Iterated Integrals and genus-one open-string amplitudes

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Abstract

Over the last few decades the prevalence of multiple polylogarithms and multiple zeta values in low order Feynman diagram computations of quantum field theory has received increased attention, revealing a link to the mathematical theories of Chen's iterated integrals and periods. More recently, a similar ubiquity of multiple zeta values was observed in the α'-expansion of genus-zero string theory amplitudes. Inspired by these developments, this work is concerned with the systematic appearance of iterated integrals in scattering amplitudes of open superstring theory. In particular, the focus will be on studying the genus-one amplitude, which requires the notion of iterated integrals defined on punctured elliptic curves.

We introduce the notion of twisted elliptic multiple zeta values that are defined as a class of iterated integrals naturally associated to an elliptic curve with a rational lattice removed. Subsequently, we establish an initial value problem that determines the expansions of twisted elliptic multiple zeta values in terms of the modular parameter τ of the elliptic curve. Any twisted elliptic multiple zeta value degenerates to cyclotomic multiple zeta values at the cusp $\tau \to i\infty$, with the corresponding limit serving as the initial condition of the initial value problem. Finally, we describe how to express genus-one open-string amplitudes in terms of twisted elliptic multiple zeta values and study the four-point genus-one open-string amplitude as an example. For this example we find that up to third order in α' all possible contributions in fact belong to the subclass formed by elliptic multiple zeta values, which is equivalent to the absence of unphysical poles in Gliozzi-Scherk-Olive projected superstring theory.

Zusammenfassung

In den vergangenen Jahrzehnten rückte das häufige Auftreten von multiplen Polylogarithmen und multiplen Zeta-Werten, in Feynman-Diagramm Rechnungen niedriger Ordnung, verstärkt in den wissenschaftlichen Fokus. Hierbei offenbarte sich eine Verbindung zu den mathematischen Theorien der Perioden und der iterierten Integrale von Chen. Eine ähnliche Allgegenwärtigkeit von multiplen Zeta-Werten wurde jüngst auch in der α'-Entwicklung von Genus-Null Stringtheorie Amplituden beobachtet. Inspiriert durch diese Entwicklung befasst sich diese Arbeit mit der Systematik der iterierten Integralen in den Streuamplituden der offenen Stringtheorie. Unser Fokus liegt insbesondere auf der Genus-Eins Amplitude, für die wir zeigen, dass sie sich vollständig durch iterierte Integrale ausdrücken lässt, welche bezüglich einer punktierten elliptischen Kurve definiert sind.

Wir führen den Begriff der getwisteten elliptischen multiplen Zeta-Werte ein. Dieser Begriff beschreibt eine Klasse von iterierten Integralen, die auf einer elliptischen Kurve definiert sind, bei welcher ein rationales Gitter entfernt wurde. Anschließend zeigen wir, dass die Entwicklung eines jeden getwisteten elliptischen multiplen Zeta-Wertes, bezüglich des modularen Parameters der elliptischen Kurve τ, durch ein Anfangswertproblem beschrieben werden kann. Weiterhin präsentieren wir ein Argument dafür, dass sich im Limes $\tau \to i\infty$ jeder getwistete elliptische multiple Zeta-Wert durch zyklotomische multiple Zeta-Werte ausdrücken lässt, wobei dieser Grenzwert auch die Anfangsbedingung des Anfangswertproblemes darstellt. Schließlich beschreiben wir wie sich Genus-Eins Amplituden in offener Stringtheorie mithilfe von getwisteten elliptischen multiplen Zeta-Werten ausdrücken lassen und illustrieren dies an dem Beispiel der Vier-Punkt Genus-Eins Amplitude. Bei der Betrachtung des zuvor genannten Beispiels zeigt es sich, dass bis zu dritter Ordnung in α' alle Beiträge vollkommen durch die Unterklasse der elliptischen multiplen Zeta-Werte ausgedrückt werden können. Diese Tatsache ist wichtig, da sie äquivalent zu der Abwesenheit unphysikalischer Pole in Gliozzi-Scherk-Olive projizierter Superstringtheorie ist.

Contents

List of own publications

This work is based on:

[1] J. Broedel, N. Matthes, G. Richter and O. Schlotterer, *"Twisted elliptic multiple zeta values and non-planar one-loop open-string amplitudes"*, `arxiv:1704.03449`.

Most of the equations are taken directly from this publication, whereas most of the text has been rewritten and the figures were partially redrawn.

Chapter 1

Introduction and Overview

1.1 Introduction

The concept of symmetry is one of the most prevalent notions throughout the endeavor of describing phenomena in physics. In general the presence of a symmetry implies functional relations among observables, thus constraining physical theories. In quantum field theory (QFT), symmetry is one of the main principles in the form of Poincaré covariance together with the concepts of locality and causality. Historically, QFT was developed for the phenomenological description of high-energy physics culminating in the standard model of particle physics but was furthermore found to have applications e.g. in cosmology, statistical physics and condensed matter physics. Despite the prominent role that QFT plays in modern theoretical physics several technical aspects still pose theoretical challenges. While the combinatorial aspects of QFT calculations are mostly understood, it is the understanding of the analytic aspects of correlation functions and amplitudes, that is rather limited. Specifically, the class of special functions in several variables needed to fully describe perturbative QFT is not known and their classification seems to be a problem that in full generality is infinitely complex. However, there are glimpses of some underlying structure in the large amount of available results in perturbative QFT calculations.

Computations in perturbative QFT can be formulated in terms of Feynman diagrams, which are combinatorial labels for terms in perturbation theory and are associated to integral expressions by Feynman rules.[1] On general grounds it is known that such integrals evaluate to multi-valued functions on the space of kinematic invariants, yet the exact special functions relevant are only known for a limited class of Feynman diagrams. On a technical level, most calculations of Feynman integrals are performed after substituting the integration over momentum space with an integration over Feynman or Schwinger parameters leading to integrals of Symanzik polynomials over (projective) simplices. The divergencies present in these integrals are taken care of by renormalization, which upon removal of subdivergencies amounts to subtracting the pole terms (and possibly a finite term), whose residue is a so-called period. Periods were introduced by Kontsevich as the subalgebra of $\mathbb{C}$ formed by numbers, which admit a representation as integrals of algebraic functions and an integration domain described via algebraic equalities or inequalities [2]. While the occurrence of periods for renormalized Feynman integrals seems

[1] The leading order terms of the pertubative expansion, called tree-level Feynman diagrams, constitute an exception in this regard as they do not involve integral expressions when evaluated in momentum space.

to be not mysterious due to their parametric realization (see e.g. [3] for a recent treatment), at least at low loop order there seems to be more structure present in Feynman integrals. A first hint of the additional structure of the set of periods present in Feynman integral calculations was the observation that a significant portion of the periods are in fact given by multiple zeta values (MZVs), see e.g. [4, 5] for a discussion in the context of ϕ^4 theory. This observation together with the fact that most Feynman integrals that could be explicitly computed evaluate to multiple polylogarithms, hints at a curious connection to Chen's iterated integrals (the canonical reference is [6]) on the thrice-punctured Riemann sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

The additional structure Feynman integrals possess due to their incarnation as Chen's iterated integrals led to a plethora of new results and methods in their study. Notable examples include algorithms based on the notion of hyperlogarithms [7–9] or graphical functions [10, 11] allowing for efficient computations of the periods of ϕ^4 theory up to seven loops, where the latter also has an intriguing link to genus-zero closed-string amplitudes [12]. A further example is the symbol map [13–15] allowing for an efficient treatment of functional relations among multiple polylogarithms. Furthermore, the symbol map allows to get insights into specific quantities like the remainder function of $\mathcal{N} = 4$ super Yang-Mills amplitudes, whose symbol may be constructed from rather general considerations without knowing the actual function [16–18].

It is however known that the framework of multiple polylogarithms and their associated periods is not sufficient to describe all Feynman integrals. There are known examples of Feynman integrals that contain MZVs at second root of unity, which may be interpreted as iterated integrals on $\mathbb{P}^1 \setminus \{0, -1, 1, \infty\}$, see [19, 20] and references therein. More generally, it was observed that there are Feynman diagrams with associated periods given by MZVs at sixth root of unity [9, 21, 22]. While both of these instances may be described as iterated integrals on a sufficiently punctured Riemann sphere, there also exist Feynman integrals demanding the consideration of iterated integrals on higher-genus Riemann surfaces. As for genus one such elliptic iterated integrals appear as early as two loops. A specific example is the sunrise diagram with three distinct internal masses, which evaluates to an elliptic dilogarithm [23–25]. It is understood that the occurrence of elliptic iterated integrals is intimately related to the fact that the vanishing locus of the second Symanzik polynomial prescribes an elliptic curve [23] leading to the description of the relevant Feynman integrals via iterated integrals of modular forms of congruence subgroups [26].

Intriguingly, the origin of iterated integrals is more apparent in the very symmetric setup of two-dimensional conformal field theory (CFT). Two-dimensional CFTs are in fact constrained by the infinite-dimensional Virasoro algebra, which is related to infinitesimal conformal transformations [27]. The key information of two-dimensional CFTs is encoded in a subset of local operators that behave tensorially under infinitesimal conformal transformations, the so-called conformal primaries. For a CFT defined on a (punctured) Riemann sphere the corresponding correlation functions of conformal primaries satisfy Knizhnik-Zamolodchikov equations, which were initially derived in the context of the Wess-Zumino-Novikov-Witten (WZNW) model [28]. These Knizhnik-Zamolodchikov equations are nonlinear partial differential equations that ascertain the corresponding correlation functions to be analytic on a simplex in the configuration space of distinct points on $\mathbb{P}^1$. Importantly, the Knizhnik-Zamolodchikov equations organize the occurrences of iterated integrals in two-dimensional CFTs on the Riemann sphere. Specifically, the one-variable case of the Knizhnik-Zamolodchikov equations on the thrice-punctured

Riemann sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ has the generating series of multiple polylogarithms (in one variable) as a solution. Additionally, the comparator of solutions regular at zero and those that are regular at one is related to the Drinfeld associator [29], which is known to be the generating series of MZVs, cf. [30, 31]. From a geometric perspective Knizhnik-Zamolodchikov equations may be thought of as flat connection on the corresponding trivial vector bundle, which is intimately related to the statement that they are equivalent to a rational solution of the classical Yang-Baxter equation [32, 33].

The Knizhnik-Zamolodchikov equations were later generalized (again in the context of the WZNW model) to the case of closed compact Riemann surfaces by Bernard [34, 35]. Moreover, for genus one a generalized notion of multiple elliptic polylogarithms [36–39] was introduced that is related to the (universal) Knizhnik-Zamolodchikov-Bernard (KZB) connection [40] in a similar sense as in the genus zero case. Correspondingly, an elliptic associator and elliptic multiple zeta values (EMZVs) were introduced [41, 42], which turn out to be relevant to both string theory and QFT.

Another intriguing angle on iterated integrals in QFT is to consider string amplitudes, which may be obtained by averaging worldsheet CFT correlation functions over the space of conformal structures on the worldsheet. Historically, the first example of a string amplitude was found by Veneziano from phenomenological considerations in an attempt to model hadronic high-energy behaviour [43]. It was later realized that this so-called Veneziano amplitude may in fact be obtained as a four-point genus-zero amplitude in bosonic open string theory. Yet it turns out that already at genus one, perturbative bosonic string theory suffers from divergencies that are associated to tachyon modes and massless tadpoles. This singular behaviour is remedied by superstring theory, which is free of tachyons after the Gliozzi-Scherk-Olive (GSO) projection and also leads to the cancellation of tadpoles at genus one given the Chan-Paton factors are chosen to be in the group $SO(32)$ [44, 45].

Amplitudes in string theory provide a direct geometric realization of iterated integrals on specific Riemann surfaces (at least for low genus) and thus are a convenient setup to get some insight into the classes of iterated integrals that may occur in field theory computations. At genus zero, the scattering of open-string states is related to integrals over the boundary of the upper half-plane eventually leading to the Drinfeld associator [12, 46–48]. For closed-string amplitudes one needs the additional notion of single-valued multiple polylogarithm, which gives rise to single-valued MZVs [49]. This appearance of iterated integrals on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ in genus-zero string theory amplitudes follows from their formulation as integrals over the moduli space of the punctured Riemann sphere for closed strings and the punctured disk for open strings, respectively.

The generalized notion of iterated integrals relevant for genus-one surfaces is that of elliptic multiple polylogarithms extensively studied in [38–42]. It was then realized that the single trace contributions of the α'-expansion of the scattering of open-string states is in fact expressible via (A-cycle) EMZVs [50]. Furthermore, the closed string amplitude may be related to so-called modular graph functions, which are themselves related to single-valued elliptic multiple polylogarithms [51]. In recent work it was found that the elliptic integrals appearing in Feynman diagram computations and the elliptic polylogarithms that naturally occur from the integration over insertions on the string worldsheet are in fact related [52, 53].

As for higher genus, the genus-two amplitude in closed superstring theory has been worked out for four points [54, 55] and is complemented by further results concerning the low-energy limits of the genus-two five-point [56] and genus-three four-point [57] amplitudes. Several of the contributing moduli space integrals have been studied extensively [58–60], yet a putative relation between closed superstring amplitudes and iterated integrals on higher-genus Riemann surfaces has not been worked out.

Finally, string amplitudes also provide a convenient setup to obtain novel insights into QFT beyond the scope of special functions and numbers. In general string amplitudes exhibit several attractive features such as the absence of ultraviolet divergencies and a comparatively tame combinatorial aspect (for closed oriented strings there is exactly one topology per order in the genus expansion). Moreover, the field theory limit of superstring scattering is related to supersymmetric field theories, with the particular examples of the type I superstring leading to $\mathcal{N} = 4$ super Yang-Mills theory and type II leading to $\mathcal{N} = 8$ supergravity, at least for low genus [61, 62]. Additionally, string theory amplitudes provide a geometrically intuitive understanding for the field-theoretically rather nebulous statement that perturbative quantum gravity may be thought of as the square of perturbative Yang-Mills theory. At tree-level, this connection linking gauge theory and gravity can be understood via the KLT relations between open and closed-string amplitudes at genus zero [63, 64]. The closely related BCJ relations for colour ordered Yang-Mills amplitudes [65] may be understood via contour integral arguments in string theory [66, 67]. Furthermore, loop integrands of gravity theories admit a representation via so-called double copies of suitably arranged gauge-theory building blocks [68,69], and explicit realizations have been constructed via string-theory methods [70–73]. However, currently there seems to be no direct phenomenological relevance of string scattering (cf. also [74–76]).

1.2 Overview

The aim of this work is to contribute to our understanding of the geometric picture behind iterated integrals in string theory. To that end our objective is twofold. Firstly, we give a self-contained introduction to iterated integrals in string amplitudes providing the necessary background to the geometry behind the integrals in question. Our second goal is to present an overview of several results concerning iterated integrals in open-string amplitudes, where our main focus lies on the genus-one amplitude. Most of the results on the genus-one amplitude we present here are due to the authors work published in [1]. These results can be found in subsections 3.2 - 3.6 as well as appendix E. However, our exposition is significantly more detailed than in said publication.

We briefly outline the structure of this work.

Chapter 2 gives a focussed exposition on scattering amplitudes of strings aiming to provide the necessary background for the study of amplitudes in open string theory. To that end we will briefly elaborate on amplitudes in bosonic string theory, which already entails most of the geometric features we want to stress. We comment on the additional structure present in scattering amplitudes in superstring theory and furthermore give additional details concerning the genus-one amplitude that is the main focus of the subsequent chapter.

Chapter 3 gives a self-contained exposition on the occurrence of iterated integrals in open-string amplitudes with the main focus on the genus-one amplitude. Concretely, we briefly summarize a few facts of iterated integrals relevant to the genus-zero amplitude mainly in order to illustrate the striking similarities to the genus-one setup we subsequently develop. We then introduce the general framework of twisted elliptic multiple zeta values (TEMZVs) and discuss algorithms how to determine their expansions in terms of the modular parameter as well as the properties of such expansions. This discussion is then complemented by an detailed application to the four-point genus-one open-string amplitude. Finally, we briefly discuss a more general setup of TEMZVs not relevant to open-string scattering but related to MZVs at roots of unity.

Chapter 4 concludes our discussion, summarizing this work and briefly addressing open questions.

Appendix A consists of some remarks on certain complex analytic aspects of the free bosonic string.

Appendix B comprises the necessary background on two-dimensional conformal field theory.

Appendix C provides some additional context to the Drinfeld associator used in chapter 3.

Appendix D briefly summarizes the definition of Chen's iterated integral. Moreover, we give the formulation of multiple zeta values as iterated integrals and comment on some of their properties.

Appendix E contains the conventions used for Jacobi θ functions as well as several computations we deemed to technical for the main text. Most of these computations are performed via manipulations on generating series and thus while conceptually not important, are included to provide the actual details of the derivation of important equations.

Chapter 2

Introduction to string amplitudes

From a geometric perspective bosonic string theory encompasses the embeddings of surfaces, the so-called worldsheets, into a D-dimensional ambient spacetime via bosonic fields X^μ. The corresponding classical system is described by an area functional, which is solved by minimal surfaces and then quantized by the path integral approach. If one then considers a fixed metric on the worldsheet, string theory may essentially be thought of as a two-dimensional conformal field theory on the worldsheet and one may make use of this fact to employ conformal field theory techniques to derive the spectrum of the free string. However, the resulting state space of the corresponding quantized bosonic string turns out to contain negative norm states unless the dimension is chosen to be $D = 26$. Furthermore, there is the additional issue that the spectrum includes a tachyon, which is at odds with causality and leads to inconsistencies in perturbation theory. The situation is improved by considering an $\mathcal{N} = (1,1)$ supersymmetric worldsheet theory, that not only reduces the consistent dimension of spacetime to $D = 10$ but also provides means as to consistently remove states from the spectrum (in particular the tachyon) via the GSO projection.

For the scattering of strings the path integral implies that we should integrate over the space of inequivalent conformal structures on the worldsheet $\Sigma_{g,n}$, which turns out to be the moduli space of (punctured) Riemann surfaces $\mathrm{Mod}(\Sigma_{g,n})$. Roughly, one may divide the parameters describing $\mathrm{Mod}(\Sigma_{g,n})$ into moduli parametrizing inequivalent conformal structures and (for a given conformal structure) the additional moduli related to the positions of punctures on the corresponding Riemann surface. It is this structure that already hints at a link to iterated integrals on Riemann surfaces, which is the main focus of this work.

Accordingly, the raison d'être of this chapter is to summarize known results to provide background for the treatment of the genus-one open-string amplitude via iterated integrals in the next chapter in order to make this treatment more accessible. In particular, our main goal will be to provide some intuition for the statement that "string amplitudes are integrals over moduli space". We note, however, that our discussion merely aspires to be a rough sketch of the relevant concepts, and omits details whenever we feel they get in the way of a succinct exposition. Discussions on the relevant concepts with the technical details in all their glory can be found e.g. in [77–92], which we also draw heavy inspiration from.

2.1 A review of free bosonic strings

The study of the free bosonic string will allow us to obtain several important insights relevant to string scattering in a geometrically less involved setup. Morally, this statement is consistent with the geometric intuition that an interacting worldsheet (to be discussed in the next section) locally looks like a free string. Specifically, the worldsheet of the free string is essentially the punctured complex plane $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ or the upper half-plane with the punctured real line $\mathbb{H} \cup \mathbb{R}^\times = \{z \in \mathbb{C} \,|\, \mathrm{Im}(z) \geq 0 \text{ and } z \neq 0\}$ and most of the relevant functions turn out be holomorphic or anti-holomorphic on the worldsheet. This allows us to use techniques of one-dimensional complex analysis without paying to much attention to questions concerning global existence and continuation issues, enabling us to streamline the discussion.

The classical motion of a one-dimensional object propagating on some D-dimensional spacetime with Lorentzian metric $\eta^{\mu\nu}$ is given by a two-dimensional worldsheet, generalizing the notion of worldline of point particles. Such a worldsheet is a two-dimensional manifold (possibly with boundary) Σ parametrized by coordinates σ, that is embedded into D-dimensional Minkowski spacetime via the embedding coordinates $X^\mu(\sigma), \mu = 0, 1, \ldots, D-1$. Furthermore, a worldsheet is restricted by the requirement that the pull back of $\eta^{\mu\nu}$, i.e. the induced metric $\eta_{\mu\nu}\partial^a X^\mu \partial^b X^\nu$, has Lorentzian signature. Hence, Σ may be considered as a Lorentzian manifold and one usually calls the coordinates the strings proper time σ^0 and it's spatial extension σ^1. From now on we will consider this metric to be Euclidean, for the sake of simplicity. Relevance of such an Euclidean theory may be motivated via Wick rotation of the Lorentzian worldsheet with coordinates $\sigma^1 = \sigma^1, \sigma^2 = i\sigma^0$. The dynamics of a string propagating through spacetime may then be described via the (Euclidean) Polyakov action

$$S_P[h, X] = \frac{1}{4\pi\alpha'} \int_\Sigma \mathrm{d}^2\sigma \sqrt{\det(h)} h^{ab} \partial_a X^\mu \partial_b X_\mu \,, \tag{2.1.1}$$

where h^{ab} is the Euclidean worldsheet metric interpreted as an independent field and α' describes the inverse string tension. Note that the equation of motion for h_{ab} implies proportionality of h_{ab} to the induced metric $\eta_{\mu\nu}\partial_a X^\mu \partial_b X^\nu$.

The Polyakov action is invariant under any diffeomorphism of the worldsheet Σ by construction. Furthermore, we find invariance under Weyl transformations, i.e. the field redefinition $h_{ab} \mapsto \Omega^2 h_{ab}$,[2] which may be seen by noting that in two dimensions h^{ab} scales inversely to $\sqrt{\det(h)}$. Note that from Weyl invariance it directly follows that the Polyakov action (2.1.1) is invariant under conformal transformations, i.e. diffeomorphisms φ whose pullback of the worldsheet metric take the form $\varphi^*(h) = e^{2\omega} h$, where $e^{2\omega}$ is some positive-definite scaling function. Their physical meaning is that in the Euclidean case these leave the notion of angle invariant (up to a possible change of orientation), while in the Lorentzian case they leave the light cone invariant. In the remainder we mean by conformal map the orientation-preserving conformal maps. Finally, we note that the action is invariant under D-dimensional Poincaré transformations of the embedding coordinates X^μ.

The study of the local aspects of the bosonic string can be greatly simplified by choosing convenient coordinates. To that end it is known that any two-dimensional Riemannian manifold is locally conformally flat, i.e. there exist so-called *isothermal coordinates* in which the worldsheet

[2] We stress that this is not the induced action of some diffeomorphism but really just a redefinition of h_{ab}.

metric is locally given by the Euclidean metric up to some positive-definite scaling function $h_{ab} = e^{2\omega}\delta_{ab}$.[3] In isothermal coordinates the Polyakov action takes the form

$$S_P[h = e^{2\omega}\delta, X^\mu] = \frac{1}{4\pi\alpha'} \int\limits_{\Sigma} \mathrm{d}^2\sigma\, \partial_a X^\mu \partial^a X_\mu \ , \tag{2.1.2}$$

where the dependence on $e^{2\omega}$ drops out due to Weyl invariance of the action. Note that there might exist combined transformations of diffeomorphisms and Weyl rescaling $\mathrm{d}s^2 \to e^{2\omega}|\Omega|^2\mathrm{d}s^2$ with $\omega = \log(|\Omega|)$ such that our choice $h_{ab} = e^{2\omega}\delta_{ab}$ is left invariant. This will be important when quantizing the theory below, as it basically tells us that, in their action on the space of metrics, the spaces of Weyl transformations and diffeomorphisms might overlap.

Choosing isothermal coordinates will simplify the equations of motion for the embedding coordinates X^μ. These in turn restrict the class of functions to which (on-shell) X^μ belongs, together with certain additional constraints depending on whether we consider open or closed strings. Explicitly, let us for concreteness choose $\sigma^1 \in [0, 2\pi]$ for closed strings and $\sigma^1 \in [0, \pi]$ for open strings.[4] Now it will be convenient to introduce two more sets of variables. Firstly, we define the complexified coordinates

$$w = \sigma^1 + i\sigma^2 \ , \quad \bar{w} = \sigma^1 - i\sigma^2 \ . \tag{2.1.3}$$

In these complexified coordinates a locally conformally flat metric is (locally) equivalent with a Hermitian form $e^{2\omega(\sigma)}((\mathrm{d}\sigma^1)^2 + (\mathrm{d}\sigma^2)^2) = e^{2\omega(w,\bar{w})}\mathrm{d}w\mathrm{d}\bar{w}$. Moreover, for complexified coordinates as above, we may consider maps (locally) of the form

$$w \mapsto f(w) \ , \quad \bar{w} \mapsto \bar{f}(\bar{w}) \ , \tag{2.1.4}$$

such that $\partial_w f(w) \neq 0$ (necessary for f being invertible), leading to the metric $\mathrm{d}f\,\mathrm{d}\bar{f} = |\partial_w f(w)|^2\,\mathrm{d}w\,\mathrm{d}\bar{w}$. This implies that locally holomorphic maps correspond to conformal transformations. In fact the converse is true as well and locally all conformal transformations can be regarded as holomorphic functions with $\partial_w f(w) \neq 0$ and vice-versa. Now, given two overlapping charts with complexified coordinates, one may infer from the requirement that the metric agrees on the overlap that the transition functions are holomorphic. Hence, we may locally describe the worldsheet by holomorphic charts with holomorphic transition functions on the overlaps. Furthermore, as holomorphic maps preserve orientation we may assign a holomorphic atlas to the worldsheet if it is orientable. The above discussion may be summarized by noting that on an orientable two-manifold a holomorphic atlas is equivalent to a metric up to a conformal factor, which is referred to as *conformal structure*. Secondly, we will use coordinates defined by the exponential map

$$z = e^{-iw} = e^{\sigma^2 - i\sigma^1} \ , \qquad \bar{z} = e^{i\bar{w}} = e^{\sigma^2 + i\sigma^1} \ , \tag{2.1.5}$$

under which a locally Hermitian metric transforms as $\mathrm{d}w\mathrm{d}\bar{w} = |z|^2\mathrm{d}z\mathrm{d}\bar{z}$, i.e. the exponential

[3] The existence of isothermal coordinates is equivalent to the existence of the solution of a partial differential equation, in particular in the Euclidean case the differential equation involved is the Beltrami equation. Strictly speaking such coordinates a priori only exist in an open neighbourhood of any point but may be extended to any simply-connected chart [78], cf. also e.g. [85, 93] for a discussion in the case of Lorentzian signature.

[4] This might seem arbitrary but in a very specific sense the closed string may be considered to be some sort of double of the open string. We will discuss this for the case of the genus-one open-string worldsheets in section 2.3.

map is a conformal map, which was to be expected as it is holomorphic. These coordinates (bijectively) map an infinitely long cylinder to the punctured plane $\mathbb{C}^\times$, where the superscript denotes removal of zero. Analogously, an infinitely long strip is mapped by the exponential map to the upper half-plane (together with the punctured real line) $\mathbb{H} \cup \mathbb{R}^\times$. Consequently, we are led to the study of functions in $z, \bar{z}$ on $\mathbb{C}^\times$ resp. $\mathbb{H} \cup \mathbb{R}^\times$, enabling us to use techniques from complex analysis in a straight-forward manner.

Let us return to the equations of motion for the embedding coordinates. Accordingly, for the closed string we have the constraint that X^μ is 2π-periodic in σ^1, and by the variational principle we find the equation of motion to be the Laplace equation

$$-\frac{1}{\sqrt{\det(h)}}\partial_a\left(\sqrt{\det(h)}h^{ab}\partial_b X^\mu\right) = \Delta_h X^\mu = 0 \tag{2.1.6}$$

which in isothermal coordinates is equivalent to

$$\partial_a \partial^a X^\mu = 0 \; . \tag{2.1.7}$$

In the coordinates $z, \bar{z}$ the Laplace equation takes the form $\partial_{\bar{z}}\partial_z X^\mu = 0$ suggesting that $\partial_{\bar{z}} X^\mu$ is holomorphic and analogously $\partial_z X^\mu$ is anti-holomorphic. Correspondingly, we can describe the kernel of the Laplacian via the expansion

$$X^\mu = x^\mu - i\frac{\alpha'}{2}p^\mu \log(z\bar{z}) + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m\in\mathbb{Z}^\times} \frac{1}{m}\left(\alpha_m^\mu z^{-m} + \tilde{\alpha}_m^\mu \bar{z}^{-m}\right) \; , \tag{2.1.8}$$

where the third term stems from the Laurent expansions of $\partial_{\bar{z}} X^\mu$ and $\partial_z X^\mu$. Furthermore, requiring reality of the (pre Wick rotated) X^μ we find the constraints $\alpha_m^\mu = \overline{\alpha}_{-m}^\mu$ and $\tilde{\alpha}_m^\mu = \overline{\tilde{\alpha}}_{-m}^\mu$. The expansion coefficients will become operators in the quantized theory. Specifically, we can extract the following commutation relations from the OPE[5]

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} \; , \qquad [\alpha_m^\mu, \alpha_n^\nu] = [\tilde{\alpha}_m^\mu, \tilde{\alpha}_n^\nu] = m\delta_{m,-n}\eta^{\mu\nu} \; . \tag{2.1.9}$$

Now when deriving the equations of motion in the case of the open string we encounter contributions from the boundary stemming from integration by parts. These boundary contributions individually are proportional to integrals of the form

$$\int_{I_i} \mathrm{d}\sigma^i \left[(\partial_{\sigma^j} X_\mu)\delta X^\mu\right]_{\partial I_j} \; , \qquad i \neq j \; , \tag{2.1.10}$$

where I_i denotes the interval in which σ^i takes values in. As σ^2 corresponds to the proper time τ, the integral constrained to ∂I_2 vanishes by definition of the variational principle. Moreover, the integral constrained to ∂I_1 turns out to vanish for both Neumann and Dirichlet boundary conditions

$$\begin{aligned} (\partial_{\sigma^1} X_\mu)|_{\sigma^1\in\partial I_1} &= 0 \qquad \text{Neumann} \\ (\delta X_\mu)|_{\sigma^1\in\partial I_1} &= 0 \qquad \text{Dirichlet} \end{aligned} \; . \tag{2.1.11}$$

[5] This is a standard technique in CFT, which we will frequently use throughout this section; an exposition on such CFT specifics can be found in appendix B. Equivalently, we may impose canonical (equal time) commutation relations to arrive at the same result, cf. [81, 87, 89].

With the above in mind the equation of motion for the embedding coordinates is again the Laplace equation (on the worldsheet with boundary) and hence $X^\mu(\sigma^1,\sigma^2)$ is a harmonic function. Then using complex coordinates $z,\bar z$ as above the solution, for Neumann boundary conditions at both ends,[6] takes the form

$$X^\mu = x^\mu - i\alpha' p^\mu \log(z\bar z) + i\left(\frac{\alpha'}{2}\right)^{1/2} \sum_{m\in\mathbb{Z}^\times} \frac{\alpha^\mu_m}{m}\left(z^{-m}+\bar z^{-m}\right) , \tag{2.1.12}$$

where reality of X^μ leads to the requirement $\overline{\alpha}^\mu_n = \alpha^\mu_{-n}$. Analogously to the closed string we find the commutation relations

$$[x^\mu, p^\nu] = i\eta^{\mu\nu} , \qquad [\alpha^\mu_m, \alpha^\nu_n] = m\delta_{m,-n}\eta^{\mu\nu} . \tag{2.1.13}$$

There are several related important remarks to be made here. Note that in the following discussion we will mainly focus on the open string, and restrict our treatment of the closed string to some comments regarding the differences. These differences basically boil down to the observation that the closed string receives two copies of the relevant equations related to the distinction between the two families of Laurent modes α_m and $\tilde\alpha_m$. Firstly, recall that the equations of motion for the worldsheet metric dictate proportionality $h_{ab} \sim \partial_a X^\mu \partial_b X_\mu$. From this we may deduce that for the classical theory, the choice of isothermal coordinates imposes additional constraints on the embedding coordinates X^μ. In particular, the requirement that $(\partial_z X)^2 = 0$ is equivalent to quadratic constraints on the coefficients of the expansion (2.1.12) that may be conveniently expressed via

$$L^{cl}_{X;m} = \frac{1}{2}\sum_{n\in\mathbb{Z}} \eta_{\mu\nu}\alpha^\mu_{m-n}\alpha^\nu_n , \qquad m\in\mathbb{Z} , \tag{2.1.14}$$

where we define $\alpha^\mu_0 = \sqrt{2\alpha'}\,p^\mu$ for the open string and $\alpha^\mu_0 = \sqrt{\alpha'/2}\,p^\mu$ for the closed string. Then we may succinctly formulate the classic constraints as $L^{cl}_{X;m} = 0\,, \forall m\in\mathbb{Z}$. The other constraint $(\partial_{\bar z} X)^2 = 0$ leads to the same constraints for the open string, while for the closed string we get an additional family $\tilde L^{cl}_{X;m} = 0\,, \forall m$ containing the $\tilde\alpha^\mu_m$. Note that only $L_{X;0}$ has ordering ambiguities w.r.t. quantization, as one may infer from the commutation relations (2.1.13). Quite importantly the (quantized) $L_{X;m}$ are also the coefficients of the expansion of the (holomorphic part of the quantized) energy-momentum tensor

$$T_X(z) = -\frac{1}{\alpha'}\eta_{\mu\nu} :(\partial_z X^\mu)(\partial_z X^\nu): = \sum_{m\in\mathbb{Z}} \frac{L_{X;m}}{z^{m+2}} , \tag{2.1.15}$$

where the colons denote normal ordering.[7] For the open string the anti-holomorphic part of the energy-momentum tensor is constrained to coincide with the holomorphic part on the boundary

[6] Other choices of boundary conditions will result in different expressions, which we omit here; cf. e.g. [81] for detailed expressions.

[7] The normal ordering prescription usually used in CFTs is determined by the non-singular part of the corresponding OPE

$$:A(z)B(z): \;=\; \lim_{w\to z}\left[A(w)B(z) - \text{singular terms of the } A(w)B(z) \text{ OPE}\,\right] ,$$

where we assumed A and B to be so-called chiral primaries (i.e. they are holomorphic and transform as tensors under local holomorphic transformations). In particular, this means that normal ordering of the corresponding Laurent modes is w.r.t. to the conformal weights of the chiral primaries; cf. [81, 84, 86, 89] for more details.

of the worldsheet (i.e. the upper half-plane)

$$T_X(z) = \tilde{T}_X(\bar{z}) \quad \text{for} \;\; z \in \mathbb{R}^\times \; . \tag{2.1.16}$$

Furthermore, for the closed string T_X and $\tilde{T}_X$ are independent and the latter has an expansion in $\bar{z}$ with coefficients $\tilde{L}_{X;m}$ completely analogous to (2.1.15). In the quantized theory the corresponding OPE of T_X with itself takes the form

$$T_X(z_1)T_X(z_2) = \frac{c_X/2}{(z_1-z_2)^4} + \frac{2T_X(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}T_X(z_2)}{(z_1-z_2)} + \text{ non-singular terms} \;\; , \tag{2.1.17}$$

from which we might infer the commutation relations for the (quantized) $L_{X;m}$

$$[L_{X;m}, L_{X;n}] = (m-n)L_{X;m+n} + \frac{c_X}{12}(m^3-m)\delta_{m,-n} \; . \tag{2.1.18}$$

The algebra that is constituted by the $L_{X;m}$ is called Virasoro algebra and c_X is called central charge. Note that we opted to explicitly depict the dependence on the central charge in formulas (2.1.17,2.1.18), while its actual value for a theory of D uncoupled free scalar fields is $c_X = D$. For the closed string $\tilde{T}_X$ satisfies an analogous OPE and we therefore get a second copy of the Virasoro algebra for the $\tilde{L}_{X;m}$ with central charge $\tilde{c}_X = D$. A slightly different approach to the Virasoro algebra is to consider it as the (unique) central extension of the Witt algebra, which has an incarnation as the algebra generating infinitesimal holomorphic transformations via $l_n = -z^{n+1}\partial_z$ satisfying the commutation relations $[l_n, l_m] = (n-m)l_{n+m}$.[8] For the (punctured) complex plane we get a second copy generating the infinitesimal anti-holomorphic transformations, whereas for the upper half-plane we require the real line to be left invariant. From this we may infer that L_0 in the case of the upper half-plane, resp. $L_0 + \tilde{L}_0$ for the complex plane, generate radial dilatation in the coordinates $z, \bar{z}$, corresponding to translation in the (imaginary) time σ^2, i.e. L_0 resp. $L_0 + \tilde{L}_0$ can be thought of as Hamiltonian.

Secondly, we may certainly quantize the action (2.1.2) as it is just the Lagrangian of D free scalar fields. Now the scale invariance of the action usually does not survive the quantization, which is referred to as Weyl anomaly.[9] Classically a necessary condition for scale invariance is tracelessness of the energy-momentum tensor $T^a_a = 0$. Furthermore, the equations of motion demanded $T_{ab} = 0$. In the quantized theory we want tracelessness of the energy-momentum tensor to hold for expectation values $\langle T^a_a \rangle$. However, it turns out that for a CFT defined on a curved two-manifold

$$\langle T^a_a \rangle \sim cR \; , \tag{2.1.19}$$

where c is the central charge of the corresponding CFT and R is the Ricci scalar of the underlying Riemannian two-manifold, cf. [83, 84]. We certainly cannot demand $R = 0$, leaving us only the hope that we may convince our CFT to satisfy $c = 0$, as we will discuss at the end of this section. Then we still have to satisfy the quantized version of the Virasoro constraints $L_{X;m} = 0$, which may be done via BRST quantization to be discussed momentarily.

Let us briefly sketch the quantization of the free bosonic string. Using the path integral

[8] In the case of the Riemann sphere, the subalgebra generated by l_{-1}, l_0, l_1 exponentiates to the Möbius group, which is the group of globally defined conformal transformations on the Riemann sphere.

[9] An important example is Yang-Mills theory in four dimensions, which has a scale invariant action but the IR and UV singularities of the quantized theory (luckily for us) inevitably introduce scales, breaking scale invariance.

approach, the basic problem is to understand what a physically consistent measure is on the "space of fields" defined on Σ. As stated above, the redundancy in the description of the action is given by diffeomorphisms and Weyl transformations and we certainly do not want a quantized theory to depend on any fixing of such a redundancy. Correspondingly, a sensible measure should not integrate over these redundancies. Then we may start with the naive measure denoted $\mathcal{D}h\mathcal{D}X^\mu$ and attempt to find a decomposition separating the redundancy degrees of freedom from their physical counterparts.

We begin with the study of the space of Riemannian metrics $\mathcal{M}(\Sigma)$ on Σ, which we equip with a Riemannian structure as described in [82, 94]. Specifically, we may define at any point $\hat{h} \in \mathcal{M}(\Sigma)$ a Riemannian metric in the tangent space $T_{\hat{h}}\mathcal{M}(\Sigma)$ via

$$(\delta g_1, \delta g_2)_{\hat{h}} = \int_\Sigma \mathrm{d}^2\sigma \sqrt{\det(\hat{h})}(\delta g_1)_{ab}(\delta g_2)_{cd}\hat{h}^{ac}\hat{h}^{bd} \, , \tag{2.1.20}$$

which is invariant under pullbacks of diffeomorphisms but not under Weyl rescaling. This metric will provide us with the notion of orthogonality for local variations of the metric and we will use it to deduce a "sensible" measure below. Now due to local conformal flatness we actually want to study the action of diffeomorphisms on the space of conformal structures $\mathcal{M}_c(\Sigma)$, which is the quotient of $\mathcal{M}(\Sigma)$ w.r.t. equivalence up to conformal factors on Σ. Accordingly, the physical degrees of freedom correspond to the orbits of the action of $\mathrm{Diff}(\Sigma)$ on $\mathcal{M}_c(\Sigma)$. As explained above, in the case of the free string the worldsheet is basically $\mathbb{C}^\times$ or $\mathbb{H}\cup\mathbb{R}^\times$, which has a unique conformal structure.[10] We conclude that in the case of the free string there are no physical degrees of freedom of the metric to integrate over. In order to get a handle on the redundancy degrees of freedom we want to find a basis of the tangent space $T_{\hat{h}}\mathcal{M}(\Sigma)$ at some metric $\hat{h}$, that is equivalent to the generators of (local) diffeomorphisms and Weyl rescaling. To that end we note that an infinitesimal diffeomorphism generated by some vector v acts on the metric $\hat{h}$ via the Lie derivative $\mathcal{L}_v\hat{h}$. Furthermore, infinitesimal Weyl transformation correspond to multiplication of the metric h by some infinitesimal factor. Consequently, the tangent space $T_{\hat{h}}\mathcal{M}(\Sigma)$ admits the following decomposition

$$\delta h_{ab} = \phi\hat{h}_{ab} + (P_1 v)_{ab} \, , \tag{2.1.21}$$

where $(P_1 v)_{ab}$ denotes the traceless part of the Lie derivative

$$(P_1 v)_{ab} = (\mathcal{L}_v\hat{h})_{ab} - \hat{h}_{ab}(\nabla_c v^c) = \nabla_a v_b + \nabla_b v_a - \hat{h}_{ab}(\nabla_c v^c) \, , \tag{2.1.22}$$

and ∇ is the covariant derivative w.r.t. the worldsheet metric $\hat{h}$. Additionally, we can deduce from eq. (2.1.22) that the kernel of P_1 is constituted by conformal Killing vectors. Note that the space on which P_1 acts depends on the worldsheet Σ. In particular for open strings the diffeomorphisms in question have to leave the boundary invariant. Geometrically, this just means that (at the boundary) v has no component orthogonal to the boundary. For example in the case where our worldsheet is the upper half-plane, the kernel of P_1 may be obtained by restricting the Möbius group $PGL(2;\mathbb{C})$ to the subgroup $PSL(2;\mathbb{R})$ leaving the real line invariant. The decomposition (2.1.21) is orthogonal w.r.t. to the metric on the space of metrics (2.1.20). Hence, we may deduce the following orthogonal decomposition of the redundancy

[10] For simplicity we assume the range of the imaginary proper time to be $\mathbb{R}$, however a finite range will still lead to a unique conformal structure on the corresponding punctured (semi-)disk.

degrees of freedom via the formal expression

$$\mathcal{D}h = \mathcal{D}\phi\,\mathcal{D}(P_1 v) = \mathcal{D}\phi\,\mathcal{D}v\,|\det(P_1)|\;. \tag{2.1.23}$$

For the free string (without the introduction of any additional distinguished point) the functional determinant of this expression is actually zero, due to a non-trivial kernel of P_1.[11] However, when considering scattering amplitudes in the next section the non-trivial kernel can be taken care of by fixing positions of the so-called vertex operators, so we will not dwell on this point any further. Eventually, the redundancy degrees of freedom v will be used to fix a conformal structure, but some care needs to be taken as the above decomposition is not Weyl invariant, an issue we will return to momentarily.

Using the Faddeev-Popov method, the functional determinant $|\det(P_1)|$ may be interpreted as a path integral of a field theory of Graßmann valued ghost fields. We note that the introduction of ghost fields is not necessary but allows us to make contact with BRST quantization below. In our specific case the field theory for ghosts is described by

$$|\det(P_1)| = \det(P_1^\dagger P_1)^{1/2} = \int \mathcal{D}b\mathcal{D}c \exp\Big(-\frac{1}{2\pi}\int\limits_\Sigma \mathrm{d}^2\sigma\,\sqrt{\det(h)}h^{de}b_{ef}\nabla_d c^f\Big)\;, \tag{2.1.24}$$

with b symmetric and traceless.[12] Similarly to the above discussion the action takes a convenient form in isothermal coordinates

$$S_{gh}[h = e^{2\omega}\delta, b, c] = \frac{1}{2\pi}\int\limits_\Sigma \mathrm{d}z\mathrm{d}\bar{z}\,\left(b_{zz}\partial_{\bar{z}}c^z + b_{\bar{z}\bar{z}}\partial_z c^{\bar{z}}\right)\;. \tag{2.1.25}$$

The classical equations of motion imply that b_{zz} and c^z are holomorphic leading to the expansions

$$c^z(z) = \sum_{m\in\mathbb{Z}} c_m z^{-m+1}\;, \qquad b_{zz}(z) = \sum_{m\in\mathbb{Z}} b_m z^{-m-2}\;, \tag{2.1.26}$$

with hermiticity implying the additional relations $(c_n)^\dagger = c_{-n}$ and $(b_n)^\dagger = b_{-n}$. Analogously we find that $b_{\bar{z}\bar{z}}$ and $c^{\bar{z}}$ are anti-holomorphic. For the open string we have $c_n = \tilde{c}_n$ as we require $c^z = c^{\bar{z}}$ on the boundary $\mathrm{Im}(z) = 0$ and analogously for the b fields. Correspondingly, for the closed string we find two families of oscillators denoted b_n, c_n and $\tilde{b}_n, \tilde{c}_n$. The classical equation of motion for the c ghost field is the conformal Killing equation and consequently the zero modes correspond to conformal Killing vectors. Moreover, the classical equation of motion for b is solved by holomorphic quadratic differentials, which are directly related to the physical degrees of freedom that are encoded in the so-called Teichmüller space.[13] However, there are no Teichmüller parameters for $\Sigma = \mathbb{C}^\times$ and hence no zero modes need to be taken into consideration. With these expansions we may deduce the anti-commutation relations

$$\{b_m, b_n\} = 0\;, \quad \{c_m, c_n\} = 0\;, \quad \{b_m, c_n\} = \delta_{m,-n}\;. \tag{2.1.27}$$

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[11] Note that for example in the case of the punctured plane $\mathbb{C}^\times$ (or infinitely long cylinder) the relevant transformations are generated by $z \mapsto az$ and $z \mapsto 1/z$, i.e. rescaling and rotations. Of course we might compute the two-point function in this context but two-point functions in a CFT are not invariant under rescaling. Or to rephrase it: possible issues related to the presence of conformal Killing vectors in the path integral quantization of the free string are not conceptual but rather of semantic nature.

[12] $P_1^\dagger$ is defined to be the adjoint of P_1 w.r.t. the inner product (2.1.20).

[13] We briefly comment on some additional complex analytic properties of the free open string in appendix A.

Moreover, the (holomorphic part of the) energy-momentum tensor of the bc-ghost system is given by

$$T_{gh} = - :(2\, b_{zz}\partial_z c^z + (\partial_z b_{zz})c^z): \,, \tag{2.1.28}$$

from which we may deduce the OPE

$$T_{gh}(z_1)T_{gh}(z_2) = \frac{-26/2}{(z_1-z_2)^4} + \frac{2T_{gh}(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}T_{gh}(z_2)}{(z_1-z_2)} + \text{ non-singular terms } , \tag{2.1.29}$$

and hence the central charge of the bc-ghost system is $c_{gh} = -26$. Correspondingly, the mode expansion of the ghost Virasoro generators is given by

$$L_{gh;m} = \sum_{n\in\mathbb{Z}} (m-n) :b_{n+m}c_{-n}: \ . \tag{2.1.30}$$

The ghost system may be used to describe our initial path integral measure via two uncoupled field theories (we denote our choice of metric by $\hat{h}$)

$$\int \mathcal{D}h\mathcal{D}X^\mu \exp\big(-S_P[h,X^\mu]\big) \to \int \mathcal{D}b\mathcal{D}c\mathcal{D}X^\mu \exp\big(-S_{gh}[\hat{h},b,c] - S_P[\hat{h},X^\mu]\big) \ . \tag{2.1.31}$$

Note that there is still the issue of how to normalize the path integral. However, we postpone a discussion of the normalization to the next section as it will not play a role in the remainder of this section.

Revisiting the issue of a possible Weyl anomaly we note that the energy-momentum tensor of two uncoupled theories is just the sum $T_X + T_{gh}$ and therefore the same is true for the Virasoro generators $L_{X;m} + L_{gh;m}$, implying that for uncoupled theories the central charges are additive. Hence, noting that $c_X = 1$ we find that for a theory of D uncoupled scalars, the total central charge is given by

$$c_{tot} = D + c_{gh} = D - 26 \ , \tag{2.1.32}$$

and we find the famous result that $D = 26$ for a consistent conformal theory without Weyl anomaly.[14]

It remains to give a meaning to the integration over the space of embedding coordinates X^μ, where similarly to the case of the worldsheet metric we introduce an inner product in order to orthogonally separate degrees of freedom. Such an orthogonal decomposition will again be our guiding principle to write down a "sensible" measure. We equip the space formed by the X^μ with the L^2 inner product (w.r.t. the measure $\mathrm{d}^2\sigma\sqrt{\det\hat{h}}$)

$$(\delta X_1^\mu, \delta X_2^\mu)_{\hat{h}} = \int_\Sigma \mathrm{d}^2\sigma \sqrt{\det(\hat{h})}\,\delta X_1^\mu \delta X_2^\mu \ , \tag{2.1.33}$$

which is invariant w.r.t. diffeomorphisms of Σ but not under Weyl rescaling. It is known that

[14] From the viewpoint of the path integral measure the Weyl anomaly manifests itself in the property that the notion of orthogonal decomposition (2.1.21) is not Weyl invariant. The same is true for the measure of the embedding coordinates to be discussed, cf. equation (2.1.34). However, it is known that when subjected to a Weyl rescaling both measures pick up a factor of e^{cS_L}, where c is the corresponding central charge and S_L denotes the so-called Liouville action depending solely on the conformal factor. Hence, we can again deduce that the measure in question is Weyl invariant if the overall central charge vanishes. A detailed discussion on this matter may be found in [78, 95].

eigenfunctions of the Laplacian $\Delta_{\hat{h}}$ form a complete orthogonal basis on $L^2(\Sigma, \mathrm{d}^2\sigma\sqrt{\det(\hat{h})})$, which suggests the orthogonal decomposition[15]

$$X^\mu = X_0^\mu + \xi^\mu \;, \qquad \Delta_{\hat{h}} X_0^\mu = 0 \tag{2.1.34}$$

and ξ^μ is a sum over all eigenfunctions of $\Delta_{\hat{h}}$ with non-zero eigenvalue. This suggests the following orthogonal decomposition of the path integral

$$\int \mathcal{D}X^\mu \exp\big(-S_P[\hat{h}, X^\mu]\big) \to \int \mathcal{D}X_0^\mu \exp\big(-S_P[\hat{h}, X_0^\mu]\big) \int \mathcal{D}\xi^\mu \exp\big(-S_P[\hat{h}, \xi^\mu]\big) \;. \tag{2.1.35}$$

Now that we have convinced ourselves that there is a sensible path integral measure we may use the mode expansions for the fields involved to construct the corresponding space of states. Yet the corresponding Fock space will also contain states for the bc ghost system and we have to understand which states are physical. Moreover there is the issue that we ignored a redundancy of the embedding coordinates related to diffeomorphism invariance. However, we may conveniently address both issues simultaneously via BRST quantization. To that end we note that the embedding coordinates transform under diffeomorphisms via

$$\delta X^\mu \sim v^z \partial_z X^\mu \;, \tag{2.1.36}$$

which for the closed string is supplemented with the anti-holomorphic counterpart. The usual remedy to this redundancy in our description is the introduction of the BRST charge Q_B that may be expressed via the Virasoro generators (i.e. the generators of the local redundancy)

$$Q_B = \sum_{m\in\mathbb{Z}} :c_{-m}(L_{X;m} + 1/2 L_{gh;m}): \;, \tag{2.1.37}$$

which generates infinitesimal transformations in the sense that the action of Q_B on the fields is given by

$$[Q_B, X^\mu] = c^z\partial_z X^\mu \;, \quad \{Q_B, c^z\} = c^z\partial_z c^z \;, \quad \{Q_B, b_{zz}\} = T_{gh}(z) + T_X(z) \;. \tag{2.1.38}$$

Consequently, we want physical states to lie in the kernel of the BRST charge also referred to as BRST closed states. Furthermore, the BRST charge has the important property that it is nilpotent $(Q_B)^2 = 0$.[16] This implies that states of the form $Q_B|sth.\rangle$ are automatically in the kernel of Q_B but also that any such state has zero norm as Q_B is Hermitian. Hence, the states we are ultimately interested in live in the cohomology associated to Q_B. The individual

[15] There are several technicalities w.r.t. regularity at the boundary. Essentially, it is possible to choose $\xi|_{\partial\Sigma} = 0$, i.e. the boundary conditions are basically encoded in the harmonic function X_0. Detailed discussions can be found in [82, 96].

[16] Note that requiring the BRST charge to be nilpotent, again implies the necessity of a vanishing central charge. Denoting $L_m = L_{X;m} + L_{gh;m}$ one may compute [81, 87]

$$(Q_B)^2 \sim \sum_{m,n\in\mathbb{Z}} \Big([L_m, L_n] - (m-n)L_{m+n}\Big) c_{-m}c_{-n}$$

and hence every term in the above formal power series of operators vanishes iff the overall central charge vanishes by virtue of the Virasoro algebra Lie bracket (2.1.18).

cohomology groups are labeled by the ghost number, i.e. the eigenvalue of

$$N_{gh} = \frac{1}{2}(c_0 b_0 - b_0 c_0) + \sum_{m=1}^{\infty} (c_{-m} b_m - b_{-m} c_m) \ , \tag{2.1.39}$$

where Q_B has ghost number +1. Finally, we note that quite importantly $\{Q_B, b\} \sim T$ and $Q_B^2 = 0$ imply that Q_B commutes with the Virasoro generators $L_{X;m} + L_{gh;m}$ and therefore is consistent with CFT representation theory.

The space of states for the whole theory is the tensor product of the corresponding space for the embedding coordinates and the ghost system. We begin by considering the ghost system for which one finds that c_0, b_0 commute with the ghost Hamiltonian $L_{gh;0}$ and satisfy $c_0^2 = 0$, $b_0^2 = 0$ and $\{c_0, b_0\} = 1$. Hence, there are two ground states for the ghost system defined by

$$c_0|\uparrow\rangle = 0 \ , \quad c_0|\downarrow\rangle = |\uparrow\rangle \ , \quad b_0|\uparrow\rangle = |\downarrow\rangle \ , \quad b_0|\downarrow\rangle = 0 \ , \tag{2.1.40}$$

with ghost numbers $N_{gh}|\downarrow\rangle = -1/2|\downarrow\rangle$ and $N_{gh}|\uparrow\rangle = 1/2|\uparrow\rangle$. Moreover, these ground states are in the kernel of the annihilation operators $c_m, b_m, m \geq 1$. Additionally we require the physical ground state to be in the kernel of b_0 (the complex structure on the worldsheet is essentially unique), which due to the anti-commutation relations (2.1.27) extends to states with ghost excitations. Consequently, a state that is in the kernel of both Q_B and b_0 will due to (2.1.38) also satisfy

$$\{Q_B, b_0\}|\psi\rangle = (L_{X;0} + L_{gh;0})|\psi\rangle = 0 \ , \tag{2.1.41}$$

with the explicit expression

$$L_{X;0} + L_{gh;0} = -1 + \alpha' p^2 + \underbrace{\sum_{m\geq 1} \left[\eta_{\mu\nu}\alpha^\mu_{-m}\alpha^\nu_m + m(c_{-m}b_m + b_{-m}c_m)\right]}_{=:N_l} \ , \tag{2.1.42}$$

where the eigenvalue of N_l is called level number.[17]

Furthermore, D-dimensional Poincaré covariance implies that in the quantized theory the space of states decomposes into representations of the D-dimensional Poincaré algebra. These may be specified by the eigenvalue of the quadratic Casimir $m^2 = -p^2$ together with a representation of the stabilizer subgroup $SO(D-1)$ (after specifying a Lorentz frame) with the exception of massless states where the corresponding subgroup is $SO(D-2)$. Correspondingly, we may deduce from (2.1.41,2.1.42) that for open strings the eigenvalue of m^2 is determined by

[17] BRST quantization of the bosonic string is originally due to [97]. Quite importantly, most BRST cohomology groups are actually trivial, which is proven in [98]. Moreover, the uniqueness of the complex structure on the (punctured) upper half-plane led us to the requirement that a physical state is in the kernel of b_0 implying that we should consider the cohomology group related to ghost number $-1/2$. Consequently, if we consider states of the form $|\psi\rangle \sim |\phi\rangle \otimes |\downarrow\rangle$, BRST closedness is the requirement that

$$Q_B|\psi\rangle = \left(c_0(L_{X;0} - 1) + \sum_{m>0} c_{-m} L_{X;m}\right)|\psi\rangle = 0 \ ,$$

leading to the physical state conditions one finds from other quantization methods

$$(L_{X;0} - 1)|\phi\rangle = 0 \ , \quad L_{X;m}|\phi\rangle = 0 \ \text{ for } \ m \geq 1 \ ,$$

cf. also [77].

the level number and takes values

$$m^2 = \frac{n-1}{\alpha'} \quad \text{with } n \in \mathbb{N} \, . \tag{2.1.43}$$

Hence, the lowest mass state of the open string has mass $m^2 = -(\alpha')^{-1}$ (called tachyon), which after a choice of Lorentz frame is denoted as $|0; p^\mu\rangle$. The next state in the mass hierarchy is massless $m^2 = 0$ and is of the form $e_\mu \alpha^\mu_{-1}|0; p^\mu\rangle$ with the additional constraint $e_\mu p^\mu = 0$ suggesting that e_μ is defined only up to addition with p_μ.[18]

For the closed string, states are generated by two copies of the Laurent mode algebras $\alpha^\mu_m, \tilde{\alpha}^\mu_m$ as well as two corresponding copies of the bc algebra and thus belong to a fourfold tensor product of the individual representation spaces. The above discussion may then be adapted with requiring physical states to lie in the kernel of (the adapted) BRST charge Q_B as well as b_0 and $\tilde{b}_0$. This suggest that a physical state has to satisfy

$$(L_{X;0} + L_{gh;0})|\psi\rangle = 0 \, , \quad (\tilde{L}_{X;0} + \tilde{L}_{gh;0})|\psi\rangle = 0 \, , \tag{2.1.44}$$

independently and we may obtain closed string states as tensor products of open-string states. For such a tensor product both factors have to belong to representations with the same level number $N_l = \tilde{N}_l$, which follows from the fact that $\alpha^\mu_0 = \tilde{\alpha}^\mu_0 = \sqrt{\alpha'/2}\, p^\mu$ both of which are present in the corresponding Virasoro generators. Moreover, in the definition of α^μ_0 and $\tilde{\alpha}^\mu_0$ is an additional factor of $1/2$ compared to the open string, leading to the mass spectrum

$$m^2 = \frac{4(n-1)}{\alpha'} \quad \text{with } n \in \mathbb{N} \, . \tag{2.1.45}$$

Consequently, a state without any excitation has mass $m^2 = -4(\alpha')^{-1}$ and may be denoted $|0, 0; p^\mu\rangle$. The next state has mass $m^2 = 0$ and is the tensor product of two massless open-string states.

For non-oriented strings we have the further requirement that states need to be well-behaved when subjected to worldsheet parity P. The corresponding map needs to be involutive $P^2 = 1$ and thus has eigenvalues ± 1. In the parametrization used above these transformations may be realized as $\sigma^2 \mapsto \pi - \sigma^2$ for open strings and $\sigma^2 \mapsto 2\pi - \sigma^2$ for the closed analogue. Accordingly, the action of P on the embedding coordinates is given by

$$P X^\mu(z, \bar{z}) P^{-1} = \begin{cases} X^\mu(-\bar{z}, -z) & \text{for open strings} \\ X^\mu(\bar{z}, z) & \text{for closed strings} \end{cases} \, . \tag{2.1.46}$$

This action can be rephrased in terms of the Laurent modes, where for the open-string mode expansion (2.1.12) worldsheet parity acts as[19]

$$P \alpha^\mu_m P^{-1} = (-1)^m \alpha^\mu_m \, , \tag{2.1.47}$$

[18] BRST closedness requires the absence of a b_{-1} excitation as well as the constraint $e_\mu p^\mu$. Furthermore, excitations from c_{-1} as well as $\alpha^\mu_{-1} p_\mu$ are BRST exact, the latter due to $p^2 = 0$.

[19] The action of worldsheet parity on the Laurent modes depends on the exact choice of boundary conditions. As stated above we exclusively consider Neumann Boundary conditions at both ends; the relevant formulas for other choices of boundary conditions may be found e.g. in [81].

and correspondingly for the closed string analogue (2.1.8) the action is

$$P\alpha_m^\mu P^{-1} = \tilde{\alpha}_m^\mu \ , \quad P\tilde{\alpha}_m^\mu P^{-1} = \alpha_m^\mu \ . \tag{2.1.48}$$

From the action of P on the Laurent modes we deduce that holomorphic and anti-holomorphic factor of the closed string state space are exchanged. Furthermore, for the open string the eigenvalue of P for any state depends only on the level number or equivalently m^2. The conditions above fix the action of P on the state space only up to a sign. We fix this sign ambiguity by demanding $P|0;p^\mu\rangle = |0;p^\mu\rangle$. Then one finds that the eigenvalue of P of an open-string state with mass m is given by $(-1)^{1+\alpha' m^2}$.

Given the spectrum of the free string we now relate string states to so-called vertex operators, which will be relevant in the next section when we study the scattering of strings. The rough statement is that vertex operators are local operators that correspond to asymptotic states of the worldsheet CFT. Quite importantly, the form of vertex operators is essentially determined by symmetry and will here be the same as in the study of the S-matrix, with the distinction that in the interacting theory the vertex operators will depend on fields defined on the "interacting worldsheet". Still, considering vertex operators for the free string has the advantage that the correspondence between vertex operators and asymptotic states happens to be very explicit. Let us start by considering a semi-infinite cylinder with imaginary proper time parametrized by $\sigma^2 \in (-\infty, 0]$, which we may map to the (punctured) unit disk using the exponential map (2.1.5). In these coordinates past infinity corresponds to zero and therefore a physical state generated by a "suitable" operator $V(z,\bar{z})$ in the limit $z, \bar{z} \to 0$, may be given the interpretation of an asymptotic in-state.[20] The precise notion of "suitable" operator is that of a conformal primary, which is defined by a "tensorial" behaviour under local conformal transformations $z \mapsto f(z), \bar{z} \mapsto \bar{f}(\bar{z})$ (i.e. transformations generated by the Virasoro algebra) in the sense that

$$\phi(f(z), \bar{f}(\bar{z})) = (\partial_z f)^{-h_\phi} (\partial_{\bar{z}} \bar{f})^{-\tilde{h}_\phi} \phi(z,\bar{z}) \ , \tag{2.1.49}$$

where the quantities $h_\phi, \tilde{h}_\phi$ defining the transformation property of ϕ are called it's conformal weights. The relevance of conformal primaries to the CFT state space stems from the fact that they correspond to highest-weight states of the Virasoro algebra and hence a state space consistent with (local) conformal symmetry. Moreover, in our setup BRST quantization led to the additional constraint that physical states should be in the kernel of $L_{X;0} + L_{gh;0}$ and $\tilde{L}_{X;0} + \tilde{L}_{gh;0}$. This requirement in turn restricts the value of the conformal weight such that the conformal weights of the bosonic and ghost part add up to zero and will turn out to lead to the mass spectrum we found above.[21] Finally, we note that the action of some local operator, say Q, on a state corresponding to some vertex operator V is determined by the singular part of the QV OPE.

In order to give explicit expressions we note that vertex operators are required to be covariant w.r.t. the symmetries of the theory (as they correspond to string states which are covariant). Firstly, D-dimensional Poincaré covariance constrains (the bosonic part of) a vertex operator to

[20] Note that it is a peculiarity of the geometric properties of the free string worldsheet that we may associate past infinity to a point rather than a subspace.

[21] Alternatively, we might again use Q_B to formulate the physical state condition. Specifically, BRST-closedness may be formulated as $[Q_B, V] = 0$, while BRST exactness means there exists a Graßmann odd W such that $V = \{Q_B, W\}$. Finally, we need to require $[b_n, V] = 0, \ n \geq 0$.

the form

$$V(z,\bar{z};p_\mu) \sim :f(\partial_z X, \partial_{\bar{z}} X)\exp\left(ip_\mu X^\mu\right): , \tag{2.1.50}$$

where f is some polynomial in (covariant) derivatives of X^μ of uniform conformal weight, with coefficients such that all D-dimensional Lorentz indices are contracted.[22] Now as for ghosts there is a subtlety we neglected so far related to the fact that the ground state $|0\rangle_{bc}$ of the ghost Virasoro algebra and the ground states of the algebra of Laurent modes of the bc-ghosts (2.1.40) are different. Yet the argument that vertex operators can be related to highest weight representations of the Virasoro algebra depends on the fact that we consider the ground state of the ghost Virasoro algebra; cf. appendix B. Here this technicality can be solved by noting the relation $c(0)|0\rangle_{bc} = |\downarrow\rangle$. Then for a state whose ghost content is described by $|\downarrow\rangle$ the conformal weight of the ghost part of the vertex operator is -1 implying that the bosonic part had better conformal weight $+1$. Note that the issue that the ground states do not coincide is in fact related to the existence of zero modes of the differential operators P_1 and $P_1^\dagger$ and will be quite relevant when we consider the scattering of strings. For the rest of the section however we will be content with ignoring the ghosts and consider only the bosonic part of vertex operators. As example, consider a vertex operator á la (2.1.50) with polynomial $f = 1$, corresponding to the closed string tachyon state

$$\lim_{z,\bar{z}\to 0} :\exp(ip_\mu X^\mu(z,\bar{z})):|0\rangle_X = |0,0;p^\mu\rangle \ . \tag{2.1.51}$$

In fact, the conformal weight of $:\exp(ip_\mu X^\mu(z,\bar{z})):$ can be computed to be $h = \tilde{h} = \alpha' p^2/4$, which together with the requirement $h = \tilde{h} = 1$ leads to the tachyon mass we found above.

In the case of the open string we can map a worldsheet that is a semi-infinite strip (with imaginary time parametrized as in the closed string case) to the unit semi-disk in the upper half-plane. Again past infinity corresponds to zero but this time has the additional property that it lies on the boundary of the worldsheet, the real unit interval in our parametrization. Accordingly, the notion of vertex operators is analogous to the closed string case. Note that there is a slight alteration in notion of conformal primary for a CFT with boundary as the holomorphic and anti-holomorphic components of the energy-momentum tensor have to coincide on the boundary. This implied that the modes (the Virasoro generators) coincide, which in turn generate infinitesimal conformal transformation. Hence, there is only one conformal weight describing the primary on the boundary CFT. Note that the open-string analogue of the tachyon vertex operator (2.1.51) has conformal weight $h = \alpha' p^2$, again leading to the correct mass.

We conclude the study of the free string with a brief discussion of the so-called Chan-Paton degrees of freedom. The idea is that we may enrich the state space by attaching degrees of freedom with trivial worldsheet dynamics to the distinguished points of an open-string worldsheet, i.e. the boundary. Due to the trivial worldsheet dynamics this is automatically consistent with all the symmetries of the Polyakov action (apart from possibly worldsheet parity). Specifically, if we consider some open-string state we may attach an additional label $i = 1, \ldots, N$ to each of the two endpoints. Correspondingly, the additional degrees of freedom of a given open-string

[22] The normalization of vertex operators is for the free string determined by the normalization of the corresponding states. However, for the interacting theory, to be discussed in the next section, we do not know the state space, leading to some arbitrariness in the normalization. Still the normalization of vertex operators corresponding to different mass states, are related by unitarity. We won't discuss this detail any further, cf. [89] for details.

state may be encoded by an $N \times N$ matrix, which may be expanded in some basis of $N \times N$ matrices

$$|\psi\rangle \otimes |a\rangle = |\psi\rangle \otimes \left(\sum_{i,j=1}^{N} (T^a)_{ij} |i,j\rangle \right) . \tag{2.1.52}$$

These additional degrees of freedom are called Chan-Paton degrees of freedom.[23] Vertex operators will simply get the corresponding matrix as an additional factor. Now, in an instance of plot-foreshadowing we note that the amplitude is related to integrals over the space of possible vertex operator insertions. For open strings these will be constrained to lie on boundary components homeomorphic to S^1. As the Chan-Paton degrees of freedom accompanying the vertex operators are non-dynamic, we may organize any such integral in terms of (products of) cyclically inequivalent traces of the corresponding matrices. The possible choices for T^a are then constrained by demanding certain factorization properties of such amplitudes. We will be content with stating the result and referring to [99] for details. It turns out that for oriented strings the T^a are generators of $U(N)$, while for unoriented strings the possible choices are either $SO(N)$ or for even N also $USp(N)$. Finally, we note that eventually our group of choice will turn out to be $SO(32)$ as this leads to the cancellation of divergencies for genus-one scattering amplitudes in type I superstring theory.

2.2 The string S-matrix

Similarly to the case of the free string, the scattering of strings should be an analogue of point particle scattering. Heuristically, one suspects that strings cannot have local interactions, yet "blowing up" the point-particle worldline to a worldsheet led to a consistent theory of free strings. This will be the rationale when studying string scattering, i.e. the S-matrix should be described by (smooth) worldsheets with prescribed (asymptotic) initial and final string states. Consequently, the usual path integral mantra of summation over all histories weighted appropriately suggests that we should sum over all genus g surfaces, compatible with the type of strings one considers, with all possible ways of attaching external strings to them. Accordingly, individual contributions to the scattering of n open strings should be described by surfaces of a definite genus g with boundary and n semi-infinite strips (smoothly) attached to its boundary components. Similarly, for the closed string we have genus g surfaces with holes to which we (smoothly) glue semi-infinite cylinders. As for the weighting of distinct surfaces, it turns out that this may be conveniently taken care of by the topological properties of the corresponding surface. Specifically, we may complement the Polyakov action with the topological invariant $S_{top} = \lambda\chi(\Sigma_g)$, where $\chi(\Sigma_g)$ is the Euler characteristic of the surface Σ_g and λ is some coupling.[24] This leads to the weighting of different worldsheets by their respective genera g via a factor $\sim (e^{\lambda})^{ag}$, where the value of the constant a depends on the specifics of the worldsheets

[23] The modern interpretation of these degrees of freedom is that they label so-called Dp-branes stacked on top of each other. In a nutshell Dp-branes can be thought of as open string with $p+1$ of the D X^μ satisfying Neumann boundary conditions and the remaining X^μ satisfy Dirichlet boundary conditions (at both ends). However, we will not consider any Dp-branes and refer the reader to [81, 89, 91].

[24] The Euler characteristic may be expressed as an integral over the corresponding worldsheet surface via the Gauss-Bonnet theorem. In this sense adding S_{top} to the Polyakov action may be considered as a generalization of the action.

under consideration and is irrelevant for our discussion.[25]

Recall that local conformal flatness holds in fact for all two-dimensional smooth manifolds. Hence, on such a worldsheet we may conformally map any semi-infinite strip (resp. cylinder) to a punctured semi-disk (resp. disk), where the appropriate map is (locally) given by the exponential map discussed in the previous section. In the case of the open string this leaves us with a genus g surface with punctures on the boundary components as remnants of conformally shrinking the semi-infinite strips. Equivalently, for closed strings we have closed surfaces with punctures. Reducing external string states via local conformal transformations comes at the price of vertex operators at the puncture. These vertex operators in turn encode the quantum numbers of the external strings. Their general form is determined by the local symmetries of the worldsheet theory as well as the global D-dimensional Poincaré covariance as was discussed in the previous section. Correspondingly, we are left with the task of computing vertex operator correlators of the form

$$\int [\mathcal{D}X\mathcal{D}h]_{\Sigma_{g,n}} V_1 \ldots V_n \exp(-S_P[X,h]) , \tag{2.2.1}$$

where we integrate over the "space of fields" defined on the genus g surface $\Sigma_{g,n}$ with n distinct punctures. From the above discussion we deduce that n-string scattering may be described by

$$A_n^{\text{string}} \sim \sum_{g\in\mathbb{N}_0} (e^{a\lambda})^g \sum_a \int [\mathcal{D}X\mathcal{D}h]_{\Sigma_{g,n;a}} V_1 \ldots V_n \exp(-S_P[X,h]) , \tag{2.2.2}$$

where the second sum is over the family of all relevant worldsheets of genus g with n distinct punctures.

As mentioned above, the classification of worldsheets-to-be-summed-over depends on the types of strings under consideration. To that end one may classify the worldsheets without punctures and then consider, for any worldsheet we deem appropriate, all consistent distributions of punctures. Then in the case of closed strings the relevant surfaces are just the closed compact genus g surfaces, which may be obtained by gluing spheres S^2, tori T^2 and projective planes $\mathbb{R}P^2$ together. Note that $\mathbb{R}P^2$ is non-orientable and therefore is only relevant in the case of unoriented strings. For open and orientable strings the worldsheets may essentially be obtained from the closed oriented surfaces by removing open disks. In the case of non-orientable open strings relevant to type I superstrings, we also have to consider worldsheets that are non-orientable surfaces with boundaries, where the exact classification is more involved. However, we will be only interested in genus ≤ 1 where the relevant surfaces for open strings are the compact disk at genus zero, complemented at genus one by the cylinder and the Möbius strip. Then we still have to sum over all inequivalent distributions of distinct vertex operator insertions, with corresponding punctures on the boundary components for asymptotic open-string states and on the interior of the worldsheet for asymptotic closed string states, respectively.[26] Furthermore, as all boundary components are homeomorphic to S^1 the order of punctures on a given boundary is only relevant up to cyclic shifts, which is reflected in that they come with a trace over Lie algebra

[25] Note that the Euler characteristic will also dependent on the number n of external strings; e.g. for closed strings the asymptotic states correspond to closed boundary curves. However, this particular contribution to the Euler characteristic will be the same for all worldsheets with n external string states and thus leads to an overall n-dependent prefactor we omit in our discussion as it can be easily restored.

[26] We note that removing points from a compact surface generally leads to non-compact surfaces, which we might illustrate with the prominent example of removing the point ∞ (or any other point) from the Riemann sphere leading to the complex plane.

generators related to the Chan-Paton degrees of freedom accompanying the vertex operators.

Now we basically know what we want to compute and we are left with the task of giving a precise meaning to the measure $[\mathcal{D}h\mathcal{D}X]_{\Sigma_{g,n}}$. This was originally done by Polyakov [95] and is the higher-genus analogue of the discussion of the previous section, albeit with new features. We briefly sketch the argument but refer to the original article as well as [77–79, 81, 87, 89] for the computational details. Our discussion will commence with the consideration of the global structure of the space of metrics up to pull-backs of diffeomorphisms. Subsequently, we will write down explicit expressions deduced from local properties as well as our global knowledge. As a disclaimer we note that the deduction of the following statements concerning global properties generally needs very heavy mathematical machinery, which we deem beyond the scope of our exposition; however details on the mathematical background can be found e.g. in [77–79, 82, 100–104].

Instead of considering compact surfaces with boundary that may or may not be orientable, it turns out to be convenient to consider certain covering surfaces and subsequently consider the (induced) action of the covering map on the structures we are interested in. In particular, for non-orientable surfaces we consider the orientation double cover with fibres given by $\mathbb{Z}_2$, i.e. the two possible choices of orientation. Moreover, for compact surfaces with boundary we can consider the double obtained by taking two copies and gluing them along their boundary components. Hence, using either or both of these two construction we end up with a closed compact surface, where the relation to the original surface is described by an anti-holomorphic involution on the cover and the boundary components are given by the connected components of the fixed-point set of said involution, cf. [82, 100]. We will discuss these cover constructions in some detail for the genus-one case in section 2.3.[27] This is convenient as we may discuss the measure in the case of closed oriented strings, where a worldsheet equipped with a conformal structure is equivalent to a closed compact Riemann surface. From now on $\Sigma_{g,n}$ will denote a surface obtained by removing n distinct points from the closed compact genus g surface Σ_g. Moreover, for the rest of this exposition we assume that

$$2 - 2g - n < 0 \tag{2.2.3}$$

unless otherwise stated, which is in particular the case for $n \geq 4$, i.e. scattering of four or more strings.

So far we have established that (the cover of) our string worldsheet equipped with some conformal structure is equivalent to a Riemann surface with punctures. Correspondingly, we know that globally defined conformal mappings on our worldsheet correspond to automorphisms (i.e. biholomorphic mappings) of Riemann surfaces. Let us briefly make some comments on the automorphism groups of closed compact Riemann surfaces.[28] Due to a theorem of Hurwitz it is known that the automorphism group is of finite order for genus $g \geq 2$, cf. e.g. [101]. This is however not true for genus zero and one. For genus zero the automorphism group is the Möbius

[27] Note that these concepts permeate the whole discussion of the free string in the previous section. Specifically, the double of the upper half-plane is of course just the plane and the corresponding anti-holomorphic involution on the double is given by $\rho(z) = \bar{z}$, with fixed point set the real line.

[28] To be precise we do not mean automorphisms of punctured surfaces but rather their unpunctured counterpart. Note that automorphism groups of punctured surfaces will be smaller than their unpunctured incarnation, as the automorphisms have to map punctures to punctures. A simple example is the once-punctured Riemann sphere equivalent to the complex plane, whose automorphisms are of the form $az+b$ leading to a two parameter subgroup of the Möbius group.

group $PSL(2;\mathbb{C})$, while in the case of the torus the automorphisms are given by translations on the torus. Hence, the dimension of the component of the automorphism group connected to the identity for closed compact Riemann surfaces is given by[29]

$$\dim_{\mathbb{C}}(\mathrm{CKG}(\Sigma_g)) = \begin{cases} 3 & \text{for } g = 0 \\ 1 & \text{for } g = 1 \\ 0 & \text{for } g \geq 2 \end{cases} . \tag{2.2.4}$$

The presence of these groups for genus zero and one suggests that in these cases we need a minimal amount of punctures to have a well-defined notion of configurations of punctures. Specifically, if we have a genus-zero surface with three punctures their individual coordinates do not really matter as we may send them to three other coordinates say $\{0, 1, \infty\}$ via some automorphism. Equivalently we may say that all $\Sigma_{0,3}$ are mutually biholomorphic and we usually choose $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$. However, if we consider four distinct punctures one may consider the cross ratio $\frac{(z_1-z_3)(z_2-z_4)}{(z_2-z_3)(z_1-z_4)}$ that is invariant under the action of the Möbius group and therefore a meaningful quantity. This is in fact our first example of moduli space. At genus one life becomes more complicated as not all Riemann surfaces associated to Σ_1 are mutually biholomorphic, while this was certainly the case for Σ_0. The different complex structures on a torus can be described by a complex number $\tau \in \mathbb{H}$, which is the second appearance of a moduli in our discussion; cf. section 2.3. Furthermore, if we have two tori described by the same τ, then their once-punctured versions are also biholomorphic. Conversely, if we have $\Sigma_{0,n\geq 3}$ or $\Sigma_{1,n\geq 1}$ we might view the automorphisms (of the corresponding unpunctured Riemann surface) as means to fix three points in the case of genus zero and one in the case of the torus. Finally we note that, for genus $g \geq 2$ the finite number of automorphisms is not sufficient to biholomorphically relate configurations of distinct points that are in generic positions.

Now our aim is to understand the space of conformal structures compatible with the topology of our surface $\Sigma_{g,n}$. A first statement one may make can be deduced from the uniformization theorem due to Koebe and Poincaré, which implies that the universal cover of $\Sigma_{0,n\geq 3}$, $\Sigma_{1,n\geq 1}$ and $\Sigma_{g\geq 2}$ is the disk suggesting that all these surfaces admit only hyperbolic metrics. Therefore, the task at hand may be formulated as understanding the space of hyperbolic metrics on $\Sigma_{g,n}$ up to diffeomorphisms (acting by pullback). As a first step it seems logical to consider the group of diffeomorphisms connected to the identity $\mathrm{Diff}_0(\Sigma_{g,n})$ as this will be what we may study via infinitesimal deformations of our metric. The corresponding space

$$\mathrm{Teich}(\Sigma_{g,n}) = \{\text{Hyperbolic metrics on } \Sigma_{g,n}\}/\mathrm{Diff}_0(\Sigma_{g,n}) \tag{2.2.5}$$

is an incarnation of the so-called Teichmüller space. Furthermore, the distinction between the infinitesimal diffeomorphisms and their discrete counterparts is captured by the mapping class group defined (in the context of smooth surfaces) by

$$\mathrm{MCG}(\Sigma_{g,n}) = \mathrm{Diff}(\Sigma_{g,n})/\mathrm{Diff}_0(\Sigma_{g,n}) , \tag{2.2.6}$$

[29] In this context the connected component of the automorphism group containing the identity is also called conformal Killing group. Note that in an abuse of notation we denote the conformal Killing group of some genus g Riemann surface by $\mathrm{CKG}(\Sigma_g)$, although in our notation Σ_g denotes the underlying surface without specification of a complex structure.

where $\mathrm{Diff}(\Sigma_{g,n})$ denotes the whole diffeomorphism group. Eventually, we may consider Teichmüller space up to the action of the mapping class group, which is the famous moduli space of Riemann surfaces of genus g with n punctures

$$\mathrm{Mod}(\Sigma_{g,n}) = \mathrm{Teich}(\Sigma_{g,n})/\mathrm{MCG}(\Sigma_{g,n}) \ . \tag{2.2.7}$$

This is basically the statement we wanted to motivate in this chapter: (in the absence of anomalies regarding the mapping class group) amplitudes in bosonic string theory may be considered as integrals over the moduli space of genus g surfaces.

Before we dirty our hands with local coordinates and infinitesimal deformations thereof we want to provide a bit more context to the spaces just defined. Let us briefly discuss the genus-zero case, there the moduli space has the nice description

$$\begin{aligned} \mathrm{Mod}(\Sigma_{0,n+3}) &= \{(z_1, \ldots, z_{n+3}) | z_i \in \mathbb{C}P^1 \text{ and } z_i \neq z_j \ , \forall i,j\}/PSL(2;\mathbb{C}) \\ &= \{(w_1, \ldots, w_n) | w_i \in \mathbb{C}P^1 \setminus \{0,1,\infty\} \text{ and } w_i \neq w_j \ , \forall i,j\} \end{aligned} \ . \tag{2.2.8}$$

Next we consider the once punctured torus $\Sigma_{1,1}$, relevant to the study of genus-one string amplitudes in the next chapter, for which one finds [102]

$$\mathrm{Teich}(\Sigma_{1,1}) = \mathbb{H} \ , \qquad \mathrm{MCG}(\Sigma_{1,1}) = SL(2;\mathbb{Z}) \ , \tag{2.2.9}$$

and correspondingly the moduli space is given by

$$\mathrm{Mod}(\Sigma_{1,1}) = \mathbb{H}/SL(2;\mathbb{Z}) \ . \tag{2.2.10}$$

Our interest in $\mathrm{Mod}(\Sigma_{1,1})$ is due to the fact that we may "forget" punctures. Specifically, there exist $n+1$ maps $\pi_i : \mathrm{Mod}(\Sigma_{g,n+1}) \to \mathrm{Mod}(\Sigma_{g,n})$ with generic fibres given by the corresponding Riemann surface $\Sigma_{g,n}$, this is e.g. nicely discussed in [105, 106].[30] This basically tells us that we may roughly think of points in the moduli space $\mathrm{Mod}(\Sigma_{1,n})$ as given by some moduli of $\tau \in \mathrm{Mod}(\Sigma_{1,1})$ together with $n-1$ copies of the corresponding Riemann surface associated to $\Sigma_{1,1}$ (without the diagonal of the product). Finally, for the generic case we state that the dimension of the Teichmüller spaces in question is given by

$$\dim_{\mathbb{C}}(\mathrm{Teich}(\Sigma_{g,n})) = 3g - 3 + n \ . \tag{2.2.11}$$

With the global properties of the worldsheet in mind we now proceed by revisiting the path integral measure, where we in the same vein as in the previous section start of with considering the space of metrics. Again our aim is to find a decomposition of the tangent space of metrics at the point $\hat{h} \in \mathcal{M}(\Sigma_{g;n})$, which is orthogonal w.r.t. the inner product (2.1.20) on $T_{\hat{h}}\mathcal{M}(\Sigma_{g;n})$. But in contradistinction to the free string decomposition (2.1.21), there are now degrees of freedom corresponding to different complex structures that are equivalent to points in Teichmüller space. Then after reabsorbing traces into the multiplicative factor ϕ, the decomposition of the tangent

[30] Note that this is only a sensible thing to do if we, by forgetting the puncture, do not leave the realm of hyperbolic geometry. Explicitly, we should stop at the "minimal hyperbolic case", i.e. $\Sigma_{0,3}$, $\Sigma_{1,1}$ and $\Sigma_g, g \geq 2$ respectively.

space $T_{\hat{h}}\mathcal{M}(\Sigma_{g;n})$ has to take the form

$$\delta h_{ab} = \phi h_{ab} + (P_1 v)_{ab} + \sum_{i\in I} t_i \theta_{i;ab} \ , \tag{2.2.12}$$

where the map P_1 again labels the traceless part of the Lie derivative and is identical to the expressions for the free string (2.1.22). However, we reiterate that the space of zero modes of P_1 depends on the global properties of the worldsheet and therefore depends on $\Sigma_{g;n}$.

The novel aspect of this expression is the appearance of infinitesimal deformations of the complex structure denoted $\theta_{ab} = \sum_i t_i \theta_{i;ab}$ with $i \in I$ labeling the degrees of freedom of the corresponding deformation. These deformations have to be symmetric in the indices a, b and demanding θ_{ab} to be orthogonal to rescalings renders θ_{ab} traceless. Furthermore, requiring orthogonality with $(P_1 v)_{ab}$ leads to

$$0 = (\theta, P_1 v)_{\hat{h}} = \int_{\Sigma_g} \mathrm{d}^2\sigma \sqrt{\det(\hat{h})}\theta_{ab}(P_1 v)^{ab} = \int_{\Sigma_g} \mathrm{d}^2\sigma \sqrt{\det(\hat{h})} v^a(-2\nabla^b)\theta_{ab} \tag{2.2.13}$$

which holds in the absence of boundary terms. Note that from (2.2.13) we can infer that the adjoint of P_1 is given by $P_1^\dagger = -2\mathrm{div}_{\hat{h}}$ (acting on symmetric traceless θ_{ab}) and hence θ_{ab} needs to be divergence free in order to be orthogonal to $P_1 v$. Choosing isothermal coordinates such that $z = \sigma^1 + i\sigma^2, \bar{z} = \sigma^1 - i\sigma^2$, these requirements can be summarized as

$$\begin{gathered} \theta_{ab}\mathrm{d}\sigma^a\mathrm{d}\sigma^b \mapsto \frac{1}{2}(\theta_{11} - i\theta_{12})(\mathrm{d}z)^2 + \frac{1}{2}(\theta_{11} + i\theta_{12})(\mathrm{d}\bar{z})^2 \ , \\ \text{with} \quad \partial_{\sigma^1}\theta_{11} = -\partial_{\sigma^2}\theta_{12} \ , \quad \partial_{\sigma^2}\theta_{11} = \partial_{\sigma^1}\theta_{12} \ , \end{gathered} \tag{2.2.14}$$

i.e. θ is the real part of a holomorphic quadratic differential. Subsequently, we want to find an orthogonal basis of the kernel of $P_1^\dagger$. To that end denoting a basis of $\ker(P_1^\dagger)$ by $\xi_{i;ab}$, we may formulate the orthogonal decomposition of the space of θ_{ab} as

$$\theta_{ab} = \sum_{i,j,k\in I} t_k\, \xi_{i;ab}\, \pi_{ij}^{-1}(\xi_j, \theta_k)_{\hat{h}} \ , \tag{2.2.15}$$

with the matrix $\pi_{ij} = (\xi_i, \xi_j)_{\hat{h}}$. Thus noting the orthogonal decomposition as described in (2.2.12,2.2.15) we deduce the measure to decompose as

$$\mathcal{D}h = \frac{\det(\theta_j, \xi_k)_{\hat{h}}}{\left[\det(\xi_j, \xi_k)_{\hat{h}}\right]^{1/2}} \left[\det{}'(P_1^\dagger P_1)\right]^{1/2} \mathcal{D}\phi\, \mathcal{D}'v\, \mathrm{d}^{|I|}t \tag{2.2.16}$$

where the prime denotes the omission of possible zero modes of P_1. From the above discussions we infer that the space of quadratic differentials is related to Teichmüller space. In fact, the space of quadratic differentials can be thought of as cotangent space of Teichmüller space $T^*_S\mathrm{Teich}(\Sigma_{g;n})$ at a given Riemann surface S and consequently we find $|I| = \dim(\mathrm{Teich}(\Sigma_{g;n}))$; cf. [102, 104, 107–109] for details. Furthermore, we note that in isothermal coordinates the inner product (2.1.20) for holomorphic quadratic differentials of the form $\phi_i(z)(\mathrm{d}z)^2, i = 1, 2$ is given by

$$(\phi_1, \phi_2)_{\hat{h}} = 2\int_{\Sigma_g} \mathrm{d}z\mathrm{d}\bar{z}\, e^{-2\omega}\, \mathrm{Re}(\phi_1\bar{\phi}_2) \ , \tag{2.2.17}$$

which is (twice the real part) of the so-called Weil-Petersson metric. Importantly, the Weil-Petersson metric is invariant under the mapping class group and therefore extends to the moduli space, cf. [110]. The volume form induced by the Weil-Petersson metric is called Weil-Petersson measure and can be identified with[31]

$$\mathrm{d}(WP) = \frac{\det(\theta_j, \xi_k)_{\hat{h}}}{(\det(\xi_j, \xi_k)_{\hat{h}})^{1/2}} \mathrm{d}^{|I|}t \ . \tag{2.2.18}$$

The degrees of freedom corresponding to $\mathcal{D}'v$ are essentially taken care of by choosing isothermal coordinates, hence we might drop them. Moreover, zero modes of P_1 only exist for surfaces $\Sigma_{0,n\leq 2}$ and $\Sigma_{1,0}$ and therefore their omission does not cause any issue.[32] As for the scaling degrees of freedom labeled by ϕ there is a technical issue related to the fact that the inner products (2.1.20,2.1.33) we used to deduce orthogonal decompositions are not Weyl invariant. This failure of being Weyl invariant will be related to certain issues related to the functional determinants to be discussed momentarily. However, one eventually finds that for $D = 26$ the Weyl anomaly of individual terms cancel exactly and thus we also omit integration over ϕ.

For the embedding coordinates we still have the decomposition

$$\int \mathcal{D}X^\mu \exp\big(-S_P[\hat{h}, X^\mu]\big) \to \int \mathcal{D}X_0^\mu \exp\big(-S_P[\hat{h}, X_0^\mu]\big) \int \mathcal{D}\xi^\mu \exp\big(-S_P[\hat{h}, \xi^\mu]\big) \ , \tag{2.2.19}$$

with the distinction that X_0, ξ encode eigenfunctions of the Laplace operator $\Delta_{\hat{h}}$ defined on $\Sigma_{g;n}$ for some specified conformal structure with the choice $\hat{h}$. The corresponding normalization may be inferred by the decomposition (2.2.19) together with the fact that

$$S_P[h, X] \sim (X_\mu, \Delta_h X^\mu)_h \tag{2.2.20}$$

up to boundary terms. Accordingly, one interprets the rhs. of (2.2.19) (in the absence of vertex operators) as some infinite-dimensional Gaussian integral, leading to the normalization

$$V_D \left[8\pi^2 \det{}'(\Delta_{\hat{h}}) \left(\int_{\Sigma_g} \mathrm{d}^2\sigma \sqrt{\det(\hat{h})} \right)^{-1} \right]^{-D/2} , \tag{2.2.21}$$

where the prime denotes omission of zero modes of the Laplacian and V_D is the volume of spacetime stemming from the integration over X_0^μ.

[31] We note that $\mathrm{Teich}(\Sigma_g)$ is for $g \geq 2$ actually homeomorphic (but not biholomorphic) to $\mathbb{H}^{3g-3}$, which may be seen by constructing so-called Fenchel-Nielsen coordinates, which we denote $(l_i, \theta_i) \in \mathbb{R}_+ \times \mathbb{R}, i = 1, \ldots, 3g-3$. These Fenchel-Nielsen coordinates only give rise to smooth charts but surprisingly they allow for a simple formula of the Weil-Petersson measure

$$\mathrm{d}(WP) = \prod_{i=1}^{3g-3} l_i \mathrm{d}l_i \mathrm{d}\theta_i \ .$$

Details can be found in [78, 102, 106, 108].

[32] We might also approach the zero mode issue with a different mindset and consider unpunctured surfaces as is usual in most of the literature, cf. [77–79,81,82,87–92]. The N_c zero modes of P_1 that are present in this setup, lead to c ghosts of the form $c = \sum_i c_{Z;i} f_i(z) + c_R(z,\bar{z})$, where $c_{Z;i}$ and c_R are Graßmann valued and the f_i are global solutions to $\partial_{\bar{z}} f_i = 0$. Accordingly, the path integral might integrate to zero due to the definition of Graßmann valued integrals $\int \mathrm{d}c_{Z;i} = 0$. This issue is usually countered by using the conformal Killing group to fix punctures such that the integral is non-zero, usually implemented by altering N_c vertex operators $V(z,\bar{z}) \mapsto (c\tilde{c}V)(z,\bar{z})$ and omitting integration over their coordinates. Then by virtue of $\int \mathrm{d}c_z c_z \neq 0$ the corresponding integral might be non-zero. Note that for genus zero these two interpretations are encoded in the equality (2.2.8).

Now for the functional determinant $\det'(\Delta_{\hat{h}})$ we note that the interpretation as infinite-dimensional Gaussian integral, should be equivalent to an interpretation as product over all eigenvalues of $\Delta_{\hat{h}}$, which however doesn't seem incredibly well-defined either way. Moreover, the functional determinant $\det'(P_1^\dagger P_1)$ is plagued by similar issues. In order to give some meaning to these determinants we need to introduce a regulator. Examples of regulators are heat-kernel regularization and zeta-function regularization; cf. [77, 78, 82] for detailed definitions. Although the technical details of both regulators are different, it turns out that for both regulators the functional determinants in question are not Weyl invariant but in fact pick up a factor of the form e^{cL}, where L depends solely on the Weyl transformation and is therefore the same for $\det(\Delta_{\hat{h}})$ and $\det(P_1^\dagger P_1)^{1/2}$. Moreover, $c = D$ for $\det(\Delta_{\hat{h}})$ and $c = -26$ for $\det(P_1^\dagger P_1)^{1/2}$, i.e. the central charges of the D uncoupled scalar fields and the bc ghost system. Rephrasing the last statement, we note that for $D = 26$ we may regulate the theory in a Weyl invariant way and hence in the critical dimension the determinants vary nicely over Teichmüller space.[33] Explicit formulas for the above functional determinants can be found in [77, 78].

Finally, we note that the averaging of vertex operator insertions over the embedding coordinates X may be readily computed.[34] For example for the scattering of n-tachyons one finds

$$\begin{aligned}\int [\mathcal{D}X]_{\Sigma_g} \Big(\prod_{i=1}^{n} \exp(ip_i^\nu X_\nu(z_i, \bar{z}_i)) \Big) \exp(-S_P[\hat{h}, X]) \\ \sim \delta^{(D)}(\sum_i p_i) \exp\Big(-1/2 \sum_{i\neq j} \eta_{\mu\nu} p_i^\nu p_j^\mu G_g(z_i, z_j) + \sum_i p_i^2 \text{ regularized } \Big) ,\end{aligned} \tag{2.2.22}$$

where the second sum in the exponential is related to the regularization of the logarithmically singular behaviour of the genus g Green function $G_g(z, w)$ in the limit $z_j \to z_i$ (and vice-versa), which we omit; cf. for the regularized expression [78]. The corresponding Green function is the solution of the differential equation

$$\Delta_h G_g(z_i, z_j) = -4\pi\delta^2(z_i - z_j) + 4\pi\Big(\int_{\Sigma_g} \mathrm{d}^2\sigma \sqrt{\det(h)} \Big)^{-1} . \tag{2.2.23}$$

2.3 Specifics on genus-one worldsheets

As the structure of the genus-one open-string amplitude will be the main focus of the next chapter we discuss the complex structures on the torus (the double of the cylinder and Möbius strip) and subsequently consider the covering maps briefly hinted at in section 2.2. The ensuing discussion is heavily inspired by [77, 79, 99, 102, 114–117].

We begin by briefly discussing the space of complex structures on the torus. The torus may be considered as some parallelogram in $\mathbb{C}$ with opposite edges identified, i.e. the coset $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ where we choose $\omega_2/\omega_1 \in \mathbb{H}$.[35] Note that the lattice $(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ is invariant by the action of $SL(2; \mathbb{Z})$, conversely one may show that two lattices are isomorphic (as embedded subgroups of

[33] Specifically, it was shown in [111] that for $D = 26$ the product of the two functional determinants involved decomposes into holomorphic and anti-holomorphic parts on Teichmüller space, cf. also [77, 82, 112].

[34] To be precise we need to demand BRST exact vertex operators to decouple from the path integral. Specifically, we demand correlation functions including vertex operators of the form $\{Q_B, W\}$ to vanish when integrated over the corresponding moduli space; cf. the expositions in [80, 113] for details.

[35] The corresponding projection $\mathbb{C} \to \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ is a covering map, implying that $\mathbb{C}$ is the universal cover of the torus.

$\mathbb{C}$) iff they are related by an $SL(2;\mathbb{Z})$ transformation. Now writing down charts will inevitably depend on the lattice but different lattices might still lead to compatible holomorphic atlases. So we have to understand under which circumstances two tori defined by different lattices can be related by a biholomorphic map. The automorphisms of $\mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2)$ necessarily descend from the automorphisms of the plane $z \mapsto az + b$. Subsequently, if we consider two embedded lattices, say L_1 and L_2, compatibility with the corresponding projections demands $L_2 = aL_1$ and furthermore $b \in L_2$. This implies that all lattices with equal $\tau = \omega_2/\omega_1$ define the same complex structure. For the remainder of this work we will therefore consider tori of the form $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$ with $\tau \in \mathbb{H}$. The corresponding Teichmüller space is $\mathrm{Teich}(T^2) = \mathbb{H}$. Now in the above discussion we implicitly fixed zero as it is the unique neutral element of the additive group of complex numbers, which is necessary for the coset constructions and led to the constraint $b \in L_2$. However, if we consider the quotients topologically we are allowed to have affine transformations and we therefore may choose the parameter b to lie in the fundamental domain of the lattice, usually denoted by $b \in U(1) \times U(1)$ with factors related to the two homology cycles on the torus, i.e. the conformal Killing group. Finally, the space of inequivalent tori, the moduli space, is given by $\mathbb{H}/SL(2;\mathbb{Z})$, where the mapping class group acts as subgroup of $PSL(2;\mathbb{R})$ on $\tau \in \mathbb{H}$. This space may be represented via the set $\{\tau \in \mathbb{H} \mid -1/2 < \mathrm{Re}(\tau) \leq 1/2 \text{ and } |\tau| > 1\}$; cf. e.g. [118]. Hence, provided we have invariance under the mapping class group, we may restrict our attention to $|\tau| > 1$.

We now go on to relate cylinder and Möbius strip to the torus $\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. Recall that the involutions describing our initial spaces have to be anti-holomorphic. It turns out that there are only finitely many such anti-holomorphic involutions on a torus with a given complex structure, cf. the discussion in [119], which we briefly summarize here. First note that the notion of anti-holomorphic map has to descend from the universal cover and due to the requirement of being an involution is at worst linear $\tilde{\rho}(z) = a\bar{z} + b$, where the tilde denotes the lift to the plane. Subsequently, from compatibility with the lattice $\rho(z+m+n\tau) = \rho(z)$ and the involutive character of the map $\rho(\rho(z)) = z$ one can deduce that $a, a\bar{\tau}, (a\bar{b} + b)$ all have to be points in the lattice $(\mathbb{Z}+\mathbb{Z}\tau)$ and furthermore $|a|^2 = 1$. For $|\tau| > 1$ this implies $a = \pm 1$ but more importantly can only be consistent if

$$\mathrm{Re}(\tau) = 0 \quad \text{or} \quad \mathrm{Re}(\tau) = 1/2\ . \tag{2.3.1}$$

The different choices for a lead only to minor alterations in the definition of the involved maps, so we will exclusively consider $a = 1$ from now on. Then $a = 1$ implies that $2\mathrm{Re}(b)$ is an element of the lattice. Moreover, one may use an automorphism of the plane (acting by conjugation on the involution) such that b is real. Hence, for $\mathrm{Re}(\tau) = 0$ we may choose $b = 0$ or $b = 1/2$, while for $\mathrm{Re}(\tau) = 1/2$ both choices lead to the same involution hence we choose $b = 0$. Finally, we note that in the case of $\mathrm{Re}(\tau) = 0$ the choice $b = 1/2$ leads to involutions without fixed point set, i.e. a Klein bottle.

We begin by considering the case of a torus with $\mathrm{Re}(\tau_C) = 0$ denoted $\tau_C = it\ ,\ t \in \mathbb{R}_+$, which will turn out to have a fixed point set consisting of two connected components, i.e. the cylinder. In order to be explicit we choose a convenient chart on the torus given by $z = s + r\tau_C$ with $r, s \in [0, 1)$ equivalent to the fundamental domain of the lattice. In the case of the cylinder the double cover may be obtained by taking two copies and gluing them along their boundary components and the corresponding anti-holomorphic involution ρ_C exchanges the two copies.

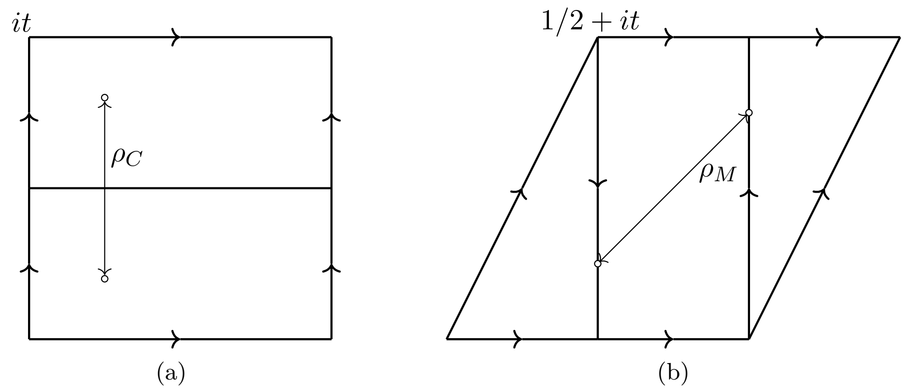

Figure 2.1: A topological cartoon for the corresponding covers for (a) the cylinder and (b) the Möbius strip.

If we choose coordinates on the torus as stated above and choose the involution[36] descending from $\tilde{\rho}_C(z) = \bar{z}$, then the two corresponding cylinders may be described by the domains $(s, r) \in [0, 1] \times [0, 1/2]$ and $[0, 1] \times [1/2, 1]$ respectively. Furthermore, ρ_C is (in the fundamental domain of the lattice) given by

$$\rho_C(s, r) = (s, 1 - r) \; ; \tag{2.3.2}$$

cf. the left panel of fig. 2.1. The fixed point set of ρ_C decomposes into two connected sets $\{z = s + r\tau_C \,|\, s \in (0, 1),\, r = 0\}$ and $\{z = s + r\tau_C \,|\, s \in (0, 1),\, r = 1/2\}$, which describe the two boundary components of the cylinder. With this representation of ρ_C we now may take the quotient $(\mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau))/\mathbb{Z}_2$, where $\mathbb{Z}_2$ is meant to be generated by the action of ρ_C.

Similarly, we may consider the case where $\mathrm{Re}(\tau_M) = 1/2$ i.e. $\tau_M = 1/2 + it$, which will lead to the Möbius strip. The corresponding parallelogram may be described by $z = s + r\tau_M$ with $(s, r) \in [0, 1)$. As stated above the corresponding involution descends again from $\tilde{\rho}_M(z) = \bar{z}$ and we may describe the Möbius strip as the rectangular $r \in (0, 1), s \in (1/2 - r/2, 1 - r/2)$, cf. the right panel of fig. 2.1. Correspondingly, the involution is given by

$$\rho_M(s, r) = \begin{cases} (s + r, 1 - r) & \text{if } s + r \in (r/2, r/2 + 1) \\ (s + r - 1, 1 - r) & \text{else} \end{cases} \;, \tag{2.3.3}$$

where we note that this time there is only one fixed point set, as should be the case for the Möbius strip. The corresponding boundary is then described by $\{z = s + r\tau_M \,|\, s \in (1/2, 1),\, r = 0\}$.

2.4 The NSR formulation of the superstring

So far we were content with discussing a purely bosonic theory. We will now go on to add supersymmetry and discuss the novel features this brings about. Specifically, we will consider a theory with local superconformal symmetry on the worldsheet, called NSR formalism, which is

[36] The other choice $\tilde{\rho}'_C(z) = -\bar{z}$ gives the partitioning "orthogonal" to the one we use; our choice is depicted in the left panel of fig. 2.1. Note that the choice $\rho(z) = \bar{z}$ leads to one boundary component that is real, which will make our life easier when considering iterated integrals in the next chapter. In particular this choice will lead to so-called A-cycle TEMZVs while the other choice of anti-involution leads to B-cycle TEMZVs that are more challenging to deal with on a technical level.

what we will exclusively consider throughout this work. For different formulations of superstring theories, cf. [87, 88, 92, 120] and the references therein. Our exposition follows [77–81, 100, 121–123].

Now that we have settled on a way of introducing supersymmetry we need to raise the question whether we can consistently define spinors on our worldsheet of choice. We start by making some comments on the essence of spinors. Generically, on a d-dimensional vector space with $d > 2$, real spinors may be regarded as representations of the double cover $Spin(d)$ of the special orthogonal group $SO(d)$, which is equivalent to the physical property that the transformation of a spinor subjected to a rotation of 2π has a sign ambiguity if the corresponding curve is not contractible. Consequently, when considering some connected orientable manifold the corresponding generalized notion is related to the principal $Spin(d)$ bundle, covering the corresponding principal $SO(d)$ bundle (the orthonormal frame bundle).[37] The existence of such a double cover was found to be tied to the so-called second Stiefel-Whitney class being trivial (as element of the second cohomology group), cf. [78, 79, 123]. Importantly the second Stiefel-Whitney class is trivial for oriented closed surfaces [79] and therefore we know that spinors on such worldsheets have no choice but to exist. Now there is a small terminology issue in two dimensions as $Spin(2)$ is isomorphic to $SO(2)$. Nevertheless, one still wants transition functions in some connected double cover of $SO(2)$ (such that the kernel of the covering map is ± 1). This is essentially the notion of spin structure in two dimensions. In the case of a non-orientable worldsheet one defines the spin structure on the orientation double cover of the surface, cf. [100].

For a given surface there are several spin structures corresponding to the sign ambiguities when parallel transporting a spinor around a non-contractible loop. Mathematically, spin structures are related to the first cohomology group of the principal $SO(2)$ bundle with coefficients in $\mathbb{Z}_2$. Moreover, due to the vanishing of the second Stiefel-Whitney class we may relate spin structures to a more intuitive object, the explicit statement being that spin structures are in on-to-one correspondence to cohomology classes of

$$H^1(\Sigma_g, \mathbb{Z}_2) \cong \mathrm{Hom}(H_1(\Sigma_g, \mathbb{Z}_2), \mathbb{Z}_2) , \tag{2.4.1}$$

cf. [123] for details on how this comes about. For orientable closed compact surfaces the rhs. is isomorphic to $\mathbb{Z}_2^{2g}$, from which one may deduce that on such surfaces there exist 2^{2g} inequivalent spin structures. Let us consider the example of the torus.[38] Choosing a basis for the homology of the torus we have four possible distributions of signs, two for each homology cycle, i.e. there are four distinct spin structures on the torus. A key observation is now that the action of the mapping class group might exchange distinct spin structures. This is most apparent in the case of the torus, where the mapping class group $SL(2;\mathbb{Z})$ induces a change of basis of the homology, and thus relates (bijectively) different homology classes. So naively we deduce that if we strive for invariance under the action of the mapping class group we need to include all spin structures that may be related by the mapping class group.[39]

[37] Let us make two remarks concerning these notions. Firstly, we note that $Spin(d)$ is not defined as the double cover of $SO(d)$, although it turns out to be exactly that for $d \geq 3$. The precise definition is as a peculiar subgroup of the group of units of the corresponding Clifford algebra. Secondly, to be precise spinors are elements of a so-called spinor bundle, which roughly can be thought of as the bundle associated to the principal *Spin* bundle by some representation of the *Spin* group. For a precise and comprehensive discussion, we recommend [123].

[38] As we will discuss in section 2.3 all possible genus-one open-string worldsheets have as double cover the torus.

[39] Still there are possible regularization issues for the involved functional determinants, and one requires the cancellation of singular terms in the summation over different spin structures (usually referred to as modular

Up to now we have been vague about what kind of spinors we actually consider, whence the notion of supersymmetry is also somewhat nebulous. The exact details of spinor representations depend on the signature (p, q) of the worldsheet metric and we briefly summarize the results; detailed expositions can be found in [124–127]. In even dimensions there are two indecomposable spinor representations given by Weyl-spinors of definite chirality, which we denote $\psi^\mu_\pm$.[40] Moreover, as there is no preferred chirality we should take both options into account. It turns out that for Lorentz signature the Clifford algebra is isomorphic to the matrix algebra of real 2×2 matrices eventually leading to two inequivalent real one-component Majorana-Weyl spinor representations. Contrarily, for Euclidean signature the two inequivalent Weyl-spinor representations are complex one-component objects and if we want to match the degrees of freedom of the Lorentz case we need to impose some reality condition usually chosen $\bar{\psi}^\mu_+ = \psi^\mu_-$.[41] Now let us consider supersymmetry in Euclidean signature with supercharges living in the minimal spinor representations denoted $S_\pm = \mathbb{C}Q_\pm$. We then take as odd part of the supersymmetry algebra $S_+ \oplus S_-$ and demand the corresponding super Lie bracket constrained to $S_+ \oplus S_-$, to map into the space of complexified momentum $\mathbb{C}^2$ with components denoted $P_z, P_{\bar{z}}$. Moreover, we note that generally the Lorentz generators are related to the Clifford algebra via $M^{ab} \sim [\gamma^a, \gamma^b]$ up to similarity transform, whence in our case $M \sim \gamma^1\gamma^2$ implying $[Q_\pm, M] \sim \pm Q_\pm$. To be more explicit, using the normalization as in [128], the super Lie bracket is given by

$$\begin{gathered}[Q_\pm, M] = \pm\frac{i}{2}Q_\pm\ ,\quad [Q_\pm, P_z] = [Q_\pm, P_{\bar{z}}] = 0\ ,\quad [P_z, M] = iP_z\ ,\quad [P_{\bar{z}}, M] = -iP_{\bar{z}}\ ,\\ \{Q_+, Q_+\} = 2P_z\ ,\quad \{Q_-, Q_-\} = 2P_{\bar{z}}\quad \text{and}\quad \{Q_+, Q_-\} = 0\ ,\end{gathered} \tag{2.4.2}$$

where we furthermore impose $Q_+^\dagger = Q_-$ (w.r.t. the Hermitian form on some putative state space) leading to $P_z = (P_{\bar{z}})^\dagger$, i.e. real momentum. This supersymmetry algebra is referred to as minimal supersymmetry or $\mathcal{N}=(1,1)$ (or sometimes even $\mathcal{N}=1$) supersymmetry.[42]

Now that we obtained the conviction that spinors may be consistently defined on the worldsheet our aim is to find a version of the Polyakov action (2.1.1) with manifest $\mathcal{N} = (1,1)$ supersymmetry on the worldsheet. Let us briefly sketch the field content of a putative super-

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invariance) cf. [78] for details.

[40] Chirality refers to the eigenvalue of the volume element of the Clifford algebra for even dimension $d = p + q$. Noting that the volume element may be chosen to be $\gamma^{d+1} = \gamma^1 \dots \gamma^d$ (note that we used "Euclidean labeling" of the Clifford algebra generators), one may infer that for even dimensions γ^{d+1} satisfies

$$\{\gamma^{d+1}, \gamma^i\} = 0 \quad \text{and} \quad (\gamma^{d+1})^2 = (-1)^{(p-q)/2}\mathbb{1}\ ,$$

leading to eigenvalues of γ^{d+1} given by ± 1 for $p - q = 0\,(\text{mod }4)$ and analogously $\pm i$ for $p - q = 2\,(\text{mod }4)$. Note that in the latter case the corresponding decomposition necessitates complexification.

[41] In two dimensions with Euclidean metric the corresponding Clifford algebra is equivalent to the quaternions. Quaternions may be expressed as the four-dimensional real matrix algebra, generated by the identity matrix $\mathbb{1}_2$ and

$$\gamma^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\ ,\quad \gamma^2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\ ,\quad \gamma^1\gamma^2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\ .$$

Spinors are related to the so-called even subalgebra of the Clifford algebra generated by $\mathbb{1}_2$ and $\gamma^1\gamma^2$, which happen to generate a matrix representation of the complex numbers. In the light of the previous footnote, we note that complexification of the Clifford algebra leads to the algebra of complex 2×2 matrices and the complexified subalgebra generated by $\mathbb{1}_2$ and $\gamma^1\gamma^2$ is then just diagonal complex matrices, i.e. $\mathbb{C} \oplus \mathbb{C}$.

[42] We note that one may also consider a superstring theory that exhibits $\mathcal{N} = (1,0)$ worldsheet supersymmetry rather than the $\mathcal{N} = (1,1)$ supersymmetric action (2.4.3) we consider below. The resulting theory is referred to as heterotic string theory, which is however outside the scope of our exposition and we will not discuss heterotic strings any further; cf. [77, 90] for details.

symmetric theory.[43] A local supersymmetry generated by some spinor field ε should relate the embedding coordinates X^μ to worldsheet fermions (belonging to some spin structure) which we denote $\delta_\epsilon X^\mu \sim \bar{\epsilon}\psi^\mu$.[44] Analogously, the worldsheet metric h_{ab} should be related to a field χ_a carrying both a worldsheet vector and worldsheet spinor index. The relation should be of the form $\delta_\varepsilon h_{ab} \sim \varepsilon(\gamma_a\chi_b + \gamma_b\chi_a)$, where γ^a denotes the generators of the Clifford algebra. With these comments in mind we will be content with merely stating the result, namely the action is of the form[45]

$$\begin{aligned} S_{NSR} =& \frac{1}{8\pi}\int\limits_\Sigma \mathrm{d}^2\sigma\sqrt{\det(h)}\Big[\frac{2}{\alpha'}h^{ab}\partial_a X^\mu \partial_b X_\mu + \psi^\mu\gamma_a\nabla^a\psi_\mu \\ & \qquad - (\psi^\mu\gamma^a\gamma^b\chi_a)\Big(\sqrt{\frac{2}{\alpha'}}\partial_b X_\mu + \frac{1}{4}\chi_b\psi_\mu\Big)\Big] , \end{aligned} \tag{2.4.3}$$

where the covariant derivative on spinors acts via the spin connection. The absence of a kinetic term for χ_a is due to the dimension of the worldsheet, as any such Rarita-Schwinger-like term would have to be contracted with $\gamma^{[a}\gamma^b\gamma^{c]} = 0$.

Let us briefly list the symmetries of the $\mathcal{N} = (1,1)$ super action (2.4.3). Firstly, the action is invariant under diffeomorphisms of the worldsheet. Next we note that the action is invariant under D-dimensional Poincaré transformations provided we demand that all $\psi^\mu_\pm$ are defined w.r.t. the same spin structure. Additionally, we require χ_a and ψ^μ to be defined w.r.t. the same spin structure, as otherwise going around a non-contractible cycle of Σ would lead to relative signs in the action, resulting in an action that would not be a well-defined functional integral on Σ. Moreover, we note that the action of Weyl rescaling $h_{ab} \mapsto \Omega^2 h_{ab}$ has to be augmented by $\gamma_a \mapsto \Omega\gamma_a$ and $\psi^\mu \mapsto \Omega^{-1/2}\psi^\mu$ in order to leave the kinetic term of the fermion invariant and hence also $\chi_a \mapsto \Omega^{1/2}\chi_a$ to leave the remaining action invariant. To show that the action (2.4.3) is in fact invariant under supersymmetry is a bit more involved, which we omit here in favour of a focused exposition and instead refer to [81, 87] for a detailed account. However, we note the supersymmetry variation of the gravitino χ_a

$$\delta_\varepsilon\chi_a = -2\nabla_a\varepsilon = (2h_a{}^b - \gamma_a\gamma^b)\nabla_b\varepsilon + \gamma_a(\gamma^b\nabla_b\varepsilon) , \tag{2.4.4}$$

as it will be relevant to the ensuing discussion. Finally, we note that in two dimensions the Clifford algebra generators satisfy $\gamma^a\gamma^b\gamma_a = 0$ and hence the redefinition $\chi_a \mapsto \chi_a + \gamma_a\kappa$, called super-Weyl transformation, leaves the action invariant. We now may use a supersymmetry variation of the gravitino (2.4.4) to locally choose $\chi_a = \gamma_a\zeta$, which simplifies the action due to $\gamma^a\gamma^b\gamma_a = 0$. Furthermore, as the worldsheet is still a two-dimensional smooth manifold we

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[43] In fact one may formulate the superconformal theory on the worldsheet on a so-called supermanifold. Such a reformulation is not only notationally quite economic but also provides a language in which supersymmetry acts naturally. Mathematically a supermanifold may be described as a locally ringed space that is locally isomorphic to $\mathbb{R}^{p|q}$ (as ringed spaces). This statement is just a fancy way of formulating a supermanifold as a space that is locally modelled after a super vector space. Detailed expositions on the supermanifold formulation of the NSR superstring can be found e.g. in [77, 78, 80]. For an overview on supermanifolds we refer to [129].

[44] Notational disclaimer: Here and in the following we occasionally suppress (worldsheet) spinor indices as to not unnecessarily clutter the notation with even more indices.

[45] In general, writing down an action for a supersymmetric theory comes with some minor annoyances as bosonic and fermionic degrees of freedom usually only match if one imposes the equations of motion. This issue is usually fixed via the introduction of auxiliary fields into the action, which can then be eliminated via the equations of motion, cf. the discussion in [128, 130, 131] for a general account from the viewpoint of supersymmetry and [81] for a discussion on auxiliary fields for the NSR formulation of the superstring.

still may use local conformal flatness to choose isothermal coordinates. This combined choice is called *superconformal gauge* and leads to the simpler expression

$$S_{NSR} = \frac{1}{8\pi} \int\limits_{\Sigma} \mathrm{d}z\, \mathrm{d}\bar{z}\, \eta_{\mu\nu} \left[\frac{2}{\alpha'} \partial_z X^\mu \partial_{\bar{z}} X^\nu + \psi_+^\mu \partial_{\bar{z}} \psi_+^\nu + \psi_-^\mu \partial_z \psi_-^\nu \right] . \tag{2.4.5}$$

Let us consider a worldsheet that is an infinite cylinder mapped to the punctured complex plane $\mathbb{C}^\times$. The bosonic field X^μ has the same equations of motion and the same algebra formed by the Laurent modes as in section 2.1. As for spinor fields, they are defined on the double cover of $\mathbb{C}^\times$ and have to satisfy the equations of motion

$$\partial_{\bar{z}} \psi_+^\mu = 0 \quad \text{and} \quad \partial_z \psi_-^\mu = 0 \, . \tag{2.4.6}$$

Now from the above discussion on spin structures we infer that on the punctured complex plane there are two inequivalent spin structures referred to as[46]

$$\psi_\pm^\mu(e^{2\pi i} z) = \begin{cases} +\psi_\pm^\mu(z) & \text{Neveu-Schwarz sector} \\ -\psi_\pm^\mu(z) & \text{Ramond sector} \end{cases} . \tag{2.4.7}$$

Hence, the equations of motion (2.4.6) are solved by meromorphic functions on the double cover, i.e. they admit expansions

$$\psi_+^\mu(z) = \begin{cases} \sum\limits_{s \in \frac{1}{2} + \mathbb{Z}} \psi_{+;s}^\mu z^{-s-1/2} & \text{Neveu-Schwarz sector} \\ \sum\limits_{s \in \mathbb{Z}} \psi_{+;s}^\mu z^{-s-1/2} & \text{Ramond sector} \end{cases} , \tag{2.4.8}$$

and correspondingly for ψ_-^μ expanded in $\bar{z}$. From these expansions and the OPE we may deduce the anti-commutation relations of the "Laurent" modes

$$\{\psi_{+;r}^\mu, \psi_{+;s}^\nu\} = \eta^{\mu\nu} \delta_{r+s,0} \, , \quad \{\psi_{-;r}^\mu, \psi_{-;s}^\nu\} = \eta^{\mu\nu} \delta_{r+s,0} \quad \text{and} \quad \{\psi_{+;r}^\mu, \psi_{-;s}^\nu\} = 0 \, , \tag{2.4.9}$$

where we note that in the Ramond sector we find for $r = s = 0$ (two copies of) the D-dimensional Clifford algebra. From the algebra (2.4.9) we deduce that the state space is the tensor product of two copies of a fermionic Fock space. However, similarly to the free bosonic string there is some dependence on the redundancy degrees of freedom in such a state space leading to negative norm states and we will want to introduce a BRST charge for the superstring to get rid of those.

Analogously to the bosonic string, choosing superconformal gauge leads to constraints among the fields on the classical level. For the bosonic string these constraints were related to the vanishing of the Laurent modes of the holomorphic and anti-holomorphic parts of the energy-momentum tensor T^{ab}, i.e. the Virasoro constraints. Now for the superstring these are augmented by an additional Graßmann valued constraint G^a that we may obtain explicitly via

[46] Note that holomorphic/meromorphic spinors ψ_+^μ can be thought of as sections of a root $K^{1/2}$ of the canonical line bundle. Such a root has to satisfy $K^{1/2} \otimes K^{1/2} \cong K$ and is in general not unique. In particular, for a closed compact Riemann surface of genus g there are exactly 2^{2g} inequivalent roots of K, corresponding to the 2^{2g} distinct spin structures; cf. [132]. Furthermore, as we may describe a section of $K^{1/2}$ by $\psi_+(\mathrm{d}z)^{1/2}$ there will be an additional sign in the periodicity properties (2.4.7) for both sectors that comes from an additional factor $e^{-iw/2}$ due to $(\mathrm{d}z)^{1/2}$.

a variation w.r.t. χ_a. Furthermore, we again require traceless $T^a_a = 0$ for scale invariance of the quantized theory, which is furthermore augmented by the super analogue $\gamma^a G_a = 0$. Both of these requirements will not be true in general for the quantized theory, but the associated anomalies happen to cancel in the critical dimension of the superstring, which will turn out to be $D = 10$. The holomorphic parts of T^{ab} and G^a are given by

$$\begin{aligned} T(z) &= \eta_{\mu\nu} : \Big[-\frac{1}{\alpha'}(\partial_z X^\mu)(\partial_z X^\nu) + \frac{1}{2}(\partial_z \psi^\mu_+)\psi^\nu_+ \Big] : \\ G(z) &= \eta_{\mu\nu} : \psi^\mu_+ \partial_z X^\nu : \end{aligned} \tag{2.4.10}$$

with the anti-holomorphic counterpart featuring the negative chirality spinor ψ^μ_-. Note that $G(z)$ is defined on the double cover of $\mathbb{C}^\times$ and accordingly the "Laurent" mode expansions are given by[47]

$$\begin{aligned} T(z) &= \sum_{m\in\mathbb{Z}} \frac{L_m}{z^{m+2}} \ , \\ G(z) &= \begin{cases} \sum_{s\in\frac{1}{2}+\mathbb{Z}} \frac{G_s}{z^{s+3/2}} & \text{Neveu-Schwarz sector} \\ \sum_{s\in\mathbb{Z}} \frac{G_s}{z^{s+3/2}} & \text{Ramond sector} \end{cases} . \end{aligned} \tag{2.4.11}$$

Moreover, the corresponding OPEs are of the form

$$\begin{aligned} T(z_1)T(z_2) &= \frac{c/2}{(z_1-z_2)^4} + \frac{2T(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}T(z_2)}{(z_1-z_2)^1} + \text{ non-singular terms} \ , \\ G(z_1)G(z_2) &= \frac{2c/3}{(z_1-z_2)^2} + \frac{2T(z_2)}{(z_1-z_2)^1} + \text{ non-singular terms} \ , \\ T(z_1)G(z_2) &= \frac{3/2G(z_2)}{(z_1-z_2)^2} + \frac{\partial_{z_2}G(z_2)}{(z_1-z_2)^1} + \text{ non-singular terms} \ , \end{aligned} \tag{2.4.12}$$

with central charge given by $c = 3D/2$. Finally, from the OPE (2.4.12) one may deduce that the "Laurent" modes of the expansions (2.4.11) give rise to the supersymmetric generalization of the Virasoro algebra

$$\begin{aligned} [L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0} \ , \\ \{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}(4r^2-1)\delta_{r+s,0} \ , \\ [L_m, G_r] &= \Big(\frac{1}{2}m - r\Big)G_{r+m} \ . \end{aligned} \tag{2.4.13}$$

In the path integral formulation of the superstring we need to integrate over the gravitino field. A decomposition of degrees of freedom of the gravitino field will include the degrees of freedom corresponding to the supersymmetry variation (2.4.4). Accordingly, choosing superconformal gauge allows us to eliminate the redundancy degrees of freedom of the gravitino but we instead need to deal with the functional determinant of the corresponding differential operator $P_{1/2}$.[48] Now we note that an algebraic change of variables for a measure of Graßmann variables

[47] Similarly to the bosonic case, the "Laurent" modes L_n, G_r are related to infinitesimal superconformal transformations of some supermanifold, cf. e.g. [80] for the explicit relation in the Neveu-Schwarz sector.

[48] Similarly, to the pure bosonic case, depending on the worldsheet there will be non-redundancy degrees of freedom that will result into an integration over Graßmann valued moduli; cf. [77, 80].

is accompanied by an inverse of the "Jacobian"; cf. for example [129, 133] for comprehensive discussions on the topic. Hence the corresponding ghosts, say β and γ,[49] have to be Graßmann even in order to get the correct sign of the exponent of the corresponding functional determinant. Moreover, β and γ have to be defined on the double cover of $\mathbb{C}^\times$ and w.r.t. the same spin structure on which the differential operator $P_{1/2}$ acts, i.e. the same spin structure as ψ^μ_+. Summarizing the above discussion we note that the ghost system in question may be described by the action

$$S_{sgh} = \frac{1}{2\pi} \int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}(b\partial_{\bar{z}}c + \beta\partial_{\bar{z}}\gamma + c.c.) \ , \tag{2.4.14}$$

where we employed superconformal gauge. The corresponding (holomorphic part of the) stress energy tensor and its supersymmetric analogue are given by

$$\begin{aligned} T_{sgh} &= - : 2b\partial_z c + (\partial_z b)c + \frac{3}{2}\beta\partial_z\gamma + \frac{1}{2}(\partial_z\beta)\gamma : \ , \\ G_{sgh} &= - : \frac{3}{2}\beta\partial_z c + (\partial_z\beta)c - \frac{1}{2}b\gamma : \ . \end{aligned} \tag{2.4.15}$$

We note that the central charge of the $\beta\gamma$ ghost system turns out to be $c_{\beta\gamma} = 11$. Accordingly, noting $c_{bc} = -26$ and the central charge of the system described by $\psi^\mu_\pm$ and X^μ (2.4.5), we find

$$c = c_{bc} + c_{\beta\gamma} + \frac{3}{2}D = \frac{3}{2}(D - 10) \ , \tag{2.4.16}$$

and hence we need $D = 10$ in order to get rid of the Weyl anomaly, which we from now on implicitly assume.

Now that we have introduced ghosts we will want to introduce a nilpotent BRST charge Q_B in order to have some notion of physical state.[50] The exact form of Q_B is actually not that important for our discussion but we give it here for completeness anyway

$$Q_B = \oint \frac{\mathrm{d}z}{2\pi i}\left[c(T + 1/2T_{sgh}) - \gamma(G + 1/2G_{sgh})\right] \ . \tag{2.4.17}$$

What is important however is the property that

$$\{Q_B, b\} = T + T_{sgh} \quad \text{and} \quad [Q_B, \beta] = G + G_{sgh} \ . \tag{2.4.18}$$

Together with nilpotence $Q_B^2 = 0$ this implies that Q_B commutes with $L_m + L_{sgh;m}$ and $G_r + G_{sgh;r}$ and thus is compatible with the superconformal field theory representation theory. Similarly, to the bosonic string studied in the previous sections we are ultimately interested in the BRST cohomology associated to the Fock space. However, for the superstring the BRST

[49] We note a slight yet unfortunate ambiguity in our notation, namely the ghost γ has no relation whatsoever with the Clifford algebra generators we denoted above as γ^a. However, no confusion should arise as the Clifford algebra generators (on the worldsheet) are effectively eliminated by the chirality decomposition.

[50] We also note that Q_B may be interpreted as generating superconformal transformations of the fields X^μ and $\psi^\mu_\pm$ in an analogous manner to the bosonic case (2.1.38). The corresponding relations take the form

$$\begin{aligned} [Q_B, X^\mu] &= a_0 c\partial X^\mu + a_1\gamma\psi^\mu \ , \\ \{Q_B, \psi^\mu\} &= b_0(\partial c)\psi^\mu + b_1 c\partial\psi^\mu + b_2\gamma\partial X^\mu \ , \end{aligned}$$

where the exact value of the coefficients a_i and b_i depends on several conventions but are essentially immaterial for this remark and hence omitted.

cohomology is accompanied by several technicalities, which we aspire to circumvent by basically using the logic of canonical quantization throughout the remaining part of our discussion of the state space.[51]

Now for the sake of simplicity, the following consideration of the superstring state space is restricted to "physical" excitations (as opposed to superghost excitations) and thus all states are meant with some implicit constraints imposed by BRST. Similar to the bosonic case it will be enough to just discuss the holomorphic sector as the anti-holomorphic sector will look exactly the same, albeit occasional occurrences of a tilde. Before we give more details on the mass spectrum we briefly make some comments on the structure of the state space. For that purpose we denote the state space generated by the bosonic Laurent modes α^μ_m as $\mathcal{F}_X$ (with a choice of Lorentz frame implicit) and it's counterpart for fermionic Laurent modes $\psi^\mu_{+;r}$ as $\mathcal{F}_\psi$. Then due to the fact that the Laurent modes of X^μ and $\psi^\mu_\pm$ commute, the state space decomposes as

$$\mathcal{F}_X \otimes \mathcal{F}_\psi \ , \tag{2.4.19}$$

and correspondingly for the anti-holomorphic modes.[52] Furthermore, we note that both factors are graded by Level number

$$\mathcal{F}_X = \bigoplus_{n\geq 0} \mathcal{F}_{X;n} \ , \quad \mathcal{F}_\psi = \bigoplus_{r\geq 0} \mathcal{F}_{\psi;\varsigma r} \ , \tag{2.4.20}$$

where $\varsigma = 1/2$ for the Neveu-Schwarz sector and $\varsigma = 1$ for the Ramond sector. From the grading of the factors (2.4.20) we deduce that the holomorphic sector of the state space (2.4.19) inherits a grading by level number taking values in $\frac{1}{2}\mathbb{N}$ for the Neveu-Schwarz sector and $\mathbb{N}$ for the Ramond sector, respectively. More concretely, for the Neveu-Schwarz sector the expression for the (chiral) Hamiltonian in terms of the Laurent modes is given by

$$L_0 = \frac{1}{2}\eta_{\mu\nu}\alpha^\mu_0\alpha^\nu_0 + \sum_{m\geq 1}\eta_{\mu\nu}\alpha^\mu_{-m}\alpha^\nu_m + \sum_{r\in\frac{1}{2}+\mathbb{N}}\eta_{\mu\nu} r\psi^\mu_{+;-r}\psi^\nu_{+;r} = \frac{1}{2}2^c\alpha' p^2 + N_{NS} \ , \tag{2.4.21}$$

where $c = +1$ for open and $c = -1$ for closed strings. Here N_{NS} denotes the (additive) level number, which induces a $\frac{1}{2}\mathbb{N}$ grading on $(\mathcal{F}_X \otimes \mathcal{F}_\psi)_{NS}$ with the subscript NS denoting the Fock space build on the Neveu-Schwarz ground state. Noting that if one changes from the α^μ_m to actual oscillators one gets an additional factor of m in the first sum of the above equation and thus e.g. α^μ_{-1} and $\psi^\mu_{+;-1/2}\psi^\nu_{+;-1/2}$ lead to states of the same grading, although they belong to different representations of the ten-dimensional Poincaré algebra. Analogously, the Ramond sector has the (chiral) Hamiltonian

$$L_0 = \frac{1}{2}\eta_{\mu\nu}\alpha^\mu_0\alpha^\nu_0 + \sum_{m\geq 1}\eta_{\mu\nu}\alpha^\mu_{-m}\alpha^\nu_m + \sum_{r\geq 1}\eta_{\mu\nu} r\psi^\mu_{+;-r}\psi^\nu_{+;r} = \frac{1}{2}2^c\alpha' p^2 + N_R \ , \tag{2.4.22}$$

[51] The study of BRST cohomology for the superstring turns out to be more subtle than the bosonic version studied in section 2.1. Essentially, the issue is due to the commutative nature of the $\beta\gamma$ ghosts and the ensuing consequences for the representation theory of the corresponding Laurent modes [134] leading to several technicalities when studying the BRST cohomology, cf. e.g. [135, 136]. Intriguingly there is a relation between the superstring BRST complex and the de Rham complex of the super Riemann surface associated to the worldsheet [137–139]; cf. also [80] for an overview in the context of perturbation theory.

[52] This decomposition should be augmented by two factors corresponding to the bc and $\beta\gamma$ ghost systems respectively, which will subsequently be constrained by BRST symmetry; cf. [77,81,90] for details on the superghost state space.

where N_R denotes the Ramond sector level number, inducing an integer grading on $(\mathcal{F}_X \otimes \mathcal{F}_\psi)_R$ with the subscript R denoting the Fock space build on the Ramond ground state. Accordingly, states in the Ramond sector that are build by acting with e.g. either α^μ_{-1} or $\psi^\mu_{+;-1}$ have the same grading.

In the Neveu-Schwarz sector physical states are prescribed by the conditions

$$(L_n - 1/2\delta_{n,0})|\phi; NS\rangle = 0 \ , \quad G_r|\phi; NS\rangle = 0 \ , \qquad n \in \mathbb{N} \ , \ r \in \frac{1}{2} + \mathbb{N} \ , \tag{2.4.23}$$

where the specific eigenvalue of the (chiral) Hamiltonian L_0 encodes the normal-ordering ambiguity and is required for the absence of negative norm states (for $D = 10$); we will not elaborate any further on this topic but refer to the textbooks [81,90] for a treatment using the superghost system. The ground state of the Fock space representing the Neveu-Schwarz sector $|0; p^\mu; NS\rangle$ is annihilated by α^μ_n for $n \geq 1$ and $\psi^\mu_{+;r}$ for $r > 0$. Moreover the ground state is a scalar w.r.t. the ten-dimensional Poincaré algebra with mass $(\alpha_0)^2 = -2\alpha' m^2 = 1$,[53] i.e. again a tachyon. Analogously to the discussion regarding the state space of the bosonic string in section 2.1, further states are then constructed by acting with creation operators α^μ_{-n} and $\psi^\mu_{+;-r}$ with $n, r > 0$ on the ground state with additional constraints imposed by the physical state condition (2.4.23). The mass spectrum for the Neveu-Schwarz sector is then given by

$$m^2 = \begin{cases} \frac{1}{2}(n-1)\alpha'^{-1} & \text{for open strings} \\ 2(n-1)\alpha'^{-1} & \text{for closed strings} \end{cases} \ , \quad n \in \mathbb{N} \ . \tag{2.4.24}$$

Accordingly, the next state in the mass hierarchy $e_\mu \psi^\mu_{+;-1/2}|0; p^\mu; NS\rangle$ is massless $p^2 = 0$ and transforms under the fundamental representation of the stabilizer subgroup $SO(8)$ (e_μ is constrained by $e_\mu p^\mu = 0$).

The Ramond sector analogue of the above is given by states defined by

$$L_n|\phi; R\rangle = 0 \ , \quad G_r|\phi; R\rangle = 0 \ , \qquad n, r \in \mathbb{N} \ , \tag{2.4.25}$$

where the absence of any normal-ordering constant is due to $L_0 \sim G_0^2$ and the fact that G_0 does not have any normal-ordering ambiguity. Again states may be constructed by acting with the creation operators α^μ_{-m} and $\psi^\mu_{+,-r}$ with $m, r \geq 1$. Consequently, the ground state of the Ramond sector $|0; p_\mu; R\rangle$ is massless and we find the mass spectrum of the Ramond sector to be given by

$$m^2 = \begin{cases} n\alpha'^{-1} & \text{for open strings} \\ 4n\alpha'^{-1} & \text{for closed strings} \end{cases} \ , \quad n \in \mathbb{N} \ . \tag{2.4.26}$$

As stated above we know that the Laurent modes $\psi^\mu_{\pm;0}$ form an incarnation of the Clifford algebra associated to the Minkowski metric $\eta^{\mu\nu}$ of ten-dimensional spacetime. Let us denote the corresponding Clifford algebra generators by Γ^μ that may be realized as 32×32 matrices. Now the Ramond sector ground state has to satisfy the Dirac equation

$$G_0|0; p_\mu; R\rangle \sim \eta_{\mu\nu}\alpha^\mu_0 \psi^\nu_{+;0}|0; p_\mu; R\rangle \sim p_\mu \Gamma^\mu |0; p_\mu; R\rangle = 0 \ , \tag{2.4.27}$$

[53] The mass is given for the open string, where $\alpha^\mu_0 = \sqrt{2\alpha'}p^\mu$. For the closed string we have the mass $(\alpha_0)^2 = -2^{-1}\alpha' m^2 = 1$ due to the difference in $\alpha^\mu_0 = \sqrt{\alpha'/2}p^\mu$.

and hence $|0; p_\mu; R\rangle$ has the interpretation of an 32 component Dirac spinor. Furthermore, in the case at hand we may decompose the Dirac spinor representation into the two distinct eigenspaces of $\Gamma^{11} = \Gamma^0 \dots \Gamma^9$. Thus there are two distinct chirality ground states for the Ramond sector, which we denote by $|0, +; p^\mu; R\rangle$ and $|0, -; p^\mu; R\rangle$ respectively.

Still the superstring spectrum contains a tachyon in the Neveu-Schwarz sector, which we want to get rid of for the sake of causality. The key observation is that we may distinguish states by the number of worldsheet fermion excitations. Specifically, for the Neveu-Schwarz sector we denote

$$F_{NS} = \sum_{s \in \frac{1}{2}+\mathbb{N}} \eta_{\mu\nu} \psi^\mu_{+;-r} \psi^\nu_{+;r} \ , \tag{2.4.28}$$

which counts the number of fermion excitations on the worldsheet and additionally satisfies $\{(-1)^{F_{NS}}, \psi^\mu_\pm\} = 0$. This implies furthermore that $[(-1)^{F_{NS}}, T] = 0$ as well as $\{(-1)^{F_{NS}}, G\} = 0$ reassuring us that the superconformal field theory state space will decompose into subspaces of definite $(-1)^{F_{NS}}$ eigenvalue. Hence, we may make use of the so-called GSO projection

$$\pi^{GSO}_{NS} = \frac{1}{2}\Big[1 - (-1)^{F_{NS}}\Big] \ , \tag{2.4.29}$$

to get rid of the tachyon and consequently all other states with an even number of worldsheet fermion excitations. Note that as this projection eliminates states of half-integer level number, the GSO projected Neveu-Schwarz sector has a mass spectrum that is identical to the mass spectrum of the Ramond sector.[54]

Now although the Ramond sector is not plagued by a tachyon we still introduce a GSO projection similar to the above, as this will lead to the state space of the NSR superstring to exhibit ten-dimensional supersymmetry. As supersymmetry has to relate states of the same mass the massless $SO(8)$ vector of the Neveu-Schwarz sector needs the same amount of degrees of freedom as the ground state of the Ramond sector, yet above we established that the Ramond ground state is a 32 component Dirac spinor. A matching of degrees of freedom can then be achieved by decomposing the 32 component Dirac spinor into the two eight complex component Weyl spinors and furthermore imposing a Majorana condition. Then defining

$$F_R = \sum_{s \geq 1} \eta_{\mu\nu} \psi^\mu_{+;-r} \psi^\nu_{+;r} \ , \tag{2.4.30}$$

which counts the number of worldsheet fermion excitations together with a choice of chirality for the ground state one may define the GSO projection via

$$\pi^{GSO}_{R;\pm} = \frac{1}{2}\Big[1 - (-1)^{F_R}(\pm\Gamma^{11})\Big] \ . \tag{2.4.31}$$

Thus after the GSO projection the massless $SO(8)$ vector of the Neveu-Schwarz sector has as fermionic counterpart an eight component Majorana-Weyl spinor as required by ten-dimensional supersymmetry.

Now for the closed string we have two independent but structurally identical copies of the above state spaces. Thus the state space is the tensor product of the two Fock spaces corresponding to holomorphic and anti-holomorphic parts, with the additional constraint given by

[54] This amounts to the replacement $n \to 2n+1$ in the Neveu-Schwarz mass spectrum (2.4.24).

the level-matching condition, i.e. states should be in the kernel of $L_0 - \tilde{L}_0$. However, we note that there is no need for the two independent copies to be defined w.r.t. the same spin structure.[55] Hence, there are the four subsectors build on ground states à la NS-NS, NS-R, R-NS and R-R. Furthermore, as we explained above (after the GSO projection) we have a choice of chirality for the ground state in the Ramond sector. Hence for GSO projected closed superstring theory there are two possibilities for pairings of holomorphic with anti-holomorphic Ramond ground states (parity in the ten-dimensional spacetime changes the chirality of the Ramond ground state therefore reducing the number of distinct choices from four to two). These two choices of GSO projected superstring theory are called type II A for two Ramond ground states of different chirality and type II B for two Ramond ground states of the same chirality. Still our focus is open string theory and we will not discuss closed superstrings any further, but refer to [77, 80, 81, 87, 88, 90] for details.

Finally, if want to allow for unoriented superstrings we again need to consider the action of worldsheet parity P. The action of worldsheet parity on the embedding coordinates X^μ is identical to the bosonic string discussion in section 2.1, additionally the chirality of worldsheet fermions is exchanged under P and we find

$$\begin{aligned} P X^\mu(z,\bar{z}) P^{-1} &= \begin{cases} X^\mu(-\bar{z},-z) & \text{for open strings} \\ X^\mu(\bar{z},z) & \text{for closed strings} \end{cases} , \\ P \psi^\mu_\pm(z,\bar{z}) P^{-1} &= \begin{cases} \psi^\mu_\mp(-\bar{z},-z) & \text{for open strings} \\ \psi^\mu_\mp(\bar{z},z) & \text{for closed strings} \end{cases} . \end{aligned} \tag{2.4.32}$$

Consequently, the action of worldsheet parity is determined by the Laurent expansions of the individual fields and a choice of P eigenvalue for Ramond and Neveu-Schwarz ground state. Furthermore, we note that P exchanges the holomorphic with the anti-holomorphic sector, as P exchanges both the chiralities of the worldsheet fermions as well as α^μ_m with $\tilde{\alpha}^\mu_m$. Thus of the two type II theories mentioned above, only the type II B string admits a worldsheet parity symmetry. Accordingly, we may arrive at a state space for unoriented strings via the projection

$$\pi_{-P} = \frac{1}{2}[1 + P] , \tag{2.4.33}$$

leading to the closed string sector of type I superstring theory. The type I theory is further augmented by the inclusion of open superstrings with possible choices for Chan-Paton factors as explained in section 2.1. However, it turns out that we want the Chan-Paton degrees of freedom to live in the fundamental representation of $SO(32)$ as this leads to cancellations of divergencies [45].

Now the remaining task is to adapt the discussion of the S-matrix in section 2.2 to our supersymmetric setup. This turns out to be quite involved at least within the framework of the component formalism used throughout this exposition and we will be content with making several remarks on the topic closely following [77, 80]; further technical details may be found in [100, 140–142].

For a concise formulation of superstring perturbation theory it is convenient to employ the

[55] Note that this is at odds with the above claim that for Euclidean worldsheets ψ_+ and ψ_- are mutually complex conjugates. However, one usually argues that this condition may be relaxed via so-called chiral splitting, cf. [77].

language of supergeometry. The idea is to consider the worldsheet superconformal field theory as defined on a $1|1$-dimensional complex supermanifold that is locally described by holomorphic coordinates denoted $(z|\zeta)$. To be precise, one actually wants the worldsheet superconformal field theory to be defined on a super Riemann surface. A super Riemann surface $\mathcal{S}$ can be roughly thought of as a complex $1|1$-dimensional supermanifold together with a subbundle $\langle D\rangle \subset T\mathcal{S}$ generated by the vector field $D = \partial_\zeta + \zeta\partial_z$ (D generates supersymmetry), which satisfies that D^2 is nowhere $\sim D$.[56] This subbundle is called superconformal structure and superconformal transformations have to leave $\langle D\rangle$ invariant.

Now for interacting bosonic strings one of the key concepts was to integrate over the space of complex structures leading us (assuming invariance under the mapping class group) to the moduli space of punctured Riemann surfaces. Now we want to integrate over the space of complex structures on the super Riemann surface corresponding to the relevant worldsheet (and a choice of spin structure). It turns out that the notion of moduli is again related to the kernel of the adjoint of the differential operators involved in the definition of superghosts leading to the occurrence of Graßmann odd moduli related to the kernel of $P^\dagger_{1/2}$, cf. [77, 80, 81]. The corresponding super moduli space has dimension

$$\dim(\mathrm{sMod}_g) = \begin{cases} 0|0 \text{ for } g = 0 \\ 1|0 \text{ for even spin structure } g = 1 \\ 1|1 \text{ for odd spin structure } g = 1 \\ 3g-3|2g-2 \text{ for } g \geq 2 \end{cases} , \tag{2.4.34}$$

where for a comprehensive motivation of this result we refer to [100]. Note however that super moduli space is in general rather complicated, in the sense that in general it cannot be (holomorphically) projected to the reduced moduli space, which we will not expand upon here but instead refer to [100, 140, 141] and references therein.

A further key ingredient was the idea of conformally mapping asymptotic string states to punctures on the worldsheet accompanied by vertex operators localized at said punctures. The generalized notion of puncture in the supersymmetric setup is rather subtle and we are content with making a few remarks closely following [80, 100, 137–139], which we also recommend for further details. For states from the Neveu-Schwarz sector the worldsheet fields are still locally meromorphic and the puncture may thus be described by their coordinates $(z_0|\zeta_0)$. So intuitively adding a Neveu-Schwarz puncture increases the dimension of super moduli space by $1|1$. Note that due to local superconformal symmetry points related by the combined transformation $z \mapsto z + \alpha\zeta$, $\zeta \mapsto \zeta + \alpha$ are equivalent and one may interpret a Neveu-Schwarz puncture as an one parameter subspace (also called a divisor). However, the situation for Ramond sector states is more involved due to the fact that a fermionic field in the Ramond sector has branch points; cf. also (2.4.8). Similarly to the above discussion the idea is again to describe the puncture as a divisor.[57] For Ramond punctures this divisor describes the singular regions of

[56] This suggests that D^2 and D are a basis of $T\mathcal{S}$ and that the superbracket gives rise to an isomorphism $T\mathcal{S}/\langle D\rangle \cong \langle D\rangle \otimes \langle D\rangle$. In local coordinates this isomorphism takes the form

$$\{\partial_\zeta + \zeta\partial_z, \partial_\zeta + \zeta\partial_z\} = 2\partial_z ,$$

which is a realization of the supersymmetry algebra, cf. [100, 142, 143] and references therein for more details.

[57] Another way to treat Ramond sector states is via the concept of bosonization and spin fields, cf. [77,81,90,92].

the superconformal structure in the sense that on the divisor $D^2 = 0$, while everywhere else one still has $D^2 \not\sim D$. Locally, (i.e. close to a Ramond divisor) we may choose coordinates such that the divisor is described by $z = 0$ or equivalently the singular conformal structure is generated by $D_s = \partial_\zeta + z\zeta\partial_z$ with square $D_s^2 = z\partial_z$ singular at $z = 0$. For the sake of completeness we note that a Ramond puncture increases the dimension of the supermoduli space by $1|\frac{1}{2}$, which we will not discuss any further; cf. [100] for details. Finally, in order to establish a link between asymptotic string states and punctures of the supermanifold we need the superconformal analogue of the exponential map (that leaves the subbundle $\langle D\rangle$ invariant). For a Neveu-Schwarz puncture corresponding to $(z|\zeta) = (0|0)$ this map is given by

$$z = e^\rho \ , \qquad \zeta = e^{\rho/2}\theta \ , \tag{2.4.35}$$

with domain the equivalence classes

$$(\rho|\theta) \sim (\rho + 2\pi i| - \theta) \ . \tag{2.4.36}$$

Note that the bosonic coordinates describe a cylinder. For a Ramond puncture described by the divisor $z = 0$ (i.e. a superconformal structure $\partial_\zeta + z\zeta\partial_z$) we may choose the map

$$z = e^\rho \ , \tag{2.4.37}$$

with domain the equivalence classes

$$(\rho|\zeta) \sim (\rho + 2\pi i| + \zeta) \ , \tag{2.4.38}$$

again with bosonic coordinates describing a cylinder. Now to see that the Graßmann odd parameters describe the corresponding sectors, we look at the superfield

$$X^\mu + \zeta\psi_+^\mu + \bar{\zeta}\psi_-^\mu + \zeta\bar{\zeta}F^\mu \ , \tag{2.4.39}$$

where F is a non-dynamical auxiliary field needed for off-shell supersymmetry and may essentially be ignored for our purposes. The above transformations lead then to the periodicity conditions on the cylinder

$$\psi_\pm^\mu(\rho + 2\pi i) = \begin{cases} -\psi_\pm^\mu(\rho) & \text{Neveu-Schwarz sector} \\ +\psi_\pm^\mu(\rho) & \text{Ramond sector} \end{cases} \ , \tag{2.4.40}$$

as needed.

Chapter 3

Iterated integrals and open-string amplitudes

As discussed in the previous chapter the calculation of amplitudes in open string theory leads to iterated integrals on Riemann surfaces. Accordingly, for the genus-zero case the associated class of iterated integrals is given by multiple polylogarithms, i.e. iterated integrals composed of the one forms $\frac{\mathrm{d}z}{z}$ and $\frac{\mathrm{d}z}{z-1}$ that are associated to the thrice-punctured Riemann sphere $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The corresponding physical setup in open string theory is the α'-expansion of the disk amplitude. This expansion is expressible via MZVs, which describe the monodromy of multiple polylogarithms.

In contradistinction to the case of genus zero, at genus one the corresponding iterated integrals are composed of one forms from an infinite family $\mathrm{d}z f^{(n)}(z)\,,\, n \geq 0$ that are defined on an once punctured elliptic curve $E_\tau^\times$. This family is defined by the expansion coefficients of the doubly-periodic completion of the Eisenstein-Kronecker series. The associated family of elliptic iterated integrals evaluated along a specific path gives rise to elliptic multiple zeta values. These EMZVs turn out to be sufficient to describe the α'-expansion of the single-trace contributions to the genus-one open-string amplitude. In order to also treat the double-trace terms one needs to consider an additional family such that the resulting class of integrals is associated to the twice punctured elliptic curve $E_\tau^\times \setminus \{\tau/2\}$. This generalized setup gives rise to a subclass of twisted elliptic multiple zeta values. While the special case of the twice-punctured elliptic curve is adequate to describe the whole genus-one open-string amplitude, one might contemplate a further generalization to integrals defined on an elliptic curve with a lattice removed $E_\tau \setminus \Lambda$. Quite interestingly, it turns out that these are related to modular forms of congruence subgroups, which in turn appear in Feynman diagram calculations.

In this chapter we will briefly sketch how the open-string genus-zero amplitude can be expressed via MZVs in section 3.1. Subsequently, in section 3.2 we discuss how the genus-one counterpart exhibits an analogous structure. This leads directly to a certain subclass of iterated integrals on elliptic curves, called TEMZVs, which will be considered in detail in section 3.3. In particular any such iterated integral depends on the modulus τ of the corresponding elliptic curve and admits an expansion in $q = e^{2\pi i\tau}$. Therefore, section 3.4 will be devoted to studying the properties of these expansions and how to compute them. With all this machinery at hand we return in section 3.5 to the genus-one open-string amplitude and elaborate on how it can be expressed in terms of TEMZVs. Finally, this chapter will be closed with an exposition on a more general class of TEMZVs in section 3.6, which although absent in open string theory context, is related to interesting numbers in the form of MZVs at roots of unity.

3.1 The genus-zero open-string amplitude

In this section we briefly review the tree-level open-string amplitude and its relation to multiple zeta values. Our exposition will only comment on a few specific features of those genus-zero integrals, most of which are originally due to [12,46–48,144–147]. We stress that our focus in this section is on how iterated integrals on the punctured Riemann sphere and related notions are related to the structure of the tree-level amplitude. A self-contained treatment of tree-level string amplitudes is beyond the scope of this thesis and we refer the interested reader to [87,88,92,148] and the references therein as a starting point.

As discussed in the previous chapter in the case of genus zero, any metric on the open-string worldsheet $\mathbb{H}$ is globally conformally equivalent to the flat metric and we are left to integrate correlators of vertex operators over the space of the corresponding insertions on the boundary up to the action of the conformal Killing group, i.e. $PSL(2,\mathbb{R})$. Hence, it is possible to fix the coordinates of three of the vertex operator insertions, where we make the canonical choice $z_1 = 0, z_{n-1} = 1, z_n = \infty$. Furthermore, we may disentangle the polarization degrees of freedom present in vertex operator correlators from the actual integration. In particular in [149,150] the genus-zero amplitude in open superstring theory was shown to factorize as

$$A_n^{\text{string}}(1,2,\dots n) = \sum_{\sigma\in S_{n-3}} A_n^{\text{YM}}(1,\sigma(2),\dots\sigma(n-2),n-1,n)W_{n;\sigma} \tag{3.1.1}$$

where A_n^{YM} denotes the n-point tree-level $\mathcal{N}=4$ super Yang-Mills amplitude with gauge group determined by the Chan-Paton degrees of freedom and $W_{n;\sigma}$ is a worldsheet integral (to be discussed below) that may be interpreted as encoding string corrections to point particle Yang-Mills amplitudes. In order to keep the discussion as concise as possible we now restrict our attention to the four-point case. Then there are no permutations to consider and the worldsheet integral in question can be expressed via Gamma functions (for convergence we require $\text{Re}(s_{12}) > 0, \text{Re}(s_{23}) > -1$)

$$\begin{aligned} W_4 &= s_{12}\int_0^1 \mathrm{d}z_2\, {z_2}^{s_{12}-1}(1-z_2)^{s_{23}} = \frac{\Gamma(1+s_{12})\Gamma(1+s_{23})}{\Gamma(1+s_{12}+s_{23})} \\ &= 1-\zeta_2 s_{12}s_{23}+\zeta_3 s_{12}s_{23}(s_{12}+s_{23}) - \frac{1}{4}\zeta_4 s_{12}s_{23}(4s_{12}^2+s_{12}s_{23}+4s_{23}^2)+\mathcal{O}(\alpha'^5)\,, \end{aligned} \tag{3.1.2}$$

where ζ_n denotes Riemann zeta values. The higher point analogues of the above worldsheet integral evaluates to multivariate generalizations of hypergeometric functions [146].

Although the worldsheet integral (3.1.2) above is known, we take a slight detour to study it in a way that will closely mirror the genus-one case we consider in section 3.5. To that end the worldsheet integral in question may be rewritten as

$$W_4 = s_{12}\int_0^1 \frac{\mathrm{d}z_2}{z_2}\, \exp\left(\frac{1}{2}\Big(s_{12}G_0(z_2)+s_{23}G_0(1-z_2)\Big)\right)\,, \tag{3.1.3}$$

where $G_0(z) = \log|z|^2$ is the genus-zero Green function. This function may be expressed via the integral

$$G_0(z_2) = \mathrm{R}\int_0^{z_2} \frac{\mathrm{d}y}{y} \,, \tag{3.1.4}$$

with regularization prescription R for the pole at zero such that the equality holds.[58] If we now expand the integrand of (3.1.3) in α' we get[59]

$$\mathrm{R}\sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 \mathrm{d}z_2 \frac{s_{12}}{z_2} \Big(\mathrm{R}\int_0^{z_2} \mathrm{d}x \Big[\frac{s_{12}}{x} + \frac{s_{23}}{x-1} \Big] \Big)^n \,. \tag{3.1.5}$$

These integrals may be rewritten into iterated integrals over the forms $\frac{\mathrm{d}x}{x}$ and $\frac{\mathrm{d}x}{1-x}$ by considering the product of integrals as an integral over the product space, i.e. over the n-dimensional unit cube (up to regularization issues), which may be decomposed into n-simplices. Such integrals are known to lead to MZVs, cf. appendix D.2. Finally, we note that the expression (3.1.5) already hints at the relation between the worldsheet integral and the Drinfeld associator, which can be roughly thought of as the path-ordered exponential of the one-form

$$\omega_{KZ} = \Big(\frac{x_0}{z} + \frac{x_1}{z-1} \Big) \mathrm{d}z \,, \tag{3.1.6}$$

along the path from zero to one with formal non-commutative variables x_0, x_1.[60] The precise relation was derived in [47] and gives the worldsheet integral as a specific component of the matrix-valued Drinfeld associator

$$W_4 = \left[\tilde{\mathcal{P}} \exp \left(\int_0^1 \mathrm{d}z \left(\begin{pmatrix} s_{12} & -s_{12} \\ 0 & 0 \end{pmatrix} \frac{1}{z} + \begin{pmatrix} 0 & 0 \\ s_{23} & -s_{23} \end{pmatrix} \frac{1}{1-z} \right) \right) \right]_{11} \,, \tag{3.1.7}$$

where the index on the square brackets indicates the relevant matrix component.

3.2 The genus-one open-string amplitude and iterated integrals

We proceed by motivating how at genus one we find a similar structure as in the genus-zero case. In particular we sketch how iterated integrals on an elliptic curve arise in the context of

[58] The regularized integral is defined by altering the integration path to start at ε close to zero (instead of zero), rendering the integral convergent and subsequently projecting onto the ε^0 term. In order to streamline the discussion we will omit any further exposition here but instead revisit this concept in more detail, when we study iterated integrals on an elliptic curve in section 3.3.

[59] Note that this is a schematic expression, as we deliberately omit issues related to regularization.

[60] To be more precise we have to spell out the regularization properly as discussed for example in [151, 152]

$$\lim_{\varepsilon\to 0} e^{x_0 \log(\varepsilon)} \tilde{\mathcal{P}} \exp \Big(\int_{\varepsilon}^{1-\varepsilon} \mathrm{d}z \Big[\frac{x_0}{z} + \frac{x_1}{z-1} \Big] \Big) e^{-x_1 \log(\varepsilon)} \,,$$

where $\tilde{\mathcal{P}}$ denotes path-ordering albeit with reversed order of non-commutative products in x_0 and x_1. For our purposes this expression of the Drinfeld associator as path-ordered exponential is sufficient. However, note that the Drinfeld associator appears in various contexts in mathematics and physics, which we briefly comment on in appendix C.

the genus-one worldsheet integrals of open string theory scattering amplitudes. Note that this section will mostly serve the purpose of motivating the study of TEMZVs in the subsequent sections, while, for the sake of dramaturgy, we postpone a detailed treatment of the open-string amplitude to the latter half of this chapter.

Recall that in section 2.3 we discussed how at genus one we may describe all the open-string worldsheets by their double - a two-torus - with boundary component(s) given by the fixed point set of a specific involution. Moreover, we argued that the family of conformally inequivalent metrics on the two torus is parametrized by the modular parameter τ living in the upper half-plane $\mathbb{H}$. Consequently, for a given modulus τ we are left with an integration over the vertex operator coordinates on the boundary component(s) up to cyclic shifts of the insertions. Finally, one needs to integrate over the modulus τ, where the exact range of integration depends on the topology of the initial open-string worldsheet.

From now on we will restrict the discussion to the cylinder topology as the Möbius strip may conveniently be related to the cylinder case [45], also cf. section 2.3. Then the cylinder may be described by a torus or *elliptic curve*[61] represented as $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$ such that $\tau = it$ with $t \in \mathbb{R}_+$ and the boundaries are described by coordinates with either $\mathrm{Im}(z) = 0$ or $\mathrm{Im}(z) = t/2$. Therefore, as all insertions have constant imaginary parts integration will be over the real parts only.[62] As in the genus-zero case the polarization degrees of freedom decouple from the integration [44]. As an example let us consider the double-trace contribution of cylinder topology with the two pairs of insertions z_1, z_2 and z_3, z_4 lying on different boundary components. Then choosing $z_1 = 0, z_2 \in (0,1)$ and $\mathrm{Im}(z_3) = \mathrm{Im}(z_4) = t/2$ the integral in question is proportional to

$$\int_0^{i\infty} \mathrm{d}\tau \int_0^1 \mathrm{d}z_2 \int_0^1 \mathrm{d}z_3 \int_0^1 \mathrm{d}z_4 \exp\left(\frac{1}{2}\sum_{i<j} s_{ij} G_1(z_i - z_j;\tau)\right) , \tag{3.2.1}$$

where $G_1(z;\tau)$ denotes the genus-one Green function. In order to evaluate the worldsheet integral of (3.2.1) for a given complex structure on the two-torus, we may expand the integrand in α' and exchange expansion and integration, leading to integrals of the form

$$\int_0^1 \mathrm{d}z_2 \int_0^1 \mathrm{d}z_3 \int_0^1 \mathrm{d}z_4 \prod_{i<j} G_1(z_{ij};\tau)^{n_{ij}} . \tag{3.2.2}$$

Analogously to the genus zero discussion of the previous section, we establish the link to iterated integrals by representing the Green function as a definite integral of a well-defined one-form on the worldsheet. For the genus-one Green function $G_1(z;\tau)$ we use the convention [44]

$$G_1(z_{ij};\tau) = \log\left|\frac{\theta_1(z_{ij};\tau)}{\theta_1'(0;\tau)}\right|^2 - \frac{2\pi}{\mathrm{Im}(\tau)}\mathrm{Im}(z_{ij})^2 , \tag{3.2.3}$$

where $\theta_1(z;\tau)$ denotes a Jacobi θ function.[63] The main idea is to rewrite the Green function as

[61] It turns out that over $\mathbb{C}$ and for any τ there exists a bijection between the corresponding two-torus and some elliptic curve (cf. [153, 154]), hence we will use these terms interchangeably.

[62] If we had chosen the anti-holomorphic inversion descending from $\tilde{\rho} = -\bar{z}$ (cf. section 2.3) we would have ended up with insertions with constant real parts. This would lead to so-called B-cycle EMZVs as opposed to the so-called A-cycle EMZVs we exclusively consider throughout this work, where A and B label (a specific choice) for the homology cycles of the torus, cf. [41].

[63] Our conventions for the Jacobi θ functions relevant to our discussions can be found in Appendix E.1.

an integral over some real interval $(0, \mathrm{Re}(z_{ij}))$, i.e. a contour coinciding with the integration over the boundary insertions. Naively, one would consider an expression of the form $\int \mathrm{d}z \partial_z G_1(z;\tau)$, but this may differ from $G_1(z;\tau)$ by a function of q. To that end it is convenient to note that the expression for the genus-one Green function itself is unique only up to a function of q. We will use this ambiguity such that $\int \mathrm{d}z \partial_z G_1(z;\tau)$ agrees with $G_1(z;\tau)$ (up to a q-independent constant) for the case where both insertions $z_i - z_j$ are on the same boundary. Note also that adding a function of q to (3.2.3) does not alter the amplitude, as it is multiplied by $\sum_{i<j} s_{ij} = 0$.

In the aforementioned case of insertions on the same boundary the argument of the Green function z is real and may be chosen such that $z \in (0,1)$. For this choice we find $\frac{\theta_1(z;\tau)}{\theta_1'(0;\tau)} > 0$, hence the absolute value may be dropped. Then we may write the genus-one Green function as an integral over the function

$$f^{(1)}(z;\tau) = \frac{\theta_1'(z;\tau)}{\theta_1(z;\tau)} + 2\pi i \frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)} \,, \tag{3.2.4}$$

with domain of integration the real interval $(0, z)$. Specifically, we have the integral representation

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathrm{Reg} \int_\varepsilon^z dy\, f^{(1)}(y;\tau) &= \lim_{\varepsilon\to 0} \mathrm{Reg} \left[\log(\theta_1(z;\tau)) - \log(\theta_1'(0;\tau)) - \log(\varepsilon) + \mathcal{O}(\varepsilon)\right] \\ &= \frac{1}{2} G_1(z;\tau)|_{z\in\mathbb{R}} - \underbrace{\lim_{\varepsilon\to 0} \mathrm{Reg}(\log(\varepsilon))}_{=:c_P} \,, \end{aligned} \tag{3.2.5}$$

where Reg projects onto the non-singular part and thus defines the constant c_P, i.e. the exact value of c_P is determined by fixing the ambiguity involved in the definition of the map Reg. Our precise definition of Reg and the corresponding value for c_P can be found in the next section, when we discuss regularization in the context of the class of elliptic iterated integrals under consideration here (cf. eq. (3.3.28)).

For the case of two insertions z_i and z_j on different boundaries, it is also possible to write the Green function as an integral over $f^{(1)}(z)$ albeit with shifted argument. Before we give the exact relation, we note the identity

$$\theta_1(z+\tau/2;\tau) = i \exp(-i\pi z) q^{-1/8} \theta_4(z;\tau) \,, \tag{3.2.6}$$

which will allow us to disentangle the absolute value and the argument of the complex-valued function, due to $q^{-1/8}\theta_4(z;\tau) > 0$ for $z \in (0,1)$ and $\tau \in i\mathbb{R}_+$. We may then state the explicit relation

$$\begin{aligned} \int_0^z \mathrm{d}y\, f^{(1)}(y+\tau/2;\tau) &= \Big[\underbrace{\log(\theta_1(z+\tau/2;\tau)) + i\pi z}_{=\log|\theta_1(z+\tau/2;\tau)| - i\pi/2} - \log(\theta_1'(0;\tau))\Big] \\ &\qquad + \log(\theta_1'(0;\tau)) - \log(\theta_1(\tau/2;\tau)) \\ &= \log\left|\frac{\theta_1(z+\tau/2;\tau)}{\theta_1'(0;\tau)}\right| + \log\left(\frac{\theta_1'(0;\tau)}{q^{-1/8}\theta_4(0;\tau)}\right) \\ &= \frac{1}{2} G_1(z+\tau/2;\tau)|_{z\in\mathbb{R}} + \log\left(\frac{\theta_1'(0;\tau)}{\theta_4(0;\tau)}\right) \,, \end{aligned} \tag{3.2.7}$$

where we chose the principal branch of the logarithm. Note that the difference between the integral and the Green function is $\frac{1}{2}G_1(\tau/2)$. Furthermore, using the product expressions for the Jacobi θ functions as in (E.1.1) we might give an explicit q-expansion for the aforementioned difference denoted $c_Q(q)$

$$\begin{aligned} c_Q(q) = \log\left(\frac{\theta_4(0;\tau)}{\theta_1'(0;\tau)}\right) &= -\log(2\pi q^{1/8}) - 2\sum_{n=1}^{\infty}\left[\log(1-q^n) - \log(1-q^{n-1/2})\right] \\ &= -\log(2\pi) - \frac{\log(q)}{8} + 2\sum_{m,n=1}^{\infty}\left[\frac{q^{mn}}{m} - \frac{q^{m(n-1/2)}}{m}\right]\,. \end{aligned} \tag{3.2.8}$$

Note that we could have absorbed $c_Q(q)$ into the definition of the Green function (3.2.3), which would lead to an equality between the integral over $f^{(1)}(y-\tau/2)$ and $G_1(z+\tau/2)$. Such a redefinition however would complicate the integral representation of the equal boundary Green function (3.2.5), where $c_Q(q)$ would reappear.

Let us return to our motivational example (3.2.2), for the case where we have two insertions on each boundary. In this case we may choose the coordinates of the insertions such that $0 = z_1 < z_2 < 1$ and $0 < \mathrm{Re}(z_3), \mathrm{Re}(z_4) < 1$, $\mathrm{Im}(z_3) = \mathrm{Im}(z_4) = \tau/2$. Then for example at first order in α' we encounter the integral

$$\int_0^1 \mathrm{d}z_2 \int_0^1 \mathrm{d}z_3 \int_0^1 \mathrm{d}z_4 \frac{1}{2}G_1(z_{12};\tau) = \lim_{\varepsilon\to 0}\mathrm{Reg}\int_0^1 \mathrm{d}z_4 \int_0^1 \mathrm{d}z_3 \int_\varepsilon^{1-\varepsilon} \mathrm{d}z_2 \left[c_P + \int_\varepsilon^{x_2} \mathrm{d}y\, f^{(1)}(y;\tau)\right]\,, \tag{3.2.9}$$

where we rewrote the Green function as an integral from 0 to z_2 and extended the regularization to the iterated integral. This is the prototype of the integrals we will study below, namely iterated integrals w.r.t. the one-forms $f^{(1)}\mathrm{d}z$ as well as the silent protagonist $f^{(0)}\mathrm{d}z = \mathrm{d}z$.

3.3 Iterated integrals on an elliptic curve

In the previous section we motivated how expanding the genus-one worldsheet integral of cylinder topology in α' leads to expansion coefficients given by iterated integrals on an elliptic curve. We now go on to give a more precise overview of the relevant class of iterated integrals and give some of their properties. For the single-trace contribution of the opne-string amplitude the corresponding class of iterated integrals is given by elliptic multiple zeta values [50, 155], which in turn is related to multiple elliptic polylogarithms, cf. [38–42]. Here we give the more general class of iterated integrals called *twisted elliptic multiple zeta values* (TEMZVs),[64] which will allow us to also treat the double-trace contributions of the worldsheet integral with cylinder topology. Our exposition will closely follow [1, 50, 155] but also draws heavily in its logical structure from the beautiful introduction on iterated integrals by Francis Brown in [157].

We begin our exposition by defining a family of well-defined functions on an once-punctured elliptic curve. This family is denoted by $\{f^{(n)}(z)\}_{n\in\mathbb{N}}$ and will include the function $f^{(1)}(z)$ needed for the integral representation of the Green function as given in (3.2.5) and (3.2.7). The corresponding functions $f^{(n)}$ will be called *weighting functions* (of weight n) and will provide us with the notion of weight for TEMZVs below. For our purposes it will be convenient to consider

[64] Disclaimer: As of now it is not clear whether the TEMZVs as defined by us are related to the monodromy of the twisted elliptic KZB connection as currently studied by Calaque and Gonzales in [156].

the elliptic curve in question, as the coset

$$E_\tau^\times = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)\setminus\{0\} \ . \tag{3.3.1}$$

This implies, that in order for a function $h(z)$ to be well-defined on $E_\tau^\times$ it needs to be doubly-periodic[65]

$$h(z;\tau) = h(z+1;\tau) = h(z+\tau;\tau) \ , \tag{3.3.2}$$

with possible singularities only at points in $\mathbb{Z}+\mathbb{Z}\tau$. The aforementioned family of functions may be conveniently encoded in a doubly-periodic generating function containing $f^{(1)}(z;\tau)$ as one of it's coefficients. Before we actually give this generating function we take a slight detour and consider the Eisenstein-Kronecker series [39, 158, 159]

$$F(z,\alpha;\tau) = \frac{\theta_1'(0;\tau)\theta_1(z+\alpha;\tau)}{\theta_1(z;\tau)\theta_1(\alpha;\tau)} \ , \tag{3.3.3}$$

where z and α are coordinates on $\mathbb{C}$. The variable α may be considered as a formal expansion parameter for our purposes. This function is quasi-periodic in z

$$F(z+1,\alpha;\tau) = F(z,\alpha;\tau) \ , \quad F(z+\tau,\alpha;\tau) = \exp(-2\pi i\alpha)F(z,\alpha;\tau) \ , \tag{3.3.4}$$

which follows from the periodicity properties of it's constituent $\theta_1(z;\tau)$

$$\theta_1(z+1;\tau) = -\theta_1(z;\tau) \ , \quad \theta_1(z+\tau;\tau) = -q^{-1/2}\exp(-2\pi i z)\theta_1(z;\tau) \ . \tag{3.3.5}$$

Furthermore, due to the odd parity of the Jacobi θ-function $\theta_1(-z) = -\theta_1(z)$, the Kronecker-Eisenstein series satisfies

$$F(-z,-\alpha;\tau) = F(z,\alpha;\tau) \ . \tag{3.3.6}$$

Considering the expansion of $F(z,\alpha;\tau)$ in α gives rise to a family of functions

$$F(z,\alpha;\tau) = \sum_{n\in\mathbb{N}} \alpha^{n-1} g^{(n)}(z;\tau) \ , \tag{3.3.7}$$

with the properties

$$\begin{aligned} g^{(n)}(z+1;\tau) &= g^{(n)}(z;\tau) \ , \\ g^{(n)}(z+\tau;\tau) &= \sum_{k=0}^{n} g^{(k)}(z;\tau)\frac{(-2\pi i)^{n-k}}{(n-k)!} \ , \\ g^{(n)}(-z;\tau) &= (-1)^n g^{(n)}(z;\tau) \ , \end{aligned} \tag{3.3.8}$$

which are a consequence of quasi-periodicity (3.3.4) and parity (3.3.6) of $F(z,\alpha;\tau)$. Additionally, the Eisenstein-Kronecker series satisfies two important functional relations necessary in the derivation of the differential equation of subsection 3.4.3; cf. appendix E.4 for the derivation. Specifically, $F(z,\alpha;\tau)$ satisfies the Fay identity [160]

$$\begin{aligned} F(z_1,\alpha_1;\tau)F(z_2,\alpha_2;\tau) = {} & F(z_1,\alpha_1+\alpha_2;\tau)F(z_2-z_1,\alpha_2;\tau) \\ & + F(z_2,\alpha_1+\alpha_2;\tau)F(z_1-z_2,\alpha_1;\tau) \ , \end{aligned} \tag{3.3.9}$$

[65] Considered as a function on $\mathbb{C}\setminus(\mathbb{Z}+\mathbb{Z}\tau)$.

which implies quadratic relations among the $g^{(n)}(z)$, and furthermore $F(z,\alpha;\tau)$ satisfies the mixed heat equation

$$2\pi i\partial_\tau F(z,\alpha;\tau) = \partial_z\partial_\alpha F(z,\alpha;\tau) \ . \tag{3.3.10}$$

As we have seen in (3.3.4) the Eisenstein-Kronecker series deviates from double-periodicity in a mild way, suggesting the definition of the following generating function [37, 38, 159]

$$\Omega(z,\alpha;\tau) = \exp\left(2i\pi\alpha\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right) F(z,\alpha;\tau) \ , \tag{3.3.11}$$

which is doubly-periodic in z

$$\Omega(z,\alpha;\tau) = \Omega(z+1,\alpha;\tau) = \Omega(z+\tau,\alpha;\tau) \ . \tag{3.3.12}$$

Expanding Ω in α gives a family of doubly-periodic functions,

$$\Omega(z,\alpha;\tau) = \sum_{n\in\mathbb{N}} f^{(n)}(z;\tau)\alpha^{n-1} \ , \tag{3.3.13}$$

including $f^{(1)}(z)$ as α^0 coefficient. Furthermore, the generating function $\Omega(z,\alpha)$ also satisfies the Fay identity

$$\begin{aligned}\Omega(z_1,\alpha_1;\tau)\Omega(z_2,\alpha_2;\tau) = \Omega(z_1,\alpha_1+\alpha_2;\tau)\Omega(z_2-z_1,\alpha_2;\tau)\\ + \Omega(z_2,\alpha_1+\alpha_2;\tau)\Omega(z_1-z_2,\alpha_1;\tau) \ ,\end{aligned} \tag{3.3.14}$$

as a consequence of (3.3.9). Again the Fay identity leads to relations among products of $f^{(n)}(z)$ and ultimately the iterated integrals we study below; cf. appendix E.2 for the relation on the level of the weighting functions.

We now have a family of well-defined functions on $E_\tau^\times$ implicitly defined via the generating series (3.3.3) and (3.3.13). Our next step will be to find explicit expressions for those functions. From the definition of $\Omega(z,\alpha;\tau)$ (3.3.11) we deduce the relation between the two families of functions $f^{(n)}(z;\tau)$ and $g^{(n)}(z;\tau)$ to be given by

$$f^{(n)}(z;\tau) = \sum_{k=0}^{n} g^{(k)}(z;\tau)\frac{1}{(n-k)!}\left(2\pi i\,\frac{\mathrm{Im}(z)}{\mathrm{Im}(\tau)}\right)^{n-k} \ . \tag{3.3.15}$$

Hence, we may express $f^{(n)}(z;\tau)$ in terms of $g^{(n)}(z;\tau)$. In order to get explicit expressions for the $g^{(n)}(z;\tau)$ we note that the Eisenstein-Kronecker series (3.3.3) has the equivalent representation [37–39, 159]

$$\begin{aligned}\alpha F(z,\alpha;\tau) = 1 + \pi\alpha\,\cot(\pi z) - 2\sum_{k=1}^{\infty}\zeta_{2k}\alpha^{2k}\\ - 2\pi i\alpha\sum_{m,n=1}^{\infty}\left[\exp(2\pi i(mz+n\alpha)) - \exp(-2\pi i(mz+n\alpha))\right]q^{mn} \ ,\end{aligned} \tag{3.3.16}$$

from which, upon equating coefficients, follows that $g^{(0)}(z;\tau)=1$ and

$$\begin{aligned} g^{(2k-1)}(z;\tau) &= \pi\,\cot(\pi z)\delta_{k,1} - 2i\frac{(2\pi i)^{2k-1}}{(2k-2)!}\sum_{n,m=1}^{\infty}\sin(2\pi m z)\,n^{2k-2}q^{mn} \\ g^{(2k)}(z;\tau) &= -2\zeta_{2k} - 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n,m=1}^{\infty}\cos(2\pi m z)\,n^{2k-1}q^{mn}\,, \end{aligned} \tag{3.3.17}$$

for $k \geq 1$.

We note that the Laurent expansion of $g^{(1)}(z;\tau)$ around $z=0$ takes the form $1/z+\mathcal{O}(z)$, while all other $g^{(n)}(z;\tau)$ are holomorphic in z on $\mathbb{C}$. This implies that of the $f^{(n)}(z;\tau)$ only $f^{(1)}(z;\tau)$ has a singular behaviour and in fact also behaves as $1/z+\mathcal{O}(z)$ around 0 and its orbit under $\mathbb{Z}+\mathbb{Z}\tau$. Moreover, from the parity property $g^{(n)}(-z;\tau)=(-1)^n g^{(n)}(z;\tau)$ together with $\mathrm{Im}(-z)=-\mathrm{Im}(z)$ follows

$$f^{(n)}(-z;\tau) = (-1)^n f^{(n)}(z;\tau)\,. \tag{3.3.18}$$

Note that in order to make contact with integrals of the form (3.2.9) it is important that there is a constant weighting function $f^{(0)}(z;\tau)=1$.

Using the relations (3.3.15,3.3.17) we can write down q-expansions[66] for the weighting functions. In particular, we consider $f^{(n)}(z-s-r\tau;\tau)$ for real z shifted by some $s+r\tau$, which we may choose in the fundamental domain of the lattice $\mathbb{Z}+\mathbb{Z}\tau$. These shifted weighting functions will be sufficient to describe the class of iterated integrals relevant to open-string amplitudes. Specifically,

$$f^{(n)}(z-s-r\tau;\tau) = \sum_{j=0}^{n} g^{(j)}(z-s-r\tau;\tau)\frac{(2\pi i r)^{n-j}}{(n-j)!}\,. \tag{3.3.19}$$

Furthermore, we may express any $g^{(n)}(z-s-r\tau;\tau)$ via an expansion in non-negative rational powers of q by rewriting the trigonometric functions in eq. (3.3.17) as follows

$$\begin{aligned} g^{(2k-1)}(z-s-r\tau;\tau) &= i\pi(1-2\sum_{n=0}^{\infty}(q^r e^{2\pi i(z-s)})^{n+1})\delta_{k,1} - 2i\frac{(2\pi i)^{2k-1}}{(2k-2)!}\sum_{n,m=1}^{\infty} n^{2k-2}q^{mn}\times \\ &\quad\times\left[\cos(2\pi m(z-s))(q^{mr}-q^{-mr}) - i\sin(2\pi m(z-s))(q^{mr}+q^{-mr})\right] \\ g^{(2k)}(z-s-r\tau;\tau) &= -2\zeta_{2k} - 2\frac{(2\pi i)^{2k}}{(2k-1)!}\sum_{n,m=1}^{\infty} n^{2k-1}q^{mn}\times \\ &\quad\times\left[\cos(2\pi m(z-s))(q^{mr}+q^{-mr}) - i\sin(2\pi m(z-s))(q^{mr}-q^{-mr})\right]\,. \end{aligned} \tag{3.3.20}$$

The preceding eqs. (3.3.19,3.3.20) establish that the weighting functions $f^{(n)}(z-s-r\tau;\tau)$ have an expansion in non-negative powers of q^r and q^{1-r}, which as $r\in[0,1)$ are non-negative powers of q. This important fact will be relevant when discussing the q-dependence of TEMZVs in section 3.4.

Given the above family of doubly-periodic functions[67] $f^{(n)}(z)$ we consider the class of iterated

[66] Note that here and throughout this whole chapter we indiscriminately brand any power series of the form $\sum_{n\geq 0} a_n q^{rn}$ with $r\in\mathbb{Q}_+$ as q-expansion, although they might be expansions in some root of q.

[67] For the sake of a slicker presentation we will from now on suppress the τ dependence in the notation of the weighting functions $f^{(n)}(z)$.

integrals on $E_\tau^\times \setminus \{b_1,\dots,b_l\}$ along the path $\gamma_z(t)=tz$ with $t,z\in[0,1]$, given by the recursion

$$\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & b_1 \end{smallmatrix};z\right)=\int_0^z \mathrm{d}z_l f^{(n_l)}(z_l-b_l)\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_1 \\ b_{l-1} & \dots & b_1 \end{smallmatrix};z_l\right)\ , \tag{3.3.21}$$

where the recursion starts with $\Gamma(\ ;z)=1$ and the b_i, called *twists*, may be chosen to lie in the fundamental domain[68] of the lattice $\mathbb{Z}+\mathbb{Z}\tau$. In the context of open-string amplitudes we found in the previous section that the relevant Green functions may be described via $f^{(1)}(z)$ and $f^{(1)}(z-\tau/2)$. Correspondingly, we have to admit that this sections setup is more general than needed in the computation of genus-one open-string amplitudes, for which it is completely sufficient to consider iterated integrals on the twice-punctured elliptic curve $E_\tau^\times\setminus\{\frac{\tau}{2}\}$. However, the more general setup causes only minor notational annoyances and has the advantage of leading to a more complete picture of the class of iterated integral under consideration, whence we use the general setup throughout this exposition.

The recursive definition of (3.3.21) can be rewritten as an integral over the l-simplex

$$\Delta_{i,j;z}=\{(z_i,\dots,z_j)|0\le z_i\le\dots\le z_j\le z, j-i+1=l\}\ , \tag{3.3.22}$$

taking the form

$$\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & b_1 \end{smallmatrix};z\right)=\int_{\Delta_{1,l;z}} \mathrm{d}z_l\dots\mathrm{d}z_1\, f^{(n_l)}(z_l-b_l)\dots f^{(n_1)}(z_1-b_1)\ . \tag{3.3.23}$$

Given an iterated integral as in eq. (3.3.23) we associate the notion of *length* $l_\Gamma=l$ and *weight* $w_\Gamma=\sum_i n_i$ to it. A few remarks concerning the definition are in order. Firstly, for the time being we explicitly restrict to the case $b_i\notin(0,z)$ as divergences on the integration contour may occur due to the singular behaviour of $f^{(1)}(z-b)$ at b. A possible generalization of definition (3.3.21) for real twists will be postponed to section 3.6, where we comment on such iterated integrals and their properties. Furthermore, we note that although the above definition is completely fine for general b_i on $E_\tau^\times\setminus(0,z)$ it will be convenient to restrict to $b_i=s_i+r_i\tau$ with $s_i,r_i\in[0,1)\cap\mathbb{Q}$. The motivation for this restriction is that relations among iterated integrals may involve differences of several twists, hence restricting to rational twists keeps the set of combined letters $\begin{smallmatrix} n_i \\ b_i\end{smallmatrix}$, needed to describe all relations, countable. Finally, we note that the family of iterated integrals (3.3.23) is defined with respect to a specific path, namely $\gamma_z(t)=tz$ with $t,z\in(0,1)$. From a mathematical point of view our choice of path is completely arbitrary, hence it would be favourable if the definition would only depend on the path in a mild way. Specifically, one usually would like the class of iterated integrals under consideration to be homotopy functionals, i.e. they only depend on the homotopy type of the path. We will omit any discussion on the matter as it does not seem to play a role in the study of genus-one open-string amplitudes but the intrigued reader may consult [39].

Iterated integrals posses a rich structure in the form of several identities among integrals or products thereof (cf. appendix D.1). For the convenience of the reader we formulate some of

[68] The points b_i on $E_\tau^\times$ may be considered as equivalence classes of points in $\mathbb{C}$, where points are equivalent if they belong to the same orbit of the lattice $\mathbb{Z}+\mathbb{Z}\tau$. In an abuse of notation we do not distinguish between the equivalence class and its members, but instead consider the twists as well as the integration path to lie in the fundamental domain of the lattice, i.e. $b_i=s_i+r_i\tau$ with $r_i,s_i\in[0,1)$.

those identities in the nomenclature of (3.3.21). Elliptic iterated integrals satisfy the reflection identity, which follows from path inversion and the parity property of the weighting functions (3.3.18)

$$\Gamma\left(\begin{smallmatrix} n_1 & n_2 & \dots & n_l \\ b_1 & b_2 & \dots & b_l \end{smallmatrix}; z\right) = (-1)^{\sum_i n_i}\,\Gamma\left(\begin{smallmatrix} n_l & \dots & n_2 & n_1 \\ z-b_l & \dots & z-b_2 & z-b_1 \end{smallmatrix}; z\right) . \tag{3.3.24}$$

Furthermore, they satisfy the shuffle product formula

$$\begin{aligned}\Gamma\left(\begin{smallmatrix} n_k & \dots & n_1 \\ b_k & \dots & b_1 \end{smallmatrix}; z\right)\Gamma\left(\begin{smallmatrix} n_{k+l} & \dots & n_{k+1} \\ b_{k+l} & \dots & b_{k+1} \end{smallmatrix}; z\right) &= \sum_{\sigma\in\Sigma(k,l)} \Gamma\left(\begin{smallmatrix} n_{\sigma(k+l)} & \dots & n_{\sigma(1)} \\ b_{\sigma(k+l)} & \dots & b_{\sigma(1)} \end{smallmatrix}; z\right) \\ &=: \Gamma\left(\left(\begin{smallmatrix} n_k & \dots & n_1 \\ b_k & \dots & b_1 \end{smallmatrix}\right) ⧢ \left(\begin{smallmatrix} n_{k+l} & \dots & n_{k+1} \\ b_{k+l} & \dots & b_{k+1} \end{smallmatrix}\right); z\right) ,\end{aligned} \tag{3.3.25}$$

where $\Sigma(k,l)$ is the set of (k,l)-shuffles.

Above we noted that $f^{(1)}(z-b)$ has a simple pole at b and thus if any $n_i = 1$ the corresponding iterated integral might be ill-defined if $b_i \in \{0, z\}$. In fact problems are caused exclusively by $n_1 = 1, b_1 = 0$ and $n_l = 1, b_l = z$. To see this, note that by the recursive definition (3.3.21) a twist $b_i = z$ only causes a problem for the last integration. Furthermore, in the case $n_1 \neq 1$ we find $\Gamma\left(\begin{smallmatrix} n_1 \\ b_1 \end{smallmatrix}; z\right) = \mathcal{O}(z)$, which then extends to any iterated integral without any label $n_i = 1$. Consider now an iterated integral with some label $n_i = 1$, $b_i = 0$ and $i > 1$. By the recursive definition (3.3.21) we see that the corresponding length i integral will not have a pole at 0, as the length $i-1$ integral is $\mathcal{O}(z_i)$ in the integration variable and hence the length i integral will be $\mathcal{O}(z_{i+1})$ in its endpoint coordinate. By the same reasoning we see that additional forms $n_i = 1, b_i = 0$ do not cause any problems.

Strictly speaking, if we allow for $b_i \in \{0, z\}$, we should consider iterated integrals defined on a punctured elliptic curve $E_\tau^\times \setminus \{z\}$, i.e. our path starts and ends on points that do not belong to the space we consider. These kind of integrals are defined using the notion of tangential basepoint.[69] The idea is to consider the iterated integral on $(\varepsilon, z-\varepsilon)$ for some arbitrarily small $\varepsilon > 0$ and then expand this integral in ε. The aforementioned expansion satisfies

$$\int\limits_{\varepsilon<z_1<\dots<z_l<z-\varepsilon} \mathrm{d}z_l \dots \mathrm{d}z_1\, f^{(n_l)}(z_l - b_l)\dots f^{(n_1)}(z_1 - b_1) = \sum_{i=0}^{l} a_i(\varepsilon)\log^{l-i}(-2\pi i\varepsilon) , \tag{3.3.26}$$

where the $a_i(\varepsilon)$ are well-defined at $\varepsilon = 0$. To see this note that only $f^{(1)}(z)$ is singular at zero and in fact has a simple pole at that point. In particular, the singular behaviour comes exclusively from the meromorphic part, i.e. $g^{(1)}(z)$. Then expanding $g^{(1)}(z)$ in a Laurent series around zero implies the logarithmic behaviour as stated in eq. (3.3.26). A similar argument holds in the case of singularities at the endpoint z.

The regularized iterated integral is then defined to be the constant term in the ε-expansion, specifically

$$\lim_{\varepsilon\to 0} \operatorname{Reg} \int\limits_{\varepsilon<z_1<\dots<z_l<z-\varepsilon} \mathrm{d}z_l \dots \mathrm{d}z_1\, f^{(n_l)}(z_l - b_l)\dots f^{(n_1)}(z_1 - b_1) := a_0(0) . \tag{3.3.27}$$

[69] This name stems from the observation that the below procedure does not depend on the coordinate ε but rather on its tangent ∂_ε, which is due to the logarithmic behaviour of the integrals in the neighbourhood of the singularities. The corresponding path of the iterated integral then lives on the elliptic curve with blow-ups at the singularities such that the endpoints are the chosen tangential vectors; cf. the discussion following the introduction of the regularized expression (3.3.27). Note that as we ultimately want to integrate along the interval $(0,1)$ we choose the regularization at z in such a way that we have a closed path starting and ending with the same tangential vector.

Note that there is some ambiguity involved in the above regularization, as we formally set $\log(-2\pi i\varepsilon)$ to zero, but we could equally well choose $\log(2\pi\varepsilon)$, which would differ by $-i\pi/2$ on the principal branch of the logarithm. Our choice is to set $\log(-2\pi i\varepsilon)$ to zero[70] leading to the explicit relation between the equal boundary Green function and the following length one elliptic iterated integral as teased in eq. (3.2.5)

$$\frac{1}{2}G_1(z)|_{z\in\mathbb{R}} = \underbrace{\frac{i\pi}{2} - \log(2\pi)}_{=c_P} + \lim_{\varepsilon\to 0}\mathrm{Reg}\int_{\varepsilon}^{z} dy\, f^{(1)}(y)\ . \tag{3.3.28}$$

We note that strictly speaking the shuffle relations (3.3.25) and the reflection identity (3.3.24) have been defined only for convergent iterated integrals so far. However the regularization procedure above enables us to extend these relations/identities to the regulated iterated integrals.[71]

Using the regularization prescription (3.3.27) we gave meaning to iterated integrals that contain twists depending on the endpoint coordinate z. In order to see how such iterated integrals fit into the pantheon of elliptic iterated integrals, we give a slight generalization of an algorithm of [50] that relates iterated integrals with endpoint dependent twists to linear combinations of iterated integrals with only constant twists (w.r.t. the integration); cf. also [39]. The aforementioned relations also conveniently fall in line with the recursive definition (3.3.21), which only contained iterated integrals with constant twists. Here we will be content with illustrating how such relations may be derived for length ≤ 2 as these cases exhibit all relevant issues; further cases may be found in appendix E.3. Note that the integrals in question generally occur in the string theory context when considering a Green function depending on two vertex operator insertion points that both have not been fixed to zero, e.g. $z_3 - z_2$

$$\int_{\varepsilon}^{z_3-z_2} \mathrm{d}y f^{(1)}(y) = -\int_{0}^{z_3-\varepsilon} \mathrm{d}w f^{(1)}(w-z_3) + \int_{0}^{z_2} \mathrm{d}w f^{(1)}(w-z_3)\ , \tag{3.3.29}$$

where we assumed the boundary insertions to be ordered such that $z_2 < z_3$.

Deriving relations between elliptic iterated integrals with endpoint dependent twists and those with endpoint independent twists will be carried out by recursion on the length. Starting at length one we find

$$\Gamma\left(\begin{smallmatrix} n_1 \\ z+b_{1;0} \end{smallmatrix};z\right) = (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1 \\ -b_{1;0} \end{smallmatrix};z\right)\ , \tag{3.3.30}$$

as a consequence of the reflection identity (3.3.24). In fact using the reflection identity we might rewrite all iterated integrals where all twists depend on the endpoint coordinate

$$\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ z+b_{l;0} & \dots & z+b_{1;0} \end{smallmatrix};z\right) = (-1)^{\sum_i n_i}\,\Gamma\left(\begin{smallmatrix} n_1 & \dots & n_l \\ -b_{1;0} & \dots & -b_{l;0} \end{smallmatrix};z\right)\ . \tag{3.3.31}$$

[70] The choice of regulator is made such that the constant terms in the q-expansion of EMZVs will be $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs as discussed in section 3.4. Other choices may involve coefficients containing $\log(\pi)$ which is unfavorable from a mathematical point of view, cf. [41, 152].

[71] Shuffle relations are a general property of iterated integrals defined with respect to the same path γ, cf. appendix D.1. Then, as a priori convergent iterated integrals may also be defined in terms of the path $\gamma_\varepsilon(t) = \varepsilon + (z - 2\varepsilon)t$, regulated iterated integrals satisfy shuffle relations. Note that the regulated expansion in ε of convergent iterated integrals reduces to the unregulated integral in the limit $\varepsilon \to 0$.

At length two we also have to consider

$$\Gamma\left(\begin{smallmatrix} n_2 & n_1 \\ b_2 & z+b_{1;0} \end{smallmatrix};z\right) , \tag{3.3.32}$$

where the case with a z-dependent b_2 is related to the above iterated integral by reflection. This elliptic iterated integral may be re-expressed as

$$\Gamma\left(\begin{smallmatrix} n_2 & n_1 \\ b_2 & z+b_{1;0} \end{smallmatrix};z\right) = \lim_{z\to 0}\Gamma\left(\begin{smallmatrix} n_2 & , & n_1 \\ b_2 & , & z+b_{1;0} \end{smallmatrix};z\right) + \int_0^z \mathrm{d}y \frac{\mathrm{d}}{\mathrm{d}y}\Gamma\left(\begin{smallmatrix} n_2 & , & n_1 \\ b_2 & , & y+b_{1;0} \end{smallmatrix};y\right) , \tag{3.3.33}$$

where the first term usually vanishes as the integration range shrinks to a point in the limit, apart from certain singular contributions. Specifically, let us consider the most singular case possible, i.e. all $n_i = 1$. Then we have to account for the poles $f^{(1)}(z-b) = \frac{1}{z-b} + \mathcal{O}(z^0)$, which leads to the expression

$$\lim_{z\to 0}\int_0^z \mathrm{d}y_2 \int_0^{y_2} \mathrm{d}y_1 \frac{1}{y_1 - z - b_{1;0}}\frac{1}{y_2} = -\lim_{z\to 0}\mathrm{Li}_2\left(\frac{z}{b_{1;0}+z}\right) = -\delta(b_{1;0})\zeta_2 , \tag{3.3.34}$$

which only contributes for $b_{1;0} = 0$. The less singular terms in the expansion of $f^{(1)}$ are at least of order z and therefore no additional contributions arise. Note that the expression in eq. (3.3.34) is given by a Goncharov polylogarithm. In fact the non-vanishing of the corresponding expression in the limit is related to the scaling property of such iterated integrals (cf. appendix E.3).

The total derivative in eq. (3.3.33) is given by

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}y}\Gamma\left(\begin{smallmatrix} n_2 & , & n_1 \\ b_2 & , & y+b_{1;0} \end{smallmatrix};y\right) &= f^{(n_2)}(y-b_2)\,\Gamma\left(\begin{smallmatrix} n_1 \\ y+b_{1;0} \end{smallmatrix};y\right) + f^{(n_1)}(-y-b_{1;0})\,\Gamma\left(\begin{smallmatrix} n_2 \\ b_2 \end{smallmatrix};y\right) \\ &\quad -\int_0^y \mathrm{d}x f^{(n_1)}(x-y-b_{1;0})f^{(n_2)}(x-b_2) ,\end{aligned} \tag{3.3.35}$$

which can be obtained using the Leibnitz integration rule and the following identity of derivatives

$$\partial_z f^{(n)}(y-z-s-r\tau) = -\partial_y f^{(n)}(y-z-s-r\tau) . \tag{3.3.36}$$

The product of the form $f^{(n_1)}(y-a_1)f^{(n_2)}(y-a_2)$ with both functions depending on the integration variable y can be rewritten into a sum of products of weighting functions with only one function depending on y using the following relation

$$\begin{aligned} f^{(n_1)}(x-y-b_{1;0})f^{(n_2)}(x-b_2) &= -(-1)^{n_2} f^{(n_1+n_2)}(b_2-y-b_{1;0}) \\ &\quad + \sum_{j=0}^{n_2}\binom{n_1+j-1}{j} f^{(n_1+j)}(x-y-b_{1;0})f^{(n_2-j)}(y+b_{1;0}-b_2) \\ &\quad + \sum_{j=0}^{n_1}\binom{n_2+j-1}{j} f^{(n_2+j)}(x-b_2)f^{(n_1-j)}(b_2-y-b_{1;0}) ,\end{aligned} \tag{3.3.37}$$

which may be obtained by equating coefficients on both sides of the Fay identity eq. (3.3.14).[72] We are then led to the relation

$$\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_2 & n_1 \\ b_2 & z+b_{1;0} \end{smallmatrix};z\right) &= -\delta_{n_1,1}\delta_{n_2,1}\delta(b_{1;0})\zeta_2 + \int_0^z \mathrm{d}y f^{(n_2)}(y-b_2)\,\Gamma\left(\begin{smallmatrix} n_1 \\ y+b_{1;0} \end{smallmatrix};y\right) \\
&\quad + (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1 & n_2 \\ -b_{1;0} & b_2 \end{smallmatrix};z\right) + (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1+n_2 & 0 \\ b_2-b_{1;0} & 0 \end{smallmatrix};z\right) \\
&\quad - \sum_{j=0}^{n_2}\binom{n_1+j-1}{j}\int_0^z \mathrm{d}y f^{(n_2-j)}(y-(b_2-b_{1;0}))\,\Gamma\left(\begin{smallmatrix} n_1+j \\ y+b_{1;0} \end{smallmatrix};y\right) \\
&\quad - \sum_{j=0}^{n_1}\binom{n_2+j-1}{j}(-1)^{n_1-j}\,\Gamma\left(\begin{smallmatrix} n_1-j & n_2+j \\ b_2-b_{1;0} & b_2 \end{smallmatrix};z\right) \\
&= -\delta_{n_1,1}\delta_{n_2,1}\delta(b_{1;0})\zeta_2 + (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_2 & n_1 \\ b_2 & -b_{1;0} \end{smallmatrix};z\right) \\
&\quad + (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1 & n_2 \\ -b_{1;0} & b_2 \end{smallmatrix};z\right) + (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1+n_2 & 0 \\ b_2-b_{1;0} & 0 \end{smallmatrix};z\right) \\
&\quad - \sum_{j=0}^{n_2}\binom{n_1+j-1}{j}(-1)^{n_1+j}\,\Gamma\left(\begin{smallmatrix} n_2-j & n_1+j \\ b_2-b_{1;0} & -b_{1;0} \end{smallmatrix};y\right) \\
&\quad - \sum_{j=0}^{n_1}\binom{n_2+j-1}{j}(-1)^{n_1-j}\,\Gamma\left(\begin{smallmatrix} n_1-j & n_2+j \\ b_2-b_{1;0} & b_2 \end{smallmatrix};z\right) ,
\end{aligned} \tag{3.3.38}$$

using the length one result for the second equality. The key point of the above procedure is the structure of the total derivative as in eq. (3.3.35), which allows us to rewrite an elliptic iterated integral into a sum containing only iterated integrals with z-independent twists as well as iterated integrals with z-dependent twist albeit of lower length. Hence, we might eliminate any z-dependent twist recursively.

Ultimately, our aim is to compute integrals with integration domain coinciding with the boundary of the cylinder worldsheet cf. (3.2.9), i.e. we integrate along the whole interval $[0,1]$. This gives rise to *twisted elliptic Multiple Zeta Values* defined by

$$\omega\left(\begin{smallmatrix} n_1 & , & \dots & , & n_l \\ b_1 & , & \dots & , & b_l \end{smallmatrix}\right) = \lim_{z\to 1}\Gamma\left(\begin{smallmatrix} n_l & , & \dots & , & n_1 \\ b_l & , & \dots & , & b_1 \end{smallmatrix};z\right) . \tag{3.3.39}$$

Note that in the special case where all $b_i = 0$, these are referred to as *elliptic Multiple Zeta Values*. The TEMZVs inherit the reflection identity and the shuffle product formula. For the reflection identity we find

$$\omega\left(\begin{smallmatrix} n_1 & , & n_2 & , & \dots & , & n_l \\ b_1 & , & b_2 & , & \dots & , & b_l \end{smallmatrix}\right) = (-1)^{\sum_i n_i}\,\omega\left(\begin{smallmatrix} n_l & , & n_{l-1} & , & \dots & , & n_1 \\ -b_l & , & -b_{l-1} & , & \dots & , & -b_1 \end{smallmatrix}\right) . \tag{3.3.40}$$

Note that for $b_i = r_i + s_i\tau$ with $r_i, s_i \in \{0, 1/2, 1\}$ this equation may have fixed points. Otherwise we get relations of TEMZV with twists b_i and $-b_i$. We conclude this section by returning to our motivational example of eq. (3.2.9) and translating it into the language of TEMZVs

$$\int_0^1 \mathrm{d}z_2 \int_0^1 \mathrm{d}z_3 \int_0^1 \mathrm{d}z_4 \frac{1}{2} G_1(z_{12};\tau) = c_P + \omega\left(\begin{smallmatrix} 1 & ; & 0 \\ 0 & , & 0 \end{smallmatrix}\right) . \tag{3.3.41}$$

[72] The derivation of this can be found in appendix E.2.

3.4 The q-expansion of TEMZVs

So far we have established that worldsheet integrals give rise to TEMZVs as coefficients of the α'-expansion, however as of now this might seem like a mere renaming, postponing the eventual computation to section 3.5. To remedy this we go on to establish that any given TEMZV satisfies an initial value problem, which will turn out to reduce the computational endeavour to a combinatorial problem, solvable by computer algebra systems. The idea is to study the expansion of a given TEMZV in fractional powers of the (exponentiated) modular parameter q, inherited from the analogous expansions of the weighting functions. Subsequently, we will discuss how the corresponding coefficients of such an expansion are obtainable via an initial value problem [1], generalizing the results for EMZVs [41, 155]. The corresponding differential equation will give the logarithmic q derivative of any TEMZV as TEMZVs of shorter length multiplied by weighting functions evaluated at some twists. Hence, the non-constant part of any TEMZV may be computed as iterated integrals of power series in fractional powers of q, whose computational complexity is lower than integrating over the trigonometric constituents of the weighting functions. Furthermore, the corresponding initial condition is determined by the q^0 term of the TEMZV under consideration, which may be conveniently extracted from the degeneration behaviour of a generating series of TEMZVs in the limit $q \to 0$.

3.4.1 The general structure of q-expansions of TEMZVs

We start by establishing some facts about TEMZVs as functions of q. To that end recall that the weighting functions $f^{(n)}(z-s-r\tau)$ admit expansions in non-negative powers of q^r and q^{1-r} (cf. eqs. (3.3.19,3.3.20)). Then by exchanging the q-expansion with integration over the simplex in the definition (3.3.23) we deduce that a given TEMZV admits an expansion in some q^p

$$\omega\left(\begin{smallmatrix} n_1 & , \dots , & n_l \\ s_1+r_1\tau & , \dots , & s_l+r_l\tau \end{smallmatrix}\right) = c_0\left(\begin{smallmatrix} n_1 & , \dots , & n_l \\ s_1+r_1\tau & , \dots , & s_l+r_l\tau \end{smallmatrix}\right) + \sum_{j=1}^{\infty} c_j\left(\begin{smallmatrix} n_1 & , \dots , & n_l \\ s_1+r_1\tau & , \dots , & s_l+r_l\tau \end{smallmatrix}\right) q^{pj} \, , \tag{3.4.1}$$

where p is the least common denominator of all r_i. The complex number $c_0(\dots)$ is called the *constant term* of the TEMZV.

From eqs. (3.3.19,3.3.20) we infer that the coefficients of the q-expansion may be computed as integrals of products of trigonometric functions over an l-simplex, while we need to be careful with the cotangent terms from $g^{(1)}$ as they give rise to logarithms and may need regularization. At length one the integrals over trigonometric functions all vanish, hence the length one TEMZVs are constants given by

$$\omega\left(\begin{smallmatrix} n \\ s+r\tau \end{smallmatrix}\right) = -\frac{i\pi(2\pi i r)^{n-1}}{(n-1)!} - \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{\zeta_{2k}(2\pi i r)^{n-2k}}{(n-2k)!} \, , \quad n \geq 1 \, . \tag{3.4.2}$$

Note that this does not depend on s, which is due to the q^0 term in the q-expansion of the weighting functions (3.3.19,3.3.20) being independent of s.

3.4.2 Constant term procedure

We now go on to discuss how one may compute the q^0 term of any TEMZV, purely combinatorially. Let us start by elaborating on the case with only zero twists as studied in [41]. There the protagonist was the elliptic associator $A(\tau; x_0, y)$,[73] originally introduced in [40]

$$
\begin{aligned}
e^{-i\pi\, \mathrm{ad}_{x_0}(y)} A(\tau; x_0, y) &= \tilde{\mathcal{P}} \exp\Big(- \int_0^1 \mathrm{d}z \sum_{n=0}^{\infty} f^{(n)}(z) \mathrm{ad}_{x_0}^n(y) \Big) \\
&= 1 + \sum_{l\geq 1} (-1)^l \sum_{n_1,\dots,n_l \geq 0} \omega\left(\begin{smallmatrix} n_1 & ; & \dots & ; & n_l \\ 0 & ; & \dots & ; & 0 \end{smallmatrix} \right) \mathrm{ad}_{x_0}^{n_l}(y) \dots \mathrm{ad}_{x_0}^{n_1}(y) \ ,
\end{aligned}
\tag{3.4.3}
$$

where x_0, y are non-commutative variables, $\tilde{\mathcal{P}} \exp(\dots)$ denotes the path ordered exponential with reversed order of the products in $\mathrm{ad}_{x_0}(y) = [x_0, y]$ and the factor of $e^{i\pi\, \mathrm{ad}_{x_0}(y)}$ is related to our choice of regularization prescription.[74] The crux of equation (3.4.3) is that the elliptic associator is a generating series for EMZVs[75] and therefore we may study the limit $q \to 0$ of any EMZV by expanding the path-ordered exponential. This might not seem like a simplification at first, but we may use the following result of Enriquez [41]

$$
A(\tau; x_0, y) = \Phi(\tilde{y}, t) e^{2\pi i \tilde{y}} \Phi(\tilde{y}, t)^{-1} + \mathcal{O}(q) \ , \tag{3.4.4}
$$

relating the $q \to 0$ limit of the elliptic associator, and hence EMZVs, to the Drinfeld associator $\Phi(t, \tilde{y})$, where $\tilde{y} = -\frac{\mathrm{ad}_{x_0}}{\exp(2\pi i\, \mathrm{ad}_{x_0}) - 1}(y)$ and $t = -\mathrm{ad}_{x_0}(y)$. Combining eqs. (3.4.3,3.4.4) we find the relation

$$
1 + \sum_{l\geq 1} (-1)^l \sum_{n_1,\dots,n_l \geq 0} c_0\left(\begin{smallmatrix} n_1 & ; & \dots & ; & n_l \\ 0 & ; & \dots & ; & 0 \end{smallmatrix} \right) \mathrm{ad}_{x_0}^{n_l}(y) \dots \mathrm{ad}_{x_0}^{n_1}(y) = e^{i\pi t} \Phi(\tilde{y}, t) e^{2\pi i \tilde{y}} \Phi(\tilde{y}, t)^{-1} \ , \tag{3.4.5}
$$

which is our basis for computing the constant terms of EMZVs. These results might seem somewhat nebulous but we will give a more detailed motivation in the context of TEMZVs below. However, before we do so we note some consequences that may be inferred from eq. (3.4.5) for the constant terms of EMZVs. As the Drinfeld associator is known to be the generating series of MZVs (see [30, 31])

$$
\begin{aligned}
\Phi(e_0, e_1) = 1 + \sum_{l\geq 1} (-1)^l \sum_{\substack{k_1,\dots,k_{l-1} \geq 1 \\ k_l \geq 2}} \zeta_{k_1,\dots,k_l} e_0^{k_l - 1} e_1 \dots e_0^{k_1 - 1} e_1 \\
+ \text{regulated terms} \ ,
\end{aligned}
\tag{3.4.6}
$$

the constant terms of EMZVs includes MZVs. Moreover, as the expansion

$$
\tilde{y} = -\frac{\mathrm{ad}_{x_0}}{\exp(2\pi i\, \mathrm{ad}_{x_0}) - 1}(y) = -\frac{1}{2\pi i} \left(1 - i\pi\, \mathrm{ad}_{x_0} - \sum_{n=1}^{\infty} 2\zeta_{2n}\, \mathrm{ad}_{x_0}^{2n} \right)(y) \tag{3.4.7}
$$

has coefficients in $\mathbb{Q}[(2\pi i)^{-1}]$ we find the TEMZV constant terms to be $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs.[76] Note that using eqs. (3.4.6) and (3.4.7) we reduced the computation of the

---

[73] To be more precise we exclusively consider the A-part of Enriquez' elliptic associator, related to the A-cycle on the elliptic curve.

[74] Note that we deliberately omit all regularization issues in order to not unnecessarily clutter the notation. For a precise treatment we refer to [41, 152], cf. eq. (3.3.26).

[75] Again we note the analogy to the genus-zero case, where the Drinfeld associator is the generating series for MZVs.

[76] In fact it is even true that all coefficients of the q-expansion of an EMZV $\sum_{k\geq 0} c_k q^k$ are $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs, cf. [152].

constant term of EMZVs to a combinatorial problem which may be solved using a computer, i.e. equating coefficients of the monomials $\mathrm{ad}_{x_0}^{n_l}(y)\ldots\mathrm{ad}_{x_0}^{(n_1)}(y)$ in eq. (3.4.5).

Before we go on to derive a similar result to the above for TEMZVs it is convenient to restrict the set of twists under consideration. In the preceding section we already restricted our attention to twists $b = s + r\tau$ with rational coefficients, we now further restrict the set of twists to a finite set. For this purpose we introduce the notation

$$\Lambda_N = \left\{0, \frac{1}{N}, \ldots, \frac{N-1}{N}\right\} , \qquad \Lambda_N^\times = \Lambda_N \setminus \{0\} , \qquad N \geq 2 , \tag{3.4.8}$$

and restrict the twists to $b \in (\Lambda_N + \Lambda_N\tau) \setminus \Lambda_N^\times$. There is no loss of generality due to the choice of a square lattice, as any non-square lattice can be considered as a sub-lattice of some square lattice of appropriate size.

We now go on to discuss how the analogues of the above results come about in the context of TEMZVs. The idea is to utilize the degeneration of a generating series for TEMZVs in the limit $q \to 0$ and derive a result akin to the EMZV case given in eq. (3.4.4). The generalization of the generating series in question is given by

$$\begin{aligned} e^{-i\pi\,\mathrm{ad}_{x_0}(y)} A_{(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N^\times}(\tau) &= \tilde{\mathcal{P}} \exp\left(-\int_0^1 \mathrm{d}z \sum_{b\in(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N^\times} \sum_{n=0}^\infty f^{(n)}(z-b)\mathrm{ad}_{x_b}^n(y)\right) \\ &= 1 + \sum_{l\geq 1}(-1)^l \sum_{\substack{n_1,\ldots,n_l\geq 0 \\ b_1,\ldots,b_l\in(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N^\times}} \omega\left(\begin{smallmatrix} n_1 & ; & \ldots & ; & n_l \\ b_1 & ; & \ldots & ; & b_l \end{smallmatrix}\right) \mathrm{ad}_{x_{b_l}}^{n_l}(y)\ldots\mathrm{ad}_{x_{b_1}}^{n_1}(y) . \end{aligned} \tag{3.4.9}$$

In order to understand the degeneration of this generating series in the limit $q \to 0$, we consider the behaviour of the weighting functions $f^{(n)}$ in the limit $\tau \to i\infty$. To that end, we start by considering the degeneration limit of the weighting functions constituents the $g^{(n)}$. Then for $k \geq 0$ we find that

$$\lim_{\tau\to i\infty} g^{(2k)}(z-s-r\tau)\mathrm{d}z = \frac{\mathrm{d}w}{w}\frac{-2\zeta_{2k}}{2\pi i} \tag{3.4.10}$$

$$\lim_{\tau\to i\infty} g^{(1)}(z-s-r\tau)\mathrm{d}z = \begin{cases} -\frac{1}{2}\frac{\mathrm{d}w}{w} + \frac{\mathrm{d}w}{w-\exp(-2\pi i s)} & \text{for } r=0 \\ \frac{1}{2}\frac{\mathrm{d}w}{w} & \text{for } r\neq 0 \end{cases} \tag{3.4.11}$$

$$\lim_{\tau\to i\infty} g^{(2k+3)}(z-s-r\tau)\mathrm{d}z = 0 , \tag{3.4.12}$$

where we introduced $w = \exp(2\pi i z)$. For the time being we only consider $r = 0$ if $s = 0$ and therefore no non-unit root of unity stemming from $g^{(1)}$ appears in our considerations. However, we will change this premise and consider twists in a whole square lattice $\Lambda_N + \Lambda_N\tau$ in section 3.6, where we will find MZVs at roots of unity. The $q \to 0$ limit of the weighting functions $f^{(n)}(z-b)$ may now be deduced from the corresponding limit of their constituents above

$$\begin{aligned} \lim_{\tau\to i\infty} f^{(n)}(z-s-r\tau)\mathrm{d}z &= \left(\frac{i\pi(-2\pi i r)^{n-1}}{(n-1)!}\theta_{n\geq 1} - \sum_{k=0}^{\lfloor n/2\rfloor}\frac{2\zeta_{2k}(-2\pi i r)^{n-2k}}{(n-2k)!}\right)\frac{1}{2\pi i}\frac{\mathrm{d}w}{w} \\ &= -\frac{\mathrm{d}w}{w}\left(\sum_{k=0}^{n}\frac{B_k(-2\pi i)^{k-1}}{k!}\frac{(-2\pi i r)^{k-m}}{(k-m)!}\right) , \qquad n \geq 1 , \end{aligned} \tag{3.4.13}$$

where B_k are the Bernoulli numbers with $B_1 = -1/2$. In the special case $r = \frac{1}{2}$, relevant to the double-trace contributions from a cylindrical worldsheet, this expression reduces to

$$\lim_{\tau\to i\infty} f^{(n)}(z - \frac{\tau}{2})\mathrm{d}z = \begin{cases} \frac{2^{n-1}-1}{2^{n-2}} \frac{\zeta_n}{2\pi i} \frac{\mathrm{d}w}{w} & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} . \end{cases} \quad (3.4.14)$$

In order to rewrite the exponent we note that the limit (3.4.13) is identical to the expansion coefficients of

$$\frac{\exp(-2\pi ir\,\mathrm{ad}_{x_b})\mathrm{ad}_{x_b}}{\exp(-2\pi i\,\mathrm{ad}_{x_b}) - 1} = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{B_k(-2\pi i)^{k-1}}{k!} \frac{(-2\pi ir)^{k-m}}{(k-m)!} \right) \mathrm{ad}_{x_b}^n . \quad (3.4.15)$$

Then we may rewrite the exponent of the generating function eq. (3.4.9) as

$$\lim_{\tau\to i\infty} -\int_0^1 \mathrm{d}z \sum_{n=0}^{\infty} \sum_{b\in(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N^\times} f^{(n)}(z-b)\mathrm{ad}_{x_b}^n(y) = \int_{C(0;1)} \left(\tilde{y}_N \frac{\mathrm{d}w}{w} + t\frac{\mathrm{d}w}{w-1} \right) , \quad (3.4.16)$$

where $C(0;1)$ is the unit circle around zero, i.e. the image of the path under the exponential map[77] $w = \exp(2\pi iz)$ and we introduced the shorthands

$$\tilde{y}_N = -\frac{\mathrm{ad}_{x_0}}{\exp(2\pi i\,\mathrm{ad}_{x_0}) - 1}(y) \quad + \sum_{b\in(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N} \frac{\exp(-2\pi ir\,\mathrm{ad}_{x_b})\mathrm{ad}_{x_b}}{\exp(-2\pi i\,\mathrm{ad}_{x_b}) - 1}(y) , \quad t = -\mathrm{ad}_{x_0}(y) . \quad (3.4.17)$$

Note that this also contains the case with all $b_i = 0$ described above.

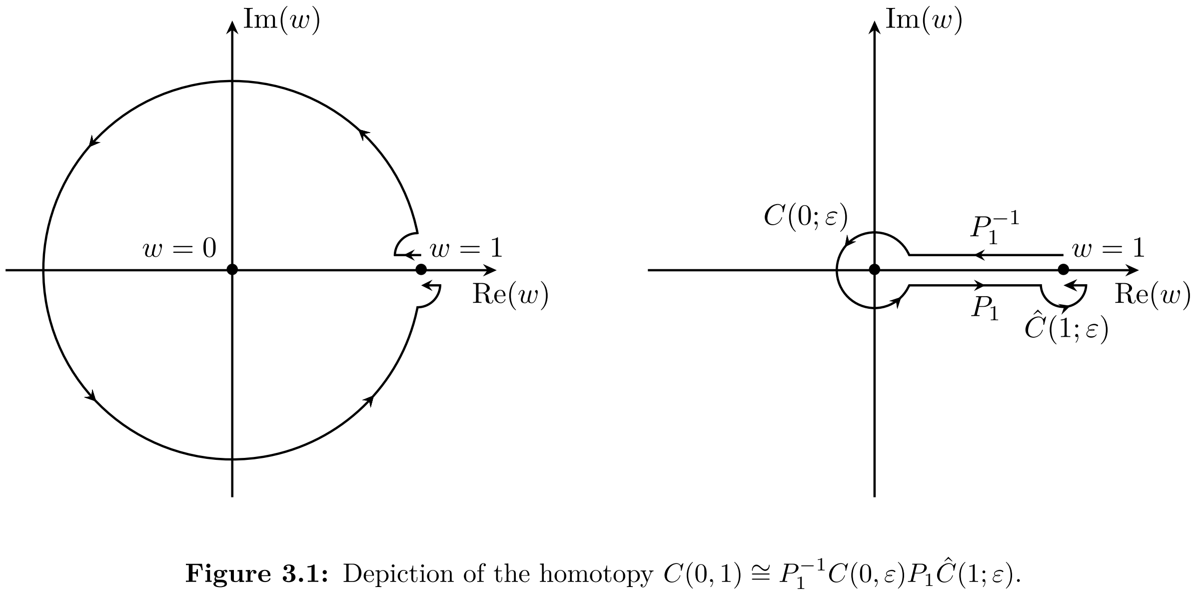

Figure 3.1: Depiction of the homotopy $C(0,1) \cong P_1^{-1}C(0,\varepsilon)P_1\hat{C}(1;\varepsilon)$.

As the integrand of eq. (3.4.16) is holomorphic on $\mathbb{C} \setminus \{0,1\}$ we may replace the integration

[77] Note that due to the necessity for regularization this image is not exactly the unit circle. The path for the regularized iterated integrals as defined in eq. (3.3.27) lives on the elliptic curve with a blow-up at zero and the tangential at zero is the basepoint of the corresponding path. Correspondingly, the image of the exponential map $w = \exp(2\pi iz)$ is a deformed unit circle with fixed tangential at one; cf. figure 3.1. For details see [152].

contour $C(0;1)$ by the homotopic path[78]

$$C(0;1) \cong P_1^{-1} C(0;\varepsilon) P_1 \hat{C}(1;\varepsilon) , \tag{3.4.18}$$

where P_1 denotes a straight-line segment from zero to one, $C(0;\varepsilon)$ is a circle around zero with radius ε and $\hat{C}(1;\varepsilon)$ is a semicircle in the lower half-plane around one with radius ε; cf. fig. 3.1. Then we may use that the path-ordered exponential w.r.t. a composed path $\alpha\beta$ decomposes into a (reversed) concatenation

$$\tilde{\mathcal{P}} \exp\Big(\int_{\alpha\beta} \omega\Big) = \tilde{\mathcal{P}} \exp\Big(\int_{\beta} \omega\Big) \tilde{\mathcal{P}} \exp\Big(\int_{\alpha} \omega\Big) . \tag{3.4.19}$$

Thus using the path decomposition in eq. (3.4.18) we may rewrite the rhs. of eq. (3.4.16) as

$$\begin{aligned} \lim_{\tau\to i\infty} e^{i\pi t} A_{(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N^\times}(\tau) &= \int\limits_{P_1^{-1}C(0;\varepsilon)P_1\hat{C}(1;\varepsilon)} \Big(\tilde{y}_N \frac{\mathrm{d}w}{w} + t\frac{\mathrm{d}w}{w-1}\Big) \\ &= \exp(i\pi t)\Phi(\tilde{y}_N,t)\exp(2\pi i\tilde{y}_N)\Phi^{-1}(\tilde{y}_N,t) , \end{aligned} \tag{3.4.20}$$

where we used the following representation of the Drinfeld associator

$$\Phi(\tilde{y}_N,t) = \tilde{\mathcal{P}} \exp\Big(\int_{P_1} \Big(\tilde{y}_N \frac{\mathrm{d}w}{w} + t\frac{\mathrm{d}w}{w-1}\Big)\Big) . \tag{3.4.21}$$

Accordingly, we may now extract the constant term of a given TEMZV by equating coefficients of monomials $\mathrm{ad}_{x_{b_l}}^{n_l}(y)\ldots\mathrm{ad}_{x_{b_1}}^{n_1}(y)$ in the equation

$$1+\sum_{l\geq 1}(-1)^l \sum_{\substack{n_1,\ldots,n_l\geq 0\\ b_1,\ldots,b_l\in(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N^\times}} c_0\left(\begin{smallmatrix} n_1 & , & \ldots & , & n_l \\ b_1 & , & \ldots & , & b_l\end{smallmatrix}\right) \mathrm{ad}_{x_{b_l}}^{n_l}(y)\ldots\mathrm{ad}_{x_{b_1}}^{n_1}(y) = e^{i\pi t}\Phi(t,\tilde{y}_N)e^{2\pi i\tilde{y}_N}\Phi(t,\tilde{y}_N)^{-1} . \tag{3.4.22}$$

As a first example we want to use eq. (3.4.22) to compute $c_0\left(\begin{smallmatrix} 1 & ; & 0 \\ 0 & ; & 0\end{smallmatrix}\right)$. According to the lhs. of eq. (3.4.22) we have to extract the coefficient of the monomial $-yt$. Expanding the rhs. of eq. (3.4.22) up to all terms that potentially might contribute gives

$$(1+\ldots)(1+\zeta_2[t,\frac{1}{2\pi i}y]+\ldots)(1-y-i\pi t+\frac{1}{2}(-y-i\pi t)^2+\ldots)(1-\zeta_2[t,\frac{1}{2\pi i}y]+\ldots) , \tag{3.4.23}$$

and hence

$$c_0\left(\begin{smallmatrix} 1 & ; & 0 \\ 0 & ; & 0\end{smallmatrix}\right) = -\frac{i\pi}{2} . \tag{3.4.24}$$

The above algorithm can be put on a computer to effectively handle the equating of coefficients as explicitly given in eq. (3.4.23). As mentioned before our main focus lies on the case of twists $b \in \{0, \frac{\tau}{2}\}$ relevant to the study of the genus-one open-string amplitude. In this case the

[78] The semicircle part is due to the fixed tangential vector at one, which results into the factor of $\exp(i\pi t)$ in eq. (3.4.20).

degeneration of the weighting functions $f^{(n)}$ eq. (3.4.14) gives the length one TEMZVs

$$\omega\left(\begin{smallmatrix} n \\ \frac{\tau}{2} \end{smallmatrix}\right) = c_0\left(\begin{smallmatrix} n \\ \frac{\tau}{2} \end{smallmatrix}\right) = \begin{cases} \dfrac{2^{n-1}-1}{2^{n-2}}\zeta_n & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}, \tag{3.4.25}$$

which is a special case of eq. (3.4.2). Furthermore, we may analogously infer the constants for the following length two TEMZVs

$$c_0\left(\begin{smallmatrix} n & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) = \begin{cases} \frac{2^{n-1}-1}{2^{n-1}}\zeta_n & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}, \quad c_0\left(\begin{smallmatrix} 1 & , & n \\ 0 & , & \frac{\tau}{2} \end{smallmatrix}\right) = \begin{cases} (-i\pi)\frac{2^{n-1}-1}{2^{n-1}}\zeta_n & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}, \tag{3.4.26}$$

and more generally the constant of any TEMZV involving the combined letter $\begin{smallmatrix} 2k-1 \\ \frac{\tau}{2} \end{smallmatrix}$ vanishes

$$c_0\left(\begin{smallmatrix} \dots & , & 2k-1 & , & \dots \\ \dots & , & \frac{\tau}{2} & , & \dots \end{smallmatrix}\right) = 0\,, \quad k \geq 1\,. \tag{3.4.27}$$

Further examples of TEMZVs relevant to the main text are given by

$$\begin{gathered} c_0\left(\begin{smallmatrix} 1,1,0 \\ 0,0,0 \end{smallmatrix}\right) = -\frac{\pi^2}{12} \qquad c_0\left(\begin{smallmatrix} 2,0,0 \\ 0,0,0 \end{smallmatrix}\right) = -\frac{\pi^2}{18}\,, \quad c_0\left(\begin{smallmatrix} 1,0,2 \\ 0,0,\frac{\tau}{2} \end{smallmatrix}\right) = -\frac{i\pi^3}{24}\,, \\ c_0\left(\begin{smallmatrix} 0,1,0,0 \\ 0,0,0,0 \end{smallmatrix}\right) = \frac{3\zeta_3}{4\pi^2}\,, \quad c_0\left(\begin{smallmatrix} 0,3,0,0 \\ 0,0,0,0 \end{smallmatrix}\right) = 0\,, \quad c_0\left(\begin{smallmatrix} 0,1,0,2 \\ 0,0,0,\frac{\tau}{2} \end{smallmatrix}\right) = \frac{\zeta_3}{8}\,, \\ c_0\left(\begin{smallmatrix} 0,1,0,1,0 \\ 0,0,0,0,0 \end{smallmatrix}\right) = -\frac{\zeta_2}{60}\,, \quad c_0\left(\begin{smallmatrix} 0,1,1,0,0 \\ 0,0,0,0,0 \end{smallmatrix}\right) = \frac{\zeta_2}{15}\,. \end{gathered} \tag{3.4.28}$$

Finally, for illustrative purposes we give a few examples of the kind of expressions obtained, when considering more general lattices $b \in (\Lambda_N + \Lambda_N\tau) \setminus \Lambda_N^\times$

$$c_0\left(\begin{smallmatrix} 1 \\ \frac{1}{2}+\frac{\tau}{3} \end{smallmatrix}\right) = -\frac{i\pi}{3}\,, \quad c_0\left(\begin{smallmatrix} 5 & , & 1 & , & 2 \\ \frac{\tau}{5} & , & \frac{\tau}{3} & , & 0 \end{smallmatrix}\right) = \frac{2072}{1875}\zeta_8\,, \quad c_0\left(\begin{smallmatrix} 2 & , & 1 & , & 0 & , & 1 \\ \frac{\tau}{2} & , & \frac{\tau}{4} & , & 0 & , & 0 \end{smallmatrix}\right) = -\frac{i\pi}{48}\zeta_3 + \frac{5}{8}\zeta_4\,, \tag{3.4.29}$$

$$c_0\left(\begin{smallmatrix} 1 & , & 1 \\ \frac{\tau}{3} & , & \frac{\tau}{5} \end{smallmatrix}\right) = -\frac{3}{5}\zeta_2\,, \quad c_0\left(\begin{smallmatrix} 1 & , & 1 & , & 1 \\ \frac{\tau}{7} & , & \frac{\tau}{5} & , & \frac{\tau}{3} \end{smallmatrix}\right) = \frac{i\pi}{7}\zeta_2\,, \quad c_0\left(\begin{smallmatrix} 3 & , & 1 & , & 0 & , & 1 \\ \frac{2\tau}{5} & , & 0 & , & 0 & , & \frac{\tau}{4} \end{smallmatrix}\right) = -\frac{9}{125}\zeta_2\zeta_3\,. \tag{3.4.30}$$

3.4.3 Differential equation

Now that we have an efficient handle on the q^0-term of the q-expansion of any TEMZV, our next step is to motivate how we may compute the remaining part of the q-expansion via some differential equation. Specifically, we will discuss how the logarithmic derivative (w.r.t. q) of any length l TEMZV may be exclusively expressed via linear combinations of length $l-1$ TEMZVs with coefficients given by weighting functions $f^{(n)}$ evaluated at some twist b_i (or differences thereof). As the $f^{(n)}(b)$ are power series in fractional powers of q just as the TEMZVs themselves, we might obtain the q-dependence as an integral over power series in fractional powers of q by "inverting" the derivative. This process may be iterated down to the constant length one TEMZVs. Eventually, this procedure then leads to iterated integrals over power series in fractional powers of q. Note that the initial condition of this differential equation may be fixed by the algorithm as discussed in the preceding subsection.

Let us briefly sketch the ingredients needed in the derivation of the aforementioned differential equation, while relegating the details of the derivation to the Appendix E.4. The idea is to define a generating object for length l TEMZVs

$$\begin{aligned}
\mathrm{T}\left[\begin{smallmatrix}\alpha_1,\alpha_2,\dots,\alpha_l\\ b_1,b_2,\dots,b_l\end{smallmatrix}\right] &= \int\limits_{0<z_1<\dots<z_l<1} dz_l\dots dz_2 dz_1 \prod_{i=1}^{l}\Omega(z_i-b_i,\alpha_i)\\
&= \sum_{n_i\geq 0}\alpha_1^{n_1-1}\alpha_2^{n_2-1}\dots\alpha_l^{n_l-1}\,\omega\left(\begin{smallmatrix}n_1,n_2,\dots,n_l\\ b_1,b_2,\dots,b_l\end{smallmatrix}\right)\ ,
\end{aligned}\tag{3.4.31}$$

and then study it's derivative with respect to the modular parameter τ. Using a generating object allows to treat all TEMZVs simultaneously but more importantly the Fay relations among the doubly-periodic Ω are more compactly formulated than the corresponding quadratic relations among the $f^{(n)}$. Furthermore, the τ derivative of Ω is conveniently encapsulated in the following consequence of the mixed heat equation (3.3.10) for real z_i

$$\begin{aligned}
\partial_\tau\Omega(z_i-s_i-r_i\tau,\alpha_i;\tau) &= \exp(-2\pi i r_i\alpha_i)\partial_\tau F(z_i-s_i-r_i\tau,\alpha_i;\tau)\\
&= \exp(-2\pi i r_i\alpha_i)(-r_i\partial_{z_i}+\frac{1}{2\pi i}\partial_{\alpha_i}\partial_{z_i})F(z_i-s_i-r_i\tau,\alpha_i;\tau)\\
&= \frac{1}{2\pi i}\partial_{\alpha_i}\partial_{z_i}\Omega(z_i-s_i-r_i\tau,\alpha_i;\tau)\ .
\end{aligned}\tag{3.4.32}$$

The τ derivative of the generating function T may then be obtained by exchanging the derivative with the integration over the simplex, followed by using the Leibniz rule and the consequence of mixed heat equation (3.4.32) to replace derivatives with respect to τ by derivatives with respect to the α_i and the simplex coordinates z_i. Subsequently, the latter derivative may be taken care of using integration by parts. Finally, using the Fay identity (3.3.14) to rewrite products of two Ω depending on the same simplex coordinate z_i, we arrive at the differential equation

$$\begin{aligned}
2\pi i\partial_\tau\,\mathrm{T}\left[\begin{smallmatrix}\alpha_1,\alpha_2,\dots,\alpha_l\\ b_1,b_2,\dots,b_l\end{smallmatrix}\right] &= \partial_{\alpha_l}\Omega(-b_l,\alpha_l)\,\mathrm{T}\left[\begin{smallmatrix}\alpha_1,\dots,\alpha_{l-1}\\ b_1,\dots,b_{l-1}\end{smallmatrix}\right]-\partial_{\alpha_1}\Omega(-b_1,\alpha_1)\,\mathrm{T}\left[\begin{smallmatrix}\alpha_2,\dots,\alpha_l\\ b_2,\dots,b_l\end{smallmatrix}\right]\\
&+\sum_{i=2}^{l}\Big(\mathrm{T}\left[\begin{smallmatrix}\alpha_1,\dots\alpha_{i-2}\ ,\ \alpha_{i-1}+\alpha_i\ ,\ \alpha_{i+1},\dots\alpha_l\\ b_1,\dots b_{i-2}\ ,\ b_i\ ,\ b_{i+1},\dots b_l\end{smallmatrix}\right]\partial_{\alpha_{i-1}}\Omega(b_i-b_{i-1},\alpha_{i-1})\\
&\quad-\mathrm{T}\left[\begin{smallmatrix}\alpha_1,\dots\alpha_{i-2}\ ,\ \alpha_{i-1}+\alpha_i\ ,\ \alpha_{i+1},\dots\alpha_l\\ b_1,\dots b_{i-2}\ ,\ b_{i-1}\ ,\ b_{i+1},\dots b_l\end{smallmatrix}\right]\partial_{\alpha_i}\Omega(b_{i-1}-b_i,\alpha_i)\Big)\ .
\end{aligned}\tag{3.4.33}$$

Importantly, the rhs. only features generating functions of length $l-1$ TEMZVs. The above differential equation for the generating function then implies a differential equation for any given TEMZV by

$$2\pi i\partial_\tau\,\mathrm{T}\left[\begin{smallmatrix}\alpha_1,\alpha_2,\dots,\alpha_l\\ b_1,b_2,\dots,b_l\end{smallmatrix}\right] = \sum_{n_i\geq 0}\alpha_1^{n_1-1}\dots\alpha_r^{n_l-1}2\pi i\partial_\tau\,\omega\left(\begin{smallmatrix}n_1,\dots,n_l\\ b_1,\dots,b_l\end{smallmatrix}\right)\ ,\tag{3.4.34}$$

which may be explicitly extracted by equating coefficients of the monomials $\alpha_1^{n_1-1}\dots\alpha_l^{n_l-1}$ and is given by ($l \geq 2$)

$$\begin{aligned}2\pi i\partial_\tau \,\omega\left(\begin{smallmatrix} n_1,\ldots,n_l \\ b_1,\ldots,b_l\end{smallmatrix}\right) &= h^{(n_l+1)}(-b_l)\,\omega\left(\begin{smallmatrix} n_1,\ldots,n_{l-1} \\ b_1,\ldots,b_{l-1}\end{smallmatrix}\right) - h^{(n_1+1)}(-b_1)\,\omega\left(\begin{smallmatrix} n_2,\ldots,n_l \\ b_2,\ldots,b_l\end{smallmatrix}\right) \\ &+ \sum_{i=2}^{l}\Big[-\theta_{n_{i-1}\geq 1}\theta_{n_i\geq 1}\,\omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & 0 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & * & ,b_{i+1},\ldots b_l\end{smallmatrix}\right) h^{(n_{i-1}+n_i+1)}(b_i-b_{i-1})(-1)^{n_i} \\ &+ \theta_{n_i\geq 1}\sum_{k=0}^{n_{i-1}+1}\binom{n_i+k-1}{k}\,\omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & n_i+k & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l\end{smallmatrix}\right) h^{(n_{i-1}-k+1)}(b_i-b_{i-1}) \\ &- \theta_{n_{i-1}\geq 1}\sum_{k=0}^{n_i+1}\binom{n_{i-1}+k-1}{k}\,\omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & n_{i-1}+k & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_{i-1} & ,b_{i+1},\ldots b_l\end{smallmatrix}\right) h^{(n_i-k+1)}(b_{i-1}-b_i)\Big] \ ,\end{aligned} \tag{3.4.35}$$

where we introduced $h^{(n)} = (n-1)f^{(n)}$ and $\theta_{n\geq 1} = 1-\delta_{n,0}$. Again we stress that the rhs. of (3.4.35) consists exclusively of length $l-1$ TEMZVs. Therefore, we may compute the q dependence via an integral over lower length TEMZVs

$$\omega\left(\begin{smallmatrix} n_1 & ,\ \ldots\ , & n_l \\ b_1 & ,\ \ldots\ , & b_l\end{smallmatrix}\right) = c_0\left(\begin{smallmatrix} n_1 & ,\ \ldots\ , & n_l \\ b_1 & ,\ \ldots\ , & b_l\end{smallmatrix}\right) + \int_0^q \frac{\mathrm{d}\log(q')}{-4\pi^2}\Big[2\pi i\partial_{\tau'}\,\omega\left(\begin{smallmatrix} n_1 & ,\ \ldots\ , & n_l \\ b_1 & ,\ \ldots\ , & b_l\end{smallmatrix}\right)\Big](q')\ . \tag{3.4.36}$$

The above recursion eventually leads to the constant length one TEMZVs given in (3.4.2). Hence we eventually end up with iterated integrals in q over the weighting functions $f^{(n)}$ evaluated at some twist b. The crux of this whole method is that the weighting functions at some twist $f^{(n)}(b)$ are power series in fractional powers of q, which are easier to integrate over a simplex than trigonometric functions, as everything reduces to the integral

$$\int_0^q \frac{\mathrm{d}q'}{q'}(q')^a = \frac{q^a}{a}\ , \qquad a>0\ . \tag{3.4.37}$$

As the weighting functions $f^{(n)}(b)$ turn out to be the protagonists of this method, we now go on to provide some explicit formulas for the special cases relevant to the open-string amplitude. The case $f^{(n)}(0)$ is related to holomorphic Eisenstein series denoted $E_k(\tau)$; specifically, we have the relation

$$f^{(k)}(0) = -E_k(\tau) = \begin{cases} -2\zeta_k - \frac{2(2\pi i)^k}{(k-1)!}\sum\limits_{n,m=1}^{\infty} n^{k-1}q^{mn} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd}\end{cases}\ . \tag{3.4.38}$$

For the other case relevant to open-string amplitudes, i.e. $f^{(n)}(\tau/2)$ we find the expression

$$f^{(k)}(\tau/2) = \begin{cases} \frac{2^{k-1}-1}{2^{k-2}}\zeta_k - \frac{2(2\pi i)^k}{(k-1)!}\sum\limits_{n,m=1}^{\infty}(n-1/2)^{k-1}q^{m(n-1/2)} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd}\end{cases}\ , \tag{3.4.39}$$

cf. appendix E.5 for a derivation of this formula.

We conclude this exposition by illustrating the above algorithm for some simple examples relevant to the four-point genus-one open-string amplitude. At length two for example, we have $\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0\end{smallmatrix}\right)$, relevant to the double-trace contribution (see for example eq. (3.5.40)). The differential equation (3.4.35) for this TEMZV gives

$$2\pi i\partial_\tau\,\omega\left(\begin{smallmatrix}1\,,\,0\\ \frac{\tau}{2}\,,\,0\end{smallmatrix}\right) = -f^{(2)}(\frac{\tau}{2})\underbrace{\omega\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}_{=1}+\underbrace{f^{(0)}(\frac{\tau}{2})}_{=1}\underbrace{\omega\left(\begin{smallmatrix}2\\ \frac{\tau}{2}\end{smallmatrix}\right)}_{=\zeta_2}\,. \tag{3.4.40}$$

Then performing the integration, we arrive at

$$\begin{aligned}\omega\left(\begin{smallmatrix}1\,,\,0\\ \frac{\tau}{2}\,,\,0\end{smallmatrix}\right) &= c_0\left(\begin{smallmatrix}1\,,\,0\\ \frac{\tau}{2}\,,\,0\end{smallmatrix}\right)+\frac{1}{6}\int_0^q \frac{\mathrm{d}\log(q_1)}{-4\pi^2}\left[\pi^2-6f^{(2)}(\frac{\tau_1}{2})\right]\\ &= 2\sum_{n,m=1}^{\infty}\frac{q^{m(n-1/2)}}{m}\,,\end{aligned} \tag{3.4.41}$$

where the constant was computed to be zero in eq. (3.4.26). Similarly, we find

$$\begin{aligned}\omega\left(\begin{smallmatrix}1\,,\,0\\ 0\,,\,0\end{smallmatrix}\right) &= c_0\left(\begin{smallmatrix}1\,,\,0\\ 0\,,\,0\end{smallmatrix}\right)-\frac{1}{6}\int_0^q \frac{\mathrm{d}\log(q_1)}{-4\pi^2}\left[2\zeta_2+f^{(2)}(0)\right]\\ &= -\frac{i\pi}{2}+2\sum_{n,m=1}^{\infty}\frac{q^{mn}}{m}\,.\end{aligned} \tag{3.4.42}$$

Note that these two TEMZVs appear in the expression for $c_Q(q)$ eq. (3.2.8).

An example at length three relevant to the double-trace contributions of a cylindrical worldsheet (cf. eq. (3.5.42)) is the TEMZV $\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ 0\,,\,0\,,\,0\end{smallmatrix}\right)$. The corresponding τ derivative is given by

$$2\pi i\partial_\tau\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ 0\,,\,0\,,\,0\end{smallmatrix}\right) = -f^{(2)}(0)\,\omega\left(\begin{smallmatrix}1\,,\,0\\ 0\,,\,0\end{smallmatrix}\right)+f^{(0)}(0)\,\omega\left(\begin{smallmatrix}1\,,\,2\\ 0\,,\,0\end{smallmatrix}\right)\,, \tag{3.4.43}$$

where in addition to the already known $\omega\left(\begin{smallmatrix}1\,,\,0\\ 0\,,\,0\end{smallmatrix}\right)$ we need

$$2\pi i\partial_\tau\,\omega\left(\begin{smallmatrix}1\,,\,2\\ 0\,,\,0\end{smallmatrix}\right) = -3f^{(4)}(0)\underbrace{\omega\left(\begin{smallmatrix}0\\0\end{smallmatrix}\right)}_{=1}-f^{(2)}(0)\underbrace{\omega\left(\begin{smallmatrix}2\\0\end{smallmatrix}\right)}_{=-2\zeta_2}-2\underbrace{f^{(0)}(0)}_{=1}\underbrace{\omega\left(\begin{smallmatrix}4\\0\end{smallmatrix}\right)}_{=-2\zeta_4}\,. \tag{3.4.44}$$

Then keeping track of the order in which to integrate over the q_i (suppressing $f^{(0)}=1$) we find the expression

$$\begin{aligned}\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ 0\,,\,0\,,\,0\end{smallmatrix}\right) &= c_0\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ 0\,,\,0\,,\,0\end{smallmatrix}\right)+\int_0^q \frac{\mathrm{d}\log(q_2)}{-4\pi^2}\Big[c_0\left(\begin{smallmatrix}1\,,\,2\\ 0\,,\,0\end{smallmatrix}\right)-c_0\left(\begin{smallmatrix}1\,,\,0\\ 0\,,\,0\end{smallmatrix}\right)f^{(2)}(0;\tau_2)+\\ &\qquad +\int_0^{q_2}\frac{\mathrm{d}\log(q_1)}{-4\pi^2}\Big(4\zeta_4+2\zeta_2(f^{(2)}(0;\tau_1)+f^{(2)}(0;\tau_2))+\\ &\qquad\qquad +f^{(2)}(0;\tau_1)f^{(2)}(0;\tau_2)-3f^{(4)}(0;\tau_1)\Big)\Big]\\ &= -\frac{\pi^2}{12}+\int_0^q\frac{\mathrm{d}\log(q_2)}{-4\pi^2}\Big[\frac{i\pi^3}{6}+\frac{i\pi}{2}f^{(2)}(0;\tau_2)-\\ &\qquad -\sum_{n_1,m_1=1}^{\infty}\frac{4\pi^2}{m_1}q_2^{n_1m_1}\Big(n_1^2+4\sum_{n_2,m_2=1}^{\infty}n_2q_2^{n_2m_2}\Big)\Big]\end{aligned}$$

$$= -\frac{\pi^2}{12} + \sum_{n,m=1}^{\infty} \frac{(n - i\pi m)q^{nm}}{m^2} + 4 \sum_{n_i,m_i=1}^{\infty} \frac{n_2 q^{n_1 m_1 + n_2 m_2}}{m_1^2 n_1 + m_1 m_2 n_2} \, . \tag{3.4.45}$$

3.5 The four-point genus-one open-string amplitude as elliptic iterated integrals

We now go on to connect the subplots exposited in sections 3.2 and 3.3 and elucidate on how they happen to be part of a bigger narrative, i.e. how TEMZVs appear in the computation of the coefficients of the α'-expansion of the four-point genus-one open-string amplitude in superstring perturbation theory.[79] The single-trace contributions of cylindrical topology were originally given in terms of EMZVs in [50]. This picture was completed by the complementary study of TEMZVs for the double-trace terms in [1].

For the sake of a slick presentation we will restrict our exposition to the four-point amplitude, although there is no conceptual difficulty in extending the below treatment to higher point amplitudes.[80] The four-point amplitude is proportional to [45]

$$\begin{aligned} A_4^{1-\text{loop}} \sim \int_0^1 \frac{\mathrm{d}q}{q} \Big[&\operatorname{tr}(T_1T_2T_3T_4)(N I_{1234}(q) - 32 I_{1234}(-q)) \\ &+ \operatorname{tr}(T_1T_2)\operatorname{tr}(T_3T_4) I_{12|34}(q) + \operatorname{cyclic}(2,3,4) \Big] \, , \end{aligned} \tag{3.5.1}$$

where I_{1234} and $I_{12|34}$ denote worldsheet integrals with all vertex operator insertions on one boundary or two on each respectively. The two occurrences of I_{1234} are related to the two inequivalent configurations for single-trace contributions, i.e. cylinder topology with all insertions on one boundary and the Möbius strip. Note that the relative factor of N between these two is due to a trace $\operatorname{tr}(1) = N$ associated to the second boundary of the single-trace cylinder contribution. In principle we might also consider configurations with three vertex operator insertions one one boundary and the remaining one on its counterpart. Such contributions would be accompanied by traces of the form $\operatorname{tr}(T_1T_2T_3)\operatorname{tr}(T_4)$, which for traceless generators T_i vanish. However, the corresponding integrals $I_{123|4}$ prominently feature in the study of monodromy relations [161, 162]. In the subsequent we will study the worldsheet integrals $I_{1234}, I_{12|34}, I_{123|4}$ and give their explicit translation to TEMZVs up to third order in α', using the relations (3.2.5,3.2.7) between the Green function and $f^{(1)}$. Note that whenever we consider a Green function with insertions on the same boundary, we have to take care of divergences by introducing a regulator as in (3.2.5). In order to not unnecessarily clutter the notation, we do not explicitly spell out this regularization, but consider it to be implied where necessary.

3.5.1 Cylinder topology – single-trace contributions

When it comes to the relation of string amplitudes to TEMZVs, the easiest case to study is the single-trace terms of the four-point amplitude originally studied in [50], whose results we briefly

[79] Note that in the literature often the following terminology is used: single-trace contributions from the worldsheet integral of cylinder topology are referred to as planar cylinder contributions, while double-trace contributions are called non-planar in this context.

[80] We note that there are partial results for the single-trace contributions for the five point genus-one case in terms of EMZVs in [50].

summarize here for the sake of completeness. Recall that the worldsheet integral of cylindrical topology with insertions on one boundary, is given by

$$I_{1234} = \int\limits_{1234} [\mathrm{d}z] \exp\left(\sum_{i<j} s_{ij}\tilde{P}(z_{ij})\right) , \tag{3.5.2}$$

where $\tilde{P}(z) = \frac{1}{2}G_1(z)|_{\mathrm{Im}(z)=0} - c_P$ (cf. eq. (3.2.5)). The subtraction of c_P leads to simpler expressions in the α'-expansion below, while not altering the corresponding expressions at a given order in α', as c_P is multiplied by $\sum_{i<j} s_{ij} = 0$. Note that by subtracting c_P, individual contributions of the form

$$\int\limits_{1234} [\mathrm{d}z] \prod_{i<j} \tilde{P}(z_{ij})^{n_{ij}} , \tag{3.5.3}$$

acquire imaginary parts, which will cancel out among all iterated integrals contributing to the same monomial in the s_{ij}.

The integration is over the coordinates of the ordered boundary insertions up to the action of the conformal Killing group, which may be used to fix the coordinate of one insertion, where we choose $z_1 = 0$. With this choice the integration is then over the coordinates of the remaining insertions $0 = z_1 < z_2 < z_3 < z_4 < 1$, taking the form

$$\int\limits_{1234} [\mathrm{d}z] = \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \int_0^{z_2} \mathrm{d}z_1 \delta(z_1) . \tag{3.5.4}$$

Furthermore, we note that several integrals may be identified due to the invariance of the measure under cyclic index shifts, e.g. we find the equivalence

$$\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12}) = \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{23}) . \tag{3.5.5}$$

Note that the choice (3.5.4) will make such relations rather opaque as we generally find equivalent integrals to be given by linear combinations of TEMZVs of a priori distinct appearances, but they will turn out to be equal by virtue of relations among TEMZVs, which we briefly discuss for the case of the example (3.5.5). To that end we recall how to treat Green functions depending on z_{ji} with $i, j \neq 1$, i.e. by considering the identity of integrals

$$\tilde{P}(z_{ij}) = \lim_{\varepsilon\to 0} \mathrm{Reg} \int_{\varepsilon}^{z_{ij}} \mathrm{d}y f^{(1)}(y) = \lim_{\varepsilon\to 0} \mathrm{Reg} \int_{\varepsilon}^{z_j} \mathrm{d}w f^{(1)}(w) + \int_0^{z_i} \mathrm{d}w f^{(1)}(w - z_j) , \tag{3.5.6}$$

valid as long $z_i < z_j$ such that we do not integrate over the simple pole of $f^{(1)}$. Then, using the shuffle relation and reflection (3.3.24), we find

$$\int\limits_{1234} [\mathrm{d}z]\tilde{P}(z_{23}) = \int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \Big[\Gamma\left(\begin{smallmatrix}0\\0\end{smallmatrix}; z_3\right) \Gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix}; z_3\right) + \Gamma\left(\begin{smallmatrix}0 & 1\\0 & z_3\end{smallmatrix}; z_3\right) \Big] = \omega\left(\begin{smallmatrix}1 & ; & 0 & ; & 0 & ; & 0\\0 & ; & 0 & ; & 0 & ; & 0\end{smallmatrix}\right) , \tag{3.5.7}$$

which is equal to

$$\int\limits_{1234} [\mathrm{d}z]\tilde{P}(z_{12}) = \int\limits_0^1 \mathrm{d}z_4 \int\limits_0^{z_4} \mathrm{d}z_3 \int\limits_0^{z_3} \mathrm{d}z_2\, \Gamma\left(\begin{smallmatrix}1\\0\end{smallmatrix}; z_2\right) = \omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) , \tag{3.5.8}$$

i.e. the left hand side of eq. (3.5.5).

Using the aforementioned invariance under cyclic shifts of indices we may classify all inequivalent integrals at a given order in the α'-expansion. At first order we have

$$c_{1;1} = \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12}) , \quad c_{1;2} = \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{13}) . \tag{3.5.9}$$

Then at second order there are six different integrals

$$\begin{aligned}
c_{2;1} &= \frac{1}{2}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})^2 , & c_{2;2} &= \frac{1}{2}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{13})^2 ,\\
c_{2;3} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{14}) , & c_{2;4} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{13})\tilde{P}(z_{24}) ,\\
c_{2;5} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{34}) , & c_{2;6} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13}) ,
\end{aligned} \tag{3.5.10}$$

and finally at third order we get the twelve integrals

$$\begin{aligned}
c_{3;1} &= \frac{1}{6}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})^3 , & c_{3;2} &= \frac{1}{6}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{13})^3 ,\\
c_{3;3} &= \frac{1}{2}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{P}(z_{23}) , & c_{3;4} &= \frac{1}{2}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{13})^2\tilde{P}(z_{24}) ,\\
c_{3;5} &= \frac{1}{2}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{P}(z_{34}) , & c_{3;6} &= \frac{1}{2}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{P}(z_{13}) ,\\
c_{3;7} &= \frac{1}{2}\int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13})^2 , & c_{3;8} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{23})\tilde{P}(z_{34}) ,\\
c_{3;9} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13})\tilde{P}(z_{23}) , & c_{3;10} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13})\tilde{P}(z_{14}) ,\\
c_{3;11} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13})\tilde{P}(z_{34}) , & c_{3;12} &= \int\limits_{1234} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13})\tilde{P}(z_{24}) .
\end{aligned} \tag{3.5.11}$$

According to the classification above, at order α' the two relevant integrals are given by the TEMZVs

$$c_{1;1} = \omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) , \tag{3.5.12}$$

$$c_{1;2} = \omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\ . \tag{3.5.13}$$

The translation of the higher order contributions into TEMZVs is generally more involved but completely algorithmic. At second order we have

$$c_{2;1} = \omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.14}$$

$$c_{2;2} = \omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}0\,,\,1\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.15}$$

$$c_{2;3} = -\,\omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.16}$$

$$c_{2;4} = 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) - \omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,1\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.17}$$

$$c_{2;5} = -\,\omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.18}$$

$$c_{2;6} = 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\ . \tag{3.5.19}$$

Note the relations $c_{2;5} = c_{2;3}$, $2c_{2;6} = c_{2;3} + c_{2;4}$ where the latter may be shown using shuffle relations.[81] Finally, at third order

$$c_{3;1} = \omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.20}$$

$$\begin{aligned}c_{3;2} = {}&\omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,0\,,\,1\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\\ &+ \omega\left(\begin{smallmatrix}0\,,\,1\,,\,1\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\end{aligned} \tag{3.5.21}$$

$$c_{3;3} = -\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.22}$$

$$\begin{aligned}c_{3;4} = {}&6\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + 3\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,0\,,\,1\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\\ &+ \omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\end{aligned} \tag{3.5.23}$$

$$c_{3;5} = -\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.24}$$

$$c_{3;6} = 3\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.25}$$

$$c_{3;7} = 3\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}1\,,\,0\,,\,1\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) \tag{3.5.26}$$

$$\begin{aligned}c_{3;8} = {}&2\,\omega\left(\begin{smallmatrix}2\,,\,0\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}0\,,\,2\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\\ &- \zeta_2\,\omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\end{aligned} \tag{3.5.27}$$

$$\begin{aligned}c_{3;9} = {}&2\,\omega\left(\begin{smallmatrix}2\,,\,0\,,\,0\,,\,0\,,\,1\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix}2\,,\,0\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + \omega\left(\begin{smallmatrix}0\,,\,2\,,\,0\,,\,0\,,\,1\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\\ &+ \omega\left(\begin{smallmatrix}0\,,\,2\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,1\,,\,0\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\,,\,0\,,\,0\,,\,1\\0\,,\,0\,,\,0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\\ &- \zeta_2\,\omega\left(\begin{smallmatrix}1\,,\,0\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) - \zeta_2\,\omega\left(\begin{smallmatrix}0\,,\,1\,,\,0\,,\,0\\0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\end{aligned} \tag{3.5.28}$$

[81] These may be found e.g. via integration by parts of five point integrals as explained in [50].

$$c_{3;10} = -2\,\omega\begin{pmatrix}1\,,1\,,0\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} - \omega\begin{pmatrix}1\,,0\,,1\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} \tag{3.5.29}$$

$$c_{3;11} = -2\,\omega\begin{pmatrix}1\,,1\,,0\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} - \omega\begin{pmatrix}1\,,0\,,1\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} \tag{3.5.30}$$

$$\begin{aligned}c_{3;12} = -2\,\omega\begin{pmatrix}2\,,0\,,0\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} - \omega\begin{pmatrix}0\,,2\,,0\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} - 2\,\omega\begin{pmatrix}1\,,0\,,1\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix}\\ - 2\,\omega\begin{pmatrix}1\,,1\,,0\,,0\,,0\,,1\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} + \zeta_2\,\omega\begin{pmatrix}1\,,0\,,0\,,0\\0\,,0\,,0\,,0\end{pmatrix}\,.\end{aligned} \tag{3.5.31}$$

The appearances of terms like ζ_2 multiplied by some shorter length TEMZVs as well as the combined letters $n_i = 2, b_i = 0$ in $c_{3;8}, c_{3;9}, c_{3;12}$ are due to the endpoint removal identities as discussed in section 3.3 (cf. also appendix E.3).

With the above inequivalent integrals we can express the expansion of the worldsheet integral (3.5.2) as

$$\begin{aligned}I_{1234} =& \frac{1}{6} + 2(s_{12}+s_{23})\big[c_{1;1} - c_{1;2}\big]\\ &+ (s_{12}^2+s_{23}^2)\big[2c_{2;1}+2c_{2;2}-c_{2;3}-c_{2;4}\big] + s_{12}s_{23}\big[4c_{2;2}-2c_{2;4}\big]\\ &+ (s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3)\big[2c_{3;1}-2c_{3;2}+6c_{3;3}+2c_{3;4}-8c_{3;6}-2c_{3;8}+2c_{3;10}\big]\\ &+ s_{12}s_{23}(s_{12}+s_{23})\big[-4c_{3;1}-2c_{3;2}+4c_{3;3}+2c_{3;4}-4c_{3;9}+2c_{3;10}\big] + \mathcal{O}(\alpha'^4)\,,\end{aligned} \tag{3.5.32}$$

where the absence of certain integrals is due to additional relations among these integrals as mentioned above. Then using the above results and several identities among TEMZVs we arrive at the expression

$$\begin{aligned}I_{1234} =& \frac{1}{6} + (s_{12}+s_{23})\left[-2\,\omega\begin{pmatrix}0\,,1\,,0\,,0\\0\,,0\,,0\,,0\end{pmatrix}\right] + (s_{12}^2+s_{23}^2)\left[\frac{\zeta_2}{6} + 2\,\omega\begin{pmatrix}2\,,0\,,0\,,0\,,0\\0\,,0\,,0\,,0\,,0\end{pmatrix}\right]\\ &+ s_{12}s_{23}\left[-2\,\omega\begin{pmatrix}0\,,2\,,0\,,0\,,0\\0\,,0\,,0\,,0\,,0\end{pmatrix}\right] + (s_{12}^3+2s_{12}^2s_{23}+2s_{12}s_{23}^2+s_{23}^3)\beta_5\\ &+ s_{12}s_{23}(s_{12}+s_{23})\beta_{2,3} + \mathcal{O}(\alpha'^4)\,,\end{aligned} \tag{3.5.33}$$

with the shorthands

$$\begin{aligned}\beta_5 &= \frac{4}{3}\left[\omega\begin{pmatrix}0\,,0\,,1\,,0\,,0\,,2\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} + \omega\begin{pmatrix}0\,,1\,,1\,,0\,,1\,,0\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} - \omega\begin{pmatrix}2\,,0\,,1\,,0\,,0\,,0\\0\,,0\,,0\,,0\,,0\,,0\end{pmatrix} - \zeta_2\,\omega\begin{pmatrix}0\,,1\,,0\,,0\\0\,,0\,,0\,,0\end{pmatrix}\right]\\ \beta_{2;3} &= \frac{\zeta_3}{12} + \frac{8\zeta_2}{3}\,\omega\begin{pmatrix}0\,,1\,,0\,,0\\0\,,0\,,0\,,0\end{pmatrix} - \frac{5}{18}\,\omega\begin{pmatrix}0\,,3\,,0\,,0\\0\,,0\,,0\,,0\end{pmatrix}\,.\end{aligned} \tag{3.5.34}$$

3.5.2 Cylinder topology – double-trace contributions

In the previous subsection EMZVs were sufficient to express every term in the α'-expansion. We will now go on to consider the double-trace contributions were TEMZVs with non-zero twists appear. The worldsheet integral of cylindrical topology with insertions on both boundaries, is given by[82]

[82] Note that in an instance of notational abuse we denote the integration measure also as $[\mathrm{d}z]$ as in the previous subsection, while those are strictly speaking not the same. However the exact meaning of $[\mathrm{d}z]$ is made precise by the attached $\int_{B_1}^{B_2}$ such that no confusion should arise.

$$I_{12|34} = e^{-2s_{12}(c_Q(q)-c_P)} \int_{12}^{34} [\mathrm{d}z] \exp\Big(s_{12}\tilde{P}(z_{12}) + s_{34}\tilde{P}(z_{34}) + s_{13}\tilde{Q}(z_{13}) + \\ + s_{14}\tilde{Q}(z_{14}) + s_{23}\tilde{Q}(z_{23}) + s_{24}\tilde{Q}(z_{24})\Big) \,, \tag{3.5.35}$$

where we introduced $\tilde{Q}(z) = \frac{1}{2}G_1(z)|_{\mathrm{Im}(z)=\pm t/2} - c_Q(q)$ in order to pull the factor $e^{-2s_{12}(c_Q(q)-c_P)}$ out of the integral. This factor may be rewritten as

$$e^{-2s_{12}(c_Q(q)-c_P)} = q^{\frac{s_{12}}{4}} \exp\Big(-2s_{12}\Big[\omega\left(\begin{smallmatrix}1\,,\,0\\0\,,\,0\end{smallmatrix}\right) - \omega\left(\begin{smallmatrix}1\,,\,0\\\frac{\tau}{2}\,,\,0\end{smallmatrix}\right)\Big]\Big) \,, \tag{3.5.36}$$

which may be seen by comparing eq. (3.2.8) with eqs. (3.4.41,3.4.42). Note also that the factor $q^{s_{12}/4}$ results in poles upon integrating over $\mathrm{d}\log(q)$. These poles are interpreted as closed-string exchange as their contribution is $\sim 1/s_{12}$ upon integrating over q.

In this case the boundary insertions are not ordered, which is a simplification due to the restriction to the four-point amplitude. Because the imaginary part of the coordinates of the boundary punctures is constant (either zero or $\tau/2$), the integration is over the real parts of said coordinates only (up to the action of the conformal Killing group). Again we may use the conformal Killing group to fix the coordinate of one insertion, where we again choose $z_1 = 0$. With the aforementioned choice the integration is over the remaining insertions, i.e. $z_1 = 0$ and $z_2, z_3, z_4 \in [0,1]$, and we may write

$$\int_{12}^{34} [\mathrm{d}z] = \int_0^1 \mathrm{d}z_4 \int_0^1 \mathrm{d}z_3 \int_0^1 \mathrm{d}z_2 \int_0^1 \mathrm{d}z_1 \delta(z_1) \,. \tag{3.5.37}$$

In this case the integral is invariant under the action of $\mathbb{Z}_2$ on each pair of insertions sharing a boundary, as well as under exchanging those pairs. Then again we may classify integrals to be inequivalent if we cannot relate them by the aforementioned $(\mathbb{Z}_2)^3$.

Before we give the translation of the relevant integrals, we note that the integration region as explained above leads to some issues when translating the coefficients of the α'-expansion to TEMZVs. Specifically, the main issue is treating Green functions depending on two coordinates, which are not set to zero e.g. $\tilde{P}(z_{34})$. In contrast to the single-trace case the application of eq. (3.5.6) causes problems as the coordinates are no longer ordered and we therefore might need to integrate over singularities. However, we may solve this problem by decomposing the integration region such that eq. (3.5.6) will not run into any problems. Specifically, if e.g. $\tilde{P}(z_{34})$ occurs we decompose the integration region as[83]

$$\{0 \le z_2, z_3, z_4 \le 1\} \sim \{0 \le z_2 \le 1; 0 \le z_3 \le z_4 \le 1\} \cup \{0 \le z_2 \le 1; 0 \le z_4 \le z_3 \le 1\} \,. \tag{3.5.38}$$

Then on each constituent of the union on the rhs. we can use eq. (3.5.6) and the techniques discussed in 3.3 for removing any endpoint-dependent twist. This will be illustrated for a simple example below, while a more involved computation relevant to the third order in the α' example may be found in appendix E.6.

As in the case of single-trace contributions above, we start by addressing the combinatoric

[83] The $\sim$ is used to denote equality up to regions of codimension at least one, which are of measure zero for the integral and hence do not contribute.

problem of classifying the relevant inequivalent integrals. Again those may be found by computing the orbits of the discrete symmetry of the configuration of boundary insertions, which in this case is $(\mathbb{Z}_2)^3$. Eventually, we find at first order in α'

$$d_{1;1} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12}) \ , \quad d_{1;2} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13}) \ ,$$

at second order

$$d_{2;1} = \frac{1}{2}\int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})^2 \ , \qquad d_{2;2} = \frac{1}{2}\int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13})^2 \ ,$$

$$d_{2;3} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{13}) \ , \qquad d_{2;4} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{34}) \ ,$$

$$d_{2;5} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13})\tilde{Q}(z_{14}) \ , \qquad d_{2;6} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13})\tilde{Q}(z_{24}) \ ,$$

and finally at third order

$$d_{3;1} = \frac{1}{6}\int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})^3 \ , \qquad d_{3;2} = \frac{1}{6}\int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13})^3 \ ,$$

$$d_{3;3} = \frac{1}{2}\int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{Q}(z_{13}) \ , \qquad d_{3;4} = \frac{1}{2}\int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{13})^2 \ ,$$

$$d_{3;5} = \frac{1}{2}\int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{P}(z_{34}) \ , \qquad d_{3;6} = \frac{1}{2}\int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13})^2\tilde{Q}(z_{14}) \ ,$$

$$d_{3;7} = \frac{1}{2}\int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13})^2\tilde{Q}(z_{24}) \ , \qquad d_{3;8} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{34})\tilde{Q}(z_{13}) \ ,$$

$$d_{3;9} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{13})\tilde{Q}(z_{24}) \ , \qquad d_{3;10} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{13})\tilde{Q}(z_{14}) \ ,$$

$$d_{3;11} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{34})\tilde{Q}(z_{13})\tilde{Q}(z_{14}) \ , \qquad d_{3;12} = \int_{12}^{34} [\mathrm{d}z]\, \tilde{Q}(z_{13})\tilde{Q}(z_{14})\tilde{Q}(z_{23}) \ .$$

The corresponding TEMZVs at first order in α' are given by

$$d_{1;1} = \int_0^1 \mathrm{d}z_2 \int_0^{x_2} \mathrm{d}y\ f^{(1)}(y) = \omega\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix};\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right) \ , \tag{3.5.39}$$

$$d_{1;2} = \int_0^1 \mathrm{d}z_3 \int_0^{x_3} \mathrm{d}y\, f^{(1)}(y-\tau/2) = \omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) \,. \tag{3.5.40}$$

Again we note that it should not matter whether we integrate $\tilde{P}(z_{12})$ or $\tilde{P}(z_{34})$ over the boundary insertions, but having used translation invariance this equality is not at all manifest. Specifically, the Green function may again be expressed via (3.3.29), but we do not have an ordering of the boundary insertions. Therefore, in order to avoid poles inside the integration domain we decompose the integration region into the regions $0 < z_3 < z_4 < 1$ and $0 < z_4 < z_3 < 1$ such that any possible pole is at the endpoint of the corresponding intermediate integrations step.[84] Then we compute

$$\begin{aligned}
\int_{12}^{34} P(z_{34}) &= -\int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \int_{z_3}^{z_4} \mathrm{d}y f^{(1)}(y-z_4) + (z_3 \leftrightarrow z_4) \\
&= -\int_0^1 \mathrm{d}z_4 \int_0^{z_4} \mathrm{d}z_3 \Big(\underbrace{\Gamma\left(\begin{smallmatrix} 1 \\ z_4 \end{smallmatrix}; z_4\right)}_{=-\Gamma\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}; z_4\right)} - \Gamma\left(\begin{smallmatrix} 1 \\ z_4 \end{smallmatrix}; z_3\right) \Big) + (z_3 \leftrightarrow z_4) \\
&= 2\,\omega\left(\begin{smallmatrix} 1 & , & 0 & , & 0 \\ 0 & , & 0 & , & 0 \end{smallmatrix}\right) + \Big[\int_0^1 \mathrm{d}z_4\, \underbrace{\Gamma\left(\begin{smallmatrix} 0 & , & 1 \\ 0 & , & z_4 \end{smallmatrix}; z_4\right)}_{=-\Gamma\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}; z_4\right)} + (z_3 \leftrightarrow z_4) \Big] \\
&= \omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right) - 2\,\underbrace{\omega\left(\begin{smallmatrix} 0 & , & 1 & , & 0 \\ 0 & , & 0 & , & 0 \end{smallmatrix}\right)}_{=0} = d_{1;1} \,,
\end{aligned} \tag{3.5.41}$$

where the reflection identity (3.3.24) is required to show equivalence of this result to the TEMZV of (3.5.39). By similar computations one may show that all other choices of $\tilde{Q}(z_{ij})$ in the second integral have the same result, although it is not obvious from the integral expressions. At second order in α' there are a priori six inequivalent configurations, although it will turn out that two integrals are equal. Specifically, the integrals are given by

$$d_{2;1} = \omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ 0 & , & 0 & , & 0 \end{smallmatrix}\right) \,, \tag{3.5.42}$$

$$d_{2;2} = \omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ \frac{\tau}{2} & , & \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) \,, \tag{3.5.43}$$

$$d_{2;3} = d_{1;1} d_{1;2} = \omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right) \omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) \,, \tag{3.5.44}$$

$$d_{2;4} = (d_{1;1})^2 = \left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right)\right)^2 \,, \tag{3.5.45}$$

$$d_{2;5} = (d_{1;2})^2 = \left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\right)^2 \,, \tag{3.5.46}$$

$$d_{2;6} = (d_{1;2})^2 = \left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\right)^2 = d_{2;5} \,. \tag{3.5.47}$$

[84] Note that this corresponds to a Cauchy principal value prescription for the logarithmic divergence on the diagonal.

At third order we have

$$d_{3;1} = \omega\left(\begin{smallmatrix} 1 & , & 1 & , & 1 & , & 0 \\ 0 & , & 0 & , & 0 & , & 0 \end{smallmatrix}\right) , \tag{3.5.48}$$

$$d_{3;2} = \omega\left(\begin{smallmatrix} 1 & , & 1 & , & 1 & , & 0 \\ \frac{\tau}{2} & , & \frac{\tau}{2} & , & \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) , \tag{3.5.49}$$

$$d_{3;3} = d_{2;1}d_{1;2} = \omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ 0 & , & 0 & , & 0 \end{smallmatrix}\right)\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) , \tag{3.5.50}$$

$$d_{3;4} = d_{1;1}d_{2;2} = \omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right)\omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ \frac{\tau}{2} & , & \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) , \tag{3.5.51}$$

$$d_{3;5} = d_{1;1}d_{2;1} = \omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ 0 & , & 0 & , & 0 \end{smallmatrix}\right)\omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right) , \tag{3.5.52}$$

$$d_{3;6} = d_{1;2}d_{2;2} = \omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ \frac{\tau}{2} & , & \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) , \tag{3.5.53}$$

$$d_{3;7} = d_{1;2}d_{2;2} = d_{3;6} = \omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ \frac{\tau}{2} & , & \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) , \tag{3.5.54}$$

$$d_{3;8} = d_{1;1}d_{2;3} = (d_{1;1})^2 d_{1;2} = \left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right)\right)^2\left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\right) , \tag{3.5.55}$$

$$d_{3;9} = d_{1;2}d_{2;3} = d_{1;1}(d_{1;2})^2 = \omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right)\left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\right)^2 , \tag{3.5.56}$$

$$d_{3;10} = d_{1;1}(d_{1;2})^2 = d_{3;9} = \omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right)\left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\right)^2 , \tag{3.5.57}$$

$$\begin{aligned} d_{3;11} = {} & 2\,\omega\left(\begin{smallmatrix} 2 & , & 0 & , & 0 & , & 0 & , & 1 \\ 0 & , & 0 & , & 0 & , & 0 & , & \frac{\tau}{2} \end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix} 2 & , & 0 & , & 0 & , & 0 & , & 1 \\ \frac{\tau}{2} & , & 0 & , & 0 & , & 0 & , & \frac{\tau}{2} \end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix} 0 & , & 2 & , & 0 & , & 0 & , & 1 \\ 0 & , & \frac{\tau}{2} & , & 0 & , & 0 & , & \frac{\tau}{2} \end{smallmatrix}\right) \\ & - 2\,\omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 & , & 0 & , & 1 \\ 0 & , & \frac{\tau}{2} & , & 0 & , & 0 & , & \frac{\tau}{2} \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 & , & 0 & , & 1 \\ \frac{\tau}{2} & , & \frac{\tau}{2} & , & 0 & , & 0 & , & \frac{\tau}{2} \end{smallmatrix}\right) , \end{aligned} \tag{3.5.58}$$

$$d_{3;12} = d_{1;2}d_{2;5} = (d_{1;2})^3 = \left(\omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)\right)^3 . \tag{3.5.59}$$

Then upon expansion of the integrand in eq. (3.5.35) and subsequently using momentum conservation and several identities among the $d_{i;j}$ found above (e.g. $d_{2;5} = d_{2;6}$), we obtain

$$\begin{aligned} q^{-\frac{s_{12}}{4}} I_{12|34} = {} & \exp\Big(-2s_{12}(d_{1;1} - d_{1;2})\Big)\Big(1 \; + \; 2s_{12}(d_{1;1} - d_{1;2}) \\ & + \; s_{12}^2(2d_{2;1} + 2d_{2;2} - 4d_{2;3} + d_{2;4} + d_{2;5}) + \; s_{13}s_{23}(2d_{2;5} - 4d_{2;2}) \\ & + \; 2s_{12}^3(d_{3;1} - d_{3;2} - 2d_{3;3} + 2d_{3;4} + d_{3;5} - d_{3;7} - d_{3;8} + d_{3;9}) \\ & + \; s_{12}s_{13}s_{23}(d_{3;12} - 3d_{3;2} + 4d_{3;4} - d_{3;7} - 2d_{3;11}) \; + \; \mathcal{O}(\alpha'^4)\Big) . \end{aligned} \tag{3.5.60}$$

From this expression we infer that the order α' contribution of $q^{-s_{12}/4}I_{12|34}(q)$ vanishes in accordance with [162]. The corresponding contributions at second order in α' are given by

$$\begin{aligned} q^{-\frac{s_{12}}{4}} I_{12|34}\big|_{s_{12}^2} &= 2\,\omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ 0 & , & 0 & , & 0 \end{smallmatrix}\right) - \omega\left(\begin{smallmatrix} 1 & , & 0 \\ 0 & , & 0 \end{smallmatrix}\right)^2 + 2\,\omega\left(\begin{smallmatrix} 1 & , & 1 & , & 0 \\ \frac{\tau}{2} & , & \frac{\tau}{2} & , & 0 \end{smallmatrix}\right) - \omega\left(\begin{smallmatrix} 1 & , & 0 \\ \frac{\tau}{2} & , & 0 \end{smallmatrix}\right)^2 \\ &= \frac{7\zeta_2}{6} + 2\,\omega\left(\begin{smallmatrix} 0 & , & 0 & , & 2 \\ 0 & , & 0 & , & 0 \end{smallmatrix}\right) , \end{aligned} \tag{3.5.61}$$

$$q^{-\frac{s_{12}}{4}} I_{12|34}\big|_{s_{13}s_{23}} = 2\,\omega\left(\begin{smallmatrix}1\,,\,0\\ \frac{\tau}{2}\,,\,0\end{smallmatrix}\right)^2 - 4\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ \frac{\tau}{2}\,,\,\frac{\tau}{2}\,,\,0\end{smallmatrix}\right) = -2\,\omega\left(\begin{smallmatrix}0\,,\,0\,,\,2\\ 0\,,\,0\,,\,0\end{smallmatrix}\right) - \frac{2\zeta_2}{3}\,, \tag{3.5.62}$$

and finally, the third order contributions are

$$\begin{aligned} q^{-\frac{s_{12}}{4}} I_{12|34}\big|_{s_{12}^3} &= \frac{2}{3}\omega\left(\begin{smallmatrix}1\,,\,0\\ 0\,,\,0\end{smallmatrix}\right)^3 - 2\,\omega\left(\begin{smallmatrix}1\,,\,0\\ 0\,,\,0\end{smallmatrix}\right)\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ 0\,,\,0\,,\,0\end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\\ &\quad - \frac{2}{3}\omega\left(\begin{smallmatrix}1\,,\,0\\ \frac{\tau}{2}\,,\,0\end{smallmatrix}\right)^3 + 2\,\omega\left(\begin{smallmatrix}1\,,\,0\\ \frac{\tau}{2}\,,\,0\end{smallmatrix}\right)\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ \frac{\tau}{2}\,,\,\frac{\tau}{2}\,,\,0\end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\\ \frac{\tau}{2}\,,\,\frac{\tau}{2}\,,\,\frac{\tau}{2}\,,\,0\end{smallmatrix}\right)\\ &= -4\zeta_2\,\omega\left(\begin{smallmatrix}0\,,\,1\,,\,0\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\,, \end{aligned} \tag{3.5.63}$$

$$\begin{aligned} q^{-\frac{s_{12}}{4}} I_{12|34}\big|_{s_{12}s_{23}s_{13}} &= \frac{4}{3}\omega\left(\begin{smallmatrix}0\,,\,3\,,\,0\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix}1,0\\ \frac{\tau}{2},0\end{smallmatrix}\right)^3 - 6\,\omega\left(\begin{smallmatrix}1\,,\,0\\ \frac{\tau}{2}\,,\,0\end{smallmatrix}\right)\omega\left(\begin{smallmatrix}1\,,\,1\,,\,0\\ \frac{\tau}{2}\,,\,\frac{\tau}{2}\,,\,0\end{smallmatrix}\right)\\ &\quad + 6\,\omega\left(\begin{smallmatrix}1\,,\,1\,,\,1\,,\,0\\ \frac{\tau}{2}\,,\,\frac{\tau}{2}\,,\,\frac{\tau}{2}\,,\,0\end{smallmatrix}\right)\\ &= \frac{5}{3}\omega\left(\begin{smallmatrix}0\,,\,3\,,\,0\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + 4\zeta_2\,\omega\left(\begin{smallmatrix}0\,,\,1\,,\,0\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) - \frac{1}{2}\zeta_3\,, \end{aligned} \tag{3.5.64}$$

where several relations among TEMZVs have been used. Collecting the contributions we have for the α'-expansion of the worldsheet integral

$$\begin{aligned} q^{-\frac{s_{12}}{4}} I_{12|34} &= 1 + s_{12}^2\left[\frac{\zeta_2}{6} - \omega\left(\begin{smallmatrix}0\,,\,2\,,\,0\\ 0\,,\,0\,,\,0\end{smallmatrix}\right)\right] + s_{13}s_{23}\left[\frac{\zeta_2}{3} + \omega\left(\begin{smallmatrix}0\,,\,2\,,\,0\\ 0\,,\,0\,,\,0\end{smallmatrix}\right)\right] - 4s_{12}^3\,\zeta_2\,\omega\left(\begin{smallmatrix}0\,,\,1\,,\,0\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right)\\ &\quad + s_{12}s_{13}s_{23}\left[\frac{5}{3}\omega\left(\begin{smallmatrix}0\,,\,3\,,\,0\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) + 4\zeta_2\,\omega\left(\begin{smallmatrix}0\,,\,1\,,\,0\,,\,0\\ 0\,,\,0\,,\,0\,,\,0\end{smallmatrix}\right) - \frac{1}{2}\zeta_3\right] + \mathcal{O}(\alpha'^4)\,. \end{aligned} \tag{3.5.65}$$

From this equation we may extract the q^0 term of the worldsheet integral, using the constant terms of the TEMZVs as given in (3.4.28)

$$I_{12|34} = q^{s_{12}/4}\Big[1 + \frac{1}{2}\zeta_2 s_{12}^2 - \frac{1}{2}\zeta_3 s_{12}^3 + \mathcal{O}(q,\alpha'^4)\Big]\,. \tag{3.5.66}$$

This result is consistent with the all-order expression given in [162]

$$I_{12|34} = \frac{2^{2s_{12}}q^{s_{12}/4}}{\pi}\left(\frac{\Gamma(\frac{1}{2}+\frac{s_{12}}{2})}{\Gamma(1+\frac{s_{12}}{2})}\right)^2 + \mathcal{O}(q)\,. \tag{3.5.67}$$

We note that the contributions given above do not contain any TEMZV with non-zero twists implying that the contributions can be expanded in positive integer powers of q only. In fact this is an important observation as it signifies the absence of unphysical poles in the string amplitude after integration over q. Specifically, as we argued above the worldsheet integral $I_{12|34}$ may generally be expressed by TEMZVs with twists $b_i \in \{0, \tau/2\}$, apart from the prefactor $q^{s_{12}/4}$. Hence, $I_{12|34}$ admits an expansion in powers of $q^{1/2}$ à la

$$I_{12|34} = q^{s_{12}/4}\sum_{n=0}^{\infty}(a_n q^n + c_n q^{n+1/2})\,, \tag{3.5.68}$$

where the expansion coefficients are formal power series in $\alpha' s_{ij}$ with coefficients consisting of $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs. Hence, the integration of $I_{12|34}$ over the modular parameter can be schematically expressed as

$$\int_0^1 \frac{\mathrm{d}q}{q}\, I_{12|34} = \sum_{n=0}^{\infty} \left\{ \frac{4a_n}{s_{12}+4n} + \frac{4c_n}{s_{12}+4n+2} \right\} . \tag{3.5.69}$$

This expression leads to kinematic poles at $s_{12} = -4n$ and $s_{12} = -4n-2$ stemming from the integer and half-integer powers of the q-expansion of $I_{12|34}$, respectively. Such poles of s_{12} correspond to internal masses of the closed-string exchange with values $m^2 = \frac{4n}{\alpha'}$ and $m^2 = \frac{4n+2}{\alpha'}$. However, it is known that in GSO projected string theories the closed-superstring spectrum only contains masses $m^2 = \frac{4n}{\alpha'}$ but not $m^2 = \frac{4n+2}{\alpha'}$, as explained in section 2.4; cf. also [87, 88, 90]. Correspondingly, a non-vanishing c_n would imply the propagation of unphysical states and violate unitarity.

We note that this does not necessarily mean that the α'-expansion is required to be completely void of TEMZVs with twits $b_i = \tau/2$. Esentially, the issue at hand is to understand whether the physically required absence of half-integer powers in the q-expansion of some linear combinations of TEMZVs implies that this linear combination is expressible exclusively with EMZVs. The appropriate language to study this problem is believed to be the reformulation of TEMZVs as iterated integrals over the weighting functions $f^{(n)}(b)$ [1], generalizing the analogous decomposition of EMZVs into linear combinations of iterated Eisenstein integrals [155, 163].

3.5.3 Double-trace terms for "3+1" – shuffling boundaries

We now go on to consider double-trace terms with $\mathrm{tr}(T_1T_2T_3)\mathrm{tr}(T_4)$. Although, such double-trace terms usually do not contribute as the generators of the Lie groups in question are traceless, they play a role in monodromy relations [162]. Here we will find that the relevant integrals may in fact be related to the integrals studied above. The worldsheet integral in question reads

$$\begin{aligned} I_{123|4} = \int_{123}^{4} [\mathrm{d}z] \exp\Big(& s_{12}\tilde{P}(z_{12}) + s_{13}\tilde{P}(z_{13}) + s_{23}\tilde{P}(z_{23}) + \\ & + s_{14}\tilde{Q}(z_{14}) + s_{24}\tilde{Q}(z_{24}) + s_{34}\tilde{Q}(z_{34})\Big) , \end{aligned} \tag{3.5.70}$$

where both $c_Q(q)$ as well as c_P drop out due to momentum conservation. The space of boundary configurations may be parametrized by the coordinates of the insertions (up to the action of the conformal Killing group), where the three insertions sharing a boundary are ordered. Hence, the corresponding classification of inequivalent integrals is given by the orbits of the action of $\mathbb{Z}_3$ on the boundary with three insertions. Moreover, the coordinates of the insertions have constant imaginary part and we may therefore integrate over the real parts only. The conformal Killing group may again be used to fix the coordinate of one insertion, where we either choose $z_1 = 0$ or $z_4 = 0$ depending on which is more convenient for the computation at hand. These two choices result in the specific expressions

$$\begin{aligned} \int_{123}^{4} [\mathrm{d}z] &= \int_0^1 \mathrm{d}z_4 \int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \int_0^{z_2} \mathrm{d}z_1 \delta(z_1) \\ &= \int_0^1 \mathrm{d}z_4 \delta(z_4) \left(\int_0^1 \mathrm{d}z_3 \int_0^{z_3} \mathrm{d}z_2 \int_0^{z_2} \mathrm{d}z_1 + \mathrm{cylic}(1,2,3) \right) . \end{aligned} \tag{3.5.71}$$

With the above in mind we find the following inequivalent integrals at first order in α'

$$e_{1;1} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12}) \,, \quad e_{1;2} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{Q}(z_{14}) \,,$$

which both do not contributed as they are multiplied by a vanishing kinematic invariant. At second order there are six different integrals

$$e_{2;1} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})^2 \,, \qquad e_{2;2} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{Q}(z_{14})^2 \,,$$

$$e_{2;3} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13}) \,, \qquad e_{2;4} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{Q}(z_{14})\tilde{Q}(z_{24}) \,,$$

$$e_{2;5} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{34}) \,, \qquad e_{2;6} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{14}) \,,$$

and finally at third order we get the integrals

$$e_{3;1} = \frac{1}{6}\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})^3 \,, \qquad e_{3;2} = \frac{1}{6}\int_{123}^{4} [\mathrm{d}z]\, \tilde{Q}(z_{14})^3 \,,$$

$$e_{3;3} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{P}(z_{13}) \,, \qquad e_{3;4} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{23})\tilde{P}(z_{13}) \,,$$

$$e_{3;5} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{Q}(z_{14})\tilde{Q}(z_{24})\tilde{Q}(z_{34}) \,, \qquad e_{3;6} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{Q}(z_{14})^2\tilde{Q}(z_{24}) \,,$$

$$e_{3;7} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{Q}(z_{14}) \,, \qquad e_{3;8} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})^2\tilde{Q}(z_{34}) \,,$$

$$e_{3;9} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{14})^2 \,, \qquad e_{3;10} = \frac{1}{2}\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{34})^2 \,,$$

$$e_{3;11} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13})\tilde{Q}(z_{14}) \,, \qquad e_{3;12} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{P}(z_{13})\tilde{Q}(z_{24}) \,,$$

$$e_{3;13} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{14})\tilde{Q}(z_{24}) \,, \qquad e_{3;14} = \int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})\tilde{Q}(z_{14})\tilde{Q}(z_{34}) \,.$$

Instead of directly translating those into TEMZVs with the methods above we argue that we may relate them to the integrals studied in the previous two subsections. Specifically, if only $\tilde{P}$

appears in the integrand it is possible to rewrite the integration domain by “shuffling” the two boundaries in the sense that

$$\int_{123}^{4} [\mathrm{d}z]\, \tilde{P}(z_{12})^{n_1} \tilde{P}(z_{23})^{n_2} \tilde{P}(z_{13})^{n_3} = \left(\int_{1234} [\mathrm{d}z] + \int_{1243} [\mathrm{d}z] + \int_{1423} [\mathrm{d}z] \right) \tilde{P}(z_{12})^{n_1} \tilde{P}(z_{23})^{n_2} \tilde{P}(z_{13})^{n_3} \ . \tag{3.5.72}$$

After relabeling we are left with integrals that all appear in the study of the single-trace contributions of the worldsheet integral of cylinder topology. In particular we may rewrite the dauntingly looking integral

$$e_{3;4} = \int_{123}^{4} [\mathrm{d}z]\, P(z_{12}) P(z_{23}) P(z_{13}) = 3c_{3;9} \ , \tag{3.5.73}$$

which was already translated. Note that this may seemingly complicate certain results, e.g.

$$e_{3;1} = 2c_{3;1} + c_{3;2} = \frac{1}{2} \omega \left(\begin{smallmatrix} 1 \, , \, 1 \, , \, 1 \, , \, 0 \\ 0 \, , \, 0 \, , \, 0 \, , \, 0 \end{smallmatrix} \right) \tag{3.5.74}$$

where the last equality needs repeated application of the shuffle relation, which is more work than obtaining it directly from the definition of $e_{3;1}$. Conversely, we may obtain more compact results for the integrals studied in the previous two subsections, e.g.

$$d_{3;1} = \int_{12}^{34} [\mathrm{d}z] \tilde{P}(z_{12})^3 = \left(\int_{123}^{4} [\mathrm{d}z] + \int_{132}^{4} [\mathrm{d}z] \right) \tilde{P}(z_{12})^3 = 2e_{3;1} \ . \tag{3.5.75}$$

Furthermore, we might treat the case where the integrand involves $\tilde{Q}$ in a similar manner and relate such integrals to the integrals of subsection 3.5.2.

Using this rewriting of the integration region we obtain the following relations at first order in α'

$$e_{1;1} = \frac{1}{2} d_{1;1} = 2c_{1;1} + c_{1;2} \ , \qquad e_{1;2} = \frac{1}{2} d_{1;2} \ , \tag{3.5.76}$$

and correspondingly at second

$$\begin{aligned} e_{2;1} &= \frac{1}{2} d_{2;1} = 2c_{2;1} + c_{2;2} \ , & e_{2;2} &= \frac{1}{2} d_{2;2} \ , \\ e_{2;3} &= \frac{1}{2} d_{2;4} = 2c_{2;6} + c_{2;3} \ , & e_{2;4} &= \frac{1}{2} d_{2;5} \ , \\ e_{2;5} &= \frac{1}{2} d_{1;1} d_{1;2} = \frac{1}{2} d_{2;3} \ , & e_{2;6} &= \frac{1}{2} d_{2;3} \ , \end{aligned} \tag{3.5.77}$$

and third order

$$\begin{aligned}
e_{3;1} &= \frac{1}{2}d_{3;1} = 2c_{3;1} + c_{3;2}\ , & e_{3;2} &= \frac{1}{2}d_{3;2}\ ,\\
e_{3;3} &= \frac{1}{2}d_{3;5} = 2c_{3;6} + c_{3;3}\ , & e_{3;4} &= 3c_{3;9}\ ,\\
e_{3;5} &= \frac{1}{2}d_{3;12} = \frac{1}{2}d_{2;3}\ , & e_{3;6} &= \frac{1}{2}d_{3;6}\ ,\\
e_{3;7} &= \frac{1}{2}d_{3;3}\ , & e_{3;8} &= \frac{1}{2}d_{1;2}d_{2;1} = \frac{1}{2}d_{3;3}\ ,\\
e_{3;9} &= \frac{1}{2}d_{3;4}\ , & e_{3;10} &= \frac{1}{2}d_{1;1}d_{2;2} = \frac{1}{2}d_{3;4}\ ,\\
e_{3;11} &= \frac{1}{2}d_{1;2}d_{2;4} = \frac{1}{2}d_{3;8}\ , & e_{3;12} &= \frac{1}{2}d_{1;2}d_{2;4} = \frac{1}{2}d_{3;8}\ ,\\
e_{3;13} &= \frac{1}{2}d_{3;11}\ , & e_{3;14} &= \frac{1}{2}d_{3;9}\ .
\end{aligned} \tag{3.5.78}$$

Hence, the expansion of the worldsheet integral (3.5.70) up to third order in α', together with the relations among the $e_{i;j}$ found above, is given by

$$\begin{aligned}
I_{123|4} =& \frac{1}{2} + (s_{12}^2 + s_{12}s_{23} + s_{23}^2)\big[2e_{2;1} + 2e_{2;2} - e_{2;3} - e_{2;4}\big]\\
&+ s_{12}s_{23}(s_{12} + s_{23})\big[- 3e_{3;1} - 3e_{3;2} + 3e_{3;3} - e_{3;4} - e_{3;5}\\
&\qquad + 3e_{3;6} - 3e_{3;13} + 3e_{3;14}\big] + \mathcal{O}(\alpha'^4)\ ,
\end{aligned} \tag{3.5.79}$$

which, via the use of several identities among TEMZVs, may be brought into the nice form

$$\begin{aligned}
I_{123|4} =& \frac{1}{2} + (s_{12}^2 + s_{12}s_{23} + s_{23}^2)\left[\frac{\zeta_2}{12} - \omega\left(\begin{smallmatrix} 0\ ,\ 2\ ,\ 0 \\ 0\ ,\ 0\ ,\ 0 \end{smallmatrix}\right)\right]\\
&+ s_{12}s_{23}(s_{12} + s_{23})\left[2\zeta_2\,\omega\left(\begin{smallmatrix} 0\ ,\ 1\ ,\ 0\ ,\ 0 \\ 0\ ,\ 0\ ,\ 0\ ,\ 0 \end{smallmatrix}\right) - \frac{5}{6}\,\omega\left(\begin{smallmatrix} 0\ ,\ 3\ ,\ 0\ ,\ 0 \\ 0\ ,\ 0\ ,\ 0\ ,\ 0 \end{smallmatrix}\right) + \frac{\zeta_3}{4}\right] + \mathcal{O}(\alpha'^4)\ ,
\end{aligned} \tag{3.5.80}$$

consistent with the zeroth and first order results found in [162]. Moreover, we note that the q^0 term of this expression is consistent with results in [44]

$$\begin{aligned}
\lim_{q\to 0} I_{123|4} &= -\frac{1}{\pi^2}\left[\frac{\Gamma(s_{12})\Gamma(s_{23})}{\Gamma(1+s_{12}+s_{23})} + \frac{\Gamma(s_{23})\Gamma(-s_{12}-s_{23})}{\Gamma(1-s_{12})} + \frac{\Gamma(s_{12})\Gamma(-s_{12}-s_{23})}{\Gamma(1-s_{23})}\right]\\
&= \frac{1}{2} + \frac{\zeta_2}{4}(s_{12}^2 + s_{12}s_{23} + s_{23}^2) + \frac{\zeta_3}{2}s_{12}s_{23}(s_{12}+s_{23}) + \mathcal{O}(\alpha'^4)\ .
\end{aligned} \tag{3.5.81}$$

3.6 Remarks on the case of proper rational twists

So far we explicitly required the twists to be either zero or of the form $b = s + r\tau$ with $r \neq 0$. The reason for this restriction was that certain iterated integrals may be ill-defined as then $f^{(1)}$ has a simple pole on the integration domain. In this Section we will make some comments on how one may generalize the definition of TEMZVs to include also the corresponding iterated integrals for the case of so-called *proper rational twists*, i.e. non-zero twists with $r = 0$. Furthermore, we comment on the properties of their q-expansions along the lines of section 3.4. Although such objects do not arise in the study of the open-string amplitude, they have interesting properties. In particular, we motivate that the constant term of TEMZVs with real twists is given by MZVs at roots of unity. These numbers are known to appear in field theory computations (at least in intermediate steps), cf. e.g. [9, 22, 164, 165].

As stated above we intend to find a good definition for twists in the whole square lattice $b_i \in \Lambda_N + \Lambda_N\tau$. Then in order to circumvent possible divergencies that originate from poles on the integration contour we propose the following generalized definition of TEMZV as iterated integral [1]

$$\omega\left(\begin{smallmatrix} n_1 \, , \, \dots \, , \, n_l \\ b_1 \, , \, \dots \, , \, b_l \end{smallmatrix}\right) = \lim_{\varepsilon\to 0} \int_{\gamma_R} f^{(n_1)}(z_1 - b_1)\mathrm{d}z_1 \dots f^{(n_l)}(z_l - b_l)\mathrm{d}z_l \,, \tag{3.6.1}$$

where γ_R is a path composed of small semicircles of radius ε (with $0 < \varepsilon < (2N)^{-1}$) around the points in Λ_N, with neighbouring semicircles connected by a straight line segments as depicted in fig. 3.2.

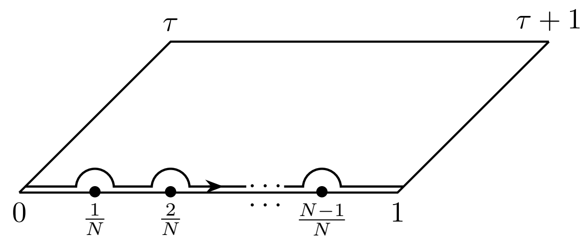

Figure 3.2: The path γ_R, avoiding the possible singularities of $f^{(1)}$. Note that for the sake of visibility we shifted γ_R slightly up.

The existence of this integral may be seen as follows. First, we note that on the semicircles there are additional contributions from the non-meromorphic terms of the weighting function $f^{(n)}$ (cf. eq. (3.3.15)). These additional contributions are generally proportional to $\mathrm{Im}(z)^j$ with $1 \leq j \leq n$ and come multiplied with a meromorphic function in z, which at most has a simple pole in z. Hence these additional contributions are non-singular in the limit $\varepsilon \to 0$. Then using the path composition formula for iterated integrals eq. (D.1.3) we find that these additional non-meromorphic contributions are at least of order ε and therefore do not contribute in the limit. Eventually, we may restrict our considerations to integrals over meromorphic functions only. These will be independent of ε as all path γ_R are homotopic (for ε small enough) and in particular are non-singular in the limit $\varepsilon \to 0$. Hence, the limit exists and is unique.

We note that the more general TEMZVs defined by (3.6.1) also satisfy shuffle product formulas as given above, inherited from their definition as iterated integrals. However, the reflection identity (3.3.40) fails as soon as a combined letter with $n_i = 1, b_i \in \Lambda_N^\times$ appears. This is due to the fact that

$$(\gamma_R^{-1})^*(f^{(n)}(z-b)\mathrm{d}z) \neq (-1)^{n+1}\gamma_R^*\Big(f^{(n)}(z-(-b))\mathrm{d}z\Big) \,, \tag{3.6.2}$$

as was true for the path along the real interval $[0, 1]$. This relation (3.6.2) fails to hold on the semicircle parts of γ_R, which contribute to the integral only for $n = 1, b \in \Lambda_N^\times$. The way that reflection fails can be exemplified by studying the following length-one TEMZV with twist

$b \in \Lambda_N^\times$

$$\omega\left(\begin{smallmatrix}1\\b\end{smallmatrix}\right) = \int_{\gamma_R} f^{(1)}(z-b)\mathrm{d}z = \int_0^{b-\varepsilon} \mathrm{d}z \frac{\theta_1'(z-b;\tau)}{\theta_1(z-b;\tau)} + \int_{\beta_\varepsilon} \mathrm{d}z f^{(1)}(z-b) + \int_{b+\varepsilon}^{1} \mathrm{d}z \frac{\theta_1'(z-b;\tau)}{\theta_1(z-b;\tau)} \\ = \log(\theta_1(-\varepsilon;\tau)) - \log(\theta_1(-b;\tau)) - i\pi + \log(\theta_1(1-b;\tau)) - \log(\theta_1(\varepsilon;\tau)) = -i\pi \ , \tag{3.6.3}$$

where the sole contribution comes from the semicircle β_ε around b. Now if we would consider the integral w.r.t. the inverted path the pole is at $1-b$ (w.r.t. the path parameter), which according to the above formula should also give $-i\pi$ but reflection would require $+i\pi$. This suggests that the reason why reflection does not hold stem from the residue nature of the semicircle contribution, as the change of orientation and the odd parity of $f^{(1)}$ cancel the sign changes, while we will not find such a behaviour for other combined letters.[85]

In general, TEMZVs with twists $b \in \Lambda_N + \Lambda_N\tau$ will depend on the modular parameter τ. Now we argued above that the non-meromorphic terms of the $f^{(n)}$ are at worst of $\mathcal{O}(\varepsilon^0)$ and hence of order ε upon integration. As these terms are the only sources of a possible $\log(q)$ dependence, we infer that the more general class of TEMZVs as defined in (3.6.1), also admits an expansion in q^r and q^{1-r}. Thus the expansion (3.4.1) also holds for the TEMZVs with twists on the whole lattice $\Lambda_N + \Lambda_N\tau$.

3.6.1 A length two example

The raison d'être of this subsection is to illustrate the definition of TEMZVs in the case of real twists for a more involved example, via an explicit computation of $\omega\left(\begin{smallmatrix}0\ ,\ 1\\0\ ,\ 1/2\end{smallmatrix}\right)$. Our starting point is the definition of TEMZVs as given in eq. (3.6.1) for twists $b \in \{0, 1/2\}$, corresponding to the integral[86]

$$\omega\left(\begin{smallmatrix}n_1\ ,\ \dots\ ,\ n_l\\b_1\ ,\ \dots\ ,\ b_l\end{smallmatrix}\right) := \lim_{\varepsilon\to 0} \int_{\alpha_1\beta_\varepsilon\alpha_2} \omega_{b_1}^{(n_1)} \dots \omega_{b_l}^{(n_l)} \ , \tag{3.6.4}$$

where for the complex variables z and b in the fundamental domain of the lattice $\mathbb{Z}+\mathbb{Z}\tau$ we use the notation $\omega_b^{(n)} = f^{(n)}(z-b)\mathrm{d}z$. For the individual segments of the path γ_R, we choose the parametrization

$$\begin{aligned} \alpha_1(t) &= (1/2-\varepsilon)t \\ \beta_\varepsilon(t) &= 1/2 - \varepsilon\exp(-i\pi t) \\ \alpha_2(t) &= 1/2+\varepsilon+(1/2-\varepsilon)t \ . \end{aligned} \tag{3.6.5}$$

Then we may compute the iterated integral using the composition of path formula

$$\int_{\alpha\beta} \omega_{b_1}^{(n_1)} \dots \omega_{b_l}^{(n_l)} = \sum_{k=0}^{l} \int_\alpha \omega_{b_1}^{(n_1)} \dots \omega_{b_k}^{(n_k)} \int_\beta \omega_{b_{k+1}}^{(n_{k+1})} \dots \omega_{b_l}^{(n_l)} \ . \tag{3.6.6}$$

for paths α, β such that $\alpha(1) = \beta(0)$ and the empty integral is defined to be one.

As the $\omega_b^{(n)}$ admit an expansion in q we may treat the q^0 term separately from the rest

[85] Still the TEMZVs satisfy relations derived from the path inversion formula eq. (D.1.1), which will lead to an reflection type identity, with extra terms for every occurrence of the combined letter $n_i = 1, b_i \in \Lambda_N^\times$.

[86] For the sake of notational clarity we omit the possibly necessary endpoint regularization.

assuming we can exchange the q-expansion with the integration. Then as the coefficients of the q^j for $j \neq 0$ are well-defined on the real line we may exchange the limit $\varepsilon \to 0$ with the integral and compute this part of the integral over the much more mundane path $\gamma(t) = t$. Specifically, the q dependent part is given by

$$I_q = -2i(2\pi i) \int\limits_{0<t_1<t_2<1} \mathrm{d}t_1 \mathrm{d}t_2 \sum_{n,m=1}^{\infty} q^{mn} \sin(2\pi m(t_2 - 1/2)) = -2 \sum_{n,m=1}^{\infty} \frac{(-1)^m q^{mn}}{m} \,. \tag{3.6.7}$$

We note that a naive application of the differential equation (3.4.35) for real twists gives

$$\begin{aligned} \omega\left(\begin{smallmatrix} 0 \,, & 1 \\ 0 \,, & 1/2 \end{smallmatrix}\right) &= c\left(\begin{smallmatrix} 0 \,, & 1 \\ 0 \,, & 1/2 \end{smallmatrix}\right) + \int_0^q \frac{\mathrm{d}\log(q_1)}{-4\pi^2} \left[\omega\left(\begin{smallmatrix} 2 \\ 1/2 \end{smallmatrix}\right) - f^{(2)}(1/2; q_1)\right] \\ &= c\left(\begin{smallmatrix} 0 \,, & 1 \\ 0 \,, & 1/2 \end{smallmatrix}\right) + \int_0^q \frac{\mathrm{d}\log(q_1)}{-4\pi^2} \left[8\pi^2 \sum_{m,n=1}^{\infty} (-1)^m n q_1^{mn}\right] \,, \end{aligned} \tag{3.6.8}$$

which coincides with the q-dependence of the expansion (3.6.7). In subsection 3.6.3 we will argue why the definition (3.6.1) does not alter the differential equation we studied in subsection 3.4.3.

Computing the constant term is more involved. Specifically, using the path composition formula (3.6.6), the constant term of the q-expansion may be extracted as follows

$$\begin{aligned} I_0 = \int\limits_{\alpha_1\beta_\varepsilon\alpha_2} [q^0]\omega_0^{(0)}\omega_{1/2}^{(1)} &= \int\limits_{\alpha_1} [q^0]\omega_0^{(0)}\omega_{1/2}^{(1)} + \int\limits_{\beta_\varepsilon} [q^0]\omega_0^{(0)}\omega_{1/2}^{(1)} + \int\limits_{\alpha_2} [q^0]\omega_0^{(0)}\omega_{1/2}^{(1)} \\ &+ \int\limits_{\alpha_1} \omega_0^{(0)} \int\limits_{\beta_\varepsilon} [q^0]\omega_{1/2}^{(1)} + \int\limits_{\alpha_1} \omega_0^{(0)} \int\limits_{\alpha_2} [q^0]\omega_{1/2}^{(1)} + \int\limits_{\beta_\varepsilon} \omega_0^{(0)} \int\limits_{\alpha_2} [q^0]\omega_{1/2}^{(1)} \,, \end{aligned} \tag{3.6.9}$$

where $[q^0]$ denotes the projection onto the constant term of the q-expansion and we used that $[q^0]\omega_b^{(0)} = \omega_b^{(0)}$. The individual integrals are given by

$$\begin{aligned} \int\limits_{\alpha_1} [q^0]\omega_0^{(0)}\omega_{1/2}^{(1)} &= \int\limits_{0<t_1<t_2<1} \mathrm{d}t_1 \mathrm{d}t_2 (1/2-\varepsilon)^2 \pi \cot(\pi((1/2-\varepsilon)t_2 - 1/2))) \\ &= \frac{\log(2)}{2} + \frac{\log(\pi\varepsilon)}{2} + \mathcal{O}(\varepsilon) \end{aligned} \tag{3.6.10}$$

$$\begin{aligned} \int\limits_{\beta_\varepsilon} [q^0]\omega_0^{(0)}\omega_{1/2}^{(1)} &= \int\limits_{0<t_1<t_2<1} \mathrm{d}t_1 \mathrm{d}t_2 (i\pi)^2 \varepsilon^2 e^{-i\pi(t_1+t_2)} \pi \cot(-\pi\varepsilon e^{-i\pi t_2}) \\ &= \mathcal{O}(\varepsilon) \end{aligned} \tag{3.6.11}$$

$$\begin{aligned} \int\limits_{\alpha_2} [q^0]\omega_0^{(0)}\omega_{1/2}^{(1)} &= \int\limits_{0<t_1<t_2<1} \mathrm{d}t_1 \mathrm{d}t_2 (1/2-\varepsilon)^2 \pi \cot(\pi((1/2-\varepsilon)t_2 + \varepsilon))) \\ &= \frac{\log(2)}{2} + \varepsilon\log(\pi\varepsilon) + \mathcal{O}(\varepsilon) \end{aligned} \tag{3.6.12}$$

$$\int\limits_{\alpha_1} \omega_0^{(0)} \int\limits_{\beta_\varepsilon} [q^0]\omega_{1/2}^{(1)} = (1/2-\varepsilon)(-i\pi) \tag{3.6.13}$$

$$\int\limits_{\alpha_1} \omega_0^{(0)} \int\limits_{\alpha_2} [q^0]\omega_{1/2}^{(1)} = (1/2-\varepsilon)(-\log(\sin(\pi\varepsilon))) \tag{3.6.14}$$

$$\int_{\beta_\varepsilon} \omega_0^{(0)} \int_{\alpha_2} [q^0]\omega_{1/2}^{(1)} = 2\varepsilon(-\log(\sin(\pi\varepsilon))) \tag{3.6.15}$$

We note that due to $\lim_{\varepsilon\to 0} \varepsilon \log(\sin(\pi\varepsilon)) = 0$ the only singular contributions come from the integrals of eqs. (3.6.10) and (3.6.14), which cancel in the sum. Then collecting all contributions we find the constant term to be given by

$$\lim_{\varepsilon\to 0} I_0 = -\frac{i\pi}{2} + \log(2)\ . \tag{3.6.16}$$

Note the occurrence of an MZV at second root of unity! This is in fact a more general property if we allow for twists $b_i \in \Lambda_N^\times$, as we will see in the next subsection. Finally, the q-expansion of the TEMZV in question is given by

$$\omega\left(\begin{smallmatrix} 0 & , & 1 \\ 0 & , & 1/2 \end{smallmatrix}\right) = \lim_{\varepsilon\to 0}(I_0 + I_q) = -\frac{i\pi}{2} + \log(2) - 2\sum_{n,m=1}^{\infty} \frac{(-1)^m q^{mn}}{m}\ . \tag{3.6.17}$$

3.6.2 Constant terms for proper rational twists

We will now go on to study the q-expansions of TEMZVs with proper rational twists in more detail. Conveniently, the combinatorial procedure described in subsection 3.4.2 can be extended to the case at hand. Recall, that in the study of constant terms in subsection 3.4.2 we found the corresponding class of numbers to be given by $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs. The more intricate degeneration behaviour of $f^{(n)}(z-b)$ for twists $b \in \Lambda_N$ leads to larger class of numbers, namely MZVs at roots of unity.

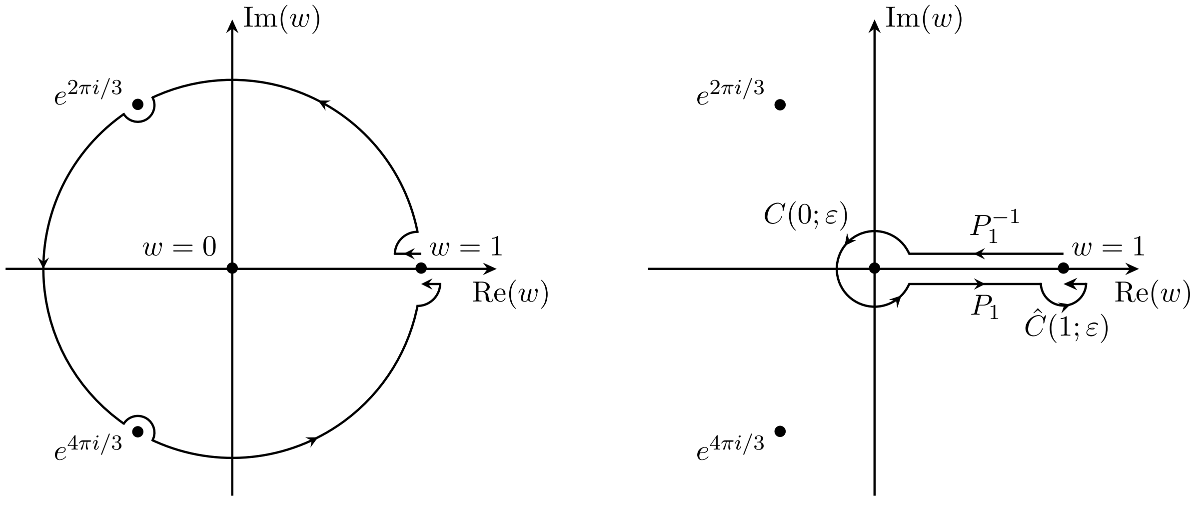

Figure 3.3: Depiction of the homotopy $C_R(0;\varepsilon) \cong P_1^{-1}C(0;\varepsilon)P_1\hat{C}(1;\varepsilon)$ in the case of real twists for $N = 3$. Note the occurrence of roots of unity $e^{2\pi i s}$ corresponding to real twists $s = b \in \Lambda_N$.

Our starting point is the generating series of TEMZVs with twists $b \in \Lambda_N + \Lambda_N\tau$

$$\begin{aligned} e^{-i\pi\,\mathrm{ad}_{x_0}(y)} A_{\Lambda_N+\Lambda_N\tau}(\tau) &= \tilde{\mathcal{P}} \exp\Big(- \int_{\gamma_R} \mathrm{d}z \sum_{b\in\Lambda_N+\Lambda_N\tau} \sum_{n=0}^{\infty} f^{(n)}(z-b)\mathrm{ad}^n_{x_b}(y)\Big) \\ &= 1 + \sum_{l\geq 1}(-1)^l \sum_{\substack{n_1,\ldots,n_l\geq 0 \\ b_1,\ldots,b_l\in\Lambda_N+\Lambda_N\tau}} \omega\left(\begin{smallmatrix} n_1 ,& \ldots ,& n_l \\ b_1 ,& \ldots ,& b_l \end{smallmatrix}\right) \mathrm{ad}^{n_l}_{x_{b_l}}(y)\ldots \mathrm{ad}^{n_1}_{x_{b_1}}(y) \;, \end{aligned} \tag{3.6.18}$$

generalizing eq. (3.4.9). Now the crucial difference to the setup above is the degeneration of $f^{(1)}(z-b)$ at $b \in \Lambda_N$

$$\lim_{\tau\to i\infty} f^{(1)}(z-s)\mathrm{d}z = -\frac{1}{2}\frac{\mathrm{d}w}{w} + \frac{\mathrm{d}w}{w-\exp(-2\pi i s)} \;, \tag{3.6.19}$$

which leads to additional one-forms with poles at roots of unity in the w-plane. These additional one-forms will lead to the larger class of numbers for constant terms. To see this we consider the degeneration of the exponent of eq. (3.6.18) given by

$$\lim_{\tau\to i\infty} - \int_{\gamma_R} \mathrm{d}z \sum_{n=0}^{\infty} \sum_{b\in\Lambda_N+\Lambda_N\tau} f^{(n)}(z-b)\mathrm{ad}^n_{x_b}(y) = \int_{C_R(0;1)} \Big(\tilde{y}_N \frac{\mathrm{d}w}{w} + \sum_{s\in\Lambda_N} t_s \frac{\mathrm{d}w}{w-\exp(-2\pi i s)}\Big) \;, \tag{3.6.20}$$

where $C_R(0,\varepsilon)$ is the image of γ_R under the exponential map and we introduced the following shorthands for the bookkeeping variables

$$\begin{aligned} \tilde{y}_N &= - \sum_{b\in\Lambda_N} \frac{\mathrm{ad}_{x_b}}{\exp(2\pi i\,\mathrm{ad}_{x_b})-1}(y) \;+ \sum_{b\in(\Lambda_N+\Lambda_N\tau)\setminus\Lambda_N} \frac{\exp(-2\pi i r\,\mathrm{ad}_{x_b})\mathrm{ad}_{x_b}}{\exp(-2\pi i\,\mathrm{ad}_{x_b})-1}(y) \;, \\ t_s &= -\mathrm{ad}_{x_b}(y) \;, \qquad \text{with } s = b \in \Lambda_N \;. \end{aligned} \tag{3.6.21}$$

Now it is important to note that the semicircles of γ_R have a small positive imaginary part and therefore give rise to an inwards "dented" unit circle $C_R(0;\varepsilon)$ under the exponential map. Conveniently, this means that $C_R(0;1)$ is homotopic to the same composed path as in the case studied in subsection 3.4.2. Namely, we have the homotopy

$$C_R(0;1) \cong P_1^{-1} C(0;\varepsilon) P_1 \hat{C}(1;\varepsilon) \;, \tag{3.6.22}$$

as there are no additional poles obstructing the homotopy; cf. fig. 3.3. However, due to the more complex degeneration behaviour (3.6.19) the integrand is quite different, leading to the occurrence of MZVs at roots of unity. This may be seen by using the homotopy of paths (3.6.22) allowing us to decompose the rhs. of eq. (3.6.20) as

$$\begin{aligned} \lim_{\tau\to i\infty} e^{i\pi t_0} A_{\Lambda_N+\Lambda_N\tau}(\tau) &= \int_{P_1^{-1}C(0;\varepsilon)P_1\hat{C}(1;\varepsilon)} \Big(\tilde{y}_N \frac{\mathrm{d}w}{w} + \sum_{s\in\Lambda_N} t_s \frac{\mathrm{d}w}{w-\exp(-2\pi i s)}\Big) \\ &= \exp(i\pi t_0)\Phi_N(\tilde{y}_N,\{t_s\}_s)\exp(2\pi i\tilde{y}_N)\Phi_N^{-1}(\tilde{y}_N,\{t_s\}_s) \;, \end{aligned} \tag{3.6.23}$$

where Φ_N denotes the so-called cyclotomic Drinfeld associator [166]

$$\Phi_N(\tilde{y}_N, \{t_s\}_s) = \int\limits_{P_1} \left(\tilde{y}_N \frac{\mathrm{d}w}{w} + \sum_{s \in \Lambda_N} t_s \frac{\mathrm{d}w}{w - \exp(-2\pi i s)} \right) . \tag{3.6.24}$$

Now it is known that Φ_N is the generating series of MZVs at N-th root of unity [167–170], hence we deduce from eqs. (3.6.21,3.6.23) that the constant terms of TEMZVs with twists $b \in \Lambda_N + \Lambda_N \tau$ are in fact $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs at roots of unity.

Again, we are now able to extract the constant term of a given TEMZV by equating coefficients of monomials $\mathrm{ad}^{n_l}_{x_{b_l}}(y) \ldots \mathrm{ad}^{n_1}_{x_{b_1}}(y)$ in the eq. (3.6.23). For the simplest case we may consider twists $b \in \{0, \frac{1}{2}\}$, which will feature constant terms involving MZVs at second root of unity

$$c_0 \left(\begin{smallmatrix} 1 \\ \frac{1}{2} \end{smallmatrix} \right) = -i\pi \, , \qquad c_0 \left(\begin{smallmatrix} 2 \, , \, 0 \, , \, 1 \\ \frac{1}{2} \, , \, 0 \, , \, \frac{1}{2} \end{smallmatrix} \right) = \frac{i\pi^3}{24} - \frac{\pi^2 \log(2)}{6} \, , \tag{3.6.25}$$

$$c_0 \left(\begin{smallmatrix} 0 \, , \, 1 \\ 0 \, , \, \frac{1}{2} \end{smallmatrix} \right) = -\frac{i\pi}{2} + \log(2) \, , \qquad c_0 \left(\begin{smallmatrix} 1 \, , \, 0 \, , \, 0 \\ \frac{1}{2} \, , \, 0 \, , \, 0 \end{smallmatrix} \right) = -\frac{i\pi}{8} - \frac{\log(2)}{2} \, . \tag{3.6.26}$$

For the sake of illustration we give some examples for $b \in \Lambda_3^\times$, i.e. constant terms including MZVs at third root of unity, cf. appendix D.2 for our notation conventions. Specifically, we have

$$c_0 \left(\begin{smallmatrix} 1 \\ \frac{1}{3} \end{smallmatrix} \right) = -i\pi \, , \;\; c_0 \left(\begin{smallmatrix} 1 \, , \, 1 \\ \frac{1}{3} \, , \, \frac{2}{3} \end{smallmatrix} \right) = i\pi \left(\zeta \left(\begin{smallmatrix} 1 \\ e^{\frac{4\pi i}{3}} \end{smallmatrix} \right) - \zeta \left(\begin{smallmatrix} 1 \\ e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) \right) - 3\zeta_2 \, , \tag{3.6.27}$$

$$c_0 \left(\begin{smallmatrix} 1 \, , \, 1 \, , \, 2 \\ 0 \, , \, 0 \, , \, \frac{1}{3} \end{smallmatrix} \right) = \frac{5}{2}\zeta_4 \, , \;\; c_0 \left(\begin{smallmatrix} 1 \, , \, 1 \, , \, 0 \\ \frac{1}{3} \, , \, 0 \, , \, 0 \end{smallmatrix} \right) = -\frac{i\pi}{2} \zeta \left(\begin{smallmatrix} 1 \\ e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) + \frac{1}{2} \zeta \left(\begin{smallmatrix} 2 \\ e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) - \zeta \left(\begin{smallmatrix} 1 \, , \, 1 \\ e^{\frac{4\pi i}{3}} \, , \, e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) \, , \tag{3.6.28}$$

$$\begin{aligned} c_0 \left(\begin{smallmatrix} 1 \, , \, 1 \, , \, 0 \, , \, 1 \\ \frac{1}{3} \, , \, 0 \, , \, 0 \, , \, 0 \end{smallmatrix} \right) = \; & \frac{i\pi}{4} \zeta_2 + \zeta_2 \zeta \left(\begin{smallmatrix} 1 \\ e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) + \frac{i\pi}{4} \zeta \left(\begin{smallmatrix} 2 \\ e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) + \frac{1}{4} \zeta \left(\begin{smallmatrix} 3 \\ e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) \\ & - \frac{i\pi}{2} \zeta \left(\begin{smallmatrix} 1 \, , \, 1 \\ e^{\frac{4\pi i}{3}} \, , \, e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) - \frac{1}{2} \zeta \left(\begin{smallmatrix} 1 \, , \, 2 \\ e^{\frac{4\pi i}{3}} \, , \, e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) - \frac{1}{2} \zeta \left(\begin{smallmatrix} 2 \, , \, 1 \\ e^{\frac{4\pi i}{3}} \, , \, e^{\frac{2\pi i}{3}} \end{smallmatrix} \right) . \end{aligned} \tag{3.6.29}$$

3.6.3 Differential equation including proper rational twists

When we discussed the differential equation for TEMZVs in subsection 3.4.3 we briefly pointed out the ingredients needed in the derivation of the corresponding differential equation for the generating series (3.4.33). Specifically, we needed the generating series of the weighting functions Ω to satisfy a mixed heat type differential equation (3.4.32) as well as the Fay identity. Furthermore, we needed integration by parts to get rid of derivatives w.r.t. to the coordinate z_i (of the image of the path) on the elliptic curve. We now go on to argue that the exact same relations hold for appropriately chosen parts of the involved formulas, with the remaining parts being of $\mathcal{O}(\varepsilon)$ and thus irrelevant in the limit $\varepsilon \to 0$. Specifically, we show that the formulas needed in the derivation in appendix E.4 are the same despite the occurrence of pullbacks along the path γ_R.

As in the case studied above it is convenient to consider a generating function for length l TEMZVs as defined in (3.6.1)

$$\mathrm{T}^R \left[\begin{smallmatrix} \alpha_1 \, , \, \ldots \, , \, \alpha_l \\ b_1 \, , \, \ldots \, , \, b_l \end{smallmatrix} \right] = \lim_{\varepsilon \to 0} \mathrm{T}^R_\varepsilon \left[\begin{smallmatrix} \alpha_1 \, , \, \ldots \, , \, \alpha_l \\ b_1 \, , \, \ldots \, , \, b_l \end{smallmatrix} \right] \, , \tag{3.6.30}$$

$$\mathrm{T}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_l\\ b_1 & , & \dots & , & b_l\end{smallmatrix}\right] = \int\limits_{0<t_i<t_{i+1}<1} (\gamma_R^*\Omega(z_1-b_1,\alpha_1;\tau)\mathrm{d}z_1)\dots(\gamma_R^*\Omega(z_l-b_l,\alpha_l;\tau)\mathrm{d}z_l)\ , \tag{3.6.31}$$

where γ_R^* denotes the pullback of the path and the t_i parametrize the curve. We note that in the case with no $b_i \in \Lambda_N$ we may use the path $\gamma(t) = t$ homotopic to γ_R and arrive at the definition (3.4.31). Now it will turn out to be convenient to separate $\gamma_R^*\Omega\mathrm{d}z$ into a meromorphic and non-meromorphic part. Specifically, as $\mathrm{Im}(z_i)$ is zero on the straight line segments and of order ε on the semicircles, we have

$$\gamma_R^*\Omega(z_i-b_i,\alpha_i;\tau)\mathrm{d}z_i = \gamma_R^*\underbrace{e^{-2\pi i r_i\alpha_i}F(z_i-b_i,\alpha_i;\tau)\mathrm{d}z_i}_{=:\tilde\Omega(z_i-b_i,\alpha_i;\tau)\mathrm{d}z_i} + \mathcal{O}(\varepsilon)\ , \tag{3.6.32}$$

where $\tilde\Omega$ is meromorphic in z_i and the non-meromorphic parts are of order ε.[87] Therefore, we may express the generating function T^R via the meromorphic parts only

$$\begin{aligned}\mathrm{T}^R\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_l\\ b_1 & , & \dots & , & b_l\end{smallmatrix}\right] &= \lim_{\varepsilon\to0}\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_l\\ b_1 & , & \dots & , & b_l\end{smallmatrix}\right]\ ,\\ \tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_l\\ b_1 & , & \dots & , & b_l\end{smallmatrix}\right] &= \int\limits_{0<t_i<t_{i+1}<1}(\gamma_R^*\tilde\Omega(z_1-b_1,\alpha_1;\tau)\mathrm{d}z_1)\dots(\gamma_R^*\tilde\Omega(z_l-b_l,\alpha_l;\tau)\mathrm{d}z_l)\ .\end{aligned} \tag{3.6.33}$$

Note that due to meremorphicity of $\tilde\Omega$ the integral (3.6.33) will not depend on ε. We also need to define the intermediate object integrated up to $0 < t_{i+1} < 1$

$$\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_i\\ b_1 & , & \dots & , & b_i\end{smallmatrix};z_{i+1}\right] = \int\limits_{0<t_1<\dots t_i<t_{i+1}}(\gamma_R^*\tilde\Omega(z_1-b_1,\alpha_1;\tau)\mathrm{d}z_1)\dots(\gamma_R^*\tilde\Omega(z_i-b_i,\alpha_i;\tau)\mathrm{d}z_i) \tag{3.6.34}$$

$$= \int_0^{t_{i+1}}\gamma_R^*\left(\tilde\Omega(z_i-b_i,\alpha_i;\tau)\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_i\\ b_1 & , & \dots & , & b_i\end{smallmatrix};z_i\right]\mathrm{d}z_i\right)\ , \tag{3.6.35}$$

which satisfies

$$\partial_{z_{i+1}}\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_i\\ b_1 & , & \dots & , & b_i\end{smallmatrix};z_{i+1}\right] = \tilde\Omega(z_{i+1}-b_i,\alpha_i;\tau)\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_{i-1}\\ b_1 & , & \dots & , & b_{i-1}\end{smallmatrix};z_{i+1}\right]\ . \tag{3.6.36}$$

With the above setup we now argue that the expressions vital to the computation (3.4.33) are structurally unchanged. Firstly, we study the action of ∂_τ on $\gamma_R^*\tilde\Omega$. Specifically, we find the equation

$$\begin{aligned}2\pi i\partial_\tau\gamma_R^*\left(\tilde\Omega(z_i-b_i,\alpha_i;\tau)\mathrm{d}z_i\right) &= 2\pi i\partial_\tau\left(e^{-2\pi i r_i\alpha_i}F(\gamma_R(t_i)-b_i,\alpha_i;\tau)\mathrm{d}\gamma_R(t_i)\right)\\ &= e^{-2\pi i r_i\alpha_i}\left[(-2\pi i r_i\partial_{\gamma_R(t_i)}+\partial_{\gamma_R(t_i)}\partial_{\alpha_i})F(\gamma_R(t_i)-b_i,\alpha_i;\tau)\right]\mathrm{d}\gamma_R(t_i)\\ &= \gamma_R^*\left(\partial_{z_i}\partial_{\alpha_i}\tilde\Omega(z_i-b_i,\alpha_i;\tau)\mathrm{d}z_i\right)\ ,\end{aligned} \tag{3.6.37}$$

that is the pullback of $\tilde\Omega$ satisfies a mixed-heat type equation akin to eq. (3.4.32). Secondly, we

[87] Note that this is in fact the same claim we used in the argument of existence and uniqueness of TEMZVs with twists $b \in \Lambda_N^\times$ as iterated integrals along γ_R (cf. eq. (3.6.1)).

may exchange the τ derivative with the integration

$$2\pi i\partial_\tau \tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_l\\ b_1 & , & \dots & , & b_l\end{smallmatrix}\right] = \sum_{i=1}^{l} \int\limits_{0<t_{j-1}<t_j<1} \prod_{j>i}(\gamma_R^*\tilde{\Omega}(z_j-b_j,\alpha_j;\tau)\mathrm{d}z_j) \\ \times \int\limits_0^{t_{i+1}} \gamma_R^*\Big((\partial_{z_i}\partial_{\alpha_i}\tilde{\Omega}(z_i-b_i,\alpha_i;\tau))\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_{i-1}\\ b_1 & , & \dots & , & b_{i-1}\end{smallmatrix}; z_i\right]\mathrm{d}z_i\Big)\ , \tag{3.6.38}$$

which we are allowed to because the integrand is holomorphic in some open neighbourhood of $[0,1]$, implying that the τ derivative of the integrand is bounded on the simplex (integration domain). Moreover, we again might rewrite the i-th integration using integration by parts (of the pullbacks)

$$\int\limits_0^{t_{i+1}} \gamma_R^*\Big((\partial_{z_i}\partial_{\alpha_i}\tilde{\Omega}(z_i-b_i,\alpha_i;\tau))\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_{i-1}\\ b_1 & , & \dots & , & b_{i-1}\end{smallmatrix}; z_i\right]\mathrm{d}z_i\Big) \\ = \gamma_R^*\Big(\partial_{\alpha_i}\tilde{\Omega}(z_{i+1}-b_i,\alpha_i;\tau))\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_{i-1}\\ b_1 & , & \dots & , & b_{i-1}\end{smallmatrix}; z_{i+1}\right]\Big) \\ - \int\limits_0^{t_{i+1}} \gamma_R^*\Big((\partial_{\alpha_i}\tilde{\Omega}(z_i-b_i,\alpha_i;\tau))\tilde{\Omega}(z_{i-1}-b_{i-1},\alpha_{i-1};\tau)\tilde{\mathrm{T}}^R_\varepsilon\left[\begin{smallmatrix}\alpha_1 & , & \dots & , & \alpha_{i-2}\\ b_1 & , & \dots & , & b_{i-2}\end{smallmatrix}; z_i\right]\mathrm{d}z_i\Big)\ . \tag{3.6.39}$$

Finally, we note that the Fay identity (3.3.14) is valid for all z_i as long as the argument is not zero. Therefore, all the formulas we needed in the derivation of (3.4.33) are the same apart from strategically placed pullbacks and replacements $\Omega \to \tilde{\Omega}$. Hence, We may mimic the computation in eq. (E.4.4) and arrive at the same result simply by virtue of the replacements $\mathrm{T} \to \mathrm{T}^R_\varepsilon$ and $\Omega \to \tilde{\Omega}$. Also note that due to the structure of the differential equation, constant terms of lower length TEMZVs will be part of the coefficients of the q-expansion, suggesting that the coefficients of TEMZVs with twists $b_i \in \Lambda_N^\times$ consist of MZVs at roots of unity as well.

Chapter 4

Conclusion

4.1 Summary and Conclusion

In this work we considered the geometric structure underlying the occurrence of elliptic iterated integrals in genus-one open-string amplitudes. The relevant class of elliptic iterated integrals is completely determined by the underlying punctured elliptic curve, leading us to the notion of TEMZVs as discussed in section 3.3. These TEMZVs admit expansions as formal power series in the modular parameter q and we studied the structure of such q-expansions as well as explicit algorithms for computing said expansions via the initial value problem of section 3.4. Furthermore, in section 3.6 we proposed a generalization to accommodate TEMZVs with twists lying on the integration path and established an initial value problem for the corresponding q-expansions. We found that the constant term of the q-expansion of any such generalized TEMZV (containing proper rational twists) has coefficients that are $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs at roots of unity, as opposed to $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of MZVs for TEMZVs without proper rational twists.

Crucially, the appearance of TEMZVs in genus-one open-string amplitudes follows from the genus-one Green function being expressible via integrals of specific one-forms $\mathrm{d}z f^{(1)}(z-b)$ defined on the relevant punctured elliptic curve and the formulation of amplitudes via integrals over the corresponding moduli space of the punctured elliptic curve in question. Specifically, we argued in sections 3.2 and 3.3 that TEMZVs defined w.r.t. $E_\tau^\times \setminus \{\frac{\tau}{2}\}$ are sufficient to describe all terms in the α'-expansion of the genus-one open-superstring amplitude before integration over the modular parameter q, which we subsequently illustrated with the explicit example of the four-point amplitude in section 3.5. Our results extend the work on the occurrence of EMZVs in the single-trace contribution of the genus-one open-string amplitude of [50, 155] to all contributions. In fact EMZVs may be considered as TEMZVs w.r.t. the once-punctured elliptic curve $E_\tau^\times$ forming a subclass of TEMZVs defined w.r.t. $E_\tau^\times \setminus \{\frac{\tau}{2}\}$. Moreover, we observed that for the double-trace contributions to the four-point amplitude TEMZVs combine in such a way that the overall result only involves the subclass formed by EMZVs, which we explicitly checked up to third order in α'. This observation is actually crucial, as it is equivalent to the absence of unphysical poles in GSO projected superstring theory, cf. subsection 3.5.2.

Finally, we again note the striking structural similarity between the genus-zero amplitude and the genus-one counterpart. In particular, in both cases the link may be established by representing the Green function via an integral over a one-form living in the corresponding de

Rham cohomology. Intriguingly, the similarities between genus zero and one are also present for closed-string amplitudes where in both cases single-valued projections of the corresponding (elliptic) multiple polylogarithms are relevant [49, 171]. However, it is not known whether this very systematic picture generalizes to higher genus.

4.2 Outlook and possible future directions

Understanding the structure behind the occurrence of classes of iterated integrals that are naturally associated with punctured Riemann surfaces in computations in QFT and string theory constitutes a compelling field of research in both physics and mathematics. Despite a lack of phenomenological relevance (as of now), string theory amplitudes remain an active field of research in theoretical physics as they are considered as laboratory to get novel insights into the amplitudes of the corresponding field theories associated with them in the low-energy limit and hence also provide implications for the relevant class of iterated integrals. Yet our picture concerning the relevant classes of iterated integrals that appear in string theory amplitudes and their properties is far from complete and there are several intriguing research directions to pursue in both physics and mathematics.

Regarding the mathematical structure of TEMZVs, there is still a plethora of interesting open questions that require addressing. One issue that needs clarification is the question if and/or how TEMZVs fit into the framework of the twisted (universal) elliptic KZB connection considered in [156]. Moreover, it is an open problem to work out the classification of all relations among TEMZVs and understand the underlying structure, possibly via some generalization of the derivation algebra used in the case of EMZVs [155]. Another very interesting direction to generalize the aforementioned notions is the technically rather demanding task to extend the work of [40] on the genus-one (universal) KZB connection to higher genus. Similarly, one may attempt to extend the genus-one results of [39, 41, 42] to higher-genus incarnations of multiple polylogarithms as well as the corresponding periods and associators.

As for applications to string theory one may generalize the results presented in chapter 3 to higher numbers of external string states and/or orders in α', which is conceptually not very difficult and all necessary concepts were presented in our treatment. Another open question is whether the genus-zero result that the all order α'-expression of the amplitude may be expressed via the Drinfeld associator [46, 47] may be generalized to genus one. Note that the possible existence of such an expression via (potentially the A-part of) Enriquez' elliptic associator would also answer the question which TEMZVs appear in genus-one open-string amplitudes. Also, given an all order α'-expression one may deduce properties of the corresponding remaining integral over the modular parameter q and thus of the full amplitude. Yet another direction is to consider genus-one closed-string amplitudes that are known to be expressible via modular graph functions [51]. As vaguely mentioned above recent work shows that those can in fact be rewritten in terms of single-valued (B-cycle) EMZVs [171]. A further direction to pursue is the technically very challenging question whether the aforementioned curious similarities between genus-zero and genus-one amplitudes generalize to higher genus. There are partial results for genus-two closed-string amplitudes [54, 55, 172] but it is not known whether the α'-expansion leads to (possibly single-valued) genus-two generalizations of TEMZVs.

Leaving the realm of string theory, one may study the point particle limit of the type I string amplitudes we considered throughout this work, leading to amplitudes in $\mathcal{N} = 4$ super

Yang-Mills theory at least for low genus [61]. Yet the point particle limit does not give rise to elliptic iterated integrals in these instances, which is not surprising as elliptic integrals only appear in field theory from two loops onwards. Recently there was a lot of progress relating the elliptic integrals appearing in e.g. the two-loop sunrise with three distinct internal masses as considered in [23–25] with elliptic multiple polylogarithms. This work recently culminated in the link to iterated integrals over modular forms [26] as well as the systematic formulation of elliptic iterated integrals adapted to Feynman diagrams in [52, 53]. Finally, we note that there seems to be some interesting structure behind the integrable model associated to $\mathcal{N} = 4$ super Yang-Mills theory. In the formulation of the spectral problem (of composite local operators) via the so-called quantum spectral curve it was observed that only MZVs appear (see e.g. [173, 174]), which is however at odds with observations that generically these periods should not suffice (cf. also [175]). A better understanding of this disparity might provide novel insights in the organizational principle underlying the occurrence of periods in QFT.

Appendix A

Remarks on certain complex analytic properties of free bosonic strings

In this appendix we briefly make some additional remarks concerning some complex analytic aspects of the bosonic string. Detailed discussions on this topic can be found in [77–79, 81, 82], which we closely follow.

As noted in chapter 2 isothermal coordinates lead to a Hermitian metric $\sqrt{\det(h)} = h_{z\bar{z}} = e^{2\omega}$, which in turn results in the Christoffel symbols $\Gamma^z_{zz} = 2\partial_z\omega$, $\Gamma^{\bar{z}}_{\bar{z}\bar{z}} = 2\partial_{\bar{z}}\omega$ with all others vanishing. Correspondingly, the covariant derivative induces maps between sections of the holomorphic tensor product bundles

$$\begin{aligned} \nabla^{(n)}_z &: K^n \to K^{n+1} \ , \qquad \nabla^{(n)}_z(T\mathrm{d}z^n) = (e^{2\omega})^n\partial_z\big[(e^{-2\omega})^n T\big]\mathrm{d}z^{n+1} \\ \nabla^z_{(n)} &: K^n \to K^{n-1} \ , \qquad \nabla^z_{(n)}(T\mathrm{d}z^n) = e^{-2\omega}(\partial_{\bar{z}}T)\mathrm{d}z^{n-1} \ , \end{aligned} \tag{A.0.1}$$

where K denotes the canonical line bundle of Σ. In order to discuss real tensors we need to also consider the anti-holomorphic line bundle $\bar{K}$, which turns out to be isomorphic to the holomorphic line bundle K^{-1}, where the isomorphism depends on the metric h on Σ. K^{-1} is defined such that $K^{-1} \otimes K$ is isomorphic to the trivial bundle. Then real vectors live in $K \oplus K^{-1}$ and accordingly we may describe $P_1 = \nabla^{(1)}_z \oplus \nabla^z_{(-1)}$. Hence, the kernel of P_1 can be described by the kernel of $\nabla^{(1)}_z$ (and its complex conjugate) and therefore the diffeomorphisms, which are not conformal transformations are related to the image $\mathrm{im}(\nabla^{(1)}_z)$. Moreover, one may define a Hermitian form on K^n via

$$(T_1(\mathrm{d}z)^n, T_2(\mathrm{d}z)^n)_h = \int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}e^{2(1-n)\omega}\bar{T}_1T_2 \ , \tag{A.0.2}$$

w.r.t. which we have the notion of adjoint action $(\nabla^{(n)}_z)^\dagger = -\nabla^z_{(n+1)}$ determined by demanding integration by parts. This suggests $P_1^\dagger = -(\nabla^z_{(2)} \oplus \nabla^{(-2)}_z)$ and thus the kernel of $P_1^\dagger$ is related to the kernel of $\nabla^z_{(2)}$ and its complex conjugate. As stated in the main text the kernel of $\nabla^z_{(2)}$ is given by holomorphic quadratic differentials that will correspond to the degrees of freedom encoded in Teichmüller space or in the context of the ghost system to the zero modes of the b field. In this complex setup the decomposition of the tangent space of the space of metrics at $h_{z\bar{z}}$ is of the form

$$\{\phi h_{z\bar{z}}\} \oplus \mathrm{im}(\nabla^{(1)}_z) \oplus \ker(\nabla^z_{(2)}) \oplus \ \text{c.c.} \ \ . \tag{A.0.3}$$

Appendix B

CFT Specifics

In this appendix we collect a few standard results concerning two-dimensional CFTs defined on an infinite cylinder, in order to increase the degree of self-containedness of our string theory introduction. The restriction to an infinite cylinder will spare us the trouble of introducing line bundles, however we note that most concepts we are going to discuss are inherently local and carry over to more general two dimensional smooth manifolds, albeit with a plethora of technicalities regarding the description. Our main sources of inspiration are [27, 33, 81, 83–86, 91, 92, 176], whereas for treatments of CFTs on higher-genus Riemann surfaces we refer to [177, 178].

B.1 Consequences of local conformal symmetry

We consider a smooth two-dimensional manifold diffeomorphic to $S^1 \times \mathbb{R}$ together with a Riemannian metric g_{ab} of signature $(+,+)$. Recall that a coordinate transformation f is called conformal if its pullback leaves the metric invariant up to a positive-definite scaling function $e^{2\omega}$. Employing complex coordinates $x + iy$ on some coordinate chart this requirement is equivalent to f being holomorphic or anti-holomorphic with non-vanishing first derivative. From now on we restrict our attention to orientation preserving conformal transformations, i.e. holomorphic maps. Thus it is natural to formulate a two-dimensional CFT on a complex one-dimensional manifold and we may use the words conformal and holomorphic (with non-vanishing first derivative) synonymously. Accordingly, any map we deem conformal in some open set may be represented by a meromorphic function with poles outside the open set under consideration, which may be rephrased by stating that any conformal map f (locally) admits a convergent Laurent expansion with the corresponding conformal Killing field generated by the differential operators $l_n = -z^{n+1}\partial_z$. These operators form the Witt algebra with Lie bracket given by

$$[l_n, l_m] = (n-m)l_{n+m} \; . \tag{B.1.1}$$

Now given any such locally holomorphic map f we may consider a quantity that behaves "tensorial" under such local conformal transformations, in the sense that

$$\phi(f(z), \bar{f}(\bar{z})) = (\partial_z f)^{-h} (\partial_{\bar{z}} f)^{-\tilde{h}} \phi(z, \bar{z}) \; , \tag{B.1.2}$$

which is called *conformal primary* of *conformal weight* $(h, \tilde{h})$.

As a disclaimer we note that throughout this appendix we restrict our attention to theories that have a Lagrangian description. Then for most applications, the general rationale of (perturbative) quantum field theory is that we may formulate any quantity of interest via a (formal) path integral. The most important notions are the partition function

$$Z[g] = \int \mathcal{D}\varphi \; e^{-S[g,\varphi]} \;, \tag{B.1.3}$$

and the correlation function of local operators ϕ_i

$$\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle_g = Z[g]^{-1} \int \mathcal{D}\varphi \; \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n) e^{-S[g,\varphi]} \;. \tag{B.1.4}$$

We note that an action $S[g,\varphi]$ quadratic in φ may be rewritten as the $L^2(S^1\times\mathbb{R}, \sqrt{\det(g)}\mathrm{d}z\mathrm{d}\bar{z})$ inner product

$$S[g,\varphi] \sim (\varphi, D\varphi)_g \;, \tag{B.1.5}$$

where D is some differential operator (e.g. the Laplacian Δ_g or the Dirac operator). Then one may formally consider the partition function as a (formal) Gaussian integral[88] and deduce that basically $Z[g] \sim \det(D)^{-1/2}$, as we have done throughout the main text; cf. the discussions around equations (2.1.24) or (2.2.20). However, as a word of caution we note that the spectrum of D is usually not bounded, necessitating regularization of some sort. Here we will be content with merely assuming well-defined expressions due to some choice of regulator, instead referring to [78, 82, 179, 180] and the references therein for detailed expositions.

Now the implications of (local) conformal symmetry may be deduced from the repercussions the said symmetry has for the partition function $Z[g]$ and the correlation functions of physical operators. Symmetries then lead to relations among correlation functions called Ward identities. Specifically, for any CFT one demands (local) Weyl covariance in the sense that correlation functions behave as

$$\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle_{\Omega(z_i,\bar{z}_i)^2 g} = \prod_{i=1}^{n} e^{-(h_i+\tilde{h}_i)\log(\Omega(z_i,\bar{z}_i)^2)} \langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle_g \;, \tag{B.1.6}$$

and for the partition function

$$Z[\Omega^2 g] = e^{cS_L[g,\log(\Omega^2)]} Z[g] \;, \tag{B.1.7}$$

where Ω^2 is a globally well-defined positive definite function, c is the central charge and S_L is the so-called Liouville action. Furthermore, we note that an infinitesimal variation w.r.t. the metric yields the energy-momentum tensor T_{ab}, which describes the response of the classical system. Correspondingly, one introduces the concept of energy-momentum tensor insertion into correlation functions in order to find the corresponding Ward identities. Explicitly,[89]

$$\langle T_{z\bar{z}}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle = -\sum_{i=1}^{n} h_i\Big(\partial_z \frac{1}{\bar{z}-\bar{z}_i}\Big)\langle \phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle \;, \tag{B.1.8}$$

[88] This also assumes that the eigenfunctions of D form a basis on the space of the functions $\varphi : S^1\times\mathbb{R}\to\mathbb{R}$.

[89] Note that in general there will be additional terms in the Ward identities related to the Weyl anomaly

$$\langle T_{z\bar{z}}\rangle \sim cR \;,$$

which we however may omit in the case of a CFT on $\mathbb{C}^\times$; cf. [178] for the general expression.

and

$$\partial_{\bar{z}}\Big(\langle T_{zz}(z,\bar{z})\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle - \sum_{i=1}^{n}\Big(\frac{h_i}{(z-z_i)^2}+\frac{1}{(z-z_i)}\partial_{z_i}\Big)\langle\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle\Big)=0\ , \tag{B.1.9}$$

suggesting we may consider correlations functions containing the local operator T_{zz} as holomorphic in some open neighbourhood of z and thus we use the notation $T_{zz}(z,\bar{z})=T(z)$. Moreover, we may (at least locally) solve the differential equation (B.1.9)

$$\langle T(z)\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle = \sum_{i=1}^{n}\Big(\frac{h_i}{(z-z_i)^2}+\frac{1}{(z-z_i)}\partial_{z_i}\Big)\langle\phi_1(z_1,\bar{z}_1)\dots\phi_n(z_n,\bar{z}_n)\rangle + \text{ non-singular terms in } z\ . \tag{B.1.10}$$

Analogous results hold for $T_{\bar{z}\bar{z}}$, which we from now on denote $\tilde{T}(\bar{z})$.

Now in order to make contact with the usual QFT language we give the second factor of $S^1\times\mathbb{R}$ the interpretation of an euclidean (imaginary) time, whence on the punctured plane past and future infinity correspond to the limit points 0 and ∞ respectively. Accordingly, the QFT notion of time ordering will be radial ordering in this setup and one often speaks of radial quantization. Correspondingly, one identifies

$$\langle\phi_1\dots\phi_n\rangle = \langle 0|R(\phi_1\dots\phi_n)|0\rangle\ , \tag{B.1.11}$$

where R orders the ϕ_i by the absolute value of $|z_i|$ and $|0\rangle$ denotes the vacuum state of the CFT. Moreover, we note that the two limit points 0 and ∞, corresponding to past and future infinity, may be related by the map $z\mapsto 1/\bar{z}$, which induces the following map on fields

$$\varsigma(\phi(z,\bar{z})) = \bar{z}^{-2h}z^{-2\tilde{h}}\phi(\bar{z}^{-1},z^{-1})\ , \tag{B.1.12}$$

and we extend ς by $\mathbb{C}$-antilinearity. Then the notion of adjoint w.r.t. to the hermitian form on the CFT state space is related to CFT correlators via

$$\langle 0|R(\phi(z_f,\bar{z}_f)^\dagger\psi(z_i,\bar{z}_i))|0\rangle = \langle\varsigma(\phi)\psi\rangle\ . \tag{B.1.13}$$

It is now customary to give identities of correlation functions à la (B.1.10), an operator interpretation called operator product expansion (OPE)

$$T(z_1)\phi(z_2,\bar{z}_2) = \frac{h\phi(z_2,\bar{z}_2)}{(z_1-z_2)^2}+\frac{\partial_{z_2}\phi(z_2,\bar{z}_2)}{(z_1-z_2)} + \text{ non-singular terms }\ , \tag{B.1.14}$$

which is meant to hold within correlation functions (as long as the coordinates of other operators are sufficiently far away) and describes the singular behaviour of correlations functions in the limit that the coordinates of two operators approach each other. As the energy-momentum tensor generates infinitesimal conformal transformations the following OPEs

$$T(z_1)T(z_2) = \frac{c/2}{(z_1-z_2)^4}+\frac{2T(z_2)}{(z_1-z_2)^2}+\frac{\partial_{z_2}T(z_2)}{(z_1-z_2)} + \text{ non-singular terms }\ , \tag{B.1.15}$$

$$T(z_1)\tilde{T}(\bar{z}_2) = -\frac{\pi c}{12}\partial_{z_1}\partial_{\bar{z}_1}\delta^{(2)}(z_1 - z_2) + \text{ non-singular terms } , \tag{B.1.16}$$

are of particular relevance for any CFT. Furthermore, as correlation functions with a $T(z)$ insertion are locally holomorphic in z, the corresponding Laurent modes may also be interpreted as operators denoted

$$L_n = \frac{1}{2\pi i}\oint_{|z|=\varepsilon} \mathrm{d}z z^{n+1} T(z) \ . \tag{B.1.17}$$

Hence, we can relate commutation relations in the operator interpretation to contour integrals of correlation functions in the sense that, e.g.

$$\langle 0|[L_n, \psi(z_2)]Y|0\rangle = \frac{1}{2\pi i}\Bigg(\oint_{|z_1|=|z_2|+\varepsilon} \mathrm{d}z_1 \quad - \oint_{|z_1|=|z_2|-\varepsilon} \mathrm{d}z_1 \Bigg) z_1^{n+1}\langle T(z_1)\psi(z_2)Y\rangle \ , \tag{B.1.18}$$

which in the mindset of radial quantization can be considered an equal-radius commutator. In particular, if one considers an correlation function with $T(z_1)T(z_2)$ insertion the above contour integral formula (B.1.18) together with the corresponding OPE (B.1.15) gives the following commutation relations of the Laurent modes

$$[L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m,-n} \ , \tag{B.1.19}$$

i.e. the Virasoro algebra. It is important to note that $L_0 + \tilde{L}_0$ generates radial dilation and thus has the interpretation of a Hamiltonian.

Finally, we briefly comment on the situation for the open string, i.e. a CFT on the upper half-plane. Certainly, we might still consider contour integrals in a neighbourhood of the boundary albeit with contours that are semi-circles. However, to make contact with most of the CFT literature we briefly discuss the so-called doubling trick. Specifically, if we consider quantities with boundary conditions $A(z) = \tilde{A}(\bar{z})\,, \mathrm{Im}(z) = 0$ (e.g. the energy-momentum tensor, cf. (2.1.16)) we may encapsulate both A and $\tilde{A}$ into a single holomorphic function defined on an open subset of the complex plane

$$A^S(z) = \begin{cases} A(z) & \text{for } \mathrm{Im}(z) \geq 0 \\ \tilde{A}(\bar{z}) & \text{for } \mathrm{Im}(z) < 0 \end{cases} , \tag{B.1.20}$$

which is the above mentioned doubling trick. Accordingly, one has the identity of integrals

$$\int_\alpha \mathrm{d}z A(z) - \int_\alpha \mathrm{d}\bar{z}\tilde{A}(\bar{z}) = \oint_{|z|=\varepsilon} \mathrm{d}z A^S(z) \ , \tag{B.1.21}$$

where the path $\alpha(t) = \varepsilon e^{it}\,,\ t \in [0, 1]$ is supposed to have the usual orientation.

B.2 The CFT state space

We established how the Virasoro algebra generates the local conformal transformations of a given CFT. Correspondingly, a state space of some CFT should be related to a unitary representation of the Virasoro algebra. Note that in an abuse of notation we do not distinguish in the following between the Virasoro generators and their representations. More details on representations of Virasoro algebras may be found in [83–86] and the references therein.

Let us make a few quick comments on the structure of the Virasoro algebra Vir. First we note that the Virasoro algebra is graded by integers with $gr(L_n) = n$ and $gr(c) = 0$ suggesting the decomposition

$$Vir = \Big(\bigoplus_{n\leq -1} \mathbb{C}L_n\Big) \oplus (\mathbb{C}c \oplus \mathbb{C}L_0) \oplus \Big(\bigoplus_{n\geq 1} \mathbb{C}L_n\Big)\ . \tag{B.2.1}$$

Moreover, we note that the three terms above form subalgebras of the Virasoro algebra, which we denote $\mathcal{N}_\pm = \bigoplus_{n\geq 1} \mathbb{C}L_{\pm n}$ and $\mathcal{N}_0 = \mathbb{C}c \oplus \mathbb{C}L_0$. With these two statements in mind we from now on consider modules over the universal enveloping algebra $\mathcal{U}(Vir)$ of the Virasoro algebra.[90] More specifically, given complex numbers h, c one defines a *highest weight module* $M_{c,h}$ of *highest weight* h via

$$L_0|\phi\rangle = h|\phi\rangle\ , \quad L_n|\phi\rangle = 0\ , n \geq 1 \quad \text{and} \quad M_{c,h} = \mathcal{U}(\mathcal{N}_-)|\phi\rangle\ , \tag{B.2.2}$$

where $|\phi\rangle$ is called highest weight state and c is the eigenvalue of the central charge represented by $c\,\mathrm{Id}$. We may think of elements of $M_{c,h}$ as linear combinations of words in the alphabet $L_{-n}\ ,\ n \geq 1$. Then due to the commutation relations (B.1.19), different words of different length may be related, however we note that any such relation will only be among words of the same grading. Correspondingly, using the commutation relations (B.1.19) we might re-express any word in L_{-n_i} by linear combinations of words $L_{-n_k} \dots L_{-n_1}$ that are ordered in the sense $1 \leq n_1 \leq \cdots \leq n_k$ and hence we might consider $M_{c,h}$ as the $\mathbb{C}$-span of such ordered words. Moreover, if these ordered words are linearly independent, i.e. the set

$$\Big\{L_{-n_k} \dots L_{-n_1}|\phi\rangle \,|\, 1 \leq n_1 \leq \cdots \leq n_k\ ,\ k \geq 0\Big\} \tag{B.2.3}$$

is a basis of $M_{c,h}$, the corresponding module is called a *Verma module*. Such a Verma module is known to exist for arbitrary complex c, h and is furthermore (for given c, h) essentially unique, cf. the argument in [85]. Furthermore, the grading of $\mathcal{U}(\mathcal{N}_-)$ induces a grading on $M_{c,h}$ with graded components

$$M^{(N)}_{c,h} = \mathrm{span}_{\mathbb{C}}\bigg(\Big\{L_{-n_k} \dots L_{-n_1}|\phi\rangle \,|\, 1 \leq k \leq N\,,\, \sum_{i=1}^{k} n_i = N \text{ and } 1 \leq n_1 \leq \cdots \leq n_k\Big\}\bigg)\ , \tag{B.2.4}$$

where N is called *level number*. In particular at level zero we have $M^{(0)}_{c,h} = \mathbb{C}|\phi\rangle$. Note that for a Verma module and a given level number $N \geq 1$ the cardinality of the basis of $M^N_{c,h}$ is given by the number of integer partitions $p(N)$ of N. As for the physical content of the above constructions, we recall that L_0 is in radial quantization basically the Hamiltonian, then a highest weight module directly leads to a Hamiltonian bounded from below by the highest weight h, as we may

[90] The universal enveloping algebra of some Lie algebra $\mathfrak{g}$, is defined as the quotient of the tensor algebra (of the underlying vector space) $T(\mathfrak{g})$ with the ideal generated by elements of the form $a \otimes b - b \otimes a - [a, b]$. Note that for a graded Lie algebra the grading induces a grading on the universal enveloping algebra. Moreover, due to the decomposition $\mathcal{N}_+ \oplus \mathcal{N}_0 \oplus \mathcal{N}_-$ of the Virasoro algebra one may employ the Poincaré-Birkhoff-Witt theorem to deduce the following decomposition on the corresponding universal enveloping algebra

$$\mathcal{U}(Vir) = \mathcal{U}(\mathcal{N}_+) \otimes \mathcal{U}(\mathcal{N}_0) \otimes \mathcal{U}(\mathcal{N}_-)\ ,$$

cf. [33].

infer from the commutation relations (B.1.19) that

$$L_0 L_{-n}|\phi\rangle = (L_{-n}L_0 + [L_0, L_{-n}])|\phi\rangle = (h+n)L_{-n}|\phi\rangle \ , \tag{B.2.5}$$

which by induction generalizes to

$$L_0 L_{-n_k} \dots L_{-n_1}|\phi\rangle = \Big(h + \sum_{i=1}^{k} n_i\Big) L_{-n_k} \dots L_{-n_1}|\phi\rangle \ . \tag{B.2.6}$$

This last statement may be rephrased as the statement that $M_{c,h}^{(N)}$ are the eigenspaces of L_0 with (chiral energy) eigenvalue $h+N$. Concerning irreducibility of $M_{c,h}$ we need to understand what proper submodules $M_{c,h}$ possesses. To that end suppose we have some non-zero element $|\psi\rangle \in M_{c,h}$, that is not the highest weight state of $M_{c,h}$ and satisfies $L_n|\psi\rangle = 0\,,\ \forall n \geq 1$. Such a state $|\psi\rangle$ is called *singular vector* of $M_{c,h}$ and it may be expanded in singular vectors of definite level number $|\psi\rangle = \sum_N |\psi^{(N)}\rangle$. Correspondingly, any such singular vector of definite level number $|\psi^{(N)}\rangle$ may be interpreted as highest weight state of some Virasoro submodule isomorphic to $M_{c,h+N}$. In fact it is known that every submodule of $M_{c,h}$ is generated by singular vectors [181], an assertion we will come back to momentarily.

Now in order for $M_{c,h}$ to have the interpretation of a consistent state space of some quantum theory we additionally need a positive-definite hermitian form ω on $M_{c,h}$, such that for all $|\phi_1\rangle, |\phi_2\rangle \in M_{c,h}$ we have

$$\omega\Big(|\phi_1\rangle, L_{-n}|\phi_2\rangle\Big) = \omega\Big(L_n|\phi_1\rangle, |\phi_2\rangle\Big) \ , \quad \forall n \in \mathbb{Z} \ , \tag{B.2.7}$$

and furthermore $\omega(|\phi\rangle, |\phi\rangle) = 1$ for the highest weight state. A Verma module admitting a positive-definite hermitian form satisfying (B.2.7) is also referred to as unitary representation of the Virasoro algebra. We note that the above statements basically determine any such putative hermitian form on the Verma module $M_{c,h}$, which we may illustrate by expressing it on our choice of basis (B.2.3), leading to

$$\omega\Big(L_{-m_l} \dots L_{-m_1}|\phi\rangle, L_{-n_k} \dots L_{-n_1}|\phi\rangle\Big) = \omega\Big(|\phi\rangle, L_{m_1} \dots L_{m_l} L_{-n_k} \dots L_{-n_1}|\phi\rangle\Big). \tag{B.2.8}$$

As a first consequence of the definition of ω we deduce that a (not necessarily positive-definite) hermitian form satisfying (B.2.7), implies that L_0 and c have to be self-adjoint (w.r.t. ω), whence their eigenvalues, c and $h+\mathbb{N}$, have to be real. Moreover, states of different level number have to be orthogonal as otherwise for non-zero $|\psi_i\rangle \in M_{c,h}^{(N_i)}, i = 1,2$, the relation

$$(h+N_2)\omega(|\psi_1\rangle, |\psi_2\rangle) = \omega(|\psi_1\rangle, L_0|\psi_2\rangle) = \omega(L_0|\psi_1\rangle, |\psi_2\rangle) = (h+N_1)\omega(|\psi_1\rangle, |\psi_2\rangle) \ , \tag{B.2.9}$$

would lead to inconsistencies. Similarly, considering any singular vector $|\psi\rangle$ and some basis element of $M_{c,h}$ we find

$$\omega\Big(L_{-m_l} \dots L_{-m_1}|\phi\rangle, |\psi\rangle\Big) = \omega\Big(|\phi\rangle, L_{m_1} \dots L_{m_l}|\psi\rangle\Big) = 0 \ , \tag{B.2.10}$$

i.e. singular vectors are orthogonal to all basis vectors and hence also to any state (including themselves). It follows that, provided non-zero singular vectors exist, they collectively form a

Virasoro submodule that can be described as

$$N_{c,h} = \Big\{|\psi\rangle \in M_{c,h} \mid \text{such that } \omega(|\lambda\rangle, |\psi\rangle) = 0\,,\ \forall|\lambda\rangle \in M_{c,h}\Big\}\ . \tag{B.2.11}$$

Quite importantly one may argue that any proper submodule of $M_{c,h}$ is in fact contained in $N_{c,h}$. To see this note that for any proper submodule S that is allegedly not contained in $N_{c,h}$, there necessarily exists an element $|\sigma\rangle \in S$ such that for some basis vector of $M_{c,h}$ we have

$$\omega(L_{-n_k} \dots L_{-n_1}|\phi\rangle, |\sigma\rangle) = \omega(|\phi\rangle, L_{n_1} \dots L_{n_k}|\sigma\rangle) \neq 0\ . \tag{B.2.12}$$

This implies that $L_{n_1} \dots L_{n_k}|\sigma\rangle \neq 0$ (it would also be an element of S) and due to orthogonality of levels additionally has to be of level zero and thus is proportional to the highest weight state $|\phi\rangle$. But any proper submodule cannot contain the highest weight sate as $M_{c,h} = \mathcal{U}(\mathcal{N}_-)|\phi\rangle$. This means that $N_{c,h}$ is the maximal proper submodule of $M_{c,h}$ and thus the quotient module $M_{c,h}/N_{c,h}$ is a simple module leading to a sensible notion of irreducible representation.[91]

Yet we still have to answer the question whether ω as defined above is positive-definite and we feel it is a good point to switch to the more common notation

$$\langle\phi|L_{m_1} \dots L_{m_l} L_{-n_k} \dots L_{-n_1}|\phi\rangle := \omega\Big(|\phi\rangle, L_{m_1} \dots L_{m_l} L_{-n_k} \dots L_{-n_1}|\phi\rangle\Big)\ . \tag{B.2.13}$$

As a first constraint we found above that for ω to be hermitian and satisfying (B.2.7) we needed h, c to be real. A second constraint may be derived by considering

$$\langle\phi|L_m L_{-m}|\phi\rangle = \langle\phi|[L_m, L_{-m}]|\phi\rangle = 2mh + \frac{c}{12}(m^3 - m)\ , \quad m \geq 1\ , \tag{B.2.14}$$

implying that the requirement of positive-definiteness needs $c \geq 0$ and $h > 0$. To be more systematic we note that due to the orthogonality of states of different level number, one may check positive-definiteness level by level. As we stated above for a Verma module at level N the basis (B.2.3) has $p(N)$ elements, which we label $|\beta_i\rangle\,, i = 1, \dots, p(N)$ for the purpose of the ensuing discussion. In order to probe whether ω is (semi-)definite in $M_{c,h}^{(N)}$ one considers the matrix with components

$$(m_{c,h}^{(N)})_{ij} = \langle\beta_i|\beta_j\rangle\ . \tag{B.2.15}$$

Accordingly, ω is positive (semi-)definite on $M_{c,h}^{(N)}$ if the matrix $m_{c,h}^{(N)}$ is positive (semi-)definite and the existence of zero eigenvalues is equivalent to the existence of singular vectors. Moreover, as ω is hermitian the same will be true for $m_{c,h}^{(N)}$ and thus the eigenvalues and the determinant have to be real. Also from the form of the commutation relations of the Virasoro algebra (B.1.19) we infer that the elements of $m_{c,h}^{(N)}$ are polynomials in c, h with real coefficients, suggesting that the eigenvalues are continuous functions of c, h. A necessary condition for positive-definiteness of the matrix $m_{c,h}^{(N)}$ is that its determinant is positive. Conveniently, a formula for the determinant of $m_{c,h}^{(N)}$ was found by Kac, specifically

$$\det(m_{c,h}^{(N)}) = \kappa \prod_{\substack{p,q\in\mathbb{N} \\ 1\leq pq\leq N}} (h - h_{p,q}(c))^{p(N-pq)}\ , \tag{B.2.16}$$

[91] A simple module is a module that does not contain any proper submodule.

with κ some positive number and roots described by

$$h_{p,q}(c) = \frac{1}{48}\Big[(13-c)(p^2+q^2) + \sqrt{(c-1)(c-25)}(p^2-q^2) - 24pq + 2(c-1)\Big] \,. \qquad \text{(B.2.17)}$$

Let us briefly discuss the roots of the determinant for $c, h > 0$. First we note that the root $h_{q,q}(c)$ is real and in fact given by

$$h_{q,q}(c) = -\frac{1}{24}(q^2-1)(c-1) \,. \qquad \text{(B.2.18)}$$

It follows that $h_{1,1}(c) = 0$ and $h_{q,q}(c)$ with $q > 1$ is negative (positive) for $c > 1$ ($0 < c < 1$). Additionally, for $1 < c < 25$ all other roots are complex with $h_{p,q} = \bar{h}_{q,p}$ meaning that $(h - h_{p,q})(h - h_{q,p}) > 0$ and thus the determinant in question is positive. Furthermore, due to $\sqrt{(c-1)(c-25)} < (c-13)$ for $c > 25$, we see that for such central charges the root $h_{p,q}(c)$ is negative and hence the determinant has to be positive again. Now a sufficient condition for positive-definiteness is that all eigenvalues of $m^{(N)}_{c,h}$ are positive. To that end we reiterate that the eigenvalues are real continuous functions of c, h, but we just established that the real roots of the determinant for $c > 1$ are negative, so no eigenvalue is allowed to have a root for these value of $c > 1, h > 0$, hence it's sign cannot change. Consequently, it should be enough to show that for some $c > 1$ and $h > 0$ all eigenvalues are positive (for all $m^{(N)}_{c,h}$). Then the usual argument goes along the lines that asymptotically $h \to \infty$ the matrix $m^{(N)}_{c,h}$ will become diagonal with all non-zero entries positive (cf. [83,85]) and therefore for $c > 1, h > 0$ any Verma module may be equipped with a positive-definite hermitian form. Note that for the case $c > 1, h = 0$ all determinants will vanish due to $h_{1,1} = 0$ but these zeros are in fact related to the existence of singular vector and we might avoid a vanishing determinant by considering $M_{c,h}/N_{c,h}$ instead.

As we have seen above for $0 \leq c < 1$ there are positive roots rendering these regions of c, h more complicated. We will be content with just giving the result, i.e. for $0 \leq c < 1$ unitary representations exist only for the following rational values of the central charge

$$c = 1 - \frac{6}{m(m+1)} \,, \quad m \geq 2 \,,\ m \in \mathbb{N} \,, \qquad \text{(B.2.19)}$$

with allowed conformal weights

$$h(m; r, s) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)} \,, \qquad \text{(B.2.20)}$$

where $1 \leq r \leq s \leq m-1$; cf. [85, 182, 183] for details.

To conclude the discussion on representations of the Virasoro algebra, we will briefly sketch the relation to the spectrum of the free open string as discussed in section 2.1; the ensuing argument follows [85, 98, 184]. We will start by considering the Fock space of the α^μ_m and consider the physical state conditions imposed by BRST later. Recall that the Laurent modes α^μ_m satisfy the commutation relations

$$[\alpha^\mu_m, \alpha^\nu_n] = m\delta_{n,-m}\eta^{\mu\nu} \,. \qquad \text{(B.2.21)}$$

Accordingly, the Laurent modes admit a Fock space representation with ground state labeled by the eigenvalue of α_0^μ, such that

$$\alpha_n^\mu|0;\alpha_0^\mu\rangle = 0 \quad \text{for} \quad n \geq 1 \ , \tag{B.2.22}$$

and thus for $n \geq 1$ ($n \leq -1$), the α_n^μ have the interpretation of annihilation (creation) operators.[92] As a reminder we note $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$ and we find the same ground state as discussed in section 2.1. Then for this particular CFT the Virasoro generators were related to the Laurent modes via

$$L_{X;m} = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\eta_{\mu\nu}\alpha_{m-n}^\mu\alpha_n^\nu: \ , \qquad m \in \mathbb{Z} \ , \tag{B.2.23}$$

and essentially we want to argue that these $L_{X;m}$ in fact satisfy the commutation relations of the Virasoro algebra on the Fock space of the α_n^μ (therefore deserving the name Virasoro generators). Taking normal ordering into account it follows from the commutation relations of the Laurent modes (B.2.21), that on Fock space we have

$$[L_{X;m}, \alpha_n^\mu] = -n\alpha_{m+n}^\mu \ . \tag{B.2.24}$$

This result implies that the $L_{X;m}$ as defined via the Laurent modes α_m^μ (cf. equation (B.2.23)) satisfy the Virasoro algebra on the Fock space. Moreover, from (B.2.23) we see that for $m > 0$ every term in $L_{X;m}$ contains at least one annihilation operator and thus $L_m|0;\alpha_0^\mu\rangle = 0$ for $m > 0$ as needed for a highest weight module (B.2.2). However, so far we have not considered ghosts and the corresponding constraints coming from BRST symmetry and one has to expect some sort of ramification of this negligence. The issue manifests itself in the fact that the usual hermitian form one defines on Fock space (such that $(\alpha_n)^\dagger = \alpha_{-n}$) will allow for negative norm states

$$\langle 0;\alpha_0^\mu|\alpha_n^0\alpha_{-n}^0|0;\alpha_0^\mu\rangle = -n \ . \tag{B.2.25}$$

To put this issue into the context of the Verma modules studied above, note that we may (after a choice of Lorentz frame) roughly interpret the bosonic string as two CFTs with $c_1 = 25$, $h_1 > 0$ and $c_2 = 1$, $h_2 < 0$, of which the former allows for irreducible unitary Verma modules while the latter is at least void of any submodules. Now one has to demand additional constraints on the state space as we did via BRST in section 2.1 in order to get rid of negative norm states; cf. [98] for the detailed expression of the free open-string state space via Verma modules.

[92] This space might be interpreted as infinitely many uncoupled harmonic oscillators. It has a very concrete realization as polynomials in infinitely many variables $\mathbb{C}[x_1, x_2, \ldots]$, s.t. for $n \geq 1$ we have $\alpha_{-n}/\sqrt{n} = x_n$ (multiplication operator) and $\alpha_n/\sqrt{n} = \partial_{x_n}$; cf. [85] for details.

Appendix C

Remarks on the Drinfeld associator

In this appendix we briefly discuss certain aspects of the Drinfeld associator closely following [29, 33, 152, 157, 185–187]. However, we stress that our account is limited in scope focusing mainly on a few remarks on the curious connection to Knizhnik-Zamolodchikov equations and quasi-Hopf algebras. For more detailed accounts on the Drinfeld associator including proofs of the statements below as well as the relevance of the Drinfeld associator to other fields in mathematics and physics we refer the reader to the works just mentioned and the references therein.

We consider the one-variable case of the Knizhnik-Zamolodchikov differential equations

$$\partial_z G(z) = \left(\frac{x_0}{z} + \frac{x_1}{z-1}\right) G(z) \ , \tag{C.0.1}$$

where x_0, x_1 denote formal non-commutative variables and $z \in \mathbb{P} \setminus \{0, 1, \infty\}$. Solutions to this equation are multi-valued functions on $z \in \mathbb{P} \setminus \{0, 1, \infty\}$ taking values in the ring of formal power series in x_0, x_1 with coefficients in $\mathbb{C}$. Of particular interest are the two solutions of the form

$$G_0 \sim \exp(x_0 \log(z)) \qquad \text{as } z \to 0 \ , \tag{C.0.2}$$

$$G_1 \sim \exp(x_1 \log(1-z)) \qquad \text{as } z \to 1 \ . \tag{C.0.3}$$

The Drinfeld associator Φ is given by $G_0 = \Phi G_1$ or alternatively $\Phi = G_0 {G_1}^{-1}$. This suggests that Φ is independent of z and may be expressed via the path-ordered exponential as given in the main text

$$\Phi = \tilde{\mathcal{P}} \exp\left(\int_0^1 \left[\frac{x_0}{z} + \frac{x_1}{z-1}\right] \mathrm{d}z\right) \ , \tag{C.0.4}$$

cf. [29, 152, 157, 186] for further details.[93]

Intriguingly, the Drinfeld associator also appears in the context of quasi-Hopf algebras over the complex numbers. Let us briefly comment on this relation. Consider an (unital) $\mathbb{C}$-algebra A with algebra morphisms $\Delta : A \to A \otimes A$ and $\varepsilon : A \to \mathbb{C}$ and invertible $\tilde{\Phi} \in A^{\otimes 3}$ satsifying

$$(\mathrm{id} \otimes \Delta)(\Delta(a)) = \tilde{\Phi} \cdot (((\Delta \otimes \mathrm{id})(\Delta(a))) \cdot \tilde{\Phi}^{-1} \ , \tag{C.0.5}$$

$$(\Delta \otimes \mathrm{id} \otimes \mathrm{id})(\tilde{\Phi}) \cdot (\mathrm{id} \otimes \mathrm{id} \otimes \Delta)(\tilde{\Phi}) = (\tilde{\Phi} \otimes 1) \cdot (\mathrm{id} \otimes \Delta \otimes \mathrm{id})(\tilde{\Phi}) \cdot (1 \otimes \tilde{\Phi}) \ , \tag{C.0.6}$$

[93] Note that we omitted the necessary regularization at the singular points 0 and 1, which may be carried out by introducing tangential basepoints; cf. [152] for a detailed exposition for the Drinfeld associator.

$$(\varepsilon \otimes \mathrm{Id}) \circ \Delta = (\mathrm{Id} \otimes \varepsilon) \circ \Delta \; , \tag{C.0.7}$$

$$(\mathrm{id} \otimes \varepsilon \otimes \mathrm{Id})(\tilde{\Phi}) = 1 \; . \tag{C.0.8}$$

This structure is called a quasi-bialgebra and in a slight abuse of terminology Δ is called coproduct.[94] The definition of quasi-bialgebra ascertains that $\tilde{\Phi}$ gives rise to an associativity isomorphism for tensor products of representations of A, cf. e.g. proposition 16.1.2 of [185] for details on this statement. Drinfeld showed in [29] that in the context of quasi-triangular quasi-Hopf algebras[95] over $\mathbb{C}$, $\tilde{\Phi}$ may be explicitly constructed using Knizhnik-Zamolodchikov equations and is given by the Drinfeld associator; cf. also the concise treatment in chapter 16 of [185].

[94] A quasi-Hopf algebra may then be obtained by introducing an antipode on the quasi-bialgebra, which is however not relevant for our discussion and we omit it here; cf. [29, 33, 185, 186] for details.

[95] Note that one may describe the coproduct of a quasi-Hopf algebra as coassociative up to similarity transform. A quasi-triangular quasi-Hopf algebra has a coproduct, which is additionally cocommutative up to similarity transform. We will not comment any further on these structures as they are beyond the scope of this work, cf. [29, 33, 185, 186] for details.

Appendix D

Chen's iterated integrals and MZVs

This appendix provides the definition and most basic properties of Chen's iterated integrals as used throughout chapter 3. Moreover, we give the representations of MZVs as iterated integrals on the Riemann sphere and briefly summarize some of their properties.

D.1 Chen's iterated integrals

In this appendix we give Chen's iterated integral and collect some of its properties, closely following [157, 188]. Given a smooth manifold M, some piecewise smooth path $\gamma : [0,1] \to M$ and smooth one-forms ω_i on M, Chen's iterated integral is defined as

$$\int_\gamma \omega_{i_1} \dots \omega_{i_n} = \int_{0<t_1<\dots<t_n<1} f_{i_1}(t_1)\mathrm{d}t_1 \dots f_{i_n}(t_n)\mathrm{d}t_n \ ,$$

where $f_i(t)\mathrm{d}t$ is the pullback of ω_i along the path γ. This integral is parametrization independent and has several additional important properties used throughout the main text, which we summarize here for the convenience of the reader.

- The path inversion formula relates the iterated integral defined on the inverse path γ^{-1} to the iterated integral defined via γ

$$\int_{\gamma^{-1}} \omega_1\omega_2 \dots \omega_{n-1}\omega_n = (-1)^n \int_\gamma \omega_n\omega_{n-1} \dots \omega_2\omega_1 \ . \tag{D.1.1}$$

- The shuffle product is given by

$$\int_\gamma \omega_{i_1} \dots \omega_{i_n} \int_\gamma \omega_{j_1} \dots \omega_{j_m} = \sum_{\sigma\in\Sigma(n,m)} \int_\gamma \omega_{\sigma(i_1)} \dots \omega_{\sigma(i_n)}\omega_{\sigma(j_1)} \dots \omega_{\sigma(j_m)} \ , \tag{D.1.2}$$

  where $\Sigma(n,m)$ is the set of n,m shuffles.

- The composition of path formula for two piecewise smooth paths α, β, satisfying the additional property $\alpha(1) = \beta(0)$, is given by

$$\int_{\alpha\beta} \omega_1 \dots \omega_n = \sum_{i=0}^{n} \int_\alpha \omega_1 \dots \omega_i \int_\beta \omega_{i+1} \dots \omega_n \ , \tag{D.1.3}$$

  where the empty integral is defined to be one.

D.2 MZVs and MZVs at roots of unity

This appendix gives the definitions of both MZVs and MZVs at roots of unity. Both of these may be expressed via nested sums or iterated integrals, with the latter representation being particularly convenient to make the connection of MZVs with the Drinfeld associator and eventually to the constant terms of TEMZVs as presented in subsections 3.4.2 and 3.6.2.

We begin by giving the definition of Multiple Zeta Values as nested sums

$$\zeta_{k_1,\dots,k_l} = \sum_{0<n_1<\dots<n_l} \frac{1}{n_1^{k_1}\dots n_l^{k_l}} \,, \tag{D.2.1}$$

where $k_i \geq 1$ with the exception of $k_l \geq 2$ needed for convergence. Another prominent form is to write MZVs as iterated integrals on the thrice-punctured Riemann sphere $\mathbb{P}\setminus\{0,1,\infty\}$. Specifically, introducing the one-forms

$$\omega_0 = \frac{\mathrm{d}z}{z} \,, \quad \omega_1 = \frac{\mathrm{d}z}{z-1} \,, \tag{D.2.2}$$

we may write any MZV defined by (D.2.1) via the iterated integral

$$\zeta_{k_1,\dots,k_l} = (-1)^l \int_0^1 \omega_1\omega_0^{k_1-1}\dots\omega_1\omega_0^{k_l-1} \,. \tag{D.2.3}$$

Note that we also need $k_l \geq 2$ otherwise we would have a singular expression as ω_1 has a pole at 1. The above definitions can be extended to also contain the singular cases $n_l = 1$ as well as iterated integrals which start with a form ω_0. For this purpose we again make use of the notion of tangential basepoint, where we choose to regulate such that

$$\lim_{\varepsilon\to 0} \mathrm{Reg} \int_{\varepsilon}^{1-\varepsilon} \omega_0 = 0 \,, \quad \lim_{\varepsilon\to 0} \mathrm{Reg} \int_{\varepsilon}^{1-\varepsilon} \omega_1 = 0 \,. \tag{D.2.4}$$

The integrals starting with ω_0 may then be expressed by MZVs using the shuffle relations.

From the definitions of MZVs as iterated integrals in the one-forms (D.2.2) one can surmise the connection[96] to the Drinfeld associator

$$\Phi(e_0,e_1) = \tilde{\mathcal{P}}\exp\left(\int_0^1 \left[\frac{e_0}{z} + \frac{e_1}{z-1}\right]\mathrm{d}z\right) \,. \tag{D.2.5}$$

However making this connection precise is beyond the scope of this appendix. For a detailed discussion on this topic we refer the intrigued reader to the review of Francis Brown in [157] and references therein.

Multiple Zeta values at N-th root of unity are defined by the nested sum

$$\zeta\left(\begin{smallmatrix} k_1 \,, & \dots \,, & k_l \\ \epsilon_1 \,, & \dots \,, & \epsilon_l \end{smallmatrix}\right) = \sum_{0<n_1<\dots<n_l} \frac{\epsilon_1^{n_1}\dots\epsilon_l^{n_l}}{n_1^{k_1}\dots n_l^{k_l}} \,, \tag{D.2.6}$$

[96] Note that the path ordered exponential (D.2.5) also needs to be regularized, as it is plagued by singularities at the points zero.

where $\epsilon_i^N = 1$. These may be expressed as iterated integrals of one-forms with poles at roots of unity and zero respectively, given by

$$\omega_0 = \frac{\mathrm{d}z}{z} , \quad \omega_i = \frac{\mathrm{d}z}{z - \epsilon_i} . \tag{D.2.7}$$

Then we may write MZVs at roots of unity by iterated integral in these forms

$$\zeta\left({k_1 \atop \epsilon_{i_2}/\epsilon_{i_1}} , {k_2 \atop \epsilon_{i_3}/\epsilon_{i_2}} , {\ldots \atop \ldots} , {n_{l-1} \atop \epsilon_{i_l}/\epsilon_{i_{l-1}}} , {k_l \atop \epsilon_l} \right) = (-1)^l \int_0^1 \omega_{i_1} \omega_0^{n_1 - 1} \omega_{i_2} \omega_0^{n_2 - 1} \ldots \omega_{i_l} \omega_0^{n_l - 1} . \tag{D.2.8}$$

As before this might be singular if $\epsilon_{i_l} = 1$ and $k_l = 1$ and needs to be regularized, which works analogously to the case above. The iterated integrals as defined in (D.2.8) are related to the expansion of the cyclotomic Drinfeld associator

$$\Phi(e_0, e_1, \ldots, e_N) = \tilde{\mathcal{P}} \exp\left(\int_0^1 \left[\frac{e_0}{z} + \sum_{i=1}^{N} \frac{e_i}{z - \epsilon_i} \right] \mathrm{d}z \right) . \tag{D.2.9}$$

Both MZVs as well as MZVs at roots of unity satisfy a plethora of relations, we will not use any of those in the main text but briefly mention them; for the case of MZVs a thorough explanation can be found in [157]. The study of such relations is an active field of research in mathematics, and we will be content with reducing our exposition to some remarks and point the reader to possible starting points in the literature.

Firstly, we note that due to their incarnation as iterated integrals MZVs satisfy shuffle relations. Secondly, the realization as nested sums leads to so-called stuffle relations, which are obtained by decomposing products of summation ranges. These stuffle relations can be regarded as a discrete analogue of the shuffle relations, where in contradistinction to the iterated integrals the diagonals of the space of summation indices matter as they are not of measure zero. Finally, there is another source of relations, namely the Hoffmann relations of which the simplest example is $\zeta_{1,2} = \zeta_3$. However, we will not discuss Hoffmann relations any further, but refer to [157, 169] for details. These three types of relations are referred to as standard relations and are conjectured to generate all relations of MZVs (over $\mathbb{Q}$). For MZVs at roots of unity there are additional relations; specifically, it is known that the standard relations are only sufficient to describe all relations up to $N = 2$ [169]. At $N \geq 3$ there are relations that come from the fact that different roots of unity may be related by complex conjugation, as well as so called distribution relations that stem from the fact that different sums of roots of unity may be related to each other. These additional relations together with the standard relations are conjectured to be enough for $N = 3, 4$. Finally, note that for MZVs and MZVs at second root of unity there are online databases [19, 189, 190]. Furthermore, there is work for some cases $N \geq 3$ see [22, 169, 170] and references therein.

Appendix E

Explicit computations

This appendix contains the conventions we use for the Jacobi θ functions as well as detailed derivations for several equations concerning TEMZVs used throughout chapter 3.

E.1 Jacobi theta functions

We use the following conventions for θ functions [191]

$$\begin{aligned}\theta_1(z;\tau) &= 2q^{1/8}\,\sin(\pi z)\prod_{n=1}^{\infty}(1-q^n)(1-2q^n\,\cos(2\pi z)+q^{2n})\\ \theta_1'(0;\tau) &= 2\pi q^{1/8}\prod_{n=1}^{\infty}(1-q^n)^3\\ \theta_4(z;\tau) &= \prod_{n=1}^{\infty}(1-q^n)(1-2q^{(n-1)/2}\,\cos(2\pi z)+q^{2n-1})\;,\end{aligned} \tag{E.1.1}$$

which all turn out to be positive given $z\in[0,1]$ and $q\in[0,1]$, i.e. in the case relevant for the genus-one open-string amplitude. The positivity of these functions on $z,q\in[0,1]$ will allow us to ignore the absolute values in the Green function; cf. section 3.2. We consider θ_4 as it is related to θ_1 by a shift of $\tau/2$ and thus allows us to rewrite the Green function for insertions on different boundaries as a function of $\mathrm{Re}(z_i-z_j)$. The precise relation between θ_1 and θ_4 is given by,

$$\theta_1(z+\tau/2;\tau) = i\exp(-i\pi z)q^{-1/8}\theta_4(z;\tau)\;. \tag{E.1.2}$$

E.2 Fay identity for weighting functions

In this appendix we elaborate on the relations among the weighting functions $f^{(n)}$ as given in (3.3.37), which follow from the Fay identity for the generating function $\Omega(z,\alpha;\tau)$. For our purposes we will be specifically interested in the case where both functions depend on the same variable z, i.e. the Fay identity specialized to $\Omega(z-a_1,\alpha_1;\tau)\Omega(z-a_2,\alpha_2;\tau)$, which we repeat here for purely altruistic reasons

$$\begin{aligned}\Omega(z-a_1,\alpha_1;\tau)\Omega(z-a_2,\alpha_2;\tau) &= \Omega(z-a_1,\alpha_1+\alpha_2;\tau)\Omega(a_1-a_2,\alpha_2;\tau)\\ &\quad+\Omega(z-a_2,\alpha_1+\alpha_2;\tau)\Omega(a_2-a_1,\alpha_1;\tau)\;.\end{aligned} \tag{E.2.1}$$

Then equating coefficients of the monomials $\alpha_1^n \alpha_2^m$ in the Fay identity (E.2.1), leads to

$$\begin{aligned}&\sum_{n,m\in\mathbb{N}} \alpha_1^{n-1}\alpha_2^{m-1} f^{(n)}(z-a_1) f^{(m)}(z-a_2)\\ &= \sum_{i,j\in\mathbb{N}} (\alpha_1+\alpha_2)^{i-1}\left(f^{(i)}(z-a_1)f^{(j)}(a_1-a_2)\alpha_2^{j-1} + f^{(i)}(z-a_2)f^{(j)}(a_2-a_1)\alpha_1^{j-1}\right)\ .\end{aligned} \tag{E.2.2}$$

The contribution from the $i = 0$ term in the double sum needs to be considered separately. Using $f^{(0)}(z-a_j) = 1$ we have

$$\begin{aligned}&\sum_{j\in\mathbb{N}}(\alpha_1+\alpha_2)^{-1}\left(\alpha_1^{j-1} - (-1)^{j-1}\alpha_2^{j-1}\right) f^{(j)}(a_2-a_1)\\ &= \alpha_1^{-1}\alpha_2^{-1} + \sum_{j\geq 2}(\alpha_1+\alpha_2)^{-1}\left(\alpha_1^{j-1} - (-1)^{j-1}\alpha_2^{j-1}\right) f^{(j)}(a_2-a_1)\\ &= \alpha_1^{-1}\alpha_2^{-1} + \sum_{j\in\mathbb{N}}\sum_{a=0}^{j}(-1)^{j-a}\alpha_1^a\alpha_2^{j-a} f^{(j+2)}(a_2-a_1)\\ &= \alpha_1^{-1}\alpha_2^{-1} - \theta_{n\geq 1}\theta_{m\geq 1}\sum_{n,m\in\mathbb{N}}(-1)^m \alpha_1^{n-1}\alpha_2^{m-1} f^{(n+m)}(a_2-a_1)\ ,\end{aligned} \tag{E.2.3}$$

where in the third line we used

$$(\alpha_1^j - (-1)^j\alpha_2^j) = (\alpha_1+\alpha_2)\sum_{k=0}^{j-1}(-1)^{j-1-k}\alpha_1^k\alpha_2^{j-1-k}\ ,\ j>0\ . \tag{E.2.4}$$

For the remaining terms with $i > 0$ we may use the binomial theorem

$$\begin{aligned}&\sum_{i,j\in\mathbb{N}} (\alpha_1+\alpha_2)^i f^{(i+1)}(z-a_1) f^{(j)}(a_1-a_2)\alpha_2^{j-1}\\ &= \sum_{i,j\in\mathbb{N}}\sum_{k=0}^{i}\binom{i}{k}\alpha_1^{i-k}\alpha_2^{j+k-1} f^{(i+1)}(z+a_1)f^{(j)}(a_1-a_2)\\ &= \sum_{n,m\in\mathbb{N}}\alpha_1^n\alpha_2^{m-1}\sum_{p=0}^{n}\binom{n+m}{p} f^{(n+m+1)}(z+a_1)f^{(n-p)}(a_1-a_2)\\ &= \sum_{n,m\in\mathbb{N}}\alpha_1^n\alpha_2^{m-1}\sum_{q=0}^{m}\binom{n+q}{q} f^{(n+q+1)}(z+a_1)f^{(m-q)}(a_1-a_2)\ ,\end{aligned} \tag{E.2.5}$$

and similarly for the remaining contribution

$$\begin{aligned}&\sum_{i,j\in\mathbb{N}} (\alpha_1+\alpha_2)^i f^{(i+1)}(z-a_2) f^{(j)}(a_2-a_1)\alpha_2^{j-1}\\ &= \sum_{n,m\in\mathbb{N}}\alpha_1^n\alpha_2^{m-1}\sum_{q=0}^{m}\binom{n+q}{q} f^{(n+q+1)}(z-a_2)f^{(m-q)}(a_2-a_1)\ .\end{aligned} \tag{E.2.6}$$

Combining equations (E.2.2,E.2.3,E.2.5,E.2.6) we have

$$\begin{aligned} f^{(n)}(z-a_1)f^{(m)}(z-a_2) = &-(-1)^m f^{(n+m)}(a_2-a_1) \\ &+\sum_{j=0}^{m}\binom{n+j-1}{j} f^{(n+j)}(z-a_1)f^{(m-j)}(a_1-a_2) \\ &+\sum_{j=0}^{n}\binom{m+j-1}{j} f^{(m+j)}(z-a_2)f^{(n-j)}(a_2-a_1)\ . \end{aligned} \tag{E.2.7}$$

E.3 Endpoint removal

In the main text we illustrated how, for low length, we may rewrite any elliptic iterated integral Γ with endpoint dependent twists in terms of several integrals without any endpoint dependent twist. We now go on to give formulas for a more general setup, closely following [50]. Note however that any such formulas are obtained by recursion w.r.t. the length of the corresponding iterated integrals. Thus the length two expressions presented in section 3.3 can be thought of as the starting point of the recursion.

We begin by sketching how the "endpoint removal" works for the case where one b_i depends on z and then comment on the case with multiple labels of this form. In this context it is convenient to split $b_i = z + b_{i;0} = z + s + r\tau$, where $b_{i;0}$ is the z independent part of b_i. As for the length two case discussed in the main text the starting point is the equation

$$\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & b_1 \end{smallmatrix}; z\right) = \lim_{z\to 0}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & b_1 \end{smallmatrix}; z\right) + \int_0^z \mathrm{d}y \frac{\mathrm{d}}{\mathrm{d}y}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & b_1 \end{smallmatrix}; y\right)\ . \tag{E.3.1}$$

For the boundary term we again start by considering the most singular case possible, i.e. all $n_i = 1$. Then upon q-expansion and considering exclusively the pole parts we find a Goncharov polylogarithm[97]

$$\int_{0<t_1<\dots<t_l<z} \mathrm{d}t_l \dots \mathrm{d}t_1 \frac{1}{t_l - b_l}\dots\frac{1}{t_1-b_1} = G(b_1,\dots,b_l;z)\ . \tag{E.3.2}$$

These have the scaling property $G(b_1,\dots,b_l;z) = G(b_1/z,\dots,b_l/z;1)$, which suggests that iff all $b_i = c_i z$ we have

$$\lim_{z\to 0} G(b_1,\dots,b_l;z) = G(c_1,\dots,c_l;1)\ . \tag{E.3.3}$$

All other possible b_i will result in a vanishing integral in the limit, cf. also the example in the main text eq. (3.3.34). Hence, in the context of string amplitudes we will have to deal only with $c_i \in \{0,1\}$ and therefore MZVs. If we consider higher q orders the corresponding (regularized) integrals, will be of order at least z and hence vanish in the limit. Analogously, iterated integrals with at least one $n_i \neq 1$ will vanish in the limit $z \to 0$.

The total derivative may be computed by the Leibniz integration rule, i.e. as partial derivative

[97] Again we omit explicitly spelling out the endpoint regularization for the sake of a slicker presentation.

with respect to the endpoint and the integrand

$$\frac{\mathrm{d}}{\mathrm{d}z}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & b_1 \end{smallmatrix};z\right) = f^{(n_l)}(z-b_l)\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_1 \\ b_{l-1} & \dots & b_1 \end{smallmatrix};z\right) + \int_0^z \mathrm{d}y\partial_z f^{(n_l)}(y-b_l)\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_1 \\ b_{l-1} & \dots & b_1 \end{smallmatrix};y\right)\;, \tag{E.3.4}$$

where in the second line all z-dependence is in the b_i. We consider the cases of the first and last label separately

$$\partial_w\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ w+b_{l;0} & \dots & b_1 \end{smallmatrix};z\right) = -f^{(n_l)}(z-w-b_{l;0})\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_1 \\ b_{l-1} & \dots & b_1 \end{smallmatrix};z\right) + \int\limits_0^z \mathrm{d}y f^{(n_l)}(y-w-b_{l;0})f^{(n_{l-1})}(y-b_{l-1})\,\Gamma\left(\begin{smallmatrix} n_{l-2} & \dots & n_1 \\ b_{l-2} & \dots & b_1 \end{smallmatrix};y\right) \tag{E.3.5}$$

$$\partial_w\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & w+b_{1;0} \end{smallmatrix};z\right) = -\int\limits_{\Delta_{2,l;z}} \mathrm{d}y_2 f^{(n_1)}(y_2-w-b_{1;0})f^{(n_2)}(y_2-b_2)\prod_{j=3}^{l}\mathrm{d}y_j f^{(n_j)}(y_j-b_j) + f^{(n_1)}(-w-b_{1;0})\,\Gamma\left(\begin{smallmatrix} n_l & \dots & n_2 \\ b_l & \dots & b_2 \end{smallmatrix};z\right)\;, \tag{E.3.6}$$

where we used integration by parts. For the generic case we have

$$\begin{aligned}
&\partial_w\Gamma\left(\begin{smallmatrix} n_l & \dots & n_i & \dots & n_1 \\ b_l & \dots & w+b_{i;0} & \dots & b_1 \end{smallmatrix};z\right) \\
&= \int\limits_{\Delta_{i+1,l;z}} \prod_{j=i+1}^{l}\mathrm{d}y_j f^{(n_j)}(y_j-b_j)\partial_w\Gamma\left(\begin{smallmatrix} n_i & \dots & n_1 \\ w+b_{i;0} & \dots & b_1 \end{smallmatrix};y_{i+1}\right) \\
&= \int\limits_{\Delta_{i+1,l;z}} \prod_{j=i+1}^{l}\mathrm{d}y_j f^{(n_j)}(y_j-b_j)\Big[-f^{(n_i)}(y_{i+1}-w-b_{i;0})\,\Gamma\left(\begin{smallmatrix} n_{i-1} & \dots & n_1 \\ b_{i-1} & \dots & b_1 \end{smallmatrix};y_{i+1}\right) \\
&\quad + \int\limits_0^{y_{i+1}} \mathrm{d}y f^{(n_i)}(y-w-b_{i;0})f^{(n_{i-1})}(y-b_{i-1})\,\Gamma\left(\begin{smallmatrix} n_{i-2} & \dots & n_1 \\ b_{i-2} & \dots & b_1 \end{smallmatrix};y\right)\Big]\;,
\end{aligned} \tag{E.3.7}$$

using (E.3.5) in the last equality. As previously stated we may compute the total derivative as the sum of the contributions of the partial derivatives with respect to the endpoint (E.3.4) and the partial derivative w.r.t. the label depending on the endpoint (E.3.5,E.3.6,E.3.7). Collecting the contributions from the partial derivatives with respect to the endpoint and some arbitrary label, we find for the cases, where either the first or the last label depends on z, the following expressions

$$\frac{\mathrm{d}}{\mathrm{d}z}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ z+b_{l;0} & \dots & b_1 \end{smallmatrix};z\right) = \int\limits_0^z \mathrm{d}y f^{(n_l)}(y-z-b_{l;0})f^{(n_{l-1})}(y-b_{l-1})\,\Gamma\left(\begin{smallmatrix} n_{l-2} & \dots & n_1 \\ b_{l-2} & \dots & b_1 \end{smallmatrix};y\right) \tag{E.3.8}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & z+b_{1;0} \end{smallmatrix};z\right) = f^{(n_l)}(z-b_l)\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_1 \\ b_{l-1} & \dots & z+b_{1;0} \end{smallmatrix};z\right) + f^{(n_1)}(-z-b_{1;0})\,\Gamma\left(\begin{smallmatrix} n_l & \dots & n_2 \\ b_l & \dots & b_2 \end{smallmatrix};z\right) - \int\limits_{\Delta_{2,l;z}} \mathrm{d}y_2 f^{(n_1)}(y_2-z-b_{1;0})f^{(n_2)}(y_2-b_2)\prod_{j=3}^{l}\mathrm{d}y_j f^{(n_j)}(y_j-b_j)\;, \tag{E.3.9}$$

whereas for the case, where some label $b_i = z + r\tau$ with $i \neq 1, r$ we find

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}z}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_i & \dots & n_1 \\ b_l & \dots & z+b_{i;0} & \dots & b_1 \end{smallmatrix};z\right) &= f^{(n_l)}(z-b_l)\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_i & \dots & n_1 \\ b_{l-1} & \dots & z+b_{i;0} & \dots & b_1 \end{smallmatrix};z\right) \\
&+ \int\limits_{\Delta_{i+1,l;z}} \prod_{j=i+1}^{l} \mathrm{d}y_j f^{(n_j)}(y_j-b_j)\Big[-f^{(n_i)}(y_{i+1}-z-b_{i;0})\,\Gamma\left(\begin{smallmatrix} n_{i-1} & \dots & n_1 \\ b_{i-1} & \dots & b_1 \end{smallmatrix};y_{i+1}\right) \\
&+ \int_0^{y_{i+1}} \mathrm{d}y f^{(n_i)}(y-z-b_{i;0}) f^{(n_{i-1})}(y-b_{i-1})\,\Gamma\left(\begin{smallmatrix} n_{i-2} & \dots & n_1 \\ b_{i-2} & \dots & b_1 \end{smallmatrix};y\right)\Big]\,. \qquad \text{(E.3.10)}
\end{aligned}
$$

As in the main text we rewrite products of the form $f^{(n_1)}(y-a_1)f^{(n_2)}(y-a_2)$ using the following consequence of the Fay identity

$$
\begin{aligned}
&f^{(n)}(y-a_1)f^{(m)}(y-a_2) \\
&\quad= -(-1)^m f^{(n+m)}(a_2-a_1) + \sum_{j=0}^{m}\binom{n+j-1}{j} f^{(n+j)}(y-a_1)f^{(m-j)}(a_1-a_2) \\
&\quad+ \sum_{j=0}^{n}\binom{m+j-1}{j} f^{(m+j)}(y-a_2)f^{(n-j)}(a_2-a_1)\,. \qquad \text{(E.3.11)}
\end{aligned}
$$

and eventually arrive at the following explicit formulas

$$
\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ z+b_{l;0} & \dots & b_1 \end{smallmatrix};z\right) &= \lim_{z\to 0}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ z+b_{l;0} & \dots & b_1 \end{smallmatrix};z\right) + (-1)^{n_l+1}\,\Gamma\left(\begin{smallmatrix} n_l+n_{l-1} & 0 & n_{l-2} & \dots & n_1 \\ b_{l-1}-b_{l;0} & 0 & b_{l-2} & \dots & b_1 \end{smallmatrix};z\right) \\
&+ \sum_{j=0}^{n_{l-1}}\binom{n_l+j-1}{j}\int_0^z \mathrm{d}y f^{(n_{l-1}-j)}(y-(b_{l-1}-b_{l;0}))\,\Gamma\left(\begin{smallmatrix} n_l+j & n_{l-2} & \dots & n_1 \\ y+b_{l;0} & b_{l-2} & \dots & b_1 \end{smallmatrix};y\right) \\
&+ \sum_{j=0}^{n_l}\binom{n_{l-1}+j-1}{j}(-1)^{n_l-j}\,\Gamma\left(\begin{smallmatrix} n_l-j & n_{l-1}+j & n_{l-2} & \dots & n_1 \\ b_{l-1}-b_{l;0} & b_{l-1} & b_{l-2} & \dots & b_1 \end{smallmatrix};z\right)
\end{aligned}
\qquad \text{(E.3.12)}
$$

$$
\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & z+b_{1;0} \end{smallmatrix};z\right) &= \lim_{z\to 0}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_1 \\ b_l & \dots & z+b_{1;0} \end{smallmatrix};z\right) + \int_0^z \mathrm{d}y f^{(n_l)}(y-b_l)\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_1 \\ b_{l-1} & \dots & y+b_{1;0} \end{smallmatrix};y\right) \\
&+ (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1 & n_l & \dots & n_2 \\ -b_{1;0} & b_l & \dots & b_2 \end{smallmatrix};z\right) + (-1)^{n_1}\,\Gamma\left(\begin{smallmatrix} n_1+n_2 & n_l & \dots & n_3 & 0 \\ b_2-b_{1;0} & b_l & \dots & b_3 & 0 \end{smallmatrix};z\right) \\
&- \sum_{j=0}^{n_2}\binom{n_1+j-1}{j}\int_0^z \mathrm{d}y f^{(n_2-j)}(y-(b_2-b_{1;0}))\,\Gamma\left(\begin{smallmatrix} n_l & \dots & n_3 & n_1+j \\ b_l & \dots & b_3 & y+b_{1;0} \end{smallmatrix};y\right) \\
&- \sum_{j=0}^{n_1}\binom{n_2+j-1}{j}(-1)^{n_1-j}\,\Gamma\left(\begin{smallmatrix} n_1-j & n_l & \dots & n_3 & n_2+j \\ b_2-b_{1;0} & b_l & \dots & b_3 & b_2 \end{smallmatrix};z\right)
\end{aligned}
\qquad \text{(E.3.13)}
$$

$$
\begin{aligned}
\Gamma\left(\begin{smallmatrix} n_l & \dots & n_i & \dots & n_1 \\ b_l & \dots & z+b_{i;0} & \dots & b_1 \end{smallmatrix};z\right) &= \lim_{z\to 0}\Gamma\left(\begin{smallmatrix} n_l & \dots & n_i & \dots & n_1 \\ b_l & \dots & z+b_{i;0} & \dots & b_1 \end{smallmatrix};z\right) \\
&+ \int_0^z \mathrm{d}y f^{(n_l)}(y-b_l)\,\Gamma\left(\begin{smallmatrix} n_{l-1} & \dots & n_i & \dots & n_1 \\ b_{l-1} & \dots & y+b_{i;0} & \dots & b_1 \end{smallmatrix};y\right) \\
&+ (-1)^{n_i}\,\Gamma\left(\begin{smallmatrix} n_i+n_{i+1} & n_l & \dots & n_{i+2} & 0 & n_{i-1} & \dots & n_1 \\ b_{i+1}-b_{i;0} & b_l & \dots & b_{1+2} & 0 & b_{i-1} & \dots & b_1 \end{smallmatrix};z\right) \\
&- \sum_{j=0}^{n_{i+1}} \binom{n_i+j-1}{j} \int_0^z \mathrm{d}y f^{(n_{i+1}-j)}(y-(b_{i+1}-b_{i;0}))\,\Gamma\left(\begin{smallmatrix} n_l & \dots & n_{i+2} & n_i+j & n_{i-1} & \dots & n_1 \\ b_l & \dots & b_{1+2} & y+b_{i;0} & b_{i-1} & \dots & b_1 \end{smallmatrix};y\right) \\
&- \sum_{j=0}^{n_i} \binom{n_{i+1}+j-1}{j} (-1)^{n_i-j}\,\Gamma\left(\begin{smallmatrix} n_i-j & n_l & \dots & n_{i+2} & n_{i+1}+j & n_{i-1} & \dots & n_1 \\ b_{i+1}-b_{i;0} & b_l & \dots & b_{1+2} & b_{i+1} & b_{i-1} & \dots & b_1 \end{smallmatrix};z\right) \\
&- (-1)^{n_i}\,\Gamma\left(\begin{smallmatrix} n_i+n_{i-1} & n_l & \dots & n_{i+1} & 0 & n_{i-2} & \dots & n_1 \\ b_{i-1}-b_{i;0} & b_l & \dots & b_{1+1} & 0 & b_{i-2} & \dots & b_1 \end{smallmatrix};z\right) \\
&+ \sum_{j=0}^{n_{i-1}} \binom{n_i+j-1}{j} \int_0^z \mathrm{d}y f^{(n_{i-1}-j)}(y-(b_{i-1}-b_{i;0}))\,\Gamma\left(\begin{smallmatrix} n_l & \dots & n_{i+1} & n_i+j & n_{i-2} & \dots & n_1 \\ b_l & \dots & b_{1+1} & y+b_{i;0} & b_{i-2} & \dots & b_1 \end{smallmatrix};y\right) \\
&+ \sum_{j=0}^{n_i} \binom{n_{i-1}+j-1}{j} (-1)^{n_i-j}\,\Gamma\left(\begin{smallmatrix} n_i-j & n_l & \dots & n_{i+1} & n_{i-1}+j & n_{i-2} & \dots & n_1 \\ b_{i-1}-b_{i;0} & b_l & \dots & b_{1+1} & b_{i-1} & b_{i-2} & \dots & b_1 \end{smallmatrix};z\right)\,,
\end{aligned}
\tag{E.3.14}
$$

with boundary terms as explained above.

The crucial point regarding these formulas is that the above relations only contain iterated integrals with z independent twists as well as iterated integrals with z dependent twist of lower length. Hence, we may recurse this procedure down to iterated integrals of length one

$$
\Gamma\left(\begin{smallmatrix} n_i \\ z+b_{i;0} \end{smallmatrix};z\right) = (-1)^{n_i}\,\Gamma\left(\begin{smallmatrix} n_i \\ -b_{i;0} \end{smallmatrix};z\right)\,. \tag{E.3.15}
$$

Hence we might completely rewrite integrals where one twist depends on the endpoint z into a sum of iterated integrals without any z dependent twist. Note that such a rewriting only involves iterated integrals of the same length and weight. While writing down such formulas by hand is nothing short of impractical, they may be easily put into ones computer algebra system of choice to obtain formulas like

$$
\begin{aligned}
\Gamma\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & z & \frac{\tau}{2} \end{smallmatrix};z\right) + \Gamma\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & \frac{\tau}{2} & z \end{smallmatrix};z\right) &= -\Gamma\left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \end{smallmatrix};z\right) - \Gamma\left(\begin{smallmatrix} 0 & 2 & 0 \\ 0 & \frac{\tau}{2} & 0 \end{smallmatrix};z\right) - \Gamma\left(\begin{smallmatrix} 0 & 0 & 2 \\ 0 & 0 & \frac{\tau}{2} \end{smallmatrix};z\right) \\
&\quad - \Gamma\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 0 & \frac{\tau}{2} \end{smallmatrix};z\right) - \Gamma\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & 0 & \frac{\tau}{2} \end{smallmatrix};z\right) + \Gamma\left(\begin{smallmatrix} 0 & 1 & 1 \\ 0 & \frac{\tau}{2} & \frac{\tau}{2} \end{smallmatrix};z\right)\,.
\end{aligned}
\tag{E.3.16}
$$

In the case where several twists depend on the endpoint z we have to alter the formulas for the partial derivative w.r.t. the twist (E.3.5,E.3.6,E.3.7). For this purpose one needs to apply the Leibniz rule to the integrand of (3.3.23), rewrite the derivatives with respect to the labels and then apply integration by parts to any term. The derivation of such formulas is conceptually not anymore involved than the above, but their explicit form is not exactly enlightening and we therefore omit them here, see [50] for some additional cases.

E.4 Differential equation

In this appendix we motivate how one may derive the differential equation for TEMZVs (3.4.35) given in the main text. For simplicity we assume that the twists $b_1, b_l \neq 0$, such that we may circumvent the issue of endpoint regularization. The corresponding derivation, which takes the endpoint regularization into account, turns out to be mathematically much more involved, but was done in [41].

We begin by defining a generating object for length l TEMZVs ($0 \in \mathbb{N}$)

$$\begin{aligned} \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\alpha_2,\dots,\alpha_l \\ b_1,b_2,\dots,b_l \end{smallmatrix}\right] &= \int\limits_{X_{1;l}} dz_l \dots dz_2 dz_1 \prod_{i=1}^{l} \Omega_\tau(z_i - b_i, \alpha_i) \\ &= \sum_{n_i \in \mathbb{N}} \alpha_1^{n_1-1}\alpha_2^{n_2-1} \dots \alpha_l^{n_l-1}\, \omega\left(\begin{smallmatrix} n_1,n_2,\dots,n_l \\ b_1,b_2,\dots,b_l \end{smallmatrix}\right) , \end{aligned} \tag{E.4.1}$$

where we introduced the following notation for the integration domain

$$X_{i;l} = \{(z_i, z_{i+1}, \dots, z_l) | 0 < z_i < z_{i+1} < \dots < z_l < 1\} . \tag{E.4.2}$$

Recall that we may write any twist as $b_i = s_i + r_i\tau$ with $s_i, r_i \in (0, 1)$. Then by application of the chain rule (for real z_i) we find that Ω satisfies a differential equation akin to the mixed heat equation (3.3.10)

$$\begin{aligned} \partial_\tau \Omega_\tau(z_i - s_i - r_i\tau, \alpha_i) &= \exp(-2\pi i r_i \alpha_i)\partial_\tau F_\tau(z_i - s_i - r_i\tau, \alpha_i) \\ &= \exp(-2\pi i r_i \alpha_i)(-r_i\partial_{z_i} + \frac{1}{2\pi i}\partial_{\alpha_i}\partial_{z_i})F_\tau(z_i - s_i - r_i\tau, \alpha_i) \\ &= \frac{1}{2\pi i}\partial_{\alpha_i}\partial_{z_i}\Omega_\tau(z_i - s_i - r_i\tau, \alpha_i) . \end{aligned} \tag{E.4.3}$$

Using this differential equation for Ω we can rewrite the τ derivative of the generating function as

$$\begin{aligned} 2\pi i \frac{\partial}{\partial \tau} \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\dots,\alpha_l \\ b_1,\dots,b_l \end{smallmatrix}\right] &= \int\limits_{X_{1;l}} dz_l \dots dz_2 dz_1 \sum_{i=1}^{l} \partial_{\alpha_i}\partial_{z_i}\Omega_\tau(z_i - b_i, \alpha_i) \prod_{j \neq i} \Omega_\tau(z_j - b_j, \alpha_j) \\ &= \sum_{i=1}^{l} \int\limits_{X_{i;l}} dz_l \dots dz_i \prod_{j>i} \Omega_\tau(z_j - b_j, \alpha_j)(\partial_{z_i}\partial_{\alpha_i}\Omega_\tau(z_i - b_i, \alpha_i))\, \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\dots,\alpha_{i-1} \\ b_1,\dots,b_{i-1} \end{smallmatrix}; z_i\right] \\ &= \int\limits_{X_{2;l}} dz_l \dots dz_2 \partial_{\alpha_1}\Omega_\tau(z_2 - b_1, \alpha_1) \prod_{j>1} \Omega_\tau(z_j - b_j, \alpha_j) \\ &\quad - \partial_{\alpha_1}\Omega_\tau(-b_1, \alpha_1)\, \mathrm{T}\left[\begin{smallmatrix} \alpha_2,\dots,\alpha_l \\ b_2,\dots,b_l \end{smallmatrix}\right] + \sum_{i=2}^{l-1} \int\limits_{X_{i+1;l}} dz_l \dots dz_{i+1} \prod_{j>i} \Omega_\tau(z_j - b_j, \alpha_j) \times \\ &\quad \times \Big(\partial_{\alpha_i}\Omega_\tau(z_{i+1} - b_i, \alpha_i)\, \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\dots,\alpha_{i-1} \\ b_1,\dots,b_{i-1} \end{smallmatrix}; z_{i+1}\right] \\ &\qquad - \int_0^{z_{i+1}} dz_i \partial_{\alpha_i}\Omega_\tau(z_i - b_i, \alpha_i)\, \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\dots,\alpha_{i-2} \\ b_1,\dots,b_{i-2} \end{smallmatrix}; z_i\right] \Omega_\tau(z_i - b_{i-1}, \alpha_{i-1}) \Big) \\ &\quad + \partial_{\alpha_l}\Omega_\tau(-b_l, \alpha_l)\, \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\dots,\alpha_{l-1} \\ b_1,\dots,b_{l-1} \end{smallmatrix}\right] - \int\limits_{X_{l;l}} dz_l \partial_{\alpha_l}\Omega_\tau(z_l - b_l, \alpha_l) \prod_{j>1} \Omega_\tau(z_j - b_j, \alpha_j) \\ &= \partial_{\alpha_l}\Omega_\tau(-b_r, \alpha_l)\, \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\dots,\alpha_{l-1} \\ b_1,\dots,b_{l-1} \end{smallmatrix}\right] - \partial_{\alpha_1}\Omega_\tau(-b_1, \alpha_1)\, \mathrm{T}\left[\begin{smallmatrix} \alpha_2,\dots,\alpha_l \\ b_2,\dots,b_l \end{smallmatrix}\right] \end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=2}^{l}\int_{X_{i;l}} dz_l \dots dz_i \prod_{j>i} \Omega_\tau(z_i - b_i, \alpha_i)\times \\
&\quad \times \Big((\partial_{\alpha_{i-1}}\Omega_\tau(z_i - b_{i-1}, \alpha_{i-1}))\Omega_\tau(z_i - b_i, \alpha_i) \\
&\qquad - \Omega_\tau(z_i - b_{i-1}, \alpha_{i-1})(\partial_{\alpha_i}\Omega_\tau(z_i - b_i, \alpha_i))\Big) \operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots,\alpha_{i-2}\\ b_1,\dots,b_{i-2}\end{smallmatrix}; z_i\right] \\
&= \partial_{\alpha_l}\Omega_\tau(-b_l, \alpha_l)\operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots,\alpha_{l-1}\\ b_1,\dots,b_{l-1}\end{smallmatrix}\right] - \partial_{\alpha_1}\Omega_\tau(-b_1, \alpha_1)\operatorname{T}\left[\begin{smallmatrix}\alpha_2,\dots,\alpha_l\\ b_2,\dots,b_l\end{smallmatrix}\right] \\
&\quad + \sum_{i=2}^{l}\Big(\operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots\alpha_{i-2}, & \alpha_{i-1}+\alpha_i & ,\alpha_{i+1},\dots\alpha_l\\ b_1,\dots b_{i-2}, & b_i & ,b_{i+1},\dots b_l\end{smallmatrix}\right] \partial_{\alpha_{i-1}}\Omega_\tau(-b_{i-1}+b_i, \alpha_{i-1}) \\
&\qquad - \operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots\alpha_{i-2}, & \alpha_{i-1}+\alpha_i & ,\alpha_{i+1},\dots\alpha_l\\ b_1,\dots b_{i-2}, & b_{i-1} & ,b_{i+1},\dots b_l\end{smallmatrix}\right] \partial_{\alpha_i}\Omega_\tau(-b_i+b_{i-1}, \alpha_i)\Big)\,,
\end{aligned} \tag{E.4.4}
$$

where we used integration by parts in the third equality and the Fay identity

$$
\begin{aligned}
\Omega_\tau(z_i - b_{i-1}, \alpha_{i-1})\Omega_\tau(z_i - b_i, \alpha_i) &= \Omega_\tau(z_i - b_{i-1}, \alpha_{i-1}-\alpha_i)\Omega_\tau(-b_i + b_{i-1}, \alpha_i) \\
&\quad - \Omega_\tau(z_i - b_i, \alpha_{i-1}-\alpha_i)\Omega_\tau(-b_{i-1}+b_i, \alpha_{i-1})\,,
\end{aligned} \tag{E.4.5}
$$

in the last equality.

From the above differential equation for the generating function we may deduce a differential equation for TEMZVs

$$
\begin{aligned}
2\pi i\partial_\tau \operatorname{T}\left[\begin{smallmatrix}\alpha_1,\alpha_2,\dots,\alpha_l\\ b_1,b_2,\dots,b_l\end{smallmatrix}\right] &= \sum_{n_i\in\mathbb{N}} \alpha_1^{n_1-1}\dots\alpha_l^{n_l-1} 2\pi i\partial_\tau\, \omega\left(\begin{smallmatrix}n_1,\dots,n_l\\ b_1,\dots,b_l\end{smallmatrix}\right) \\
&= \partial_{\alpha_l}\Omega_\tau(-b_l, \alpha_l)\operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots,\alpha_{l-1}\\ b_1,\dots,b_{l-1}\end{smallmatrix}\right] - \partial_{\alpha_1}\Omega_\tau(-b_1, \alpha_1)\operatorname{T}\left[\begin{smallmatrix}\alpha_2,\dots,\alpha_l\\ b_2,\dots,b_l\end{smallmatrix}\right] \\
&\quad + \sum_{i=2}^{l}\Big(\underbrace{\operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots\alpha_{i-2}, & \alpha_{i-1}+\alpha_i & ,\alpha_{i+1},\dots\alpha_l\\ b_1,\dots b_{i-2}, & b_i & ,b_{i+1},\dots b_l\end{smallmatrix}\right] \partial_{\alpha_{i-1}}\Omega_\tau(-b_{i-1}+b_i, \alpha_{i-1})}_{A_{i,+}} \\
&\qquad - \underbrace{\operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots\alpha_{i-2}, & \alpha_{i-1}+\alpha_i & ,\alpha_{i+1},\dots\alpha_l\\ b_1,\dots b_{i-2}, & b_{i-1} & ,b_{i+1},\dots b_l\end{smallmatrix}\right] \partial_{\alpha_i}\Omega_\tau(-b_i+b_{i-1}, \alpha_i)}_{A_{i,-}}\Big)\,.
\end{aligned} \tag{E.4.6}
$$

Eventually we want to equate coefficients of a monomial $\alpha_1^{n_1-1}\dots\alpha_l^{n_l-1}$ to extract the τ derivative of the corresponding TEMZV. In order to do so we will consider the terms in (E.4.6) separately. Introducing the shorthand $h^{(m)}(z) = (n-1)f^{(m)}(z)$ we start by considering the boundary term, which may be rewritten as

$$
\begin{aligned}
\partial_{\alpha_l}\Omega_\tau(-b_l, \alpha_l)\operatorname{T}\left[\begin{smallmatrix}\alpha_1,\dots,\alpha_{l-1}\\ b_1,\dots,b_{l-1}\end{smallmatrix}\right] &= \sum_{n_l\in\mathbb{N}} h^{(n_l)}(-b_l)\alpha_l^{n_l-2}\sum_{n_i\in\mathbb{N}}\alpha_1^{n_1-1}\dots\alpha_{l-1}^{n_{l-1}-1}\,\omega\left(\begin{smallmatrix}n_1,\dots,n_{l-1}\\ b_1,\dots,b_{l-1}\end{smallmatrix}\right) \\
&= \sum_{n_i\in\mathbb{N}}\alpha_1^{n_1-1}\dots\alpha_{l-1}^{n_{l-1}-1}\alpha_l^{n_l-1}h^{(n_l+1)}(-b_l)\,\omega\left(\begin{smallmatrix}n_1,\dots,n_{l-1}\\ b_1,\dots,b_{l-1}\end{smallmatrix}\right) \\
&\quad + h^{(0)}(-b_l)\alpha_l^{-2}\sum_{n_i\in\mathbb{N}}\alpha_1^{n_1-1}\dots\alpha_{l-1}^{n_{l-1}-1}\,\omega\left(\begin{smallmatrix}n_1,\dots,n_{l-1}\\ b_1,\dots,b_{l-1}\end{smallmatrix}\right)\,,
\end{aligned} \tag{E.4.7}
$$

and we get a similar expression for the second boundary term. The sum proportional to α_r^{-2} will be canceled by some contribution of the intermediate terms $A_{i,\pm}$. Specifically, for $A_{i,+}$ we have

$$
\begin{aligned}
A_{i,+} &= \mathrm{T}\left[\begin{smallmatrix} \alpha_1,\dots\alpha_{i-2}, & \alpha_{i-1}+\alpha_i & ,\alpha_{i+1},\dots\alpha_l \\ b_1,\dots b_{i-2}, & b_{i-1} & ,b_{i+1},\dots b_l \end{smallmatrix}\right] \partial_{\alpha_{i-1}} \Omega_\tau(-b_{i-1}+b_i, \alpha_{i-1}) \\
&= \sum_{n_1,\dots,n_{i-2},n_{i+1},\dots,n_l \in \mathbb{N}} \alpha_1^{n_1-1} \dots \alpha_{i-2}^{n_{i-2}-1} \alpha_{i+1}^{n_{i+1}-1} \dots \alpha_l^{n_l-1} \times \\
&\qquad \Big[\underbrace{(\alpha_{i-1}+\alpha_i)^{-1} \,\omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & 0 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & b_i & ,b_{i+1},\dots b_l \end{smallmatrix}\right) \sum_{j\in\mathbb{N}} h^{(j)}(-b_{i-1}+b_i)\alpha_{i-1}^{j-2}}_{B_{i,+}} \\
&\qquad + \underbrace{\sum_{j,k\in\mathbb{N}} \sum_{i=0}^{k} \binom{k}{i} \alpha_{i-1}^{k-i+j-2} \alpha_i^i \,\omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & k+1 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & b_i & ,b_{i+1},\dots b_l \end{smallmatrix}\right) h^{(j)}(-b_{i-1}+b_i)}_{C_{i,+}} \Big] ,
\end{aligned}
\tag{E.4.8}
$$

and the analogous result for $A_{i,-}$ may be obtained by exchanging the labels $i-1$ and i. In the following the manipulations only affect pairs (α_{i-1}, α_i) and we will suppress the remaining α_j. Then, as $h^{(0)}(b)$ is independent on b we have the equivalence of iterated integrals

$$
\omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & 0 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & b_{i-1} & ,b_{i+1},\dots b_l \end{smallmatrix}\right) = \omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & 0 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & b_i & ,b_{i+1},\dots b_l \end{smallmatrix}\right) . \tag{E.4.9}
$$

Hence, we might rewrite the terms, not expressible by the binomial law $B_{i,\pm}$, as

$$
\begin{aligned}
B_{i,+} - B_{i,-} &= (\alpha_{i-1}+\alpha_i)^{-1} \,\omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & 0 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & * & ,b_{i+1},\dots b_l \end{smallmatrix}\right) \Big(\sum_{j\in\mathbb{N}} h^{(j)}(-b_{i-1}+b_i)\alpha_{i-1}^{j-2} \\
&\qquad\qquad - \sum_{j\in\mathbb{N}} h^{(j)}(-b_i+b_{i-1})\alpha_i^{j-2} \Big) \\
&= \omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & 0 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & * & ,b_{i+1},\dots b_l \end{smallmatrix}\right) \Big(-(\alpha_{i-1}+\alpha_i)^{-1}(\alpha_{i-1}^{-2} - \alpha_i^{-2}) \\
&\qquad\qquad + \sum_{j\in\mathbb{N}} h^{(j+3)}(-b_{i-1}+b_i) \sum_{a=0}^{j} (-1)^{j-a} \alpha_{i-1}^a \alpha_i^{j-a} \Big) ,
\end{aligned}
\tag{E.4.10}
$$

where we used

$$
(\alpha_{i-1}^j - (-1)^j \alpha_i^j) = (\alpha_{i-1}+\alpha_i) \sum_{a=0}^{j-1} (-1)^{j-1-a} \alpha_{i-1}^a \alpha_i^{j-1-a} , \qquad j > 0 , \tag{E.4.11}
$$

and $h^{(0)} = -1$, $h^{(1)} = 0$, $h^{(2)}(b) = h^{(2)}(-b)$. The term of homogeneity -3 in α_{i-1}, α_i cancels among adjacent $B_{i,+} - B_{i,-}$. Next we rewrite the remaining contributions of $B_{i,+} - B_{i,-}$ into a form where we can directly read of the coefficient of a given α monomial

$$
\begin{aligned}
&\omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & 0 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & * & ,b_{i+1},\dots b_l \end{smallmatrix}\right) \sum_{j=0}^{\infty} h^{(j+3)}(-b_{i-1}+b_i) \sum_{a=0}^{j} (-1)^{j-a} \alpha_{i-1}^a \alpha_i^{j-a} \\
&= \omega\left(\begin{smallmatrix} n_1,\dots n_{i-2}, & 0 & ,n_{i+1},\dots n_l \\ b_1,\dots b_{i-2}, & * & ,b_{i+1},\dots b_l \end{smallmatrix}\right) \sum_{m,n\in\mathbb{N}} h^{(m+n+3)}(-b_{i-1}+b_i)(-1)^n \alpha_{i-1}^m \alpha_i^n .
\end{aligned}
\tag{E.4.12}
$$

Moreover, for the contributions from $C_{i,+}$ we have

$$
\begin{aligned}
C_{i,+} &= \sum_{j,k\in\mathbb{N}} \sum_{p=0}^{k} \binom{k}{p} \alpha_{i-1}^{k+j-p-2} \alpha_i^p \, \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & k+1 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(j)}(-b_{i-1}+b_i) \\
&= \sum_{j,k\in\mathbb{N}} \sum_{p=0}^{k-1} \binom{k}{p} \alpha_{i-1}^{k+j-p-1} \alpha_i^p \, \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & k+1 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(j+1)}(-b_{i-1}+b_i) \\
&\quad + \sum_{k\in\mathbb{N}} \sum_{p=0}^{k} \binom{k+1}{p} \alpha_{i-1}^{k-p-1} \alpha_i^p \, \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & k+2 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(0)}(-b_{i-1}+b_i) \\
&\quad + \alpha_{i-1}^{-2} \sum_{a\in\mathbb{N}} h^{(0)}(-b_{i-1}+b_i) \alpha_i^a \, \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & a+1 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) ,
\end{aligned}
\tag{E.4.13}
$$

and $C_{i,-}$ may be obtained via exchanging the labels $i-1$ and i. Note that this implies that the sums proportional to α_{i-1}^{-2} cancel among $C_{i,+}$ and $C_{i-1,-}$, with the exception of the cases $C_{2,+}$ and $C_{l,-}$, where these sums cancel with the corresponding sums of the boundary terms; cf. (E.4.7). The remaining part of $C_{i,+}$ can then be rewritten as

$$
\begin{aligned}
&\sum_{j,k\in\mathbb{N}} \sum_{p=0}^{k-1} \binom{k}{p} \alpha_{i-1}^{k+j-p-1} \alpha_i^p \, \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & k+1 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(j+1)}(-b_{i-1}+b_i) \\
&\quad + \sum_{k\in\mathbb{N}} \sum_{p=0}^{k} \binom{k+1}{p} \alpha_{i-1}^{k-p-1} \alpha_i^p \, \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & k+2 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(0)}(-b_{i-1}+b_i) \\
&= \sum_{m,n\in\mathbb{N}} \alpha_{i-1}^{m-1} \alpha_i^n \sum_{j=0}^{m} \binom{m+n-j}{n} \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & m+n+1-j & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(j+1)}(-b_{i-1}+b_i) \\
&\quad + \sum_{m,n\in\mathbb{N}} \alpha_{i-1}^{m-1} \alpha_i^n \binom{m+n+1}{n} \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & m+n+2 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(0)}(-b_{i-1}+b_i) \\
&= \sum_{m,n\in\mathbb{N}} \alpha_{i-1}^{m-1} \alpha_i^n \sum_{k=0}^{m+1} \binom{n+k}{k} \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & n+k+1 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(m-k+1)}(-b_{i-1}+b_i) .
\end{aligned}
\tag{E.4.14}
$$

Finally, we may use the above expressions to equate coefficients of α monomials, leading to the differential equation for TEMZVs ($r \geq 2$)

$$
\begin{aligned}
2\pi i \partial_\tau \, \omega\left(\begin{smallmatrix} n_1,\ldots,n_l \\ b_1,\ldots,b_l \end{smallmatrix}\right) &= h^{(n_l+1)}(-b_l)\, \omega\left(\begin{smallmatrix} n_1,\ldots,n_{l-1} \\ b_1,\ldots,b_{l-1} \end{smallmatrix}\right) - h^{(n_1+1)}(-b_1)\, \omega\left(\begin{smallmatrix} n_2,\ldots,n_l \\ b_2,\ldots,b_l \end{smallmatrix}\right) \\
&+ \sum_{i=2}^{l} \Big[-\theta_{n_{i-1}\geq 1} \theta_{n_i\geq 1} \, \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & 0 & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & * & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(n_{i-1}+n_i+1)}(-b_{i-1}+b_i)(-1)^{n_i} \\
&+ \theta_{n_i\geq 1} \sum_{k=0}^{n_{i-1}+1} \binom{n_i+k-1}{k} \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & n_i+k & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_i & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(n_{i-1}-k+1)}(-b_{i-1}+b_i) \\
&- \theta_{n_{i-1}\geq 1} \sum_{k=0}^{n_i+1} \binom{n_{i-1}+k-1}{k} \omega\left(\begin{smallmatrix} n_1,\ldots n_{i-2}, & n_{i-1}+k & ,n_{i+1},\ldots n_l \\ b_1,\ldots b_{i-2}, & b_{i-1} & ,b_{i+1},\ldots b_l \end{smallmatrix}\right) h^{(n_i-k+1)}(-b_i+b_{i-1}) \Big] ,
\end{aligned}
\tag{E.4.15}
$$

where we introduced $\theta_{n\geq 1} = 1 - \delta_{n,0}$.

E.5 Weighting functions at twists in $\Lambda_2 + \Lambda_2\tau$

One of the main ingredients in the computation of TEMZVs is the weighting functions $f^{(n)}$ evaluated at some twist b. Here we give the derivation for concise formulas in the case $b \in \Lambda_2 + \Lambda_2\tau$; cf. equations (3.4.38,3.4.39). As we will be mainly dealing with manipulations on formal power series, we denote $[q^0]$ resp. $[q^{>0}]$ the projector onto the q^0 term resp. the non-constant part of the formal power series.

As already noted in the main text, the constant term of $f^{(n)}(s + r\tau)$ only depends on r. Specifically, we have for the case $r = 0$

$$[q^0]f^{(k)}(s) = \begin{cases} -2\zeta_k & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases} . \tag{E.5.1}$$

The computation of the constant part in the case $r = 1/2$ is more involved. To that end we start by considering the formula

$$[q^0]f^{(k)}(s + \tau/2) = \begin{cases} \frac{2^{k-1}-1}{2^{k-2}}\zeta_k & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases} , \tag{E.5.2}$$

which in the subsequent we will argue to be correct. This expression should be equivalent to[98]

$$\begin{aligned} [q^0]f^{(k)}(s + \tau/2) &= -\frac{(i\pi)^k}{(k-1)!} - \sum_{j=0}^{\lfloor k/2 \rfloor} \frac{(i\pi)^{k-2j}}{(k-2j)!}\zeta_{2j} = \frac{(2\pi i)^k}{2^{k-1}} \sum_{j=0}^{k} \frac{2^{j-1}B_j}{j!(k-j)!} \\ &= \frac{(2\pi i)^k}{2^{k-1}}(1 - 2^{k-1})\frac{B_k}{k!} , \end{aligned} \tag{E.5.3}$$

where B_i are the Bernoulli numbers such that $B_1 = -1/2$ and the last equality is the assertion under consideration. This equality may then be seen by considering the generating function of Bernoulli numbers $\frac{1}{e^x-1} = \sum_{j=0}^{\infty} \frac{B_j}{j!} x^{j-1}$, which satisfies

$$\frac{e^x}{e^{2x}-1} = \frac{1}{e^x-1} - \frac{1}{e^{2x}-1} . \tag{E.5.4}$$

Expanding both sites of this equation in x and equating coefficients leads to the last equality in (E.5.3).

The non-constant part of the power series for $b = 0$ and $b = 1/2$ may be directly inferred from (3.3.17)

$$\begin{aligned} [q^{>0}]f^{(k)}(0) &= \begin{cases} -2\frac{(2\pi i)^k}{(k-1)!} \sum_{n,m=1}^{\infty} n^{k-1}q^{mn} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases} , \qquad k \geq 2 , \\ [q^{>0}]f^{(k)}(1/2) &= \begin{cases} -2\frac{(2\pi i)^k}{(k-1)!} \sum_{n,m=1}^{\infty} (-1)^m n^{k-1}q^{mn} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases} . \end{aligned} \tag{E.5.5}$$

[98] While the guess in eq. (E.5.2) is originally due to this thesis' author, the following elegant argument for its veracity is due to Nils Matthes.

Next we consider the case $b = \tau/2$, which may be obtained as follows

$$\begin{aligned}[q^{>0}]f^{(k)}(\tau/2) &= \sum_{j=1}^{k} \frac{(i\pi)^{k-j}}{(k-j)!}[q^{>0}]g^{(j)}(\tau/2) \\ &= -\frac{2(i\pi)^k}{(k-1)!}\sum_{m=1}^{\infty} q^{m/2} - \sum_{j=1}^{k}\frac{(i\pi)^{k-j}}{(k-j)!}\frac{(2\pi i)^j}{(j-1)!}\sum_{n,m=1}^{\infty} n^{j-1}q^{mn}(q^{m/2}+(-1)^j q^{-m/2}) \\ &= -\frac{2(i\pi)^k}{(k-1)!}\left[\sum_{m=1}^{\infty}\left(q^{m/2}+\sum_{n=1}^{\infty}q^{mn}((1+2n)^{k-1}q^{m/2}-(1-2n)^{k-1}q^{-m/2})\right)\right] \\ &= \frac{(1+(-1)^k)}{2}\left[-\frac{2(2i\pi)^k}{(k-1)!}\sum_{n,m=1}^{\infty}\frac{q^{m(n-1/2)}}{(n-1/2)^{1-k}}\right] .\end{aligned} \tag{E.5.6}$$

Finally, we have $b = 1/2 + \tau/2$ which yields

$$\begin{aligned}[q^{>0}]f^{(k)}(1/2+\tau/2) &= \sum_{j=1}^{k}\frac{(i\pi)^{k-j}}{(k-j)!}[q^{>0}]g^{(j)}(1/2+\tau/2) \\ &= -\frac{2(i\pi)^k}{(k-1)!}\sum_{m=1}^{\infty}(-1)^m q^{m/2} \\ &\quad - \sum_{j=1}^{k}\frac{(i\pi)^{k-j}}{(k-j)!}\frac{(2\pi i)^j}{(j-1)!}\sum_{n,m=1}^{\infty}(-1)^m n^{j-1}q^{mn}(q^{m/2}+(-1)^j q^{-m/2}) \\ &= -\frac{2(i\pi)^k}{(k-1)!}\Bigg[\sum_{m=1}^{\infty}(-1)^m\Big(q^{m/2}+ \\ &\qquad + \sum_{n=1}^{\infty}q^{mn}((1+2n)^{k-1}q^{m/2}-(1-2n)^{k-1}q^{-m/2})\Big)\Bigg] \\ &= \frac{(1+(-1)^k)}{2}\left[-\frac{2(2i\pi)^k}{(k-1)!}\sum_{n,m=1}^{\infty}\frac{(-1)^m q^{m(n-1/2)}}{(n-1/2)^{1-k}}\right] .\end{aligned} \tag{E.5.7}$$

Summarizing, we have for twists $b = (s + r\tau) \in (\Lambda_2 + \Lambda_2\tau)$ the following result for $k \geq 2$

$$f^{(k)}(b) = \begin{cases} (-2+2^3(1-2^{-k})r)\zeta_k - 2\frac{(2\pi i)^k}{(k-1)!}\sum\limits_{n,m=1}^{\infty} e^{2\pi i s m}(n-r)^{k-1}q^{m(n-r)} & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd} \end{cases} . \tag{E.5.8}$$

Finally, we note that these four functions of q are not linearly independent

$$f^{(k)}(0) + \frac{2^{k-2}}{2^{k-2}-1}[f^{(k)}(1/2) + f^{(k)}(\tau/2) + f^{(k)}(1/2+\tau/2)] = 0 . \tag{E.5.9}$$

E.6 The integral $d_{3;11}$

In this appendix we illustrate the translation of the integrals occurring in the α' expansion into TEMZVs. Note that while the below formulas generally lack in refinement all manipulations involved are practically combinatoric and can be automatized.

The enfant terrible of the double-trace contribution up to third order in the α'-expansion is the integral

$$
\begin{aligned}
d_{3;11} &= \int_{12}^{34} [\mathrm{d}z]\, \tilde{P}(z_{34})\tilde{Q}(z_{13})\tilde{Q}(z_{14}) \\
&= \int\limits_{0<z_3<z_4<1} \mathrm{d}z_3\mathrm{d}z_4\, \Gamma\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix};z_3\right)\Gamma\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix};z_4\right)\left[\Gamma\left(\begin{smallmatrix}1\\ 0\end{smallmatrix};z_4\right)+\Gamma\left(\begin{smallmatrix}1\\ z_4\end{smallmatrix};z_3\right)\right] + (z_3 \leftrightarrow z_4) \\
&= 2\int_0^1 \mathrm{d}z_4\, \Gamma\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix};z_4\right)\Gamma\left(\begin{smallmatrix}1\\ 0\end{smallmatrix};z_4\right)\int_0^{z_4}\mathrm{d}z_3\,\Gamma\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix};z_3\right) \\
&\quad + 2\int_0^1 \mathrm{d}z_4\, \Gamma\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix};z_4\right)\int_0^{z_4}\mathrm{d}z_3\,\Gamma\left(\begin{smallmatrix}1\\ z_4\end{smallmatrix};z_3\right)\Gamma\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix};z_3\right) \\
&= 2\,\Gamma\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup \begin{smallmatrix}1\\ 0\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}0&1\\ 0&\frac{\tau}{2}\end{smallmatrix}\right)\right);1\right) + \\
&\quad + 2\int_0^1 \mathrm{d}z_4\, \Gamma\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix};z_4\right)\left[\Gamma\left(\begin{smallmatrix}0&1&1\\ 0&\frac{\tau}{2}&z_4\end{smallmatrix};z_4\right)+\Gamma\left(\begin{smallmatrix}0&1&1\\ 0&z_4&\frac{\tau}{2}\end{smallmatrix};z_4\right)\right] ,
\end{aligned}
\tag{E.6.1}
$$

which is the first instance for the double-trace contributions at four points, where we need to use the endpoint removal identities discussed in section 3.3 (as well as in appendix E.3) in order to get rid of twists $b = z_4$. Specifically, we may use

$$
\begin{aligned}
\Gamma\left(\begin{smallmatrix}0&1&1\\ 0&\frac{\tau}{2}&z_4\end{smallmatrix};z_4\right)+\Gamma\left(\begin{smallmatrix}0&1&1\\ 0&z_4&\frac{\tau}{2}\end{smallmatrix};z_4\right) &= -\Gamma\left(\begin{smallmatrix}0&2&0\\ 0&\frac{\tau}{2}&0\end{smallmatrix};z_4\right)-\Gamma\left(\begin{smallmatrix}0&0&2\\ 0&0&0\end{smallmatrix};z_4\right)-\Gamma\left(\begin{smallmatrix}0&0&0\\ 0&0&\frac{\tau}{2}\end{smallmatrix};z_4\right) \\
&\quad -\Gamma\left(\begin{smallmatrix}1&0&1\\ 0&0&\frac{\tau}{2}\end{smallmatrix};z_4\right)-\Gamma\left(\begin{smallmatrix}0&1&1\\ 0&0&\frac{\tau}{2}\end{smallmatrix};z_4\right)+\Gamma\left(\begin{smallmatrix}0&1&1\\ 0&\frac{\tau}{2}&\frac{\tau}{2}\end{smallmatrix};z_4\right) ,
\end{aligned}
\tag{E.6.2}
$$

which follows from (E.3.12,E.3.13,E.3.14). This equality allows us to explicitly translate the remaining integral of the last line of (E.6.1) into TEMZVs. Hence, the integral in question reads

$$
\begin{aligned}
d_{3;11} = 2\,\Gamma&\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup \begin{smallmatrix}1\\ 0\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}0&1\\ 0&\frac{\tau}{2}\end{smallmatrix}\right)\right);1\right) - 2\,\Gamma\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}0&2&0\\ 0&\frac{\tau}{2}&0\end{smallmatrix}\right)\right);1\right) \\
- 2\,\Gamma&\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}0&0&2\\ 0&0&0\end{smallmatrix}\right)\right);1\right) - 2\,\Gamma\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}0&0&2\\ 0&0&\frac{\tau}{2}\end{smallmatrix}\right)\right);1\right) \\
- 2\,\Gamma&\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}1&0&1\\ 0&0&\frac{\tau}{2}\end{smallmatrix}\right)\right);1\right) - 2\,\Gamma\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}0&1&1\\ 0&0&\frac{\tau}{2}\end{smallmatrix}\right)\right);1\right) \\
+ 2\,\Gamma&\left(\begin{smallmatrix}0\\ 0\end{smallmatrix}\left(\begin{smallmatrix}1\\ \frac{\tau}{2}\end{smallmatrix}\sqcup\!\sqcup\left(\begin{smallmatrix}0&1&1\\ 0&\frac{\tau}{2}&\frac{\tau}{2}\end{smallmatrix}\right)\right);1\right) .
\end{aligned}
\tag{E.6.3}
$$

Explicitly, expanding the shuffle products and changing to the notation for TEMZVs we get the aesthetically unpleasing expression

$$
\begin{aligned}
d_{3;11} = {} & 4\,\omega\left(\begin{smallmatrix} 1, & 1, & 1, & 0, & 0 \\ 0, & \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix} 1, & 1, & 1, & 0, & 0 \\ \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix} 1, & 1, & 0, & 1, & 0 \\ 0, & \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0 \end{smallmatrix}\right) \\
& + 6\,\omega\left(\begin{smallmatrix} 1, & 1, & 1, & 0, & 0 \\ \frac{\tau}{2}, & \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix} 1, & 1, & 0, & 1, & 0 \\ \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0 \end{smallmatrix}\right) \\
& - 2\,\omega\left(\begin{smallmatrix} 1, & 2, & 0, & 0, & 0 \\ \frac{\tau}{2}, & 0, & 0, & 0, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 2, & 1, & 0, & 0, & 0 \\ 0, & \frac{\tau}{2}, & 0, & 0, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 2, & 0, & 1, & 0, & 0 \\ 0, & 0, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) \\
& - 2\,\omega\left(\begin{smallmatrix} 2, & 0, & 0, & 1, & 0 \\ 0, & 0, & 0, & \frac{\tau}{2}, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 2, & 1, & 0, & 0, & 0 \\ \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 2, & 0, & 1, & 0, & 0 \\ \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) \\
& - 2\,\omega\left(\begin{smallmatrix} 2, & 0, & 0, & 1, & 0 \\ \frac{\tau}{2}, & 0, & 0, & \frac{\tau}{2}, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 1, & 2, & 0, & 0, & 0 \\ \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 1, & 0, & 2, & 0, & 0 \\ \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) \\
& - 2\,\omega\left(\begin{smallmatrix} 0, & 1, & 2, & 0, & 0 \\ 0, & \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 0, & 2, & 1, & 0, & 0 \\ 0, & \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 0, & 2, & 0, & 1, & 0 \\ 0, & \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0 \end{smallmatrix}\right) .
\end{aligned}
\tag{E.6.4}
$$

This may be rewritten into a much more compact expression, using identities among TEMZVs à la

$$
\begin{aligned}
0 = \omega\left(\begin{smallmatrix} 1, & 1, & 0, & 0 \\ 0, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) \omega\left(\begin{smallmatrix} 1 \\ \frac{\tau}{2} \end{smallmatrix}\right) &= 2\,\omega\left(\begin{smallmatrix} 1, & 1, & 1, & 0, & 0 \\ 0, & \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) + \omega\left(\begin{smallmatrix} 1, & 1, & 1, & 0, & 0 \\ \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0, & 0 \end{smallmatrix}\right) \\
&\quad + \omega\left(\begin{smallmatrix} 1, & 1, & 0, & 1, & 0 \\ 0, & \frac{\tau}{2}, & 0, & \frac{\tau}{2}, & 0 \end{smallmatrix}\right) + \omega\left(\begin{smallmatrix} 1, & 1, & 0, & 0, & 1 \\ 0, & \frac{\tau}{2}, & 0, & 0, & \frac{\tau}{2} \end{smallmatrix}\right) ,
\end{aligned}
\tag{E.6.5}
$$

which follows from the shuffle product formula (3.3.25) and $\omega\left(\begin{smallmatrix} 1 \\ \frac{\tau}{2} \end{smallmatrix}\right) = 0$. Eventually, we then arrive at the following formula for $d_{3;11}$

$$
\begin{aligned}
d_{3;11} = {} & 2\,\omega\left(\begin{smallmatrix} 2, & 0, & 0, & 0, & 1 \\ 0, & 0, & 0, & 0, & \frac{\tau}{2} \end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix} 2, & 0, & 0, & 0, & 1 \\ \frac{\tau}{2}, & 0, & 0, & 0, & \frac{\tau}{2} \end{smallmatrix}\right) + 2\,\omega\left(\begin{smallmatrix} 0, & 2, & 0, & 0, & 1 \\ 0, & \frac{\tau}{2}, & 0, & 0, & \frac{\tau}{2} \end{smallmatrix}\right) \\
& - 2\,\omega\left(\begin{smallmatrix} 1, & 1, & 0, & 0, & 1 \\ 0, & \frac{\tau}{2}, & 0, & 0, & \frac{\tau}{2} \end{smallmatrix}\right) - 2\,\omega\left(\begin{smallmatrix} 1, & 1, & 0, & 0, & 1 \\ \frac{\tau}{2}, & \frac{\tau}{2}, & 0, & 0, & \frac{\tau}{2} \end{smallmatrix}\right) ,
\end{aligned}
\tag{E.6.6}
$$

as presented in the main text (3.5.58).

E.7 Some all order contributions

An interesting question is whether there exists a closed expression for the inequivalent integrals discussed in section 3.5 in terms of TEMZVs, at all orders in α'. Although we did not succeed in finding such an expression, below we give a list of some special cases.

Single-trace terms

$$
c_{n;1} = \frac{1}{n!} \int\limits_{1234} [\mathrm{d}z] \tilde{P}(z_{12})^n = 2^n\, \omega\left(\underbrace{\begin{smallmatrix} 1, & \dots, & 1 \\ 0, & \dots, & 0 \end{smallmatrix}}_{n \text{ times}} \begin{smallmatrix} , & 0, & 0, & 0, & 0 \\ , & 0, & 0, & 0, & 0 \end{smallmatrix}\right)
\tag{E.7.1}
$$

Double-trace terms

$$
d_{n;1} = \frac{1}{n!} \int\limits_{12}^{34} [\mathrm{d}z] \tilde{P}(z_{12})^n = \frac{2^n}{n!} \int\limits_0^1 \mathrm{d}z_2 \prod_{i=1}^{n} \int\limits_0^{z_{(gc2}} \mathrm{d}y_i\, f^{(1)}(y_i) = \omega\left(\underbrace{\begin{smallmatrix} 1, & \dots, & 1 \\ 0, & \dots, & 0 \end{smallmatrix}}_{n \text{ times}} \begin{smallmatrix} , & 0 \\ , & 0 \end{smallmatrix}\right)
\tag{E.7.2}
$$

$$d_{n;2} = \frac{1}{n!}\int_{12}^{34}[\mathrm{d}z]\tilde{Q}(z_{13})^n = \omega\left(\underbrace{\begin{smallmatrix}1 & , & \cdots & , & 1\\ \frac{\tau}{2} & , & \cdots & , & \frac{\tau}{2}\end{smallmatrix}}_{n\text{ times}}\begin{smallmatrix}, & 0\\ , & 0\end{smallmatrix}\right) \tag{E.7.3}$$

$$d_{n,m;1} = \frac{1}{n!m!}\int_{12}^{34}[\mathrm{d}z]\tilde{P}(z_{12})^n\tilde{P}(z_{34})^m = d_{n;1}d_{m;1} = \omega\left(\underbrace{\begin{smallmatrix}1 & , & \cdots & , & 1\\ 0 & , & \cdots & , & 0\end{smallmatrix}}_{n\text{ times}}\begin{smallmatrix}, & 0\\ , & 0\end{smallmatrix}\right)\omega\left(\underbrace{\begin{smallmatrix}1 & , & \cdots & , & 1\\ 0 & , & \cdots & , & 0\end{smallmatrix}}_{m\text{ times}}\begin{smallmatrix}, & 0\\ , & 0\end{smallmatrix}\right) \tag{E.7.4}$$

$$d_{n,m;2} = \frac{1}{n!m!}\int_{12}^{34}[\mathrm{d}z]\tilde{P}(z_{12})^n\tilde{Q}(z_{13})^m = d_{n;1}d_{m;2} = \omega\left(\underbrace{\begin{smallmatrix}1 & , & \cdots & , & 1\\ 0 & , & \cdots & , & 0\end{smallmatrix}}_{n\text{ times}}\begin{smallmatrix}, & 0\\ , & 0\end{smallmatrix}\right)\omega\left(\underbrace{\begin{smallmatrix}1 & , & \cdots & , & 1\\ \frac{\tau}{2} & , & \cdots & , & \frac{\tau}{2}\end{smallmatrix}}_{m\text{ times}}\begin{smallmatrix}, & 0\\ , & 0\end{smallmatrix}\right) \tag{E.7.5}$$

$$d_{n,m;3} = \frac{1}{n!m!}\int_{12}^{34}[\mathrm{d}z]\tilde{Q}(z_{13})^n\tilde{Q}(z_{14})^m = d_{n;2}d_{m;2} \tag{E.7.6}$$

$$d_{n,m,p;1} = \frac{1}{n!m!p!}\int_{12}^{34}[\mathrm{d}z]\tilde{P}(z_{12})^n\tilde{P}(z_{34})^m\tilde{Q}(z_{13})^p = d_{n;1}d_{m,p;2} = d_{n;1}d_{m;1}d_{p;2} \tag{E.7.7}$$

$$d_{n,m,p;2} = \frac{1}{n!m!p!}\int_{12}^{34}[\mathrm{d}z]\tilde{P}(z_{12})^n\tilde{Q}(z_{13})^m\tilde{Q}(z_{14})^p = d_{n;1}d_{m,p;3} = d_{n;1}d_{m;2}d_{p;2} \tag{E.7.8}$$

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126/2014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin 11.04.2018, Gregor Richter