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Modular Spacetime and Metastring Theory

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Abstract. In this talk we review our recent work on metastring theory and its habitat, a new form of quantum spacetime, called modular spacetime. We emphasize that the geometry underlying modular spacetime, i.e. the background geometry of metastring theory is also the geometry underlying any quantum theory as formulated in terms of Aharonov's modular variables. Thus the metastring sheds light on the foundations of quantum theory, and it represents a new formulation of string theory and quantum gravity based on the principle of relative locality.

1. Introduction

Quantum gravity is arguably the theoretical conundrum *par excellence*. There are many approaches to this problem, and little in the way of broad consensus. For example, one might quantize metric fluctuations as one quantizes other quantum fields, but then one is faced with the problem of non-renormalizability, that some researchers try to avoid by looking for a UV fixed point (asymptotic safety) [1]. One might attempt to quantize geometry, as is done in the context of loop quantum gravity [1], where one emphasizes background independence and where space is fundamentally discrete, but where the challenge is in the continuum limit and in dynamics, as well as in the reconciliation with the general relativity principle outside the classical limit. One might attempt the extrapolation of quantum field theory, as offered by canonical string theory [2], where spacetime is probe-dependent, but where the quantum or discrete notion of spacetime is hard to grasp. One might attempt emergent models of quantum gravity based on causal dynamical triangulation [3], or causal sets [4], or condensed matter inspired approaches [5], or violation of Lorentz symmetry [6], or holography via the AdS/CFT correspondence [7]. One might attempt to formulate intrinsic non-commutative geometry of spacetime and matter [8], or look at auxiliary spaces, such as twistor spaces, which encode space-time in a non-local, but intrinsically geometric manner [9].

We take the lessons of all these approaches very seriously in our new formulation of string theory and quantum gravity that we call *metastring theory*, developed in [10, 11, 12, 13, 14]. In particular, we take into account the lessons of geometry (the view of general relativists interested in quantum gravity) and the lessons of the quantum (the view of quantum field theorists and string theorists interested in quantum gravity) very seriously and proceed with a fresh look at the problem. One message that is common to all these various approaches to quantum gravity is that of *non-locality*. There are non-local observables in background independent approaches, there are non-local probes such as strings or twistors, there is non-locality of holography or



discrete space-time etc etc. Moreover, there is intrinsic non-locality in general relativity because of diffeomorphism symmetry, and there is quite different non-locality in quantum theory due to its linearity (leading to interference and entanglement).

In this talk we will see that the language of string theory is particularly appropriate for our unified approach, because it offers a view of both quantum theory of gravity and quantum theory of matter. According to canonical, textbook [2], string theory the underlying theory is quantum mechanically consistent (it is a two-dimensional worldsheet theory) and the worldsheet fields are interpreted as coordinatizing a target spacetime. This means, that the geometrical structures (such as the target space metric) are viewed as states of the string. In this approach to quantum gravity one uses a built-in interpretation: there is an emergent target space-time theory that at low energies reproduces local effective quantum field theories, and also includes (quantum) gravity. This is nice, but it cannot be the whole story. Our first insight towards a novel formulation of quantum gravity is the following observation. Any theory of gravity contains two parameters \hbar and G_N which can be viewed as conversion parameters, based on two scales: the Planck constant is the product of fundamental length and fundamental energy scales, and the Newton constant is the ratio of the same. If we want to probe quantum gravity in all regimes, we have to take this into account. (Note that in string theory, the Newton constant involves both a fundamental scale of the string, α' , as well as the string coupling. In that case, it is α' that is given as the ratio of the fundamental length and fundamental energy.) Also, usually G_N is treated as a coupling, and not a conversion parameter. However, in general relativity due to the equivalence principle, G_N can be understood as a universal inverse tension, and thus, as a conversion parameter between spacetime and energy-momentum space. This in turn leads to observer-dependent spacetimes and energy-momentum spaces and, in principle, to their unification in a new geometric structure associated with a relativistic phase space of individual quantum probes of spacetime physics. The running of a dimensionful coupling such as G_N is always fixed in terms of another dimensionful parameter. We choose Planck units as units of energy and time, where G_N does not run. Thus, we are extending the equivalence principle to quantum probes.

Given this insight, we also need to take into account that the concept of locality might be observer-dependent. We speak of *absolute locality* when spacetime is treated as an arena, independent of physical probes. Likewise, we speak of *relative locality* [15, 16] when spacetime depends on the nature of a probe (e.g. its energy). In metastring theory, relative locality is implemented by regarding spacetime as a subspace of the target space of the string. In other words, spacetime can be thought of as a choice of basis, that is, a choice of polarization of phase space. This means that *Born reciprocity* [17] can be built into the theory: a given space-time is only one choice of polarization, and energy-momentum space is on the same footing as spacetime, within a fully consistent quantum gravity. This ties nicely with the insight about G_N being a conversion parameter (which puts it on the same footing as \hbar and c). Finally, we will see that there is a particular geometric structure that takes into account these insights, the one we call *Born geometry*.

In what follows, we present an implementation of these insights in string theory, in what we call a metastring formulation and then find, quite surprisingly, that the habitat of metastring theory is a new quantum space, called modular spacetime, which reveals novel geometric structures behind quantum theory, thus shedding light on the origin of quantum theory, string theory and quantum gravity.

2. Born Reciprocity, String theory and T-duality

Born reciprocity is the simple observation that the change of basis from the position to the energy basis ($P \rightarrow Q$ and $Q \rightarrow -P$) is a fundamental kinematical symmetry of quantum mechanics (QM). However, Born reciprocity is broken in general relativity (GR). In GR there

is a preferred polarization: the spacetime one. Thus Born reciprocity is broken by the choice of the principle of absolute locality, the statement that the same spacetime is shared by all probes independently of their energy state or history. From this viewpoint there exists a fundamental tension between GR and QM. On one hand GR admits curvature and the spacetime geometry can “flex” at will. On the other hand QM is algebraic and its underlying structure is rigid. From the geometrical point of view, the superposition principle of QM implies that its geometry is projective. That is, the QM geometry is almost linear and constantly curved. Unifying the two geometries (the dynamical GR geometry and the rigid QM geometry) requires to “gravitize quantum mechanics”. In order to render QM more flexible, we need to relax its geometry. We will see that this is one of the messages of metastring theory.

We start by emphasizing the role of Born reciprocity, or duality (as a motivation for relative locality) in string theory. We begin our discussion [10] by examining the Polyakov action coupled to a flat metric h [2]

$$S_P(X) = \frac{1}{4\pi} \int_{\Sigma} h_{\mu\nu} (*dX^{\mu} \wedge dX^{\nu}), \quad (1)$$

where $*$, d denote the Hodge dual and exterior derivative on the worldsheet, respectively. We generally will refer to local coordinates on Σ as σ, τ , while it is traditional to interpret X^{μ} as local coordinates on a target space M , here with Minkowski metric $h_{\mu\nu}$. Since we are in Lorentzian signature, $*d\tau = d\sigma$ and $*^2 = 1$. Note that S_P has dimensions of length-squared if we take X^{μ} to have dimension of length, so appears in the path integral as e^{iS_P/λ^2} . λ is the string length which is related to the slope parameter by $\lambda^2 \equiv \alpha' \hbar$, where \hbar is the Planck constant of the worldsheet quantum theory. With this definition S_P/λ^2 has the usual coefficient $\frac{1}{4\pi\alpha'}$ in units of \hbar . In order for the Polyakov action to be well-defined, one must demand that the integrand be single-valued on Σ . For example, on the cylinder $(\sigma, \tau) \in [0, 2\pi] \times \mathbb{R}$ it would be sufficient that $dX^{\mu}(\sigma, \tau)$ is periodic¹ with respect to σ with period 2π . However, and this is a crucial point, this does not mean that $X^{\mu}(\sigma, \tau)$ has to be a periodic function, even if M is non-compact. Instead, it means that X^{μ} must be a *quasi-periodic* function which satisfies $X^{\mu}(\sigma + 2\pi, \tau) = X^{\mu}(\sigma, \tau) + \delta^{\mu}$. Here δ^{μ} is the quasi-period, or monodromy, of X^{μ} . If δ^{μ} is not zero, there is no *a priori* geometrical interpretation of a closed string propagating in a flat space-time – periodicity goes hand-in-hand with a space-time interpretation. Of course, such monodromies are encountered in textbook string theory [2] if the background is compact. However, we will see later in this talk that in the metastring formulation such monodromies are generic, even for non-compact backgrounds.

Now, let us consider the dimensionless momentum and position densities:

$$P \equiv \partial_{\tau} X / \lambda, \quad Q \equiv \partial_{\sigma} X / \lambda. \quad (2)$$

The dynamics of the canonical free string is characterized by the following Hamiltonian and differemorphism constraints:

$$H \equiv P^2 + Q^2 = 0, \quad D \equiv P \cdot Q = 0. \quad (3)$$

This is symmetric under the exchange $P \rightarrow Q, Q \rightarrow -P$, as noticed by Veneziano [18]. But this is just the statement of Born reciprocity: physics should be formulated in a way that is democratic from the position and momentum point of view [17]. This concept of Born reciprocity can be precisely related to one of the main features of string theory, to wit, that of T-duality [2]. In toroidal compactifications, the string does not distinguish radius R from α'/R : the spectrum of the string is simply, relabeled, with momentum and winding modes being exchanged. The

¹ The most general condition would be to ask that $dX^{\mu}(\sigma+2\pi) = \Lambda^{\mu}_{\nu} dX^{\nu}(\sigma)$ where Λ is a Lorentz transformation. For simplicity, in this talk we only consider the case where $\Lambda = 1$.

usual string solutions (the solution of the 2d wave equation) are chiral (i.e. left (L) and right (R)) combinations

$$X(\tau, \sigma) = X_L(\tau - \sigma) + X_R(\tau + \sigma). \quad (4)$$

In order to discuss T-duality we consider the other, dual, combination

$$\tilde{X}(\tau, \sigma) = X_L(\tau - \sigma) - X_R(\tau + \sigma). \quad (5)$$

Thus T-duality exchanges momentum density (P) with the position density (Q). As such, T-duality is a Born duality.

We can make this observation more precise. T-duality can be thought of as acting non-trivially on a choice of polarization. In fact, it can be thought of as a ‘‘Fourier transform’’ [10]. Usually in toroidal compactifications, we interpret short distance (radius) as long distance in a dual spacetime. However, we can also directly relate this to a Fourier transform. In particular, consider a string state

$$\Psi[x(\sigma)] = \int_{X|\partial\Sigma=x} [DXDg] e^{iS_P[X]/\lambda^2}. \quad (6)$$

Define a Fourier transform of this state by

$$\tilde{\Psi}[y(\sigma)] \equiv \int [Dx] e^{i/\hbar \int_{\partial\Sigma} x^\mu dy_\mu} \Psi[x(\sigma)]. \quad (7)$$

Extending $y(\sigma)$ to the worldsheet, we integrate out X to obtain a *dual* Polyakov path integral

$$\tilde{\Psi}[y(\sigma)] = \int_{Y|\partial\Sigma=y} [DYDg] e^{-iS_P[Y]/\epsilon^2}. \quad (8)$$

Note that $\alpha'p = \int_C *dX$ and $\delta = \int_C dX$ exchange roles, so that in the dual description $\delta = \alpha' \int_C *dY$ and $p = \int_C dY$.

Metastring theory developed in our papers [10, 11, 12, 13, 14], represents a reformulation of the Polyakov (bosonic) string theory, written in such a way that T-duality acts linearly on coordinates and that by so doing spacetime and energy-momentum space are put on equal footing in a unified phase space formulation. This re-interpretation of the string path integral is well defined as is the usual quantum string theory. The principal difference, however, is that in the case of metastring theory, the boundary conditions are relaxed, allowing for general monodromies $X^\mu(\sigma + 2\pi, \tau) = X^\mu(\sigma, \tau) + \delta^\mu$.

3. Metastring Theory: classical and quantum

At the classical level, metastring theory [10, 11, 12, 13, 14] can be thought of as a formulation in which the target space is doubled in such a way that T-duality acts linearly on the coordinates. This doubling means that momentum and winding modes appear on an equal footing. We refer to the target space as a phase space since the metastring action requires the presence of a background symplectic form ω . The metastring formulation also requires the presence of geometrical structures that generalize to phase space the spacetime metric and the B -field. In fact, in the metastring we have not one but *two* notions of a metric. The first metric η is a neutral metric that defines a bi-Lagrangian structure and allows to define the classical spacetime as a Lagrangian sub-manifold² — more precisely, the classical space-time is defined as a null subspace for η which is also Lagrangian for ω . The second metric H is a metric of signature

² We remind the reader that in symplectic geometry, a Lagrangian subspace is a half-dimensional submanifold of phase space upon which the symplectic form pulls back to zero. In simple terms, a Lagrangian submanifold might be the subspace coordinatized by the q 's within the phase space coordinatized by q 's and p 's.

$(2, 2(D-1))$ that encodes the geometry along the classical spacetime (of dimension D) as well as the transverse energy-momentum space geometry. In this formulation, T-duality exchanges the Lagrangian sub-manifold with its image under $J = \eta^{-1}H$. Classical metastring theory is defined by Tseytlin's action [19, 20, 21]

$$\hat{S} = \frac{1}{4\pi} \int_{\Sigma} d^2\sigma \left(\partial_{\tau} \mathbb{X}^A (\eta_{AB} + \omega_{AB}) \partial_{\sigma} \mathbb{X}^B - \partial_{\sigma} \mathbb{X}^A H_{AB} \partial_{\sigma} \mathbb{X}^B \right), \quad (9)$$

where \mathbb{X}^A are dimensionless coordinates on phase space and the fields η, H, ω are all dynamical (i.e., in general dependent on \mathbb{X}) phase space fields. In the context of a flat metastring we have constant η_{AB} , H_{AB} and ω_{AB}

$$\eta_{AB} \equiv \begin{pmatrix} 0 & \delta \\ \delta^T & 0 \end{pmatrix}, \quad H_{AB} \equiv \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix}, \quad \omega_{AB} = \begin{pmatrix} 0 & \delta \\ -\delta^T & 0 \end{pmatrix}, \quad (10)$$

where δ_{ν}^{μ} is the d -dimensional identity matrix and $h_{\mu\nu}$ is the d -dimensional Lorentzian metric, T denoting transpose.

In this formulation it is convenient, as suggested by the double field formalism [22, 23], to introduce dimensionless coordinates $\mathbb{X}^A \equiv (X^{\mu}/\lambda, P_{\mu}/\epsilon)^T$ on phase space, where λ and ϵ represent the fundamental spacetime and energy-momentum scales. As already stated, $\hbar = \lambda\epsilon$ and $\alpha' = \frac{\lambda}{\epsilon}$. Given a pair (H, η) it is natural to consider the operator $J \equiv \eta^{-1}H$. The consistency of string theory requires J to be a chiral structure, that is, a real structure ($J^2 = 1$) compatible with η , implying that J is an $O(D, D)$ transformation (realizing generalized T-duality in target space). These three structures, the symplectic $Sp(2D)$ ω , the $O(D, D)$ η and the $SO(2, 2(D-1))$ H , define the new concept of Born geometry [10, 11, 12, 13, 14] which unifies the complex geometry of quantum theory with the metrical geometry of general relativity and the symplectic geometry of canonical Hamiltonian dynamics [24, 25, 26, 27, 28]. Note that in the phase space formulation the local phase space coordinates \mathbb{X} are *quasiperiodic*

$$\mathbb{X}^A(\sigma + 2\pi) = \mathbb{X}^A(\sigma) + \Delta^A, \quad (11)$$

where Δ^A is the corresponding quasiperiod (which either vanishes for the canonical Polyakov string or is given by the winding number in the usual treatment of T-duality on compact spaces [2].)

The worldsheet Σ is a topological surface of genus g which possesses n boundaries. The first important point is that, in order to formulate the Tseytlin action we need to distinguish between σ and τ . This means that (the Lorentzian worldsheet) Σ is equipped with what we call a *causal structure*: That is, we assume that there exists a time function $\tau : \Sigma \rightarrow \mathbb{R}$ such that τ is a Morse function and such that $\partial\Sigma = \partial\Sigma_{-} \cup \partial\Sigma_{+}$ where $\partial\Sigma_{\pm} = \tau^{-1}(\pm\infty)$ [12]. By demanding that Tseytlin's action is well defined on Σ we have to impose that $\partial_{\tau}\mathbb{X}$ and $\partial_{\sigma}\mathbb{X}$ are single valued on Σ , i.e. they are periodic. But as we already emphasized, this *does not mean* that \mathbb{X} is a periodic function. This is a crucial point on which our generalization rests. The proper mathematical implementation of this idea is that instead of parametrizing the action by a sets of coordinates \mathbb{X}^A on \mathcal{P} , we need to parametrize it by a *closed one form* $\delta^A = \delta_{\sigma}^A d\sigma + \delta_{\tau}^A d\tau$ valued into the tangent space $T\mathcal{P}$, $d\delta^A = 0$. Such a form possesses *monodromies*. For each cycle γ we have

$$\Delta_{\gamma}^A = \int_{\gamma} \delta^A. \quad (12)$$

Since δ^A is closed the monodromy depends only on the homology of C . In particular, this means that if the string goes through an interaction point and splits, then the monodromy after the

splitting is the sum of the monodromies before the split, $\Delta_{\gamma_{12}} = \Delta_{\gamma_1} + \Delta_{\gamma_2}$. Therefore, the set of monodromies should form a lattice Γ . That is, if $\Delta, \Delta' \in \Gamma$ then $\Delta + \Delta' \in \Gamma$. This implies that the Tseytlin action on a generic surface depends on monodromies $\Delta \in \Gamma$ and that the flat Tseytlin action on a generic surface should be written as

$$S_{\Delta} \equiv \frac{1}{2\pi} \int_{\Sigma} ((\eta_{AB} + \omega_{AB})\delta_{\tau}^A \delta_{\sigma}^B - H_{AB}\delta_{\sigma}^A \delta_{\sigma}^B), \quad (13)$$

where δ^A is a closed form with fixed monodromy $\gamma \in H_1(\Sigma) \rightarrow \Delta_{\gamma}^A$.

Now we can state the quantum definition of the metastring: The quantum metastring amplitude associated with Σ is a functional that depends on n_+ initial configurations \mathbf{x}_i and n_+ out configurations \mathbf{x}_o , where $\mathbf{x}(\sigma) = \int_0^{\sigma} \delta_{\sigma}^A d\sigma$ for each boundary circle. It is defined as

$$A_{\Sigma}(\mathbf{x}_i, \mathbf{x}_o) \equiv \sum_{\Delta \in \Gamma} \int_C D\tau D\sigma \int_{H^1(\Sigma, \Delta)} e^{iS_{\Delta}(\delta)} D\delta. \quad (14)$$

Here the sum is over all the monodromies in Γ and the first integral is over the moduli space of causal structure, which is the Lorentzian analog of the moduli space of complex structure by the GWKN theorem [29, 30]. The last integration is over all closed one forms with prescribed holonomies. This prescription is the metastring generalization of Polyakov's prescription for the fundamental string amplitude. There are two interesting things to notice about this amplitude. The first one is that $A_{\Sigma}(\mathbf{x}_i, \mathbf{x}_o)$ is necessarily a function periodic with respect to translations along the lattice Γ

$$A_{\Sigma}(\mathbf{x}_i + \Delta, \mathbf{x}_o + \Delta') = A_{\Sigma}(\mathbf{x}_i, \mathbf{x}_o), \quad \forall \Delta, \Delta' \in \Gamma. \quad (15)$$

The second one is that the form of the lattice Γ is restricted by the demand of worldsheet diffeomorphism invariance. The requirement that the coupling of the 2-dimensional causal structure to the metastring is invariant under worldsheet diffeomorphisms leads to two constraints, the Hamiltonian and diffeomorphism constraints. These can be expressed simply at the classical level as

$$H = \delta_{\sigma}^A H_{AB} \delta_{\sigma}^B = 0, \quad D = \delta_{\sigma}^A \eta_{AB} \delta_{\sigma}^B = 0. \quad (16)$$

At the quantum level they imply that

$$\frac{1}{2} \Delta^A \eta_{AB} \Delta^B = N - \tilde{N}, \quad \frac{1}{2} \Delta^A H_{AB} \Delta^B = N + \tilde{N} - 2, \quad (17)$$

where $N, \tilde{N} \in \mathbb{N}$ are positive integers. Imposing this conditions forces the lattice Γ to be a self-dual even lattice. Note that the constant metric η defines a non-degenerate pairing $X \cdot Y = \mathbb{X}^A \eta_{AB} \mathbb{Y}^B$. Then the first condition of (17) implies that if $\Delta, \Delta' \in \Gamma$ and if we define $\Delta_A \equiv \eta_{AB} \Delta^B$ and $\Delta \cdot \Delta' = \Delta^A \Delta'_A$, then $\Delta \cdot \Delta' = \frac{1}{2}(\Delta + \Delta') \cdot (\Delta + \Delta') - \frac{1}{2}\Delta \cdot \Delta - \frac{1}{2}\Delta' \cdot \Delta' \in \mathbb{Z}$. This means that Δ_A belongs to the dual lattice Γ^* . Note that the second constraints implies that the map $\Delta \mapsto J(\Delta)$ is an endomorphism of Γ to itself [31]. In fact we can show that the condition of modular invariance implies that $\Gamma = \Gamma^*$ [12].

Here is a summary of the crucial features of the metastring:

1) The worldsheet formulation of the metastring is chiral. Thus, even though the fields are doubled the central charges (left and right) are $c_L = c_R = D$ and we still have $D = 26$ for criticality.

2) The metastring is not manifestly invariant under the worldsheet Lorentz transformations.

3) The metastring contains monodromies $\mathbb{X}^A(\sigma + 2\pi) = \mathbb{X}^A(\sigma) + \Delta^A$.

4) The usual Polyakov string can be obtained by integrating out the dual \tilde{X} , for constant η and H backgrounds, and by supposing that the monodromies are in the kernel of $(\eta - \omega)$.

5) T-duality is implemented in target space by the action of the chiral J operator ($J \equiv \eta^{-1}H$, $J^2 = 1$): $\mathbb{X} \rightarrow J(\mathbb{X})$.

6) The target space of the metastring is not spacetime, but, to first order, a chiral phase space \mathcal{P} equipped by the symplectic structure ω , and the bilagrangian structure, and in particular, the polarization metric η which relates to the symplectic connection of the Fedosov deformation quantization [32] and thus leads to the star product of deformation quantization, and finally, the quantum H metric which relates to the complex structure in the context of geometric quantization [33], leading to the concept of Hilbert spaces. This classical Born geometry implements the ideas of Born duality in string theory.

7) The classical equations of motion of the metastring reads

$$\partial_\tau \mathbb{X}^A - (J\partial_\sigma \mathbb{X})^A = 0, \quad (18)$$

also implies the relation between momenta and monodromies $2\pi P = J(\Delta)$. There is *soldering* between worldsheet null coordinates $\sigma^\pm \equiv \sigma \pm \tau$ and the chiral target space structure $\partial_\pm \mathbb{X}^A - (P_\pm \mathbb{X})^A = 0$, where the chiral projector is defined as $2P_\pm = (1 \pm J)$. This allows us to liberate the left geometry from the right geometry (which is reminiscent of twistor theory).

8) The careful analysis of the metastring action [12] shows that its symplectic form is

$$\Omega = \frac{1}{4\pi} \int \delta \mathbb{X}^A \eta_{AB} \nabla_\sigma \delta \mathbb{X}^B, \quad (19)$$

where ∇ is the generalized Fedosov connection found in the Fedosov deformation quantization approach [32].

9) The operator product expansion of the metastring vertex operators

$$V_k = \epsilon_k e^{i\mathbb{K}\mathbb{X}}, \quad (20)$$

leads to the restriction of \mathbb{K} on a double Lorentzian integral lattice Γ , that by modular invariance, must be self-dual. These exist in $D = 2 \bmod(8)$, and are unique. Criticality gives a very unique lattice

$$\Gamma = \Pi_{1,25} \times \Pi_{1,25}. \quad (21)$$

This fact, in turn, leads to the large symmetry structure found by Borchers in the study of the monstrous moonshine [34, 35]³.

10) As already noted, the metastring is chiral. This requires the introduction of a preferred worldsheet time coordinate which is fundamentally Lorentzian [12]. How can this be consistent with modular invariance? The answer is given by employing the Giddings-Wolpert-Krichever-Novikov (GWKN) construction [29, 30]: given a Riemann surface, provided a choice of local coordinates around punctures is labeled by one scalar, there exists a unique Abelian differential e with imaginary periods. The real part of this Abelian differential is the modular invariant time $\tau = \text{Re}(e)$. The zeros of e represent interaction points where the worldsheet Lorentzian cones double. Cutting the Riemann surface along the real trajectory of e we obtain a string decomposition of the surface. The Nakamura graphs [31] encode this decomposition and give a very effective cell decomposition of moduli space. Thus Nakamura graphs are the natural Feynman diagrams for closed strings [37].

³ For a string theory related discussion, see [36].

3.1. More General Backgrounds and Born Geometries

Although in this talk we will mainly address the flat metastring, it is instructive to briefly consider more general structures. In fact, it is quite straightforward to include a (constant) background in the Polyakov action

$$S_P(X) = \frac{1}{4\pi} \int (G_{\mu\nu}(X) * dX^\mu \wedge dX^\nu + B_{\mu\nu}(X) dX^\mu \wedge dX^\nu). \quad (22)$$

The simplest way to incorporate this curved background is to apply to the flat Tseytlin action an $O(d, d)$ transformation of the form

$$O^T = \begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^T & 0 \\ 0 & e^{-1} \end{pmatrix}. \quad (23)$$

This acts trivially on the constant η and has the effect of mapping

$$H \mapsto H = O^T H O \equiv \begin{pmatrix} [G - BG^{-1}B] & [BG^{-1}] \\ -[G^{-1}B] & [G^{-1}] \end{pmatrix}, \quad \omega \mapsto \omega = O^T \omega O \equiv \begin{pmatrix} -2B & \delta \\ -\delta & 0 \end{pmatrix}, \quad (24)$$

resulting in the general metastring action (9). Alternatively, this can be derived from the first order formulation of the Polyakov action

$$\hat{S} = \frac{1}{2\pi} \int \left(\frac{1}{\lambda\varepsilon} \mathbf{P}_\mu \wedge dX^\mu + \frac{1}{2\varepsilon^2} (\hat{G}^{\mu\nu} * \mathbf{P}_\mu \wedge \mathbf{P}_\nu + \hat{B}^{\mu\nu} \mathbf{P}_\mu \wedge \mathbf{P}_\nu) \right), \quad (25)$$

where we have defined

$$[\hat{G} + \hat{B}] \equiv [(G + B)^{-1}], \quad (26)$$

although one must be careful to preserve boundary terms in order to obtain the correct modification of ω .

Thus, the usual string theory in curved backgrounds corresponds to making ω and the H dynamical (but not the η). Let us discuss further generalizations. Given the existence of ω and H , there is a natural way to understand this geometrical structure (ω, H) from the point of view of quantum mechanics. If one takes the point of view of geometric quantization [33], the construction of the Hilbert space associated with a phase space (\mathcal{P}, ω) requires the introduction of a complex structure I compatible with ω . Such a complex structure defines the notion of coherent states as holomorphic functionals and equips the phase space with a quantum-metric via the relation $HI = \omega$ [10]. This structure is, in effect, what Born suggested to be part of quantum gravity in the 1930's [17]. In the string case, H is related to ω via a complex structure.

However, the Born proposal is not enough. As we have pointed out in [10], in metastring theory we must take η to be dynamical as well. As we have seen above, it is η that governs the splitting of phase space into space and momentum space. In particular, one can think of spacetime as a Lagrangian submanifold, that is a manifold of maximal dimension on which the symplectic structure vanishes. Analogously, momentum space is just another Lagrangian submanifold \tilde{L} in this description, which is transverse to the spacetime Lagrangian submanifold. Thus we end up with a bilagrangian structure on \mathcal{P} . That is a decomposition of \mathcal{P} into two transverse Lagrangian manifolds: $T\mathcal{P} = \mathcal{TL} \oplus \mathcal{T}\tilde{L}$ and $TL \cap T\tilde{L} = \{0\}$. What is remarkable is the fact that a bilagrangian structure is uniquely characterized by a polarization metric η . This metric is characterized by the fact that $L = \ker(\eta + \omega)$ and $\tilde{L} = \ker(\eta - \omega)$. In other words, the geometrical notion of η is to provide a bilagrangian decomposition of phase space. The neutral metric η that seems like a purely stringy metric is in fact a very natural object from the point of view of phase space, in that it labels its decomposition into space and momentum. In order to

prove this, let us introduce a structure K which is $+1$ on the vectors tangent to the spacetime Lagrangian L and -1 on the momentum space Lagrangian \tilde{L} . This is a real structure which satisfies $K^2 = 1$. Since L and \tilde{L} are Lagrangians K also satisfies an anti-compatibility condition with ω : $K^T \omega K = -\omega$. These two properties in turn show that $\eta = \omega K$ is a neutral metric. We have already emphasized the importance of the endomorphism $J = \eta^{-1} H$, which relates the two metrics. Its properties enforce the chirality of the σ -model. We thus suppose that the geometry of \mathcal{P} should be constrained by the property $J^2 = 1$.

It is relative locality that suggests that both η and H be dynamical. In particular, in canonical quantum theory H is a purely kinematical structure and η , which describes the choice of polarization, can be modified by unitary dynamics. Conversely, in the context of gravitational dynamics, η is a purely kinematical structure (because spacetime provides the preferred basis or polarization), while H , through its spacetime part, can be made dynamical. According to Born, when we introduce gravity into quantum theory we have to make H into a truly dynamical quantity. When we introduce quantum theory into gravity, we have to make the neutral metric η dynamical, and thus in the context of quantum gravity, both H and η have to be dynamical.

The neutral metric η is, together with the generalized phase space metric H , indispensable for the definition of space-time as a maximally null subspace of η with the spacetime metric given by the restriction of the H metric to this η -null subspace [10]. The structure (ω, η, K) can also be described in terms of the two real structures J, K and the map I . We can check that the relation between these maps is given by

$$JK = I. \quad (27)$$

If, in addition, we assume that B vanishes we have that I is a complex structure and that $JK = -KJ$. Phase space geometries that have I, J, K satisfying these conditions were referred to as Born geometries in [10]. These possess para-quaternionic structure (because $I^2 = -1$, but $J^2 = 1 = K^2$ and they anticommute). Born geometry represents a natural unification of quantum and spacetime and phase space geometries, and it implies a new view on the kinematical and dynamical structure of quantum gravity.

Note that the above is a traditional sigma-model discussion, with some apparent relation to generalized geometry. However, in metastring theory we speak in terms of phase space \mathcal{P} instead of a double spacetime, as is the case in generalized geometry and double field theory. In fact, now we think that the proper interpretation of the phase space \mathcal{P} equipped with a triplet of structures, the symplectic structure ω , the polarization metric η and the generalized metric H , can be found in the *geometry of modular quantization* [12, 13, 14]. Thus we turn to the subject of geometry of quantum theory.

4. Quantum Geometry: Phase Space, Heisenberg Group and Modular Variables

In this section we re-examine the geometry of quantization, in order to find a very revealing and unsuspected connection with metastring theory. We focus on the Heisenberg (or Weyl-Heisenberg) group, which is generated, on the level of the corresponding algebra, by familiar position \hat{q}^a and momentum \hat{p}_b operators:

$$[\hat{q}^a, \hat{p}_b] = i\hbar\delta_b^a. \quad (28)$$

It will be convenient to introduce a length scale λ and a momentum scale ϵ , with $\lambda\epsilon = \hbar$. Then, let us introduce the following suggestive notation

$$\hat{x}^a \equiv \hat{q}^a/\lambda, \quad \hat{x}_a \equiv \hat{p}_a/\epsilon, \quad [\hat{x}^a, \hat{x}_b] = i\delta_b^a. \quad (29)$$

Even more compactly let us, once again, suggestively write

$$\mathbb{X}^A \equiv (x^a, \tilde{x}_a)^T, \quad [\hat{\mathbb{X}}^a, \hat{\mathbb{X}}^b] = i\omega^{AB}, \quad (30)$$

with $\frac{1}{2}\omega_{AB}dX^AdX^B = \frac{1}{\hbar}dp_a \wedge dq^a$, where $\omega_{AB} = -\omega_{BA}$ is the canonical symplectic form on phase space \mathcal{P} . The Heisenberg group $H_{\mathcal{P}}$ is generated by Weyl operators [38]

$$W_{\mathbb{K}} \equiv e^{2pii\omega(\mathbb{K},\mathbb{X})}. \quad (31)$$

These form a central extension of the translation algebra

$$W_{\mathbb{K}}W_{\mathbb{K}'} = e^{2pii\omega(\mathbb{K},\mathbb{K}')}W_{\mathbb{K}+\mathbb{K}'}. \quad (32)$$

The projection $\pi : H_{\mathcal{P}} \rightarrow \mathcal{P}$ (where $\pi : W_{\mathbb{K}} \rightarrow \mathbb{K}$) defines a line bundle over \mathcal{P} (in principle a covariant phase space of quantum probes). In this formulation, states are sections of degree one

$$W_{\mathbb{K}'}\Phi(\mathbb{K}) = e^{2\pi i\omega(\mathbb{K},\mathbb{K}')} \Phi(\mathbb{K} + \mathbb{K}'). \quad (33)$$

In this language, geometric quantization means to take a Lagrangian $L \in \mathcal{P}$, so that states descent to square integrable functions on L .

A Lagrangian submanifold L is a maximally isotropic subspace L with $\omega|_L = 0$, and thus $\{\partial/\partial q^a\} \in T\mathcal{P}$ defines a Lagrangian submanifold, or “space”. (Indeed, $\omega(\partial/\partial q^a, \partial/\partial q^b) = 0$.) This can be understood as a classical characterization of space (and in the covariant context, of spacetime), as a “slice” of phase space. How about a purely quantum characterization of space?

Note that for space-like separations the operators of a local quantum field theory commute. Thus in order to understand the meaning of quantum spacetime (quantum Lagrangian), we need to look at a maximally commuting subalgebra of the Heisenberg algebra and the representation that diagonalizes it. Thus, borrowing from notions of non-commutative algebra and non-commutative geometry [39] (such as the theorem of Gelfand-Naimark [40]), we can say that a Lagrangian submanifold is a maximally commutative subgroup of the Heisenberg group. If we accept this notion of a Lagrangian, then the quantum regime is very different from the classical regime. In particular the vanishing Poisson bracket $\{f(q), g(p)\}$ requires either f or g to be constant. However, the vanishing commutator $[f(\hat{q}), g(\hat{p})] = 0$ requires only that the functions be commensurately periodic

$$e^{i\alpha\hat{p}}e^{i\beta\hat{q}} = e^{i\hbar\alpha\beta}e^{i\beta\hat{q}}e^{i\alpha\hat{p}}, \quad \alpha\beta = 2\pi/\hbar. \quad (34)$$

What is interesting here is that similar considerations led Aharonov to introduce *modular variables* to describe purely quantum phenomena, such as interference [41].

4.1. Modular variables

Modular variables are described in great detail in the very insightful book by Aharonov and Rohrlich [41], where one can find detailed bibliography on this subject⁴. The fundamental question posed there was as follows: how does one capture interference effects (due to the fundamental linearity of quantum theory) in terms of Heisenberg operators? For example, what are the quantum observables that can measure the relative phase responsible for interference in a double-slit experiment? No polynomial functions of the operators \hat{q} and \hat{p} can detect such phases, but operators that translate in space, such as $e^{iR\hat{p}/\hbar}$, do. Thus the modular variables denoted $[\hat{q}]$ and $[\hat{p}]$, which are defined modulo a length scale R (the slit spacing being a natural choice), play a central role, where

$$[p]_{h/R} = p \bmod (h/R), \quad [q]_R = q \bmod (2\pi R), \quad (35)$$

and $h = 2\pi\hbar$. The shift operator $e^{iR\hat{p}/\hbar} = e^{iR[\hat{p}]/\hbar}$ shifts the position of a particle state (say an electron in the double-slit experiment) by a distance R and is a function of the modular momenta.

⁴ See also [42] and [43].

These modular variables satisfy non-local operator equations of motion. For example, given the Hamiltonian, $\hat{H} = \hat{p}^2/2m + V(\hat{q})$, the Heisenberg equation of motion for the shift operator is,

$$e^{-iR\hat{p}/\hbar} \frac{d}{dt} e^{iR\hat{p}/\hbar} = -\frac{iR}{\hbar} \left(\frac{V(\hat{q} + R) - V(\hat{q})}{R} \right). \quad (36)$$

Modular variables are fundamentally non-local in a non-classical sense, since we see here that their evolution depends on the value of the potential at distinct locations. Remarkably, thanks to the uncertainty principle, this dynamical non-locality does not lead to a violation of causality [41]. One of the characteristic features of these variables is that they do not have classical analogues; indeed, the limit $\hbar \rightarrow 0$ of $[p]_{\hbar/R}$ is ill-defined.

Note that modular variables are contextual in general⁵. In other words, they carry specific experimental information, such as the length R between the two-slits. However, in the context of quantum gravity such scales are automatically built in, and the contextuality is in principle removed. Also, the fundamental dynamical equations for modular variables are non-local in quantum gravity because of the presence of the fundamental length.

When exponentiated (i.e. when understood as particular Weyl operators), the modular variables naturally commute. In other words, given $[x^a, \tilde{x}_b] = \frac{i}{2\pi} \delta_b^a$, the following commutator of modular operators vanishes [14]

$$[e^{2\pi i x}, e^{2\pi i \tilde{x}}] = 0. \quad (37)$$

Thus a quantum algebra of modular variables possesses more commutative directions than a classical Poisson algebra, because the Poisson bracket of modular variables does not vanish, $\{e^{2\pi i x}, e^{2\pi i \tilde{x}}\} \neq 0$.

Here we make a historical note [44]: The above non-local equations of motion were essentially written by Max Born, in the very first paper which used the phrase ‘‘Quantum Mechanics’’ in its title, in 1924, one year before the Heisenberg breakthrough paper. Actually, Heisenberg crucially used Born’s prescription of replacing classical equations by the corresponding difference equations, in order to derive what we now call the canonical commutation relations (properly written by Born and Jordan) from the Bohr-Sommerfeld quantization conditions.

5. Modular Space and Geometry of Quantum Theory

Returning to the subject of quantum Lagrangians, note that the quantum Lagrangian is analogous to a Brillouin cell in Condensed Matter physics. The volume and shape of the cell are given by λ and ϵ (i.e. \hbar and $G_N(\alpha')$). The uncertainty principle is implemented in a subtle way: we can specify a point in modular cell, but if so, we can’t say *which* cell we are in.

This means that there is a more general notion of quantization, beyond that of geometric quantization. Instead of selecting a classical polarization L (the arguments of the wave function, or the arguments of a local quantum field) we can choose a *modular polarization*. In terms of the Heisenberg group all that is happening is that in order to have a commutative algebra, we need only

$$\omega(\mathbb{K}, \mathbb{K}') \in 2Z, \quad W_{\mathbb{K}} W_{\mathbb{K}'} = e^{2\pi i i \omega(\mathbb{K}, \mathbb{K}')} W_{\mathbb{K}+\mathbb{K}'} = W_{\mathbb{K}'} W_{\mathbb{K}}. \quad (38)$$

This defines a lattice Λ in phase space \mathcal{P} . Finally, we specify a ‘‘lift’’ of the lattice from the phase space \mathcal{P} to the Heisenberg group $H_{\mathcal{P}}$.

Maximally commuting subgroups $\hat{\Lambda}$ of the Heisenberg group correspond to lattices that are integral and self-dual with respect to ω [45, 46]. Given W_{λ} where $\lambda \in \Lambda$ there is a lift to $\hat{\Lambda}$ which defines ‘‘modular polarization’’

$$U_{\lambda} = \alpha(\lambda) W_{\lambda}, \quad (39)$$

⁵ Aharonov and collaborators have pushed the logic associated with modular variables to argue for a new kind of weak measurements of such non-local variables that capture the superposition principle of quantum theory. Similarly, Aharonov and collaborators argue for a time symmetric formulation of quantum theory [41].

where $\alpha(\lambda)$ satisfies the co-cycle condition

$$\alpha(\lambda)\alpha(\mu)e^{pii\omega(\lambda,\mu)} = \alpha(\lambda + \mu), \quad \lambda, \mu \in \Lambda. \quad (40)$$

One can parametrize a solution to the co-cycle condition by introducing a symmetric bilinear form from η and setting

$$\alpha_\eta(\lambda) \equiv e^{i\frac{pi}{2}\eta(\lambda,\lambda)}. \quad (41)$$

Finally, when we choose a classical Lagrangian L , there is a special state that we associate with the vacuum: it is translation invariant (which in our context can be interpreted as “empty space”). In modular quantization, there is no such translation invariant state (because of the lattice structure). The best we can do is to choose a state that minimizes an “energy”, which requires the introduction of another symmetric bilinear form, that we call, again suggestively, H . This means, first, that we are looking for operators such that

$$[\hat{\mathbb{P}}_A, \Phi] = \frac{i}{2\pi} \partial_A \Phi, \quad \Phi(\hat{\mathbb{X}} + \lambda) = \Phi(\hat{\mathbb{X}}), \quad (42)$$

where the modular observables $\Phi(\hat{\mathbb{X}} + \lambda) = \Phi(\hat{\mathbb{X}})$ are generated by the lattice observables U_λ with $\lambda \in \Lambda$. Translation invariance would be the condition $\hat{\mathbb{P}}|0\rangle = 0$. Since this is not possible the next natural choice, is to minimize the translational energy. Therefore we pick a positive definite metric H_{AB} on \mathcal{P} , and we define

$$\hat{E}_H \equiv H^{AB} \hat{\mathbb{P}}_A \hat{\mathbb{P}}_B, \quad (43)$$

and demand that $|0\rangle_H$ be the ground state of \hat{E}_H . This is indeed the most natural choice and it shows that we cannot fully disentangle the kinematics (i.e., the definition of translation generators) from the dynamics. In the Schrödinger case, since the translation generators commute, the vacuum state $\hat{E}|0\rangle = 0$ is also the translation invariant state and it carries no memory of the metric H needed to define the energy. In our context, due to the non-commutativity of translations, the operators \hat{E}_H and $\hat{E}_{H'}$ do not commute. As a result the vacuum state depends on H , in other words $|0\rangle_H \neq |0\rangle_{H'}$, and it also possesses a non-vanishing zero point energy.

Thus, modular quantization involves the introduction of three quadratic forms (ω, η, H) , i.e. Born geometry. Therefore, quite surprisingly, Born geometry underlies the concept of Aharonov’s modular variables.

5.1. Born Geometry and Quantum Theory

This triple of structures has appeared in the context of metastring theory. In that context a choice of polarization is a choice of a spacetime within \mathcal{P} but the most general choice is a *modular polarization* that we have discussed above. From this new point of view Born geometry (ω, η, H) arises as a parametrization of such quantizations, which results in a notion of quantum spacetime, that we call *modular spacetime*. Finally, large spacetimes of canonical string theory and general relativity result as a “many-body” phenomenon, through a process of tensoring of unit modular cells, that we refer to as “extensification” [14].

In particular, the symplectic structure ω

$$ds_\omega^2 = \frac{1}{2} \omega_{AB} d\mathbb{X}^A d\mathbb{X}^B = \frac{1}{\hbar} dp_a \wedge dq^a, \quad (44)$$

is encoded in the canonical Heisenberg commutator between q^a and p_a . The generalized, quantum, metric H comes from the Born rule in quantum theory

$$ds_H^2 = H_{AB} d\mathbb{X}^A d\mathbb{X}^B = \frac{1}{\hbar} \left(\frac{dq_a dq^a}{G_N} + G_N dp_a dp^a \right). \quad (45)$$

For weak gravity, this metric reduces to the spacetime metric (where spacetime can be viewed as a slice of phase space). Due to gravity's extreme weakness, we only see spacetime metric at low energies. (The ratio ϵ/λ defines a tension; if this is identified with c^3/G_N , it is enormous, $\sim 10^{32}kg/sec.$) Therefore, in this formulation the usual dynamical spacetime metric is the low energy leftover of the quantum metric. Finally, the polarization (or locality metric) η encodes the distinction between spacetime-like and energy-momentum-like aspects of phase space (and in this sense it defines an analog of the "causal" structure in phase space)

$$ds_\eta^2 = \eta_{AB}d\mathbb{X}^A d\mathbb{X}^B = \frac{2}{\hbar} dp_a dq^a. \quad (46)$$

This new metric captures the essence of relative locality - when η is constant we have absolute locality. Curving η also means "gravitizing the quantum". In general all three elements of Born geometry, ω , η and H are dynamical and curved in metastring theory, as we will discuss in section 7.

Before doing so, let us end by a few comments regarding the Stone-von Neumann theorem [47, 48, 49] which asserts that all representations of the Heisenberg group are unitarily equivalent. Normally, we think of this as a choice of basis in phase space (a choice of polarization or classical Lagrangian), and all such choices are related by Fourier transform. Similarly, one can pass from a classical polarization (such as the Schrödinger representation) to a modular polarization via the Zak transform [50]. Note that, there is a connection on the line bundle over phase space that has unit flux through a modular cell. (This is very similar to integer quantum Hall effect systems.) A modular wave function is quasi-periodic

$$\Psi(x + a, \tilde{x}) = e^{2i\pi a \tilde{x}} \Psi(x, \tilde{x}), \quad \Psi(x, \tilde{x} + \tilde{a}) = \Psi(x, \tilde{x}). \quad (47)$$

The quasi-periods correspond to the tails of an Aharonov-Bohm [51] potential attached to a unit flux. In particular, vacuum states must have at least one zero in a cell, which leads to theta functions (the Zak transforms of Gaussians). Note that from the point of modular polarization, the familiar Schrödinger polarization is just a singular limit.

6. Modular Spacetime and Metastring Theory

Now we come to our main point. Is there any real evidence that we should regard the above discussion of modular polarization as connected to quantum gravity in some way? In fact, the triple of data $(\mathcal{P}; \omega, \eta, \mathcal{H})$ is in one-to-one correspondence with the geometric data underlying metastring, which indeed has the string length built in (λ). As well, the vertex operator algebra of the metastring gives copies of the Heisenberg group (in the zero mode sector, with decorations by oscillator modes) [12].

When gravity is turned on, we should regard modular quantization as a gravitization of the quantum, which makes the whole of Born geometry dynamical. According to the picture presented above, the modular cell is a quantum unit of spacetime - the spacetime qubit. More generally, we can extensify the modular cell, by tensoring many units together, resulting in the large flux. The procedure of extensification can lead to large spacetimes, and in principle, to arbitrary spacetime backgrounds (including known and unknown compactifications).

To summarize: Usually, the notion of space determines what set of commuting measurement can be performed. We have reversed this logic and defined modular space as the maximal set of commutative operations ($[e^{2\pi i x}, e^{2\pi i \tilde{x}}] = 0$). In this sense modular spaces are fundamentally quantum. Also, modular spaces turn out to have, by construction, built in length and energy scales. Note that in contrast to a one dimensional Euclidean space, which is non-compact and simply connected, one dimensional modular space is actually two dimensional, it is compact and not simply-connected (it carries Aharonov-Bohm flux [51]). One can view the unit phase

space cells equipped with the unit Aharonov-Bohm flux lines, as the spacetime qubits, the basic building blocks of quantum spacetime. In principle, we now have superpositions and entanglements of quantum spacetimes.

The surprising fact is that modular spaces are realized in string theory, in its metastring formulation, because the target space of a closed metastring is a modular space. (That is, it is not a doubled space, and it is not a compactified space. Also, modularity is the target space realization of T-duality.) In the canonical Polyakov string, the target space is interpreted as spacetime. In the metastring, the target space is a modular space that comes equipped with the triple structure of Born geometry $(\mathcal{P}; \omega, \eta, \mathcal{H})$. From this point of view, the usual spacetime backgrounds are just Lagrangian submanifolds. Note that if we suppose this to be a modular Lagrangian, it is associated with a lattice which turns out to be unique. This comes from a careful study of the operator algebra of the string, closely associated with the analysis of the Heisenberg group [12]. In particular, mutual locality of physical states and modular invariance, require that lattice be Lorentzian even and self-dual. This is the unique Lorentzian lattice associated with the famous mathematical work of Borcherds [34, 35] $\Lambda = \Pi_{1,25} \times \Pi_{1,25}$. Note that this is the same as “Narain lattice of fully compactified space-time” [36]. However, the traditional interpretation of this lattice, is nonsensical because there is no apparent causal interpretation - i.e. the time is compact as well as space [36]. Our point is that in the modular spacetime interpretation, such problems are lifted. Moreover, the modular quantization, reconciles the apparent lack of Lorentz symmetry of this lattice structure, taken as the basic building block of quantum spacetime, as we will discuss below.

The metastring possesses a huge symmetry group generated by all dimension $(1, 0) + (0, 1)$ operators, that is, the products of Borcherds algebras. (In principle, all the usual string backgrounds, a presumably many more, are extensions of these algebras.) The simple roots of the Borcherds algebra in the lattice L are either tachyonic or in L . The tachyonic roots generate a compact subgroup. Also, T-duality is represented by a rotation by an angle π . The lattice L is given as

$$L \equiv \{K \in I_{1,25} | K^2 = 2, K \cdot \rho = -1\} \quad (48)$$

and is isomorphic to the Leech lattice $\rho \equiv (0, 1, 2, \dots, 24|70)$. The null roots generate the Heisenberg group needed for the modular construction. The Borcherds symmetry is effectively broken in the usual Polyakov string by choosing various subsectors. How does one recover the usual string? The monodromies of the usual string are $\Delta^A = (0, \tilde{\delta}_\mu)$. The momenta of the usual string are $K^A \equiv J(\Delta)/2\pi = (\tilde{\delta}_\mu/2\pi, 0)$. For the metastring we have $K_A = (k_\mu, \tilde{k}^\mu)$. The usual string spectrum is recovered if we truncate the metastring spectrum to operators such that $|k| \gg |\tilde{k}|$. The usual locality (absolute locality) limit is a limit of large quantum numbers, i.e. it is a semi-classical limit.

6.1. Modular space, Lorentz symmetry and Relative locality

So how does our construction resolve the fundamental conundrum of any quantum theory of gravity: how to make continuous symmetries (such as the Lorentz symmetry) consistent with the existence of a fundamental scale that is characteristic of quantum (modular) spaces [14] ?

A useful analogy for the reconciliation of discreteness of quantum spacetime and continuity of Lorentz symmetry is provided by angular momentum in quantum mechanics. (In some sense, the spacetime qubits are the direct analog of spin.) Classically, the angular momentum describes a point on a sphere, and its canonical actions are given by rotations which are symmetries of the sphere. Quantum mechanically, representations are discrete; the eigenvalue of the angular momentum is evenly spaced and this can be interpreted as a discretization of the sphere, in which it is replaced by a discrete set of circles equally spaced along, say, the z-axis. The lattice apparently destroys the rotational symmetry of the classical description. However, we know that

is not true: quantum theory restores the rotational symmetry by arranging for a superposition of spin states which in turn parametrize a sphere's-worth of states. The spin states are merely a basis for the entire state space, and the basis is not invariant under rotations.

How does this analogy work in the context of modular spacetime? In the Schrödinger representation, rotations act within the set of states $|x\rangle$ (using a fully covariant notation), giving back a linear combination of these states. In the modular polarization, clearly, we should regard the Lorentz rotations as acting on the basis, resulting in a new choice of modular cell, i.e., a new 'quantization axis'. Such a transformation corresponds to a canonical transformation.

As shown in [14], in the context of spatial rotations but fully applicable to Lorentz rotations, once we introduce a fundamental scale, the natural polarization that respects the presence of this scale introduces a bilagrangian lattice [45] $\Lambda = \ell \oplus \tilde{\ell}$ embedded in phase space \mathcal{P} , and equipped with a neutral metric η . Moreover, the Hilbert space is given by a space of sections of a line bundle L_Λ over the torus $T_\Lambda = \mathcal{P}/\Lambda$ and that the embedding $T_\Lambda \hookrightarrow L_\Lambda$ is characterized by a lift, $\alpha_\eta \in U(1)$. Lorentz rotations act on this fully covariant phase space as symplectic transformations

$$\mathbb{X} = (x, \tilde{x}) \mapsto O \cdot \mathbb{X} = (Ox, O^T \tilde{x}), \quad O \in O(1, d-1). \quad (49)$$

These transformations not only preserve ω , but they also preserve the polarization metric η and the quantum metric H . Remarkably, we can show that Lorentz rotations are in fact the *only* transformations that preserve the given Born geometry (ω, η, H) of the modular space. Let us consider the basic triplet of structures associated with Born geometry, $(\mathcal{P}, \omega, \eta, H)$, where ω is a symplectic structure, η a polarisation metric and H a quantum metric. In Darboux coordinates, we have

$$\omega^0 = dx^a \wedge d\tilde{x}_a, \quad \eta^0 = dx^a d\tilde{x}_a, \quad H^0 = dx^a h_{ab} dx^b + d\tilde{x}_a \tilde{h}^{ab} d\tilde{x}_b, \quad (50)$$

where $h_{ab} = \text{diag}(-1, 1, \dots, 1)$ is a flat Lorentzian metric and \tilde{h}^{ab} its inverse. We now consider a linear coordinate transformation

$$\mathbb{U} := \begin{pmatrix} \alpha^a_b & \gamma^{ab} \\ \beta_{ab} & \delta_a^b \end{pmatrix}, \quad \mathbb{X} \mapsto \mathbb{U}\mathbb{X} = \begin{pmatrix} \alpha x + \gamma \tilde{x} \\ \beta x + \delta \tilde{x} \end{pmatrix}. \quad (51)$$

Under such a transformation the geometrical structure is modified to $(\mathcal{P}, \omega, \eta, H)$, where

$$\omega = [\alpha^T \beta]_{[ab]} dx^a \wedge dx^b + [\gamma^T \delta]^{[ab]} d\tilde{x}_a \wedge d\tilde{x}_b + [\alpha^T \delta - \beta^T \gamma]_a^b dx^a \wedge d\tilde{x}_b, \quad (52)$$

as well as

$$\eta = [\alpha^T \beta]_{(ab)} dx^a dx^b + [\gamma^T \delta]^{(ab)} d\tilde{x}_a d\tilde{x}_b + [\alpha^T \delta + \beta^T \gamma]_a^b dx^a d\tilde{x}_b, \quad (53)$$

and finally,

$$H = [\alpha^T h \alpha + \beta^T \tilde{h} \beta]_{(ab)} dx^a dx^b + [\delta^T \tilde{h} \delta + \gamma^T h \gamma]^{(ab)} d\tilde{x}_a d\tilde{x}_b + 2[\alpha^T h \gamma + \beta^T \tilde{h} \delta]_a^b dx^a d\tilde{x}_b. \quad (54)$$

The subset of transformations that preserves ω , η and H satisfy

$$\beta = \gamma = 0, \quad \alpha^T = \delta^{-1}, \quad \alpha^T h \alpha = h. \quad (55)$$

So we conclude that the group of transformations that preserves the background Born geometry is the Lorentz group! In other words, we have that the Lorentz group lies at the intersection of the symplectic, neutral and doubly orthogonal groups,

$$O(1, d-1) = \text{Sp}(2d) \cap O(d, d) \cap O(2, 2(d-1)). \quad (56)$$

Here $\text{Sp}(2d)$ is the symplectic group preserving ω , $\text{O}(d, d)$ is the neutral group preserving η and $\text{O}(2, 2(d - 1))$ the orthogonal (conformal) group preserving H . It is also interesting to understand the pairwise intersections. For instance, the group of transformations that preserves the symplectic structure and the polarization metric η is given by

$$\text{GL}(d) = \text{Sp}(2d) \cap \text{O}(d, d). \quad (57)$$

Here $\text{GL}(d)$ is the general linear group that represents the sets of frames e_i^a determining the vacuum metric

$$H(x, \tilde{x}) = x^a q_{ab} x^b + \tilde{x}_a q^{ab} \tilde{x}_b, \quad (58)$$

with $q^{ab} = e_i^a \delta^{ij} e_j^a$ and q_{ab} its inverse, while the space of vacuum metrics associated with a given Hilbert space is encoded in $\text{GL}(d)/\text{O}(d)$. The other pairwise intersections are

$$\text{U}(d) = \text{Sp}(2d) \cap \text{O}(2d), \quad \text{O}(d) \times \text{O}(d) = \text{O}(d, d) \cap \text{O}(2d). \quad (59)$$

The space $\text{U}(d)/\text{O}(d)$ represents the choice of inequivalent lifts η that preserve the symplectic structure and the vacuum state, while the quotient of the double orthogonal group by $\text{O}(d)$ represent the choice of inequivalent symplectic structures, hence Weyl groups, that possess the same Hilbert space with the same vacuum.

Given this new view of the Lorentz symmetry, as defined by the intersection of the triplet of geometric structures of Born geometry, how does modular quantization avoid breaking the Lorentz group into a discrete subgroup? The answer is offered by relative locality. Relative locality implies that this breaking cannot take place: a spacetime is a choice of polarization, and thus each observer makes his/her own choice. Each observer sees the discreteness, but Lorentz symmetry acts to change the polarization, restoring Lorentz invariance. (This is precisely what happens in the above spin analogy: choose a quantization axis, get discrete spin states, which viewed as a discretization of the sphere, break the rotation group, but rotations act to rotate the quantization axis, which is a choice of polarization, and because the spin operators do not commute the states of one axis are unitarily related to those of the other.) Thus, for Lorentz symmetry to be unbroken, we need, non-commutative torus in phase space and relative locality, that is, spacetime not being an arena shared by all observers, the arena being only a ‘‘many-body’’ effect in this interpretation/approximation.

7. Metastring theory and the two scale Renormalization Group (RG)

Finally, we explain how metastring theory makes the Born geometry dynamical by extending the usual argument in canonical string theory, according to which the Einstein equations are derived from the requirement of conformal invariance of the worldsheet sigma model [2].

One of the central features of the metastring is the following phase space commutator:

$$\{X^A(\sigma), X^B(\sigma')\} = \eta^{AB} \theta(\sigma - \sigma') + \omega^{AB} \quad (60)$$

where θ is the staircase distribution and ω^{AB} term, that is indicative of the phase space interpretation, captures the non-commutativity of the zero modes. This can be derived from the Tseytlin action [12] (and will be elaborated on in our upcoming publications). Thus at the quantum level the metastring target variables are non-commutative. In particular, both the zero mode and oscillator sectors are non-commutative. This fact is important because it is known that in the context of non-commutative field theory one needs to generalize the concept of the renormalization group (RG) to a two scale RG in order to address the question of the existence of the continuum limit [52]. Thus, in the context of the metastring in which the doubled coordinates do not commute, one expects a similar (however in this case, covariant) two scale RG.

First, we recall the fundamental idea of Friedan [53, 54]: Demanding invariance under scale transformation of the probe is equivalent to Einstein's equations. Let us look at a relativistic particle (with the action $W_p = e^{i\frac{Length}{2\lambda}}$), coupled to a non trivial metric g [55]. In order to define the propagator let us specify a discretization of the path where the proper time steps are taken to be $\Delta T = \epsilon$, so that $T = N\delta$. We have that

$$G(y, x) = \int dw \lim_{N \rightarrow \infty, N\delta = w} \int \prod_{i=1}^N dx_i \prod_{i=0}^N G_\delta(x_{i+1}, x_i), \quad (61)$$

where the particle Green's function is

$$G_\delta(x_{i+1}, x_i) = \left(\sqrt{\frac{m}{2\pi\hbar\delta}}\right)^D \exp\left[-\frac{m}{\hbar\delta} d_\delta^2(x_{i+1}, x_i)\right]. \quad (62)$$

If we coarse-grain by a factor of two, we effectively get a change of scale $\delta \rightarrow 2\delta$. Under this change the effective metric changes by

$$g_\delta \rightarrow g_{2\delta} = g_\delta - \frac{\hbar\delta}{6m} R, \quad (63)$$

where R denotes the Ricci scalar. Therefore we see that if the vacuum Einstein equations are satisfied, then we are at a fixed point of the RG flow. Or, we may reverse the logic and say that if we demand that the physics of the probe is independent of its finer details, then Einstein's equations are satisfied [53]. Friedan's idea is of particular relevance for string theory, because the string propagates in the background that is a condensation of the string's own excitations.

In the Polyakov string theory case the quantum weight of a string is $W_s = e^{i\frac{Area(\Sigma)}{2\lambda^2}}$, where λ is the string length. Note that in order to define the quantum string we only need a length scale, whereas to define quantum gravity we need two: \hbar and G_N . If we discretize the string along a square lattice with the condition $N^2\delta^2 = A$, the important point is that the scale δ drops out. If we perform a decimation we find that the metric is renormalized by

$$g_{ab} \rightarrow g_{ab} - \frac{\lambda^2}{12} R_{ab} + \dots \quad (64)$$

Thus, the vacuum Einstein equation follows from demanding the independence of the physics under the probe's small distance structure (or discretization) [53].

This procedure has to be generalized in the context of metastring theory. The admissible backgrounds are now Lorentz invariant CFT's that correspond to metastring sigma models invariant under Weyl (W) and Lorentz (L) transformations. If we write the metric in terms of frame fields $ds^2 = e_+e_-$, we have

$$W : (e_+, e_-) \rightarrow (e^\sigma e_+, e^\sigma e_-), \quad L : (e_+, e_-) \rightarrow (e^\theta e_+, e^{-\theta} e_-). \quad (65)$$

In the metastring sigma model the first symmetry is kinematical, while the second one is implemented as the end point symmetry of the RG flow. Indeed, it is well known that even if the generic theories are not conformally invariant, the end point of the RG flow admits an enhanced, conformal symmetry. The general formulation of metastring theory necessarily needs to involve non-Lorentz covariant sigma-models and thus we should find an RG flow whose endpoint admits an enhanced symmetry in two ways: it is enhanced under Weyl transformations and it is enhanced under Lorentz transformations. This means that we need to double the RG flow into the following transformations of the metastring partition function Z

$$\delta_{\sigma(x)} Z = 0 \rightarrow (\partial^2 + \tilde{\partial}^2)\delta\eta = 0, \quad (66)$$

as well as

$$\delta_{\theta(x)}Z = 0 \rightarrow (\partial \cdot \tilde{\partial})\delta\eta + 2(J\delta\eta) = 0. \quad (67)$$

These flow equations determine the fixed points under the double RG. In the context of the metastring this should be the proper formulation of closed string field theory. Fixed points which are Lorentz invariant but not free (i.e. have non-trivial Lorentz dimensions) correspond to backgrounds where η is curved, in the same way that fixed points which are conformally invariant but not free, correspond to backgrounds where H is curved. Now why do we expect a double flow in relativistic theories? In a relativistic theory if we want to put a cutoff we need generically two scales

$$E < \Lambda, \quad |\vec{p}| < \Lambda'. \quad (68)$$

In 2d this means that we have one scale for the left momenta and one for the right momenta, and we do not need to assume that they are equal. The conformal invariance should then be doubled to a fixed point symmetry that dynamically implements both the conformal (Weyl) and Lorentz symmetries.

Following the usual RG derivation of Einstein's equations in the sigma model approach to string theory, this new formulation of quantum gravity via metastring theory gives new equations for η_{AB} and H_{AB} , for both dynamical space-time and dynamical momentum space, which, in turn, consistently define a dynamical phase space. At the linearized level, these equations can be also obtained from conformal perturbation techniques⁶, again, following the usual string theory technique. The resulting equations for H and η read as follows:

$$H^{AB}\partial_A\partial_B H_{CD} = 0, \quad H^{AB}\partial_A\partial_B\eta_{CD} = 0, \quad \eta^{AB}\partial_A\partial_B H_{CD} = 0. \quad (69)$$

The first two equations show that both metrics H and η are indeed dynamical, and they reduce to the linearized Einstein equations in the spacetime limit. At the non-linear level, unless we force η to be flat, we have a coupling between the phase space metric H and the polarization/localization metric η . Finally, note that, in general, even the symplectic form ω becomes dynamical, which is not taken into account in this calculation.

8. Outlook

In this presentation we have given a reformulation of string theory and quantum gravity that generalizes the concept of locality (and causality) and incorporates T-duality. It also provides a specific quantum version of spacetime, the modular spacetime. This formulation possesses deep links with the geometry of quantization and exhibits a large symmetry algebra of Borchers. The metastring integrates many of either new or recently developed ideas: generalized geometry, relative locality, Nakamura strips, modular variables and double RG flow. The formulation is essentially non-commutative (even though consistent with Lorentz symmetry) and non-local (but consistent with causality), and it sheds new light on the foundations and origins of quantum theory, including quantum field theory. Using the double RG analysis of the metastring we have obtained a generalization of linearized Einstein equation for modular spacetime with implications for fully non-linear Born geometry. Also, in this formulation, there should exist a new effective modular quantum field theory that is consistent with the two scale RG.

In this talk we have concentrated our attention on a new concept of quantum spacetime, called modular spacetime that also appears as a habitat for metastring theory. Note that this concept stems from a quantization of spacetime, and not from quantization of gravitational field/metric. Even the flat space is quantized according to our approach to quantum gravity. This allows for superposition and entanglement of spacetimes. Also, this formulation provides for an explicit construction of spacetime quanta or qubits.

⁶ Such computations were performed in detail by Barak Shoshany, a student at Perimeter Institute.

We conclude with a few remarks. A very basic question is the role of causality in this formalism. For example, in the context of ordinary quantum mechanics, Aharonov has argued that the presence of modular time ($[t], [E]$) does not lead to a violation of causality [41]. We think it is plausible that the same is true in our more general context; at the very least, the usual field-theoretic notion of causality emerges with extensification. Nevertheless, it is intriguing to contemplate the purely stringy/quantum effects that modular spacetime offers. In ordinary quantum mechanics, modular variables were introduced in order to describe purely quantum phenomena in a simple way, diffraction gratings and Aharonov-Bohm effects being paradigmatic examples. One might say facetiously that modular spacetime is the diffraction grating of quantum gravity. Are there such effects that might be observable? Do they become important in extreme situations, such as in black hole physics or cosmology? For example, non-local effects have been expected for long time in the context of the vacuum (dark) energy problem [56], and more recently in the context of black hole physics [57]. This is the natural arena for the application of the metastring physics. Similarly, quantum effects associated with “fluctuation” of spacetime are being sought for in the context of gravitational interferometry (LIGO, Fermilab’s Holometer). Such experiments provide an observational platform needed for detecting modular spacetime. Similarly, gravitization of the quantum should be constrained in the experimental tests of quantum theory in gravitational environments [58]. Finally, we think that there are implications of the metastring for the problems of dark matter and the origin of the standard model of particle physics or the physics of the early universe. We believe that modular space-time may illuminate some of the long-standing puzzles in the foundations of quantum theory, such as the measurement problem [59], as well as the outstanding questions of string theory concerning the short distance [60], high energy and high temperature limits [61] and target space properties of strings [62].

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