{bc+} + 4\,\varepsilon^{ij} T^{ac}{}_{ij} \mathcal{D}^b \hat{F}^+_{bc} \\ & + w\big\{ \tfrac{2}{3}\mathcal{D}^a\mathcal{R} - 4\,\mathcal{D}^a D - \mathcal{D}^b (T^{acij}T_{bcij}) \big\} \; . \end{aligned} \tag{4.2.5}$$

Here the derivatives $\mathcal{D}_a$ are covariant with respect to the linearly acting bosonic transformations. Hence they do not contain the connection field or the conformal boosts $f_\mu{}^a$. Note that we have kept the K-connection $f_\mu{}^a$ within the fully covariant derivatives in the last term of (4.2.4) for later convenience, but there is no obstacle in extracting it here as well.

Performing a similar decomposition in $B_{ij}|_{\mathbb{T}(\ln\bar\Phi)}$ and $F^-_{ab}|_{\mathbb{T}(\ln\bar\Phi)}$ and dropping a number of total derivatives, we find

$$\begin{aligned}
e^{-1}\mathcal{L} = {}& 4\,\mathcal{D}^2 A'\,\mathcal{D}^2\hat{\bar{A}} + 8\,\mathcal{D}^a A'\,\big[\mathcal{R}_{ab} - \tfrac{1}{3}\mathcal{R}\,\eta_{ab}\big]\mathcal{D}^b\hat{\bar{A}} + C'\,\hat{\bar{C}} \\
& - \mathcal{D}^\mu B'_{ij}\,\mathcal{D}_\mu\hat{B}^{ij} + (\tfrac{1}{6}\mathcal{R} + 2\,D)\,B'_{ij}\hat{B}^{ij} \\
& - \big[\varepsilon^{ik}\,B'_{ij}\,\hat{F}^{+\mu\nu}\,R(\mathcal{V})_{\mu\nu}{}^j{}_k + \varepsilon_{ik}\,\hat{B}^{ij}\,F'^{-\mu\nu}R(\mathcal{V})_{\mu\nu j}{}^k\big] \\
& - 8\,D\,\mathcal{D}^\mu A'\,\mathcal{D}_\mu\hat{\bar{A}} + \big(8\,\mathrm{i}\,R(A)_{\mu\nu} + 2\,T_\mu{}^{cij}\,T_{\nu cij}\big)\mathcal{D}^\mu A'\,\mathcal{D}^\nu\hat{\bar{A}} \\
& - \big[\varepsilon^{ij}\mathcal{D}^\mu T_{bcij}\mathcal{D}_\mu A'\,\hat{F}^{+bc} + \varepsilon_{ij}\mathcal{D}^\mu T_{bc}{}^{ij}\mathcal{D}_\mu\hat{\bar{A}}\,F'^{-bc}\big] \\
& - 4\big[\varepsilon^{ij}T^{\mu b}{}_{ij}\,\mathcal{D}_\mu A'\,\mathcal{D}^c\hat{F}^+_{cb} + \varepsilon_{ij}T^{\mu bij}\,\mathcal{D}_\mu\hat{\bar{A}}\,\mathcal{D}^c F'^-_{cb}\big] \\
& + 8\,\mathcal{D}_a F'^{-ab}\,\mathcal{D}^c\hat{F}^+_{cb} + 4\,F'^{-ac}\,\hat{F}^+_{bc}\,\mathcal{R}_a{}^b + \tfrac{1}{4}T_{ab}{}^{ij}\,T_{cdij}F'^{-ab}\hat{F}^{+cd} \\
& + w\Big\{ - \tfrac{2}{3}\mathcal{D}^a A'\,\mathcal{D}_a\mathcal{R} + 4\mathcal{D}^a A'\,\mathcal{D}_a D - T^{acij}T_{bcij}\,\mathcal{D}^b\mathcal{D}_a A' \\
& \quad - 2\mathcal{D}^a F'^-_{ab}\,\mathcal{D}_c T^{cbij}\varepsilon_{ij} + \mathrm{i}\,F'^{-ab}R(A)^-_{ad}\,T_b{}^{dij}\varepsilon_{ij} + F^-_{ab}T^{abij}\varepsilon_{ij}(\tfrac{1}{12}\mathcal{R} - \tfrac{1}{2}D) \\
& \quad + A'\,\big[\tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ab}\mathcal{R}_{ab} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^+_{ab}{}^j{}_i \\
& \qquad + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}D_a D^b T_{bcij}\big]\Big\}\ . \qquad (4.2.6)
\end{aligned}$$

The above Lagrangian is the central result of this chapter and can be used to construct a large variety of invariants in the same way as has been done in [65]. Three brief comments should be made about it. First, in the limit $w = 0$, we recover exactly (4.2) of [65]. Second, the w-terms appear not only explicitly in the final four lines of (4.2.6) but also implicitly within the covariant derivatives of $\hat{\bar{A}}$, as we have already stressed earlier. Finally, we argued in section 4.1.1 that if Φ' is set to a constant, then the action cannot actually depend on the components of $\ln\bar\Phi$. This is apparent in (4.2.6) by inspection: only the last two lines survive in this limit and they depend on the conformal supergravity fields alone. We note in particular the appearance of the non-conformal part of the Gauss-Bonnet invariant involving $\frac{2}{3}\mathcal{R}^2 - 2\mathcal{R}^{ab}\mathcal{R}_{ab}$. This confirms our conjecture that the kinetic multiplet based upon $\ln\bar\Phi$ can be used to generate the $\mathcal{N} = 2$ Gauss-Bonnet invariant. Based on our discussion in section 4.1.1, we were led to postulate the action

$$S^-_\chi = \int \mathrm{d}^4x\,\Big(\mathcal{L}^-_{\mathrm{W}} + \mathcal{L}^-_{\mathrm{NL}}\Big) = -\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Big(W^{\alpha\beta}W_{\alpha\beta} + w^{-1}\mathbb{T}(\ln\bar\Phi)\Big) \qquad (4.2.7)$$

as the $\mathcal{N} = 2$ supersymmetric Gauss-Bonnet, based mainly on the form its component action took in the linearized limit. Using the above results we can verify explicitly that its component Lagrangian contains the combination (4.1.2) of curvature-squared terms. However, the full $\mathcal{N} = 2$ Gauss-Bonnet must not only include this combination, but must also be a topological quantity.

We will establish its topological nature in the following by analyzing its component structure, keeping only the bosonic terms, and show that it indeed reduces to a topological quantity. In principle, this should be sufficient as it is unlikely that the fermionic terms would not be a topological invariant if the bosonic terms are. For a formal proof of this statement, we refer the reader to [70] where a superspace argument, which encompasses all the terms, is presented.

We begin with the density formula for the kinetic multiplet,

$$2w \int \mathrm{d}^4x \, \mathcal{L}^-_{\mathrm{NL}} = -2 \int \mathrm{d}^4x \, \mathrm{d}^4\theta \, \mathcal{E} \, \mathbb{T}(\ln \bar{\Phi}) = \int \mathrm{d}^4x \, e \left(C|_{\mathbb{T}(\ln \bar{\Phi})} - \tfrac{1}{16} (T_{ab\,ij}\varepsilon^{ij})^2 \, A|_{\mathbb{T}(\ln \bar{\Phi})} \right) , \tag{4.2.8}$$

where $C|_{\mathbb{T}(\ln \bar{\Phi})}$ and $A|_{\mathbb{T}(\ln \bar{\Phi})}$ are given in (4.2.2). We have already discussed how the dependence on the fields of the anti-chiral multiplet must be limited to total derivative terms, but we would like to explicitly check this. Making use of (4.2.4), we easily find

$$\begin{aligned} 2w \, e^{-1} \mathcal{L}^-_{\mathrm{NL}} = {}& \mathcal{D}_a V^a - 2w \mathcal{R}^{ab} \mathcal{R}_{ab} + \tfrac{2}{3} w \mathcal{R}^2 - 6 w D^2 \\ & + 2w R(A)^{ab} R(A)_{ab} - w R(\mathcal{V})^{+\,i}_{ab\ j} R(\mathcal{V})^{ab+j}{}_i \\ & + \tfrac{1}{128} w T^{abij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cdkl} + w T^{acij} D_a D^b T_{bcij} \, , \end{aligned} \tag{4.2.9}$$

where the components of the multiplet $\ln \bar{\Phi}$ are confined to the covariant term V^a given in (4.2.5).

The well-known conformal supergravity invariant constructed from the square of the superconformal Weyl tensor is

$$\begin{aligned} e^{-1} \mathcal{L}^-_{\mathrm{W}} = {}& \tfrac{1}{2} C^{abcd} C_{abcd} - \tfrac{1}{2} C^{abcd} \tilde{C}_{abcd} - 2 R(A)^-_{ab} R(A)^{ab-} + \tfrac{1}{2} R(\mathcal{V})^{-\,i}_{ab\ j} R(\mathcal{V})^{ab-j}{}_i \\ & + 3D^2 - \tfrac{1}{2} T^{acij} D_a D^b T_{bcij} - \tfrac{1}{256} T^{abij} T_{ab}{}^{kl} T^{cd}{}_{ij} T_{cdkl} \, . \end{aligned} \tag{4.2.10}$$

Combining the expressions (4.2.9) and (4.2.10) with the appropriate coefficients leads to

$$\begin{aligned} e^{-1} \mathcal{L}^-_{\chi} = e^{-1} \mathcal{L}^-_{\mathrm{W}} + e^{-1} \mathcal{L}^-_{\mathrm{NL}} = {}& \tfrac{1}{2} C^{abcd} C_{abcd} - \mathcal{R}^{ab} \mathcal{R}_{ab} + \tfrac{1}{3} \mathcal{R}^2 - \tfrac{1}{2} C^{abcd} \tilde{C}_{abcd} \\ & + R(A)_{ab} \tilde{R}(A)^{ab} - \tfrac{1}{2} R(\mathcal{V})_{ab}{}^i{}_j \tilde{R}(\mathcal{V})^{abj}{}_i + \tfrac{1}{2} w^{-1} \mathcal{D}_a V^a \, . \end{aligned} \tag{4.2.11}$$

As required, $\mathcal{L}^-_{\chi}$ is a topological invariant. It involves respectively the Euler density, the Pontryagin density, the SU(2) and U(1) topological invariants, and an explicit total covariant derivative. It is interesting (although perhaps coincidental) that the specific combination of U(1) and SU(2) curvatures appearing in the above expression can be rewritten purely in terms of the U(2) curvature.

4.2.1 Reduction to 4D of the 5D mixed Chern-Simon term

The kinetic multiplet $\mathbb{T}(\ln\bar\Phi_w)$ discussed in the preceding sections plays a natural role in extending the known classes of chiral superspace higher derivative invariants. Evidence for the existence of a new class of higher derivative invariants was actually seen in [91] where the dimensional reduction of the supersymmetric version of the 5D Chern-Simons action $\mathrm{Tr}(W\wedge R\wedge R)$ was considered. The authors of [91] identified three distinct types of terms in the dimensional reduction: one corresponded to a usual chiral superspace integral of a holomorphic prepotential $F(X, A|_{W^2})$, another was identified as a full superspace integral $\mathcal{H}(X,\bar X)$, and a third remained a mystery. As discussed in [70], this identification was actually incorrect: the second and third invariants described in [91] are actually part of a single irreducible chiral invariant constructed from a kinetic multiplet $\mathbb{T}(\ln\bar\Phi_w)$. Our goal in this section is to back up this claim by keeping a much wider range of terms in the dimensional reduction and checking against the proposed 4D action.

The supersymmetric version of the 5D Chern-Simons action $\mathrm{Tr}(W\wedge R\wedge R)$, constructed originally in [92], is given in the conventions of [94] by

$$\begin{aligned}
E^{-1}\,\mathcal{L}_{\rm vww} &= \tfrac{1}{4}c_I Y_{ij}{}^I\, T^{AB} R_{ABk}{}^j(V)\,\varepsilon^{ki} \\
&+ c_I\sigma^I\Big[\tfrac{1}{64}R_{AB}{}^{CD}(M)\,R_{CD}{}^{AB}(M) + \tfrac{1}{96}R_{ABj}{}^i(V)\,R^{AB}{}_i{}^j(V)\Big] \\
&- \tfrac{1}{128}\mathrm{i}\,E^{-1}\,\varepsilon^{MNPQR}\,c_I W_M{}^I\,\big[R_{NP}{}^{AB}(M)\,R_{QRAB}(M) + \tfrac{1}{3}R_{NPj}{}^i(V)\,R_{QRi}{}^j(V)\big]\Big] \\
&+ \tfrac{3}{16}c_I\big(10\,\sigma^I\,T_{AB} - F_{AB}{}^I\big)\,R(M)_{CD}{}^{AB}\,T^{CD} \\
&+ c_I\sigma^I\Big[3\,T^{AB}\mathcal{D}^C\mathcal{D}_A T_{BC} - \tfrac{3}{2}\big(\mathcal{D}_A T_{BC}\big)^2 + \tfrac{3}{2}\mathcal{D}_C T_{AB}\,\mathcal{D}^A T^{CB}\Big] \\
&+ c_I\sigma^I\Big[\tfrac{8}{3}D^2 + 8\,T^2\,D - \tfrac{33}{8}(T^2)^2 + \tfrac{81}{2}(T^{AC}T_{BC})^2 + \mathcal{R}_{AB}(T^{AC}T^B{}_C - \tfrac{1}{2}\eta^{AB}T^2)\Big] \\
&+ \tfrac{3}{4}\mathrm{i}\,\varepsilon^{ABCDE}\Big[c_I F_{AB}{}^I\big(T_{CF}\mathcal{D}^F T_{DE} + \tfrac{3}{2}T_{CF}\mathcal{D}_D T_E{}^F\big) - 3\,c_I\sigma^I T_{AB}T_{CD}\,\mathcal{D}^F T_{FE}\Big] \\
&- c_I F_{AB}{}^I\Big[T^{AB}\,D + \tfrac{3}{8}T^{AB}\,T^2 - \tfrac{9}{2}\,T^{AC}T_{CD}T^{DB}\Big]\,, \qquad (4.2.12)
\end{aligned}$$

with $E=\det(E_M{}^A)$, the determinant of the 5D vielbein. The fields σ^I, $W_M{}^I$, and $Y_{ij}{}^I$ are the bosonic components of a 5D vector multiplet, with field strength $F_{MN}{}^I = 2\partial_{[M}W_{N]}{}^I$. The index I enumerates a number of such multiplets. The fields T_{AB} and D are the covariant bosonic fields of the 5D Weyl multiplet. The 5D Lorentz and SU(2) curvature tensors are given respectively by $R(M)_{MN}{}^{AB}$ and $R(V)_{MNi}{}^{j}$.

We will show that the full 4D invariant that matches the reduction of (4.2.12) is given by

$$S_{\rm vww} = \frac{\rm i}{64} \int {\rm d}^4x\, {\rm d}^4\theta\, \mathcal{E}\; c_I \frac{X^I}{X^0} \Big(W^{\alpha\beta} W_{\alpha\beta} - \tfrac{1}{3} \mathbb{T}(\ln \bar{X}^0) \Big) + {\rm c.c.} \qquad (4.2.13)$$

This corresponds to a chiral superspace action where the holomorphic function F is, in the usual normalization convention, given by

$$F = -\frac{1}{64} \frac{c_I X^I}{X^0} \Big(\tfrac{1}{32} (T_{ab}{}^{ij} \varepsilon_{ij})^2 - \tfrac{1}{3} A|_{\mathbb{T}(\ln \bar{X}^0)} \Big) \,. \qquad (4.2.14)$$

This expression involves three types of fields: the "matter" vector multiplets X^I, the Kaluza-Klein vector multiplet X^0, and the 4D Weyl multiplet superfield $W_{\alpha\beta}$ whose lowest component is $T_{ab}{}^{ij}\varepsilon_{ij}$. The expression within parentheses in (4.2.13) is composed of two chiral invariants. The first involves the square of the Weyl multiplet, and the second involves the kinetic multiplet $\mathbb{T}(\ln \bar{X}^0)$.

Before proceeding to details of the actual computation, some elucidating comments are necessary about how to organize the Lagrangian. While (4.2.12) is fairly complicated, we draw attention to one important feature: every term is linear in a component of the 5D vector multiplet. Upon dimensional reduction we must retain this feature, so the 4D Lagrangian should take the form

$$e^{-1} \mathcal{L}|_{\rm 4D} = -\tfrac{1}{2} c_I Y^{ij\,I} L_{ij} - \tfrac{1}{2} {\rm i}\, c_I F_{\mu\nu}{}^I \tilde{E}^{\mu\nu} + c_I X^I G + c_I \bar{X}^I \bar{G} \,, \qquad (4.2.15)$$

for some composite functions L_{ij}, $\tilde{E}_{\mu\nu} \equiv \tfrac{1}{2} \varepsilon^{\mu\nu\rho\sigma} E^{\rho\sigma}$, G and $\bar{G}$. It is natural to write the coefficient of $F_{\mu\nu}{}^I$ as the dual of a two-form $E_{\mu\nu}$ since the Bianchi identity on $F_{\mu\nu}{}^I$ implies that $E_{\mu\nu}$ can be defined only up to a gauge transformation, $E_{\mu\nu} \to E_{\mu\nu} + 2\partial_{[\mu} \Lambda_{\nu]}$.

We have chosen the normalizations of the composite functions in (4.2.15) in a very particular way. Supersymmetry dictates that the functions L_{ij}, $E_{\mu\nu}$, G, and $\bar{G}$, must correspond to the bosonic components of a (composite) tensor multiplet[6]. This has some deep implications when one compares two expressions of the form (4.2.15), such as those we plan to derive from (4.2.12) and (4.2.13). In particular, to show full equivalence between them, we must only prove that the two expressions

[6] We refer to section 5.2.1 for a more detailed discussion about this multiplet.

for L_{ij} are the same: as these are the lowest components of some (composite) tensor multiplet, the equality of the remaining pieces follows by supersymmetry.

Unfortunately, we cannot fully exploit this observation. A strict proof along these lines requires that the fermionic bilinears of L_{ij} be compared as well, and in the calculation of the Lagrangian (4.2.12) these would need to be restored. We will instead demonstrate a proof of equivalence between all bosonic terms of L_{ij}, as well as some characteristic bosonic terms of $E_{\mu\nu}$ and G. This establishes beyond any doubt the equivalence between (4.2.13) and the reduction of (4.2.12).

We begin by reviewing some key results of the off-shell dimensional reduction formulated in [91]. In order to avoid confusion between 4D and 5D fields, we henceforth will place a diacritic on all 5D quantities (e.g. $E_M{}^A \to \breve{E}_M{}^A$). All bosonic components of the 5D Weyl multiplet, ($\breve{E}_M{}^A$, $\breve{b}_M$, $\breve{V}_{Mi}{}^j$, $\breve{T}_{AB}$, and $\breve{D}$), must reduce to expressions involving the 4D Weyl multiplet and a Kaluza-Klein vector multiplet X^0. Below we provide a dictionary relating the 5D and 4D components. To avoid potential confusion the index 5 will refer *only* to the fifth component of the tangent space index A and *never* to the fifth coordinate.

The fundamental bosonic fields of the Weyl multiplet are given by

$$
\begin{aligned}
\breve{E}_M{}^A &= \begin{pmatrix} e_\mu{}^a & \tfrac{1}{2}W_\mu{}^0\,|X^0|^{-1} \\ 0 & \tfrac{1}{2}\,|X^0|^{-1} \end{pmatrix} , \quad \breve{b}_M = \begin{pmatrix} b_m \\ 0 \end{pmatrix} , \\
\breve{V}_{ai}{}^j &= \mathcal{V}_a{}^j{}_i \,, \quad \breve{V}_{5i}{}^j = -\frac{1}{2}\varepsilon_{ik}Y^{kj\,0}|X^0|^{-1} \,, \\
\breve{T}_{ab} &= -\tfrac{1}{24}\mathrm{i}\left(\varepsilon_{ij}T_{ab}{}^{ij}\bar{X}^0 - F_{ab}^{-0}\right)|X^0|^{-1} + \text{c.c.} \,, \qquad \breve{T}_{a5} = \tfrac{1}{12}\mathrm{i}\,\mathcal{D}_a \ln(X^0/\bar{X}^0) \,, \\
\breve{D} &= \tfrac{1}{4}D - \tfrac{1}{16}|X^0|^{-1}(\mathcal{D}^a\mathcal{D}_a + \tfrac{1}{6}\mathcal{R})|X^0| - \tfrac{3}{512}|X^0|^{-2}F_{ab}{}^0F^{ab\,0} \\
&\quad + \tfrac{1}{64}|X^0|^{-2}Y^{ij\,0}Y_{ij}{}^0 - \tfrac{3}{8}\breve{T}^{ab}\breve{T}_{ab} - \tfrac{3}{4}\breve{T}^{a5}\breve{T}_{a5} \,.
\end{aligned} \tag{4.2.16}
$$

Some derived quantities are also useful. The 5D spin connection and Riemann tensor can be found in [91], while the 5D SU(2) curvature tensor is given by

$$
\begin{aligned}
\breve{R}(V)_{ab\,i}{}^j &= R(\mathcal{V})_{ab}{}^j{}_i - \tfrac{1}{4}\varepsilon_{ik}Y^{kj\,0}\,F_{ab}{}^0\,|X^0|^{-2} \,, \\
\breve{R}(V)_{a5\,i}{}^j &= -\tfrac{1}{2}\varepsilon_{ik}|X^0|\,\mathcal{D}_a\Big(Y^{kj\,0}/|X^0|^2\Big) \,.
\end{aligned} \tag{4.2.17}
$$

The decomposition of the 5D vector multiplet is given by

$$
\begin{aligned}
\breve{\sigma}^I &= -\mathrm{i}\,|X^0|\Big(\frac{X^I}{X^0} - \frac{\bar{X}^I}{\bar{X}^0}\Big)\ , & \breve{Y}^{ij\,I} &= -\tfrac{1}{2}Y^{ij\,I} + \tfrac{1}{4}\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big)Y^{ij\,0}\ , \\
\breve{W}_a{}^I &= W_a{}^I\ , & \breve{W}_5{}^I &= -|X^0|\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big)\ , \\
\breve{F}_{ab}{}^I &= F_{ab}{}^I - \tfrac{1}{2}F_{ab}{}^0\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big)\ , & \breve{F}_{a5}{}^I &= -|X^0|\,\mathcal{D}_a\Big(\frac{X^I}{X^0} + \frac{\bar{X}^I}{\bar{X}^0}\Big)\ .
\end{aligned}
\qquad (4.2.18)
$$

It is important to note that all of these equations are invariant under the 4D U(1) R-symmetry group. This is because there is no U(1) factor in the 5D R-symmetry group; it emerges from the dimensional reduction.

Let us now analyze the first term L_{ij} of the 4D Lagrangian (4.2.15). This arises only from the first term in (4.2.12), which decomposes as

$$
\begin{aligned}
64\,L_{ij} = &-\tfrac{1}{3}\varepsilon_{ik}R(\mathcal{V})^{ab\,k}{}_j\Big(\mathrm{i}\bar{X}^0 T_{ab}{}^{mn}\varepsilon_{mn} - \mathrm{i}F_{ab}^{-0} + \text{c.c.}\Big)\,|X^0|^{-2} \\
&+ \tfrac{1}{12}Y_{ij}{}^0\Big(\mathrm{i}\,\bar{X}^0 T^{ab\,kl}\varepsilon_{kl}F_{ab}^{-0} - \mathrm{i}\,(F_{ab}^{-0})^2 + \text{c.c.}\Big)\,|X^0|^{-4} \\
&- \tfrac{2}{3}\mathrm{i}\,\mathcal{D}^a\ln(X^0/\bar{X}^0)\,\mathcal{D}_a(Y_{ij}{}^0/|X^0|^2)\ .
\end{aligned}
\qquad (4.2.19)
$$

This expression includes all the bosonic contributions to L_{ij}. Now let us calculate the same contribution from the 4D superspace action (4.2.13). It helps to rewrite the action as

$$
\frac{\mathrm{i}}{64}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\frac{c_I X^I}{X^0}\Phi\ , \qquad \Phi = W^{\alpha\beta}W_{\alpha\beta} - \tfrac{1}{3}\mathbb{T}(\ln\bar{X}^0) \qquad (4.2.20)
$$

and express the component action in terms of the components of Φ. For example, the contribution to L_{ij} is given by

$$
64\,L_{ij} = \frac{\mathrm{i}}{2}\frac{Y_{ij}{}^0}{(X^0)^2}\,A|_\Phi - \frac{\mathrm{i}}{2}\frac{1}{X^0}B_{ij}|_\Phi + \text{c.c.}\ , \qquad (4.2.21)
$$

The components of Φ can then be calculated as

$$
\begin{aligned}
A|_\Phi &= \tfrac{1}{32}(T_{ab}{}^{ij}\varepsilon_{ij})^2 - \tfrac{1}{3}A|_{\mathbb{T}(\ln\bar{X}^0)} \\
&= \tfrac{1}{96}(T_{ab}{}^{ij}\varepsilon_{ij})^2 + (\bar{X}^0)^{-1}\Big(\tfrac{2}{3}\Box_{\rm c}X^0 + \tfrac{1}{12}T^{ab\,ij}\varepsilon_{ij}F_{ab}^{-0}\Big) \\
&\quad + (\bar{X}^0)^{-2}\Big(\tfrac{1}{6}(F_{ab}^{+0} - \tfrac{1}{4}X^0 T_{ab\,ij}\varepsilon^{ij})^2 - \tfrac{1}{12}(Y_{ij}{}^0)^2\Big)\ , \\
B_{ij}|_\Phi &= \varepsilon_{ik}R(\mathcal{V})_{ab}{}^k{}_j\Big\{\tfrac{1}{2}T^{ab\,kl}\varepsilon_{kl} + \tfrac{2}{3}(F_{ab}^{+0} - \tfrac{1}{4}X^0 T_{ab\,kl}\varepsilon^{kl})\,(\bar{X}^0)^{-1}\Big\} \\
&\quad + \tfrac{2}{3}(\Box_{\rm c} + 3D)\Big(\frac{Y_{ij}{}^0}{\bar{X}^0}\Big)\ .
\end{aligned}
\qquad (4.2.22)
$$

A straightforward calculation leads to L_{ij} as in (4.2.19). As already mentioned, this nearly guarantees equivalence of the final expressions, but we will check some additional terms to marshal further evidence.

Let us now analyze the second term $E_{\mu\nu}$ of the 4D Lagrangian (4.2.15). We will check only a subset of contributions. One obvious source is terms involving $\breve{F}_{AB}{}^I$ whose decomposition in 4D tangent space indices yields $F_{ab}{}^I$. These give contributions to the 4D Lagrangian of the form

$$\begin{aligned} &-\tfrac{1}{2}\,c_I\,F_{ab}{}^I\Big[\tfrac{3}{16}\breve{R}(M)_{CD}{}^{ab}\,\breve{T}^{CD}+\breve{T}^{ab}\Big(\breve{D}+\tfrac{3}{8}(\breve{T}_{CD})^2\Big)-\tfrac{9}{2}\breve{T}^{aC}\breve{T}_{CD}\breve{T}^{Db}\Big]|X^0|^{-1}\\ &+\tfrac{3}{8}\,\mathrm{i}\,\varepsilon^{abCDE}c_IF_{ab}{}^I\Big(\breve{T}_{DF}\breve{\mathcal{D}}^F\breve{T}_{DE}+\tfrac{3}{2}\breve{T}_{CF}\breve{\mathcal{D}}_D\breve{T}_E{}^F\Big)\,|X^0|^{-1}\ . \end{aligned} \tag{4.2.23}$$

We will discuss how to simplify this expression shortly. The other contributions come from the Chern-Simons term, which gives

$$-\tfrac{1}{64}\,\mathrm{i}\,\varepsilon^{abcd}\,c_I\,W_a{}^I\Big(\breve{R}(M)_{bc}{}^{EF}\breve{R}(M)_{d5\,EF}+\frac{1}{3}\breve{R}(\mathcal{V})_{bci}{}^j\breve{R}(\mathcal{V})_{d5j}{}^i\Big)\,|X^0|^{-1}\ . \tag{4.2.24}$$

This can be rearranged to

$$\begin{aligned} &-\tfrac{1}{64}\,\mathrm{i}\,\varepsilon^{abcd}c_IF_{ab}{}^I\Big(\tfrac{1}{8}R_{cd}{}^{ef}F_{ef}{}^0|X^0|^2+\tfrac{1}{128}(F_{ef}{}^0)^2F_{cd}{}^0+\tfrac{1}{64}F^{ef\,0}F_{ce}{}^0F_{df}{}^0\Big)|X^0|^{-4}\\ &\quad+\tfrac{1}{192}\,\mathrm{i}\,\varepsilon^{abcd}c_IF_{ab}{}^I\Big(\tfrac{1}{4}\,\varepsilon^{jk}\,R(\mathcal{V})_{cd}{}^i{}_k\,Y_{ij}{}^0\,|X^0|^2+\tfrac{1}{32}F_{cd}{}^0(Y_{ij}{}^0)^2\Big)\,|X^0|^{-4} \end{aligned} \tag{4.2.25}$$

up to terms involving derivatives of $|X^0|$, which from now on we will neglect to keep our expressions simpler. It will be useful to neglect other terms in (4.2.23). For example, expressions involving $\breve{T}_{a5}$ appear in nearly every term, often in multiple ways (e.g. from the 5D spin connection), so it will be convenient to set $\breve{T}_{a5}$ to zero, which amounts to discarding $\mathcal{D}_a\ln(X^0/\bar{X}^0)$. We will also ignore all terms involving $F_{ab}{}^0$ that also contain a factor of $T_{cd}{}^{ij}$, T_{cdij} or another $F_{cd}{}^0$. These conditions together allow us to focus on only the first line of (4.2.23). Proceeding, we find that the first line reduces to

$$-\tfrac{1}{2}c_I\,F_{ab}{}^I\Big[\tfrac{3}{16}\breve{R}(M)_{cd}{}^{ab}\,\breve{T}^{cd}+\breve{T}^{ab}\Big(\breve{D}+\tfrac{3}{8}(\breve{T}_{cd})^2\Big)-\tfrac{9}{2}\breve{T}^{ac}\breve{T}_{cd}\breve{T}^{db}\Big]\,|X^0|^{-1}\ . \tag{4.2.26}$$

Now we combine this with (4.2.25) and find the coefficient of $c_IF^{ab\,I}$ to be

$$\begin{aligned} -64\,\mathrm{i}\,\tilde{E}_{ab}\sim{}&\tfrac{1}{2}\mathrm{i}\,\mathcal{C}_{abcd}\,T^{cd\,ij}\varepsilon_{ij}\,(X^0)^{-1}+\tfrac{1}{3}\mathrm{i}\,\varepsilon_{ik}\,R(\mathcal{V})^{-}_{ab}{}^k{}_j\,Y^{ij\,0}\,|X^0|^{-2}\\ &+\tfrac{4}{3}\mathrm{i}\,(\mathcal{R}_a{}^c-\tfrac{1}{4}\delta_a{}^c\mathcal{R})\,F^{+0}_{cb}\,|X^0|^{-2}+\tfrac{1}{9}\mathrm{i}\,\mathcal{R}(F^{-0}_{ab}+\tfrac{1}{2}\bar{X}^0T_{ab}{}^{ij}\varepsilon_{ij})\,|X^0|^{-2}\\ &-\tfrac{2}{3}\mathrm{i}\,D\,(F^{-0}_{ab}-\bar{X}^0T_{ab}{}^{ij}\varepsilon_{ij})\,|X^0|^{-2}-\tfrac{1}{12}\mathrm{i}\,(Y_{ij}{}^0)^2\,\big(F^{-0}_{ab}-\tfrac{1}{2}\bar{X}^0T_{ab}{}^{ij}\varepsilon_{ij}\big)\,|X^0|^{-4}\\ &-\tfrac{1}{192}\mathrm{i}\,T_{ab}{}^{ij}\varepsilon_{ij}\,(T_{cd}{}^{kl}\varepsilon_{kl})^2\,\bar{X}^0(X^0)^{-2}-\tfrac{1}{64}\mathrm{i}\,T_{ab}{}^{ij}\varepsilon_{ij}\,(T_{cd\,kl}\varepsilon^{kl})^2\,(\bar{X}^0)^{-1}+\text{c.c.} \end{aligned} \tag{4.2.27}$$

up to the terms we neglected. Keep in mind that $\tilde{E}_{ab}$ is imaginary so the above expression is actually real. To extract the corresponding terms from the 4D Lagrangian (4.2.13), we return to (4.2.20), where

$$-64\,\mathrm{i}\,\tilde{E}_{ab} = -\frac{\mathrm{i}}{X^0} F^-_{ab}|_\Phi + \frac{1}{(X^0)^2}\Big(\mathrm{i}F^{-\,0}_{ab} - \tfrac{1}{4}\mathrm{i}\,\bar{X}^0 T_{ab}{}^{ij}\varepsilon_{ij} + \tfrac{1}{4}\mathrm{i}\,X^0 T_{ab\,ij}\varepsilon^{ij}\Big)\,A|_\Phi + \text{c.c.} \tag{4.2.28}$$

The result for $A|_\Phi$ was given in (4.2.22). The expression for $F^-_{ab}|_\Phi$ is

$$\begin{aligned} F^-_{ab}|_\Phi = &-\tfrac{1}{2}\mathcal{R}(M)^{cd}{}_{ab}\,T_{cd}{}^{ij}\varepsilon_{ij} - \tfrac{1}{3}\varepsilon_{ij}T_{ab}{}^{ij}\,\Box_\mathrm{c}\ln\bar{X}^0 + \tfrac{1}{3}R(\mathcal{V})^{-\,i}_{ab\;\;k}Y^{jk\,0}\varepsilon_{ij}\,(\bar{X}^0)^{-1} \\ &- \tfrac{1}{24}T_{ab}{}^{ij}T_{cd\,ij}(F^{cd\,+\,0} - \tfrac{1}{4}X^0 T^{cd}{}_{kl}\varepsilon^{kl})(\bar{X}^0)^{-1} \\ &+ \tfrac{1}{3}(\delta_a{}^{[c}\delta_b{}^{d]} - \tfrac{1}{2}\varepsilon_{ab}{}^{cd})\Big[4D_cD^e\Big(\frac{F^{+0}_{ed} - \frac{1}{4}X^0 T_{ab\,ij}\varepsilon^{ij}}{\bar{X}^0}\Big) - D_cD^eT_{ed}{}^{ij}\varepsilon_{ij} \\ &\qquad + D^e\ln\bar{X}^0\,D_cT_{de}{}^{ij}\varepsilon_{ij} + D_c\ln\bar{X}^0 D^eT_{ed}{}^{ij}\varepsilon_{ij}\Big] \end{aligned} \tag{4.2.29}$$

A careful calculation, keeping only the terms discussed, reproduces (4.2.27).

Let us now analyze the last term G of the 4D Lagrangian (4.2.15). Because of the complexity of the full expression, we will only look at a small number of characteristic terms. We begin with all terms involving the 4D SU(2) curvature tensor, which arise only from the second and third lines of (4.2.12). These are

$$\begin{aligned} 128\,X^0\,G \sim &-\tfrac{1}{3}\,\mathrm{i}\,R(\mathcal{V})^{+\,i}_{ab\;\;j}R(\mathcal{V})^{ab+j}{}_i - \mathrm{i}R(\mathcal{V})^{-\,i}_{ab\;\;j}R(\mathcal{V})^{ab\,-\,j}{}_i \\ &+ \tfrac{1}{8}R(\mathcal{V})_{ab}{}^j{}_k\varepsilon^{ki}\,Y_{ij}{}^0\Big(\tfrac{4}{3}\,\mathrm{i}\,\bar{X}^0\,T^{ab\,mn}\varepsilon_{mn} + \tfrac{8}{3}\,\mathrm{i}F^{ab\,-\,0} + \text{c.c.}\Big)|X^0|^{-2} \end{aligned} \tag{4.2.30}$$

Next, we collect all terms involving the 4D auxiliary field D that do not involve derivatives of X^0 or $\bar{X}^0$. These arise only from 5D terms involving $\breve{D}$ and are given by

$$\begin{aligned} 128\,X^0\,G \sim -\tfrac{32}{3}\mathrm{i}\,D^2 + \mathrm{i}\,D\Big[&\tfrac{1}{6}\frac{\bar{X}^0}{X^0}(T_{ab}{}^{ij}\varepsilon_{ij})^2 + \tfrac{1}{6}\frac{X^0}{\bar{X}^0}(T_{ab\,ij}\varepsilon^{ij})^2 - \tfrac{2}{3}F^{-0}_{ab}T^{abij}\varepsilon_{ij}(X^0)^{-1} \\ &+ (F^{-0}_{ab})^2|X^0|^{-2} + \tfrac{1}{3}(F^{+0}_{ab})^2|X^0|^{-2} + \tfrac{8}{9}\mathcal{R} - \tfrac{4}{3}(Y_{ij}{}^0)^2|X^0|^{-2}\Big]\,. \end{aligned} \tag{4.2.31}$$

Finally, we include all expressions quadratic in the 4D Riemann tensor as well as the terms $(Y_{ij}{}^0)^4$ and $\mathcal{R}(Y_{ij}{}^0)^2$. These are easily deduced from the 5D Lagrangian because they arise only from the second and third lines as well as the term involving

$\breve{D}^2$. The result is

$$128\,X^0\,G \sim -2\mathrm{i}\,\mathcal{C}_{ab}^{-\,cd}\mathcal{C}_{cd}^{-\,ab} - \tfrac{2}{3}\mathrm{i}\,(\mathcal{R}_{ab})^2 + \tfrac{4}{27}\mathrm{i}\,\mathcal{R}^2 - \tfrac{1}{24}\mathrm{i}(Y_{ij}{}^0)^4|X^0|^{-4} + \tfrac{1}{18}\mathrm{i}\,\mathcal{R}\,(Y_{ij}{}^0)^2|X^0|^{-2}\,. \tag{4.2.32}$$

These three sets of terms, (4.2.30)–(4.2.32), constitute a useful characteristic set. They can be found within the 4D Lagrangian (4.2.20), for which G is given by

$$\begin{aligned}
128\,G = &-\frac{\mathrm{i}}{X^0}\,C|_\Phi - \frac{\mathrm{i}}{2(X^0)^2}Y^{ij\,0}\,B_{ij}|_\Phi - \frac{\mathrm{i}}{4\bar{X}^0}T^{ab}{}_{ij}\varepsilon^{ij}\,F^+_{ab}|_\Phi + \frac{\mathrm{i}}{(X^0)^2}\big(F^{ab-0} - \tfrac{1}{4}\bar{X}^0T^{ab\,ij}\varepsilon_{ij}\big)\,F^-_{ab}|_\Phi \\
&- \frac{\mathrm{i}}{(X^0)^2}\Big[2\Box_{\mathrm{c}}\bar{X}^0 + \tfrac{1}{4}(F^{+0}_{ab} - \tfrac{1}{4}\,X^0T_{ab\,ij}\varepsilon^{ij})T^{ab}{}_{kl}\varepsilon^{kl} - \frac{1}{2\,X^0}Y_{ij}{}^0\,Y^{ij\,0} \\
&\qquad\qquad + \frac{1}{X^0}(F^{-0}_{ab} - \tfrac{1}{4}\bar{X}^0\,T_{ab}{}^{ij}\varepsilon_{ij})^2\Big]\,A|_\Phi \\
&- 2\mathrm{i}\,\Box_{\mathrm{c}}\Big(\frac{\bar{A}|_\Phi}{\bar{X}^0}\Big) + \frac{\mathrm{i}}{4(\bar{X}^0)^2}\,T^{ab}{}_{ij}\varepsilon^{ij}(F^{+0}_{ab} - \tfrac{1}{4}X^0\,T_{ab\,kl}\varepsilon^{kl})\,\bar{A}|_\Phi\,.
\end{aligned} \tag{4.2.33}$$

The expressions for all of the bosonic components of Φ have been repeated explicitly except for $C|_\Phi$ which can be found in (4.2.2).

4.3 Gauss-Bonnet term and entropy contributions

Originally the first calculation of the entropy of BPS black holes involving higher derivative couplings was based on the supersymmetric extension of the square of the Weyl tensor [9, 11]. More precisely (4.1.14) was generalized to a holomorphic and homogeneous function F of weight two, depending on $W^2 = W_{\alpha\beta}W^{\alpha\beta}$ and the vector multiplets $\mathcal{X}^I$, i.e.

$$S \propto \int \mathrm{d}^4x\,\mathrm{d}^4\theta\;\mathcal{E}\,F(\mathcal{X}^I, W^2) + \text{h.c.} \tag{4.3.1}$$

A somewhat surprising result was that the actual contribution from the higher derivative terms did not originate from the square of the Weyl tensor, but from the terms
$T^{acij}D_aD^bT_{bcij}$ required by supersymmetry. Some time later, in a specific model [93], the entropy was calculated by replacing the square of the Weyl tensor by the Gauss-Bonnet combination

$$C^{abcd}C_{abcd} \Longrightarrow C^{abcd}C_{abcd} - 2\mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{2}{3}\mathcal{R}^2\,, \tag{4.3.2}$$

keeping the same coefficient in front of C^2 term. Since the supersymmetrization of the Gauss-Bonnet term was not known, no additional terms were included. The surprising result was that this pure Gauss-Bonnet coupling gave rise, at least in this model, to the same result as [9, 11].

With the results of this chapter [70] it is now straightforward to analyze the reasons behind this unexpected match, which holds even when including all the terms required by supersymmetry. The relevant terms in the supersymmetrization (4.3.1) of the Weyl tensor squared are

$$\begin{aligned} e^{-1}A'\,\mathcal{L}^{-}_{\mathrm{W}} \sim A'\Big\{ &\tfrac{1}{2}C^{abcd}C_{abcd} - \tfrac{1}{2}C^{abcd}\tilde{C}_{abcd} \\ &- \tfrac{1}{2}T^{acij}D_aD^bT_{bcij} - \tfrac{1}{256}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl}\Big\}\,, \end{aligned} \tag{4.3.3}$$

where A' denotes the scalar associated with the ratio of two vector multiplets. As already mentioned, the sole contribution to the BPS black hole entropy in the original calculation came from the third term above. The reason is that the Wald entropy follows in this particular case from varying the action with respect to $\mathcal{R}_{ab}{}^{cd}$ and subsequently restricting the background to ensure that the near-horizon horizon is fully supersymmetric (for further details we refer to [9, 11]). In this near-horizon background both the Weyl tensor and the Ricci scalar vanish, so that the term quadratic in the Weyl tensor cannot give a contribution to the entropy. However, it turns out that the square of the (conformally) covariant derivatives acting on T_{bcij} involve terms linear in the Ricci tensor, while the tensor fields T are non-vanishing so that this term determines the entropy.

Let us now give the relevant terms in the non-linear kinetic multiplet, which can be added to (4.3.3) to carry out the replacement (4.3.2) in the fully supersymmetric context,

$$e^{-1}A'\,\mathcal{L}^{-}_{\mathrm{NL}} \sim A'\Big\{ -\mathcal{R}^{ab}\mathcal{R}_{ab} + \tfrac{1}{3}\mathcal{R}^2 + \tfrac{1}{2}T^{acij}D_aD^bT_{bcij} + \tfrac{1}{256}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl}\Big\}\,. \tag{4.3.4}$$

Here the first and the third term do both contribute to the entropy, but as it turns out their contribution cancels in the near-horizon geometry by virtue of the relation $\mathcal{R}_{ab} = -\frac{1}{8}T_a{}^{cij}T_{bcij}$. Hence it follows that the replacement (4.3.2) *at the fully supersymmetric level* does not affect the result for the BPS black hole entropy.[7] Moreover, the terms depending on the tensor fields cancel in the sum of (4.3.3) and (4.3.4), so that in the calculation based on the Gauss-Bonnet term the

[7] Similarly, $\mathcal{L}^{-}_{\mathrm{NL}}$ contributes nothing to the electric charges of BPS black holes.

supersymmetric completion will not contribute, just as indicated by the result of [93].

In addition one may also consider the actual value of the two invariants in the supersymmetric near-horizon background. This is the reason why we also included the T^4 terms in (4.3.3) and (4.3.4), as they are the only other terms that can generate additional contributions to the action in the near-horizon geometry. Working out this particular contribution, we find that (4.3.4) vanishes, and furthermore that the T-dependent terms vanish in the sum of (4.3.3) and (4.3.4). Hence the supersymmetric completion does not contribute to the Gauss-Bonnet coupling, and the value of the action will not change under the replacement (4.3.2) at the fully supersymmetric level. We should add that this last result has a bearing on the evaluation of the logarithmic corrections to the BPS entropy in [95]. There the square of the Weyl tensor and the Gauss-Bonnet invariant were equated and their contributions summed without further information of the possible supersymmetric completion of the coupling to a Gauss-Bonnet term. This was necessary in order to obtain quantitative agreement when comparing two methods for calculating the logarithmic corrections. Our above analysis thus confirms and clarifies the earlier observations in [93, 95].

We have showed for this case that the non-linear version of the kinetic multiplet vanishes at supersymmetric field configurations and it does not contribute to the entropy of a BPS black hole. A more complete analysis, establishing the existence of a BPS non-renormalization theorem in a more general Lagrangian, would proceed along the same lines as in [65], which established that Lagrangians involving the usual kinetic multiplet $\mathbb{T}(\bar{\Phi})$ will vanish for a supersymetric background and also their first derivative with respect to fields or parameters will vanish in such a background. The latter would imply in particular that they cannot contribute to the BPS black hole entropy or to the electric charges. The proof was based on the fact that weight-zero chiral superfields must be proportional to a constant in the supersymmetric limit. For the non-linear version of the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi})$ considered here, there is a marked difference because Φ is a chiral multiplet of non-zero weight. Its supersymmetric value is therefore not necessarily proportional to a constant, which makes the corresponding BPS analysis significantly more involved, with constraints imposed on the supergravity background as well as the chiral multiplet itself. We will give a more thorough analysis of these features in the next chapter.

Chapter 5

A non-renormalization theorem in $\mathcal{N} = 2$ supergravity

In flat space-time the analysis of fully supersymmetric backgrounds is rather straightforward. In that case the supersymmetry algebra generically implies that all component fields are space-time independent, so that all derivative terms in the supersymmetry transformations can be ignored. It then follows that all fields that are in the image of the supercharges must vanish. Therefore only the lowest-dimensional field, which cannot be generated by applying a supersymmetry transformation on yet another field, can take a finite, but constant value. In terms of superfields, this means that full supersymmetry requires any superfield to be constant, i.e. independent of both the bosonic and the fermionic coordinates. In the context of non-trivial space-times, similar results can be derived as long as one is dealing with rigid supersymmetry.

The first part of this chapter deals with a systematic analysis of the supersymmetric values that certain supermultiplets can take, but now in the context of local supersymmetry which is somewhat more subtle. When considering a large variety of supersymmetric invariants, it is preferable to make use of the (off-shell) superconformal multiplet calculus, where one encounters an extended set of local gauge invariances associated with the superconformal algebra. Proper attention should be paid to all these invariances. This last aspect does not form an impediment for analyzing supersymmetric backgrounds and in fact the presence of the extra conformal (super)symmetries, as already explained, greatly improves the systematics of the analysis. But it is important to appreciate that we are now dealing with *local* gauge invariances which imply a reduction of the physical degrees of freedom. Therefore it does not make sense to just impose gauge invariance on a field configuration and it is natural that a gauge invariant orbit of solutions will remain at the end. In principle this implies that a fully supersymmetric background is only determined up to (small) gauge transformations. In practice this means that we will obtain (conformally) covariant conditions on the field configuration.

To explain the strategy that we will follow in this chapter for establishing supersymmetric backgrounds and to further elucidate some of the conceptual issues, we start in section 5.1 by discussing a single $\mathcal{N}=2$ vector supermultiplet coupled to a conformal supergravity background (whose covariant quantities comprise the Weyl multiplet $\mathbf{W}$). When deriving the consequences of supersymmetry for the resulting field configuration we naturally discover that the conformal supergravity background itself is also subject to constraints. These constraints are identical to the ones that apply to the Weyl multiplet without the presence of the vector multiplet.

In section 5.2, we briefly present three other short supermultiplets coupled to a conformal supergravity background, namely the tensor multiplet, the non-linear multiplet, and the hypermultiplet. These three multiplets are all characterized by the fact that their lowest-weight scalars transform under the SU(2) R-symmetry group. Requiring supersymmetry in the presence of any of these multiplets turns out to impose a stronger restriction on the Weyl multiplet than when only vector multiplets are present. With this additional restriction the allowed field configurations are equivalent to the ones derived in [11].

Having determined the conditions imposed by supersymmetry we turn to a large class of supersymmetric actions with higher-derivative couplings. We first concentrate on the kinetic multiplet $\mathbb{T}(\ln\bar{\Phi}_w)$, extensively discussed in the previous chapter, and derive the conditions imposed by full supersymmetry. This then facilitates our task, undertaken in section 5.4, to establish the existence of the non-renormalization theorem of the type discussed before for this class of couplings. This result thus establishes an extension of the non-renormalization theorem that was initially proven for the more restricted class of higher-derivative couplings with $w=0$ [65]. Some early indications of this extended non-renormalization theorem were already noted in section 4.2.1 of this work ([70]), where some applications were also pointed out.

5.1 Vector supermultiplets in a superconformal background

In this section we derive the conditions that follow from imposing full supersymmetry on a field configuration consisting of a single vector supermultiplet in a conformal supergravity background. We first focus on the conditions imposed by supersymmetry on the vector multiplet. This eventually leads to conditions on the Weyl multiplet, the supermultiplet that characterizes the conformal supergravity background. The same analysis for the Weyl supermultiplet without any vector multiplet present turns out to lead to identical conditions. This situation will change in the case that other supermultiplets than the vector one are present, as will be shown in section 5.2. There we will deal with the remaining short supermultiplets, namely the tensor multiplet, the so-called non-linear multiplet and the hypermultiplet. As it turns out, in the presence of either one of these multiplets, the Weyl multiplet is subject to additional restrictions.

Note that we will generally suppress terms that are of higher order in the fermions, because eventually the supersymmetric field configurations will be presented with all fermion fields set to zero.

Before beginning the actual analysis of supersymmetric field configurations, let us recall that the superconformal symmetries are realized as *local* gauge invariances, which makes the analysis conceptually rather different as compared to the rigid case. For instance, imposing rigid supersymmetry requires the scalar field X to be constant. In the present context such a result is not meaningful, because X is subject to local scale and phase transformations, so that any two non-zero values of the field X will be gauge equivalent. A similar comment applies also to the fermions, where one might expect that the fields Ω_i will be required to vanish. But here again one realizes that two different values of Ω_i can be gauge equivalent by S-supersymmetry. Obviously a gauge invariant orbit of solutions must remain, but it is often convenient to choose a particular representative of the gauge orbit, which is equivalent to adopting a gauge condition. However, we prefer to restrict this option to the fermionic symmetries and leave the bosonic superconformal gauge invariances unaffected to keep the structure of our results as transparent as possible.

Let us now point out that in certain cases the analysis of supersymmetric configurations can be more direct, which is an important result that will be relevant throughout this chapter. Rather than considering a single vector multiplet, let us briefly consider two such multiplets with fields (X^1, X^2), $({\Omega_i}^1, {\Omega_i}^2)$, etcetera. Then we may consider a (conformal primary) chiral multiplet with the components

$$\frac{X^1}{X^2}\,, \qquad \frac{X^2\,\Omega_i{}^1 - X^1\,\Omega_i{}^2}{(X^2)^2}\,, \qquad \text{etcetera}\,. \tag{5.1.1}$$

Now the analysis of full supersymmetry becomes straightforward, because the first (scalar) component is invariant under dilatations and U(1) transformations (it has weights $w = c = 0$), whereas the second fermionic component is invariant under S-supersymmetry. Therefore it is now straightforward to conclude that the scalar must be a constant, while the fermionic component must vanish. Continuing this analysis will show that this multiplet is restricted to a constant, or, equivalently, that in the supersymmetric limit the two multiplets must be proportional to one another. This is an example of a more generic result: if the lowest-weight (scalar) component of a multiplet does not transform under dilatations and U(1) transformations, then the supersymmetry algebra implies that the lowest-weight fermion into which it transforms must be invariant under S-supersymmetry. In the supersymmetric limit, this multiplet is then restricted to a constant. For a general chiral multiplet this result was proven in [65].

From the above result it is therefore clear that nothing will be learned by considering several vector multiplets at once, so we return to the original problem using a single vector multiplet. Given the fact that the local superconformal gauge invariances will naturally lead to a certain degeneracy, we will define a specific approach based on two guiding principles. First of all, we insist that the bosonic superconformal invariances are preserved so that the final result can be expressed in terms of equations that are manifestly covariant with respect to all these gauge invariances. Secondly we assume that all (supercovariant) fermionic quantities will vanish in the bosonic background. This leaves the bosonic invariance intact. The only equations that are relevant thus follow from the requirement that the supersymmetry variations of the (supercovariant) fermionic quantities should vanish under a particular set of supersymmetry transformations parametrized by eight independent spinorial parameters ϵ^i and ϵ_i. The resulting bosonic covariant equations then characterize all the supersymmetric configurations. As we shall see, this strategy amounts to choosing a certain representative of the fermionic gauge orbit. In principle one can still apply the fermionic gauge transformations, but this will then lead to a different representative for which the fermion fields do not vanish.

Hence, in order that X is invariant under full supersymmetry one naturally assumes that $\Omega_i = 0$. To ensure that the transformation of the fermions will vanish as well, one requires that a linear combination of Q- and S-supersymmetry will vanish on the spinor fields Ω_i, which can be found by expressing the parameter η_i of the S-supersymmetry transformation in terms of the parameters of the Q-supersymmetry transformations, i.e.,

$$\hat{\eta}_i = -X^{-1}\big[\not{\mathcal{D}}X\epsilon_i + \tfrac{1}{4}\varepsilon_{ij}\hat{F}^-_{bc}\gamma^{bc}\epsilon^j + \tfrac{1}{2}Y_{ij}\epsilon^j\big]\,. \tag{5.1.2}$$

Here we have replaced the supercovariant derivative D_a by the derivative $\mathcal{D}_a$ covariant with respect to only the linearly realized bosonic symmetries. [1]

In this strategy the initial vector multiplet plays a key role, but in due course we will demonstrate that the results will be independent of the choice of the particular supermultiplet from where one starts this procedure. We should also mention that all the constraints can alternatively be obtained by exploiting the observation given below (5.1.1). Namely, one can start from bosonic expressions constructed from various supermultiplet components that are invariant under dilatations and chiral transformations, and explore the fact that they must vanish under repeated supersymmetry transformations. We shall comment on this aspect when considering the specific results of our calculations.

As explained earlier we subsequently require that all supercovariant fermionic quantities vanish under supersymmetry and so must their supersymmetry variations. Hence the superconformal derivative $D_a\Omega_i$ is assumed to vanish identically. What remains is to ensure that also its variation will vanish under the particular combination of Q- and S-supersymmetry defined by (5.1.2). To investigate the invariance of $D_a\Omega_i$, let us first define the superconformal derivative,

$$D_a\Omega_i = \mathcal{D}_a\Omega_i - \not{D}X\psi_{ai} - \tfrac{1}{4}\varepsilon_{ij}\hat{F}^-_{bc}\gamma^{bc}\psi_a{}^j - \tfrac{1}{2}Y_{ij}\psi_a{}^j - X\,\phi_{ai}\,, \tag{5.1.3}$$

where $\psi_\mu{}^i$ and $\psi_{\mu i}$ denote the chiral and anti-chiral components of the gravitino field that is the gauge field associated with Q-supersymmetry. The gauge fields of S-supersymmetry are not elementary but composite fields denoted by $\phi_{\mu i}$ and $\phi_\mu{}^i$. Its explicit definition can be found in e.g. [65, 70] (see also B.5). The derivative $\mathcal{D}_\mu$ is covariant under all the linearly acting bosonic transformations, namely dilatations, local Lorentz transformations and local R-symmetry transformations. Since we assumed that the fermionic gauge field must also vanish in the supersymmetric limit we indeed have $D_a\Omega_i = 0$.

[1] It is easy to realize, by plugging in this value for the S-supersymmetry parameter η^i into the transformation rules, say, of the Weyl multiplet (B.1), that the gauge fixed version of the field transformations is highly non-linear in the fields. That justifies our choice to study conformal, instead of Poincaré, supergravity.

Now consider the supersymmetry variation of $D_a\Omega_i$, restricting ourselves to the purely bosonic terms, using that the generic supersymmetry variations of the Q- and S-supersymmetry gauge fields are given (up to terms proportional to fermionic bilinears) by

$$\begin{aligned}\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \tfrac{1}{8}T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\,,\\ \delta\phi_\mu{}^i &= -\,2\,f_\mu{}^a\gamma_a\epsilon^i + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_\mu\epsilon^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i - \tfrac{1}{8}\not{D}T^{ab\,ij}\gamma_{ab}\gamma_\mu\epsilon_j + 2\,\mathcal{D}_\mu\eta^i\,,\end{aligned} \tag{5.1.4}$$

where $f_\mu{}^a$ is the gauge field of special conformal boosts, which is a composite field whose bosonic terms take the form (B.6).

Of course, for consistency one must also determine the constraints from full supersymmetry on the conformal supergravity background. As a first step in that direction we will therefore also include the consequences of the supersymmetry invariance of the spinor χ^i, which belongs to the Weyl multiplet. An independent analysis of the supersymmetry conditions based only on the Weyl multiplet fields will be discussed at the end of this section. Under supersymmetry χ^i transforms as follows,

$$\delta\chi^i = -\tfrac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \tfrac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \tfrac{1}{3}\mathrm{i}R(A)_{\mu\nu}\gamma^{\mu\nu}\epsilon^i + D\epsilon^i + \tfrac{1}{12}\gamma_{ab}T^{abij}\eta_j\,. \tag{5.1.5}$$

In evaluating the consequences of the above results one may assume that both X and $T_{ab}{}^{ij}$ are non-vanishing. The reason is that they are the lowest-weight fields of the two multiplets, so that their vanishing would imply that the corresponding multiplets will vanish.

Upon substituting (5.1.2) it turns out that $\delta(D_a\Omega_i)=0$ and $\delta\chi^i=0$ give rise to the following conditions,

$$\begin{aligned}R(\mathcal{V})_{\mu\nu}{}^i{}_j &= R(A)_{\mu\nu} = R(D)_{\mu\nu} = Y_{ij} = 0\,,\\ D &= \tfrac{1}{48}\big[X^{-1}\,\varepsilon_{ij}T_{ab}{}^{ij}\,\hat{F}^{-ab} + \bar{X}^{-1}\,\varepsilon^{ij}T_{abij}\,\hat{F}^{+ab}\big]\,,\\ \hat{F}^{-}_{a}{}^{c}\,T_{cb}{}^{ij} &= T_{ac}{}^{ij}\,\hat{F}^{-c}{}_b\,,\\ \bar{X}\,\varepsilon_{ij}\,T_{ab}{}^{ij}\,\hat{F}^{-ab} &= X\,\varepsilon^{ij}\,T_{abij}\,\hat{F}^{+ab}\,.\end{aligned} \tag{5.1.6}$$

The third equation implies that $\hat{F}^{-ab}$ is proportional to $\bar{X}\,\varepsilon_{ij}\,T_{ab}{}^{ij}$, with a proportionality factor that is invariant under local dilatations and U(1) R-symmetry transformations. Using also the the second and fourth equation in (5.1.6), one can determine this factor and obtain the relation

$$\hat{F}^-_{ab} = \frac{24\,D\,X\,T_{ab}{}^{ij}\,\varepsilon_{ij}}{(T^{cdkl}\,\varepsilon_{kl})^2}\,. \tag{5.1.7}$$

Here we have assumed that $T_{ab}{}^{ij}$ is not null, that is, $(T_{ab}{}^{ij}\varepsilon_{ij})^2 \neq 0$. We will continue making this assumption from now on.[2]

Furthermore we also derive the following conditions involving derivatives,

$$\begin{aligned}
\mathcal{D}_a\big(X\,T^{abij}\big) &= 0\,,\\
\mathcal{D}_a\big(X\,T^{ab}{}_{ij}\big) &= 2\,\varepsilon_{ij}\,\mathcal{D}_a\hat{F}^{-ab}\,,\\
\mathcal{D}_a\hat{F}^{-ab} &= -\,\mathcal{D}_a\ln(X/\bar{X})\,\hat{F}^{-ab}\,,\\
\mathcal{D}_a\hat{F}^{-bc} - \mathcal{D}_a\ln X\,\hat{F}^{-bc} &= -\,2\left[\mathcal{D}^{[b}\ln(X\bar{X})\,\hat{F}^{-c]}{}_a - \mathcal{D}_d\ln(X/\bar{X})\,\hat{F}^{-d[b}\,\delta^{c]}{}_a\right]^{[bc]-}\,,\\
X\,D_{(a}D_{b)}X - 2\,\mathcal{D}_aX\,\mathcal{D}_bX &= \frac{X}{2\,\bar{X}}\,\hat{F}^{-c}_{a}\hat{F}^{+}_{cb} - \frac{1}{2}\eta_{ab}\Big[(\mathcal{D}_cX)^2 + \frac{1}{16}X\,\hat{F}^{-cd}\,T_{cd}{}^{ij}\varepsilon_{ij}\Big]\,,
\end{aligned} \tag{5.1.8}$$

where, in the last equation, $D_{(a}D_{b)}X \equiv \big(\mathcal{D}_{(a}\,\mathcal{D}_{b)} + f_{\mu(a}\,e_{b)}{}^{\mu}\big)X$. This equation thus leads to a condition on the field $f_\mu{}^a$ and therefore on $R(\omega,e)_\mu{}^a$. The imaginary part of the second equation is consistent with the Bianchi identity on the field strength associated with the vector gauge field W_μ. The last term in the fourth equation (5.1.6) involves an anti-selfdual projection on the indices $[bc]$. When this is taken into account, the result takes the form

$$\mathcal{D}_a\hat{F}^{-bc} - \mathcal{D}_a\ln(X\bar{X})\,\hat{F}^{-bc} + 2\,\mathcal{D}^{[b}\ln X\,\hat{F}^{-c]}{}_a - 2\,\mathcal{D}_d\ln X\,\hat{F}^{-d[b}\,\delta_a^{c]} = 0\,, \tag{5.1.9}$$

which is conformally invariant in agreement with our original assumption.

We note one more equation that follows from the first three equations of (5.1.8), namely

$$\big(\hat{F}^{-ab} + \tfrac{1}{4}X\,T^{ab}{}_{ij}\varepsilon^{ij}\big)\mathcal{A}_b = 0\,, \tag{5.1.10}$$

where

$$\mathcal{A}_\mu \equiv -\tfrac{1}{2}\mathrm{i}\mathcal{D}_\mu\ln[X/\bar{X}] = A_\mu - \tfrac{1}{2}\mathrm{i}\partial_\mu\ln[X/\bar{X}]\,. \tag{5.1.11}$$

[2] The case where $(T_{ab}{}^{ij}\varepsilon_{ij})^2$ vanishes (in spite of the fact that $T_{ab}{}^{ij} \neq 0$) is rather special but can still be dealt with by using the same method. Since the results are not substantially different, we ignore this case here.

Obviously $\mathcal{A}_\mu$ is invariant under chiral U(1) and dilatations. Because $R(A)_{\mu\nu}=0$ it follows that $\partial_{[\mu}\mathcal{A}_{\nu]}=0$. Substituting (5.1.7) into (5.1.10), one derives, after multiplication with the selfdual tensor T_{abij} and making use of the standard identities for products of (anti-)selfdual tensors,

$$\left[\varepsilon^{ij}T_{ab\,ij}\,T^{ackl}\varepsilon_{kl}+24\,D\,\delta_b{}^c\right]\mathcal{A}_c=0\,, \tag{5.1.12}$$

The first term in this equation contains the product of a selfdual and an anti-selfdual tensor which is symmetric and traceless, and whose square must be proportional to the identity matrix. In this way one can obtain the following equation,

$$\left(\frac{D^2}{\left|(T^{abij}\varepsilon_{ij})^2\right|^2}-\frac{1}{(96)^2}\right)\mathcal{A}_\mu=0\,. \tag{5.1.13}$$

At this point we have not yet evaluated all the constraints of full supersymmetry on the Weyl multiplet. Besides the spinor field χ^i that we have already considered, there exists a supercovariant tensor-spinor, $R(Q)_{ab}{}^i$, which is the superconformal field strength of the gravitini fields. It emerges as the supersymmetry variation of the tensor field T^{abij}, so that it must vanish. Under Q- and S-supersymmetry $R(Q)_{ab}{}^i$ transforms as

$$\delta R(Q)_{ab}{}^i=-\tfrac{1}{2}\not{D}T_{ab}{}^{ij}\epsilon_j+R(\mathcal{V})^{-}_{ab}{}^i{}_j\,\epsilon^j-\tfrac{1}{2}\mathcal{R}(M)_{ab}{}^{cd}\,\gamma_{cd}\epsilon^i+\tfrac{1}{8}T_{cd}{}^{ij}\gamma^{cd}\gamma_{ab}\,\eta_j\,, \tag{5.1.14}$$

where $\mathcal{R}(M)_{ab}{}^{cd}$ is a modification of the curvature associated with the spin connection field $\omega_\mu{}^{ab}$ (see (B.9) and (B.3)).

Requiring $\delta R(Q)_{ab}{}^i=0$, and using again (5.1.2), leads to two more equations,

$$\begin{aligned}\mathcal{D}_aT^{bcij}-\mathcal{D}_a\ln X\,T^{bcij}+2\,\mathcal{D}^{[b}\ln X\,T^{c]}{}_a{}^{ij}-2\,\mathcal{D}_d\ln X\,T^{d[bij}\,\delta^{c]}{}_a&=0\,,\\ \mathcal{R}(M)^{-}_{ab\,cd}-\frac{1}{2\,|X|^2}\,(\varepsilon_{ij}\bar{X}\,T_{a[c}{}^{ij})\,\hat{F}^{-}{}_{d]b}\Big|^{[ab]-}&=0\,.\end{aligned} \tag{5.1.15}$$

From the first equation we derive

$$\varepsilon_{kl}T_{ab}{}^{kl}\,\mathcal{D}_cT^{cbij}\varepsilon_{ij}=-\tfrac{1}{8}\mathcal{D}_a(T^{bckl}\varepsilon_{kl})^2\,, \tag{5.1.16}$$

by making use of the identities that hold for contractions of (anti-)selfdual tensors. Furthermore one derives, upon combining (5.1.7), (5.1.9) and the first equation of (5.1.15), that certain ratios of fields must be constant,

$$\frac{X^2}{(T^{abij}\varepsilon_{ij})^2}=\text{constant}\,,\qquad \frac{D}{\left|(T^{abij}\varepsilon_{ij})^2\right|}=\text{constant}\,. \tag{5.1.17}$$

These expressions can be regarded as the lowest-weight components of a chiral or real supermultiplet, respectively, with $w = c = 0$. According to the theorem discussed earlier in this section, such multiplets must indeed be equal to a constant in the supersymmetric limit. This observation enables an alternative derivation of the same results that we are deriving in this section.

The second equation (5.1.15) involves an anti-selfdual projection over the the index pair $[ab]$ (because of the symmetry of this term, it is also anti-selfdual in $[cd]$), while $\mathcal{R}(M)_{ab\,cd}^{-}$ is anti-selfdual in both index pairs $[ab]$ and $[cd]$. Using (5.1.7) the equation then takes the form

$$\mathcal{R}(M)_{ab\,cd}^{-} - \frac{12\,D}{(T^{abij}\varepsilon_{ij})^2}\,P_{ab,cd}^{-} = 0\,, \tag{5.1.18}$$

where[3]

$$P_{ab,cd}^{-} \equiv T_{a[c}\,T_{d]b}\Big|^{[ab]-} = \tfrac{1}{8}\big(\delta_{a[c}\,\delta_{d]b} - \tfrac{1}{2}\varepsilon_{abcd}\big)(T^{efij}\varepsilon_{ij})^2 - \tfrac{1}{2}\varepsilon_{ij}\,T_{cd}{}^{ij}\,T_{ab}{}^{kl}\,\varepsilon_{kl}\,. \tag{5.1.19}$$

By now we have obtained a number of conditions that do not explicitly involve the vector multiplet fields. A relevant question is therefore whether the Weyl multiplet alone (i.e. without being coupled to a vector multiplet) requires the same conditions when imposing supersymmetry. Therefore we repeat the same procedure but now without coupling to a vector multiplet. Hence we start with the supersymmetry variation of the field χ^i shown in (5.1.5), and choose $\hat\eta_i$ such that its supersymmetry variation vanishes.

At this point the reader may wonder whether a different choice for $\hat\eta_i$ would not affect the results of the previous analysis, so that they would become incompatible with the new ones that we are about to derive. This is actually not the case, as one can simply see by considering the supersymmetry variation of the S-supersymmetric linear combination, $T^{abij}\gamma_{ab}\Omega_j - 24\,X\,\chi^i$, whose vanishing under Q-supersymmetry is obviously independent of whether $\hat\eta_i$ is chosen such that $\delta\Omega_i$ or $\delta\chi^i$ will vanish. To base the analysis on S-supersymmetric combinations of spinors was precisely the approach followed in [11]. Hence it follows that the choice of $\hat\eta_i$ is irrelevant, and it is again obvious that the fermionic gauge orbit associated with S-supersymmetry is not affected, as was emphasized earlier. Our approach of adopting a specific $\hat\eta_i$ associated with a specific supermultiplet is thus a matter of convenience when considering separate configurations of supermultiplets.

[3]Note that we are using Pauli-Källén conventions so that the Levi-Civita symbol is effectively pseudo-real.

Using the expression for $\hat{\eta}_i$ that is found by solving $\delta\chi^i = 0$ directly, one can evaluate the variations of $D_a\chi^i$ and $R(Q)^i_{ab}$, requiring them to vanish also. This calculation is completely similar to the approach followed before. A careful evaluation then shows that all the constraints of the Weyl multiplet imposed by requiring supersymmetry coincide fully with the constraints that we have evaluated before, starting from the vector multiplet (possibly exploiting the first equation of (5.1.17)).

Let us now return the last equation of (5.1.8), which involves terms quadratic in derivatives and yields an expression for the composite connection $f_\mu{}^a$ associated with the conformal boosts,

$$\begin{aligned} f_a{}^b = & -\mathcal{D}_a\mathcal{D}^b \ln X + \mathcal{D}_a \ln X\, \mathcal{D}^b \ln X - \tfrac{1}{2}\delta_a{}^b \big(\mathcal{D}_c \ln X\big)^2 - \tfrac{3}{4}\delta_a{}^b D \\ & - \frac{288\, D^2\, \varepsilon_{ij} T_{ac}{}^{ij}\, T^{bc}{}_{kl}\varepsilon^{kl}}{\big|(T^{demn}\varepsilon_{mn})^2\big|^2}\,. \end{aligned} \tag{5.1.20}$$

Whereas the left-hand side is manifestly real, the right-hand side is not. To analyze this we note that $\mathcal{D}_\mu X = \mathcal{D}_\mu|X| + \mathrm{i}\mathcal{A}_\mu$, where $\mathcal{A}_\mu$ has been defined in (5.1.11). The reality of (5.1.20) then implies

$$\mathcal{D}_a\mathcal{A}_b - 2\,\mathcal{A}_{(a}\,\mathcal{D}_{b)} \ln|X| - \eta_{ab}\,\mathcal{A}_c\,\mathcal{D}^c \ln|X| = 0\,, \tag{5.1.21}$$

where we note that (5.1.13) implies that $\mathcal{A}_\mu = 0$ for $|D| \neq \frac{1}{96}\,|(T^{abij}\varepsilon_{ij})^2|$. Hence we obtain the following form for the real part of (5.1.20),

$$\begin{aligned} f_a{}^b = & -\mathcal{D}_a\mathcal{D}^b \ln|X| + \mathcal{D}_a \ln|X|\, \mathcal{D}^b \ln|X| - \mathcal{A}_a\,\mathcal{A}^b \\ & - \tfrac{1}{2}\delta_a{}^b \big[\big(\mathcal{D}_c \ln|X|\big)^2 - \mathcal{A}^c\mathcal{A}_c + \tfrac{3}{2}D\big] - \frac{288\, D^2\, \varepsilon_{ij} T_{ac}{}^{ij}\, T^{bc}{}_{kl}\varepsilon^{kl}}{\big|(T^{demn}\varepsilon_{mn})^2\big|^2}\,. \end{aligned} \tag{5.1.22}$$

This completes the derivation of a consistent set of covariant equations that characterize the fully supersymmetric configurations consisting of a vector and the Weyl supermultiplet. What remains is to present the results for the components of the Riemann tensor. Up to this point we have fully preserved the covariance with respect to the bosonic symmetries of the superconformal group, so that the spin-connection field $\omega_\mu{}^{ab}$ depends both on the vierbein $e_\mu{}^a$ and on the dilatational gauge field b_μ. Hence the associated curvature $R(\omega)_{\mu\nu}{}^{ab}$ is only identical to the Riemann tensor when b_μ vanishes. For a conformally invariant action b_μ will be absent, while otherwise one still has the option to impose $b_\mu = 0$ as a gauge condition. Comparing (5.1.22) to (B.6), one derives the following expression for the Ricci tensor, which is in general not symmetric in the presence of the field b_μ, and

Ricci scalar,

$$\begin{aligned}\mathcal{R}(\omega,e)_{ab} =& -2\,\mathcal{D}_a\mathcal{D}_b \ln|X| + 2\,\mathcal{D}_a \ln|X|\,\mathcal{D}_b \ln|X| - 2\,\mathcal{A}_a\,\mathcal{A}_b \\ & - \eta_{ab}\Big[\mathcal{D}^c\mathcal{D}_c \ln|X| + 2\left(\mathcal{D}_c \ln|X|\right)^2 + 2\,\mathcal{A}^c\mathcal{A}_c + 3\,D\Big] \\ & - \Big[\frac{1}{16} + \frac{576\,D^2}{\left|(T^{demn}\varepsilon_{mn})^2\right|^2}\Big]\,\varepsilon_{ij}T_{ac}{}^{ij}\,T_b{}^{ckl}\varepsilon_{kl}\,. \\ \mathcal{R}(\omega,e) =& -6\,\mathcal{D}^a\mathcal{D}_a \ln|X| - 6\,\mathcal{D}^a \ln|X|\,\mathcal{D}_a \ln|X| + 6\,\mathcal{A}^2 - 12\,D\,. \end{aligned} \qquad (5.1.23)$$

Finally we note

$$\mathcal{R}(M)_{ab}{}^{cd} = \mathcal{C}(e,\omega)_{ab}{}^{cd} + D\,\delta_{ab}{}^{cd} + \cdots\,, \qquad (5.1.24)$$

where the suppressed terms are proportional to $R(A)_{\mu\nu}$ and to fermion bilinears, which all vanish in the supersymmetric background. Making use of (5.1.18) one then derives the expression for the Weyl tensor,

$$\mathcal{C}(e,\omega)_{ab}{}^{cd} = D\,\left[2\,\delta_{ab}{}^{cd} - \frac{6\,\varepsilon_{ij}T^{ij}_{ab}\,T^{cdkl}\varepsilon_{kl}}{(\varepsilon_{mn}T^{demn})^2} - \frac{6\,\varepsilon^{ij}T_{abij}\,T^{cd}{}_{kl}\varepsilon^{kl}}{(\varepsilon^{mn}T^{de}{}_{mn})^2}\right]\,. \qquad (5.1.25)$$

5.2 Three other short multiplets

In this section, we consider the remaining $\mathcal{N}=2$ short multiplets commonly encountered. They are the tensor multiplet, the non-linear multiplet, and the (on-shell) hypermultiplet. Their distinctive feature is that their lowest-weight components are scalar fields transforming under the SU(2) R-symmetry. For the tensor multiplet these fields are the pseudo-real SU(2) vector L_{ij}, for the non-linear multiplet it is given by a space-time dependent SU(2) element $\Phi^i{}_\alpha$, and for the hypermultiplet they are represented by certain sections $A(\phi)_i{}^\alpha$ of a hyperkähler cone.[4] These quantities will be introduced shortly. We assume that their SU(2) invariant norms are non-vanishing. For the non-linear multiplet, the norm equals $\det[\Phi^i{}_\alpha] = 1$; for the tensor and the hypermultiplet, these norms are the length L of the vector L_{ij} and the so-called hyperkähler potential $\chi(\phi)$, respectively, which both have $w=2$. Their precise definitions will be given shortly.

Requiring that the scalars are invariant under supersymmetry leads to the condition that the fermion fields must vanish. We discover that the presence of SU(2) indices on the lowest-dimension scalars generically leads to stronger conditions on the Weyl multiplet than the ones found for the vector multiplet in the previous

[4] The indices α for the non-linear multiplet and the hypermultiplet sections are unrelated. For example, the former take the values $\alpha=1,2$ while the latter take the values $\alpha=1,\cdots,r$.

section. Since all the underlying principles of the analysis have already been exhibited in the previous section, we keep the presentation rather concise. Obviously the conditions on the Weyl multiplet alone may be assumed. In particular, taking $R(\mathcal{V})_{\mu\nu}{}^i{}_j = R(A)_{\mu\nu} = R(D)_{\mu\nu} = 0$ from the start will simplify the analysis. An important condition, which will play a key role in many of the formulae, is

$$\mathcal{D}_a \ln\left|(T_{bc}{}^{ij}\varepsilon_{ij})^2\right| = \mathcal{D}_a \ln(X\bar{X}) = \begin{cases} \mathcal{D}_a \ln L\ , & \text{tensor multiplet} \\ -V_a\ , & \text{non-linear multiplet} \\ \mathcal{D}_a \ln\chi\ , & \text{hypermultiplet} \end{cases} \tag{5.2.1}$$

where V_a is a vector component of the non-linear multiplet, and L and χ are the two composite real $w=2$ scalar fields introduced above. These conditions are consistent with the (now familiar) observation that any $w=c=0$ scalar field must be constant, and so $|(T_{ab}{}^{ij}\varepsilon_{ij})^2|$ must be proportional to $X\bar{X}$, L and χ for a vector multiplet, tensor multiplet and hypermultiplet, respectively. Note that the vector V_a is not invariant under special conformal boosts.

In contrast with the previous section, we will find that for the three multiplets discussed here, the $w=2$ scalar field D of the Weyl multiplet will be required to vanish. This turns out to have major consequences for both the Weyl multiplet and for any vector multiplet. Invoking (5.1.7) and (5.1.18), one derives the following constraints on the Weyl multiplet and any vector multiplet:

$$D=0 \qquad \Longrightarrow \qquad \mathcal{R}(M)_{ab\,cd}=0\,, \qquad \hat{F}_{ab}=0\,. \tag{5.2.2}$$

The second equation implies that the Weyl tensor must vanish as a result of (5.1.25). The third equation of (5.2.2) and (3.3.10), with the gravitinos put to zero, leads to a constraint on the vector multiplet field strength,

$$F_{\mu\nu} \equiv 2\,\partial_{[\mu}W_{\nu]} = \tfrac{1}{4}\big[X\,T_{\mu\nu\,ij}\,\varepsilon^{ij} + \bar{X}\,T_{\mu\nu}{}^{ij}\,\varepsilon_{ij}\big]\,. \tag{5.2.3}$$

Another consequence of $D=0$ is given by (5.1.13), which implies that

$$\mathcal{A}_\mu = -\tfrac{1}{2}\mathrm{i}\mathcal{D}_\mu \ln(X/\bar{X}) = -\tfrac{1}{4}\mathrm{i}\mathcal{D}_\mu \ln\big[(T_{bc}{}^{ij}\varepsilon_{ij})^2/(T^{de}{}_{kl}\varepsilon^{kl})^2\big] = 0\ . \tag{5.2.4}$$

This determines the U(1) gauge connection in terms of the phase of $T_{ab}{}^{ij}$ (or X). The final two conditions we will encounter are the analogues of (5.1.15) and (5.1.22), found by making the replacement (5.2.1) with the additional constraints (5.2.2) and (5.2.4).

5.2.1 The tensor multiplet

The tensor multiplet consists of a pseudo-real SU(2) triplet of scalar fields L_{ij}, which has Weyl weight $w = 2$ and satisfies the pseudo-reality constraint $(L^{ij})^* = \varepsilon_{ik}\varepsilon_{jl}L^{kl}$, a doublet of spinors φ^i, a two-form gauge field $E_{\mu\nu}$, and a complex scalar G. Their Q- and S-supersymmetry transformations are

$$\begin{aligned}
\delta L_{ij} &= 2\,\bar\epsilon_{(i}\varphi_{j)} + 2\,\varepsilon_{ik}\varepsilon_{jl}\,\bar\epsilon^{(k}\varphi^{l)}\,,\\
\delta\varphi^i &= \not{D}L^{ij}\,\epsilon_j + \varepsilon^{ij}\,\hat{\not{E}}^I\,\epsilon_j - G\,\epsilon^i + 2L^{ij}\,\eta_j\,,\\
\delta G &= -\,2\,\bar\epsilon_i\not{D}\,\varphi^i \;-\bar\epsilon_i(6\,L^{ij}\,\chi_j + \tfrac14\,\gamma^{ab}T_{abjk}\,\varphi^l\,\varepsilon^{ij}\varepsilon^{kl}) + 2\,\bar\eta_i\varphi^i\,,\\
\delta E_{\mu\nu} &= \mathrm{i}\bar\epsilon^i\gamma_{\mu\nu}\varphi^j\,\varepsilon_{ij} - \mathrm{i}\bar\epsilon_i\gamma_{\mu\nu}\varphi_j\,\varepsilon^{ij} \;+\; 2\mathrm{i}\,L_{ij}\,\varepsilon^{jk}\,\bar\epsilon^i\gamma_{[\mu}\psi_{\nu]k} - 2\mathrm{i}\,L^{ij}\,\varepsilon_{jk}\,\bar\epsilon_i\gamma_{[\mu}\psi_{\nu]}{}^k\,,
\end{aligned} \tag{5.2.5}$$

where D_a are the superconformally covariant derivatives, and $\hat E^a$ equals the dual of a supercovariant three-form field strength,

$$\hat E^\mu = \tfrac12\mathrm{i}\,e^{-1}\,\varepsilon^{\mu\nu\rho\sigma}\Big[\partial_\nu E_{\rho\sigma} - \tfrac12\mathrm{i}\bar\psi^i_\nu\gamma_{\rho\sigma}\varphi^j\varepsilon_{ij} + \tfrac12\mathrm{i}\bar\psi_{\nu i}\gamma_{\rho\sigma}\varphi_j\varepsilon^{ij} - \mathrm{i}\,L_{ij}\varepsilon^{jk}\bar\psi_\nu{}^i\gamma_\rho\psi_{\sigma k}\Big]\,. \tag{5.2.6}$$

A supersymmetric field configuration for this multiplet can be found by following the same steps as for the vector multiplet. We note the convenient identity, $L^{ij}L_{jk} = \delta^i{}_k\,L^2$, where the modulus L of the SU(2) triplet is given by $L^2 = \frac12 L^{ij}L_{ij}$. We will assume that L is non-vanishing and impose $\delta\varphi^i = 0$ by choosing

$$\hat\eta_i = -\tfrac12 L_{ij}\,L^{-2}\big[\not{D}L^{jk}\,\epsilon_k + \varepsilon^{jk}\,\hat{\not{E}}\,\epsilon_k - G\,\epsilon^j\big]\,, \tag{5.2.7}$$

where all terms containing fermionic bilinears can be dropped. Next, we impose the conditions $\delta(D_a\varphi^i) = 0$ and $\delta\chi^i = \delta R(Q)_{ab}{}^i = 0$ and analyze their consequences. Although the latter two conditions have already been investigated separately, it turns out that when combining these with the condition $\delta(D_a\varphi^i) = 0$, while using the expression (5.2.7), one more readily obtains the results (5.2.2), strongly restricting the Weyl multiplet. Assuming as before that $T_{ab}{}^{ij}$ does not vanish leads to the conditions

$$G = \hat E_a = 0\,, \qquad L_{ik}\overleftrightarrow{\mathcal{D}}_a L^{kj} = 0\,, \tag{5.2.8}$$

which force the two-form $E_{\mu\nu}$ to be pure gauge and restrict $\mathcal{D}_a L_{ij} = L_{ij}\,\mathcal{D}_a\ln L$, or

$$\mathcal{D}_a(L_{ij}\,L^{-1}) = 0\,. \tag{5.2.9}$$

We find that the derivative of $T_{ab}{}^{ij}$ is given by (5.1.15) with the replacement $\mathcal{D}_a \ln X \to \frac{1}{2}\mathcal{D}_a \ln L$, implying both (5.2.4) and (5.2.1). Similarly, the analogue of (5.1.22) is reproduced.

5.2.2 The non-linear multiplet

Next we consider the case of the 'non-linear multiplet' in a conformal supergravity background [67, 96]. This multiplet consists of a scalar SU(2) matrix $\Phi^i{}_\alpha$ with $\alpha = 1, 2$, a fermion doublet with negative (positive) chirality components λ^i (λ_i), a complex anti-symmetric tensor M^{ij} and a real vector field V^a. Because $\Phi^i{}_\alpha$ is an element of SU(2), it must have vanishing Weyl weight and its inverse matrix is given by its hermitian conjugate denoted by $\Phi^\alpha{}_i$. Under Q- and S-supersymmetry, the fields transform as

$$
\begin{aligned}
\delta\Phi^i{}_\alpha &= (2\,\bar\epsilon^i\lambda_j - \delta^i{}_j\,\bar\epsilon^k\lambda_k - \text{h.c.})\,\Phi^j{}_\alpha\ ,\\
\delta\lambda^i &= -\tfrac{1}{2}\not{V}\epsilon^i - \tfrac{1}{2}M^{ij}\epsilon_j + \Phi^i{}_\alpha\not{\mathcal{D}}\Phi^\alpha{}_j\epsilon^j + \eta^i\ ,\\
\delta M^{ij} &= 12\,\bar\epsilon^{[i}\chi^{j]} + \tfrac{1}{2}\bar\epsilon^k\gamma^{ab}\lambda_k\,T_{ab}{}^{ij} - 4\bar\epsilon^{[i}\not{V}\lambda^{j]} - 2\,\bar\epsilon^k\lambda_k M^{ij} + 8\,\bar\epsilon^{[i}\big(\not{\mathcal{D}}\lambda^{j]} + \Phi^{j]}{}_\alpha\not{\mathcal{D}}\Phi^\alpha{}_k\lambda^k\big)\ ,\\
\delta V^a &= \tfrac{3}{2}\bar\epsilon^i\gamma^a\chi_i - \tfrac{1}{8}\bar\epsilon^i\gamma^a\gamma^{bc}\lambda^j\,T_{bc\,ij} - \bar\epsilon^i\gamma^a\not{V}\lambda_i + \bar\epsilon^i\gamma^a\lambda^j M_{ij} + 2\,\bar\epsilon^i\gamma^{ab}\mathcal{D}_b\lambda_i\\
&\quad + 2\bar\epsilon_i\gamma^a\Phi^i{}_\alpha\not{\mathcal{D}}\Phi^\alpha{}_j\lambda^j - \bar\lambda_i\gamma^a\eta^i + \text{h.c.}\ ,
\end{aligned}
\tag{5.2.10}
$$

where we have suppressed terms explicitly quadratic in the fermion fields. In order for the supersymmetry algebra to close, the vector V^a must obey the non-linear constraint (up to terms quadratic in the fermion fields)

$$
D_aV^a - \tfrac{1}{2}V^2 - 3D - \tfrac{1}{4}M^{ij}M_{ij} + \mathcal{D}_a\Phi^i{}_\alpha\mathcal{D}^a\Phi^\alpha{}_i = 0\,,
\tag{5.2.11}
$$

which can be interpreted as a condition on the field D of the Weyl multiplet. An unusual feature is that V^a transforms under conformal boosts, $\delta_{\rm K}V^a = 2\,\Lambda_{\rm K}{}^a$. Therefore the bosonic terms in the covariant derivative of $D_\mu V^a$ take the form

$$
D_\mu V^a = (\partial_\mu - b_\mu)V^a - \omega_\mu{}^{ab}\,V_b - 2\,f_\mu{}^a\,.
\tag{5.2.12}
$$

Since V^a has Weyl weight $w=1$, it follows that $\delta_{\rm K}(D_aV^a) = 2\,\Lambda_{\rm K}{}^a\,V_a$, so that the combination $D_aV^a - \frac{1}{2}V^2$ is conformally invariant.

As before, the condition $\delta\lambda^i = 0$ can be implemented by making a special choice for the S-supersymmetry parameter,

$$
\hat\eta^i = \tfrac{1}{2}\not{V}\epsilon^i + \tfrac{1}{2}M^{ij}\epsilon_j - \Phi^i{}_\alpha\not{\mathcal{D}}\Phi^\alpha{}_j\,\epsilon^j\ .
\tag{5.2.13}
$$

Requiring $\delta(D_a\lambda^i) = 0$ and $\delta\chi^i = \delta R(Q)_{ab}{}^i = 0$ leads to a number of conditions. The Weyl multiplet constraints are obviously implied, and one again finds that (5.2.2) should hold, along with

$$M^{ij} = 0 \;, \qquad \Phi^i{}_\alpha\, \mathcal{D}_a \Phi^\alpha{}_j = 0 \;. \tag{5.2.14}$$

The latter equation determines the SU(2) connection in terms of $\Phi^i{}_\alpha \partial_\mu \Phi^\alpha{}_j$. In addition, one finds

$$V_a = -\mathcal{D}_a \ln(T^{bcij}\varepsilon_{ij})^2 = -\mathcal{D}_a \ln(T^{bc}{}_{kl}\varepsilon^{kl})^2 \,, \tag{5.2.15}$$

implying (5.2.4) and (5.2.1). The equations (5.1.18) and (5.1.22), upon replacing $\mathcal{D}_a \ln X \to -\frac{1}{2}V_a$, are also found.

5.2.3 The hypermultiplet sector

Unlike the previous supermultiplets, hypermultiplets are realized as an on-shell supermultiplet. Since the multiplet consists only of scalar fields and fermions, without any gauge fields, there does not exist a preferred basis for the fields, which are subject to non-linear redefinitions that take the form of target-space diffeomorphisms and frame transformations of the fermions. For this reason, the hypermultiplets tend to mix under supersymmetry and so it is necessary to consider the entire hypermultiplet sector at once.

For a system of r hypermultiplets, one is dealing with a $4r$-dimensional hyperkähler target space with local coordinates ϕ^A and a target-space metric g_{AB}, $2r$ positive-chirality spinors $\zeta^{\bar\alpha}$ and $2r$ negative-chirality spinors ζ^α. The chiral and anti-chiral spinors are related by complex conjugation as they are Majorana spinors. They are subject to field-dependent reparametrizations of the form $\zeta^\alpha \to S^\alpha{}_\beta(\phi)\,\zeta^\beta$; the fields $\zeta^{\bar\alpha}$ are then redefined with the complex conjugate of $S^\alpha{}_\beta$. The target space is subject to arbitrary diffeomorphisms and has the standard Christoffel connection $\Gamma_{AB}{}^C$. Likewise there exist connections $\Gamma_A{}^\alpha{}_\beta$ and $\Gamma_A{}^{\bar\alpha}{}_{\bar\beta}$ associated with the field-dependent redefinitions noted above. Furthermore supersymmetry implies the existence of an hermitian and a skew-symmetric covariantly constant tensor, $G^{\alpha\bar\beta}$ and $\Omega^{\alpha\beta}$, respectively. The hermitian one appears in the kinetic term for the fermions, and the skew-symmetric one is related to the canonical invariant antisymmetric tensor of Sp(r).

In order to couple the r hypermultiplets to conformal supergravity, their target-space geometry must be a $4r$-dimensional hyperkähler cone [97].[5] The hypermultiplet scalars transform under dilatations associated with a homothetic Killing vector, and under the SU(2) R-symmetry, associated with the SU(2) Killing vectors of the hyperkähler cone. The fermions transform under dilatations and the U(1) factor of the R-symmetry by scale transformations and chiral rotations, respectively.

A systematic treatment of hypermultiplets makes use of local sections $A_i{}^\alpha(\phi)$ of an $\mathrm{Sp}(r) \times \mathrm{Sp}(1)$ bundle, where $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ refers to the corresponding R-symmetry group. These sections transform covariantly under R-symmetry and scale under dilatations with $w = 1$. We refer to [97] for further details. The Q- and S-supersymmetry transformations on the sections and the fermions take the following form,

$$\begin{aligned} \delta A_i{}^\alpha &= 2\bar\epsilon_i\zeta^\alpha + 2\varepsilon_{ij}\, G^{\alpha\bar\beta}\Omega_{\bar\beta\bar\gamma}\bar\epsilon^j\zeta^{\bar\gamma} - \delta_{\mathrm{Q}}\phi^B\,\Gamma_B{}^\alpha{}_\beta A_i{}^\beta\ , \\ \delta\zeta^\alpha &= \not{D}A_i{}^\alpha\epsilon^i + A_i{}^\alpha\eta^i - \delta_{\mathrm{Q}}\phi^B\,\Gamma_B{}^\alpha{}_\beta\zeta^\beta\ , \end{aligned} \tag{5.2.16}$$

where $\delta_{\mathrm{Q}}\phi^A$ denotes the transformation rule for the target-space scalars whose form is not relevant for what follows. The covariant tensors $G_{\bar\alpha\beta}$ and $\Omega_{\bar\alpha\bar\beta}$ can be expressed as bilinears in the covariant derivatives of the sections,

$$g^{AB}\,D_AA_i{}^\alpha\,D_BA^{j\bar\beta} = \delta_i{}^j\,G^{\alpha\bar\beta}\,, \qquad g^{AB}\,D_AA_i{}^\alpha\,D_BA_j{}^\beta = \varepsilon_{ij}\,\Omega^{\alpha\beta}\,. \tag{5.2.17}$$

A supersymmetric configuration requires that both the fermions and their supersymmetry variations vanish. For $r > 1$, one cannot find a choice for $\hat\eta^i$ which immediately solves $\delta\zeta^\alpha = 0$ for all α, so it will help to first single out one specific fermion to solve for $\hat\eta^i$. We will follow a similar procedure as in [11] and first single out the $w = 2$ hyperkähler potential χ, defined by

$$\chi = \tfrac{1}{2}\varepsilon^{ij}\,\bar\Omega_{\alpha\beta}\,A_i{}^\alpha\,A_j{}^\beta\,, \tag{5.2.18}$$

and focus on the composite fermion ζ_i into which it varies,

$$\delta\chi = 2\varepsilon^{ij}\bar\epsilon_j\zeta_i + \text{c.c.}\ , \qquad \zeta_i = \bar\Omega_{\alpha\beta}\,A_i{}^\alpha\,\zeta^\beta\ . \tag{5.2.19}$$

[5] Upon fixing the dilatational and SU(2) gauges, conformal supergravity is converted to Poincaré supergravity, and correspondingly the hyperkähler cone is converted into a quaternion-Kähler target space [97, 98], in accordance with [99].

Solving $\delta\zeta_i = 0$ leads to

$$\hat{\eta}^i = \varepsilon^{ij}\,\chi^{-1} A_j{}^\beta\,\bar{\Omega}_{\beta\alpha} \not{D} A_k{}^\alpha\,\epsilon^k\,. \tag{5.2.20}$$

Subsequently one imposes the conditions $\delta\chi^i = \delta R(Q)_{ab}{}^i = 0$ and $\delta(D_a\zeta_i) = 0$. One confirms again the standard conditions on the Weyl multiplet, including the additional conditions (5.2.2) and (5.2.4). The first equation of (5.1.18) and (5.1.22) follow with $\mathcal{D}_a \ln X \to \frac{1}{2}\mathcal{D}_a \ln\chi$. In addition to these constraints, one finds

$$A_{(i}{}^\alpha \bar{\Omega}_{\alpha\beta} \mathcal{D}_a A_{j)}{}^\beta = 0\ . \tag{5.2.21}$$

For $r > 1$, one must still satisfy $\delta\zeta^\alpha = 0$. Using (5.2.21), one finds the additional condition (trivially satisfied for $r = 1$)

$$\mathcal{D}_a A_i{}^\alpha - \tfrac{1}{2}\mathcal{D}_a \ln\chi\, A_i{}^\alpha = \chi^{1/2}\mathcal{D}_a(\chi^{-1/2} A_i{}^\alpha) = 0\ . \tag{5.2.22}$$

This implies that the $w = 0$ section $\chi^{-1/2} A_i{}^\alpha$ is covariantly constant.

We should draw attention to the fact that the hypermultiplet sector is on-shell and so is associated with a specific Lagrangian. The hyperkähler potential, for instance, captures all the details of a locally supersymmetric two-derivative Lagrangian of hypermultiplets. In closing this section we should also mention that many of the equations obtained here can also be found in [11] where the results were derived in a slightly different context. In the next section we will be discussing a supermultiplet that has never been subjected to this analysis.

5.3 The chiral $\mathbb{T}(\ln\bar{\Phi}_w)$ multiplet

From supersymmetry transformation rules for a chiral multiplet in a superconformal background given in (3.3.7) ([65, 67]) it is easy to see that if a chiral multiplet has weight $w = 0$, then requiring $\delta\Psi_i = 0$ amounts to choosing A to be constant and $B_{ij} = F_{ab}^- = \Lambda_i = C = 0$, as was argued in [65]. For chiral multiplets of non-zero weight, the situation is more subtle. To give an explicit example, let us consider the kinetic multiplet $\mathbb{T}(\ln\bar{\Phi}_w)$ constructed in the previous chapter [70] and analyze the conditions for a supersymmetric configuration. Here we concern ourselves only with the bosonic components and their bosonic constituents. These are given by

$$
\begin{aligned}
A|_{\mathbb{T}(\ln\bar\Phi)} &= \hat{\hat{C}}\ ,\\
B_{ij}|_{\mathbb{T}(\ln\bar\Phi)} &= -2\,\varepsilon_{ik}\varepsilon_{jl}\big(\Box_{\rm c}+3\,D\big)\hat B^{kl} - 2\,\hat F^+_{ab}\,R(\mathcal{V})^{ab\,k}{}_i\,\varepsilon_{jk}\ ,\\
F^-_{ab}|_{\mathbb{T}(\ln\bar\Phi)} &= -\big(\delta_a{}^{[c}\delta_b{}^{d]}-\tfrac12\varepsilon_{ab}{}^{cd}\big)\\
&\qquad\times\big[4\,D_cD^e\hat F^+_{ed} + (D^e\hat{\hat A}\,D_cT_{de}{}^{ij} + D_c\hat{\hat A}\,D^eT_{ed}{}^{ij})\varepsilon_{ij} - wD_cD^eT_{ed}{}^{ij}\varepsilon_{ij}\big]\\
&\quad+\Box_{\rm c}\hat{\hat A}\,T_{ab}{}^{ij}\varepsilon_{ij} - R(\mathcal{V})^-{}_{ab}{}^i{}_k\,\hat B^{jk}\,\varepsilon_{ij} + \tfrac18 T_{ab}{}^{ij}\,T_{cdij}\hat F^{+cd}\ ,\\
C|_{\mathbb{T}(\ln\bar\Phi)} &= 4(\Box_{\rm c}+3D)\Box_{\rm c}\hat{\hat A} + 6(D_aD)\,D^a\hat{\hat A} - 16\,D^a\big(R(D)^+_{ab}D^b\hat{\hat A}\big)\\
&\quad- D^a(T_{abij}T^{cbij}D_c\hat{\hat A}) - \tfrac12 D^a(T_{abij}T^{cbij})D_c\hat{\hat A} + \tfrac1{16}(T_{abij}\varepsilon^{ij})^2\hat{\hat C}\\
&\quad+\tfrac12 D_aD^a(T_{bcij}\hat F^{bc+})\varepsilon^{ij} + 4\,D_a\big(D^bT_{bcij}\hat F^{ac+} + D^b\hat F^+_{bc}T^{ac}{}_{ij}\big)\varepsilon^{ij}\\
&\quad- w\big[R(\mathcal{V})^+_{ab}{}^i{}_jR(\mathcal{V})^{ab+j}{}_i + 8R(D)^+_{ab}R(D)^{ab+}\big]\\
&\quad- w\big[D^aT_{abij}D_cT^{cbij} + D^a(T_{abij}D_cT^{cbij})\big]\ .
\end{aligned}
\tag{5.3.1}
$$

Following the same strategy as before, let us analyze the conditions for a supersymmetric configuration. Requiring $\delta\hat\Psi_i=0$ leads to

$$
\hat\eta_i = -\frac1w\Big[\not{D}\hat A\epsilon_i + \tfrac12\hat B_{ij}\epsilon^j + \tfrac14\gamma^{ab}\hat F^-_{ab}\varepsilon_{ij}\epsilon^j\Big]\ . \tag{5.3.2}
$$

Next we sequentially impose $\delta\hat\Lambda_i=0$, $\delta\chi^i=\delta R(Q)_{ab}{}^i=0$ and finally $\delta(D_a\hat\Psi_i)=0$ using this choice for $\hat\eta_i$. We find several algebraic conditions,

$$
\begin{aligned}
&\hat B_{ij}\hat F^-_{ab} = \hat B_{ij}T_{ab}{}^{kl} = 0\ , \qquad && \hat C = -\tfrac1{2w}\hat F^-_{ab}\,\hat F^{ab-} - \tfrac1{4w}\hat B_{kl}\hat B_{mn}\varepsilon^{kn}\varepsilon^{lm}\ ,\\
&\hat F^-_{a[b}T_{c]}{}^{a\,ij} = 0\ , && D = \tfrac1{24w}\hat F^{ab-}T_{ab}{}^{ij}\varepsilon_{ij}\ ,
\end{aligned}
\tag{5.3.3}
$$

in addition to the first-order differential equations

$$
\begin{aligned}
\mathcal{D}_\mu\hat B_{ij} - \tfrac1w\mathcal{D}_\mu\hat A\,\hat B_{ij} &= 0\ ,\\
\mathcal{D}_aT^{bcij} - \tfrac1w\mathcal{D}_a\hat A\,T^{bcij} + \tfrac2w\mathcal{D}^{[b}\hat A\,T^{c]}{}_a{}^{ij} - \tfrac2w\mathcal{D}_d\hat A\,T^{d[bij}\delta^{c]}{}_a &= 0\ ,\\
\mathcal{D}_a\hat F^{bc-} - \tfrac1w\mathcal{D}_a\hat{\hat A}\,\hat F^{bc-} + \tfrac2w\mathcal{D}^{[b}\hat A\,\hat F^{c]}{}_a{}^- - \tfrac2w\mathcal{D}_d\hat A\,\hat F^{-d[b}\delta^{c]}{}_a &= 0\ ,
\end{aligned}
\tag{5.3.4}
$$

and the second-order differential equation

$$
\mathcal{D}_a\mathcal{D}_b\hat A + w\,e_a{}^\mu f_{\mu b} - \tfrac1w\mathcal{D}_a\hat A\mathcal{D}_b\hat A + \tfrac1{2w}\mathcal{D}_c\hat A\mathcal{D}^c\hat A\,\eta_{ab} + \tfrac34\,w\,D\,\eta_{ab} - \tfrac1{2w}\hat F^-_{ac}\,\hat F^{+c}{}_b = 0\ . \tag{5.3.5}
$$

One additional condition is also found:

$$\mathcal{D}^c(\hat{A} - \hat{\bar{A}})\, \hat{F}^-_{cb} = -\tfrac{1}{4}\, w\, \mathcal{D}^c(\hat{A} - \hat{\bar{A}})\, T_{cb\, ij}\, \varepsilon^{ij}\ . \tag{5.3.6}$$

From (5.3.3), we deduce that

$$\hat{B}_{ij} = 0\ , \qquad \hat{F}^-_{ab} = \frac{24\, w\, D\, T_{ab}{}^{ij}\varepsilon_{ij}}{(T_{cd}{}^{kl}\varepsilon_{kl})^2}\ , \qquad \hat{C} = -\frac{288\, w\, D^2}{(T_{ab}{}^{ij}\varepsilon_{ij})^2}\ . \tag{5.3.7}$$

Multiplying the second equation of (5.3.4) by $T_{bc}{}^{kl}$ leads to $\mathcal{D}_a\big[\hat{A} - \frac{1}{2} w \ln(T^{bcij}\varepsilon_{ij})^2\big] = 0$. Because $\hat{A} - \frac{1}{2} w \ln(T^{bcij}\varepsilon_{ij})^2$ is inert under dilatations and U(1) rotations, one recovers

$$\mathcal{D}_a\big[\hat{A} - \tfrac{1}{2} w \ln(T^{bcij}\varepsilon_{ij})^2\big] = 0 \qquad \Longrightarrow \qquad \hat{A} = \tfrac{1}{2} w \ln(T_{ab}{}^{ij}\varepsilon_{ij})^2 + \text{const}\ . \tag{5.3.8}$$

With these choices, the equations (5.3.3)–(5.3.6) are identically satisfied, once we use the conditions established for the Weyl multiplet in section 5.1. At this point we should remark that we could have immediately derived these results by noting that

$$\hat{A} - \tfrac{1}{2} w \ln(T_{ab}{}^{ij}\varepsilon_{ij})^2 = \ln\left(\frac{A}{((T_{ab}{}^{ij}\varepsilon_{ij})^2)^{w/2}}\right) \tag{5.3.9}$$

is the lowest component of a $w = 0$ chiral multiplet and therefore must be a constant. The higher components of this new $w = 0$ multiplet must vanish, which leads after some algebra to the relations (5.3.7).

Now we are in a position to evaluate the supersymmetric configuration of $\mathbb{T}(\ln \bar{\Phi}_w)$. From (5.3.7) one finds that the lowest component of the kinetic multiplet is completely determined to be

$$A|_{\mathbb{T}(\ln \bar{\Phi}_w)} = -\frac{288\, w\, D^2}{(T_{abij}\varepsilon^{ij})^2}\ . \tag{5.3.10}$$

The remainder of the components of $\mathbb{T}(\ln \bar{\Phi}_w)$ can be found by explicit use of the formulae (5.3.1), but it is much simpler to note that since $\mathbb{T}(\ln \bar{\Phi}_w)$ is a $w = 2$ chiral multiplet, it must be proportional to the square of the Weyl multiplet, schematically denoted W^2, whose lowest component is $(T_{ab}{}^{ij}\varepsilon_{ij})^2$. For example, we can relate the component B_{ij} of $\mathbb{T}(\ln \bar{\Phi}_w)$ to the same component of W^2,

$$B_{ij}|_{\mathbb{T}(\ln \bar{\Phi}_w)} = B_{ij}|_{W^2} \times \frac{A|_{\mathbb{T}(\ln \bar{\Phi}_w)}}{(T_{cd}{}^{kl}\varepsilon_{kl})^2} = 0\ . \tag{5.3.11}$$

In the last equality we have used the fact that in the supersymmetric configuration $B_{ij}|_{W^2}$ is proportional to $\varepsilon_{ik}R(\mathcal{V})_{ab}{}^k{}_j$, which vanishes. In a similar way, one finds

$$F^-_{ab}|_{\mathbb{T}(\ln\bar{\Phi}_w)} = 48D\, T_{ab}{}^{ij}\varepsilon_{ij}\,\frac{A|_{\mathbb{T}(\ln\bar{\Phi}_w)}}{(T_{cd}{}^{kl}\varepsilon_{kl})^2}\ , \quad C|_{\mathbb{T}(\ln\bar{\Phi}_w)} = 576D^2\,\frac{A|_{\mathbb{T}(\ln\bar{\Phi}_w)}}{(T_{cd}{}^{kl}\varepsilon_{kl})^2}\ . \quad (5.3.12)$$

Note that these higher components are completely determined by the lowest component $A|_{\mathbb{T}(\ln\bar{\Phi}_w)}$, given in (5.3.10). Two special cases are worthy of note. If Φ_w is actually a weight $w = 0$ multiplet, then $\mathbb{T}(\ln\bar{\Phi}_w)$ vanishes completely, as was noted in [65]. Similarly, if we apply the conditions of section 5.2 (equivalently, the conditions of [11]), then $D = 0$ causes the entire kinetic multiplet to vanish for any value of the Weyl weight. This will be a crucial point for the non-renormalization theorem presented in the next section.

5.4 A new non-renormalization theorem

The preceding sections have mainly been concerned with deriving the conditions of off-shell $\mathcal{N} = 2$ supersymmetry for various multiplets independently of any action. We devoted particular attention to the chiral multiplet $\mathbb{T}(\ln\bar{\Phi}_w)$, which has been constructed only recently. This multiplet leads to a new class of 4D higher-derivative invariants. Our goal in this section is to establish a non-renormalization theorem: in a fully supersymmetric configuration, these higher-derivative invariants always vanish, as do their first derivative with respect to any field or coupling constant. To accomplish this, we will make one assumption. In addition to the apparent field content – a non-vanishing chiral multiplet Φ_w coupled to conformal supergravity – we require at least one multiplet of the set discussed in section 5.2. The motivation for this last requirement is physical. As already explained in section 3.4.2, a Poincaré supergravity action requires both a vector multiplet and at least one other short multiplet. So even if such a multiplet is not present in the specific higher-derivative terms under discussion, it must be present in the sector of the action responsible for generating Poincaré supergravity. This means that it too must take its supersymmetric value. Making this assumption means that the restrictive conditions discussed in section 5.2 apply. In particular, we will require that $D = 0$.

It will be convenient to exploit superfield and superspace terminology as in the previous chapters. In section 3.4.2, we already pointed out that, generally, any full superspace integral can be rewritten as a chiral superspace integral, so the distinction between these two types of superspace invariants is not a sharp one.

Therefore, when we discuss chiral superspace invariants, we usually mean ones which *cannot* be converted back into full superspace invariants by removing a kinetic operator. It will be convenient to call such chiral multiplets *intrinsically chiral*.

A common example of intrinsically chiral integrands are of the form $F(X, A|_{W^2})$ where X^I are vector multiplets scalars and $A|_{W^2} = (T_{ab}{}^{ij}\varepsilon_{ij})^2$ is the lowest component of the square of the Weyl multiplet. This class $F(X, A|_{W^2})$ is actually quite important: it was shown in [9, 100] to accurately describe the subleading corrections to the Wald entropy in the limit of large charges required for matching the degeneracy of the microscopic string and brane states. This precise matching was in retrospect quite surprising since there are in principle a number of higher-derivative actions that do not fall into this class. In fact, this was the motivation in [65] where a non-renormalization theorem established that a large class of full superspace integrals (3.4.6) do not contribute to the Wald entropy.

It is now important to address what other intrinsically chiral invariants might exist and whether they might possess non-renormalization theorems as well. As discussed in section 4.1.2 [70], the kinetic multiplet $\mathbb{T}(\ln \bar{\Phi}_w)$ is actually a new contribution to intrinsically chiral functions F. In fact, the naive equality (4.1.18)

$$-\frac{1}{2}\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,\Phi'\,\mathbb{T}(\ln \bar{\Phi}_w) \stackrel{?}{=} \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}\,E\,\Phi'\,\ln \bar{\Phi}_w \tag{5.4.1}$$

(where Φ' is some $w = 0$ chiral multiplet) does not hold since the integrand on the right-hand side is not actually weight zero due to the inhomogeneous dilatation transformation of $\ln \bar{\Phi}_w$. This means that the left-hand side is actually an intrinsically chiral quantity.

It would seem that this observation might open the door for many new intrinsically chiral contributions, but it turns out this is not the case. The reason is that any two such multiplets are actually related to each other by the kinetic operator of a weight-zero multiplet. Taking Φ'_w and Φ_w to be chiral multiplets of the same nonzero weight (for simplicity), the difference

$$\mathbb{T}(\ln \bar{\Phi}'_w) - \mathbb{T}(\ln \bar{\Phi}_w) = \mathbb{T}(\ln(\bar{\Phi}'_w/\bar{\Phi}_w)) \tag{5.4.2}$$

is actually the kinetic multiplet of a weight-zero multiplet. This permits, for example, manipulations like

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\;\Phi'\;\mathbb{T}(\ln \bar{\Phi}'_w) = \int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\;\Phi'\;\mathbb{T}(\ln \bar{\Phi}_w) - 2\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{d}^4\bar{\theta}\,E\;\Phi'\;\ln(\bar{\Phi}'_w/\bar{\Phi}_w)\;, \tag{5.4.3}$$

where Φ' is a $w = 0$ chiral multiplet. This allows any operators $\mathbb{T}(\ln \bar{\Phi}'_w)$ to be traded for one universal choice $\mathbb{T}(\ln \bar{\Phi}_w)$ and the rest lifted to full superspace integrals, where the non-renormalization theorem of [65] applies.

We will now establish a new non-renormalization theorem: the contribution of $\mathbb{T}(\ln \bar{\Phi}_w)$ to any chiral integral (3.4.7) always vanishes as does the first derivative with respect to any field or coupling constant. Using the condition $D = 0$ found in section 5.2, we find that *the entire kinetic multiplet* $\mathbb{T}(\ln \bar{\Phi}_w)$ *vanishes in a supersymmetric vacuum.* In other words, in a supersymmetric vacuum, we can replace

$$F(\Phi, \mathbb{T}(\ln \bar{\Phi}_w)) \quad \longrightarrow \quad F(\Phi, 0) \tag{5.4.4}$$

in any chiral superspace integral (3.4.7). We still must be careful to analyze what happens under *variations* of the fields in a supersymmetric configuration. For simplicity, we consider first the case

$$-2 \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\; \Phi'\; \mathbb{T}(\ln \bar{\Phi}_w) \tag{5.4.5}$$

with a weight-zero chiral multiplet Φ' whose component action was constructed in the chapter 4 (see also [70]). In principle, there are three ways in which this quantity could be varied: we may vary either of the two multiplets Φ' and $\bar{\Phi}_w$ explicit in the expression, or we may vary the supergravity fields which are implicit. Variations of Φ' clearly give zero since $\mathbb{T}(\ln \bar{\Phi}_w)$ vanishes in the supersymmetric background. Variations of $\bar{\Phi}_w$ within the kinetic multiplet also give zero. This can be seen by parametrizing the variation as $\delta\bar{\Phi}_w = \bar{\Phi}_w \bar{\Lambda}$ where $\bar{\Lambda}$ is a $w = 0$ anti-chiral multiplet. This leads to $\mathbb{T}(\delta \ln \bar{\Phi}_w) = \mathbb{T}(\bar{\Lambda})$ and so we can write

$$\delta_{\Phi_w} \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\; \Phi'\; \mathbb{T}(\ln \bar{\Phi}_w) = \int \mathrm{d}^4x\, \mathrm{d}^4\theta\, \mathcal{E}\; \Phi'\; \mathbb{T}(\bar{\Lambda}) = \int \mathrm{d}^4x\, \mathrm{d}^4\bar{\theta}\, \bar{\mathcal{E}}\; \bar{\mathbb{T}}(\Phi')\; \bar{\Lambda} \tag{5.4.6}$$

where we “integrate by parts” the kinetic operator as in [65]. Since Φ' has zero Weyl weight, its supersymmetric value is a constant and so $\bar{\mathbb{T}}(\Phi') = 0$. The last possibility is to vary the components of the Weyl multiplet itself, with Φ' fixed at its supersymmetric value. Taking the result for the component action of (5.4.5) given in [70] and imposing the supersymmetry conditions on the components of

Φ', one finds

$$e^{-1}\mathcal{L} = w\,A'\Big(\tfrac{2}{3}\mathcal{R}^2 - 2\,\mathcal{R}^{ba}\mathcal{R}_{ab} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+\,j}_{ab\;\;i} + \tfrac{1}{128}T^{abij}T_{ab}{}^{kl}T^{cd}{}_{ij}T_{cdkl} + T^{acij}\mathcal{D}_a\mathcal{D}^bT_{bcij} - T^{acij}f_a{}^bT_{bcij}\Big)\ , \tag{5.4.7}$$

where A' must be a constant. Note already that the terms D^2, $(R(A)_{ab})^2$ and $(R(\mathcal{V})^{+\,i}_{ab\;\;j})^2$ are quadratic in quantities which vanish in the supersymmetric background, and so any variation of these quantities must vanish. It turns out that the same holds for the remaining terms. The Lagrangian (5.4.7) can be written as

$$e^{-1}\mathcal{L} = w\,A'\Big(2(Z_{ab}\eta^{ab})^2 - 2Z^{ba}Z_{ab} - \tfrac{1}{2}Z^1_aZ^{2a} - 6\,D^2 + 2\,R(A)^{ab}R(A)_{ab} - R(\mathcal{V})^{+abi}{}_j\,R(\mathcal{V})^{+\,j}_{ab\;\;i} + \mathcal{D}^a\mathcal{O}_a\Big) \tag{5.4.8}$$

where the three complex quantities

$$\begin{aligned}
Z_{ab} &= \mathcal{R}_{ab} - \tfrac{1}{6}\eta_{ab}\mathcal{R} + \tfrac{1}{8}T_{acij}T_b{}^{cij} + 2w^{-1}\mathcal{D}_a\mathcal{D}_b\hat{\bar{A}} - 2w^{-2}\mathcal{D}_a\hat{\bar{A}}\mathcal{D}_b\hat{\bar{A}} + w^{-2}\eta_{ab}(\mathcal{D}_c\hat{\bar{A}})^2\ ,\\
Z^1_a &= \mathcal{D}^bT_{ba\,ij}\,\varepsilon^{ij} + w^{-1}\mathcal{D}^b\hat{\bar{A}}\,T_{ba\,ij}\,\varepsilon^{ij}\ ,\\
Z^2_a &= \mathcal{D}^bT_{ba}{}^{ij}\,\varepsilon_{ij} + w^{-1}\mathcal{D}^b\hat{\bar{A}}\,T_{ba}{}^{ij}\,\varepsilon_{ij}\ ,
\end{aligned} \tag{5.4.9}$$

vanish in a supersymmetric configuration, using the supersymmetry conditions (5.3.3) – (5.3.6), along with the additional condition $D = 0$ (which implies $\mathcal{D}_a\hat{A} = \mathcal{D}_a\hat{\bar{A}}$). The last term of (5.4.8), which involves $\mathcal{D}_a\mathcal{O}^a$ for

$$\begin{aligned}
\mathcal{O}_a = T_{ac}{}^{ij}\mathcal{D}_bT^{bc}{}_{ij} + w^{-1}T_{ac\,ij}T^{bc\,ij}\,\mathcal{D}_b\hat{\bar{A}} - 4w^{-1}\mathcal{R}\,\mathcal{D}_a\hat{\bar{A}} + 8w^{-1}\mathcal{R}_{ba}\mathcal{D}^b\hat{\bar{A}}\\
- 8w^{-2}\mathcal{D}_a\hat{\bar{A}}\mathcal{D}^2\hat{\bar{A}} + 8w^{-2}\mathcal{D}^b\hat{\bar{A}}\,\mathcal{D}_b\mathcal{D}_a\hat{\bar{A}} - 8w^{-3}\mathcal{D}_a\hat{\bar{A}}\,(\mathcal{D}_c\hat{\bar{A}})^2\ ,
\end{aligned} \tag{5.4.10}$$

gives a total derivative because A' is constant. The remaining pieces are each quadratic in terms that vanish in the supersymmetric vacuum, so their variation with respect to any of the supergravity fields must vanish.

We have now established a non-renormalization theorem for the expression (5.4.5). This is straightforwardly extended to the more general class of functions

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\,F(\Phi^I, \mathbb{T}(\ln\bar{\Phi}_w))\ . \tag{5.4.11}$$

Here the superfields Φ^I are a set of chiral superfields which may possess any weight. For instance, they may consist of vector multiplets X^I and the chiral supergravity invariant $W^{\alpha\beta}W_{\alpha\beta}$. We have already observed that in a supersymmetric vacuum $\mathbb{T}(\ln\bar{\Phi}_w)$ vanishes. In this context, the functions F should be analytic at $\mathbb{T}(\ln\bar{\Phi}_w) = 0$. Therefore, we may construct a series expansion, a characteristic term of which would be

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathcal{E}\ \Phi_{2-2n}\ \left[\mathbb{T}(\ln\bar{\Phi}_w)\right]^n\ . \tag{5.4.12}$$

But any such term can always be written as (5.4.5) for the choice $\Phi' \propto \Phi_{2-2n}\left[\mathbb{T}(\ln\bar{\Phi}_w)\right]^{n-1}$. Since our treatment of (5.4.5) holds for arbitrary Φ', the non-renormalization theorem applies to this term and therefore to the broad class (5.4.11).

Chapter 6

Flat directions and supersymmetry

Up to now, we have studied very general aspects of supersymmetric higher derivative invariants in four dimensions. We have explicitly constructed a new class of such invariants, analyzed the allowed fully BPS backgrounds and proven of a non-renormalization theorem.
In this chapter, our analysis will become much more specific: we will consider explicit (BPS) black hole solutions of five- and ten-dimensional supergravity, and the behavior of scalar fields in these backgrounds. In particular we want to check whether these scalar fields are influenced by the presence of higher derivative terms. At the same time we will present explicit results for the higher derivative corrections to the entropy of the black hole configurations considered. But before doing so, it is necessary to explain in more detail some of the features of the attractor mechanism, already introduced in section 1.1.1, and what they entail.

According to the attractor mechanism, the near-horizon field configuration of an extremal black hole is insensitive to the asymptotic data on scalar fields of the theory. Also, many moduli fields of the theory are fixed at the horizon, while others remain unfixed, meaning that the black hole entropy does not depend on them. The attractor mechanism has been observed for asymptotically flat as well as asymptotically AdS black holes and in theories with higher derivative interactions. There is a long list of papers where this subject has been studied widely also in connection to black hole entropy calculations both from the macroscopic and microscopic side [7, 9, 11, 100–117]. Here, we summarize the main points, that will play a role in our analysis:

1. Let us consider a general theory of gravity, coupled to abelian gauge fields, neutral scalar fields and p-form gauge fields, with a local Lagrangian density, which is invariant under gauge and general coordinate transformation. Suppose the theory admits a rotating or spherically symmetric extremal black hole solution. It has been proved [93, 118] that the entropy of this black hole remains invariant under continuous deformation of the asymptotic data for the moduli fields[1].

   It is important to stress that this result does not depend on the supersymmetries the solution preserves but relies only on the existence of an AdS_2 component in the near-horizon geometry. This allows us to define the "entropy function" [93] of the theory, and by extremizing it, find the explicit values of the near-horizon parameters.

   In presence of higher derivative terms, even finding a generic solution of the full theory is a non-trivial task. However we can restrict our attention on the subclass of extremal black hole solutions that are not destabilized by higher derivative terms, i.e. they still admit an AdS_2 component in the near-horizon geometry (this was shown in section 1.1.1 for the Reissner-Nordström solution). If that is the case, then we can expect the results for the attractors at the two derivative level to hold even for a covariant theory of higher derivative gravity.

2. *Flat direction*: As we already pointed out, the attractor equations fix some of the moduli fields at the horizon in terms of the black hole charges. On the other hand, it is possible that certain moduli fields cannot be fixed by extremization, meaning the entropy function has a series of degenerate stationary points. In that case, the entropy will be independent of the near-horizon values of these moduli, that we will refer to as flat directions.
   The existence of flat directions is strictly related to the (super)symmetries preserved by the solution, and it is likely that the same symmetries will completely constrain the behavior of flat directions even when higher derivative terms are considered. It is however possible that the specific form of the higher derivative interactions, and, as a consequence, its symmetries, might influence the fate of flat directions in higher derivative gravity.

Generically, we expect that if the two derivative theory has a BPS black hole solution with a flat direction, then supersymmetry will protect the structure of the near horizon geometry and the flat direction will not be lifted, when supersymmetric higher derivative interactions are considered. On the other hand, nothing can

[1] It has been observed that there can be discrete jumps because of multi-centered black holes.

be said a priori for non-BPS solutions. To confirm our expectations for the BPS case and obtain some knowledge about the non-BPS case, we study two concrete examples:

1. Five-dimensional minimal gauged supergravity in presence of higher derivative corrections, which is a subclass of the theories described by the Lagrangian (4.2.12). There exists an asymptotically AdS, supersymmetric solution of two derivative gravity [119] that preserves 1/2 of the supersymmetries and the near-horizon geometry of this solution is specified in terms of a single parameter Δ. The five-dimensional field content is given by the metric and one gauge field. We perform a dimensional reduction on this five-dimensional geometry over a circle (ψ direction), obtaining a larger field content including the metric, two gauge fields and two scalars. We write down the entropy function and the attractor equations in four dimensions and find that the equations are satisfied without any knowledge of one scalar field and, moreover, the entropy is independent of its near-horizon value. This mean that this scalar field is a flat direction of our theory. We explicitly check that, even in presence of higher derivative terms in the Lagrangian, the flat direction remains flat. The details are studied in section 6.1.

2. Rotating D3 brane solution in type IIB string theory, which does not preserve any supersymmetry. The solution admits an extremal limit and the near-horizon geometry has an AdS_2 part. The near-horizon value of dilaton does not appear in the entropy function, or the entropy, therefore it is a flat direction of the theory. When higher derivative terms are considered in the action, the dilaton is lifted, i.e. its near-horizon value gets fixed, but remains independent of physical charges. The details are studied in section 6.2.

Since we are interested in higher derivative solutions for the metric and the scalar sector, we will consider, without loss of generality, only bosonic actions and neglect all the fermions. Consequently, we will use the Christoffel connections (2.5.3) to covariantize the derivative operators in curved space, instead of the spin connections ${\omega_\mu}^{ab}$. We will specify the conventions used when needed, referring to [120] for further details.

6.1 Flat direction in five-dimensional supergravity

Not long ago all possible purely bosonic supersymmetric solutions of minimal gauged supergravity in 5 dimensions were classified [121], using the properties of the Killing spinors. These solutions are known to be 1/2 BPS [2][122], i.e. they preserve 4 supercharges in the minimal theory and of course, they solve the equations of motion arising from the minimal gauged supergravity action [121, 123]. Analyzing all possible near-horizon geometries of these supersymmetric solutions, Gutowski and Reall [119] were able to find an one-parameter family of black hole solutions, which has a spatially compact horizon (squashed S^3) and is (globally) asymptotically AdS_5, in contrast with the ungauged case where the near-horizon geometry of a BPS solution is always maximally symmetric. We work in a suitable coordinate system, where the AdS_2 part of the near-horizon geometry is manifest [124]. The metric, the U(1) gauge field and its field strength have the following form,

$$
\begin{aligned}
\mathrm{d}s^2 &= v_3\left(B\cos\theta\,\mathrm{d}\chi + \frac{e_0}{r}\mathrm{d}r + e_0 r\,\mathrm{d}t + \mathrm{d}\psi\right)^2 + v_2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\chi^2\right) + v_1\left(\frac{\mathrm{d}r^2}{r^2} - r^2\mathrm{d}t^2\right), \\
A &= (e_0\varphi + e_1)\,r\mathrm{d}t + (B\varphi + P)\cos\theta\mathrm{d}\chi + \frac{e_0}{r}\mathrm{d}r + \varphi\mathrm{d}\psi, \qquad (6.1.1) \\
F &= (e_1 + e_0\varphi)\mathrm{d}r\wedge\mathrm{d}t - (P + B\varphi)\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\chi\,.
\end{aligned}
$$

We consider, in the following, minimal gauged supergravity theory coupled to a single U(1) gauge field, including some supersymmetric higher derivative terms. These higher derivative terms are all of the fourth order and they are related to the mixed gauge gravitational Chern-Simons term by supersymmetry. The supersymmetric completion of this term was first found in [92], using the superconformal formalism, which gives a complete off-shell result. An on-shell version of these higher derivative supersymmetric invariants was derived later in [125], by integrating out all the auxiliary fields. The action obtained with this method, however, can be reduced further, by means of partial integrations, field redefinitions and Bianchi identities [126]. Finally one can show that the Lagrangian density

[2] AdS_5 is the only maximally supersymmetric solution of the gauged theory with a negative cosmological constant.

includes only five bosonic higher derivative terms ($\mu, \nu, \dots = 1, \dots, 5$)

$$\sqrt{-g}\,\mathcal{L}_5 = \sqrt{-g}\left[\mathcal{R} + \frac{12}{L^2} - \frac{F^2}{4} + \frac{\kappa}{3}\,\epsilon^{\mu\nu\rho\sigma\delta} A_\mu F_{\nu\rho} F_{\sigma\delta} + L^2 \Big(c_1 \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + c_2 (F^2)^2 \right.$$
$$\left. + c_3\, F_\mu{}^\nu F_\sigma{}^\mu F_\nu{}^\rho F_\rho{}^\sigma + c_4\, \mathcal{R}_{\mu\nu\rho\sigma} F^{\mu\nu} F^{\rho\sigma} + c_5 \epsilon^{\mu\nu\rho\sigma\delta} A_\mu \mathcal{R}_{\nu\rho\gamma\eta} \mathcal{R}_{\sigma\delta}{}^{\gamma\eta} \Big) \right], \tag{6.1.2}$$

where the supersymmetric values of the coefficients c_2, c_3, c_4, c_5 and κ are given, in terms of c_1, by:

$$\kappa = \frac{1}{4\sqrt{3}}(1 - 288 c_1), \quad c_2 = \frac{c_1}{24}, \quad c_3 = -\frac{5c_1}{24}, \quad c_4 = -\frac{c_1}{2}, \quad c_5 = \frac{c_1}{2\sqrt{3}}. \tag{6.1.3}$$

The term inversely proportional to L^2 is the cosmological term, where the constant length L is the radius of the AdS space. Notice that the action (6.1.2) is not strictly gauge invariant, i.e. $N^\mu \neq 0$, due to the presence of Chern-Simons terms (see Appendix C for further details). In the following sub-section we deal with this issue, obtaining a generalization of a result known at the two derivative level for asymptotically AdS black holes: a relation between the five-dimensional and the reduced four-dimensional black hole charges.

6.1.1 Black hole charges in 5D and 4D: Noether potential and entropy function

In order to obtain any knowledge on the behavior of the moduli fields of the theory under consideration, we want to make use of Sen's entropy function formalism [106], which, however, is applicable only to strictly gauge invariant Lagrangians, unlike (6.1.2). To circumvent this problem, we dimensionally reduce our five-dimensional theory over a circle, obtaining a four-dimensional gauge invariant action (details about the reduction are presented later on in this section and appendix B of [120]). Even so, the entropy of the reduced 4D black hole solution will depend on the four-dimensional charges, thus we must first find a relation linking the lower- and higher-dimensional charges. In this way we can still get the entropy of the "original" black hole solution as a function of the five-dimensional charges. Now, at the two derivative level it was proven that the 5D and 4D black hole charges are exactly equivalent (for five-dimensional AdS solutions this result was first found in [127]). We would like to find a relation between the black hole charges when higher derivative interactions are considered. While the answer is known for asymptotically flat BPS black hole [94, 128, 129], where a mismatch, due to the gravitational Chern-Simons term, was found among the charges, in the asymptotically AdS case, no one (to the best of our knowledge) has studied this relation. In the following we intend to fill this gap. The construction is generic and in particular does not assume the supersymmetric values of the various coefficients presented in (6.1.3).

Let us start from the calculation of the 5D charges. We will use the covariant approach explained in Chapter 2, which is suited to analyze theories containing higher derivative couplings and their conserved quantities.
From (2.0.3) it is straightforward to derive the dynamical equation for the metric $g_{\alpha\beta}$ and the gauge field A_μ obtained from the Lagrangian (6.1.2). They read as follows

$$
\begin{aligned}
E_A^\mu =& \nabla_\nu\Big(- F^{\mu\nu} + 8\, c_2 (F^2) F^{\mu\nu} - 8\, c_3\, F^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} + 4\, c_4\, \mathcal{R}^{\mu\nu\rho\sigma} F_{\rho\sigma}\Big) \\
&+ \epsilon^{\mu\nu\rho\sigma\tau}\Big(\kappa\, F_{\nu\rho} F_{\sigma\tau} + c_5\, \mathcal{R}_{\nu\rho\alpha\beta} \mathcal{R}_{\sigma\tau}{}^{\alpha\beta}\Big)\,, \\
E_g^{\alpha\beta} =& \tfrac{1}{2}(\mathcal{L} - \tfrac{\kappa}{3}\, \epsilon^{\mu\nu\rho\sigma\tau} A_\mu F_{\nu\rho} F_{\sigma\tau} - c_5\, \epsilon^{\mu\nu\rho\sigma\tau} A_\mu \mathcal{R}_{\nu\rho}{}^{\gamma\delta} \mathcal{R}_{\sigma\tau\gamma\delta})\, g^{\alpha\beta} \\
&- \mathcal{R}^{\alpha\beta} + \tfrac{1}{2} F^{\alpha\gamma} F^{\beta}{}_{\gamma} + c_1\big(- 2\, \mathcal{R}^{(\alpha|\nu\rho\sigma} \mathcal{R}^{\beta)}{}_{\nu\rho\sigma} + 4 \nabla_\rho \nabla_\sigma \mathcal{R}^{\rho(\alpha\beta)\sigma}\big) \\
&- 4\, c_2\, F^2 F^{(\alpha|\gamma} F^{\beta)}{}_{\gamma} - 4\, c_3\, F^{(\alpha|\gamma} F_{\gamma\lambda} F^{\lambda\rho} F_{\rho}{}^{\beta)} \\
&+ c_4\big(3\, \mathcal{R}_{\nu\rho\sigma}{}^{(\alpha} F^{\beta)\sigma} F^{\nu\rho} + 2 \nabla_\rho \nabla_\sigma (F^{\rho(\alpha} F^{\beta)\sigma})\big) \\
&+ 2\, c_5\, \varepsilon^{\mu\nu\rho\sigma(\alpha}(\nabla_\lambda F_{\mu\nu} \mathcal{R}_{\rho\sigma}{}^{\lambda|\beta)} + 2 F_{\mu\nu} \nabla_\rho \mathcal{R}_{\sigma}{}^{\beta)})\,.
\end{aligned}
$$

Now the Noether current conserved under abelian gauge transformations $\delta_\xi A_\mu = \partial_\mu \xi$ can be derived from (2.0.3),(2.2.1), (2.2.4), and reads

$$
\begin{aligned}
J_{\mathrm{Q}}^\mu =& \sqrt{-g}\Big(- F^{\mu\nu} + \tfrac{4}{3}\, \kappa \epsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau} + 8\, c_2 (F^2) F^{\mu\nu} + 8\, c_3\, F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}\Big) \partial_\nu\, \xi + 4\, c_4 \mathcal{R}^{\mu\nu\rho\sigma} F_{\rho\sigma} \\
&- \xi\, \epsilon^{\mu\nu\rho\sigma\tau}\Big(\frac{\kappa}{3} F_{\nu\rho} F_{\sigma\tau} + c_5\, \mathcal{R}_{\nu\rho\alpha\beta} \mathcal{R}_{\sigma\tau}{}^{\alpha\beta}\Big). \qquad (6.1.4)
\end{aligned}
$$

The crucial observation is that whenever the dynamical equations (6.1.4) are satisfied the current can be written as the total derivative of the Noether potential $Q_{\mu\nu}$ (2.2.10). The conserved electric charge is then given by

$$
Q_{5\mathrm{D}} = \int \mathrm{d}\Sigma_{\mu\nu} Q^{\mu\nu}\,, \qquad (6.1.5)
$$

where $\mathrm{d}\Sigma_{\mu\nu} = \mathrm{d}S\, \sqrt{h} \epsilon_{\mu\nu}$, $\sqrt{h}$ is the determinant of the induced metric on the null surface S of the horizon, and the tensor $\epsilon_{\mu\nu}$[3] is the binormal on that surface, satisfying the normalization condition $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$, with only one non-zero component

[3]The notation we use for the binormal and the anti-symmetric epsilon tensor are similar, but they are tensors of different ranks.

in this background, $\epsilon_{tr} = v_1$.
The presence of the Chern-Simons terms complicates significantly the analysis of section 2.2, which can nevertheless be worked out precisely, following the procedure outlined in Appendix C. The gauge Noether potential reads,

$$\begin{aligned} Q^{\mu\nu} = & - F^{\mu\nu} + 2\,\kappa\,\epsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau} + 8\,c_2 (F^2) F^{\mu\nu} - 8\,c_3\, F^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} + 4\,c_4\,\mathcal{R}^{\mu\nu\rho\sigma} F_{\rho\sigma} \\ & - 4\,c_5\,\epsilon^{\mu\nu\rho\sigma\tau}\Big(\Gamma_{\rho\beta}{}^{\alpha}\partial_\sigma \Gamma_{\tau\alpha}{}^{\beta} + \tfrac{2}{3}\,\Gamma_{\rho\beta}{}^{\alpha}\Gamma_{\sigma\gamma}{}^{\beta}\Gamma_{\tau\alpha}{}^{\gamma}\Big). \end{aligned} \tag{6.1.6}$$

Now, plugging this expression in (6.1.5), we can compute the five-dimensional conserved electric charge for the background (6.1.1). The result is quite long, so we avoid writing it here, and refer to appendix A of [120] for the full expression.

The evaluation of the angular momentum is completely analogous, but this time no complications arise from the Chern-Simons terms (see Appendix C for details). The full action (6.1.2) is invariant under diffeomorphism and the current associated to this invariance reads:

$$\begin{aligned} J^\mu_\Theta = & \Big(- F^{\mu\nu} + \tfrac{4}{3}\,\kappa\epsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau} + 4\,c_4\,\mathcal{R}^{\mu\nu\rho\sigma} F_{\rho\sigma} \\ & + 8\,c_2\,(F^2) F^{\mu\nu} + 8\,c_3\, F^{\nu\rho} F_{\rho\sigma} F^{\sigma\mu}\Big)\big(\xi^\lambda F_{\lambda\nu} + \nabla_\nu(\xi^\lambda A_\lambda)\big) \\ & - 2\Big(g^{\left[\nu[\sigma\right.} g^{\left.\rho]\mu\right]} + 2\,c_5\,\varepsilon^{\mu\alpha\beta\sigma\tau} A_\tau \mathcal{R}_{\alpha\beta}{}^{\nu\rho} + 2\,c_1\,\mathcal{R}^{\mu\nu\rho\sigma} + c_4\, F^{\mu\nu} F^{\rho\sigma}\Big)\nabla_\rho\big(2\nabla_{(\sigma}\xi_{\nu)}\big) \\ & + 4\,\nabla_\rho\Big(2\,c_5\,\varepsilon^{\alpha\beta\rho\sigma\tau} A_\tau R_{\alpha\beta}{}^{\mu\nu} + 2\,c_1\,\mathcal{R}^{\mu\nu\rho\sigma} + c_4\, F^{\mu\nu} F^{\rho\sigma}\Big)\,\nabla_{(\sigma}\xi_{\nu)} - \xi^\mu \mathcal{L}_5 \end{aligned} \tag{6.1.7}$$

Now, to extract a total derivative from the current, we add a linear combination of the equations of motion of E^μ_A and $E^{\mu\nu}_g$, which will not alter in any way the final physical result, since they vanish on-shell. The Noether potential for diffeomorphism is calculated from

$$J^\mu + 2E^{\mu\nu}_g \xi_\nu + (\xi\cdot A) E^\mu_A = \nabla_\nu \Theta^{\mu\nu} \tag{6.1.8}$$

and reads

$$\begin{aligned} \Theta^{\mu\nu} = & \big(- F^{\mu\nu} + 4\,c_4\,\mathcal{R}^{\mu\nu\rho\sigma} F_{\rho\sigma} + 8\,c_2\,(F^2) F^{\mu\nu} - 8\,c_3\, F^{\mu\rho} F_{\rho\sigma} F^{\sigma\nu} + 4\,\frac{\kappa}{3}\,\epsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau}\big)(\xi\cdot A) \\ & + c_5\,\Big(4\epsilon^{\mu\nu\rho\alpha\beta} A_\rho \mathcal{R}_{\alpha\beta}{}^{\sigma\tau}\,\nabla_\tau \xi_\sigma + 2\,\epsilon^{\mu\rho\sigma\alpha\beta}\, F_{\rho\sigma} \mathcal{R}_{\alpha\beta}{}^{\nu\tau}\,\xi_\tau + 4\,\epsilon^{\rho\sigma\alpha\beta(\nu}\, F_{\rho\sigma} \mathcal{R}_{\alpha\beta}{}^{\tau)\mu}\,\xi_\tau\Big) \\ & - 2\,g^{\left[\nu[\sigma\right.} g^{\left.\rho]\mu\right]}\,\nabla_\rho \xi_\sigma + c_1\Big(- 4\,\mathcal{R}^{\mu\nu\rho\sigma}\nabla_\rho \xi_\sigma + 8\xi_\sigma \nabla_\rho \mathcal{R}^{\mu\nu\rho\sigma}\Big) \\ & + c_4\big(- 2\,F^{\mu\nu} F^{\rho\sigma}\,\nabla_\rho \xi_\sigma + 2\,\nabla_\rho (F^{\mu\rho} F^{\nu\sigma})\,\xi_\sigma + 4\,\nabla_\rho (F^{\mu(\sigma} F^{\rho|\nu)})\,\xi_\sigma\big). \end{aligned} \tag{6.1.9}$$

To evaluate the angular momentum we integrate the above expression, where $\xi^\mu = \varphi^\mu$ is the rotational Killing vector of the black hole space-time solution, as in (2.5.13),

$$\Theta_{5D} = \int d\Sigma_{\mu\nu}\Theta^{\mu\nu}. \qquad (6.1.10)$$

Again, we refer to in appendix A of [120] for the explicit result.

The four-dimensional charges for the correspondent four-dimensional system can be determined using the entropy function formalism. The derivation is, again, completely generic as it only depends on the form of the near-horizon solution presented in (6.1.1) and does not assume any particular value for any near-horizon parameters.

As explained above, to apply entropy function, we need gauge invariant Lagrangian and thus, we first need to perform a Kaluza-Klein reduction of the action (6.1.2). We take the following ansatz for the metric and gauge field for the reduction $(i, j, \cdots = 1, \ldots, 4)$,

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = h_{ij}dx^i dx^j + \phi(r)(d\psi + A_K^2), \\ A_\mu &= A_i dx^i + \sigma(r)(d\psi + A_{KK}) \end{aligned} \qquad (6.1.11)$$

and compactify along ψ direction. We denote A_{KK} as the Kaluza-Klein (KK) gauge field and the corresponding field strength is given by $(F_{KK})_{ij} = \partial_i(A_{KK})_j - \partial_j(A_{KK})_i$. The four-dimensional gauge field is A and the corresponding field strength is F. All un-hatted curvature quantities are composed of the four-dimensional metric h_{ij}. The reduction of gauge invariant terms in the action is straightforward (appendix B of [120] for more details). On the other hand, the reduction of the two Chern-Simons terms is tricky, and requires the addition of some total derivatives terms. Here, we present only the reduced four-dimensional action corresponding to these terms. Specifically the two derivative gauge Chern-Simons term takes the following form

$$\int d^5x\sqrt{-g}\,\epsilon^{\mu\nu\rho\sigma\delta}A_\mu F_{\nu\rho}F_{\sigma\delta} = 12\pi\int d^4x\sqrt{-h}\,\epsilon^{ijlm}\left[\frac{1}{3}\sigma^3(F_{KK})_{ij}(F_{KK})_{lm}+\sigma^2(F_{KK})_{ij}F_{lm}+\sigma F_{ij}F_{lm}\right],$$

while the four derivative mixed Chern-Simons term reads:

$$\int d^5x\sqrt{-g}\epsilon^{\mu\nu\rho\sigma\delta}A_\mu\mathcal{R}_{\nu\rho}{}^{\alpha\beta}\mathcal{R}_{\sigma\delta\alpha\beta} = (T_1 - 4T_2), \qquad (6.1.12)$$

where, T_1 and T_2 are given by,

$$
\begin{aligned}
T_1 &= 4\pi \int \mathrm{d}^4x\sqrt{-h}\epsilon^{ijkl}\Big\{\sigma\Big[\mathcal{R}^{mn}{}_{ij}\Big(\mathcal{R}_{klmn} - \Phi\big((F_{\mathrm{KK}})_{km}(F_{\mathrm{KK}})_{ln} + (F_{\mathrm{KK}})_{kl}(F_{\mathrm{KK}})_{mn}\big)\Big) \\
&\quad + \frac{\Phi^2}{4}\Big((F_{\mathrm{KK}})_{ij}(F_{\mathrm{KK}})_{kl}(F_{\mathrm{KK}})^2 - 2(F_{\mathrm{KK}})_{ij}(F_{\mathrm{KK}})_{km}(F_{\mathrm{KK}})^{mn}(F_{\mathrm{KK}})_{nl}\Big) \\
&\quad + \frac{\Phi}{2}\Big(\nabla_m(F_{\mathrm{KK}})_{ij}\nabla^m(F_{\mathrm{KK}})_{kl}\Big)\Big]\Big\}\,, \\
T_2 &= 2\pi \int \mathrm{d}^4x\sqrt{-h}\,\epsilon^{ijkl}F_{kl}\Big[\frac{\Phi}{2}\mathcal{R}^{mn}{}_{ij}(F_{\mathrm{KK}})_{mn} + \frac{\Phi^2}{4}(F_{\mathrm{KK}})_{im}(F_{\mathrm{KK}})_{jn}(F_{\mathrm{KK}})^{mn} \\
&\qquad - \frac{\Phi^2}{8}(F_{\mathrm{KK}})_{ij}(F_{\mathrm{KK}})^2\Big]\,,
\end{aligned}
$$

and $(F_{\mathrm{KK}})^2 = (F_{\mathrm{KK}})_{ij}(F_{\mathrm{KK}})^{ij}$. The periodicity of the compact direction is 2π.
Now, the four-dimensional action obtained is gauge invariant so that we can apply the entropy function formalism. Our ansatz for near-horizon metric and gauge field is (6.1.1).
From the four-dimensional point of view we have two gauge fields, one coming from usual five-dimensional gauge field and the other coming from the metric components $g_{\psi\mu}$ (Kaluza-Klein gauge field): to each of those gauge fields corresponds a charge, respectively Q and Θ, to which we associate a charge parameter, e_1 and e_0 respectively. The entropy function is, then, defined as follows:

$$
\mathcal{E} = 2\pi(Qe_1 + \Theta e_0 - \bar{L})\,, \tag{6.1.13}
$$

where, $\bar{L}$ is given by,

$$
\bar{L} = \frac{8\pi^2}{16\pi G_5}\int \mathrm{d}\theta \mathrm{d}\chi\,\mathcal{L}_4\,, \tag{6.1.14}
$$

and $\mathcal{L}_4$ is the reduced four-dimensional Lagrangian, including the higher derivative terms. The attractor equations are obtained by minimizing the entropy function with respect to the near-horizon parameters, while the four-dimensional physical charges are given by:

$$
Q = \frac{\partial \bar{L}}{\partial \bar{e}_1}, \qquad \Theta = \frac{\partial \bar{L}}{\partial \bar{e}_0}. \tag{6.1.15}
$$

Calculating $\bar{L}$ over the near-horizon geometry (6.1.1), we find the expression for the four-dimensional physical charges. The proper four-dimensional charges are rescaled as in [106] to $\tilde{Q} = 2Q$, $\tilde{\Theta} = 2\Theta$.

Comparing the five-dimensional charges with the corresponding four-dimensional charges, we find the expected complete match between the two sets of charges at the two derivative level. However, as already seen for asymptotically flat black hole solutions, they differ at the four derivative level and the difference is proportional to c_5 only (gravitational Chern-Simons term). We also find that the relation between five-dimensional and four-dimensional charge remains exactly the same as in the case of asymptotically flat black holes. For the angular momentum Θ_{5D}, our result is more generic than the one in [91], as in our case the black hole can carry an extra parameter P, which can be thought of as a magnetic field. Thus, we see that the difference between five- and four-dimensional charges is purely a topological effect due to the Chern-Simons term which is not gauge invariant. The details can be found in [91]. The bottom line of our analysis is that the asymptotic geometry of the space time is not relevant to account for the differences in the physical charges. The relations between 4D and 5D charges for asymptotically AdS black holes is given by:

$$\tilde{Q} = -Q_{5D} + \frac{8\pi B v_3}{G v_2} c_5 \,, \quad \tilde{\Theta} = -\Theta_{5D} + \frac{4\pi P v_3^2 (e_0^2 v_2^2 - B^2 v_1^2)}{G v_1^2 v_2^2} c_5 \,. \tag{6.1.16}$$

These expressions are one of the main results of this chapter and constitute a generalization of previous work [127] to higher-derivative gravity theory.

6.1.2 Flat directions in higher derivative gravity

So far we have studied the generic relation between four-dimensional and five-dimensional charges. Now we concentrate on a particular class of supersymmetric solutions in five dimensions and find that it exhibits a flat direction. Our goal is to study the fate of this flat direction when higher derivative interactions are taken into account.

We consider, in the following, the supersymmetric asymptotically AdS_5 black hole solution presented by Gutowski and Reall [119]. This solution is 1/2 BPS (preserves 4 supercharges) [122] and its near-horizon geometry has an AdS_2 component. We use the coordinates that make the AdS_2 part of the near-horizon geometry manifest [124],

$$\begin{aligned} \mathrm{d}s^2 &= \frac{1}{\Delta^2 + 9L^{-2}} \left(\frac{\mathrm{d}r^2}{r^2} - \mathrm{d}t^2 r^2 \right) + \frac{1}{\Delta^2 - 3L^{-2}} \left(\mathrm{d}\theta^2 + \mathrm{d}\chi^2 \sin^2(\theta) \right) \\ &\quad + \left(\frac{\Delta}{\Delta^2 - 3L^{-2}} \right)^2 \left(\mathrm{d}\psi + \cos\theta \mathrm{d}\chi - \frac{3r}{L\Delta} \frac{\Delta^2 - 3L^{-2}}{\Delta^2 + 9L^{-2}} \left(\mathrm{d}t + \frac{\mathrm{d}r}{r^2} \right) \right)^2 \,, \\ F &= \frac{\sqrt{3}\Delta}{\Delta^2 + 9L^{-2}} \mathrm{d}r \wedge \mathrm{d}t - \frac{\sqrt{3}\sin\theta}{L(\Delta^2 - 3L^{-2})} \mathrm{d}\theta \wedge \mathrm{d}\chi \,. \end{aligned} \tag{6.1.17}$$

Comparing this solution with near-horizon ansatz given in (6.1.1) we find the following values of the near-horizon parameters

$$v_1 = \frac{1}{\Delta^2 + 9L^{-2}}\,, \qquad v_2 = \frac{1}{\Delta^2 - 3L^{-2}}\,, \qquad v_3 = \left(\frac{\Delta}{\Delta^2 - 3L^{-2}}\right)^2\,,$$
$$e_0 = -\frac{3}{L\Delta}\frac{\Delta^2 - 3L^{-2}}{\Delta^2 + 9L^{-2}}\,, \qquad e_5 = e_1 + e_0\varphi = \frac{\sqrt{3}\Delta}{\Delta^2 + 9L^{-2}}\,, \qquad B = 1\,,$$
$$A_\chi = P + B\varphi = \frac{\sqrt{3}\sin\theta}{L(\Delta^2 - 3L^{-2})}. \tag{6.1.18}$$

The leading attractor equations (i.e. derived from the two derivative action) are given by,

$$\begin{aligned}
v_1 \text{ equation :} \quad & v_1^2\left(B^2\left(\varphi^2 + v_3^2\right) - 2\Lambda v_2^2 - 4v_2\right) + 2BP\varphi v_1^2 \\
& + v_2^2\left(e_0^2\left(\varphi^2 + v_3^2\right) + 2e_1e_0\varphi + e_1^2\right) + P^2v_1^2 = 0\,, \\
v_2 \text{ equation :} \quad & B^2\varphi^2v_1^2 + B^2v_3^2v_1^2 + 2BP\varphi v_1^2 + e_0^2\varphi^2v_2^2 - 4v_2^2v_1 \\
& + 2e_0e_1\varphi v_2^2 + e_1^2v_2^2 + e_0^2v_2^2v_3^2 + P^2v_1^2 + 2\Lambda v_2^2v_1^2 = 0\,, \\
v_3 \text{ equation :} \quad & v_1^2\left(B^2\left(\varphi^2 + 3v_3^2\right) - 2\Lambda v_2^2 - 4v_2\right) + 2BP\varphi v_1^2 \\
& - v_2^2\left(e_0^2\left(\varphi^2 + 3v_3^2\right) + 2e_1e_0\varphi + e_1^2\right) + P^2v_1^2 + 4v_2^2v_1 = 0\,, \\
\varphi \text{ equation :} \quad & B^2\varphi v_3v_1^2 + Bv_1\left(24\varphi v_2\left(e_0\varphi + e_1\right)\kappa + Pv_1v_3\right) \\
& - v_2\left(e_0\varphi + e_1\right)\left(e_0v_2v_3 - 24Pv_1\kappa\right) = 0\,.
\end{aligned} \tag{6.1.19}$$

Other two attractor equations (for e_0 and e_1) define the four-dimensional charges Θ_{4D} and Q_{4D} in terms of near-horizon geometry.

Substituting the leading values of near-horizon geometry (6.1.19) in the attractor equations one can check that the first three equations (corresponding to v_1, v_2 and v_3) vanish. Furthermore the φ equation vanishes for a particular value of $\kappa = \frac{1}{4\sqrt{3}}$, which is the supersymmetric value. Thus one does not need any specific φ to solve the attractor equations. It is also easy to check that the entropy of the black hole does not depend on the near-horizon value of this scalar field and it is given by:

$$S = \frac{2\pi\Delta L^4}{G\left(\Delta^2L^2 - 3\right)^2}. \tag{6.1.20}$$

Therefore we conclude that, at two derivative level, φ is a flat direction, as it was already observed in [127].

To check whether φ remains flat in presence of higher derivative terms we follow the same procedure as we did at leading order. We first find the corrections to the five-dimensional near-horizon geometry due to higher derivative interactions. Then we study the attractor equations derived from the entropy function in presence of higher derivative terms on this corrected solution.

We consider the following higher derivative correction to the five-dimensional near-horizon geometry[4]:

$$\begin{aligned} v_1 &= \frac{1}{\Delta^2 + 9L^{-2}} + \gamma V_1 \,, & v_2 &= \frac{1}{\Delta^2 - 3L^{-2}} + \gamma V_2 \,, \\ e_0 &= -\frac{3}{L\Delta}\frac{\Delta^2 - 3L^{-2}}{\Delta^2 + 9L^{-2}} + \gamma E_0 \,, & e_5 &= \frac{\sqrt{3}\Delta}{\Delta^2 + 9L^{-2}} + \gamma E_5 \,. \end{aligned} \tag{6.1.21}$$

We can solve for these higher derivative corrections (V_1, V_2, E_0 and E_5) using the five-dimensional equations of motion. The solution is given in appendix C of [120]. One can easily check that the fifth component of five-dimensional gauge field (φ) never appeared in any Einstein's equation. Therefore, we can not fix this scalar or its higher derivative correction in five dimensions. However, it is important to check whether the entropy depends on this scalar field or not. To this end, we first verify that the corrected five-dimensional solution solves the higher derivative attractor equations and then compute the entropy function on this solution. As it turns out, imposing on-shell conditions on the entropy function will make its dependence on φ disappear. In fact, the entropy reads:

$$\begin{aligned} S = \frac{2\pi\Delta L^4}{G\,(\Delta^2L^2-3)^2}\Bigg[1 + \frac{\gamma}{(\Delta^4L^4 - 6\Delta^2L^2 - 15)}\Bigg\{72\Big(2c_2(7\Delta^4L^4 - 10\Delta^2L^2 + 3) \\ + c_3(9\Delta^4L^4 + 2\Delta^2L^2 + 3) + 2\sqrt{3}c_5(\Delta^4L^4 - 7\Delta^2L^2 - 5)\Big) \\ + c_1(-16\Delta^6L^6 + 503\Delta^4L^4 + 246\Delta^2L^2 + 585) - 24c_4(\Delta^6L^6 \\ - 24\Delta^4L^4 - 53\Delta^2L^2 + 15)\Bigg\}\Bigg] + \mathcal{O}(\gamma^2) \,. \end{aligned} \tag{6.1.22}$$

As expected, the flat direction at the two derivative level is not lifted, if the solution preserves some supersymmetries. What strikes as a surprise, however, is that throughout the whole analysis we never specified the supersymmetric values for the coefficients c_i's of the higher derivative interactions in (6.1.3), meaning that

[4]We do not consider any correction for v_3, A_χ and B as one can use redundancies in the leading solution to choose the corrections to the above three parameters to be zero. This issue is addressed in appendix C of [120].

the flat direction will remain flat even for non supersymmetric deformations of the higher derivative action (6.1.2). Unexpectedly, the symmetries of the leading order black hole solution seem to protect the flat directions from being lifted independently of the symmetries of the full action.

For supersymmetric values given in (6.1.3) the entropy has the following form (remember that the AdS radius L also picks up a correction):

$$S_{\text{susy}} = \frac{2\pi\Delta K L^4}{G\,(\Delta^2L^2-3)^2} - \frac{\pi\gamma c_1 \Delta K L^4\,(2\Delta^8L^8 - 103\Delta^6L^6 - 2633\Delta^4L^4 + 8463\Delta^2L^2 - 33)}{G\,(\Delta^2L^2-3)^3\,(\Delta^4L^4 - 6\Delta^2L^2 - 15)}\,. \tag{6.1.23}$$

6.2 Flat direction in ten-dimensional supergravity

Supersymmetric, asymptotically AdS_5 black hole solutions, like the one analyzed in the previous section, have been used for a huge number of applications in the AdS/CFT correspondence. But these are obviously not the only solutions used to obtain some knowledge of the dual field theory.
The first attempt of finding BPS black holes in five-dimensional minimal gauged supergravity dates back few years [130], but all the solutions suffer from not having regular horizons or naked singularity. Later, they were found as a special limit of a more general class of non supersymmetric black hole solutions [131], which contain a non-extremality parameter μ linking solutions of the ungauged theory with supersymmetric solution of the gauged theory ($\mu = 0$ is the BPS-saturated limit). Such non supersymmetric black hole solutions of the minimal five-dimensional gauged $U(1)^3$ supergravity, which are asymptotically AdS_5, can have a regular extreme limit with zero Hawking temperature and finite entropy[131].
It is also possible to embed such solutions in type IIB supergravity [132]. The full class of solutions, which was shown to satisfy the ten-dimensional equations of motion coming from the two derivative type IIB supergravity action [133], includes the ten-dimensional black hole metric, a self-dual five form F_5, three gauge fields a_i, coming from the five-dimensional $U(1)^3$ gauged theory lifted in 10 dimensions and physical charges $\tilde{q}_i$:

$$\begin{aligned} ds_{10}^2 =& \sqrt{\triangle}\left[-(H_1H_2H_3)^{-1} f dt^2 + \left(f^{-1}dr^2 + r^2(d\mathcal{M}_3)^2\right)\right] \\ &+ \frac{1}{\sqrt{\triangle}} \sum_{i=1}^{3} L^2 H_i \left(d\mu_i^2 + \mu_i^2\left[d\phi_i + a_i\ dt\right]^2\right)\,, \end{aligned} \tag{6.2.1}$$

where $\mathcal{M}_3 = \{\mathbb{R}^3, \mathrm{S}^3\}$ is a spatial manifold corresponding to curvatures $\kappa = \{0, 1\}$,

$$a_i = \frac{\tilde{q}_i}{q_i}\, L^{-1}\left(H_i^{-1} - 1\right)\,, \qquad H_i = 1 + \frac{q_i}{r^2}\,,$$
$$\triangle = H_1 H_2 H_3 \sum_{i=1}^{3} \frac{\mu_i^2}{H_i}\,, \qquad f = \kappa - \frac{\mu}{r^2} + \frac{r^2}{L^2} H_1 H_2 H_3\,, \tag{6.2.2}$$

and

$$\mu_1 = \cos\theta_1\,, \qquad \mu_2 = \sin\theta_1\ \cos\theta_2\,, \qquad \mu_3 = \sin\theta_1\ \sin\theta_2\,. \tag{6.2.3}$$

$\kappa = 0$ corresponds to flat horizon. For $\kappa = 1$ horizon topology is S^3 and $\kappa = -1$ gives negatively curved horizon. The physical charges $\tilde{q}_i$ are related to charge parameters q_i in the following way,

$$\tilde{q}_i = \sqrt{q_i(\mu + \kappa q_i)}\,. \tag{6.2.4}$$

The five form field strength is given by:

$$F_5 = \mathcal{F}_5 + \star\mathcal{F}_5\,, \qquad \mathcal{F}_5 = dB_4\,, \tag{6.2.5}$$

where,

$$B_4 = -\frac{r^4}{L}\triangle\ \mathrm{d}t \wedge \mathrm{dVol}_{\mathcal{M}_3} - L\sum_{i=1}^{3} \tilde{q}_i \mu_i^2 \left(L\,\mathrm{d}\phi_i - \frac{q_i}{\tilde{q}_i}\ \mathrm{d}t\right) \wedge \mathrm{dVol}_{\mathcal{M}_3}\,, \tag{6.2.6}$$

where $\mathrm{dVol}_{\mathcal{M}_3}$ is a volume form on $\mathcal{M}_3$. Note that the ten-dimensional Bianchi identity on the five form $\nabla_a F^{abcde} = 0$ gives rise to the five-dimensional equations of motion for the scalars and the gauge fields. Finally, the dilaton equation of motion admit a general solution of the form $\phi(r) = c_0 + c_1 h(r)$ where the function $h(r)$ is singular at the horizon. To circumvent this problem we can set $c_1 = 0$, so that the dilaton is just an arbitrary constant.
Once again, we remind the readers that the above solution does not preserve any supersymmetry, contrarily to the solution analyzed in the previous section.

It is important to stress that the dilaton is constant and it is not possible to find its value by solving Einstein's equations of motion. The entropy of this black hole solution does not depend on it. Therefore the dilation is a flat direction at the two derivatives level. We would like to see the fate of this flat direction when we add higher derivative terms in the action. However, for our purposes it is easier to consider, without loss of generality, the extremal limit of this black hole solution and, once again, consider only its near-horizon geometry. This procedure is discussed in the following section.

6.2.1 Extremal near-horizon geometry

The extremal limit corresponds to (taking Hawking temperature to zero),

$$2r_0^6 + r_0^4(\kappa + q_1 + q_2 + q_3) - q_1 q_2 q_3 = 0 \,, \tag{6.2.7}$$

where r_0 solves the above equation given the charges.
The mass parameter μ, in the near-horizon geometry is fixed to be:

$$\mu = \kappa r_0^2 + \frac{r_0^4}{L^2} \prod (1 + \frac{q_i}{r_0^2}) \,. \tag{6.2.8}$$

For simplicity we consider only three equal charge solution: $q_1 = q_2 = q_3 = q$. In that case we see

$$\begin{aligned} H_1 = H_2 = H_3 = H = 1 + \frac{q}{r^2} \,, \\ \triangle = H^2 \,, \end{aligned} \tag{6.2.9}$$

which leads to $q = 2\,r_0^2$ and $\mu = 27\,r_0^4$. For convenience we take $L = 1$ throughout this section. Therefore the three equal charge black hole metric becomes,

$$\begin{aligned} \mathrm{d}s_{10}^2 &= \sqrt{\triangle}\left[-(H)^{-3} f \mathrm{d}t^2 + \left(f^{-1}\mathrm{d}r^2 + r^2(\mathrm{d}\mathcal{M}_3)^2\right)\right] \\ &\quad + \left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2\right) + \sum_{i=1}^{3} \mu_i^2 \left[\mathrm{d}\phi_i + a_i\ \mathrm{d}t\right]^2 . \end{aligned} \tag{6.2.10}$$

We consider $\kappa = 0$ case, i.e. flat horizon. The analysis can be repeated for the $\kappa = 1$ case, and it is completely analogous.

The extremal near-horizon metric is given by,

$$\begin{aligned} ds^2 &= \frac{1}{12}\left(-r^2\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{r^2}\right) + 3r_0^2(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) + \left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2\right) \\ &\quad + \mu_1^2\left(\mathrm{d}\phi_1 + \frac{r}{3\sqrt{2}}\mathrm{d}t\right)^2 + \mu_2^2\left(\mathrm{d}\phi_2 + \frac{r}{3\sqrt{2}}\mathrm{d}t\right)^2 + \mu_3^2\left(\mathrm{d}\phi_3 + \frac{r}{3\sqrt{2}}\mathrm{d}t\right)^2 , \end{aligned} \tag{6.2.11}$$

and the four form field B reads,

$$B_4 = -\sqrt{3}r_0^3\left(r\mathrm{d}t + \cos^2\theta_1 \mathrm{d}\phi_1 + \sin^2\theta_2(\mathrm{d}\phi_2\cos^2\theta_2 + \mathrm{d}\phi_3 \sin^2\theta_2)\right) \wedge \mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z \,. \tag{6.2.12}$$

Thus we see that at the extremal limit the near-horizon geometry admits an AdS_2 part. However, we shall determine this near-horizon geometry using again the entropy function analysis and discuss the fate of the flat direction ϕ.

6.2.2 The entropy function

We consider x, y and z direction to be compactified on a three torus. Therefore from the seven-dimensional point of view the four form R-R field B_4 appears to be an one form field $A_\mu = (B_4)_{\mu xyz}$, where μ runs over all indices except x, y and z (the $D3$ brane is a point-like object and A_1 is electrically coupled to it).

Now we would like to compactify over ϕ_i directions. Therefore from the four-dimensional ($\{t, r, \theta_1, \theta_2\}$) point of view there are three KK gauge fields $a_i = z_1 r dt$ (all of them are equal for the three equal charges case) and six scalars: three of them coming from the metric and three of them from B_4. Given the symmetry of the problem, the scalars coming from the metric are equal and will be denoted by w_1. Analogously, the scalars coming from the 4-form can be all denoted by b. In fact even starting with different values for the scalars, they will be constrained to be equal. Therefore we can write down the following near-horizon ansatz for the metric and the gauge field[5]:

$$\begin{aligned} ds^2 &= v_1\left(-\rho^2 \mathrm{d}t^2 + \frac{\mathrm{d}\rho^2}{\rho^2}\right) + v_2\ (\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \\ &\quad + w_1\left[\left(\mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2\right) + \sum_{i=1}^{3} \mu_i^2\,[\mathrm{d}\phi_i + z_1\ r\ \mathrm{d}t]^2\right] \end{aligned} \tag{6.2.13}$$

We can decompose the seven-dimensional field A_1 in terms of four-dimensional field $A^{(4)}$ and KK fields as follows

$$A = \frac{b}{2}\ \sum_i \mu_i^2(\mathrm{d}\phi_i + z_1 r \mathrm{d}t) + A^{(4)} \tag{6.2.14}$$

with

$$A^{(4)} = e_0 r \mathrm{d}t \wedge \mathrm{d}z\ . \tag{6.2.15}$$

[5] In fact the derivation is more involved. Once we break the $SO(6)$ symmetry, the lower-dimensional scalars and gauge fields depend on the angular direction of lower-dimensional space-time, in this case on θ_1 and θ_2. One has to solve the attractor equations to find the angular dependence of the lower-dimensional fields. See [134] for the details. However in this case, as the leading near-horizon geometry is known, we substitute the angular dependence of the fields from the beginning. Therefore the scalars and the different components of the gauge fields are determined by AdS_2 symmetry only.

Explicitly, dA_1, which can be thought of as a field strength in four dimensions, reads:

$$\begin{aligned}\mathrm{d}A &= \Big[q_5\ \mathrm{dr}\wedge\mathrm{dt} + b\ \sin(\theta_1)\cos(\theta_1)\left(-\mathrm{d}\phi_3\sin^2(\theta_2) - \mathrm{d}\phi_2\cos^2(\theta_2) + \mathrm{d}\phi_1\right)\wedge\mathrm{d}\theta_1 \\ &\quad -b\ \mathrm{d}\theta_2\wedge(\mathrm{d}\phi_2 - \mathrm{d}\phi_3)\sin(\theta_2)\sin^2(\theta_1)\cos(\theta_2)\Big]\ , \end{aligned} \tag{6.2.16}$$

where

$$q_5 = e_0 + \frac{z_1 b}{2}\ . \tag{6.2.17}$$

In ten dimensions the five form RR field strength F_5 is given by,

$$F_5 = \mathrm{d}B_4 + \star\mathrm{d}B_4 \tag{6.2.18}$$

where $\mathrm{d}B_4 = \mathrm{d}A\wedge\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z$. Therefore, the corresponding F_5^2 equals:

$$\frac{1}{4\ 5!}F_5^2 = \frac{1}{2v_2^3}\left(-\frac{q_5^2}{v_1^2} + \frac{2b^2}{w_1^2}\right)\ . \tag{6.2.19}$$

Hence, the final result for the on-shell action reads,

$$\begin{aligned} S &= \frac{V_3}{16\pi G_{10}}\int_0^{\pi/2}\mathrm{d}\theta_1\int_0^{\pi/2}\mathrm{d}\theta_2\sqrt{-g_{10}}\left[R_{10} - \frac{1}{2}(\partial\phi)^2 - \frac{1}{4\cdot 5!}(F_5)^2\right] \\ &= \frac{V_3}{128\pi G_{10}}\frac{v_1 v_2^{\frac{3}{2}} w_1^{\frac{5}{2}}}{2}\left(-\frac{\frac{2b^2}{w_1^2} - \frac{\left(\frac{bz_1}{2}+e_0\right)^2}{v_1^2}}{v_2^3} + \frac{w_1 z_1^2 - 4v_1}{v_1^2} + \frac{40}{w_1}\right) \end{aligned} \tag{6.2.20}$$

We define the entropy function with respect to the lower-dimensional charges,

$$\mathcal{E} = 2\pi\mathcal{A}\left(Q\ e_0 + \Theta\ z_1 - \frac{S}{\mathcal{A}}\right)\ , \tag{6.2.21}$$

where $\mathcal{A} = \frac{V_3}{128\pi G_{10}}$. Solving the attractor equations we find the following solution for the near-horizon geometry:

$$\begin{aligned} v_1 &= \frac{\sqrt{-Q}}{24}\ , \quad v_2 = \frac{(2\Theta/3)^{2/3}}{(-Q)^{5/6}}\ , \quad z_1 = \frac{1}{3\sqrt{2}}\ , \\ & b = \frac{4\Theta}{3Q}\ , \quad w_1 = \frac{\sqrt{-Q}}{2}\ . \end{aligned} \tag{6.2.22}$$

Furthermore, the entropy is given by:

$$\mathcal{S} = \frac{\Theta}{192\sqrt{2}G_{10}}\ . \tag{6.2.23}$$

Since we know the extremal near-horizon geometry exactly (6.2.11), (6.2.12), we can solve the attractor equations and find the lower-dimensional charges in terms of a single parameter r_0. We can then re-write the entropy as a function of r_0 and check it is in agreement with Bekenstein-Hawking law.
The near-horizon geometry reads:

$$\begin{aligned} v_1 &= \frac{1}{12}, \quad v_2 = 3r_0^2, \quad z_1 = \frac{1}{3\sqrt{2}} \\ & b = -6\sqrt{6}\, r_0^3, \quad w_1 = 1 \end{aligned} \tag{6.2.24}$$

Substituting these values in the attractor equations we get,

$$Q = -4, \quad \Theta = 18\sqrt{6} r_0^3 \,. \tag{6.2.25}$$

Therefore the entropy turns out to be

$$\mathcal{S} = \frac{3\sqrt{3} r_0^3}{32 G_{10}} = \frac{\text{Area}}{4 G_{10}} \,. \tag{6.2.26}$$

One should note that the near-horizon value of the dilaton does not appear in the entropy function, therefore it is a flat direction. Our goal is to check what happens to this flat direction when we add supersymmetric higher derivative terms which appear in type IIB string theory.

6.2.3 Higher derivative terms in type IIB string theory

For type IIB supergravity, which is a low-momentum expansion of type IIB superstring theory, the higher derivative corrections can be written as a series in α'. The series is of the following form:

$$\alpha'^4\, S_{\text{IIB}} = S^{(0)} + \alpha'\, S^{(1)} + \cdots + (\alpha')^n\, S^{(n)}, \tag{6.2.27}$$

The terms $n = 1$ and $n = 2$ are not expected to appear at tree-level and 1-loop in the string coupling g_S, so the first contribution to the action $S^{(0)}$ is of the order α'^3. It is an eight derivative action, containing the well known $\mathcal{R}^4$ term. Unfortunately, the standard superfield techniques ([135, 136]) can not be used for the construction of the full $S^{(3)}$ contribution to the two derivative action, that corresponds to the supersymmetric completion of the $\mathcal{R}^4$ term [137]. Nevertheless, if one considers only a subset of the full field content of type IIB theory, specifically the metric and the five-form, then a general formula for the supersymmetric higher derivative correction exists [136]. For the sake of completeness, we outline the steps taken to obtain such an invariant. First of all, U(1) gauge invariance of the theory allows us to separate all the higher derivative terms by their charge. We will then only look for terms that are neutral under U(1) and contain at most one fermion bilinear. These terms, schematically, look like:

$$S^{(3)}_{0;\mathrm{B}} = \int \mathrm{d}^{10}x \, f^{(0,0)}(\tau,\bar\tau)(C^4 + (F_5)^8 + \cdots) \,, \tag{6.2.28}$$

$$S^{(3)}_{0;\mathrm{BFF}} = \int \mathrm{d}^{10}x \, f^{(0,0)}(\tau,\bar\tau)(C^2\bar\psi\psi + (F_5)^7\bar\psi\psi \cdots) \,, \tag{6.2.29}$$

where C is the Weyl tensor, ψ is the gravitino, F_5 is the self-dual five form and $f^{(0,0)}(\tau,\bar\tau)$ is a modular function of the complex scalar fields τ and $\bar\tau$, which reads as in (6.2.32), (6.2.33). Note that the five form and the metric are the only bosonic fields neutral under U(1). Now if one starts from this restricted set of fields, considering terms only linear in the fermions in the supersymmetry variations and setting $\partial\tau = \partial\bar\tau = \lambda = 0$ (λ being the dilatino of the theory), it is possible to show that the supersymmetry variation of (6.2.28) cancels exactly against the supersymmetry variation of (6.2.29) (neglecting fermions trilinear). Of course, setting the derivative of the scalar field τ to zero we are effectively neglecting the variation of the modular form $f^{(0,0)}(\tau,\bar\tau)$. Restricting our attention to these terms, it is pretty straightforward to show that the obstruction to the existence of the chiral measure, found in [135], is circumvented. As one would expect, then, the final result for the eight derivative action [136], turns out to be exact. This construction is, however, highly non-trivial and only few explicit calculations were carried out [138, 139]. Recently, a simplified, explicit expression for the eight derivative coupling between the metric and five-form was found in [140]. In the following we will make use of this general result, together with the solutions (6.2.1),(6.2.6). The full action reads:

$$I = \frac{1}{16\pi G_N}\int_{\mathcal{M}_{10}} \mathrm{d}^{10}x\sqrt{-g}\left[R_{10} - \tfrac{1}{2}(\partial\phi)^2 - \frac{1}{4\cdot 5!}(F_5)^2 + \cdots + (\alpha')^3\gamma(\phi)\mathcal{W} + \cdots\right] \tag{6.2.30}$$

$$\gamma(\phi) = \frac{1}{16} f^{(0,0)}(\tau,\bar\tau), \qquad G_N \propto \alpha'^4 \tag{6.2.31}$$

$$f^{(0,0)}(\tau,\bar\tau) = \sum_{(m,n)\neq(0,0)} \frac{\tau_2^{3/2}}{|m+n\tau|^3}, \tag{6.2.32}$$

where

$$\tau = \tau_1 + i\tau_2 = C^{(0)} + ie^{-\phi}. \tag{6.2.33}$$

The above correction to the leading supergravity action is a complete quantum, i.e. α' as well as string-loop, correction (the string coupling $g_s \propto \exp\phi_\infty$). The first term in the expansion of $f^{(0,0)}(\tau,\bar{\tau})$, $n=0$, $m\neq 0$, appears only as a supergravity (α') correction to the leading two derivative Lagrangian (see for instance [141]). This corresponds, in the AdS/CFT language, to consider finite α' and the planar limit $N\to\infty$ limit together with the $g_s\to 0$ limit.
In the following, we will keep and consider the entire quantum correction to the leading supergravity action.

The higher derivative $\mathcal{W}$ contribution is explicitly given by [140]:

$$\mathcal{W} \equiv \frac{1}{86016}\sum_{\zeta=1}^{20} n_\zeta\, M_\zeta\ , \tag{6.2.34}$$

where n_ζ and M_ζ are given by $(a,b\cdots=1,\ldots,10)$,

n_ζ	M_ζ
-43008	$C_{abcd}C_{abef}C_{cegh}C_{dgfh}$
86016	$C_{abcd}C_{aecf}C_{bgeh}C_{dgfh}$
129024	$C_{abcd}C_{aefg}C_{bfhi}\mathcal{T}_{cdeghi}$
30240	$C_{abcd}C_{abce}\mathcal{T}_{dfghij}\mathcal{T}_{efhgij}$
7392	$C_{abcd}C_{abef}\mathcal{T}_{cdghij}\mathcal{T}_{efghij}$
-4032	$C_{abcd}C_{aecf}\mathcal{T}_{beghij}\mathcal{T}_{dfghij}$
-4032	$C_{abcd}C_{aecf}\mathcal{T}_{bghdij}\mathcal{T}_{eghfij}$
-118272	$C_{abcd}C_{aefg}\mathcal{T}_{bcehij}\mathcal{T}_{dfhgij}$
-26880	$C_{abcd}C_{aefg}\mathcal{T}_{bcehij}\mathcal{T}_{dhifgj}$
112896	$C_{abcd}C_{aefg}\mathcal{T}_{bcfhij}\mathcal{T}_{dehgij}$
-96768	$C_{abcd}C_{aefg}\mathcal{T}_{bcheij}\mathcal{T}_{dfhgij}$
1344	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdeijk}\mathcal{T}_{fghijk}$
-12096	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdfijk}\mathcal{T}_{eghijk}$
-48384	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cdfijk}\mathcal{T}_{egihjk}$
24192	$C_{abcd}\mathcal{T}_{abefgh}\mathcal{T}_{cefijk}\mathcal{T}_{dghijk}$
2386	$\mathcal{T}_{abcdef}\mathcal{T}_{abcdgh}\mathcal{T}_{egijkl}\mathcal{T}_{fijhkl}$
-3669	$\mathcal{T}_{abcdef}\mathcal{T}_{abcdgh}\mathcal{T}_{eijgkl}\mathcal{T}_{fikhjl}$
-1296	$\mathcal{T}_{abcdef}\mathcal{T}_{abcghi}\mathcal{T}_{dejgkl}\mathcal{T}_{fhkijl}$
10368	$\mathcal{T}_{abcdef}\mathcal{T}_{abcghi}\mathcal{T}_{dgjekl}\mathcal{T}_{fhkijl}$
2688	$\mathcal{T}_{abcdef}\mathcal{T}_{abdegh}\mathcal{T}_{cgijkl}\mathcal{T}_{fjkhil}$

The tensor $\mathcal{T}$ is defined by

$$\mathcal{T}_{abcdef} = P_{1050^+}\left(i\nabla_a F_{bcdef} + \frac{1}{8}F_{abcmn}F^{mn}_{def}\right) . \tag{6.2.35}$$

If we impose self-duality of the five-form, this reduces to

$$\mathcal{T}_{abcdef} = i\nabla_a F_{bcdef} + \frac{1}{16}\left(F_{abcmn}{F_{def}}^{mn} - 3F_{abfmn}{F_{dec}}^{mn}\right) ,$$

where the RHS should be anti-symmetrized in the triplets $[abc], [def]$ and symmetrized for their interchange. Here, we also note that the higher derivative correction has been given in the Einstein frame. We can also go to the string frame with proper transformation of the metric, but, obviously, the physical information of the system will not depend on the frame chosen.

6.2.4 The fate of the flat direction

As we already explained, the dilaton parametrizes a flat direction at two derivative level, that can be lifted or not by the presence of supersymmetric higher derivative interactions. To verify its fate, we first compute the entropy function in presence of these higher derivative terms (6.2.34) and then focus on the attractor equations corresponding to the two scalars, i.e. axion and dilaton. Since the higher derivative term $\mathcal{W}$ evaluated on the leading solution turns out to be constant[6], the axion-dilaton equations take the following form,

$$\left.\frac{\partial f^{(0,0)}(\tau_1,\tau_2)}{\partial \tau_1}\right|_{\tau_1=(\tau_1)_h,\tau_2=(\tau_2)_h} = 0 , \quad \left.\frac{\partial f^{(0,0)}(\tau_1,\tau_2)}{\partial \tau_2}\right|_{\tau_1=(\tau_1)_h,\tau_2=(\tau_2)_h} = 0 . \tag{6.2.36}$$

The axion equation reads:

$$\sum_{(m,n)\neq(0,0)} \frac{n(m+n\tau_1)}{|m+n\tau|^5} = 0 . \tag{6.2.37}$$

and it is easily solved by $\tau_1 = 0$. On the other hand, the dilaton equation of motion is given by (setting τ_1 to zero):

$$\sum_{(m,n)\neq(0,0)} \frac{\sqrt{\tau}_2(m^2 - n^2\tau_2^2)}{(m^2+n^2\tau_2^2)^{5/2}} = 0 . \tag{6.2.38}$$

[6] The value of $\mathcal{W}$ for the near-horizon geometry (6.2.24) considered is 14580.

One solution of the above equation is $\tau_2 = 0 \implies \phi_h \to \infty$, but this divergent behavior destabilizes the near-horizon geometry, so we will not take it into account. Another possible solution is $\tau_2 = 1$, for which

$$\sum_{(m,n)\neq(0,0)} \frac{(m^2 - n^2)}{(m^2 + n^2)^{5/2}} = 0, \tag{6.2.39}$$

is identically satisfied. Therefore, the leading near-horizon value of the dilaton is $\phi_h = 0$, so that the flat direction is lifted when we add higher derivative terms in the action.

One important observation is that considering only the leading term in the modular function ($n = 0$, $m \neq 0$), that is, only the leading higher derivative correction (α'^3) to supergravity, setting all loop corrections to zero, then the leading value of the dilaton is fixed to infinity. The thermodynamics of the system (temperature, entropy) does not receive any correction due to this leading higher derivative term (since vanishes on-shell, $\tau_2 = 0$), but the system is destabilized.

However, considering the full quantum correction, then there is a possibility of a finite dilaton solution. This is a rather interesting phenomenon, as the full quantum correction stabilizes the system again. Not only that, it seems that supersymmetries, and not just extremality of a black hole solution is necessary to protect the flat directions from being lifted. Once again, it looks as if the symmetries of the higher derivative interactions do not play any role to decide the fate of flat directions.

For completeness, we present also the higher derivative correction to the entropy: the entropy function and the attractor equations are defined as before.
We computed the full supersymmetric higher derivative term (6.2.34) for the near-horizon geometry and the lengthy expression is presented in appendix D of [120]. Solving the corrected attractor equations we get the following corrections to the near-horizon geometry:

$$\begin{aligned} v_1 &= \frac{\sqrt{-Q}}{24} - \frac{144155}{384Q}\hat{\gamma}\,, \quad v_2 = \frac{(2\Theta/3)^{2/3}}{(-Q)^{5/6}} - \frac{25115(\Theta(-Q))^{2/3}}{8\,\sqrt[3]{2}\,3^{2/3}Q^3}\hat{\gamma}\,, \\ z_1 &= \frac{1}{3\sqrt{2}} - \frac{810\sqrt{2}}{(-Q)^{3/2}}\hat{\gamma}\,, \quad b = \frac{4\Theta}{3Q} + \frac{541115\Theta}{9(-Q)^{5/2}}\hat{\gamma}\,, \quad w_1 = \frac{\sqrt{-Q}}{2} + \frac{21085}{32Q}\hat{\gamma}\,, \end{aligned} \tag{6.2.40}$$

where,

$$\hat{\gamma} = \frac{\alpha'^3}{16} \sum_{(m,n)\neq(0,0)} \frac{1}{(m^2 + n^2)^{3/2}}\,. \tag{6.2.41}$$

The entropy is given by:

$$\mathcal{S} = \frac{\Theta}{192\sqrt{2}G_{10}} - \frac{405\Theta}{16\sqrt{2}G_{10}(-Q)^{3/2}}\hat{\gamma}\ . \qquad (6.2.42)$$

6.3 Future Directions

We hope this work will pave the way for a number of possible applications and extensions. As of now, we are not aware of any asymptotically AdS solutions of supergravity in ten dimensions that preserves some supersymmetries. Thus, one interesting direction is to uplift the five-dimensional spinning AdS solution, analyzed in the first part of this chapter, to ten dimensions. Knowing the correct uplift of this class of supersymmetric five-dimensional black hole solutions, one can study the behavior of flat directions (if any) for the uplifted solution and compare the results obtained with the ones presented in this chapter.

In this chapter [120], we have seen that, for the five-dimensional case, the supersymmetric values of various higher derivative terms did not play any role. Without specifying the correct supersymmetric coefficients of various higher derivative terms, we saw the flat direction of the leading solution remains flat. As we stressed before, the supersymmetric form of the leading solution played an important role in the whole analysis. It would be interesting to find a supersymmetric black hole solution in the higher derivative theory as well although this would require an analysis of the complete off-shell formulation of minimal gauged supergravity in five dimensions. This analysis would certainly make use of the correct values of various coefficients of the higher derivative terms and give us the first supersymmetric asymptotic AdS black hole solution away from supergravity limit.

Appendix A

Conventions and useful identities

In this paper, we have used in parallel both superspace, which is conventionally written in two-component notation, and multiplet calculus, which is usually carried out in four-component notation. To aid the reader in translating any given formula between the two notations, in this appendix we summarize our conventions for both.

We use the Pauli-Källén convention. Space-time indices are denoted $\mu, \nu, \ldots$, Lorentz indices are denoted $a, b, \ldots$, and SU(2) indices are denoted $i, j, \ldots$. The Lorentz metric is $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ and the anti-symmetric tensor ε_{abcd} is imaginary, with $\varepsilon_{0123} = -\text{i}$. The four-component γ matrices, which differ from those of [31], are built out of the σ matrices and obey

$$(\gamma^a)^\dagger = \gamma_a \ , \qquad \{\gamma^a, \gamma^b\} = 2\eta^{ab} \ , \qquad \gamma_5 = -\text{i}\gamma^0\gamma^1\gamma^2\gamma^3 \ . \tag{A.1}$$

The anti-symmetric combination of γ matrices is defined analogously,

$$\gamma^{ab} := \tfrac{1}{2}[\gamma^a, \gamma^b] \ . \tag{A.2}$$

Hence, the product of two γ matrices can be written as,

$$\gamma_a\gamma_b = \eta_{ab} + \gamma_{ab} \ . \tag{A.3}$$

We note the following useful identities involving gamma matrices,

$$\begin{aligned} \gamma_{ab} &= -\tfrac{1}{2}\,\varepsilon_{abcd}\gamma^{cd}\gamma_5 \ , & \gamma^b\gamma_a\gamma_b &= -2\gamma_a \ , \\ \gamma_{ab}\gamma^{ab} &= -12 \ , & \gamma^{cd}\gamma_{ab}\gamma_{cd} &= 4\gamma_{ab} \ , \\ \gamma^c\gamma_{ab}\gamma_c &= 0 \ , & \gamma^{ab}\gamma_c\gamma_{ab} &= 0 \ , \\ [\gamma^c,\gamma_{ab}] &= 4\,\delta_{[a}{}^c\gamma_{b]} \ , & \{\gamma^c,\gamma_{ab}\} &= 2\,\varepsilon_{ab}{}^{cd}\gamma_5\gamma_d \ , \\ [\gamma_{ab},\gamma^{cd}] &= -8\,\delta_{[a}{}^{[c}\gamma_{b]}{}^{d]} \ , & \{\gamma_{ab},\gamma^{cd}\} &= -4\,\delta_{[a}{}^{[c}\delta_{b]}{}^{d]} + 2\,\varepsilon_{ab}{}^{cd}\gamma_5 \ . \end{aligned} \tag{A.4}$$

We define a charge conjugation matrix C that satisfies the following identities,

$$\begin{aligned} C^\dagger &= C^{-1} \ , & C\gamma_5 C^{-1} &= \gamma_5^T \ , \\ C^T &= -C \ , & C\gamma_\mu C^{-1} &= -\gamma_\mu^T \ . \end{aligned} \tag{A.5}$$

A Majorana fermion is a four-component Dirac fermion ψ that satisfies the reality constraint,

$$\bar{\psi} = \psi^T C \ , \tag{A.6}$$

with $\bar{\psi} = i\,\psi^\dagger\gamma^0$ Dirac conjugate.
Two Majorana spinors that do not form a bilinear can be written as a linear combination of bilinears by a Fierz rearrangement, i.e.

$$\phi\bar{\psi} = -\tfrac{1}{4}(\bar{\psi}\phi) - \tfrac{1}{4}(\bar{\psi}\gamma^a\phi)\gamma_a - \tfrac{1}{4}(\bar{\psi}\gamma_5\phi)\gamma_5 + \tfrac{1}{4}(\bar{\psi}\gamma^a\gamma_5\phi)\gamma_a\gamma_5 + \tfrac{1}{8}(\bar{\psi}\gamma^{ab}\phi)\gamma_{ab} \ . \tag{A.7}$$

We want to give some details about the chiral spinor notation used throughout this work. Consider two Majorana spinors Ψ^i_M, $i = 1, 2$. Such spinors can be decomposed in the left- and right-handed chirality as follows,

$$\Psi^i_L = \tfrac{1}{2}(\mathbf{1} + \gamma_5)\Psi^i_M \ , \qquad \Psi^i_R = \tfrac{1}{2}(\mathbf{1} - \gamma_5)\Psi^i_M \tag{A.8}$$

Now, while the original Majorana spinors were invariant under charge conjugation, defined as $(\Psi^i_M)^c \equiv C\bar{\Psi}^{iT}_M$, the left- and right-handed components do not, since

$$(\Psi^i_L)^c = \Psi^i_R \ , \qquad (\Psi^i_R)^c = \Psi^i_L \ , \tag{A.9}$$

which is equivalent to state that the left- and right-handed components of the Majorana spinors Ψ^i_M transform in the conjugate representations $\mathbf{2}$ and $\bar{\mathbf{2}}$ of the group SU(2). It is then convenient to change notation and make use of the SU(2) indices to indicate the different chiralities, i.e. $\Psi^i = \Psi^i_L$ and $\Psi_i = \varepsilon_{ij}\,\Psi^j_R$ (or viceversa, but to be specified for each fermion). This notation is consistent with the properties of conjugate representations if we associate charge conjugation with change of chirality, namely SU(2) indices are raised or lowered by charge conjugation, e.g. $(\Psi^i)^c = \Psi_i$. For different representations of the Lorentz group, the charge conjugation operation typically corresponds to complex conjugation, e.g. $(T_{abij})^* = T_{ab}{}^{ij}$. These simple definitions lead straightforwardly to the following useful identities for spinors ψ_i and ϕ_j of the same chirality,

$$
\begin{aligned}
\bar{\psi}^i \phi_j &= 0 \, , & \bar{\psi}^i \gamma_\mu \phi^j &= 0 \, , \\
\bar{\psi}^i \phi^j &= \bar{\phi}^j \psi^i \, , & (\bar{\psi}^i \phi^j)^* &= \bar{\psi}_i \phi_j \, , \\
\bar{\psi}^i \gamma_\mu \phi_j &= -\bar{\phi}_j \gamma_\mu \psi^i \, , & (\bar{\psi}^i \gamma_\mu \phi_j)^* &= \bar{\psi}_i \gamma_\mu \phi^j \, ,
\end{aligned}
\tag{A.10}
$$

and analogous equations hold for more complex bilinears. These identities can be used to simplify the Fierz rearrangement presented above for chiral spinors as follows,

$$
\begin{aligned}
\phi^i \bar{\psi}^j &= -\tfrac{1}{2}(\bar{\psi}^j \phi^i) + \tfrac{1}{8}(\bar{\psi}^j \gamma^{ab} \phi^i)\gamma_{ab} \, , \\
\phi^i \bar{\psi}_j &= -\tfrac{1}{2}(\bar{\psi}_j \gamma^a \phi^i)\gamma_a \, .
\end{aligned}
\tag{A.11}
$$

Our two-component conventions follow mainly [31] with the following modification: the spinor matrices are given by $\sigma^a = (-\mathbf{1}, -\tau^i)$ with τ^i the Pauli matrices,

$$
\tau^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \, , \qquad \tau^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \, , \qquad \tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \, . \tag{A.12}
$$

A generic four-component Dirac fermion Ψ decomposes into spinors ψ_α and $\bar{\chi}^{\dot{\alpha}}$, which are respectively left-handed and right-handed two-component spinors. The Dirac conjugate $\bar{\Psi} = \mathrm{i}\Psi^\dagger \gamma^0$ has components $\chi^\alpha = (\bar{\chi}^{\dot{\alpha}})^*$ and $\bar{\psi}_{\dot{\alpha}} = (\psi_\alpha)^*$. Spinor indices can be raised and lowered using the anti-symmetric tensor $\epsilon_{\alpha\beta}$:

$$
\psi^\beta = \epsilon^{\beta\alpha}\psi_\alpha \, , \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta \, , \qquad \epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma \, , \qquad \epsilon^{12} = \epsilon_{21} = 1 \, . \tag{A.13}
$$

Similar equations pertain for $\epsilon_{\dot{\alpha}\dot{\beta}}$ and dotted spinors. We define

$$
(\bar{\sigma}^a)^{\dot{\alpha}\alpha} := \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}(\sigma^a)_{\beta\dot{\beta}} \, , \qquad \bar{\sigma}^a = (\sigma^0, -\sigma^i) \tag{A.14}
$$

so that

$$
(\sigma^a\bar{\sigma}^b + \sigma^b\bar{\sigma}^a)_\alpha{}^\beta = -2\eta^{ab}\delta_\alpha^\beta \, , \qquad (\bar{\sigma}^a\sigma^b + \bar{\sigma}^b\sigma^a)^{\dot{\alpha}}{}_{\dot{\beta}} = -2\eta^{ab}\delta_{\dot{\beta}}^{\dot{\alpha}} \, . \tag{A.15}
$$

Our γ and C matrices can be expressed in terms of the σ matrices as follows,

$$\gamma^a = \begin{pmatrix} 0 & \mathrm{i}\,(\sigma^a)_{\alpha\dot\beta} \\ \mathrm{i}\,(\bar\sigma^a)^{\dot\alpha\beta} & 0 \end{pmatrix} , \quad \gamma_5 = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & -\delta^{\dot\alpha}{}_{\dot\beta} \end{pmatrix} , \quad C = \begin{pmatrix} -\epsilon^{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot\alpha\dot\beta} \end{pmatrix} . \tag{A.16}$$

We define anti-symmetric combinations of σ matrices as

$$(\sigma^{ab})_\alpha{}^\beta := \tfrac{1}{4}(\sigma^a\bar\sigma^b - \sigma^b\bar\sigma^a)_\alpha{}^\beta , \qquad (\bar\sigma^{ab})^{\dot\alpha}{}_{\dot\beta} := \tfrac{1}{4}(\bar\sigma^a\sigma^b - \bar\sigma^b\sigma^a)^{\dot\alpha}{}_{\dot\beta} , \tag{A.17}$$

One can check that $(\sigma^{ab})_{\alpha\beta} = \epsilon_{\beta\gamma}(\sigma^{ab})_\alpha{}^\gamma$ is *symmetric* in its spinor indices and similarly for $(\bar\sigma^{ab})_{\dot\alpha\dot\beta} = \epsilon_{\dot\alpha\dot\gamma}(\bar\sigma^{ab})^{\dot\gamma}{}_{\dot\beta}$. These obey the duality properties (see also the first of (A.4))

$$\tfrac{1}{2}\varepsilon_{abcd}\sigma^{cd} = -\sigma_{ab} , \qquad \tfrac{1}{2}\varepsilon_{abcd}\bar\sigma^{cd} = +\bar\sigma_{ab} . \tag{A.18}$$

The main difference between four-component and two-component notation (aside from the use of γ- versus σ-matrices) is that the latter usually yields more direct information about the Lorentz group representation of the field in question. For example, in four-component calculations, one must remember the chirality of all spinor quantities. This is accomplished in $\mathcal{N} = 2$ multiplet calculus by using the location of the SU(2) index to distinguish between the left-handed and right-handed fields; for example, $\gamma_5\psi_\mu{}^i = \psi_\mu{}^i$ and $\gamma_5\psi_{\mu i} = -\psi_{\mu i}$ for the Q-supersymmetry connections while $\gamma_5\phi_\mu{}^i = -\phi_\mu{}^i$ and $\gamma_5\phi_{\mu i} = \phi_{\mu i}$ for the S-supersymmetry connections. In two-component notation, the first pair are written as $\psi_{\mu\alpha}{}^i$ and $\bar\psi_\mu{}^{\dot\alpha}{}_i$ and the second pair by $\bar\phi_\mu{}^{\dot\alpha i}$ and $\phi_{\mu\alpha i}$ with the explicit spinor index denoting the chirality, so one can in principle raise or lower the SU(2) index using the anti-symmetric tensor ε_{ij}. However, we will avoid doing this to maintain maximum compatibility with four-component notation.

Similarly, vectors and tensors can be written with spinor indices to explicitly indicate their properties under the Lorentz group. A vector V^a is associated with a field $V_{\alpha\dot\alpha}$ with one dotted and one undotted index via

$$V_{\alpha\dot\alpha} = (\sigma^a)_{\alpha\dot\alpha}V_a , \qquad V_a = -2(\bar\sigma_a)^{\dot\alpha\alpha}V_{\alpha\dot\alpha} . \tag{A.19}$$

An anti-symmetric two-form F_{ab} is associated with symmetric bi-spinors $F_{\alpha\beta}$ and $F_{\dot\alpha\dot\beta}$ corresponding to its anti-selfdual and selfdual parts,

$$\begin{aligned} F^-_{ab} &= (\sigma_{ab})_\alpha{}^\beta F_\beta{}^\alpha \ , \qquad F^+_{ab} = (\bar\sigma_{ab})^{\dot\alpha}{}_{\dot\beta} F^{\dot\beta}{}_{\dot\alpha} \ , \\ F^\pm_{ab} &= \tfrac{1}{2}(F_{ab} \pm \tilde F_{ab}) \ , \qquad \tilde F_{ab} = \tfrac{1}{2}\varepsilon_{abcd}F^{cd} \ , \qquad \tilde F^\pm_{ab} = \pm F^\pm_{ab} \ . \end{aligned} \tag{A.20}$$

If F_{ab} is real, then $(F_{\alpha\beta})^* = -F_{\dot\alpha\dot\beta}$. We always apply symmetrization and anti-symmetrization with unit strength, so that $F_{[ab]} = F_{ab}$ and $F_{(\alpha\beta)} = F_{\alpha\beta}$. Furthermore, the following useful identities for products of (anti-)selfdual tensors are noted,

$$\begin{aligned} G^\pm_{[a[c}\, H^\pm_{d]b]} &= \pm\, \tfrac{1}{8} G^\pm_{ef}\, H^{\pm ef}\, \varepsilon_{abcd} - \tfrac{1}{4}(G^\pm_{ab}\, H^\pm_{cd} + G^\pm_{cd}\, H^\pm_{ab}) \,, \\ G^\pm_{ab}\, H^{\mp cd} + G^{\pm cd}\, H^\mp_{ab} &= 4\delta^{[c}_{[a} G^\pm_{b]e}\, H^{\mp d]e} \,, \\ \tfrac{1}{2}\varepsilon^{abcd}\, G^{\pm e}_{[c}\, H^\pm_{d]e} &= \pm\, G^{\pm[a}{}_e\, H^{\pm b]e} \,, \\ G^{\pm ac}\, H^{\pm b}_c + G^{\pm bc}\, H^{\pm a}_c &= -\, \tfrac{1}{2}\eta^{ab}\, G^{\pm cd}\, H^\pm_{cd} \,, \\ G^{\pm ac}\, H^{\mp b}_c &= G^{\pm bc}\, H^{\mp a}_c \,, \\ G^{\pm ab}\, H^\mp_{ab} &= 0 \,. \end{aligned} \tag{A.21}$$

Finally, we remind the reader that SU(2) indices are swapped by complex conjugation, $(T_{abij})^* = T_{ab}{}^{ij}$, and we make use of the invariant SU(2) tensor ε^{ij} and ε_{ij} defined as $\varepsilon^{\underline{12}} = \varepsilon_{\underline{12}} = 1$ with $\varepsilon^{ij}\varepsilon_{kj} = \delta^i_k$. As already stated, unlike in the superspace approaches [75, 84], we do *not* raise or lower SU(2) indices with the ε_{ij} tensor.

Appendix B

Superconformal gravity and multiplet calculus

In this appendix, we present the transformation rules for the $\mathcal{N}=2$ conformal supergravity (or Weyl) multiplet. Some useful multiplet calculus identities are presented, together with an important example of reduced chiral multiplet, the covariant Weyl multiplet $\mathbf{W}$.

B.1 Superconformal algebra - Weyl multiplet

Recall that the superconformal algebra comprises the generators of the general-coordinate, local Lorentz, dilatation, special conformal, chiral SU(2) and U(1), supersymmetry (Q) and special supersymmetry (S) transformations. The gauge fields associated with general-coordinate transformations ($e_\mu{}^a$), dilatations (b_μ), R-symmetry ($\mathcal{V}_\mu{}^i{}_j$ and A_μ) and Q-supersymmetry ($\psi_\mu{}^i$) are independent fields. The remaining gauge fields associated with the Lorentz ($\omega_\mu{}^{ab}$), special conformal ($f_\mu{}^a$) and S-supersymmetry transformations ($\phi_\mu{}^i$) are composite objects [67–69]. The multiplet also contains three other fields: a Majorana spinor doublet χ^i, a scalar D, and a selfdual Lorentz tensor T_{abij}, which is anti-symmetric in $[ab]$ and $[ij]$. The Weyl and chiral weights have been collected in table B.1. Under Q-supersymmetry, S-supersymmetry and special conformal transformations the Weyl multiplet fields

	Weyl multiplet											parameters	
field	$e_\mu{}^a$	$\psi_\mu{}^i$	b_μ	A_μ	$\mathcal{V}_\mu{}^i{}_j$	$T_{ab}{}^{ij}$	χ^i	D	ω_μ^{ab}	$f_\mu{}^a$	$\phi_\mu{}^i$	ϵ^i	η^i
w	-1	$-\frac{1}{2}$	0	0	0	1	$\frac{3}{2}$	2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$
c	0	$-\frac{1}{2}$	0	0	0	-1	$-\frac{1}{2}$	0	0	0	$-\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
γ_5		$+$					$+$				$-$	$+$	$-$

Table B.1: Weyl and chiral weights (w and c) and fermion chirality (γ_5) of the Weyl multiplet component fields and the supersymmetry transformation parameters.

transform as

$$
\begin{aligned}
\delta e_\mu{}^a &= \bar\epsilon^i\,\gamma^a\psi_{\mu i} + \bar\epsilon_i\,\gamma^a\psi_\mu{}^i\,,\\
\delta\psi_\mu{}^i &= 2\,\mathcal{D}_\mu\epsilon^i - \tfrac{1}{8}T_{ab}{}^{ij}\gamma^{ab}\gamma_\mu\epsilon_j - \gamma_\mu\eta^i\\
\delta b_\mu &= \tfrac{1}{2}\bar\epsilon^i\phi_{\mu i} - \tfrac{3}{4}\bar\epsilon^i\gamma_\mu\chi_i - \tfrac{1}{2}\bar\eta^i\psi_{\mu i} + \text{h.c.} + \Lambda_{\rm K}^a e_{\mu a}\,,\\
\delta A_\mu &= \tfrac{1}{2}\mathrm{i}\bar\epsilon^i\phi_{\mu i} + \tfrac{3}{4}\mathrm{i}\bar\epsilon^i\gamma_\mu\,\chi_i + \tfrac{1}{2}\mathrm{i}\bar\eta^i\psi_{\mu i} + \text{h.c.}\,,\\
\delta\mathcal{V}_\mu{}^i{}_j &= 2\,\bar\epsilon_j\phi_\mu{}^i - 3\bar\epsilon_j\gamma_\mu\,\chi^i + 2\bar\eta_j\,\psi_\mu{}^i - (\text{h.c. ; traceless})\,,\\
\delta T_{ab}{}^{ij} &= 8\,\bar\epsilon^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta\chi^i &= -\tfrac{1}{12}\gamma^{ab}\,\not{D}T_{ab}{}^{ij}\,\epsilon_j + \tfrac{1}{6}R(\mathcal{V})_{\mu\nu}{}^i{}_j\gamma^{\mu\nu}\epsilon^j - \tfrac{1}{3}\mathrm{i}R_{\mu\nu}(A)\gamma^{\mu\nu}\epsilon^i + D\epsilon^i + \tfrac{1}{12}\gamma_{ab}T^{abij}\eta_j\,,\\
\delta D &= \bar\epsilon^i\,\not{D}\chi_i + \bar\epsilon_i\,\not{D}\chi^i\,.
\end{aligned}
\tag{B.1}
$$

Here ϵ^i and ϵ_i denote the spinorial parameters of Q-supersymmetry, η^i and η_i those of S-supersymmetry, and $\Lambda_{\rm K}{}^a$ is the transformation parameter for special conformal boosts. The full superconformally covariant derivative is denoted by D_μ, while $\mathcal{D}_\mu$ denotes a covariant derivative with respect to Lorentz, dilatation, chiral U(1) and SU(2) transformations,

$$
\mathcal{D}_\mu\epsilon^i = \big(\partial_\mu - \tfrac{1}{4}\omega_\mu{}^{cd}\,\gamma_{cd} + \tfrac{1}{2}\,b_\mu + \tfrac{1}{2}\mathrm{i}\,A_\mu\big)\epsilon^i + \tfrac{1}{2}\,\mathcal{V}_\mu{}^i{}_j\,\epsilon^j\,. \tag{B.2}
$$

The covariant curvatures are given by

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a =& 2\,\partial_{[\mu}\, e_{\nu]}{}^a + 2\, b_{[\mu}\, e_{\nu]}{}^a - 2\,\omega_{[\mu}{}^{ab}\, e_{\nu]b} - \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]i} + \text{h.c.})\,,\\
R(Q)_{\mu\nu}{}^i =& 2\,\mathcal{D}_{[\mu}\psi_{\nu]}{}^i - \gamma_{[\mu}\phi_{\nu]}{}^i - \tfrac{1}{8}\, T^{abij}\,\gamma_{ab}\,\gamma_{[\mu}\psi_{\nu]j}\,,\\
R(A)_{\mu\nu} =& 2\,\partial_{[\mu}A_{\nu]} - \mathrm{i}\,\big(\tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \text{h.c.}\big)\,,\\
R(\mathcal{V})_{\mu\nu}{}^i{}_j =& 2\,\partial_{[\mu}\mathcal{V}_{\nu]}{}^i{}_j + \mathcal{V}_{[\mu}{}^i{}_k\,\mathcal{V}_{\nu]}{}^k{}_j + 2(\bar\psi_{[\mu}{}^i\,\phi_{\nu]j} - \bar\psi_{[\mu j}\,\phi_{\nu]}{}^i) - 3(\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_j - \bar\psi_{[\mu j}\gamma_{\nu]}\chi^i)\\
&- \delta_j{}^i(\bar\psi_{[\mu}{}^k\,\phi_{\nu]k} - \bar\psi_{[\mu k}\,\phi_{\nu]}{}^k) + \tfrac{3}{2}\delta_j{}^i(\bar\psi_{[\mu}{}^k\gamma_{\nu]}\chi_k - \bar\psi_{[\mu k}\gamma_{\nu]}\chi^k)\,,\\
R(M)_{\mu\nu}{}^{ab} =& 2\,\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\,\omega_{[\mu}{}^{ac}\omega_{\nu]c}{}^b - 4f_{[\mu}{}^{[a}e_{\nu]}{}^{b]} + \tfrac{1}{2}(\bar\psi_{[\mu}{}^i\,\gamma^{ab}\,\phi_{\nu]i} + \text{h.c.})\\
&+ (\tfrac{1}{4}\bar\psi_\mu{}^i\,\psi_\nu{}^j\,T^{ab}{}_{ij} - \tfrac{3}{4}\bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,\gamma^{ab}\chi_i - \bar\psi_{[\mu}{}^i\,\gamma_{\nu]}\,R(Q)^{ab}{}_i + \text{h.c.})\,,\\
R(D)_{\mu\nu} =& 2\,\partial_{[\mu}b_{\nu]} - 2f_{[\mu}{}^a e_{\nu]a} - \tfrac{1}{2}\bar\psi_{[\mu}{}^i\phi_{\nu]i} + \tfrac{3}{4}\bar\psi_{[\mu}{}^i\gamma_{\nu]}\chi_i - \tfrac{1}{2}\bar\psi_{[\mu i}\phi_{\nu]}{}^i + \tfrac{3}{4}\bar\psi_{[\mu i}\gamma_{\nu]}\chi^i\,,\\
R(S)_{\mu\nu}{}^i =& 2\,\mathcal{D}_{[\mu}\phi_{\nu]}{}^i - 2f_{[\mu}{}^a\gamma_a\psi_{\nu]}{}^i - \tfrac{1}{8}\not{D}T_{ab}{}^{ij}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]\,j} - \tfrac{3}{2}\gamma_a\psi_{[\mu}{}^i\,\bar\psi_{\nu]}{}^j\gamma^a\chi_j\\
&+ \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_{[\mu}\psi_{\nu]}{}^i\,,\\
R(K)_{\mu\nu}{}^a =& 2\,\mathcal{D}_{[\mu}f_{\nu]}{}^a - \tfrac{1}{4}\big(\bar\phi_{[\mu}{}^i\gamma^a\phi_{\nu]i} + \bar\phi_{[\mu i}\gamma^a\phi_{\nu]}{}^i\big)\\
&+ \tfrac{1}{4}\big(\bar\psi_\mu{}^i D_b T^{ba}{}_{ij}\psi_\nu{}^j - 3\,e_{[\mu}{}^a\psi_{\nu]}{}^i\not{D}\chi_i + \tfrac{3}{2}D\,\bar\psi_{[\mu}{}^i\gamma^a\psi_{\nu]j} - 4\,\bar\psi_{[\mu}{}^i\gamma_{\nu]}D_b R(Q)^{ba}{}_i + \text{h.c.}\big)\,.
\end{aligned}
\tag{B.3}
$$

The connections $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ are algebraically determined by imposing the conventional constraints

$$
\begin{aligned}
R(P)_{\mu\nu}{}^a = 0\,, \quad \gamma^\mu R(Q)_{\mu\nu}{}^i + \tfrac{3}{2}\gamma_\nu\chi^i = 0\,,\\
e^\nu{}_b\,R(M)_{\mu\nu a}{}^b - \mathrm{i}\tilde R(A)_{\mu a} + \tfrac{1}{8}T_{abij}T_\mu{}^{bij} - \tfrac{3}{2}D\,e_{\mu a} = 0\,.
\end{aligned}
\tag{B.4}
$$

Their solution is given by

$$
\begin{aligned}
\omega_\mu{}^{ab} =& -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} - e^{\nu[a}e^{b]\sigma}e_{\mu c}\partial_\sigma e_\nu{}^c - 2e_\mu{}^{[a}e^{b]\nu}b_\nu\\
&- \tfrac{1}{4}(2\bar\psi_\mu^i\gamma^{[a}\psi_i^{b]} + \bar\psi^{ai}\gamma_\mu\psi_i^b + \text{h.c.})\,,\\
\phi_\mu{}^i =& \tfrac{1}{2}\left(\gamma^{\rho\sigma}\gamma_\mu - \tfrac{1}{3}\gamma_\mu\gamma^{\rho\sigma}\right)\left(\mathcal{D}_\rho\psi_\sigma{}^i - \tfrac{1}{16}T^{abij}\gamma_{ab}\gamma_\rho\psi_{\sigma j} + \tfrac{1}{4}\gamma_{\rho\sigma}\chi^i\right)\,,\\
f_\mu{}^\mu =& \tfrac{1}{6}R(\omega,e) - D - \big(\tfrac{1}{12}e^{-1}\varepsilon^{\mu\nu\rho\sigma}\bar\psi_\mu{}^i\,\gamma_\nu\mathcal{D}_\rho\psi_{\sigma i} - \tfrac{1}{12}\bar\psi_\mu{}^i\psi_\nu{}^jT^{\mu\nu}{}_{ij} - \tfrac{1}{4}\bar\psi_\mu{}^i\gamma^\mu\chi_i + \text{h.c.}\big)\,.
\end{aligned}
\tag{B.5}
$$

We will also need the bosonic part of the expression for the uncontracted connection $f_\mu{}^a$,

$$
f_\mu{}^a = \tfrac{1}{2}R(\omega,e)_\mu{}^a - \tfrac{1}{4}\big(D + \tfrac{1}{3}R(\omega,e)\big)e_\mu{}^a - \tfrac{1}{2}\mathrm{i}\tilde R(A)_\mu{}^a + \tfrac{1}{16}T_{\mu b}{}^{ij}T^{ab}{}_{ij}\,,
\tag{B.6}
$$

where $R(\omega,e)_\mu{}^a = R(\omega)_{\mu\nu}{}^{ab}e_b{}^\nu$ is the non-symmetric Ricci tensor, and $R(\omega,e)$ the corresponding Ricci scalar. The curvature $R(\omega)_{\mu\nu}{}^{ab}$ is associated with the spin connection field $\omega_\mu{}^{ab}$. Note that by simply substituting the expression for

the K-gauge connection $f_\mu{}^a$ into the Lorentz curvature $R(M)_{\mu\nu}{}^{ab}$ (or equivalently $\mathcal{R}(M)_{\mu\nu}{}^{ab}$ shown below) one obtains a term proportional to the Weyl tensor $C_{\mu\nu}{}^{\rho\sigma} = \mathcal{R}_{\mu\nu}{}^{\rho\sigma} - 2\,\delta_{[\mu}{}^{[\rho}\,\mathcal{R}_{\nu]}{}^{\sigma]} + \frac{1}{3}\delta_\mu{}^{[\rho}\delta_\nu{}^{\sigma]}\,\mathcal{R}$, up to terms proportional to the dilatation gauge field b_μ. This explains how equation (4.1.14) is obtained from the covariant Weyl multiplet presented in (B.17).
The transformations of $\omega_\mu{}^{ab}$, $\phi_\mu{}^i$ and $f_\mu{}^a$ are induced by the constraints (B.4). We present their Q- and S-supersymmetry variations, as well as the transformations under conformal boosts, below,

$$\begin{aligned}
\delta\omega_\mu{}^{ab} =& -\tfrac{1}{2}\bar\epsilon^i\gamma^{ab}\phi_{\mu i} - \tfrac{1}{2}\bar\epsilon^i\psi_\mu{}^j\,T^{ab}{}_{ij} + \tfrac{3}{4}\bar\epsilon^i\gamma_\mu\gamma^{ab}\chi_i \\
&+ \bar\epsilon^i\gamma_\mu\,R(Q)^{ab}{}_i - \tfrac{1}{2}\bar\eta^i\gamma^{ab}\psi_{\mu i} + \text{h.c.} + 2\,\Lambda_{\rm K}{}^{[a}e_\mu{}^{b]}\,, \\
\delta\phi_\mu{}^i =& -2\,f_\mu{}^a\gamma_a\epsilon^i + \tfrac{1}{4}R(\mathcal{V})_{ab}{}^i{}_j\gamma^{ab}\gamma_\mu\epsilon^j + \tfrac{1}{2}\mathrm{i}R(A)_{ab}\gamma^{ab}\gamma_\mu\epsilon^i - \tfrac{1}{8}\not{D}T^{ab\,ij}\gamma_{ab}\gamma_\mu\epsilon_j \\
&+ \tfrac{3}{2}[(\bar\chi_j\gamma^a\epsilon^j)\gamma_a\psi_\mu{}^i - (\bar\chi_j\gamma^a\psi_\mu{}^j)\gamma_a\epsilon^i] + 2\,\mathcal{D}_\mu\eta^i + \Lambda_{\rm K}{}^a\gamma_a\psi_\mu{}^i\,, \\
\delta f_\mu{}^a =& -\tfrac{1}{2}\bar\epsilon^i\psi_\mu{}^i\,D_bT^{ba}{}_{ij} - \tfrac{3}{4}e_\mu{}^a\bar\epsilon^i\not{D}\chi_i - \tfrac{3}{4}\bar\epsilon^i\gamma^a\psi_{\mu i}\,D \\
&+ \bar\epsilon^i\gamma_\mu\,D_bR(Q)^{ba}{}_i + \tfrac{1}{2}\bar\eta^i\gamma^a\phi_{\mu i} + \text{h.c.} + \mathcal{D}_\mu\Lambda_{\rm K}{}^a\,.
\end{aligned} \tag{B.7}$$

The transformations under S-supersymmetry and conformal boosts reflect the structure of the underlying SU(2,2|2) gauge algebra. The presence of curvature constraints and of the non-gauge fields T_{abij}, χ^i and D induce deformations of the Q-supersymmetry algebra, as is manifest in the above results, in particular in (B.3) and (B.7).

Combining the conventional constraints (B.4) with the various Bianchi identities one derives that not all the curvatures are independent. For instance,

$$\varepsilon^{abcd}D_bR(M)_{cd}{}^{ef} = 2\,\varepsilon^{abc[e}\,R(K)_{bc}{}^{f]} + \tfrac{9}{2}\,\eta^{a[e}\bar\chi^i\gamma^{f]}\chi_i + \tfrac{3}{2}\big[\bar\chi^i\gamma^aR(Q)^{ef}_i - \text{h.c.}\big]\,. \tag{B.8}$$

Furthermore it is convenient to modify two of the curvatures by including suitable covariant terms,

$$\begin{aligned}
\mathcal{R}(M)_{ab}{}^{cd} &= R(M)_{ab}{}^{cd} + \tfrac{1}{16}\big(T_{abij}\,T^{cdij} + T_{ab}{}^{ij}\,T^{cd}{}_{ij}\big)\,, \\
\mathcal{R}(S)_{ab}{}^i &= R(S)_{ab}{}^i + \tfrac{3}{4}T_{ab}{}^{ij}\chi_j\,.
\end{aligned} \tag{B.9}$$

where we observe that $\gamma^{ab}\big(\mathcal{R}(S)-\mathcal{R}(S)\big)_{ab}{}^{i}=0$. The modified curvature $\mathcal{R}(M)_{ab}{}^{cd}$ satisfies the following relations,

$$\begin{aligned}
\mathcal{R}(M)_{\mu\nu}{}^{ab}\, e^{\nu}{}_{b} &= \mathrm{i}\tilde{R}(A)_{\mu\nu} e^{\nu a} + \tfrac{3}{2} D\, e_{\mu}{}^{a}\,,\\
\tfrac{1}{4}\varepsilon_{ab}{}^{ef}\, \varepsilon^{cd}{}_{gh}\, \mathcal{R}(M)_{ef}{}^{gh} &= \mathcal{R}(M)_{ab}{}^{cd}\,,\\
\varepsilon_{cdea}\, \mathcal{R}(M)^{cde}{}_{b} &= \varepsilon_{becd}\, \mathcal{R}(M)_{a}{}^{ecd} = 2\tilde{R}(D)_{ab} = 2\mathrm{i}R(A)_{ab}\,.
\end{aligned} \tag{B.10}$$

The first of these relations corresponds to the third constraint given in (B.4), while the remaining equations follow from combining the curvature constraints with the Bianchi identities. Note that the modified curvature does not satisfy the pair exchange property; instead we have,

$$\mathcal{R}(M)_{ab}{}^{cd} = \mathcal{R}(M)^{cd}{}_{ab} + 4i\delta_{[a}^{[c}\, \tilde{R}(A)_{b]}{}^{d]}\,. \tag{B.11}$$

We now turn to the fermionic constraint given in (B.4) and its consequences for the modified curvature defined in (B.9). First we note that the constraint on $R(Q)_{\mu\nu}{}^{i}$ implies that this curvature is anti-selfdual, as follows from contracting the constraint with $\gamma^{\nu}\,\gamma_{ab}$,

$$\tilde{R}(Q)_{\mu\nu}{}^{i} \equiv \tfrac{1}{2}\, e\, \varepsilon_{\mu\nu}{}^{\rho\sigma} R(Q)_{\rho\sigma}{}^{i} = -R(Q)_{\mu\nu}{}^{i}\,. \tag{B.12}$$

Furthermore, combination of the Bianchi identity and the constraint on $R(Q)_{\mu\nu i}$ yields the following condition on the modified curvature $\mathcal{R}(S)_{ab}{}^{i}$,

$$\gamma^{a}\tilde{\mathcal{R}}(S)_{ab}{}^{i} = 2\, D^{a}\tilde{R}(Q)_{ab}{}^{i} = -2\, D^{a}R(Q)_{ab}{}^{i}\,. \tag{B.13}$$

This identity (upon contraction with $\gamma^{b}\gamma_{cd}$) leads to the following identity on the anti-selfdual part of $\mathcal{R}(S)_{ab}{}^{i}$,

$$\mathcal{R}(S)_{ab}{}^{i} - \tilde{\mathcal{R}}(S)_{ab}{}^{i} = 2\, \not{D}\big(R(Q)_{ab}{}^{i} + \tfrac{3}{4}\gamma_{ab}\chi^{i}\big)\,. \tag{B.14}$$

B.2 Chiral multiplets calculus and the covariant Weyl multiplet

In the following, we give explicit formulas of the product rules for two chiral supermultiplets and the correspondent fields identifications for functions of these multiplet.

The product of two chiral multiplets with components $\big(A, \Psi_i, B_{ij}, F_{ab}^{-}, \Lambda_i, C\big)$ and

$\big(a, \psi_i, b_{ij}, f_{ab}^-, \lambda_i, c\big)$, respectively, leads to the following decomposition,

$$\begin{aligned}
&\big(A, \Psi_i, B_{ij}, F_{ab}^-, \Lambda_i, C\big) \otimes \big(a, \psi_i, b_{ij}, f_{ab}^-, \lambda_i, c\big) = \\
&\quad \big(A\,a\,,\ A\,\psi_i + a\,\Psi_i, A\,b_{ij} + a\,B_{ij} - \bar\Psi_{(i}\psi_{j)}\,, \\
&\quad A\,f_{ab}^- + a\,F_{ab}^- - \tfrac{1}{4}\varepsilon^{ij}\bar\Psi_i\gamma_{ab}\psi_j\,, \\
&\quad A\,\lambda_i + a\,\Lambda_i - \tfrac{1}{2}\varepsilon^{kl}(B_{ik}\,\psi_l + b_{ik}\,\Psi_l) - \tfrac{1}{4}(F_{ab}^-\gamma^{ab}\psi_i + f_{ab}^-\gamma^{ab}\Psi_i)\,, \\
&\quad A\,c + a\,C - \tfrac{1}{2}\varepsilon^{ik}\varepsilon^{jl}B_{ij}\,b_{kl} + F_{ab}^-\,f^{-ab} + \varepsilon^{ij}(\bar\Psi_i\lambda_j + \bar\psi_i\Lambda_j)\big)\,.
\end{aligned} \tag{B.15}$$

Using this, one can show that a function $\mathcal{G}(\Phi)$ of chiral superfields Φ^I defines a chiral superfield, whose component fields take the following form,

$$\begin{aligned}
A|_{\mathcal{G}} &= \mathcal{G}(A)\,, \\
\Psi_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\,\Psi_i{}^I\,, \\
B_{ij}|_{\mathcal{G}} &= \mathcal{G}(A)_I\,B_{ij}{}^I - \tfrac{1}{2}\mathcal{G}(A)_{IJ}\,\bar\Psi_{(i}{}^I\Psi_{j)}{}^J\,, \\
F_{ab}^-|_{\mathcal{G}} &= \mathcal{G}(A)_I\,F_{ab}^{-I} - \tfrac{1}{8}\mathcal{G}(A)_{IJ}\,\varepsilon^{ij}\bar\Psi_i{}^I\gamma_{ab}\Psi_j{}^J\,, \\
\Lambda_i|_{\mathcal{G}} &= \mathcal{G}(A)_I\,\Lambda_i{}^I - \tfrac{1}{2}\mathcal{G}(A)_{IJ}\big[B_{ij}{}^I\varepsilon^{jk}\Psi_k{}^J + \tfrac{1}{2}F_{ab}^{-I}\gamma^{ab}\Psi_k{}^J\big] \\
&\quad + \tfrac{1}{48}\mathcal{G}(A)_{IJK}\,\gamma^{ab}\Psi_i{}^I\,\varepsilon^{jk}\bar\Psi_j{}^J\gamma_{ab}\Psi_k{}^K\,, \\
C|_{\mathcal{G}} &= \mathcal{G}(A)_I\,C^I - \tfrac{1}{4}\mathcal{G}(A)_{IJ}\big[B_{ij}{}^I B_{kl}{}^J\,\varepsilon^{ik}\varepsilon^{jl} - 2\,F_{ab}^{-I}F^{-abJ} + 4\,\varepsilon^{ik}\bar\Lambda_i{}^I\Psi_j{}^J\big]\,, \\
&\quad + \tfrac{1}{4}\mathcal{G}(A)_{IJK}\big[\varepsilon^{ik}\varepsilon^{jl}B_{ij}{}^I\Psi_k{}^J\Psi_l{}^K - \tfrac{1}{2}\varepsilon^{kl}\bar\Psi_k{}^I F_{ab}^{-J}\gamma^{ab}\Psi_l{}^K\big] \\
&\quad + \tfrac{1}{192}\mathcal{G}(A)_{IJKL}\,\varepsilon^{ij}\bar\Psi_i{}^I\gamma_{ab}\Psi_j{}^J\,\varepsilon^{kl}\bar\Psi_k{}^K\gamma_{ab}\Psi_l{}^L\,.
\end{aligned} \tag{B.16}$$

From this formula, the components of the $\log\Phi$ multiplet (4.1.12) are easily derived. Furthermore, after some Fierz rearrangements, presented in Appendix A one can show that the supersymmetry variations of these components are indeed given by (4.2.1).

As already explained, chiral multiplet of Weyl weight $w = 1$ can be consistently reduced by imposing a reality constraint. In section 3.3 we already introduced in detail the vector multiplet, which is a reduced chiral multiplets. We also hinted at another such multiplets, the covariant Weyl multiplet, satisfying the flat superspace constraint $D^{\alpha\beta}W_{\alpha\beta} = \bar D_{\dot\alpha\dot\beta}\bar W^{\dot\alpha\dot\beta}$. This equation can be easily lifted to curved superspace, by substituting the flat superspace derivatives D with their curved counterpart ∇. The resulting chiral superfield components are given by,

$$\begin{aligned}
A_{ab}|_{\mathbf{W}} &= T_{ab}{}^{ij}\,\varepsilon_{ij}\,,\\
\psi_{abi}|_{\mathbf{W}} &= 8\,\varepsilon_{ij}R(Q)^{i}_{ab}\,,\\
B_{abij}|_{\mathbf{W}} &= -8\,\varepsilon_{k(i}R(\mathcal{V})^{-}_{ab}{}^{k}{}_{j)}\,,\\
(F^{-}_{ab})^{cd}|_{\mathbf{W}} &= -8\,\mathcal{R}(M)^{-cd}_{ab}\\
\Lambda_{abi}|_{\mathbf{W}} &= 8\,(\mathcal{R}(S)^{-}_{abi}+\tfrac{3}{4}\,\gamma_{ab}\not{D}\chi_i)\\
C_{ab}|_{\mathbf{W}} &= 4\,D_{[a}\,D^{c}T_{b]cij}\,\varepsilon^{ij}-\text{dual}\,.
\end{aligned}\tag{B.17}$$

Note that the Weyl tensor is contained inside the highest independent component $(F^{-}_{ab})^{cd}$ through the $\mathcal{R}(M)^{-cd}_{ab}$ curvature, as it is explained in Appendix B. The Q- and S-supersymmetry variations for the first few components read,

$$\begin{aligned}
\delta T_{ab}{}^{ij} =&\, 8\,\bar{\epsilon}^{[i}R(Q)_{ab}{}^{j]}\,,\\
\delta R(Q)_{ab}{}^{i} =&\, -\tfrac{1}{2}\not{D}T_{ab}{}^{ij}\,\epsilon_j+R(\mathcal{V})^{-}{}_{ab}{}^{i}{}_{j}\,\epsilon^{j}-\tfrac{1}{2}\hat{R}(M)_{ab}{}^{cd}\,\gamma_{cd}\epsilon^{i}+\tfrac{1}{8}T_{cd}{}^{ij}\,\gamma^{cd}\gamma_{ab}\,\eta_j\,,\\
\delta R(\mathcal{V})^{-}{}_{ab}{}^{i}{}_{j} =&\, 2\bar{\epsilon}_j\not{D}R(Q)_{ab}{}^{i}-2\bar{\epsilon}^{i}\big(\mathcal{R}(S)^{-}_{abj}+\tfrac{3}{4}\gamma_{ab}\not{D}\chi_j\big)\\
&+\bar{\eta}_j(2R(Q)_{ab}{}^{i}+3\gamma_{ab}\chi^{i})-(\text{traceless})\,,\\
\delta\hat{R}(M)^{-cd}_{ab} =&\, \tfrac{1}{2}\bar{\epsilon}_i\not{D}\gamma^{cd}R(Q)_{ab}{}^{i}-\tfrac{1}{2}\bar{\epsilon}^{i}\gamma^{cd}\big(\mathcal{R}(S)^{-}_{abi}+\tfrac{3}{4}\gamma_{ab}\not{D}\chi_i\big)\\
&-\bar{\eta}_i\gamma_{ab}R(Q)^{cdi}-\tfrac{1}{2}\bar{\eta}_i\gamma^{cd}R(Q)_{ab}{}^{i}-\tfrac{3}{4}\bar{\eta}_i\gamma_{ab}\gamma^{cd}\chi^{i}\,.
\end{aligned}\tag{B.18}$$

By squaring the covariant Weyl multiplet $\mathbf{W}$ a scalar chiral multiplet with $w=2$ is obtained,

$$\begin{aligned}
A =&\, (T_{ab}{}^{ij}\varepsilon_{ij})^2\,,\\
\Psi_i =&\, 16\,\varepsilon_{ij}R(Q)^{j}_{ab}\,T^{klab}\,\varepsilon_{kl}\,,\\
B_{ij} =&\, -16\,\varepsilon_{k(i}R(\mathcal{V})^{k}{}_{j)ab}\,T^{lmab}\,\varepsilon_{lm}-64\,\varepsilon_{ik}\varepsilon_{jl}\,\bar{R}(Q)_{ab}{}^{k}\,R(Q)^{l\,ab}\,,\\
F^{-ab} =&\, -16\,\mathcal{R}(M)_{cd}{}^{ab}\,T^{klcd}\,\varepsilon_{kl}-16\,\varepsilon_{ij}\,\bar{R}(Q)^{i}_{cd}\gamma^{ab}R(Q)^{cd\,j}\,,\\
\Lambda_i =&\, 32\,\varepsilon_{ij}\,\gamma^{ab}R(Q)^{j}_{cd}\,\mathcal{R}(M)^{cd}{}_{ab}+16\,(\mathcal{R}(S)_{ab\,i}+3\gamma_{[a}D_{b]}\chi_i)\,T^{klab}\,\varepsilon_{kl}\\
&-64\,R(\mathcal{V})_{ab}{}^{k}{}_{i}\,\varepsilon_{kl}\,R(Q)^{ab\,l}\,,\\
C =&\, 64\,\mathcal{R}(M)^{-cd}{}_{ab}\,\mathcal{R}(M)^{-ab}_{cd}+32\,R(\mathcal{V})^{-ab\,k}{}_{l}\,R(\mathcal{V})^{-l}_{ab}{}_{k}\\
&-32\,T^{ab\,ij}\,D_a\,D^{c}T_{cb\,ij}+128\,\bar{\mathcal{R}}(S)^{ab}{}_{i}\,R(Q)_{ab}{}^{i}+384\,\bar{R}(Q)^{ab\,i}\gamma_a D_b\chi_i\,.
\end{aligned}\tag{B.19}$$

Both the covariant Weyl multiplet $\mathbf{W}$ and its square are functions of the curvatures of the local superconformal algebra. As expected from a reduced chiral multiplet, the highest components of the Weyl multiplet are not independent.

Appendix C

Chern-Simons terms in 5D - Noether potential

In this appendix we want to analyze the five-dimensional gauge and mixed Chern-Simons Lagrangians,

$$\mathcal{L}_{\text{CS}} = \sqrt{-g}\big(k_1\,\epsilon^{\mu\nu\rho\sigma\tau}A_\mu F_{\nu\rho}F_{\sigma\tau} + k_2\,\epsilon^{\mu\nu\rho\sigma\tau}A_\mu R_{\nu\rho\alpha\beta}R_{\sigma\tau}{}^{\alpha\beta}\big)\;. \tag{C.1}$$

The presence of an explicit gauge field in the Lagrangian breaks gauge covariance and strict gauge invariance, since $N^\mu \neq 0$. It follows that the procedure explained in section 2.2 will not be applicable anymore and more general methods must be used to calculate the electric charge, as we will show shortly. On the other hand, Chern-Simons terms are still described by diffeomorphism covariant Lagrangian densities, invariant always only up to total derivative (see section 2.5). We will give the final result and skip the non-trivial manipulations involved. For simplicity, we will also calculate the *covariantly* conserved vector current J^μ, but keeping in mind that the conserved current is actually given by the vector density $\sqrt{-g}J^\mu$.

C.1 Gauge invariance

The quantities of interest are the equation of motion for the gauge field A_μ (the dynamical equations do not play a role here) and the current J^μ,

$$\theta^\mu(A_\mu, \delta A_\mu) = 4\,k_1\,\epsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau}\,,$$
$$E^\mu_A = 3\,k_1\,F_{\nu\rho}F_{\sigma\tau} + k_2\,R_{\nu\rho\alpha\beta}R_{\sigma\tau}{}^{\alpha\beta}\,,$$
$$\delta_\xi \mathcal{L} = \nabla_\mu N^\mu = \nabla_\mu\Big[\xi\epsilon^{\mu\nu\rho\sigma\tau}\Big(k_1\,F_{\nu\rho}F_{\sigma\tau} + k_2\,R_{\nu\rho\alpha\beta}R_{\sigma\tau}{}^{\alpha\beta}\Big)\Big]\,.$$

Now, to obtain an equation similar to (2.2.8) for Chern-Simons terms turns out to be a quite delicate issue. Here we will only sketch the reasoning, referring to [94] for further details. Typically, the invariance under abelian gauge transformations is realized strictly, e.g. the QED theory treated in section (2.2), so that $N^\mu = 0$. As a consequence, the conserved current will be proportional to the symmetry variations of the fields, which means that it will vanish on a symmetric background. This is not the case for Chern-Simons Lagrangians since they are gauge invariant upto a total derivative. As it turns out, one can still obtain a current which is proportional to the gauge field symmetry variations $\partial_\nu\xi$, even though $N^\mu \neq 0$ by adding specific improvement terms. At the same time, one considers suitable linear combinations of the equation of motion to write an equality linking the conserved current and the Noether potential $Q_{\mu\nu}$ (see for instance (2.2.8)). This means that the Noether potential itself will satisfy an on-shell conservation law of the form $\partial_\nu\,Q^{\mu\nu} = 0$, which after integration leads to an expression for the conserved electric charge (2.2.13). Now for the case at hand, the improvement terms to be added to the current are given by,

$$\epsilon^{\mu\nu\rho\sigma\tau}\nabla_\nu\Big[2\,k_1\,\xi A_\rho F_{\sigma\tau} - 4\,k_2\,\xi\big(\Gamma_{\rho\beta}{}^{\alpha}\partial_\sigma\Gamma_{\tau\alpha}{}^{\beta} + \tfrac{2}{3}\Gamma_{\rho\beta}{}^{\alpha}\Gamma_{\sigma\gamma}{}^{\beta}\Gamma_{\tau\alpha}{}^{\gamma}\big)\Big]\,, \tag{C.2}$$

and the Noether potential reads,

$$Q^{\mu\nu} = 6\,k_1\,\xi\epsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau} - 4\,k_2\,\xi\big(\Gamma_{\rho\beta}{}^{\alpha}\partial_\sigma\Gamma_{\tau\alpha}{}^{\beta} + \tfrac{2}{3}\Gamma_{\rho\beta}{}^{\alpha}\Gamma_{\sigma\gamma}{}^{\beta}\Gamma_{\tau\alpha}{}^{\gamma}\big)\,. \tag{C.3}$$

It should not come as a surprise that the same result can be obtained by simply rewriting the equation of motion E^μ_A as a total derivative term. The difference resides only into a constant, ξ, which can be fixed by requiring some matching condition. In short, since the current vanishes for symmetric backgrounds, $\partial_\mu\xi = 0$, one obtains the identity $\xi\,E^\mu_A = \partial_\nu\,Q^{\mu\nu}$.

C.2 Diffeomorphism invariance

The Chern-Simons Lagrangians are both covariant under general coordinate transformations. No *ad hoc* rules are needed in this case because we already explained the procedure to be followed when dealing with covariant Lagrangian densities. On the other hand, the calculations are far more involved and tricky if compared to the simple gauge invariance of the previous section. We only note the Palatini identity, from which many other useful identities can be derived,

$$\delta R^{\beta}{}_{\mu\rho\sigma} = \nabla_\rho \delta\Gamma_{\sigma\mu}{}^{\beta} - \nabla_\sigma \delta\Gamma_{\rho\mu}{}^{\beta} \, .$$

By varying the Lagrangian, one obtains the following results

$$\begin{aligned} \delta\mathcal{L} &= \sqrt{-g}(\nabla_\mu \theta^\mu + E_g^{\alpha\beta}\delta g_{\alpha\beta}) \, , \\ \theta^\mu &= 4\, k_1\, \varepsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau} \delta A_\nu + 4\, k_2\, \varepsilon^{\mu\alpha\beta\sigma\tau} A_\tau R_{\alpha\beta}{}^{\nu\rho}\, \nabla_\rho \delta g_{\sigma\nu} + 4\, k_2\, \nabla_\rho \big(\varepsilon^{\alpha\beta\rho\sigma\tau} A_\tau R_{\alpha\beta}{}^{\mu\nu}\big)\delta g_{\sigma\nu} \, , \\ E_g^{\alpha\beta} &= 2\, k_2\, \varepsilon^{\mu\nu\rho\sigma(\alpha}\big[\nabla_\lambda F_{\mu\nu} R_{\rho\sigma}{}^{\lambda|\beta)} + 2\, F_{\mu\nu}\, \nabla_\rho R_\sigma{}^{\beta)}\big] \, . \end{aligned}$$

As usual for scalar densities, $\delta_\xi \mathcal{L}_{\text{CS}} = \nabla_\mu N^\mu = \partial_\mu N^\mu = \partial_\mu(\xi^\mu \mathcal{L}_{\text{CS}})$. Now we should add a linear combination of the equations of motion, E_A^μ and $E_g^{\alpha\beta}$, to the current to extract a total derivative term from it. The coefficients of such combination are found after few easy but cumbersome manipulations. The complete expression reads,

$$J^\mu + 2\, E_g^{\mu\nu}\, \xi_\nu + (\xi \cdot A) E_A^\mu = \nabla_\nu Q^{\mu\nu} \, , \tag{C.4}$$

where the Noether potential under diffeomorphism is given by

$$\begin{aligned} Q^{\mu\nu} = k_2 & \Big(4\, \varepsilon^{\mu\nu\rho\alpha\beta} A_\rho R_{\alpha\beta}{}^{\sigma\tau} \nabla_\tau \xi_\sigma + 2\, \varepsilon^{\mu\rho\sigma\alpha\beta} F_{\rho\sigma} R_{\alpha\beta}{}^{\nu\tau} \xi_\tau + 4\, \varepsilon^{\rho\sigma\alpha\beta(\nu} F_{\rho\sigma} R_{\alpha\beta}{}^{\tau)\mu} \xi_\tau \Big) \\ & + 4\, k_1 (\xi \cdot A)\, \varepsilon^{\mu\nu\rho\sigma\tau} A_\rho F_{\sigma\tau} \, . \end{aligned} \tag{C.5}$$

Bibliography

[1] K. Schwarzschild, "Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit", *Sitzungsberichte der Kniglich Preussischen Akademie der Wissenschaften*, **1**, 424, (1916).

[2] J. Droste, "The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field", *Koninklijke Nederlandse Akademie van Wetenschappen Proceedings Series B Physical Sciences*, **19**, 197–215, (1917).

[3] J. M. Bardeen, B. Carter, and S. Hawking, "The Four laws of black hole mechanics", *Commun. Math. Phys.*, **31**, 161–170, (1973).

[4] J. Bekenstein, *Baryon Number, Entropy, and Black Hole Physics.* PhD thesis, (1972).

[5] S. Hawking, "Black hole explosions", *Nature*, **248**, 30–31, (1974).

[6] A. Strominger and C. Vafa, "Microscopic origin of the Bekenstein-Hawking entropy", *Phys. Lett.*, **B379**, 99–104, (1996), hep-th/9601029.

[7] J. M. Maldacena, A. Strominger, and E. Witten, "Black hole entropy in M theory", *JHEP*, **9712**, 002, (1997), hep-th/9711053.

[8] K. Behrndt, G. Lopes Cardoso, B. de Wit, D. Lüst, T. Mohaupt, and W. A. Sabra, "Higher order black hole solutions in N=2 supergravity and Calabi-Yau string backgrounds", *Phys.Lett.*, **B429**, 289–296, (1998), hep-th/9801081.

[9] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, "Corrections to macroscopic supersymmetric black hole entropy", *Phys.Lett.*, **B451**, 309–316, (1999), hep-th/9812082.

[10] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, "Deviations from the area law for supersymmetric black holes", *Fortsch. Phys.*, **48**, 49–64, (2000), hep-th/9904005.

[11] G. Lopes Cardoso, B. de Wit, J. Kappeli, and T. Mohaupt, "Stationary BPS solutions in N=2 supergravity with R^2 interactions", *JHEP*, **0012**, 019, (2000), hep-th/0009234.

[12] R. M. Wald, "Black hole entropy is the Noether charge", *Phys. Rev.*, **D48**, 3427–3431, (1993), gr-qc/9307038.

[13] S. Carrol, *Spacetime and geometry: an introduction to general relativity.* Addison-Wesley Longman, (2003).

[14] H. Reissner, "Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie", *Annalen der Physik*, **50**, 106–120, (1916).

[15] G. Nordström, "On the Energy of the Gravitational Field in Einstein's Theory", *Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk.*, **26**, 12011208, (1918).

[16] R. M. Wald, *General Relativity.* University of Chicago Press, (1984).

[17] W. Israel, "Event horizons in static vacuum space-times", *Phys. Rev.*, **164**, 1776–1779, (1967).

[18] W. Israel, "Event horizons in static electrovac space-times", *Commun. Math. Phys.*, **8**, 245–260, (1968).

[19] R. H. Price, "Nonspherical perturbations of relativistic gravitational collapse. 1. Scalar and gravitational perturbations", *Phys. Rev.*, **D5**, 2419–2438, (1972).

[20] R. H. Price, "Nonspherical Perturbations of Relativistic Gravitational Collapse. II. Integer-Spin, Zero-Rest-Mass Fields", *Phys. Rev.*, **D5**, 2439–2454, (1972).

[21] B. Carter, "Axisymmetric Black Hole Has Only Two Degrees of Freedom", *Phys. Rev. Lett.*, **26**, 331–333, (1971).

[22] D. Robinson, "Uniqueness of the Kerr black hole", *Phys. Rev. Lett.*, **34**, 905–906, (1975).

[23] E. Poisson, *An advanced course in general relativity.* Cambridge University Press, (2004).

[24] J. Gervais and B. Sakita, "Field Theory Interpretation of Supergauges in Dual Models", *Nucl. Phys.*, **B34**, 632–639, (1971).

[25] Y. Golfand and E. Likhtman, "Extension of the Algebra of Poincare Group Generators and Violation of P-Invariance", *JETP Lett.*, **13**, 323–326, (1971).

[26] D. Volkov and V. Akulov, "Possible universal neutrino interaction", *JETP Lett.*, **16**, 438–440, (1972).

[27] S. Ferrara, J. Wess, and B. Zumino, "Supergauge Multiplets and Superfields", *Phys.Lett.*, **B51**, 239, (1974).

[28] A. Salam and J. Strathdee, "Supergauge Transformations", *Nucl. Phys.*, **B76**, 477–482, (1974).

[29] A. Salam and J. Strathdee, "On Superfields and Fermi-Bose Symmetry", *Phys. Rev.*, **D11**, 1521–1535, (1975).

[30] S. Gates, M. T. Grisaru, M. Roček, and W. Siegel, "Superspace Or One Thousand and One Lessons in Supersymmetry", *Front. Phys.*, **58**, 1–548, (1983), hep-th/0108200.

[31] J. Wess and J. Bagger, *Supersymmetry and supergravity.* Princeton, USA: Univ. Pr., (1992).

[32] A. Bilal, "Introduction to supersymmetry", (2001), hep-th/0101055.

[33] R. Gilmore, *Lie group, Lie algebras and some of their application.* Dover Publications, (1974).

[34] S. R. Coleman and J. Mandula, "All Possible Symmetries of the S Matrix", *Phys. Rev.*, **159**, 1251–1256, (1967).

[35] J. Wess and B. Zumino, "A Lagrangian Model Invariant Under Supergauge Transformations", *Phys. Lett.*, **B49**, 52, (1974).

[36] R. Haag, J. T. Lopuszański, and M. Sohnius, "All Possible Generators of Supersymmetries of the S Matrix", *Nucl. Phys.*, **B88**, 257, (1975).

[37] D. Z. Freedman and A. Van Proeyen, *Supergravity.* Cambdrige University Press, (2012).

[38] J. Polchinski, "Effective field theory and the Fermi surface", (1992), hep-th/9210046.

[39] A. V. Manohar, "Effective field theories", (1996), hep-ph/9606222.

[40] C. P. Burgess, "Introduction to Effective Field Theory", *Ann. Rev. Nucl. Part. Sci.*, **57**, 329–362, (2007), hep-th/0701053.

[41] A. Pich, "Effective field theory: Course", (1998), hep-ph/9806303.

[42] W. Heisenber and H. H. Euler, "Folgerungen aus der Diracschen Theorie des Positrons", *Z. Phys.*, **98**, 714, (1936).

[43] B. de Wit, N. Hari Dass, S. Katmadas, and I. Lodato, "Noether's theorem and modern perspectives", *to appear.*

[44] M. Ostrogradski, "Mémoirs sur les équations differentielles relatives au probléme des isopèrimétre", *Mem. Acad. St. Petersbourg*, **IV 4**, 385, (1850).

[45] A. Pais and G. E. Uhlenbeck, "On field theories with non-localized action", *Phys. Rev.*, **79**, 145, (1950).

[46] E. Witten, "Interacting Field Theory of Open Superstrings", *Nucl.Phys.*, **B276**, 291, (1986).

[47] C. Crnković and E. Witten, "Covariant description of canonical formalism in geometric theories", (1986). *Three hundred years of gravitation*, eds. Hawking, S.W. and Israel, W.

[48] G. J. Zuckerman, "Action principles and global geometry", *Conf. Proc.*, **C8607214**, 259–284, (1986).

[49] J. Lee and R. M. Wald, "Local symmetries and constraints", *J.Math.Phys.*, **31**, 725–743, (1990).

[50] A. Ashtekar, L. Bombelli, and O. Reula, "The covariant phase space of asymptotically flat gravitational fields", (1990).

[51] S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry, Vol I.* John Wiley and Sons, (1963). Chapter I.

[52] G. Barnich, M. Henneaux, and C. Schomblond, "On the covariant description of the canonical formalism", *Phys.Rev.*, **D44**, 939–941, (1991).

[53] P. A. M. Dirac, *Lectures on Quantum Field Theory.* Academic Press, New York, (1966). Belfer Graduate School of Science, Yeshiva University, New York.

[54] R. M. Wald, "On identically closed forms locally constructed from a field", *J. Math. Phys.*, **31**, 2378, (1990).

[55] S. Weinberg, *Dynamics and algebraic symmetries*, vol. 1. eds. S. Deser, M.T. Grisaru and H. Pendleton (MIT Press), (1970).

[56] V. Iyer and R. M. Wald, "Some properties of Noether charge and a proposal for dynamical black hole entropy", *Phys. Rev.*, **D50**, 846–864, (1994), gr-qc/9403028.

[57] B. de Wit, *Introduction to supergravity.* World Scientific, Singapore, (1984). Proceedings, *Supersymmetry and Supergravity '84*, Trieste, eds. B. de Wit, P. Fayet and P. van Nieuwenhuizen.

[58] M. Nakahara, *Geometry, topology and physics.* Taylor & Francis, (2003).

[59] R. Arnowitt, S. Deser, and C. W. Misner, "Dynamical Structure and Definition of Energy in General Relativity", *Phys. Rev.*, **116**, 1322–1330, (1959).

[60] A. Komar, "Positive-Definite Energy Density and Global Consequences for General Relativity", *Phys. Rev.*, **129**, 1873–1876, (1963).

[61] M. Kaku, P. Townsend, and P. van Nieuwenhuizen, "Gauge Theory of the Conformal and Superconformal Group", *Phys.Lett.*, **B69**, 304–308, (1977).

[62] S. Ferrara, M. Kaku, P. Townsend, and P. van Nieuwenhuizen, "Gauging the Graded Conformal Group with Unitary Internal Symmetries", *Nucl.Phys.*, **B129**, 125, (1977).

[63] M. Kaku, P. Townsend, and P. van Nieuwenhuizen, "Superconformal Unified Field Theory", *Phys.Rev.Lett.*, **39**, 1109, (1977).

[64] E. Stueckelberg, "Interaction energy in electrodynamics and in the field theory of nuclear forces", *Helv.Phys.Acta*, **11**, 225–244, (1938).

[65] B. de Wit, S. Katmadas, and M. van Zalk, "New supersymmetric higher-derivative couplings: Full N=2 superspace does not count!", *JHEP*, **1101**, 007, (2011), 1010.2150.

[66] B. de Wit, J. van Holten, and A. Van Proeyen, "Transformation Rules of N=2 Supergravity Multiplets", *Nucl.Phys.*, **B167**, 186, (1980).

[67] B. de Wit, J. van Holten, and A. Van Proeyen, "Structure of N=2 Supergravity", *Nucl.Phys.*, **B184**, 77, (1981).

[68] B. de Wit, P. G. Lauwers, and A. Van Proeyen, "Lagrangians of N=2 Supergravity - Matter Systems", *Nucl.Phys.*, **B255**, 569, (1985).

[69] B. de Wit and A. Van Proeyen, "Potentials and Symmetries of General Gauged N=2 Supergravity: Yang-Mills Models", *Nucl.Phys.*, **B245**, 89, (1984).

[70] D. Butter, B. de Wit, S. M. Kuzenko, and I. Lodato, "New higher-derivative invariants in N=2 supergravity and the Gauss-Bonnet term", *JHEP*, **1312**, 062, (2013), 1307.6546.

[71] B. de Wit, R. Philippe, and A. Van Proeyen, "The Improved Tensor Multiplet in $N = 2$ Supergravity", *Nucl.Phys.*, **B219**, 143, (1983).

[72] M. de Roo, J. W. van Holten, B. de Wit, and A. Van Proeyen, "Chiral Superfields in $N = 2$ Supergravity", *Nucl.Phys.*, **B173**, 175, (1980).

[73] R. Firth and J. Jenkins, "Supersymmetry with Isospin", *Nucl.Phys.*, **B85**, 525, (1975).

[74] S. Ferrara and P. van Nieuwenhuizen, "Tensor Calculus for Supergravity", *Phys.Lett.*, **B76**, 404, (1978).

[75] D. Butter, "N=2 Conformal Superspace in Four Dimensions", *JHEP*, **1110**, 030, (2011), hep-th/1103.5914.

[76] B. de Wit, P. Lauwers, R. Philippe, S. Su, and A. Van Proeyen, "Gauge and Matter Fields Coupled to N=2 Supergravity", *Phys.Lett.*, **B134**, 37, (1984).

[77] E. Fradkin and M. A. Vasiliev, "Minimal set of auxiliary fields in SO(2) extended supergravity", *Phys.Lett.*, **B85**, 47–51, (1979).

[78] M. Henningson, "Extended superspace, higher derivatives and SL(2,Z) duality", *Nucl.Phys.*, **B458**, 445–455, (1996), hep-th/9507135.

[79] B. de Wit, M. T. Grisaru, and M. Roček, "Nonholomorphic corrections to the one loop N=2 superYang-Mills action", *Phys.Lett.*, **B374**, 297–303, (1996), hep-th/9601115.

[80] I. Buchbinder, S. Kuzenko, and A. A. Tseytlin, "On low-energy effective actions in N=2, N=4 superconformal theories in four-dimensions", *Phys.Rev.*, **D62**, 045001, (2000), hep-th/9911221.

[81] A. Banin, I. Buchbinder, and N. Pletnev, "On low-energy effective action in N=2 super Yang-Mills theories on non-Abelian background", *Phys.Rev.*, **D66**, 045021, (2002), hep-th/0205034.

[82] P. C. Argyres, A. M. Awad, G. A. Braun, and F. P. Esposito, "Higher derivative terms in N=2 supersymmetric effective actions", *JHEP*, **0307**, 060, (2003), hep-th/0306118.

[83] A. S. Galperin, E. A. Ivanov, V. I. Ogievetsky, and E. S. Sokatchev, *Harmonic Superspace.* Cambridge University Press, (2001).

[84] S. M. Kuzenko, U. Lindström, M. Roček, and G. Tartaglino-Mazzucchelli, "4D N=2 supergravity and projective superspace", *JHEP*, **0809**, 051, (2008), hep-th/0805.4683.

[85] S. M. Kuzenko, U. Lindström, M. Roček, and G. Tartaglino-Mazzucchelli, "On conformal supergravity and projective superspace", *JHEP*, **0908**, 023, (2009), hep-th/0905.0063.

[86] E. Bergshoeff, M. de Roo, and B. de Wit, "Extended Conformal Supergravity", *Nucl.Phys.*, **B182**, 173, (1981).

[87] B. de Wit and F. Saueressig, "Off-shell N=2 tensor supermultiplets", *JHEP*, **0609**, 062, (2006), hep-th/0606148.

[88] F. Moura, "Four-dimensional 'old minimal' N=2 supersymmetrization of R^4", *JHEP*, **0307**, 057, (2003), hep-th/0212271.

[89] W. Chemissany, S. Ferrara, R. Kallosh, and C. Shahbazi, "N=2 Supergravity Counterterms, Off and On Shell", *JHEP*, **1212**, 089, (2012), 1208.4801.

[90] D. Butter and S. M. Kuzenko, "New higher-derivative couplings in 4D N = 2 supergravity", *JHEP*, **1103**, 047, (2011), 1012.5153.

[91] N. Banerjee, B. de Wit, and S. Katmadas, "The Off-Shell 4D/5D Connection", *JHEP*, **1203**, 061, (2012), 1112.5371.

[92] K. Hanaki, K. Ohashi, and Y. Tachikawa, "Supersymmetric Completion of an R^2 term in Five-dimensional Supergravity", *Prog.Theor.Phys.*, **117**, 533, (2007), hep-th/0611329.

[93] A. Sen, "Entropy function for heterotic black holes", *JHEP*, **0603**, 008, (2006), hep-th/0508042.

[94] B. de Wit and S. Katmadas, "Near-Horizon Analysis of D=5 BPS Black Holes and Rings", *JHEP*, **1002**, 056, (2010), 0910.4907.

[95] A. Sen, "Logarithmic Corrections to N=2 Black Hole Entropy: An Infrared Window into the Microstates", *Gen. Relat. Gravit*, **44**, 1207–1266, (2012), hep-th/1108.3842.

[96] B. de Wit, *Formulations of N=2 supergravity theories.* Ettore Majorana Int. Science Series, Phys. Sciences Vol. 7, Plenum, (1980). *Unification of the Fundamental Particle Interactions*, Europhysics Study Conference, Erice, Italy,eds. S. Ferrara, J, Ellis and P. van Nieuwenhuizen.

[97] B. de Wit, B. Kleijn, and S. Vandoren, " Superconformal hypermultiplets", *Nucl. Phys. B*, **568**, 475, (2000), hep-th/9909228.

[98] B. de Wit, M. Roček, and S. Vandoren, "Hypermultiplets, hyperkähler cones and quaternion-Kähler geometry", *JHEP*, **0102**, 039, (2001), hep-th/0101161.

[99] J. Bagger and E. Witten, "Matter couplings in N=2 supergravity", *Nucl. Phys. B*, **222**, 1, (1983).

[100] G. Lopes Cardoso, B. de Wit, and T. Mohaupt, "Macroscopic entropy formulae and nonholomorphic corrections for supersymmetric black holes", *Nucl.Phys.*, **B567**, 87–110, (2000), hep-th/9906094.

[101] S. Ferrara, R. Kallosh, and A. Strominger, "N=2 extremal black holes", *Phys.Rev.*, **D52**, 5412–5416, (1995), hep-th/9508072.

[102] S. Ferrara and R. Kallosh, "Supersymmetry and attractors", *Phys.Rev.*, **D54**, 1514–1524, (1996), hep-th/9602136.

[103] T. Mohaupt, "Black hole entropy, special geometry and strings", *Fortsch. Phys.*, **49**, 3–161, (2001), hep-th/0007195.

[104] A. Dabholkar, "Exact counting of black hole microstates", *Phys.Rev.Lett.*, **94**, 241301, (2005), hep-th/0409148.

[105] P. Kraus and F. Larsen, "Attractors and black rings", *Phys.Rev.*, **D72**, 024010, (2005), hep-th/0503219.

[106] A. Sen, "Black hole entropy function and the attractor mechanism in higher derivative gravity", *JHEP*, **0509**, 038, (2005), hep-th/0506177.

[107] K. Goldstein, N. Iizuka, R. P. Jena, and S. P. Trivedi, "Non-supersymmetric attractors", *Phys.Rev.*, **D72**, 124021, (2005), hep-th/0507096.

[108] R. Kallosh, N. Sivanandam, and M. Soroush, "The Non-BPS black hole attractor equation", *JHEP*, **0603**, 060, (2006), hep-th/0602005.

[109] G. Cardoso, B. de Wit, and S. Mahapatra, "Black hole entropy functions and attractor equations", *JHEP*, **0703**, 085, (2007), hep-th/0612225.

[110] M. Alishahiha and H. Ebrahim, “Non-supersymmetric attractors and entropy function”, *JHEP*, **0603**, 003, (2006), hep-th/0601016.

[111] B. Sahoo and A. Sen, “BTZ black hole with Chern-Simons and higher derivative terms”, *JHEP*, **0607**, 008, (2006), hep-th/0601228.

[112] B. Chandrasekhar, S. Parvizi, A. Tavanfar, and H. Yavartanoo, “Non-supersymmetric attractors in R^2 gravities”, *JHEP*, **0608**, 004, (2006), hep-th/0602022.

[113] B. Sahoo and A. Sen, “Higher derivative corrections to non-supersymmetric extremal black holes in N=2 supergravity”, *JHEP*, **0609**, 029, (2006), hep-th/0603149.

[114] A. Ghodsi, “R^4 corrections to D1D5p black hole entropy from entropy function formalism”, *Phys.Rev.*, **D74**, 124026, (2006), hep-th/0604106.

[115] M. Alishahiha and H. Ebrahim, “New attractor, entropy function and black hole partition function”, *JHEP*, **0611**, 017, (2006), hep-th/0605279.

[116] S. Ferrara and M. Gunaydin, “Orbits and Attractors for N=2 Maxwell-Einstein Supergravity Theories in Five Dimensions”, *Nucl.Phys.*, **B759**, 1–19, (2006), hep-th/0606108.

[117] D. Astefanesei, H. Nastase, H. Yavartanoo, and S. Yun, “Moduli flow and non-supersymmetric AdS attractors”, *JHEP*, **0804**, 074, (2008), 0711.0036.

[118] P. Kraus and F. Larsen, “Holographic gravitational anomalies”, *JHEP*, **0601**, 022, (2006), hep-th/0508218.

[119] J. B. Gutowski and H. S. Reall, “Supersymmetric AdS(5) black holes”, *JHEP*, **0402**, 006, (2004), hep-th/0401042.

[120] N. Banerjee, S. Dutta, and I. Lodato, “The fate of flat directions in higher derivative gravity”, *JHEP*, **1305**, 027, (2013), 1301.6773.

[121] J. P. Gauntlett and J. B. Gutowski, “All supersymmetric solutions of minimal gauged supergravity in five-dimensions”, *Phys.Rev.*, **D68**, 105009, (2003), hep-th/0304064.

[122] J. Grover, J. Gutowski, G. Papadopoulos, and W. Sabra, “Index Theory and Supersymmetry of 5D Horizons”, (2013), 1303.0853.

[123] M. Gunaydin, G. Sierra, and P. Townsend, “Gauging the d = 5 Maxwell-Einstein Supergravity Theories: More on Jordan Algebras”, *Nucl.Phys.*, **B253**, 573, (1985).

[124] D. Astefanesei and Y. K. Srivastava, “CFT Duals for Attractor Horizons”, *Nucl.Phys.*, **B822**, 283–300, (2009), 0902.4033.

[125] S. Cremonini, K. Hanaki, J. T. Liu, and P. Szepietowski, “Black holes in five-dimensional gauged supergravity with higher derivatives”, *JHEP*, **0912**, 045, (2009), 0812.3572.

[126] R. C. Myers, M. F. Paulos, and A. Sinha, “Holographic Hydrodynamics with a Chemical Potential”, *JHEP*, **0906**, 006, (2009), 0903.2834.

[127] D. Astefanesei, N. Banerjee, and S. Dutta, “Near horizon data and physical charges of extremal AdS black holes”, *Nucl.Phys.*, **B853**, 63–79, (2011), 1104.4121.

[128] A. Castro, J. L. Davis, P. Kraus, and F. Larsen, “5D Black Holes and Strings with Higher Derivatives”, *JHEP*, **0706**, 007, (2007), hep-th/0703087.

[129] A. Castro, J. L. Davis, P. Kraus, and F. Larsen, “Precision Entropy of Spinning Black Holes”, *JHEP*, **0709**, 003, (2007), 0705.1847.

[130] K. Behrndt, A. H. Chamseddine, and W. Sabra, “BPS black holes in N=2 five-dimensional AdS supergravity”, *Phys.Lett.*, **B442**, 97–101, (1998), hep-th/9807187.

[131] K. Behrndt, M. Cvetič, and W. Sabra, “Nonextreme black holes of five-dimensional N=2 AdS supergravity”, *Nucl.Phys.*, **B553**, 317–332, (1999), hep-th/9810227.

[132] M. Cvetič, M. Duff, P. Hoxha, J. T. Liu, H. Lu, *et al.*, “Embedding AdS black holes in ten-dimensions and eleven-dimensions”, *Nucl.Phys.*, **B558**, 96–126, (1999), hep-th/9903214.

[133] A. Buchel, “Higher derivative corrections to near-extremal black holes in type IIB supergravity”, *Nucl.Phys.*, **B750**, 45–72, (2006), hep-th/0604167.

[134] A. Sen, “Black Hole Entropy Function, Attractors and Precision Counting of Microstates”, *Gen.Rel.Grav.*, **40**, 2249–2431, (2008), 0708.1270.

[135] S. de Haro, A. Sinkovics, and K. Skenderis, “On a supersymmetric completion of the R^4 term in IIB supergravity”, *Phys.Rev.*, **D67**, 084010, (2003), hep-th/0210080.

[136] A. Rajaraman, “On a supersymmetric completion of the R^4 term in type IIB supergravity”, *Phys.Rev.*, **D72**, 125008, (2005), hep-th/0505155.

[137] M. B. Green and S. Sethi, "Supersymmetry constraints on type IIB supergravity", *Phys.Rev.*, **D59**, 046006, (1999), hep-th/9808061.

[138] M. B. Green and C. Stahn, "D3-branes on the Coulomb branch and instantons", *JHEP*, **0309**, 052, (2003), hep-th/0308061.

[139] M. B. Green, K. Peeters, and C. Stahn, "Superfield integrals in high dimensions", *JHEP*, **0508**, 093, (2005), hep-th/0506161.

[140] M. F. Paulos, "Higher derivative terms including the Ramond-Ramond five-form", *JHEP*, **0810**, 047, (2008), 0804.0763.

[141] M. B. Green and M. Gutperle, "Effects of D instantons", *Nucl.Phys.*, **B498**, 195–227, (1997), hep-th/9701093.

Samenvatting

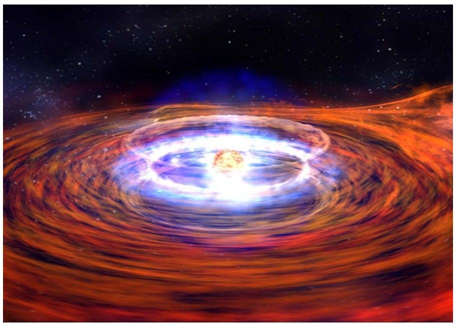
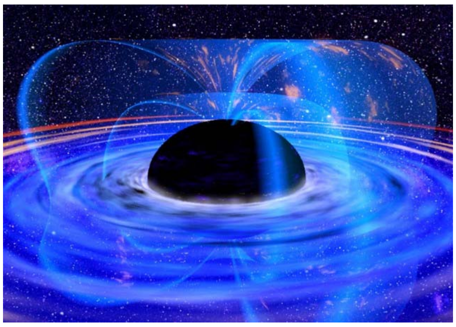
Dit proefschrift is gebaseerd op de resultaten behaald in de jaren van mijn doctoraalonderzoek en zijn, derhalve, praktisch onmogelijk te begrijpen voor mensen zonder een specifieke achtergrond in mijn werkgebied. Daarom heb ik besloten om een heel basale samenvatting te schrijven, in de hoop om op zijn minst de motivatie en de fundamentele logica achter mijn werk te verduidelijken. Hier komt het!

Elk natuurlijk verschijnsel kan, onafhankelijk van haar karakteristieke lengte- en energieschalen, in beginsel verklaard worden met de vier natuurkrachten, of interacties: de elektromagnetische kracht, de zwakke kernkracht, de sterke kernkracht en de zwaartekracht. De eerste drie interacties, elektromagnetisme en zwakke en sterke kernkracht, worden geassocieerd met de microscopische wereld van elementaire deeltjes en zijn samengevoegd in één enkele theorie: het Standaard Model. Deze theorie is gebaseerd op quantummechanica die de fysica beschrijft van zeer kleine objecten met zeer kleine massa's. Quantummechanica beschrijft bijvoorbeeld de structuur van atomen en moleculen, elektromagnetische en thermische straling (licht en warmte) en veel meer verschijnselen in de microscopische wereld. Aan de andere kant is er de theorie die de vierde interactie beschrijft, de zwaartekracht: Einsteins algemene relativiteit. Deze theorie beschrijft de dynamica van zeer zware objecten over heel grote schalen, zoals sterren en melkwegstelsels. In het ideale geval zou een theorie van alles, een unificatie van zwaartekracht en de drie quantumkrachten, de oorsprong van het heelal kunnen beschrijven waar afstanden heel klein maar, tegelijkertijd, energiewaardes enorm groot waren (de oerknal heeft in één explosie het equivalent van de totale energie aanwezig in ons universum vrijgemaakt). Los van het ongelooflijke belang die zo'n ontdekking zou hebben, is het een typisch natuurwetenschappelijke houding om te proberen om algemene theorieën op te stellen die zo veel mogelijk verschijnselen kunnen beschrijven (omgekeerd kan een theorie die niks kan verklaren niet als wetenschappelijk beschouwd worden). Echter, quantummechanica en algemene relativiteit zijn geldig in tegengestelde regimes van afstanden en massa's en dat ligt aan de basis van alle moeilijkheden om ze te verenigen in één unieke theorie

van quantumzwaartekracht.
Het is natuurlijk onmogelijk om de energieschalen typisch voor de oerknal te bereiken in een experimentele opstelling. Dus ook al zijn zowel het Standaard Model als algemene relativiteit getest tot op hoge precisie, hun vereniging zou uiteindelijk alleen wiskundig gezien moeten kloppen (wiskundig consistent moeten zijn) gezien het gebrek aan een direct experiment die haar zou kunnen testen.
In de puur theoretische aanpak om zo'n verenigde theorie van quantumzwaartekracht te vinden zou het zeer goed van pas komen om objecten te bestuderen waarvan de beschrijving gebaseerd is op zowel algemene relativiteit als quantummechanica. De vraag is dus: bestaan zulke objecten in de natuur?

Zwarte gaten
De eerste theorie die de gravitatiekracht beschrijft werd opgesteld door Newton. Zij is zeer precies als het bestudeerde object beweegt met een snelheid die laag is vergeleken met de lichtsnelheid. Later, in het begin van de twintigste eeuw, formuleerde Einstein zijn algemene relativiteit, een relativistische zwaartekrachtstheorie, die de zwaartekrachtsinteracties beschrijft van objecten waarvan de snelheid vergelijkbaar kan zijn met de lichtsnelheid. Algemene relativiteit werd in verschillende experimenten geverifieerd en voorspelde tegelijkertijd veel fascinerende verschijnselen, zoals het bestaan van zogenaamde *zwarte gaten*. Zwarte gaten, zoals beschreven door Einsteins theorie, worden gevormd door zware sterren die ineenstorten wanneer ze bezwijken onder hun eigen gewicht. Van vele, zo niet alle melkwegstelsels weten we dat ze een superzwaar zwart gat bevatten in hun centrum. Een zwart gat wordt gekarakteriseerd door zijn extreem hoge massadichtheid (een theelepeltje van een zwart gat kan vele tonnen wegen) en oefent daarom een sterke gravitationele aantrekking uit op alles om hem heen. Om precies te zijn, er bestaat zelfs een "point of no return" voor welk object dan ook (zelfs licht) dat in de buurt van een zwart gat beweegt, de zogenaamde *horizon* (zie afbeelding R1). De horizon kan worden voorgesteld als een immateriële schil (zie afbeelding L1) die het zwarte gat omringt: als dit oppervlak eenmaal gepasseerd is, is het onmogelijk om terug te gaan en alles zal onvermijdelijk naar het centrum van het zwarte gat vallen.

Vanuit een puur theoretisch oogpunt bezitten zwarte gaten vele andere unieke eigenschappen: in tegenstelling tot wat men wellicht zou vermoeden zijn ze waarschijnlijk juist de simpelste objecten die er zijn in de natuur in de "macroscopische" (niet-quantum) limiet, aangezien ze volledig beschreven worden door slechts een handvol parameters, zoals hun massa, hun ladingen en hun impulsmoment. Echter, onze wiskundig strenge beschrijving eindigt buiten de horizon. Alles wat daarbinnen valt, zal hoe dan ook naar het centrum van het zwarte gat vallen, dat een punt

Afbeelding L1 (links): Een schilderachtige (maar incorrecte) weergave van de horizon als een immateriële schil die het zwarte gat omringt.
Afbeelding R1 (rechts): Een realistische weergave van een zwart gat. Dit is alles wat we direct kunnen zien. Alles, zelfs het licht, is gevangen binnen de horizon.

is waar onze klassieke beschrijving ons in de steek laat, omdat de zwaartekrachtsinteractie daar oneindig groot, en dus onfysisch is. Dit suggereert dat de klassieke theorie van Einstein niet genoeg is om de fysica van zwarte gaten volledig te verklaren. In 1974 bevestigde Hawking dit idee toen hij ontdekte dat zwarte gaten niet alleen alles absorberen dat binnen de horizon valt, maar ook een *warmtestraling* uitzenden en dus een zekere temperatuur T ongelijk aan nul bezitten. Belangrijker nog, quantummechanica speelt een cruciale rol in de beschrijving van deze objecten omdat zij, zoals eerder vermeld, de enige theorie is die straling volledig kan verklaren. De analyse van deze quantumaspecten van een zwart gat zou niet alleen de klassieke theorie kunnen repareren (en de oneindig grote zwaartekrachtsaantrekking in het centrum van het zwarte gat kunnen verklaren), maar kan ook een podium leveren waarop zwaartekracht en quantummechanica elkaar ontmoeten. Kortom, zwarte gaten zijn zeer zware objecten en hun zwaartekrachtsaantrekking strekt zich uit over zeer grote afstanden maar tegelijkertijd zenden ze een thermische straling uit, waardoor quantumeffecten extreem relevant zijn voor hun volledige beschrijving.

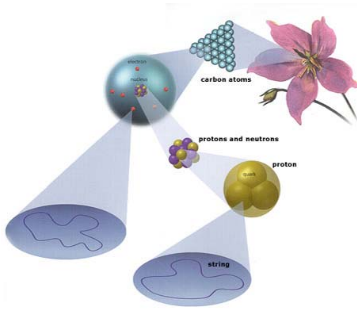
De entropie van een zwart gat als gereedschap voor quantumzwaartekracht?

Voor Hawkings ontdekking was er al een serie natuurwetten bewezen die de fysische eigenschappen van een zwart gat beschrijven. Deze *wetten van de mechanica van zwarte gaten*, gebaseerd op Einsteins klassieke (niet-quantum) zwaartekrachtstheorie, leken verrassend genoeg zeer veel op de wetten van de thermodynamica, de tak van de fysica die thermische verschijnselen beschrijft. Toen eenmaal ontdekt was dat zwarte gaten eigenlijk thermodynamische objecten zijn, was het makkelijk om de wetten van de mechanica van zwarte gaten te herkennen als de wetten die hun thermodynamische eigenschappen beschrijven. Om een voorbeeld te geven, zeer relevant voor dit proefschrift, de *entropie* van een zwart gat, ofwel, in alledaagse termen, de maat voor zijn wanorde, kan worden herkend als het oppervlak van zijn horizon. Dit is slechts een klassiek en macroscopisch resultaat, omdat het gebaseerd is op Einsteins theorie. Echter, aangezien zwarte gaten straling uitzenden, kunnen hun thermodynamische eigenschappen volledig beschreven worden door uitsluitend hun quantumeffecten. Dat betekent dat ook de klassieke entropie, het oppervlak van de horizon, gemodificeerd kan worden op een quantumniveau. De fundamentele vraag wordt dan of er een quantumtheorie bestaat die zulke effecten kan beschrijven. Het antwoord is ja: snaartheorie!

Superzwaartekracht als een effectieve beschrijving van snaartheorie
Waar het Standaard Model de dynamica en interacties beschrijft van microscopische puntdeeltjes die de wetten van de quantummechanica volgen, zijn de bouwstenen van snaartheorie, inderdaad, snaren, of gewoon ééndimensionale lijnen. Tot nu toe bestaat er geen experimentele verificatie van snaartheorie, hoewel het wordt geloofd dat alle bekende elementaire deeltjes (electronen, quarks etcetera) gecreëerd worden als de excitaties van zulke snaren. Belangrijker nog, zwaartekracht is al op zeer natuurlijke manier ingebed in snaartheorie, wat het de meest prominente kandidaat maakt voor een theorie van quantumzwaartekracht (hoewel het niet allemaal zo mooi is als het klinkt: vele wetenschappers zijn constant bezig met pogingen om de problemen van snaartheorie op te lossen. In deze samenvatting en in dit proefschrift zullen we deze aspecten volledig negeren).

Het meest relevante aspect van snaartheorie voor dit werk is dat zij in staat is om de microscopische eigenschappen van zwarte gaten te beschrijven, inclusief de quantumcorrecties op hun thermodynamische eigenschappen, zoals hun entropie. Echter, deze berekeningen in snaartheorie zijn vaak zeer ingewikkeld. Het zou erg van pas komen als we quantumeffecten op een simpelere manier konden beschouwen, via een aanpak waarin de vele moeilijkheden van de beschrijving in termen van snaren geëlimineerd of "uitgemiddeld" kunnen worden. Een zeer contra-intuïtieve oplossing voor dit probleem is: laten we die quantumeffecten vanuit een macroscopisch perspectief bekijken. Maar kan dat? Quantumeffecten horen in de wereld van het infinitesimaal kleine, hoe kunnen we dat op macroscopische manier beschrijven?

Om deze verwarrende discussie te doorgronden, is het handig en voldoende om een voorbeeld te geven. In de beschrijving van een fysisch fenomeen moet men de relevantie beschouwen van vele details die afhangen van de energie- (of afstands-) schaal waarin we geïnteresseerd zijn. Het is bijvoorbeeld bekend dat elk object (zie Afbeelding 2) samengesteld is uit vele kleine deeltjes, gegroepeerd in atomen, die weer gegroepeerd zijn in moleculen, enzovoort. Om echter de beweging van een auto te beschrijven, zou het in principe mogelijk, maar in de praktijk



Afbeelding 2: Wat zijn de elementaire componenten van alles? Men denkt dat, op zeer kleine afstanden of zeer hoge energieën, de elementaire deeltjes zijn opgebouwd uit snaren.

onuitvoerbaar en zeer onwenselijk zijn, om haar volledige atomische structuur te beschouwen. Dit betekent simpelweg dat vele details die tot de extreem kleine afstandsschaal behoren "effectief" in beschouwing worden genomen in de beschrijving op een grote afstandsschaal. De twee verschillende perspectieven, microscopisch en macroscopisch, moeten natuurlijk tot dezelfde resultaten leiden, maar de *effectieve beschrijving* is het meest geschikt voor dit geval, de studie van een rijdende auto.

Voor dit proefschrift is van belang dat de effectieve, lange-afstands (of lage-energie) beschrijving van snaartheorie bekend staat als *superzwaartekracht*. In deze limiet kunnen snaren effectief beschouwd worden als puntdeeltjes (een zeer kleine en verre snaar zou effectief waargenomen worden als een punt) en berekeningen in superzwaartekracht behouden een soort "klassieke" aanblik terwijl quantumeffecten wel rigoureus beschouwd worden.

De cruciale eigenschap van zowel snaartheorie als superzwaartekracht is hun invariantie onder een zeer speciale symmetrie, genaamd *supersymmetrie*. Supersymmetrie transformeert fermionen (de elementaire bestanddelen van materie) in bosonen (de dragers van de krachten tussen fermionen) en leidt tot zeer elegante quantummechanische modellen, waarin quantumeffecten meestal volledig onder controle zijn (als de quantumbeschrijving van een theorie niet compleet of correct is, stuit men vaak op vele problemen in de berekening van quantumeffecten).
Het is belangrijk te benadrukken dat theorieën van superzwaartekracht, die bijvoorbeeld verschillen in de hoeveelheid supersymmetrie die ze behouden, in de klassieke limiet, waarin quantumeffecten genegeerd worden, reduceren tot Einsteins zwaartekrachtstheorie, of beter tot supersymmetrische versies daarvan. Dit is een zeer belangrijke eigenschap omdat het een brug slaat tussen experimenteel geteste theorieën, zoals Einsteins zwaartekracht, en superzwaartekracht en snaartheorie, die tot nu toe nog niet experimenteel geverifieerd zijn.

Quantumeffecten in superzwaartekracht - Dit proefschrift

In theorieën van superzwaartekracht worden quantumeffecten, waarvan de microscopische beschrijving gegeven wordt door snaartheorie, effectief afgebeeld als nieuwe soorten interacties tussen de elementaire deeltjes van de theorie (onthoud dat snaren op lage energieschalen, of op grote afstanden, effectief beschouwd kunnen worden als puntdeeltjes) die *hogere afgeleide koppelingen* genoemd worden. Zodra deze nieuwe interacties bekend zijn, kan men de quantumcorrecties echt beschrijven voor verschillende klassen van zwarte gaten, gekarakteriseerd bijvoorbeeld door de verschillende hoeveelheid van supersymmetrie die ze behouden.
Het is een interessante vraag of er bepaalde klassen van zwarte gaten bestaan waarvan de thermodynamische eigenschappen, zoals de entropie, niet beïnvloed worden door quantumeffecten of, vice versa, of er quantumeffecten zijn die de klassieke eigenschappen van (sommige) zwarte gaten niet beïnvloeden.

Het voornaamste doel van dit proefschrift is om expliciet een bepaalde klasse van hogere afgeleide koppelingen de construeren, middels het gebruik van technieken die speciaal geschikt zijn voor theorieën van superzwaartekracht. Zulke koppelingen, die corresponderen met bepaalde quantumcorrecties, zijn grondig bestudeerd. Het cruciale resultaat behaald in dit proefschrift is dat deze specifieke hogere afgeleide koppelingen de klassieke thermodynamische eigenschappen van maximaal supersymmetrische zwarte gaten niet beïnvloeden. Dit betekent dat quantumeffecten praktisch genegeerd kunnen worden, in overeenstemming met de analoge resultaten behaald op het microscopisch niveau in snaartheorie.
Zoals we al opmerkten in de inleiding is het formuleren van een correcte en complete verenigde theorie van quantumzwaartekracht geen eenvoudige opgave, gegeven de onmogelijkheid van directe experimentele bevestiging. Men moet alles overlaten aan het enige gereedschap dat we hebben: logische en abstracte redenatie, wat het meest efficiënt in wiskundige termen uitgedrukt kan worden. Als een experimenteel bevestigde theorie aan bepaalde criteria voldoet dan kan men redelijkerwijs aannemen dat soortgelijke theorieën aan soortgelijke criteria voldoen. Zoals we in ons werk aangetoond hebben, vallen quantumeffecten berekend vanuit een macroscopisch perspectief in superzwaartekracht precies samen met de microscopische resultaten behaald in snaartheorie. Volg nu dezelfde redenatie gebruikt in thermodynamica, wat een experimenteel bevestigde theorie is, en deze gelijkheid is een noodzakelijke conditie voor de resultaten behaald in macroscopische en microscopische theorieën om te kunnen kloppen (anders zou één van de twee beschrijvingen, of beiden, incorrect moeten zijn). Ons belangrijkste resultaat bevestigt verder dat superzwaartekracht en snaartheorie aan sommige eisen voldoen die noodzakelijk zijn om beschouwd te kunnen worden als de correcte theorieën om zwaartekrachts- en tegelijkertijd quantumverschijnselen te beschrijven.

Dit is duidelijk slechts een kleine stap in die richting. Andere theorieën (maar tot nu toe nog geen enkele) zouden aan soortgelijke criteria kunnen voldoen en in dat geval zou het, gegeven het gebrek aan experimentele bevestiging, zeer lastig worden om uit te maken welke theorie de correcte is. Daarom is het belangrijk om de structuur te bestuderen en te proberen om ons begrip van snaartheorie en superzwaartekracht te verbreden. Het zijn uiteindelijk extreem complexe theorieën, waarvoor het om een begrip te kweken noodzakelijk is om zeer specifieke eigenschappen te analyseren. De reden hiervoor is duidelijk: om een groot kasteel te bouwen moet men goed begrijpen welk soort bakstenen te gebruiken, omdat de structuur anders dramatisch in elkaar zou kunnen storten.

In deze context hebben we ook bestudeerd hoe quantumeffecten, afkomstig van hogere afgeleide koppelingen, de deeltjes kunnen beïnvloeden die rond een zwart gat bewegen, gegeven de (super)symmetrieën die het zwarte gat behoudt. Door verschillende voorbeelden te bestuderen laten we zien dat als een (klassiek) zwart gat supersymmetrisch is, specifieke deeltjes van de theorie, *vlakke richtingen* genaamd, op klassieke wijze behandeld kunnen worden, omdat ze niet beïnvloed zullen worden door zekere klassen van quantumeffecten in superzwaartekracht. Vice versa, voor niet-supersymmetrische zwarte gaten, dreigen *enkele* quantumeffecten het klassieke gedrag van vlakke richtingen op een niet-fysische manier te verstoren. Aan de andere kant zal de *volledige verzameling* van quantumeffecten hun klassieke gedrag corrigeren, maar resulterend in fysische resultaten.

Zo is er dus een nieuwe steen op haar plaats gelegd, om het nog incomplete bouwwerk van snaartheorie en superzwaartekracht meer stabiliteit te geven, in afwachting van de vervolmaking van het kasteel dat de verenigde theorie van quantumzwaartekracht zal zijn.

Summary

This thesis is based on the results obtained during my years of doctoral research and, as such, would be nearly impossible to understand for people having no specific knowledge of my field of work. For this reason, I decided to write a very introductory summary in the hope of clarifying at least the essential reasoning and motivations behind my work. Here it goes!

Every natural phenomenon, independently from its characteristic scales of energy or distance, can be essentially explained by the known four fundamental forces or interactions: the electromagnetic force, the weak nuclear force, the strong nuclear force and the gravitational force. The first three interactions, electromagnetism, strong and weak force, are associated with the microscopic world of elementary particles and are comprised by one unique theory, the Standard Model. Such theory is based on quantum mechanics which describes the physics of very small objects of very small masses. To give few example, quantum mechanics successfully explains the structure of atoms and molecules, electromagnetic and thermal radiation (namely, light and heat) and many more phenomena of the microscopic world. On the other hand, the theory explaining the fourth interaction, the gravitational one is Einstein's general relativity. This theory describes the dynamics of very massive object over very large distances, such as stars or galaxies. Ideally, a theory of everything, unifying gravity and the other three quantum forces, could describe the origin of the Universe where the distance scales were very small but, at the same time, energies were incredibly high (the Big Bang explosion liberated the equivalent of the whole energy present nowadays in the Universe). Apart from the incredible importance that such a discovery could assume, it is a typically scientific attitude to try to construct general theories which could account for as many different physical phenomena as possible (the converse is also true: a theory able to explain nothing could not be considered scientific). But quantum mechanics and general relativity are applicable in opposite regimes of distances and masses and that is at the origin of all the difficulties in trying to unify them in one unique theory of quantum gravity.

Of course, it is impossible to reach the energies of a Big Bang explosion in an experimental settings. So, even though the Standard model and general relativity have been experimentally verified to a high degree of precision, their unification should ultimately make sense from a mathematical point of view (hence be consistent) given the lack of direct experimental testing which could confirm it.
In the purely theoretical attempt to find such unified theory of quantum gravity, it would be extremely helpful to study objects whose description is based on both general relativity and quantum mechanics theories. The obvious question then becomes: is there any such physical object in Nature?

Black Holes
The first theory describing the gravitational force is due to Newton. Its level of accuracy where high when the object considered would move at a speed very small compared to the speed of light. Later on, at the beginning of the 20th century, Einstein formulated general relativity, a relativistic theory of gravity, which describes the gravitational interactions of objects whose speed can be comparable to the speed of light. General relativity received several experimental verifications and, at the same time, predicted many fascinating results, such as the existence of the so-called *black holes*. Black holes, as described by Einstein's theory, are created from massive stars that collapse under their own weight. Many, if not all, galaxies are known to possess a super-massive black hole at their center. What characterizes a black hole is the incredibly high mass-density (a spoonful of a black hole can weigh many tons), hence the strong gravitational attraction they exert on anything around them. To be more precise, there exists even a point-of-no-return for anything, including light, that moves close to a black hole called the *event horizon* (see Figure R1). The event horizon can be imagined, for simplicity, as an immaterial shell (see Figure L1) that surrounds the black hole: once this surface has been crossed, it is impossible to go back and everything will inevitably fall toward the center of the black hole.

From a purely theoretical point of view, black hole possess many other unique features: contrarily to what one might think, in fact, they are probably the simplest object in Nature in the "macroscopic" (not quantum) limit, as they are fully characterized by only few parameters, such as their mass, their charges and their angular momentum. However, our mathematically rigorous description of black hole ends outside the event-horizon. In fact, anything that falls inside it, will necessarily move towards the center of the black hole, which is a point where our classical description breaks down, because the gravitational attraction there is infinite, hence unphysical. This suggests that the classical theory of Einstein is not enough to fully explain the physics of black holes. In 1974, Hawking confirmed

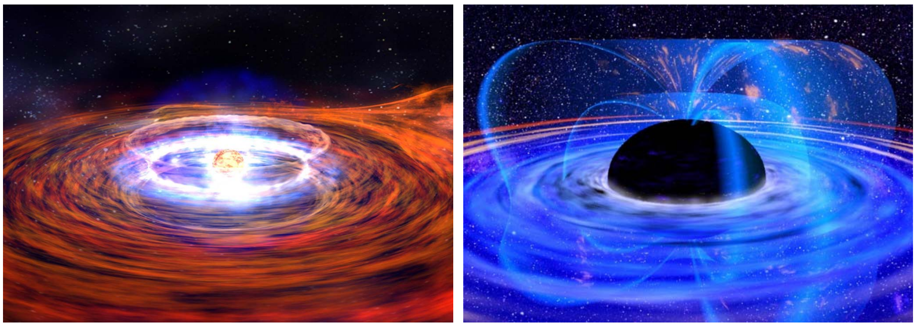


Figure L1 (left): A picturesque (albeit incorrect) image of the event horizon as an immaterial shell surrounding the black hole.
Figure R1 (right): A realistic image of a black hole. This is all we can directly see. Everything, even light, is trapped inside the horizon.

this idea when he discovered that black holes do not just absorb everything that falls inside the horizon, but they also emit a *thermal radiation*, hence they possess a certain non-zero temperature T. More importantly, quantum mechanics has a crucial role to play in describing these objects since, as we argued before, that is the only theory that fully explains radiation. The analysis of these quantum aspects of a black hole might then not only resolve the break down of the classical theory (and physically explain the infinite gravitational attraction at the black hole center) but also furnish a context which allows for a crossing between gravity and quantum mechanics. In short, black holes are very heavy objects and their gravitational attraction extends over very large distances but, at the same time, they emit thermal radiation, so quantum effects are extremely relevant for their full description.

Black hole entropy as a tools towards quantum gravity?

Before Hawking's discover, a series of laws describing the physical properties of a black hole had already been proven. These *laws of black hole mechanics*, based on the classical (not quantum) Einstein' s theory of gravity, curiously shared a deep resemblance with the laws of thermodynamics, the branch of physics describing thermal phenomena. Once it was discovered that black holes are in fact thermodynamic objects, it was easy to identify the laws of black hole mechanics as the laws governing their thermodynamic properties. To give an example, extremely relevant to this work, the *entropy* of a black hole, or in layman terms the measure of its disorder, can be identified to the area of its event-horizon. This is only a classical and macroscopic result, because based on Einstein' s theory. But, since black holes emit radiation, their thermodynamic properties can be fully explained only by considering quantum effects. This means that also the classical entropy, the horizon area, can be modified at a quantum level. The fundamental issue then becomes whether a quantum theory able to describe such effects exists. The answer is yes: *String Theory*!

Supergravity as an effective description of String theory

While the Standard model describes the dynamics and interactions of microscopic and practically point-like particles obeying the rules of quantum mechanics, the elementary building blocks of string theory are, indeed, strings, or just unidimensional lines. Up until now, no experimental verification of string theory exists, although it is believed that all the elementary particles we know (electrons, quarks, etc) are created from excitations of such strings. More importantly, gravity is already embedded in a very natural way in string theory, rendering it the most prominent candidate for a theory of quantum gravity (not everything is a bed of roses as it sounds: many scientists are constantly working to try and solve the problems, more or less relevant, of string theory. We will neglect these aspect in this summary and in the thesis completely).

The most relevant aspect for this work is that string theory is able to describe the microscopic features of black holes, including quantum corrections to their thermodynamic properties, such as the entropy. However, these calculation in string theory are often very complicated. It would be convenient if we could consider quantum effects in a simpler way, where many of the difficulties connected to the microscopic description based on strings might be eliminated or just "averaged". And the very counter-intuitive solution to this problem is: let us consider quantum effects from a macroscopic perspective. But, is there such a thing? Quantum effects belong to the world of the infinitesimally small, how can we describe it in a macroscopic way?

To explain this puzzling discussion, it is instructive and sufficient to give an example. When describing a physical phenomenon one must take into account the relevance of many details which depend on the energy (or distance)-scale we are interested in. For instance, it is known that every object (see Figure 2) is composed by very small particles, grouped in atoms, grouped in molecules, and so on. However to describe the motion of a car, it would be theoretically possible, but practically prohibitive other than extremely inconvenient to consider its full atomic structure. Instead one would consider only the important variables characterizing the car, as its shape or velocity. This does not mean that quantum mechanics is not true or cannot explain the motion of a car. It simply means that many details which belong to the extremely small distance scale are "effectively" taken into account in the large distance scale description. The two different perspectives, microscopic and macroscopic, must yield of course the same result, but the *effective description* is best suited to the problem at hand, the study of a car motion.

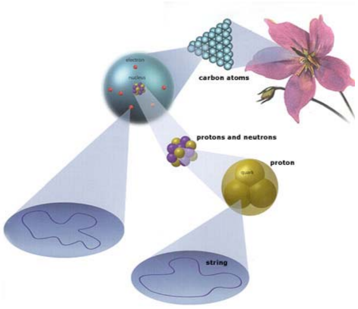


Figure 2: What are the elementary components of everything? It is believed that, at very small distances, or very high energy, even elementary particles are composed by strings.

In the context of interest to this thesis, the effective, long-distance (or low-energy) description of string theory is called *supergravity*. In this limit, strings can be effectively considered as point-particles (a very small string, very far away,would appear effectively as a point) and calculations in supergravity maintain somehow a “classical” facade while, in reality, quantum effects are rigorously considered.

The crucial aspect shared by string theory and supergravity is their invariance under a very special symmetry, called *supersymmetry*. Supersymmetry transforms fermions (which are the elementary constituents of matter) in bosons (which are the intermediary of the forces between fermions) and leads to very elegant quantum-mechanical models, where quantum effects are typically under complete control (if the quantum description of a theory is not complete or correct then often one encounters many problems when calculating quantum effects).
It is important to stress that supergravity theories, which for instance differ by the amount of supersymmetry they preserve, reduce in the classical limit, when quantum effects are neglected, to Einstein’s theory of gravity, or better at a supersymmetric version thereof. This is a very important feature as it constitutes a bridge between experimentally verified theories, such as Einstein’ s gravity, and supergravity and string theory, which have no experimental verification as of now.

Quantum effects in supergravity - This thesis
In theories of supergravity, quantum effects, whose microscopic description is given by string theory, are effectively encoded in new types of interactions between the elementary particles of the theory (remember, strings at low energies, or large distances, can be effectively considered point-like particles) called *higher derivative couplings*. Once these new interactions are known, one can actually calculate the quantum corrections for different classes of black holes characterized, for instance by the different amount of supersymmetries they preserve.
It is an interesting question to ask whether there exist particular classes of black

holes whose thermodynamic properties, such as the entropy, are not affected by (certain) quantum effects or, vice-versa, if there are quantum effects which do not influence the classical properties of (certain) black holes.

The main aim of this thesis is to explicitly construct a particular class of higher derivative couplings, by using techniques which are specifically suited to supergravity theories. Such couplings, which correspond to certain quantum corrections, have been thoroughly analyzed. The crucial result obtained is that these specific higher derivative couplings do not influence the classical thermodynamic properties of maximally supersymmetric black hole solutions. This means that quantum effects can practically be neglected, in agreement with the analogous results obtained at the microscopic level in string theory.
As we already mentioned in the introduction, the formulation of a correct and complete unified theory of quantum gravity is not an easy task, given the impossibility of a direct experimental verification. One must entrust everything to the only tools left at our disposal: logic and abstract reasoning, whose most efficient language is mathematics. If an experimentally verified theory satisfies certain properties one can reasonably assume that similar theories must satisfy similar properties. As we have shown in our work, quantum effects calculated from a macroscopic perspective in supergravity coincide exactly with the microscopic results obtained in string theory. Follow the same reasoning used in thermodynamics, which is an experimentally verified theory, this equality is a necessary condition for the results obtained in the macroscopic and microscopic theories to make logical sense (otherwise one of the two descriptions, or both, might be incorrect). Our main result further confirm that supergravity and string theory satisfy some of the conditions necessary to be considered the correct theories to describe gravitational and at the same time quantum phenomena.

Obviously, this is only a small step in that direction. Other theories (though none so far) could satisfy similar conditions, and in that case, given the lack of an experimental confirmation, it would be very difficult to decide which theory is the correct one. For this reason it is important to study the structure and try to widen our understanding of string theory and supergravity. They are, in fact, extremely complex theories, to study and understand which it is often indispensable to analyze very specific aspects. The reason is evident: even to construct a huge castle, one needs to understand properly what kind of bricks to use, otherwise the structure could drastically collapse.

In this context, we have also studied how quantum effects, due to higher derivative couplings, can influence particles that move around a black hole, given the symmetries, and in particular supersymmetries, the black hole preserves. We show, by studying two different examples, that if a (classical) black hole is supersymmetric then specific particles of the theory, called *flat directions*, can be treated classically, namely they will not be affected by certain classes of quantum effects in supergravity. Vice-versa, for non-supersymmetric black holes,*single* quantum effects tends to disrupt in a non-physical fashion the classical behaviour of flat directions. On the other hand, the *full set* of quantum effects will correct their classical behaviour, leading however to physical results.

Another brick put in place, to give the yet incomplete building of string theory and supergravity more stability, waiting for the completion of the castle which will be the unified theory of quantum gravity.

Riassunto

Questa tesi è basata sui risultati ottenuti durante il periodo di ricerca di dottorato e come tale potrebbe risultare pressoché incomprensibile per persone senza alcuna conoscenza specifica nel mio campo di lavoro. Per questa ragione, ho deciso di scrivere un riassunto molto introduttivo nella speranza di chiarire almeno i ragionamenti essenziali e le motivazioni che stanno alla base del mio lavoro. Eccovelo!

Ogni fenomeno naturale, indipendentemente dalla scale di energia o distanze che lo caratterizzano, può essere essenzialmente spiegato dalle note quattro forze o interazioni fondamentali: l'interazione elettromagnetica, l'interazione nucleare forte, l'interazione nucleare debole e l'interazione gravitazionale. Le prime tre forze, l'elettromagnetismo, la forza forte e debole sono collegate al mondo microscopico delle particelle elementari e sono incluse in un'unica teoria, il Modello Standard. Questa teoria è basata sulla meccanica quantistica che descrive la fisica di oggetti molto piccoli di masse minuscole. Per dare qualche esempio, la meccanica quantistica spiega con successo la struttura degli atomi e delle molecole, della radiazione termica ed elettromagnetica (cioè calore e luce) e molti altri fenomeni del mondo microscopico. D'altra parte la teoria della relatività generale di Einstein descrive l'interazione gravitazionale di oggetti molto pesanti a distanze enormi, come stelle o galassie. Idealmente, una teoria del tutto, che unifichi la gravità e le altre tre forze quantistiche, potrebbe descrivere l'origine dell'Universo, quando le distanze in gioco erano piccolissime ma, allo stesso tempo, le energie erano incredibilmente alte (l'esplosione del Big Bang ha liberato l'equivalente di tutta l'energia presente oggigiorno nell'Universo). A prescindere dall'incredibile importanza che una tale scoperta potrebbe rivestire, il tentativo di costruire teorie generali che possano spiegare quanti più fenomeni fisici possibili è un atteggiamento tipicamente scientifico (il contrario è anche vero: una teoria che non riesce a spiegare nulla non può essere considerata scientifica). Ma la meccanica quantistica e la relatività generale sono teorie applicabili in regimi opposti di distanze e masse e questa differenza è all'origine di tutte le difficoltà nel tentativo di unificarle in un'unica teoria della

gravità quantistica.
Ovviamente è impossibile raggiungere le energie di esplosione del Big Bang in un esperimento. Quindi, anche se le teorie del Modello Standard e della relatività generale sono state verificate sperimentalmente con un alto indice di precisione, la loro unificazione dovrebbe sostanzialmente avere senso da un punto di vista matematico (cioè essere consistente) data la mancanza di test sperimentali diretti che possano confermarla.
Nel tentativo puramente teorico di trovare questa teoria unificata di gravità quantistica, sarebbe estremamente utile studiare oggetti la cui descrizione è basata su entrambe le teorie della relatività generale e della meccanica quantistica. La domanda spontanea è quindi: esistono in Natura oggetti di questo tipo?

Buchi Neri
La prima teoria che descrive la forza gravitazionale è dovuta a Newton. Il livello di accuratezza di questa teoria è alto se gli oggetti considerati si muovono a velocità molto inferiori rispetto alla velocità della luce. In seguito, all'inizio del XX secolo, Einstein formulò la relatività generale, una teoria della gravità relativistica che descrive l'interazione gravitazionale di oggetti la cui velocità può essere comparata a quella delle luce. La relatività generale ricevette molte verifiche sperimentali e, allo stesso tempo, predisse molti risultati affascinanti, come l'esistenza dei cosidetti *buchi neri*. I buchi neri, come la relatività generale ci insegna, sono creati da pesantissime stelle che collassano a causa del loro stesso peso. Si sa che molte, se non tutte, le galassie contengono buchi neri super-massivi al loro centro. Ciò che caratterizza un buco nero è la densità di massa incredibilmente alta (un cucchiaio di buco nero può pesare parecchie tonnellate), e di conseguenza la fortissima attrazione gravitazionale che esercita attorno. Per essere più precisi esiste persino un punto di non ritorno per tutto ciò che si muove vicino ad un buco nero, inclusa la luce, chiamato *orizzonte degli eventi* (vedi Figura R1). L'orizzonte degli eventi può essere immaginato, per semplicità, come un guscio immateriale (vedi Figura L1) che circonda il buco nero: quando questa superficie viene oltrepassata, è impossibile uscirne e ogni cosa cadrà inevitabilmente verso il centro del buco nero.

Da un punto di vista puramente teorico, i buchi neri presentano molte altre caratteristiche uniche: contrariamente a ciò che si possa pensare, infatti, sono gli oggetti più semplici che esistano in Natura nel limite "macroscopico" (non quantistico), dato che sono caratterizzati completamente da pochissimi parametri, come la loro massa, carica e momento angolare. Purtroppo però, i buchi neri possono essere descritti rigorosamente solo fuori dall'orizzonte degli eventi. Infatti, tutto ciò che oltrepassa l'orizzonte si muoverà necessariamente verso il centro del buco nero, il

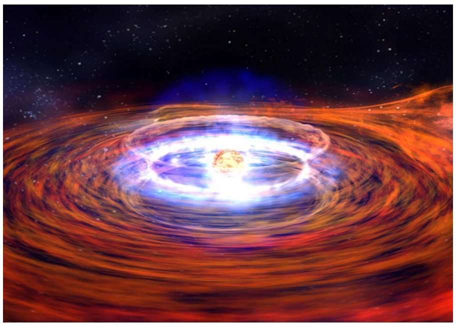
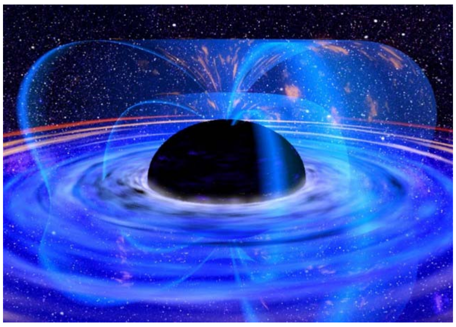
 

Figura L1 (sinistra): Un'imagine pittoresca (sebben incorretta) dell'orizzonte degli eventi visto come un guscio immateriale che circonda il buco nero.
Figura R1 (destra): Un'immagine realistica di un buco nero. Questo come appare. Tutto, anche la luce, viene intrappolato dentro l'orizzonte.

punto dove la nostra descrizione classica non è più adeguata, perché l'attrazione gravitazionale in quel punto è infinita e quindi non accettabile fisicamente. Questo fatto ci suggerisce che la teoria classica di Einstein non è sufficiente per spiegare la fisica dei buchi neri in maniera completa. Nel 1974 Hawking confermò questa idea scoprendo che i buchi neri non assorbono solamente tutto ciò che cade dentro l'orizzonte ma emettono anche una *radiazione termica*, quindi posseggono una certa temperatura T. Ancora più significativo è il fatto che la meccanica quantistica ha un ruolo fondamentale nella descrizione di questi oggetti, dato che, come abbiamo discusso in precedenza, questa è l'unica teoria che spiega correttamente il fenomeno della radiazione. L'analisi di questi aspetti quantistici di un buco nero potrebbe quindi non solo risolvere i problemi della teoria classica (e spiegare fisicamente l'infinita attrazione gravitazionale al centro di un buco nero) ma anche fornire un contesto in cui è possible incrociare gravità e meccanica quantistica. In breve, i buchi neri posseggono una massa elevatissima e la loro attrazione gravitazionale si estende su ampie distanze ma, allo stesso tempo, emettono radiazione termica, quindi gli effetti quantistici sono estremamente rilevanti per la loro completa descrizione.

L'entropia dei buchi neri come strumento vero la gravità quantistica

Antecedente la scoperta di Hawking, erano già state dimostrate delle leggi che descrivono le propietà fisiche di un buco nero. Queste *leggi della meccanica dei buchi neri*, basate sulla teoria classica (non quantistica) della gravità di Einstein, curiosamente rassomigliavano incredibilmente alle leggi della termodinamica, la branca della fisica che descrive fenomeni termici. Non appena fu scoperto che i buchi neri sono realmente oggetti termodinamici, fu semplice identificare le leggi della meccanica dei buchi neri con le leggi che governano le loro proprietà termodinamiche. Per dare un esempio, estremamente importante per questo lavoro di tesi, l'*entropia* di un buco nero, o in parole povere la misura del suo disordine, può essere identificata con l'area dell'orizzonte degli eventi. Questo è solo un risultato classico e macroscopico, perché basato sulla teoria di Einstein. Ma, dato che i buchi neri emettono radiazione, le loro proprietà termodinamiche possono essere descritte in maniera completa solo se si considerano anche effetti quantistici. Questo significa che anche l'entropia classica, l'area dell'orizzonte, può subire modifiche quantistiche. A questo punto, il problema fondamentale diventa capire se esiste una teoria quantistica in grado di descrivere questi effetti. La risposta è si: la *Teoria delle Stringhe*!

La supergravità come descrizione effettiva della teoria delle stringhe
Mentre il Modello Standard descrive la dinamica e le interazioni di particelle microscopiche e praticamente puntiformi che obbediscono alle leggi della meccanica quantistica, i mattoni elementari della teoria delle stringhe sono, appunto stringhe, cioè piccolissime corde unidimensionali (ovvero delle linee). Fino ad oggi, non esiste alcuna verifica sperimentale di questa teoria, sebbene si creda che tutte le particelle elementari conosciute (elettroni, quark, etc) siano create da eccitazioni di queste stringhe. Ancora più importante è il fatto che la forza gravitazionale è già incorporata nella teoria delle stringhe in maniera molto naturale, e ciò rende questa teoria il candidato più promettente per una teoria unificata della gravità quantistica (non tutto è rose e fiori come sembra: molti scienziati lavorano costantemente per cercare di risolvere i problemi più o meno importanti della teoria delle stringhe. Non ci occuperemo di nessuno di questi aspetti in questo riassunto o nella tesi).
L'aspetto più di interesse per questo lavoro è che la teoria delle stringhe è in grado di spiegare le caratteristiche microscopiche dei buchi neri, incluse le correzioni quantistiche alle loro quantià termodinamiche, come l'entropia. Tuttavia, questi calcoli nella teoria delle stringhe sono spesso molto complicati. Sarebbe estremamente conveniente se potessimo analizzare gli effetti quantistici in una maniera più semplice, in cui molte delle difficoltà connesse con la descrizione microscopica basate sulle stringhe possano essere eliminate o semplicemente "considerate in media". E la soluzione contro-intuitiva a questo problema è: consideriamo gli effetti quantistici da una prospettiva macroscopica. Ma, che senso ha tutto ció? Gli effetti quantistici appartengono al mondo dell'infinitamente piccolo, come possiamo descriverli in maniera macroscopica?
Per spiegare questa discussione enigmatica, è istruttivo e sufficiente dare un esempio. Quando descriviamo un fenomeno fisico bisogna considerare quanto molti dettagli, che dipendono dalle scale di energia (o distanza) in questione, siano rilevanti. Per esempio, si sa che ogni oggetto (vedi Figura 2) è composto da particelle piccolissime, raggruppate in atomi, raggruppati in molecole, e via dicendo.

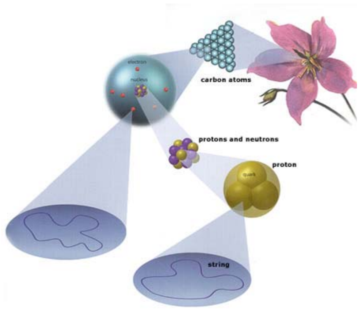


Figura 2: Quali sono le componenti elementari di ogni cosa? Molti credono che, a scale di distanza piccolissime, o ad energie elevatissime, anche le particelle elementari sono composte da stringhe.

Comunque, per descrivere il moto di una macchina, sarebbe teoricamente possibile, ma in pratica proibitivo e estremamente sconveniente considerare la sua completa struttura atomica. Invece, si considererebbero solo le variabili importanti che caratterizzano la macchina, come la forma o la velocità. Questo non vuol dire che la meccanica quantistica sia falsa o non possa spiegare il moto di una macchina. Significa semplicemente che molti dettagli che appartengono alle scale di distanza estremamente piccole sono "effettivamente" tenuti in considerazione nella descrizione a larga scala. Le due prospettive differenti, microscopica e macroscopica, devono portare ovviamente allo stesso risultato finale, ma la *descrizione effettiva* si presta meglio per risolvere il problema che ci siamo posti, lo studio del moto di una macchina.

Nel contesto di interesse per questa tesi, la descrizione effettiva, a lunga distanza (o bassa energia) della teoria delle stringhe è chiamata *supergravità*. In questo limite, le stringhe possono essere effettivamente considerate come particelle puntiformi (un linea piccolissima, posta a grande distanza, sembrerà effettivamente un punto) e i calcoli in supergravità mantengono in qualche modo una "facciata" classica anche se gli effetti quantistici sono tenuti rigorosamente in considerazione.

L'aspetto cruciale che la teoria delle stringhe e la supergravità condividono è la loro invarianza rispetto ad una simmetria speciale, chiamata *supersimmetria*. La supersimmetria trasforma i fermioni (che sono i costituenti elementari della materia) in bosoni (che sono gli intermediari delle forze tra i fermioni) e porta a modelli quanto-meccanici molto eleganti, in cui gli effetti quantistici sono tenuti tipicamente sotto stretto controllo (se la descrizione quantistica di una teoria è incompleta o incorretta spesso si incontrano problemi enormi nel calcolare effetti quantistici).
È importante sottolineare che le teorie di supergravità , che differiscono l'una dall'altra, per esempio, per il numero di supersimmetrie che preservano, si riducono

nel limite classico, dove gli effetti quantistici vengono omessi, alla teoria della gravità Einstein, o meglio ad una sua versione supersimmetrica. Questa caratteristica è molto importante perché costituisce il ponte tra una teoria sperimentalmente verificata, come la gravità di Einstein, e la supergravità e la teoria delle stringhe, che ancora non hanno avuto alcuna verifica sperimentale.

Effetti quantistici in supergravità - In questa tesi

Nelle teorie di supergravità gli effetti quantistici, la cui descrizione microscopica è data dalla teoria delle stringhe, sono effettivamente codificati in nuovi tipi di interazione fra le particelle elementari della teoria (ricorda, le stringhe a basse energie, o distanze larghe, possono essere effettivamente considerate particelle puntiformi) chiamate *accoppiamenti a più derivate*. Non appena queste nuove interazioni sono conosciute, si possono calcolare le correzioni quantistiche per diverse classi di buchi neri caratterizzate, per esempio, dal numero di supersimmetrie che preservano.
Una questione interessante da analizzare potrebbe essere l'esistenza di classi particolari di buchi neri le cui proprietà termodinamiche, come l'entropia, non risentano di (certi) effetti quantistici o, viceversa, se esistono effetti quantistici che non influenzino la proprietà classiche di certi buchi neri.

L'obiettivo principale di questa tesi è la costruzione di una classe particolare di accoppiamenti a più derivate, per mezzo di tecniche idonee alle teorie di supergravità. Questi accoppiamenti, che corrispondo a particolari effetti quantistici, sono stati analizzati a fondo. Il risultato essenziale è che questi accoppiamenti a più alte derivate non influenzano le proprietà termodinamiche classiche delle soluzioni di buco nero massimamente supersimmetriche. Questo significa che, in pratica, gli effetti quantistici possono essere trascurati, in accordo con l'analogo risultato ottenuto da un punto di vista microscopico in teoria delle stringhe.
Come avevamo già accennato nell'introduzione, la formulazione di una corretta e completa teoria unificata della gravità quantistica risulta essere un compito tutt'altro che facile, data l'impossibilità di una verifica sperimentale diretta. Risulta necessario affidarsi completamente agli unici strumenti rimasti a nostra disposizione: la logica e il ragionamento astratto, che trovano nella matematica il loro linguaggio più efficace. Se una teoria verificata sperimentalmente soddisfa certe proprietà si suppone ragionevolmente che teorie simili debbano soddisfare simili proprietà. Come abbiamo mostrato nel nostro lavoro, gli effetti quantistici calcolati nell'ambito macroscopico della supergravità coincidono esattamente con i risultati microscopici della teoria delle stringhe. Seguendo lo stesso ragionamento logico utilizzato in termodinamica, che è una teoria verificata sperimentalmente, questo eguaglianza è una condizione necessaria perché i risultati ottenuti dalle

due teorie, quella macroscopica e quella microscopica, abbiano senso logico (altrimenti una delle due descrizioni, o entrambe, potrebbero essere sbagliate). I nostri risultati principali sono un'ulteriore conferma del fatto che la supergravità e la teoria delle stringhe soddisfano alcune condizioni necessarie per essere considerate le teorie corrette in grado di descrivere fenomeni gravitazionali e allo stesso tempo quantistici.

Ovviamente, questo costituisce solo un piccolo passo in quella direzione. Altre teorie (anche se nessuna finora) potrebbero soddisfare condizioni simili, nel qual caso sarebbe difficile decidere quale teoria sia corretta, in mancanza di una conferma sperimentale. Per questo motivo è anche di fondamentale importanza studiare la struttura e cercare di ampliare la nostra comprensione delle teorie di supergravità e stringhe. Si tratta infatti di teorie estremamente complesse, per studiare e capire le quali è spesso indispensabile analizzare aspetti estremamente specifici. Il motivo è ovvio: anche per la costruzione di un enorme castello, c'è bisogno di capire bene quali tipi di mattoni si dovranno usare, altrimenti l'intera struttura potrebbe crollare drasticamente.

In questo contesto, nel nostro lavoro abbiamo anche studiato come gli effetti quantistici, dovuti alle interazioni a più alte derivate, possano influenzare particelle che si muovono attorno ad un buco nero, a seconda delle simmetrie, ed in particolare delle supersimmetrie, che il buco nero stesso rispetta. Viene mostrato, studiando due esempi differenti, che se un buco nero (classico) è supersimmetrico allora specifiche particelle della teoria, chiamate *direzioni piatte*, possono essere trattate classicamente, cioè non verranno influenzate da (certi) effetti quantistici in supergravità. Viceversa, per buchi neri non supersimmetrici, *singoli* effetti quantistici tendono a distruggere in maniera fisicamente non accettabile il comportamente classico delle direzioni piatte. D'altra parte, se si considera l'insieme *completo* di tutti gli effetti quantistici, allora il il loro comportamento classico verrà modificato, ma in maniera fisicamente accettabile.

Un altro mattone messo al suo posto, per dare stabilità alla struttura non ancora completa della teoria delle stringhe e della supergravità in attesa di completare il castello che sarà la teoria unificata della gravità quantistica.

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To Silvana, who uncovered the most beautiful truth for a hardheaded incorruptible admirer of the structure of Nature like me: whether we be puppets in the hands of Destiny or the only architects of our own destiny, time and space cease to be meaningful concepts in the face of unconditional love.

Curriculum Vitae

I was born in Enna, Italy on November 29, 1985 where I completed my school studies in July 2004. In the fall of the same year, I started my bachelor studies in Physics at the University of Catania and, at the same time, I passed the admittance exams (third place) and became an ordinary student of the Excellence School for University studies “Scuola Superiore di Catania”. In 2007 I received the University Bachelor degree with honors after which I started my master studies in Catania in 2007. In 2008 I received the Bachelor degree from the “Scuola Superiore di Catania” with honors. I then moved in January 2009 in Utrecht, first for a semester as a free mover student, to which followed one academic year as an ordinary student of the Master Program in Theoretical Physics. In the fall 2010 I received the Master Diploma in Utrecht and shortly after the Master Diploma from Catania, both with honors. The thesis entitled “The Noether potential: definition and applications”, under the supervision of Prof. B. de Wit, concerned the study of covariant methods to calculate conserved charges in the context of field theories. In 2010, I also received the Dutch Shell Prize, awarded to the best theoretical physics Master students of the year in The Netherlands. In October of the same year, I started my PhD studies at the ITP, Utrecht and then Nikhef, Amsterdam, under the supervision of Prof. B. de Wit. This work of thesis is based on (part of) the research carried out during the four years of my PhD studies.