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Counting gauge invariant operators in supersymmetric theories using Hilbert series.

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Declaration

I herewith certify that, to the best of my knowledge, all of the material in this dissertation which is not my own work has been properly acknowledged.

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Abstract

In this thesis, the problem of counting gauge invariant operators in certain supersymmetric theories is discussed.

These objects have a very important role in supersymmetric gauge theories, since they can be used to describe the space of zero-energy solutions, called *moduli space*, of such theories. In order to approach the counting problem, a technique is used based on a function known in Algebraic Geometry as the *Hilbert series*. For the examined theories, this can be considered a a partition function counting gauge invariant operators in the field theory according to their charges under quantum global symmetries.

In the first part of the thesis, particular focus will be given to the application of the Hilbert series to conformal Chern-Simons theories living on the world-volume of M2-branes probing different toric Calabi-Yau 4-fold singularities. It will be shown how the Hilbert series can be combined with the brane tiling formalism to characterise the mesonic moduli space of vacua of a given theory through its generators and the relations they satisfy. Then, toric duality for these theories will be presented, with special attention to the role played by Hilbert series in making such feature manifest between two or more theories. Finally, Chern-Simons theories living on M2-branes probing cones over smooth toric Fano 3-folds and their mesonic Hilbert series will be presented.

In the second part, it will be shown how the Hilbert series can be applied to counting gauge invariant operators in supersymmetric generalisations of Quantum Chromodynamics, known as SQCD theories. The discussion will hinge on a specific class of theories, with N multiplets transforming in the fundamental and anti-fundamental and one in the adjoint representation of the gauge group. For each classical group, the Hilbert series of the moduli space will be used to determine the dimension on the spaces, their generators and to argue that they are all Calabi-Yau manifolds.

To my family

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1 Introduction and Outline

Below, we give a brief overview of the developments in theoretical physics of the last few decades, from the rise of quantum field theory until string theory. It is not meant to be a comprehensive introduction to the subjects, but merely a motivation of the work discussed in this thesis. For a more exhaustive presentation of the subjects, the reader is referred to excellent books and review papers [1, 2, 3, 4, 5, 6, 7, 8].

Quantum field theory has played a central role in contemporary high energy physics, the most important result probably being the formulation of the Standard Model of particles using a simple gauge principle. Quantum field theories were introduced during the last century in the attempt of understanding Quantum Mechanics in the relativistic regime. One of the first achievements in this context was the introduction of Quantum Electro-Dynamics (QED), a theoretical framework that provided a generalisation of the Maxwell's theory of electromagnetism to the microscopic scale. In particular, the electromagnetic interaction was described in terms of fields propagating in space-time, with elementary particles being viewed as excitations of such fields.

Shortly after this remarkable result, quantum field theories were employed to investigate other fundamental forces, namely the weak interaction, being responsible of particle decays, and the strong interaction, believed to be the one holding protons and neutrons together in atomic nuclei. Research along these lines presented two fascinating features: the first is the existence of a unified description of the electromagnetic and the weak forces, the so-called *electroweak symmetry*; the second one is that it seemed possible to describe all three microscopic interactions by imposing that the corresponding Lagrangians possess local symmetries, also called *gauge symmetry*. In particular, the electroweak theory follows from invariance under the gauge group $SU(2) \times U(1)$ and the theory for strong interactions, called Quantum Chromo-Dynamics (QCD), from requiring invariance under $SU(3)$. Both these features should be considered as great achievements of quantum field theories, since they go very well along the spirit that has driven scientific research throughout the centuries.

One of the problems with this paradigm is that the insertion of mass terms for gauge bosons in the Lagrangian breaks gauge symmetry explicitly. At first, this would suggest that all the gauge bosons are massless and, therefore, the interactions they mediate are all long-range. However, there is a good amount of experimental evidence which contradicts this statement for the weak and the strong interactions. A solution to this puzzle was found by introducing a scalar field, called *Higgs boson*, together with a potential, which in the vacuum acquires an expectation value. This mechanism allows mass generation for gauge bosons without breaking gauge symmetry explicitly.

The picture described so far constitute what is known as the *Standard Model* of particles, one of the greatest achievements of all the scientific endeavour, which has allowed to make predictions, some of which agree with the experimental results up to the 15^{th} digit!

Having explored to some extent the power of symmetry in the context of field theory, it is natural to ask what is the maximum symmetry one can impose on a Lagrangian. In 1967 Coleman and Mandula [9] proved a *no-go theorem*, which has been named after them, that says that it is impossible to combine space-time and internal symmetries if not in a trivial manner. A way of avoiding the conclusions of the theorem is to extend the Poincaré algebra to a so-called *graded Lie algebra*, or *superalgebra*, where spinorial generators and anti-commutation relations are allowed. The symmetry one obtains following this path, called *supersymmetry*, is much less trivial and contains transformations that can exchange fermions with bosons. The discovery of supersymmetry is usually credited to Wess and Zumino through their work of 1974 [10] but, as it turns out, two other research groups independently reached the same conclusions some years before: Gol'fand and Likhtman published the first paper on the subject in 1971 [11, 12] and Volkov and Akulov published two years later, in 1973 [13].

At present, supersymmetry is an elegant idea which awaits to be proven experimentally, but there exist already many motivations to employ this theoretical tool to describe the quantum reality. Possibly, the strongest one is to solve the so-called *hierarchy problem* which has been haunting particle physics for decades. Since the Higgs boson is a scalar field, there are no symmetries in the Standard Model to protect its mass from receiving quantum corrections of the order of the Planck's mass, where a grand unification of forces is expected to take place and which sets a cut-off scale for the Standard Model. However, the Higgs mass should be of the order of the electroweak symmetry breaking scale, which is roughly 15 orders of magnitude smaller than the Planck's mass. Supersymmetry has enough power to protect the Higgs mass from receiving contributions that would increase its mass enormously and thus resolves the puzzle.

From the point of view of quantum field theories, the introduction of supersymmetry is of great importance since it stretches further the exploration of the role played by symmetries in the description of our world. Also, it provides solutions to some problems related to the Standard Model. For this reason, supersymmetric gauge theories will be one of the main focal points of this manuscript, in the hope that the research presented helps clarifying the details of such beautiful theoretical constructions.

The framework provided until this point by field theories is certainly very satisfactory, but it still leaves some open problems. In particular, we have discussed to some extent how gauge theories seem to be a natural place to treat three of the four fundamental forces, but what about gravity ?

From the point of view of quantum field theory, General Relativity is non-renormalisable. This means that a theory of quantum gravity formulated along this path should be considered valid only up to a certain scale, provided by the Planck's mass. Through the years, many proposals have been made to solve this problem, string theory being probably the most promising.

Quite remarkably, string theory was proposed in the late sixties, as a possible candidate to describe strong interactions. In those years, Quantum Chromo-Dynamics hadn't appeared yet and many were skeptics about an approach based on quantum field theories since the perturbative expansion that was so important for QED could not be used in this new context. It was believed that one of the features a theory for strong interactions should have was the *duality*, namely the idea that s- and t-channel Feynman diagrams give alternative descriptions of the same physics. It was not obvious how a scattering amplitude could be invariant under the exchange of s and t, until Veneziano proposed one in 1968 [14] based on ratios of Euler's functions. His proposal had phenomenological motivations, but it nevertheless described a consistent perturbation expansion of a quantum field theory. Veneziano's work attracted some attention but the more research was being carried out on it, the more the proposal seemed strange. The first puzzle came from the unitarity of the model, which was proved to exist only in 26 space-time dimensions [15]. Another puzzling feature was that the generalisation of the Veneziano amplitude to N particles seemed to have a natural interpretation as of scattering of modes of a relativistic string [16, 17, 18, 19]. Furthermore, the spectrum of the model contained massless bosons of spin 1 and 2, which had never been observed, and tachyons. Finally, the Veneziano amplitude in high energy fixed-angle regime had an exponential fall-off, whereas the experiments were showing a power law behaviour. Since this feature was very well explained in parton models, the Veneziano model was put aside.

It is worth pointing out that the seeds for the work by Wess and Zumino on global supersymmetry in 4 dimensions originated within the context of string theory, as a generalisation of the so-called Ramond-Neveu-Schwarz (RNS) model [20, 21], which included supersymmetry in order to produce fermionic degrees of freedom in the spectrum of the Veneziano's theory.

However, given the beauty that string theory had already shown, some authors did not surrender and decided to further investigate the theory even beyond the realm of hadrons. In particular, it was suggested [22] that the massless spin-2 particle be interpreted as a graviton and it was shown that in the low-energy limit this field obeys Einstein's equations of General Relativity. Furthermore, the massless spin-1 particle that string theory contained in the spectrum could be interpreted as the mediator of Yang-Mills interactions.

A very important contribution to the newly born string theory was given by Gliozzi, Scherk and Olive [23, 24] who introduced a modification of the RNS model that removed all the tachyons from the string spectrum. This paved the way to introducing space-time supersymmetry in string theory, giving birth to the so-called *superstring theories*. In order for these theories to be unitary, the dimensions of space-time are required to be 10, rather than the 26 of the non-supersymmetric *bosonic string*.

In the early eighties, a classification of consistent ten-dimensional superstring theories was carried out, bringing to light theories that were called Type I, Type IIA and Type IIB. At the time these theories were introduced, it wasn't clear whether they could be used to describe the world we live in. In particular, Type I and Type IIB could accommodate parity violation, but it had to be proven that they were free from gauge anomalies. In 1984, Alvarez-Gaumé and Witten [25] presented formulas to analyse different types of anomalies in an arbitrary number of dimensions. From this investigation, they concluded that indeed in Type IIB superstring theory gauge anomalies cancel and, therefore, the theory can be considered as a consistent chiral theory. Few years later, Green and Schwarz also proved that Type I is anomaly free, provided that the gauge gauge group of the theory is either $SO(32)$ or the exceptional $E_8 \times E_8$ [26, 27, 28].

Few months after these results, another kind of ten-dimensional string theories, called *heterotic*, was discovered [29, 30, 31] and was shown to possess the same gauge groups as Type I. Together, all these results gave rise to what is known as the *first superstring revolution*, where not only the subject of string theory grew immensely, but it also started to regain a good deal of attention from the scientific community.

About ten years later, another great amount of fascinating features of string theory came to light, giving rise to what has been called the *second superstring revolution.* The first achievement during this time was the understanding that string theory did not just contain fundamental strings oscillating in space-time, but it also included non-perturbative extended objects called Dp-branes [32, 33], where p indicates the number of spatial dimensions they occupy. The defining property of the D-branes is simply that they are hyperplanes where open strings can end, but there's much more to it. In fact, it was shown by Polchinski in 1995 [34] that these objects preserve half of the space-time supersymmetry and that they couple to the Ramond-Ramond fields. Furthermore, Witten showed [35] that the end points of strings on a stack of N D-branes generate a $U(N)$ Yang-Mills theory on their world-volume.

A similar extended object which preserves half the supersymmetry is the so-called Neveu-Schwarz (NS)5-brane [36]. It is a solitonic object that couples magnetically to the NS-NS two-form[1] and that lives in Type II and heterotic superstring theories only.

Another incredibly important feature of the second revolution was the introduction of dualities. A first example was introduced by Buscher [37],

[1]We could think of this brane as the magnetic dual of the fundamental string.

with the so-called *T-duality*, which connects two string theories living in different space-time geometries. In particular, if we imagine a theory in a space where one of the dimensions is compactified on a circle of radius R, then T-duality connects it with another theory with the compact dimension having radius α'/R, where α' is related to the tension of the fundamental string. Without delving into technical details, it is worth noticing that T-duality allows to connect different superstring theories: as it turns out, Type IIA and Type IIB are dual to each other, and the same applies to heterotic with gauge group $SO(32)$ and $E_8 \times E_8$.

The second discovered duality was *S-duality.* Instead of relating theories living in different space-time geometries, it connects theories at different gauge couplings. This duality was first introduced in the context of Yang-Mills theories by Montonen and Olive in 1977 [38] and its existence in string theory was later postulated for heterotic compactified on T^6 [39]. In subsequent years, S-duality was extended to Type IIB theory [40], which is self-dual, and Type I and heterotic with gauge group $SO(32)$ [41].

The question that now arises is what are the S-duals of Type IIA and heterotic $E_8 \times E_8$ string theories. Very surprisingly, as the string coupling in both theories is increased, an eleventh space-time dimension is developed and the theories admit the same limit, known as *M-theory* [42, 43]. The low-energy limit of this theory is the eleven dimensional supergravity that was introduced by Cremmer, Julia and Scherk in 1978 [44]. Within its spectrum, this supergravity theory contains an anti-symmetric three-form that, in M-theory, couples electrically to a 2-dimensional object called M2-brane and magnetically to an M5-brane. Interestingly, there is no analog of the anti-symmetric two-form $B_{\mu\nu}$ of Type II or heterotic, so we conclude that there are no strings in M-theory.

In recent years, one of the most striking insights that string theory has given is the duality between conformal field theories and string/M- theory in a background composed of an anti-de-Sitter space-time and a compact manifold of odd dimensions called Sasaki-Einstein. This idea rose from the grounds set with [45, 46, 47], it was suggested by Maldacena [48], and further refined by Gubser, Klebanov, Polyakov [49] and Witten [50]. In its original formulation, the correspondence states that Type IIB superstring theory in $AdS_5 \times S^5$ is dual to $\mathcal{N} = 4$ super-Yang-Mills theory in $(3 + 1)$ dimensions. More generally, one can consider a five-dimensional Sasaki-

Einstein manifold instead of the five-sphere and that would give rise to a gauge theory in 4 dimensions with $\mathcal{N} = 1$ supersymmetry (see e.g. [51]).

In the context of this gauge/gravity duality, a long-standing puzzle was the so-called AdS_4/CFT_3 case (see e.g. [8]), on the one side of which there is M-theory on $AdS_4 \times SE^7$ and on the other side there is a three dimensional conformal field theory. In particular, the missing ingredient in this construction was the field theory on the world-volume of the M2-branes, which remained mysterious for many years. One of the simplest approaches would be to start from a stack of N D2-branes in Type IIA string theory and then take the strong coupling limit. This way, the world-volume theory on the stack of M2-branes would be a $(2+1)$-dimensional super-Yang-Mills theory, obtainable with a dimensional reduction from the well-known 4-dimensional $\mathcal{N} = 4$ SYM. This approach already presents a problem since, upon reduction, we obtain a theory which has 7 scalars instead of 8, which is the expected number for a theory on M2-branes. In the abelian case, one could dualise a gauge vector to a scalar, thus obtaining a correct counting, but it's not clear how to do so in the non-abelian case. Also, the Yang-Mills coupling in $(2 + 1)$-dimensions is dimensionful and therefore the theory we constructed has no manifest conformal invariance.

A first hint about what the possible solution could be was given by Schwarz in [52]. He considered the field content on the world-volume of an M2-brane, eight scalars and eight two-component Majorana spinors, and asked what strategy could be adopted in order to add gauge fields without breaking the matching of the on-shell degrees of freedom and, hence, supersymmetry. He argued that the only possible way to achieve this was to introduce gauge fields with Chern-Simons couplings, and analysed the properties of a $U(N)$ theory possessing these features. However, this path presented problems since, for example, a Chern-Simons theory with only one gauge group is parity-violating whereas M-theory should conserve parity. A real breakthrough in this search came with the work of Bagger and Lambert [53] and, independently, Gustavsson [54], who constructed a theory possessing $\mathcal{N} = 8$ supersymmetry and $SO(4)$ gauge group using a mathematical object called "3-algebra" which, to some extent, can be considered as a non-associative extension of a Lie algebra. Subsequently, it was shown [55, 56] that this model is completely equivalent to a Chern-Simons theory having $SU(2)\times SU(2)$ gauge group and opposite levels. Finally, in 2008

Aharony, Bergman, Jafferis and Maldacena [57] generalised the construction by proposing a model with $U(N) \times U(N)$ gauge symmetry, Chern-Simons levels $(k, -k)$ and proved that it possesses $\mathcal{N} = 6$ supersymmetry for generic k, enhanced to $\mathcal{N} = 8$ for $k = 1, 2$. Furthermore, they interpreted the model as describing the world-volume theory of a stack of N M2-branes probing the orbifold singularity of $\mathbb{R}^8/\mathbb{Z}_2$. The model of Bagger Lambert and Gustavsson was a particular case, describing 2 M2-branes in flat space.

This manuscript is organised as follows. In Chapter 2, we will introduce the techniques that will be used to analyse the moduli spaces of supersymmetric gauge theories. First, we will present a class of theories which arise on the world-volume of D3-branes, called *quiver gauge theories*, and we will discuss a graphic technique to efficiently represent them, called *brane tilings*. These are periodic bi-partite graphs which encode all the information needed to reconstruct their Lagrangian. We will then outline the main ingredients of the so-called *forward algorithm*, which uses the data provided by the brane tilings to give the classical moduli space of vacua of a quiver gauge theory. Finally, we will introduce the *Hilbert series*, that we will employ as a partition function counting the gauge invariant operators in supersymmetric gauge theories.

In Chapter 3, we will discuss the moduli spaces of theories on M2-branes. We will do so by first introducing the ABJM model and the significance of the mini-revolution that it brought about [58], and then we will illustrate how the brane tilings and the forward algorithm can be extended from the context of D3-brane theories to be applicable to the Chern-Simons theories that live on the world-volume of the M2-branes. Next, we will use these techniques to investigate important features of these theories, such as *toric duality* [59] and the *Higgs mechanism* [60]. Finally, we will present a classification of theories which arise whenever an M2-brane is probing the singularity of a CY_4 manifold, constructed from Fano 3-folds [61].

In Chapter 4, we will apply the Hilbert series to the problem of counting gauge invariant operators in Adjoint SQCD [62]. Typically, by SQCD one denotes a class of theories with $\mathcal{N} = 1$ supersymmetry in 4 dimensions, matter fields transforming in some representations of a gauge group G, and vanishing superpotential. Among all the possibilities, we will focus on a specific subset of these theories, with N_f fields in the fundamental and N_f in the anti-fundamental and one adjoint representation of the gauge group. For all classical groups, we will use the Hilbert series to count gauge invariant operators and to make some conjectures regarding the classical moduli space of vacua of these theories.

2 An overview of quivers, brane tilings and Hilbert series

2.1 Quivers gauge theories

Quivers are commonly used tools in the investigation of $\mathcal{N} = 1$ theories realised on the world-volume of D3-branes probing a Calabi-Yau singularity. These objects are oriented graphs, formed by a set of nodes and arrows (see e.g. [63, 64, 65] for a historical introduction).

For our purposes, there exists a very simple dictionary to interpret a quiver diagram:

- Each node corresponds to a $U(N)$ gauge group; the ranks of all the gauge groups clearly need not be the same, but we shall see that the physics can impose some restrictions;
- Each arrow corresponds to a chiral matter field, with the convention that an outgoing (incoming) arrow from a node represents a field that transforms in the fundamental (anti-fundamental) representation of the gauge group that corresponds to the node. In case the head and the tail of the arrow are connected to the same node, then the corresponding chiral field transforms in the adjoint representation of the gauge group;
- Each term in the superpotential is represented by closed loops in the quiver;

Interestingly, the vice versa to the last point does not necessarily hold, which can also be rephrased by saying that a quiver is not sufficient to fully specify the Lagrangian of an $\mathcal{N} = 1$ theory in 4 dimensions: a superpotential must be given as well. We will see in the next section that this problem is removed with the use of brane tilings.

In Figure 2.1 we can see a prototypical example of what a quiver looks like.

Figure 2.1: An example of a quiver

Throughout this chapter, we will be interested in quiver gauge theories which arise on the world-volume of D3-branes placed at some Calabi-Yau singularity. Thus, the framework for our analysis is Type IIB string theory which, as is well-known, is chiral. It is well established that in chiral theories one loop corrections can introduce gauge anomalies that make the theory inconsistent. In order to avoid this situation, some constraints need to be imposed.

In particular, denoting a node of the quiver by n_α and an arrow by a_i, we can define a function $\sigma(n_\alpha, a_i)$ such that:

$$\sigma(n_\alpha, a_i) = \begin{cases} 1 & \text{if } a_i \text{ is pointing towards } n_\alpha \\ -1 & \text{if } a_i \text{ is outgoing from } n_\alpha \end{cases}. \tag{2.1.1}$$

With this definition, the condition for the gauge anomalies to cancel can be written as [66]:

$$\sum_{a_i=(n_\alpha,n_\beta)} \sigma(a_i, n_\alpha) N_\beta = 0, \tag{2.1.2}$$

where the summation is done over all the arrows that terminate on the node n_α and start from a node n_β. We denote by N_β the rank of the gauge group corresponding to the node n_β.

Another way of presenting this condition is by defining a function $\tau_i(n_\alpha, n_\beta)$ which equals 1 if the arrow a_i transforms in the bi-fundamental representation $(N_\alpha, \bar{N}_\beta)$ of the gauge group $U(N_\alpha) \times U(N_\beta)$ and -1 if it transforms in the conjugate representation. This way, we can rewrite (2.1.2) as:

$$\sum_{\beta \text{ con. } \alpha} \tau(n_\beta, n_\alpha) N_\beta = 0, \tag{2.1.3}$$

where "β con. α" means that the node n_β is connected to the node n_α. In the remaining of this thesis we will make the assumption that all the gauge groups have the same rank, say N. It is obvious that, within this framework, the condition (2.1.3) reduces to the requirement that for each node of the quiver the number of outgoing arrows equals that of the incoming ones.

2.2 Brane tilings

We now introduce one of the main characters of this thesis: brane tilings [67, 68, 69, 70, 71, 72]. In our discussion, we will treat these as periodic bi-partite graphs defined on the two-torus $\mathbb{T}^2$. The bi-partiteness condition means that every vertex can be coloured either black or white and each black node is connected to white nodes only and vice versa. The area of the tiling which contains all the information without repetition is called *fundamental domain*. Copying the domain along the cycles of the torus reproduces the complete brane tiling.

An example of a brane tiling and its fundamental domain is presented in Figure 2.2:

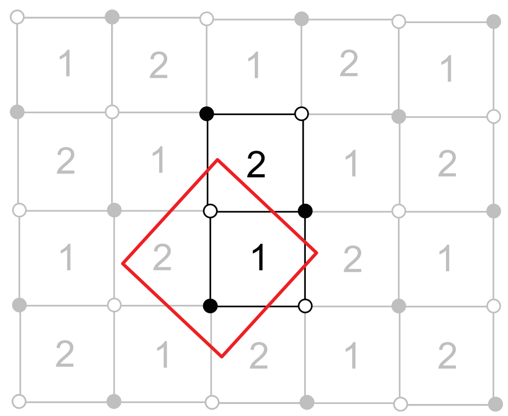


Figure 2.2: An example of a tiling. The area enclosed in red is the fundamental domain.

Just like quivers, brane tilings need to be interpreted according to a dictionary that we will now introduce:

- *White (resp. black) nodes* correspond to positive (negative) terms in the superpotential. They have an induced clockwise (anti-clockwise) orientation.

- *Edges* connect to nodes and correspond to the bifundamental fields in the superpotential. Going along the induced orientations around nodes, one can identify the matter fields associated to a specific superpotential term in the correct cyclic order.

- *Faces* correspond to $U(n_i)$ gauge groups. Every edge X_{ij} in the tiling has two neighbouring faces corresponding to $U(n_i)$ and $U(n_j)$. The quiver orientation of the bifundamental field X_{ij} is induced by that around black and white nodes at the two ends of the corresponding tiling edge.

Note that the bi-partiteness of the tiling implies that each face has an even number of edges, say $2n$. Of these, n are outgoing and n are ingoing. Hence, working under the assumption that all gauge groups have the same rank, the anomaly cancellation condition is automatically satisfied.

A great advantage of using brane tilings rather than quivers is that the information on the superpotential is encoded in the graph in a completely unambiguous manner: each superpotential term is associated to a node and vice versa. Hence, a brane tiling contains the necessary information to specify an $\mathcal{N} = 1$ supersymmetric gauge theory.

Before we go any further, it is interesting to discuss how brane tilings give a physical interpretation of the famous Euler's characteristic. Denoting by F the number of faces, N the number of vertices and E the number of edges on the tiled torus, the formula for the characteristic reads:

$$F + N - E = 2g - 2. \tag{2.2.4}$$

Interestingly, this identity can be transferred on the gauge theory side by noting that in our case $g = 1$, F is the number of gauge groups, N the number of superpotential terms and E the number of quiver fields. This result is nice in itself, since it was derived uniquely from the geometry of the brane tiling, but we can make a step further and gain a deeper insight on the interpretation of the formula.

In particular, if we assign an R-charge to each edge attached to a node in the tiling, the condition that the superpotential has R-charge equal to 2 imposes the following:

$$\sum_{i \in edge\ around\ node} R_i = 2, \tag{2.2.5}$$

with the sum running over all the edges connected to a certain node. We can now sum over all the nodes in the fundamental domain, and we would have the following condition:

$$\sum_{nodes} \sum_{i \in edges} R_i = 2N. \tag{2.2.6}$$

Since we will focus on conformal field theories, we should impose the vanishing of the beta function for every gauge group of the theory. This means we will have to impose:

$$2 + \sum_{k \in edges\ around\ face} (R_k - 1) = 0. \tag{2.2.7}$$

Summing over all the faces of the tiling and using (2.2.5), the equation above reduces to

$$2F + 2N - 2E = 0. \tag{2.2.8}$$

We can thus appreciate how the vanishing of the Euler's characteristic is tightly related to gauge theory being superconformal, which provides further motivation for the specific choice of the torus as a Riemann surface to tile [68].

2.2.1 The physical interpretation of brane tilings

In the following, we will give a brief overview on the interpretation of brane tilings as systems of 5-branes, thus providing a physical construction of the tools that we will be using in the remainder of this manuscript (see [71] for an excellent presentation of this point).

The starting point for our discussion is Type IIB string theory. In this framework, we consider a stack of N D5-branes spanning directions 012357, on which world-volume there exists a $U(N)$ gauge theory. Obviously the directions 5 and 7 will be redundant for our purposes since we are concerned only with $(3+1)$-dimensional gauge theories, and so we compactify them on a torus. This gives rise to the well-known $\mathcal{N}=4$ super-Yand-Mills in 4 dimensions, which implies that we need to add some extra ingredient in order to obtain theories with less supersymmetry and more diverse gauge groups and matter content. Thus, we add two sets of NS5-branes, one that spans the 012345 directions and the other one, that we will refer to as NS5'-branes to avoid confusion, which spans the 012367 directions. We shall assume that each set of branes wraps the torus along one of its cycles.

When each of the NS5-branes meets the stack of D5-branes at a junction, a bound state is created between the two which, by charge conservation, is an $(N,1)$- or $(N,-1)$-brane, depending on the orientation of the NS5-brane. It is important to point out that we are working in a strong-coupling regime, $g_s \to \infty$, which allows the construction to be particularly simple, with NS5-branes bending at an angle at the junction[1]. Note that the parts of the NS5-branes which are parallel to the D5-branes will merge into each other and form a single NS5-brane which extends in the directions 0123 and that wraps a 2-dimensional surface Σ in the 4567 directions.

As we said above, the $\mathbb{T}^2$ is now divided into three different types of region: $(N,0)$-, $(N,1)$-, and $(N,-1)$-branes. From the brane tilings point of view, the $(N,0)$-branes are identified with the faces of the tiling, i.e. with the gauge groups of the quiver theory. Two such regions are connected by fundamental strings that transform in bi-fundamental representations of the corresponding gauge groups. A priori, it would seem reasonable to have strings transforming in conjugate representations, forming a hypermultiplet, but the orientation of the NS5-branes selects only one of the representations, thus making the theory chiral. This is the main reason why edges on brane tilings can be given a precise orientation around a node. Note that the presence of fields transforming in conjugate representations is not forbidden, but they will have to correspond to strings that stretch across different NS5-branes.

Following [71], we can consider this point from another angle. If we remove the NS5'-branes from the construction, the remaining system will give

[1] In fact, in such a limit the NS5-brane tension goes to zero much faster than the D5-brane one.

rise to an $\mathcal{N} = 2$ supersymmetric theory, whose hypermultiplets parameterise the motion of the D5-branes in the 6789 directions. If we re-introduce the NS5'-branes, supersymmetry is broken by half and each hypermultiplet is split into two chiral multiplets. However, the presence of the NS5'-branes does not allow the D5-branes to move in the 67 directions, which implies that only one of the two chiral multiplets is left.

Finally, the $(N, 1)$- and $(N, -1)$-brane regions correspond to superpotential terms for the gauge theory, where by convention we choose the former to represent white nodes in the tilings and the latter the black ones. The content of each superpotential term is given by going around the $(N, \pm 1)$-brane with an orientation induced by the NS5-brane and looking at the strings connected to different D5-branes.

Having elucidated how the bipartite graphs introduced above can be interpreted from a physical point of view, we can make one step further and connect our set-up of D5- and NS5- branes with the 4-dimensional gauge theories we wish to investigate. In particular, if we perform two T-duality transformations along the 5- and 7-direction, the D5-branes are mapped to D3-branes and the NS5-branes are turned into pure geometry, in particular that of a Calabi-Yau manifold. The precise geometry of the manifold is given by the number of NS5-branes introduced and the cycles of the torus that they wrap.

2.3 Toric geometry

We shall now give a brief overview of toric geometry, a fundamental tool for our analysis. For a more detailed discussion, the reader is referred to excellent sources, such as [73, 74, 75, 76, 77].

2.3.1 Toric varieties

The bi-partiteness of the brane tiling implies that every edge is connected to a white node on one end and a black node on the other hand. From the point of view of the gauge theory, we can restate this by saying that each field appears in the superpotential exactly twice, once with a positive sign and once with a negative sign. This condition is crucial for our methods and is often referred to as the *toric condition*. In turn, this guarantees that the moduli space of the theory is a toric variety.

A good starting point to appreciate the usefulness and the beauty of toric geometry is, perhaps unsurprisingly, the torus, a geometrical construction which is omnipresent in string theory. As an example of the usefulness of tori, the compactification of a higher dimensional theory on a torus $\mathbb{T}^d$ ensures the maximal amount of supersymmetry is preserved in the lower dimensions. An even better situation than dealing with a d-dimensional torus, would be to have something that possesses similar features, like the $U(1)^d$ symmetry, but with a more general geometrical structure. This is precisely what toric geometry does, by focusing on varieties that indeed have a $U(1)^d$ action, like a $\mathbb{T}^d$, which is now allied to have fixed points.

To illustrate this, let's consider the complex plane $\mathbb{C}$. By representing each complex number as a real number times a phase, we immediately see that the space admits a $U(1)$ action:

$$z \to |z|e^{i\phi}. \tag{2.3.9}$$

Notice that this action has indeed a fixed point at $|z| = 0$. In a similar manner, the projective space $\mathbb{P}^1$, which is equivalent to a two-sphere S^2, can be represented as a finite segment over which a circle is fibrated, the fibers shrinking at the end points of the segment.

We can formalise the discussion by discussing the notion of *toric variety.* A very interesting way of doing so was introduced by the mathematician David Cox, which allows us to construct toric varieties in a similar way to complex projective spaces.

The starting point is the d-dimensional complex space, $\mathbb{C}^d$, on which we define the action of an algebraic c-torus, $(\mathbb{C}^*)^c$. We indicate with Z the subset of points in $\mathbb{C}^d$ which are fixed by a continuous subgroup of the torus. A toric variety X is the algebraic variety which is obtained by subtracting Z from $\mathbb{C}^d$ and by quotienting the whole by the c-torus:

$$X = (\mathbb{C}^d \backslash Z)/(\mathbb{C}^*)^c. \tag{2.3.10}$$

Notice that, since $c < d$, there is a $(\mathbb{C}^*)^{d-c}$ torus action on the variety itself. It is also clear that $(d-c)$ is the dimension of the algebraic torus and also of the toric variety.

From the definition, it is obvious how to construct a generic complex space as a toric variety, and it is also easy to show how a generic projective space $\mathbb{P}^n$ can be built in this manner. In fact, a standard way of defining such spaces is to start from the complex space $\mathbb{C}^{n+1}$, and then define the following equivalence relation:

$$(z_1, z_2, \ldots, z_{n+1}) \sim \lambda(z_1, z_2, \ldots, z_{n+1}), \qquad \lambda \in \mathbb{C}^* \tag{2.3.11}$$

Modding out by this relation yields the n-dimensional projective space. With the terminology we have introduced above, the equivalence relation is the action of a one-dimensional torus, and the subset of fixed points under this action contains the origin of the complex space as its only element. Hence, we can summarise all by the following:

$$\mathbb{P}^n = (\mathbb{C}^{n+1} \backslash \{0\})/(\mathbb{C}^*). \tag{2.3.12}$$

2.3.2 The toric diagram and the symplectic quotient

One of the main advantages of dealing with toric varieties is that all their relevant geometric information can be represented very simply by means of convex polygons, called *toric diagrams*, defined on integer lattices. The spirit is very similar to the one outlined above, where the projective space $\mathbb{P}^1$ was represented as a finite interval together with a $U(1)$ fibration.

In order to define toric diagrams, let us begin by taking the integer lattice $\mathbb{Z}^d$, and define a *rational polyhedral cone* σ therein:

$$\sigma \quad = \quad \{\alpha_1 \mathbf{t}_1 + \alpha_2 \mathbf{t}_2 + \ldots + \alpha_k \mathbf{t}_k \in \mathbf{R}^d | \alpha_1, \ldots, \alpha_k \geq 0\}, \tag{2.3.13}$$

where the vectors $\mathbf{t}_i$ are elements of $\mathbb{Z}^d$. In order to construct toric diagrams, we must focus on cones which are *strictly convex*, a condition which can be formally defined by the following:

$$\sigma \cap -\sigma = \{0\}. \tag{2.3.14}$$

The concept of a cone is central to our discussion, but even more so is that of a *fan*, call it Σ, roughly speaking a collection of convex cones glued together. The introduction of this concept will lead us towards that of a toric diagram, but first we should answer the following question: how can we associate a fan to each toric variety ?

This is quickly done by recalling the definition of toric variety that we have given. In particular, a crucial point in the definition is the quotient of a subset of a complex space of some dimension by an algebraic torus. This can be written as a set of identifications of this form:

$$(z_1, z_2, \ldots, z_d) \sim (\lambda^{Q_n^1} z_1, \lambda^{Q_n^2} z_2, \ldots, \lambda^{Q_n^d} z_d), \qquad n = 1, \ldots, d - D \tag{2.3.15}$$

where D is the dimension of the toric variety. Then, we can collect all the Q_n^k into a $(d - D) \times d$ matrix, that we shall call Q_t, and take the null space of this matrix. This will be spanned by d vectors that are defined in $\mathbb{Z}^D$, and which can be used to construct a fan.

As an example, we can construct the fan associated with $\mathbb{P}^n$, which we have discussed above. Looking at (2.3.11), it is obvious that the Q_t matrix has only one row and can be written simply as:

$$Q_t = (1, 1, \ldots, 1). \tag{2.3.16}$$

The d-dimensional fan is therefore generated by the following vectors in $\mathbb{Z}^n$:

$$t_1 = (1, 0, \ldots, 0), \qquad t_2 = (0, 1, \ldots, 0) \qquad \ldots \qquad t_d = (-1, -1, \ldots, -1). \tag{2.3.17}$$

More concretely, in Figure 2.3 we can see the toric fan for the variety $\mathbb{P}^2$.

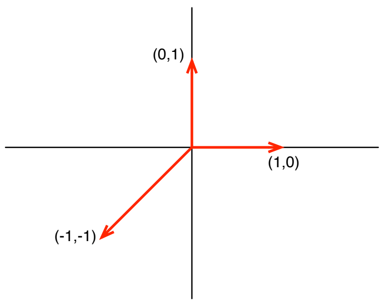


Figure 2.3: The toric fan for $\mathbb{P}^2$.

In the following chapters, we will be interested mainly in toric Calabi-

Yau manifolds of 3 and 4 complex dimensions. These objects enjoy a very nice property that can be proved in complete generality for all dimensions: the vectors that generate the fan associated to a toric Calabi-Yau lie on a common hyperplane. In the physics literature, the lattice points on this hyperplane form a so-called *toric diagram*. An interesting property that can be proved starting from this result is that the rows of Q_t sum to zero:

$$\sum_{k=1}^{d} Q_n^k = 0, \qquad n = 1, \ldots, d - D \tag{2.3.18}$$

Another way of constructing toric varieties which is often used in the physics literature is through the *symplectic quotient* [76]. Essentially, this method is completely equivalent to the one discussed above, except that the identifications of the coordinates are not made in a single step using a complex parameter λ, but rather dividing the procedure in two. First, let's consider the d-dimensional complex space again and let's define a set of functions called *moment maps*, $\mu_n : \mathbb{C}^d \to \mathbb{C}$, with n going from 1 to $(d - D)$, in the following way:

$$\sum_{i=1}^{d} Q_n^i |z_i|^2 \quad = \quad \mathrm{Re}(t_n). \tag{2.3.19}$$

The Q_n^i are the integer numbers we have introduced above and the t_n are generic complex numbers which are identified with Kähler parameters of the toric Calabi-Yau. After having defined a set of relations for the absolute values of the complex coordinates, we now pass to their phases and introduce an action of an algebraic torus given by the Q_n^j charges:

$$z_j \to e^{ik_n Q_n^j} z_j, \qquad n = 1, \ldots, d - D. \tag{2.3.20}$$

The toric variety is given by imposing the moment map equations and by modding out by the algebraic torus. This construction is particularly useful in the study of the moduli spaces of Gauged Linear Sigma Models (GLSMs) in two dimensions, where the scalar components of the chiral multiplets are identified with the complex coordinates z_i, the gauge groups correspond to the algebraic torus action and the charges of the fields under the gauge groups are given by the Q_n^j integers

2.4 The forward algorithm

The strong connection between quiver gauge theories and toric varieties offers the framework to two standard problems: the *forward problem*, which is the construction of the toric moduli space of vacua given a supersymmetric gauge theory and the *inverse problem*, which goes in the opposite direction.

In case of D3-branes, both problems have been solved efficiently with two mathematical algorithms, whereas in the case of M2-branes only for the forward problem there exists a solution. This, which we will refer to as the *forward algorithm*, is the focus of the present and following sections.

The problem of engineering a gauge theory having a certain toric variety as its moduli space is certainly not a new one, and a typical solution is through the use of 2-dimensional GLSMs, with n chiral superfields charged under a product of abelian groups, $U(1)^k$ [78]. It can be shown that the computation of the moduli space of these theories is exactly the same as the symplectic quotient discussed above, where the scalar components of the superfields play the role of the coordinates $\{z_i\}$, the gauge group is the action of the algebraic torus, the gauge charges of the fields are interpreted as the integers Q_n^j and the D-terms of the gauge theory correspond to the moment maps.

This is a very nice result and, remarkably, brane tilings allow us to extend it, associating a GLSM to every quiver gauge theory. Below, we shall give a brief overview on how this is done. A reader looking for further details and additional information is referred, for example, to [79].

The first step in establishing a connection between GLSMs and brane tilings is to determine which component of the tiling allows us to define a field in the GLSM. This problem is solved by introducing the concept of *perfect matching*, typically denoted by p_α. Perfect matchings are defined as sets of edges in the brane tiling which touch each node exactly once and correspond to chiral multiplets in the GLSM. Perfect matchings are therefore products of quiver fields and can be all found systematically using the information contained in the fundamental domain of the brane tilings. Every edge that crosses the boundary of this region acquires a winding number on the torus of the form (n_1, n_2), which we use to assign to each edge a weight $x^{n_1}y^{n_2}$. Obviously, edges that do not touch the boundaries of the fundamental domain are given weight 1. We can now define a matrix

K, called *Kasteleyn matrix* with as many rows as white nodes and as many columns as black nodes of the tiling. The generic entry K_{ij} of the matrix is given by the sum of the fields connected to the i^{th} and j^{th} node, together with their weights. The permanent of the matrix is a generating function for the perfect matchings: apart from the weights, its coefficients contain products of fields which generate each perfect matching.

We will discuss below how the permanent of the Kasteleyn matrix can be employed in the *fast forward algorithm*, but for the time being we consider it as a tool to determine systematically how perfect matchings are related to quiver fields. We then collect this information in a matrix P, called *perfect matching matrix*, with rows corresponding to chiral fields and columns to perfect matchings. The generic entry P_{ij} is 1 if the i^{th} field belongs to the j^{th} perfect matching or 0 otherwise.

The kernel of the P matrix is called Q_F and contains perfect matching charges which can be used to construct the space of solution to the F-terms, called *Master space* [90], using the symplectic quotient introduced in the preceding section. It can be shown that the rows of the Q_F matrix always sum up to zero, which allows us to conclude that the Master space is a toric Calabi-Yau manifold.

For the remainder of the section, we will focus on the D-terms of the supersymmetric gauge theory. First, we introduce a matrix that contains information about the gauge charges of the quiver fields. The matrix, which we call d, has as many columns as fields in the tiling and a row for every gauge group. Generic elements of the matrix, d_{ij}, are 1 (resp. -1) if the j^{th} field transforms in the anti-fundamental (resp. fundamental) of the i^{th} gauge group, and 0 otherwise.

Next, we define $\tilde{d}_f$ as the matrix that satisfies the following equation:

$$d = \tilde{d}_f \cdot P^T, \tag{2.4.21}$$

where P^T is the transpose of the perfect matching matrix. One could consider $\tilde{d}_f$ as the matrix which contains the gauge charges of the perfect matchings. The charge matrix for the D-terms is defined as:

$$Q_D \quad = \quad \ker\{(1,1,\ldots,1)\} \cdot \tilde{d}_D. \tag{2.4.22}$$

The multiplication by $\ker\{(1, 1, \ldots, 1)\}$ is simply to guarantee that the rows of Q_D sum to zero.

Finally, we can collect all the data obtained so far in a single charge matrix Q_t:

$$Q_t = \begin{pmatrix} Q_F \\ Q_D \end{pmatrix} \tag{2.4.23}$$

This matrix contains all the charges of the GLSM fields which allow the construction of the toric variety. Also, as we explained above, it is possible to draw a toric diagram which represents all the geometric information of the Calabi-Yau. We collect the coordinates of the vertices of the fan in the columns of the following matrix:

$$G_t = \ker Q_t. \tag{2.4.24}$$

An interesting consequence of the outlined algorithm is that to every perfect matching in the brane tiling, there corresponds a point in the toric diagram. Some of these points are on the corner of a polygon, and correspond to a divisor of the toric variety, and some are internal points, corresponding to vanishing 2-cycles in the Calabi-Yau.

2.4.1 The fast forward algorithm

A milder version of the problem we have been tackling above consists in determining just the toric diagram starting from the brane tiling. We will refer to this procedure as to the *fast forward algorithm* (see e.g. [71]). Recall the Kasteleyn matrix and the weight function we have defined for every chiral field. When we take the permanent of the matrix, each of its terms corresponds to a perfect matching and it has a coefficient that contains a product of the weights of the generic form $x^{n_1}y^{n_2}$. The exponents of the weights are precisely the coordinates of the toric point corresponding to a given perfect matching. In more rigorous terms, the toric diagram is given by the Newton polygon of the permanent of the Kasteleyn matrix.

2.5 The Hilbert series

For the remainder of this manuscript we will be interested in the problem of counting gauge invariant operators in supersymmetric gauge theories. We will see how this can be done in M2-brane theories and in SQCD-like theories. In both cases, the tool that will be used is a function called *Hilbert series* [80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92] that we will discuss below .

The topic can be introduced very elegantly in an algebraic manner [80, 81]. Let's start by considering a ring R, and let's suppose that the ring is graded and can be decomposed in the following way:

$$R = \bigoplus_{n \in \mathbb{N}} R_n, \tag{2.5.25}$$

where the elements contained in R_n are typically said *homogeneous of degree* n. The Hilbert series of the module is thus defined as:

$$g(t; M) \quad = \quad \sum_{n=0}^{\infty} \dim(R_n) t^n, \tag{2.5.26}$$

where the variable t is called *fugacity.*

This definition can be generalised to the case where the ring is given a multiple grading:

$$M = \bigoplus_{\vec{n} \in \mathbb{N}^k} R_{n_1, \dots, n_k}, \tag{2.5.27}$$

in which case the Hilbert series becomes:

$$g(t_1, \dots, t_k; M) \quad = \quad \sum_{\vec{n} \in \mathbb{N}^k} \dim(R_{n_1, \dots, n_k}) t_1^{n_1} \dots t_k^{n_k} \tag{2.5.28}$$

As an example, let's consider the ring of polynomials over the complex field, $\mathbb{C}[x]$, with grading according to the degree of x. For a generic degree n, the only element of the ring that has such a degree is x^n and, therefore, we have that $\dim(M_n) = 1$. There follows that the Hilbert series of this polynomial ring can be written as:

$$g(t;\mathbb{C}[x]) \;=\; \sum_{k=0}^{\infty} t^k = \frac{1}{1-t}. \tag{2.5.29}$$

We see that in this example the Hilbert series, that we introduced as a formal sum, converges to a very simple rational function. This result can be extended to much more general cases [80, 81], such as polynomial rings, where the Hilbert series can be shown to always take the form:

$$g(t;M) \;=\; \frac{P(t)}{Q(t)}, \tag{2.5.30}$$

where $P(t)$ is a polynomial and $Q(t)$ can always be expressed as a product of terms of the form $(1-t^{\alpha})$, for some power α. Thus, the Hilbert series has a pole for $t=1$ and it can be shown that the degree of divergence at the pole coincides with the dimension of of the polynomial ring.

In the example we have considered, the numerator of the Hilbert series was trivial, but in more general cases it is a polynomial whose meaning can be easily interpreted. Let's consider a polynomial ring $S=\mathbb{C}[x_1,x_2]$, and let's impose the relation $x_1^2=0$. Clearly, the ring only contains elements of the form x_2^n or $x_1x_2^n$, n being a generic integer. Therefore, imposing a double grading given by the powers of the two elements of S, the Hilbert series can be written generically as:

$$\begin{aligned} g(t_1,t_2;S) &= \sum_{m=0}^{\infty} t_2^m + \sum_{n=0}^{\infty} t_1 t_2^n = \frac{1+t_1}{1-t_2} \\ &= \frac{1-t_1^2}{(1-t_1)\,(1-t_2)} \end{aligned} \tag{2.5.31}$$

It appears as if the precise form of the numerator is tightly related to the relations that are imposed among the elements of the ring. In fact, setting $x_1^2=0$ implies that all the products between x_1^2 and any element of the ring have to vanish as well. Formally, this amounts to saying that all the elements belonging to the ideal I generated by the imposed relation have to be removed from the ring. It is immediate to see that, in terms of Hilbert series, the insertion of $-t_1^2$ in the numerator does exactly this job.

For more convoluted cases, the computation of the Hilbert series is quite complicated and one typically has to use computer programs like *Macaulay2* or *CoCoA*. However, the method used for the computation is exactly the same that we have described above.

There's a very interesting connection between the outlined construction of the quotient ring and Algebraic Geometry that we shall briefly discuss [82]. An algebraic variety X is basically defined as the locus of points where a certain set of polynomials, call it f, vanish. If the elements of f belong to a polynomial ring $K[x_1, \ldots, x_n]$, we can associate an ideal to the polynomials that define the variety:

$$I(X) = \{f \in K\left[x_1, \ldots, x_n\right] : f \equiv 0\} \tag{2.5.32}$$

Algebraically, the variety X is described in terms of the *coordinate ring*:

$$A(X) = K[x_1, \ldots, x_n]/I(X). \tag{2.5.33}$$

Elements of the coordinate ring of the variety are called *regular functions* and are basically all the polynomials in $K[x_1, \ldots, x_n]$ which do not vanish on X. It is important to stress that the knowledge of all the regular functions of an algebraic variety is sufficient to characterise the variety completely. From previous discussion, it is also clear that a simple method for counting the regular functions of X according to some grading is through the Hilbert series.

In this manuscript, we will be looking at moduli spaces of supersymmetric gauge theories precisely from this algebraic point of view, with the ideal $I(X)$ being generated by the F-terms, the coordinate ring being the Master space and the regular functions being the gauge invariant operators of the theory.

In this context, it is worth mentioning a very interesting result which says that if the algebraic variety X is also Calabi-Yau, then the corresponding Hilbert series is a rational function whose numerator $P(t)$ is palindromic, which means that it satisfies:

$$P(1/t) = t^w P(t), \tag{2.5.34}$$

for some w that depends on the details of the ring. Remarkably enough, the converse holds as well, which implies that if we find that the Hilbert series of a certain variety has a palindromic numerator, then we are dealing with a Calabi-Yau variety. This is a highly non-trivial result that we will be using heavily in the second part of this manuscript.

Above, we have stressed the importance of the regular functions in the study of an algebraic variety. Obviously, many of these functions can be expressed as products of other, more elementary ones, and therefore it makes sense to ask which regular functions are independent or, in other words, which are the generators of the coordinate ring. An extremely efficient tool for this task is the so called *plethystic logarithm* which, for a multi-graded ring, is defined as [84, 86]:

$$\mathrm{PL}[g(t_1,\ldots,t_n;R)] \;=\; \sum_{k=1}^{\infty}\frac{\mu(k)}{k}\log[g(t_1^k,\ldots,t_n^k;R)], \qquad (2.5.35)$$

where $\mu(k)$ is the Möbius function which equals 1 (resp. -1) if k is a square-free positive integer with an even (resp. odd) number of prime factors and 0 if k is not square-free. The plethystic logarithm of a Hilbert series is an infinite sum of terms; the first set of positive terms count the generators of the polynomial ring and the first set of negative terms counts the relations among them. In order to understand this point better, let's consider again the example discussed above, with the polynomial ring S. The plethystic logarithm of the Hilbert series is easily computed to be:

$$\mathrm{PL}[g(t_1,t_2;R)] \;=\; t_1+t_2-t_1^2. \qquad (2.5.36)$$

According to what we said above, the first two terms represent the generators of S, namely x_1 and x_2, whereas the last term accounts for the relation we have imposed, $x_1^2=0$.

We distinguish between three different scenarios: if the plethystic logarithm is a finite sum of positive terms, in which case we will say the ring is *freely generated*, it means the generators of the ring are completely independent; if it is a finite sum but contains also a set of negative terms, which we will refer to as *complete intersection*, the polynomial ring can be described by a set of generators which satisfy a number of independent relations; in the most general case, the plethystic logarithm does not terminate, which means that the relations among the generators are not independent and satisfy some higher order relations which, in turn, are not independent and so on.

2.5.1 The Molien-Weyl formula

We have discussed the importance of the Hilbert series in counting regular functions in the coordinate ring associated to some algebraic variety. Also, we have mentioned how this problem is closely related to the one of counting gauge invariant operators in supersymmetric theories. In this paragraph, we shall leverage on this "correspondence" to first learn how a Hilbert series can be associated to a given gauge theory and then to understand how it can be restricted to counting gauge invariant operators only. For a much more detailed discussion on the subject, see e.g.[93, 94, 95, 96, 97, 98, 99].

In order to count operators in the moduli space, we will use a grading imposed by the symmetries of the gauge theories. In particular, in case of an abelian symmetry, the grading is given by the charges of the chiral fields under the group, whereas for the non-abelian case, we will consider the charges under the Cartan subgroup. Each gauge theory is obviously equipped with a local symmetry group, but in order to count gauge invariant operators, we must ensure that the theory possesses global symmetries as well. Since we will be considering $\mathcal{N} = 1$ supersymmetric theories, in the IR there always exists a $U(1)_R$ group to classify operators. In addition, both the brane tiling theories and the SQCD ones will admit a larger symmetry, which we will refer to as *flavour symmetry*, to further refine our analysis.

Let us now look at an example and consider the $\mathcal{N} = 1$ SQCD with $SU(2)$ gauge group, three chiral fields in the fundamental representation, call them Q_1, Q_2 and Q_3, and zero superpotential. Calling h an element of the Cartan subgroup of the complexified gauge group[2], we have that for both chiral fields:

$$q_i \to hq_i = \begin{pmatrix} z & 0 \\ 0 & 1/z \end{pmatrix} \begin{pmatrix} q_i^1 \\ q_i^2 \end{pmatrix} = \begin{pmatrix} q_i^1 z \\ q_i^2/z \end{pmatrix}, \qquad (2.5.37)$$

where q_i is the scalar component of the chiral multiplet Q_i. Having made explicit the charges of the fields under the Cartan subgroup of the gauge

[2]It is well known that imposing D-term relations modulo gauge transformations amounts to modding the Master space by the complexified gauge group.

group helps us understand that the Hilbert series for this system is precisely:

$$\begin{aligned} g(t,z) &= 1+\left(3z+\frac{3}{z}\right)t+\left(6z^2+9+\frac{6}{z^2}\right)t^2+O(t^3) \\ &= \frac{1}{\left((1-tz)\left(1-\frac{t}{z}\right)\right)^3}. \end{aligned} \tag{2.5.38}$$

Note that the system can be refined a bit further, since it is clear that there is an $SU(3)$ flavour symmetry under which the two chiral fields transform in the fundamental representation. Introducing this new group into the game and using similar arguments to the ones above, we can write the Hilbert series of this theory as:

$$g(t,x_1,x_2,z) = \frac{1}{(1-tx_1z)\left(1-\frac{tx_1}{z}\right)\left(1-\frac{tx_2z}{x_1}\right)\left(1-\frac{tx_2}{x_1z}\right)\left(1-\frac{tz}{x_2}\right)\left(1-\frac{t}{x_2z}\right)}. \tag{2.5.39}$$

Once the general Hilbert series is constructed, a natural question is how to get rid of the gauge degrees of freedom or, otherwise stated, how can we restrict the Hilbert series to count operators which are invariant with respect to the gauge groups. In Invariant Theory, this corresponds to an old problem solved by Molien, who introduced a formula which is known as the *Molien-Weyl integral.* Generically, this formula tells us that, in order to project out the gauge degrees of freedom, one has to integrate the Hilbert series of the system by the fugacities corresponding to the gauge group together with its Haar measure. More explicitly:

$$g_{inv}(t,\{x_i\}) = \oint d\mu_G \;\; g(t,\{x_i\},\{z_j\}) \tag{2.5.40}$$

For the example we are considering, the Molien-Weyl formula reduces to:

$$\begin{aligned} g_{inv}(t,x_1,x_2) &= \oint_{|z|=1}\frac{dz}{2\pi i z}(1-z^2)\;\; g(t,x_1,x_2,z) \\ &= \frac{1}{(1-t^2x_2)\left(1-\frac{t^2x_2}{x_1}\right)\left(1-\frac{t^2}{x_1}\right)} \end{aligned} \tag{2.5.41}$$

It is interesting to consider the Taylor expansion of the Hilbert series:

$$\begin{aligned} g_{inv}(t, x_1, x_2) &= 1 + \left(x_2 + \frac{x_1}{x_2} + \frac{1}{x_1} \right) t^2 + \left(x_2^2 + x_1 + \frac{x_1^2}{x_2^2} + \frac{x_2}{x_1} + \frac{1}{x_2} + \frac{1}{x_1^2} \right) t^4 \\ &+ \left(\frac{1}{x_2^3} + x_1 x_2 + \frac{x_1^2}{x_2} + \frac{x_2^2}{x_1} + \frac{x_1^3}{x_2^3} + 1 + \frac{x_1}{x_2^2} + \frac{x_2}{x_1^2} + \frac{1}{x_1 x_2} + \frac{1}{x_1^3} \right) t^6 \\ &+ O(t^8) \end{aligned} \tag{2.5.42}$$

Notice that the coefficients in the expansion are precisely characters of $SU(3)$, which not only suggests that the gauge invariant operators transform in the flavour group, but it tells us in which representations. In particular, using the Dynkin label convention for writing the characters, we can re-write the expansion of the Hilbert series in a nice, compact form:

$$g(t, x_1, x_2) = \sum_{n=0}^{\infty} [0, n] t^{2n}. \tag{2.5.43}$$

This way of writing the Hilbert series is often called *character expansion* and it contains important information about the transformation properties of the gauge invariant operators. To conclude our example, let's determine the generators of the moduli space we are looking at. The plethystic logarithm of equation (2.5.39) can be written as:

$$\mathrm{PL}[g_{inv}(t, x_1, x_2)] = [0, 1] t^2. \tag{2.5.44}$$

The above expression tells us that the moduli space is generated by the vacuum expectation value of one single operator, given by the product of two chiral multiplets, and which transforms in the anti-fundamental representation of $SU(3)$. These conditions allow only one possibility:

$$M^i = \epsilon^{ijk} \epsilon^{\alpha\beta} (Q_\alpha)_j (Q_\beta)_k, \tag{2.5.45}$$

where the greek letters indicate gauge indices and latin letters stand for flavour indices. Notice also that the plethystic logarithm contains only one positive term, which means that the moduli space is freely generated. In particular, since the 'meson' M^i has three independent components, it is immediate to identify the moduli space with $\mathbb{C}^3$.

2.6 An Example: the $Y^{2,2}$ theory

We have given an introduction of the Hilbert series and we have provided a simple example for a particular SQCD theory. In this section, we shall see a less trivial example regarding a quiver gauge theory that belongs to a two-parameter family called $Y^{p,q}$. In particular, we will show how the forward algorithm works in practice and how it can be used to compute the Hilbert series for this particular theory.

The example we wish to discuss is the $Y^{2,2}$ theory, which corresponds to the orbifold $\mathbb{C}^3/\mathbb{Z}_4$, with action $(1, 1, -2)$ (see e.g. [100]).

The tiling and quiver of this theory are shown in Figure 2.4. As can be

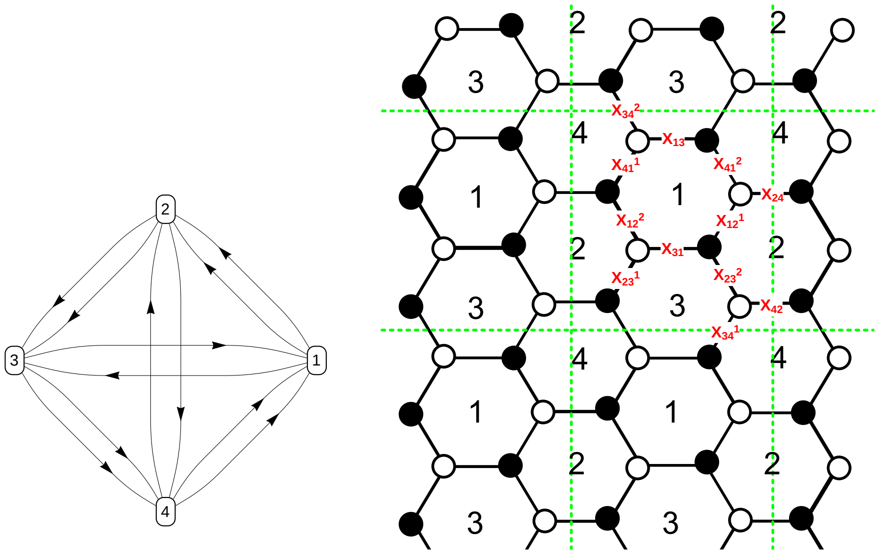


Figure 2.4: The quiver and brane tiling of the $Y^{2,2}$ theory.

seen from the tiling, the superpotential is:

$$\begin{aligned} W &= \epsilon_{ij}\,\mathrm{Tr}\left(X^i_{23}X_{31}X^j_{12}\right) + \epsilon_{kl}\,\mathrm{Tr}\left(X^k_{41}X_{13}X^l_{34}\right) \\ &+ \epsilon_{mn}\,\mathrm{Tr}\,(X^m_{34}X_{42}X^n_{23}) + \epsilon_{op}\,\mathrm{Tr}\,(X^o_{12}X_{24}X^p_{41}) \ . \end{aligned} \quad (2.6.46)$$

The Kasteleyn matrix for this theory can be written as:

$$K = \left(\begin{array}{c|cccc} & b_1 & b_2 & b_3 & b_4 \\ \hline w_1 & X_{13} & X_{41}^1 & 0 & \frac{1}{x} X_{34}^2 \\ w_2 & X_{41}^2 & y X_{24} & X_{12}^1 & 0 \\ w_3 & 0 & X_{12}^2 & X_{31} & X_{23}^1 \\ w_4 & x X_{34}^1 & 0 & X_{23}^2 & y X_{42} \end{array} \right) . \tag{2.6.47}$$

The permanent of this matrix can be written as:

$$\begin{aligned} \text{perm}(K) &= x X_{12}^1 X_{23}^1 X_{34}^1 X_{41}^1 + \frac{1}{x} X_{12}^2 X_{23}^2 X_{34}^2 X_{41}^2 + y^2 X_{42} X_{13} X_{24} X_{31} \\ &+ X_{12}^1 X_{12}^2 X_{34}^1 X_{34}^2 + X_{23}^1 X_{23}^2 X_{41}^1 X_{41}^2 + y X_{12}^1 X_{12}^2 X_{42} X_{13} \\ &+ y X_{23}^1 X_{23}^2 X_{13} X_{24} + y X_{34}^1 X_{34}^2 X_{24} X_{31} + y X_{41}^1 X_{41}^2 X_{42} X_{31} . \end{aligned} \tag{2.6.48}$$

According to this equation, the $G_K^{(2,2)}$ matrix can be written as:

$$G_K^{(2,2)} = \left(\begin{array}{ccccccccc} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 1 & 1 & 1 & 1 \end{array} \right) , \tag{2.6.49}$$

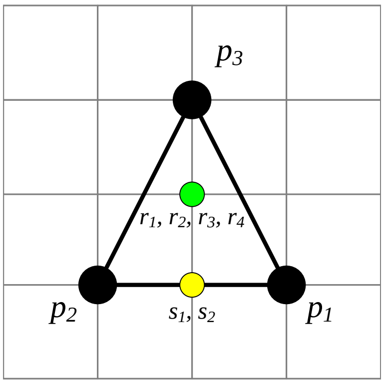


Figure 2.5: The toric diagram of the cone over $Y^{2,2}$

This sheds some light on the relation between the chiral fields and the perfect matchings. This information is collected in the P matrix:

$$P = \left(\begin{array}{c|ccccccccc} & p_1 & p_2 & p_3 & s_1 & s_2 & r_1 & r_2 & r_3 & r_4 \\ \hline X_{12}^1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ X_{12}^2 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ X_{23}^1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ X_{23}^2 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ X_{34}^1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ X_{34}^2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ X_{41}^1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ X_{41}^2 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ X_{42} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ X_{13} & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ X_{24} & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ X_{31} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \end{array} \right) . \tag{2.6.50}$$

The kernel of this matrix gives the Q_F matrix:

$$Q_F = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \end{pmatrix} , \tag{2.6.51}$$

from which one can derive the relations among the perfect matchings:

$$\begin{aligned} p_1 + p_2 - s_1 - s_2 &= 0 \\ p_3 + s_1 - r_1 - r_3 &= 0 \\ p_3 + s_2 - r_2 - r_4 &= 0. \end{aligned} \tag{2.6.52}$$

Furthermore, the Q_D matrix can be written as:

$$Q_D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} , \tag{2.6.53}$$

Combining all these pieces of information together one can write the Q_t

matrix:

$$Q_t = \begin{pmatrix} Q_F \\ Q_D \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{2.6.54}$$

Before starting with the computation of the Hilbert series for the mesonic moduli space, one needs to assign the charges of the global symmetry to the perfect matchings. The non-abelian part is easy to deal with, because in the Q_t matrix there's only one pair of repeated columns, and therefore the conclusion is that the perfect matchings p_1 and p_2 transform as a doublet under this symmetry group. Also, since this theory is just an abelian orbifold of the well-known theory for $\mathbb{C}^3$, the R-charges of the purely external perfect matchings are all equal to $2/3$. The assignment for the abelian flavour group is arbitrary, as long as the superpotential is uncharged under it and the vectors of charges are linearly independent from each others. Finally, fugacity t_1 is assigned to the perfect matchings p_1 and p_2, t_2 to p_3 and $\tilde{t}_3$ to s_1 and s_2. Not only will this help to identify the perfect matching content of each generator, but it will also allow for an easy generalization of the particular results of each sections.

All this discussion can be summirized in Table 2.1.

Having fixed all the charges of the perfect matchings, the Hilbert series of the mesonic moduli space of the theory is given by the Molien-Weyl formula:

$$\begin{aligned}
& g^{\text{mes}}(t_1, t_2, t_3, x, f; Y^{2,2}) \\
& = \prod_{i=1}^{3} \oint_{|z_i|=1} \frac{\mathrm{d}z_i}{2\pi i z_i} \prod_{j=1}^{3} \oint_{|b_j|=1} \frac{\mathrm{d}b_j}{2\pi i b_j} \frac{1}{(1 - x f z_1 t_1)\left(1 - \frac{z_1 f t_1}{x}\right)\left(1 - \frac{t_2 z_2 z_3}{f^2}\right)} \\
\times \; & \frac{1}{\left(1 - \frac{\tilde{t}_3 z_2}{z_1}\right)\left(1 - \frac{\tilde{t}_3 z_3}{z_1}\right)\left(1 - \frac{b_1}{z_2}\right)\left(1 - \frac{b_2}{b_1 z_3}\right)\left(1 - \frac{b_3}{b_2 z_2}\right)\left(1 - \frac{1}{b_3 z_3}\right)} \\
= \; & \frac{1 + [2] t_1^2 t_2 t_3 + [2] f^4 t_1^4 t_3^2 + f^4 t_1^6 t_2 t_3^3}{\left(1 - t_1^4 t_3^2 x^4 f^4\right)\left(1 - \frac{t_1^4 t_3^2 f^4}{x^4}\right)\left(1 - \frac{t_2^2}{f^4}\right)},
\end{aligned} \tag{2.6.55}$$

	$SU(2)$	$U(1)_f$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	fugacity
p_1	1	1	2/3	0	0	0	$t_1 f x$
p_2	-1	1	2/3	0	0	0	$t_1 f/x$
p_3	0	-2	2/3	0	0	0	t_2/f^2
s_1	0	0	0	0	0	0	$\tilde{t}_3$
s_2	0	0	0	0	0	0	$\tilde{t}_3$
r_1	0	0	0	1	0	0	b_1
r_2	0	0	0	-1	1	0	b_2/b_1
r_3	0	0	0	0	-1	1	b_3/b_2
r_4	0	0	0	0	0	-1	$1/b_3$

Table 2.1: Charges of the perfect matchings under the global symmetry of the $Y^{2,2}$ theory. Here t_i are the fugacities relative to different perfect matchings, x is the weight of the $SU(2)$ symmetry, f is the fugacity of the flavour symmetry and b_i is the fugacity of the $U(1)_{B_i}$ symmetry.

where $t_3 = (\tilde{t}_3)^{1/2}$.

The plethystic logarithm of this function can be written as:

$$\mathrm{PL}[g^{\mathrm{mes}}(t_1, t_2, t_3, x; Y^{2,2})] = \frac{t_2^2}{f^4} + [2]t_1^2 t_2 t_3 + [4] f^4 t_1^4 t_3^2 - O(t_1^6 t_2 t_3^3) \tag{2.6.56}$$

The generators of the mesonic moduli space corresponding to the above positive terms in the plethystic logarithm are in terms of the perfect matchings[3] defined in (2.6.50):

$$A_1 = p_1^2 p_3 \ , \ A_2 = p_1 p_2 p_3 \ , \ A_3 = p_2^2 p_3$$
$$B = p_3^2$$
$$C_0 = p_1^4 \ , \ C_1 = p_1^3 p_2 \ , \ C_2 = p_1^2 p_2^2 \ , \ C_3 = p_1 p_2^3 \ , \ C_4 = p_2^4 \tag{2.6.57}$$

[3]In the following the internal perfect matchings are omitted since they carry 0 charge under all symmetries.

The first order relations formed by the above generators are,

$$
\begin{aligned}
\{ \quad & A_2^2 = A_1 A_3, \\
& A_3 B_3 = A_2 B_4, A_3 B_2 = A_2 B_3, A_3 B_1 = A_2 B_2, A_3 B_0 = A_2 B_1, \\
& A_2 B_3 = A_1 B_4, A_2 B_2 = A_1 B_3, A_2 B_1 = A_1 B_2, A_2 B_0 = A_1 B_1, \\
& C_1^2 = C_0 C_2, C_1 C_2 = C_0 C_3, C_1 C_3 = C_0 C_4, C_2^2 = C_1 C_3, C_2 C_3 = C_1 C_4, C_3^2 = C_2 C_4, \\
& A_3^2 = B C_4, A_2 A_3 = B C_3, A_2^2 = B C_2, A_1 A_2 = B C_1, A_1^2 = B C_0 \ \} \ .
\end{aligned}
\tag{2.6.58}
$$

The above first order relations also fall into irreducible representations of $SU(2)$ and contribute the following terms to the plethystic logarithm of the mesonic Hilbert series,

$$
-t_1^4 t_2^2 t_3^2 - ([4]+[2]) \ f^4 \ t_1^6 t_2 t_3^3 - ([4]+[0]) \ f^8 \ t_1^8 t_3^4 - [4] \ t_1^4 t_2^2 t_3^2 \ . \tag{2.6.59}
$$

Looking carefully at (2.6.55), one observes that the Hilbert series depends only on two specific combinations of perfect matchings and, therefore, it can be expressed in terms of only the non-abelian fugacity x and the two variables:

$$
T_1 = t_1^2 t_2 t_3 \qquad T_2 = f^4 t_1^4 t_3. \tag{2.6.60}
$$

With these new conventions,

$$
\begin{aligned}
g^{\mathrm{mes}}(T_1, T_2, x; Y^{2,2}) &= \frac{1 + [2]T_1 + [2]T_2 + T_1 T_2}{(1 - T_2 x^4)\left(1 - \frac{T_2}{x^4}\right)\left(1 - \frac{T_1^2}{T_2}\right)} \\
&= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [4n_2] T_1^{2n_1} T_2^{n_2 - n_1} \\
&\quad + \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [4n_2 + 2] T_1^{2n_1+1} T_2^{n_2 - n_1}
\end{aligned}
\tag{2.6.61}
$$

3 M2-branes and Hilbert series

3.1 M2-branes and AdS/CFT

3.1.1 Introduction

In the following sections, we shall see how brane tilings and Hilbert series can be employed to count gauge invariant operators for theories arising on the world-volume of M2-branes. However, before we embark on this discussion, it will be beneficial to review what these theories are and how they came about in the literature. In particular, we will focus on the contribution of Aharony, Bergman, Jafferis and Maldacena [57] that played a crucial role in opening the path towards a better understanding of such theories. They did so by introducing a model, which we shall refer to as the ABJM model, which is a Chern-Simons theory with gauge groups $U(N)\times U(N)$ and integer levels $(k,-k)$. To date, there exists a considerable amount of evidence that this model describes the low-energy behaviour of a stack of M2-branes probing the singularity of the orbifold $\mathbb{R}^8/\mathbb{Z}_k$. Hence, it is conjectured that, in the framework of AdS/CFT correspondence, the ABJM model is dual to M-theory on $AdS_4 \times S^7/\mathbb{Z}_k$.

The field theory content of the ABJM model strikingly resembles that of the Klebanov-Witten theory in $(3+1)$ dimensions [51], and so this shall be our starting point. Let us consider the 10 dimensional space-time $M_4 \times \mathcal{C}$, where M_4 is the 4 dimensional Minkowski space-time and $\mathcal{C}$ is a manifold that is usually referred to as the *conifold*. Introducing 4 complex variables u_i, with $i = 1, \ldots, 4$, the conifold can be defined as the set of points satisfying the following algebraic constraint:

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0. \tag{3.1.1}$$

Obviously this constraint reduces the number of complex dimensions to 3, which therefore implies the conifold has 6 real dimensions, which is just the right number for the construction we are considering. As a next step, we can place a D3-brane near the tip of the cone defined by (3.1.1), and ask which gauge theory would arise on its world-volume in the low energy limit. As it turns out, this is an $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with gauge group $U(1)_1 \times U(1)_2$ and four chiral fields, that we denote as z_1, z_2, w_1 and w_2 [51]. The fields transform under the gauge groups according to the charges reported in Table 3.1.

	$U(1)_1$	$U(1)_2$
z_1, z_2	1	-1
w_1, w_2	-1	1

Table 3.1: Abelian charges of the chiral fields under the two gauge groups of the conifold theory.

Interestingly, all the matter fields are neutral under the diagonal gauge field[1] $A_+ = A + \hat{A}$, and are charged under the anti-diagonal gauge field $A_- = A - \hat{A}$. Furthermore, as far as the one D3-brane case is considered, the gauge theory has a vanishing superpotential.

As usual, the moduli space of vacua of the gauge theory is given by the zero-energy solutions to the F-term and D-term equations. Since the theory has a vanishing superpotential, F-term equations are automatically satisfied, and we are left only with solving the D-term equations which can be written as:

$$\frac{D^2}{g^2} + D\left(|z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2\right) = 0, \tag{3.1.2}$$

where D is the usual auxiliary field which can be integrated out. By doing so, we obtain the new equation that, moduli overall factors, can be written as:

$$g^2\left(|z_1|^2 + |z_2|^2 - |w_1|^2 - |w_2|^2\right)^2 = 0. \tag{3.1.3}$$

By definition, the moduli space of vacua is defined by imposing this equation, i.e. the vanishing of the D-terms, and by dividing by the gauge group.

[1] We choose the convention that non hatted quantities refer to gauge group 1, whereas hatted quantities refer to gauge group 2.

In turn, this amounts to imposing the following constraints:

$$\begin{aligned}&|z_1|^2+|z_2|^2-|w_1|^2-|w_2|^2=0,\\ &z_i\sim z_i e^{i\alpha}, \qquad\qquad w_i\sim w_i e^{-i\alpha}.\end{aligned} \tag{3.1.4}$$

These conditions identify a manifold which is precisely that introduced before by (3.1.1). This can also be thought of as a real cone over $(SU(2)\times SU(2))/U(1)$, which in the literature is often referred to as $T^{1,1}$. The two $SU(2)$ factors can be considered of as flavour symmetries, one acting on the z_i's, which transform as a doublet, and the other acting on the w_i's, also transforming as a doublet. The $U(1)$ group is identified with the interacting anti-diagonal subgroup A_- under which the chiral fields are charged.

Another way to define the conifold is by introducing four complex variables, u_{ij}, defined as:

$$\begin{aligned}u_{11}&=u_1+iu_2, \qquad & u_{12}&=iu_3-u_4,\\ u_{21}&=iu_3+u_4, \qquad & u_{22}&=u_1-iu_2,\end{aligned} \tag{3.1.5}$$

related to the chiral fields by the relation $u_{ij}=z_iw_j$. In terms of the new variables, the defining equation for the conifold can be written simply as:

$$\det u_{ij}=0. \tag{3.1.6}$$

This way of defining the conifold is equivalent to the one we have discussed above. Given the definition of the coordinates u_{ij} in terms of w_i and z_i, the equation (3.1.6) remains the same if we act on the chiral fields with the transformations:

$$z_i\to\lambda z_i \qquad w_i\to\lambda^{-1}w_i \qquad \lambda\in\mathbb{C}^*. \tag{3.1.7}$$

If we write $\lambda=se^{i\alpha}$, with $s\in\mathbb{R}^+$ and $\alpha\in\mathbb{R}$, the parameter s can be used to satisfy (3.1.4), while α parameterizes the gauge invariance.

A generalization to the case where we have a stack of N parallel D3-branes probing the tip of the cone is straightforward. In that case, we have a Yang-Mills theory with $\mathcal{N}=1$ supersymmetry, gauge group $U(N)_1\times U(N)_2$, and chiral matter fields which transform under these groups according to Table 3.2.

	$U(N)_1$	$U(N)_2$
z_1, z_2	$\mathbf{N}$	$\bar{\mathbf{N}}$
w_1, w_2	$\bar{\mathbf{N}}$	$\mathbf{N}$

Table 3.2: Transformations of the chiral fields under the two gauge groups. The symbol $\mathbf{N}$ indicates that a field transforms in the fundamental representation of $U(N)$, whereas $\bar{\mathbf{N}}$ indicates that a field transforms in the anti-fundamental representation of $U(N)$.

The matter content of this theory can be represented with a so called "quiver" diagram, which we show in Figure 3.1.

Figure 3.1: Quiver diagram for the conifold theory.

The red nodes correspond to the gauge groups, and each arrow corresponds to a chiral superfield of the theory. An outgoing (incoming) arrow from a given node signifies that the corresponding chiral field transforms in the fundamental (anti-fundamental) representation of the corresponding gauge group. Arrows starting and ending on the same node transform in the adjoint representation of the corresponding gauge group.

For the case with multiple branes, a superpotential needs to be added. The only superpotential quartic in the superfields and invariant under the $SU(2) \times SU(2)$ flavor symmetry is:

$$W \propto \mathrm{Tr}\left(\epsilon^{AC}\epsilon^{BD} z_A w_B z_C w_D\right) \propto \mathrm{Tr}\left(z_1 w_1 z_2 w_2 - z_1 w_2 z_2 w_1\right) \ . \quad (3.1.8)$$

Note that this superpotential vanishes in the abelian case, where all the fields commute. All the fields have R-charge equal to 1/2. Thus, at the conformal fixed point, the scaling dimension of each field is:

$$\Delta = \frac{3}{2}|R| = \frac{3}{4}. \quad (3.1.9)$$

It can be shown that the superpotential is an exactly marginal operator for this gauge theory.

The gauge group of the theory, $U(N)_1 \times U(N)_2$, contains two $U(1)$ factors. The diagonal subgroup of the gauge group, under which all the fields are neutral, decouples trivially. The anti-diagonal subroup, commonly denoted as $U(1)_B$, becomes a global symmetry far in the IR because its gauge coupling flows to zero. Therefore, the AdS/CFT correspondence for this case states that Type IIB string theory on $AdS_5 \times T^{1,1}$ with N units of RR flux on $T^{1,1}$ is dual to $\mathcal{N} = 1$ supersymmetric Yang-Mills theory with gauge group $SU(N)_1 \times SU(N)_2$.

3.1.2 One M2-brane and Abelian Chern-Simons Theory

Having briefly introduced the AdS/CFT correspondence in the context of D3-branes, we are now ready to discuss the correspondence for M2-branes. Let us recall the formulation of 3-dimensional $\mathcal{N} = 2$ supersymmetric Chern-Simons theories coupled to charged fields [52, 101]. In the superspace formalism, the Chern-Simons action for theories with a single $U(1)$ gauge group and N_f matter flavours can be written, in Wess-Zumino gauge, as:

$$S = \int \mathrm{d}^3x \int \mathrm{d}^4\theta \left(\frac{k}{4\pi} \mathcal{V}\Sigma + \sum_{i=1}^{N_f} \Phi_i e^{q_i \mathcal{V}} \Phi_i \right), \qquad k \in \mathbb{Z} \qquad (3.1.10)$$

where Φ_i are chiral matter superfields transforming under the gauge group with charge q_i, and where:

$$\begin{aligned} \mathcal{V} &= 2i\theta\bar{\theta}\sigma + 2\theta\gamma^\mu\bar{\theta}A_\mu + \sqrt{2}i\theta^2\bar{\theta}\bar{\chi} - \sqrt{2}i\bar{\theta}^2\theta\chi + \theta^2\bar{\theta}^2 D, \\ \Sigma &= \bar{D}^\alpha D_\alpha \mathcal{V}. \end{aligned} \qquad (3.1.11)$$

As we can see from above, the vector superfield $\mathcal{V}$ is composed of a gauge field A_μ, a two-component Dirac spinor χ, a scalar field σ, which comes from the A_3 component of the gauge field when we do dimensional reduction from the 3+1 dimensional theory, and another scalar field D.

The parameter k is called the Chern-Simons level: the requirement that a non-abelian theory is invariant under large gauge transformations restricts it to integer values. In components, the action can be written as:

$$\begin{aligned} S_{CS} &= \frac{k}{4\pi}\int \mathrm{d}^3x\ \Big(\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + i\bar{\chi}\chi - 2D\sigma\Big), \\ S_{matter} &= \int \mathrm{d}^3x\ \sum_{i=1}^{N_f}\Big(-\mathscr{D}_\mu\phi_i^\dagger\mathscr{D}^\mu\phi_i - i\zeta_i^\dagger \not{\mathscr{D}}\zeta_i + q_i\phi_i^\dagger D\phi_i - q_i^2\phi_i^\dagger\sigma^2\phi_i + \\ & \quad -q_i\zeta_i^\dagger\sigma\zeta_i + iq_i\phi_i^\dagger\bar{\chi}\zeta_i - iq_i\zeta_i^\dagger\chi\phi_i\Big), \end{aligned} \tag{3.1.12}$$

where $\mathscr{D}_\mu$ represents the covariant derivative, and where ϕ_i and ζ_i represent, respectively, the scalar and the fermionic part of the chiral matter field Φ_i. Note that all the fields in the vector multiplet are non-dynamical, so they are all auxiliary fields. Integrating out the scalar field D, we have that:

$$\sigma = -\frac{2\pi}{k}\sum_{i=1}^{N_f} q_i\phi_i^\dagger\phi_i, \tag{3.1.13}$$

and the D-term potential can be written as:

$$V_D \propto \sum_{i=1}^{N_f} q_i^2\phi_i^\dagger\phi_i\sigma^2. \tag{3.1.14}$$

Note that, because of (3.1.13), the D-term potential is sextic in the scalar fields ϕ_i.

Now let us consider the specific gauge theory proposed in [57] as a description of a single M2-brane. This theory has gauge group $U(1)\times U(1)$, Chern-Simons levels $(k,-k)$ and four chiral superfields transforming under these groups as given in Table 3.1. Interestingly, the quiver diagram for this Chern-Simons gauge theory looks exactly like that in Figure 3.1. The matter action for this theory can be written in the superspace formalism as:

$$S_{matter} = \int \mathrm{d}^3x \int \mathrm{d}^4\theta\ \Big(\bar{\mathcal{Z}}_A e^{-\mathcal{V}}\mathcal{Z}^A e^{\hat{\mathcal{V}}} + \bar{\mathcal{W}}^B e^{-\hat{\mathcal{V}}}\mathcal{W}_B e^{\mathcal{V}}\Big), \qquad A,B = 1,2, \tag{3.1.15}$$

where $\mathcal{Z}^A$ and $\mathcal{W}_B$ are chiral multiplets, whose lowest components are scalar fields that we denote as Z^A and W_B. Expanding the action in components, and deriving the equations of motion for the auxiliary fields, we have that:

$$\sigma = \hat{\sigma} = \frac{2\pi}{k}\left(|Z_1|^2 + |Z_2|^2 - |W_1|^2 - |W_2|^2\right) \ . \tag{3.1.16}$$

The D-term potential for this theory is proportional to $(\sigma - \hat{\sigma})^2$; therefore, it vanishes. Since the F-term potential vanishes as well (the abelian theory has no superpotential), we could be tempted to conclude that the moduli space is simply $\mathbb{R}^8$, or $\mathbb{C}^4$. However, this is true only up to a $\mathbb{Z}_k$ identification.

Let us combine the fields as follows [57, 102]:

$$Y^A = \left\{Z^A, W^{\dagger A}\right\} \qquad Y^\dagger_A = \left\{Z^\dagger_A, W_A\right\}. \tag{3.1.17}$$

The newly defined fields Y^A have the same charges as Z^A under the gauge group $U(1)_1 \times U(1)_2$.
With this definition, the bosonic part of the Chern-Simons action can be written as:

$$S_{\text{bos}} = \frac{k}{4\pi}\int \mathrm{d}^3x \ \ \epsilon^{\mu\nu\lambda}\left(A_\mu \partial_\nu A_\lambda - \hat{A}_\mu \partial_\nu \hat{A}_\lambda\right) + \int \mathrm{d}^3x \ \ \mathscr{D}_\mu Y^\dagger_A \mathscr{D}^\mu Y^A \tag{3.1.18}$$

Defining $A^\pm_\mu = A_\mu \pm \hat{A}_\mu$, the covariant derivative can be written as:

$$\mathscr{D}_\mu Y^A = \partial_\mu Y^A + i(A_\mu - \hat{A}_\mu)Y^A = \partial_\mu Y^A + iA^-_\mu Y^A. \tag{3.1.19}$$

The CS action itself can be written in a more concise form as:

$$S_{\text{CS}} = \frac{k}{4\pi}\int \mathrm{d}^3x \ \ \epsilon^{\mu\nu\lambda} A^-_\mu F^+_{\nu\lambda}. \tag{3.1.20}$$

If we define the theory on $\mathbb{R}^3$ and map it to $\mathbb{R} \times S^2$, then there are sectors with quantized monopole fluxes:[2]

$$\int_{S^2} F^+ = 4\pi n \ . \tag{3.1.21}$$

[2]For a discussion of the extra factor of 2 appearing in the quantization condition, see [103, 104].

If we now consider a gauge transformation for which:

$$A_\mu^- \to A_\mu^- + \partial_\mu \Lambda^-, \qquad Y^A \to e^{i\Lambda^-} Y^A, \tag{3.1.22}$$

this transformation brings about a boundary term that can be written as:

$$\delta S = \frac{k}{4\pi} \Lambda^- \int_{S^2} F^+. \tag{3.1.23}$$

Since the fluxes are quantized, in order for the action to shift by 2π times an integer, we must require that:

$$\Lambda^- = \frac{2\pi l}{k} \ , \qquad l \in \mathbb{Z} \ . \tag{3.1.24}$$

This provides an identification on the matter fields which implies that the moduli space for this theory with Chern-Simons levels $(k, -k)$ is $\mathbb{C}^4/\mathbb{Z}_k$.

Another way to derive this result is to note that, since A_μ^+ does not appear in the action, we can treat $F_{\mu\nu}^+$ as a basic variable, rather than A_μ^+ itself. Thus, in order to ensure that the equation $\mathrm{d}F^+ = 0$ remains valid, we need to add a Lagrange multiplier [105, 106]:

$$S_\tau = \frac{1}{4\pi} \int \mathrm{d}^3x \ \ \tau \epsilon^{\mu\nu\rho} \partial_\mu F_{\nu\rho}^+. \tag{3.1.25}$$

Having inserted this term into the action, the equations of motion of $F_{\nu\rho}^+$ can be written as:

$$A_\mu^- = \frac{1}{k} \partial_\mu \tau \ . \tag{3.1.26}$$

The quantization condition on the fluxes requires that τ be periodic with period 2π. It follows that, under the gauge transformations,

$$\tau \to \tau + k\Lambda^- \ . \tag{3.1.27}$$

We can use this gauge transformation to fix τ to be 0 but, because of its periodicity, we still have the freedom to make a transformation with $\Lambda^- = \frac{2\pi}{k}$. Examining how these "large" $\mathbb{Z}_k$ gauge transformations act on the fields Y^A, we again conclude that the moduli space of this Chern-Simons theory is $\mathbb{C}^4/\mathbb{Z}_k$.

In addition to the $\mathbb{Z}_k$ gauge symmetry, the model has a global $U(1)_b$ symmetry which corresponds to the conserved current $j_\mu \sim \epsilon_{\mu\nu\lambda}F^{+\nu\lambda}$. This symmetry acts by shifting the dual scalar τ. The monopole (anti-monopole) operators $e^{\pm i\tau}$, that are charged under the $U(1)_b$, create field configurations with magnetic flux of F^+ through the S^2 surrounding the point of insertion.

The non-abelian, $U(N)\times U(N)$ version of the model we are discussing was proposed in 2008 by Aharony, Bergman, Jafferis and Maldacena in [57], and in the literature is often referred to as the ABJM theory. It has the merit of having the conformal symmetry manifest, although the complete amount of supersymmetry is not manifest for $k = 1, 2$. A different, older description of the gauge theory on a stack of N M2-branes is in terms of the IR limit of the gauge theory on N D2-branes, i.e. the $\mathcal{N} = 8$ supersymmetric Yang-Mills theory in $2 + 1$ dimensions. For example, for a single D2-brane the action is:

$$S = \int \mathrm{d}^3x \left(-\frac{1}{4}F_{\mu\nu}^2 - \sum_{i=1}^{7} \frac{1}{2}(\partial_\mu \phi^i)^2 \right). \tag{3.1.28}$$

Dualizing the field strength to a scalar,

$$F_{\mu\nu} \sim \epsilon_{\mu\nu\lambda}\partial^\lambda \phi^8, \tag{3.1.29}$$

the $SO(8)$ symmetry acting on the 8 scalars is manifest. However, it is not known how to generalize the duality transformation to the non-abelian gauge theory on multiple D2-branes. In this approach to M2-branes, the $\mathcal{N} = 8$ supersymmetry is manifest, but the conformal invariance is not; it contains the coupling constant g_{YM} which has dimension of $\text{Energy}^{1/2}$. Presumably, the superconformal M2-brane theory is the IR sector of the theory on N D2-branes which emerges for energies much smaller than g_{YM}^2.

3.1.3 Non-Abelian Chern-Simons theory

A breakthrough in the search for a Chern-Simons matter theory with $\mathcal{N} = 8$ supersymmetry came with the work of Bagger and Lambert [53] and, independently, Gustavsson [54]. They constructed a theory, often referred to as the BLG model, using a so-called "3-algebra." Given a set of generators

T_a, such an algebra can be defined by introducing the triple product:

$$\left[T^a, T^b, T^c\right] = f^{abc}{}_d T^d, \tag{3.1.30}$$

where f_{abcd} is a fully anti-symmetric tensor. Given this algebra, a maximally supersymmetric Chern-Simons lagrangian is:

$$\begin{aligned}
\mathcal{L}_{CS} &= \frac{1}{2}\epsilon^{\mu\nu\lambda}\left(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right),\\
\mathcal{L}_{matter} &= -\frac{1}{2}\mathscr{D}^\mu x^{aI}\mathscr{D}_\mu x^I_a + \frac{i}{2}\bar{\psi}^a\Gamma^\mu\mathscr{D}_\mu\psi_a + \frac{i}{4}\bar{\psi}_b\Gamma_{IJ}x^I_c x^J_d\psi_a f^{abcd}\\
&\quad -\frac{1}{12}\operatorname{Tr}\left([x^I,x^J,x^K][x^I,x^J,x^K]\right), \qquad I,J=1,\ldots,8,
\end{aligned} \tag{3.1.31}$$

where A^μ_{ab} is the gauge boson, and ψ_a and $x^I = x^I_a T^a$ are matter fields[3]. If we let $a = 1,\ldots,4$, then we can obtain an $SO(4)$ gauge symmetry by choosing $f^{abcd} = f\epsilon^{abcd}$, f being a constant.[4] Not only does this seem like a natural choice, but it turns out to be the only one that gives a gauge theory with manifest unitarity and $\mathcal{N} = 8$ supersymmetry.

After this model was proposed, it was shown [55, 56] that this gauge theory is equivalent to an $SU(2)\times SU(2)$ Chern-Simons gauge theory with opposite Chern-Simons levels:

$$\begin{aligned}
\mathcal{L}_{CS} &= \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}\operatorname{Tr}\left(A_\mu\partial_\nu A_\lambda + \frac{2i}{3}A_\mu A_\nu A_\lambda - \hat{A}_\mu\partial_\nu\hat{A}_\lambda - \frac{2i}{3}\hat{A}_\mu\hat{A}_\nu\hat{A}_\lambda\right),\\
\mathcal{L}_{matter} &= -(\mathscr{D}^\mu X^I)^\dagger\mathscr{D}_\mu X^I + i\bar{\Psi}^\dagger\Gamma^\mu\mathscr{D}_\mu\Psi - \frac{4i\pi}{k}\bar{\Psi}^\dagger\Gamma^{IJ}\Big(X^I X^{J\dagger}\Psi + X^J\Psi^\dagger X^I +\\
&\quad +\Psi X^{I\dagger}X^J\Big) - \frac{32\pi^2}{3k^2}\operatorname{Tr}\left(X^{[I}X^{\dagger J}X^{K]}X^{\dagger[K}X^J X^{\dagger I]}\right),
\end{aligned} \tag{3.1.32}$$

where the covariant derivative can be written as:

$$\mathscr{D}_\mu X^I = \partial_\mu X^I + iA_\mu X^I - iX^I\hat{A}_\mu. \tag{3.1.33}$$

The bi-fundamental matter fields X^I are related to the ones in the $SO(4)$

[3]By convention, space-time indices are denoted by Greek letters $\mu,\nu,\lambda,\ldots$, gauge indices by $a,b,c,\ldots$, and SO(8) vector indices by capital letters $I,J,\ldots$.

[4]The invariance under large gauge transformations requires $f = \frac{2\pi}{k}$.

notation by:

$$X^I = \frac{1}{2}(x_4^I \mathbb{I}_{2\times 2} + i x_i^I \sigma^i), \tag{3.1.34}$$

where $\mathbb{I}_{2\times 2}$ is the 2×2 identity matrix and σ^i are the Pauli matrices. Also, the bi-fundamental matter fields satisfy the reality condition [56]:

$$X_a^{\hat{a}} = -\epsilon_{ab}\, X_{\hat{b}}^b\, \epsilon^{\hat{a}\hat{b}}, \qquad \epsilon^{ab} = i\sigma_2^{ab}. \tag{3.1.35}$$

An $\mathcal{N}=2$ superspace formulation of this theory was given in [57, 102]. The gauge fields A and $\hat{A}$ can be thought of as components of two vector supermultiplets, $\mathcal{V}$ and $\hat{\mathcal{V}}$, whose other components are auxiliary fields. The matter fields can be combined into chiral bi-fundamental superfields, which we denote as $\mathcal{Z}^A$. The lowest component of $\mathcal{Z}^A$ is a complex scalar field Z^A, related to the X^I fields by:

$$Z^A = X^A + iX^{A+4}, \qquad A = 1,\ldots,4. \tag{3.1.36}$$

In terms of the chiral multiplets $\mathcal{Z}_a^A$ transforming under the $SO(4)$ gauge group and carrying R-charge 1/2, the superpotential can be written as:

$$W = -\frac{\pi}{4!k}\epsilon_{ABCD}\epsilon^{abcd}(\mathcal{Z}_a^A \mathcal{Z}_b^B \mathcal{Z}_c^C \mathcal{Z}_d^D), \tag{3.1.37}$$

Thus, the theory has a manifest $U(1)_R \times SU(4)$ symmetry. By virtue of (3.1.34), this superpotential can be expressed in terms of the bi-fundamental chiral superfields $\mathcal{Z}^A$ as:

$$W = \frac{\pi}{3k}\epsilon_{ABCD}(\mathcal{Z}^A \mathcal{Z}^{\ddagger B} \mathcal{Z}^C \mathcal{Z}^{\ddagger D})\ , \tag{3.1.38}$$

where we have introduced the operations:

$$\begin{aligned} Z^{\ddagger A} &= -\epsilon (Z^A)^{\mathrm{T}}\epsilon = X^{\dagger A} + iX^{\dagger A+4}, \\ \bar{Z}_A &= -\epsilon (Z^A)^{*}\epsilon = X^A - iX^{A+4}. \end{aligned} \tag{3.1.39}$$

Although the first of these operations might look like a hermitian conjugation, we note that there is no conjugation of the imaginary part, but only of the scalar multiplets X^I. This operation is crucial because it does not break the holomorphy of the superpotential.

It is not obvious that this $\mathcal{N} = 2$ formalism describes the BLG model. The latter has $\mathcal{N} = 8$ supersymmetry, and therefore an $SO(8)_R$ invariance, whereas (3.1.38) only has a manifest $U(1)_R \times SU(4)$ symmetry. However, expressing the action of this model in components and integrating out all the auxiliary fields, the sum of the D-term potential and of the F-term potential has the desired $SO(8)_R$ global symmetry [102].

Since the reality condition (3.1.35) and the "double dagger" operation (3.1.39) are special to $SU(2) \times SU(2)$, it seems difficult to generalize the construction to gauge groups with a higher rank. A way to overcome this difficulty, proposed in [57], is to abandon the manifest global $SU(4)$ invariance by forming the following combinations of the bi-fundamental fields:

$$\begin{aligned} Z^1 &= X^1 + iX^5, \qquad W^1 = X^{3\dagger} + iX^{7\dagger} \ , \\ Z^2 &= X^2 + iX^6, \qquad W^2 = X^{4\dagger} + iX^{8\dagger} \ . \end{aligned} \tag{3.1.40}$$

If we promote the fields Z^A and W^A to chiral superfields, that we shall denote as $\mathcal{Z}^A$ and $\mathcal{W}^A$, the superpotential of the model can be written as [57, 102]:

$$W = \frac{2\pi}{k} \epsilon_{AC} \epsilon^{BD} \,\mathrm{Tr}\left(\mathcal{Z}^A \mathcal{W}_B \mathcal{Z}^C \mathcal{W}_D \right) \ . \tag{3.1.41}$$

This superpotential looks exactly the same as that for the D3-brane theory on the conifold and can be easily generalized to higher rank gauge groups of the type $SU(N) \times SU(N)$. Furthermore, there is an important *caveat*. Apart from the manifest $U(1)_R \times SU(2) \times SU(2)$ global symmetry, the superpotential has also a $U(1)$ symmetry, under which the chiral multiplets transform as:

$$\mathcal{Z}^A \to e^{i\alpha} \mathcal{Z}^A, \qquad \mathcal{W}_B \to e^{-i\alpha} \mathcal{W}_B. \tag{3.1.42}$$

In the $3+1$ dimensional theory on N D3-branes at the conifold singularity, this starts out as a gauge symmetry but in the far IR becomes global and is identified with a $U(1)$ baryonic symmetry. In the $2+1$ dimensional case at hand, the dynamics is different and we have to treat this $U(1)$ as a gauge symmetry, although eventually it becomes broken to a $\mathbb{Z}_k$ subgroup (3.1.24). Using this argument, as well as type IIB brane constructions, ABJM proposed that the gauge group on N M2-branes is $U(N) \times U(N)$, and not $SU(N) \times SU(N)$. We have already checked that the moduli space of the $N = 1$ abelian theory with Chern-Simons levels $(k, -k)$ is the $\mathbb{Z}_k$ orbifold of $\mathbb{C}^4$; thus, this theory correctly describes a single M2-brane. Additional geometrical arguments for why the ABJM theory describes N M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ are presented in [108, 60]. A type IIA reduction performed there leads to D2-branes on the conifold fibered over $\mathbb{R}$, with a 2-form RR-flux turned on. This explains the close relation between the ABJM theory and the theory [51] for D-branes on the conifold.

An important property of the classical ABJM action is that the $U(1)_R \times SU(2) \times SU(2)$ global symmetry is enhanced to $SU(4)_R$, corresponding to $\mathcal{N} = 6$ supersymmetry. The scalar potential is made of two parts, one coming from the F-terms and the other from the D-terms. The former can be written as:

$$\begin{aligned} V_F^{\rm bos} &= \left|\frac{\partial W}{\partial Z^A}\right|^2 + \left|\frac{\partial W}{\partial W_A}\right|^2 = \mathrm{Tr}\left[F_A^\dagger F^A + G^{\dagger A} G_A\right], \\ F^A &= \frac{4\pi}{k}\epsilon^{AC}\epsilon_{BD} W^{\dagger B} Z_C^\dagger W^{\dagger D}, \\ G_A &= \frac{4\pi}{k}\epsilon_{AC}\epsilon^{BD} Z_B^\dagger W^{\dagger C} Z_D^\dagger. \end{aligned} \tag{3.1.43}$$

The contribution coming from the D-terms can be written as:

$$\begin{aligned} V_D^{\rm bos} &= \mathrm{Tr}\left[N_A^\dagger N^A + M^{\dagger A} M_A\right], \\ N^A &= \sigma Z^A - Z^A \hat{\sigma}, \qquad\qquad M_A = \hat{\sigma} W_A - W_A \sigma, \\ \sigma &= \frac{2\pi}{k}(Z^A Z_A^\dagger - W^{\dagger B} W_B), \qquad \hat{\sigma} = \frac{2\pi}{k}(Z_A^\dagger Z^A - W_B W^{\dagger B}). \end{aligned} \tag{3.1.44}$$

Combining the two contributions and using the notation introduced in (3.1.17), we find that the full bosonic potential is:

$$\begin{aligned} V^{\rm bos} &= -\frac{4\pi^2}{3k^2}\,\mathrm{Tr}\Big[Y^A Y_A^\dagger Y^B Y_B^\dagger Y^C Y_C^\dagger + Y_A^\dagger Y^A Y_B^\dagger Y^B Y_C^\dagger Y^C + \\ &\quad +4 Y^A Y_B^\dagger Y^C Y_A^\dagger Y^B Y_C^\dagger - 6 Y^A Y_B^\dagger Y^B Y_A^\dagger Y^C Y_C^\dagger\Big]. \end{aligned} \tag{3.1.45}$$

The interaction terms, quadratic in the fermion fields and quartic in the scalars, possess the manifest $SU(4)_R \sim SO(6)_R$ symmetry as well [102]. Therefore, this symmetry is manifest in the classical action; it strongly suggests that, for general N and k, the ABJM theory has at least $\mathcal{N} = 6$

supersymmetry. An explicit demonstration of the $\mathcal{N}=6$ superconformal invariance of the ABJM theory was presented in [109].

3.1.4 Gravitational description of coincident M2-branes

A stack of N coincident M2-branes creates the following extremal geometry:

$$\begin{aligned}
ds_{11}^2 &= h(r)^{-2/3}\left(-dt^2+dx_1^2+dx_2^2\right)+h(r)^{1/3}\left(dr^2+r^2 d\Omega_7^2\right), \\
h(r) &= 1+\frac{L^6}{r^6}\,, \qquad L^6 = 32\pi^2 N l_p^6, \\
F_4 &= d^3x\wedge dh(r)^{-1}.
\end{aligned} \tag{3.1.46}$$

In the limit where $r\to 0$, the metric becomes that of $AdS_4\times S^7$:

$$ds_{11}^2 = L^2\left(\frac{1}{4}ds_{AdS_4}^2 + ds_{S^7}^2\right). \tag{3.1.47}$$

For the strongly coupled theory on N M2-branes, the AdS/CFT correspondence predicts an interesting essential feature that has not been completely understood yet: the number of degrees of freedom scales as $N^{3/2}$ [110], not as N^2 found for the strongly coupled gauge theories on D3-branes. A way to see this is to study the thermal gauge theory [57, 110]. The non-extremal geometry for a stack of N M2-branes is:

$$\begin{aligned}
ds_{11}^2 &= h(r)^{-2/3}\left(-f(r)dt^2+dx_1^2+dx_2^2\right)+h^{1/3}(r)\left(\frac{dr^2}{f(r)}+r^2 d\Omega_7^2\right), \\
h(r) &= 1+\frac{L^6}{r^6}, \qquad f(r) = 1-\frac{r_0^6}{r^6}\,.
\end{aligned} \tag{3.1.48}$$

In the near-horizon region, $r\ll L$, this becomes a black brane in AdS_4, and r_0 represents the Schwarzschild radius, which is related to the Hawking temperature T.

If we consider a stack of M2-branes placed at the singularity of the orbifold $\mathbb{C}^4/\mathbb{Z}_k$, which is described by the level k ABJM theory, then $d\Omega_7^2$ refers to the metric on unit $S^7/\mathbb{Z}_k$. When the 't Hooft coupling $\lambda = N/k$ is very large, so that the metric is weakly curved, the Bekenstein-Hawking entropy is written as:

$$S_{BH} = \frac{A}{4G_N} = 2^{7/2}3^{-3}\pi^2 V_2 T^2 k^{1/2} N^{3/2} + O(N^{1/2}), \tag{3.1.49}$$

where V_2 is the spatial volume of the stack of M2-branes. Thus, the AdS/CFT correspondence predicts that for large λ the number of degrees of freedom in the ABJM theory scales like $N^{3/2}k^{1/2}$. More generally, the thermal entropy of the ABJM theory can be written as:

$$\frac{S_{BH}}{V_2 T^2} \propto N^2 f(\lambda) \; . \tag{3.1.50}$$

In the perturbative regime, $\lambda \ll 1$, we expect $f(\lambda) = 1 + O(\lambda^2)$, while the AdS/CFT correspondence predicts that:

$$f(\lambda) \propto \frac{1}{\sqrt{\lambda}}, \qquad \lambda \gg 1. \tag{3.1.51}$$

This behavior is very different from the situation where a stack of D3-branes is considered. In that case the Bekenstein-Hawking entropy can be written in a similar form to (3.1.50) but, when the 't Hooft coupling becomes large, the function $f(\lambda)$ approaches 3/4 at large λ.[5]

Now, let us discuss the parity transformation. In $2+1$ dimensions, the reflection of both coordinates is simply a rotation; therefore, one has to consider the reflection of only one. Before the ABJM model was proposed, consensus began to rise around Chern-Simons theories as preferred candidates to explain the world-volume theory on stack of M2-branes in flat space. In analogy with D3-branes, a natural choice seemed a theory with one gauge group $U(N)$ and level k. However, this presented a big riddle since the Chern-Simons term breaks parity and is, thus, unreconcilable with M-theory, known to be parity-conserving [52]. To see why this is the case, let's consider the Chern-Simons term for a single gauge group $U(N)$ with level k:

$$S_{CS} \;=\; \frac{k}{4\pi}\int \mathrm{d}^3x \, \left(\epsilon^{\mu\nu\lambda}A_\mu\partial_\nu A_\lambda + \frac{2}{3}\epsilon^{\mu\nu\lambda}A_\mu A_\nu A_\lambda + i\bar{\chi}\chi - 2D\sigma\right), \tag{3.1.52}$$

From the point of view of $\mathcal{N}=2$ superspace, all the terms in the Chern-Simons action come from a single term and, therefore, should have the same

[5]The situation becomes even more interesting when we consider a stack of M5-branes, because the supergravity approximation shows that the Bekenstein-Hawking entropy scales like N^3 [110].

transformation property under parity. In particular, looking at the first and the second term, it is obvious that the field A_μ is parity odd, since it must have same behaviour as the derivative ∂_μ. Furthermore, since the first two terms are parity odd, we conclude that the Chern-Simons action is odd under parity transformation.

In the case of the ABJM model, since there are two gauge groups, the parity transformation can be made a full symmetry of the Lagrangian, provided that it is accompanied by the following transformations on the fields:

$$\begin{aligned} x^1 &\to -x^1, & A &\leftrightarrow \hat{A}, \\ Y^A &\leftrightarrow Y_A^\dagger, & \psi_\alpha^{\dagger A} &\leftrightarrow \gamma^1_{\alpha\beta}\psi_A^\beta, \end{aligned} \tag{3.1.53}$$

3.1.5 Supersymmetry Enhancement and Monopole Operators

Let us show that $\mathcal{N} = 6$ is the correct amount of supersymmetry for N M2-branes placed at the singularity of $\mathbb{C}^4/\mathbb{Z}_k$ for $k > 2$. The $\mathbb{Z}_k$ orbifold acts on the 4 complex coordinates of $\mathbb{C}^4$ as:

$$y^A \to e^{2\pi i/k} y^A. \tag{3.1.54}$$

Note that this preserves the $SU(4)$ symmetry that rotates the y^A; this is the R-symmetry from the gauge theory point of view. The generators of $\mathbb{Z}_k$ act on the $SO(8)$ spinors as:

$$\Psi \to e^{2\pi i(s_1+s_2+s_3+s_4)/k}\Psi, \tag{3.1.55}$$

where $s_i = \pm 1/2$ are the spinor weights. By chirality projection, the sum of the s_i's has to be even, giving an 8 dimensional representation. Therefore, the spinors that survive the orbifold projection must satisfy:

$$\sum_{i=1}^{4} s_i = 0 \pmod{k}. \tag{3.1.56}$$

For $k > 2$, 6 of the 8 spinors are left invariant by the orbifold action. It follows that this supersymmetric gauge theory is expected to have 12 supercharges and, accordingly, $\mathcal{N} = 6$ supersymmetry, in agreement with that in the classical action of the ABJM theory. However, for $k = 1, 2$ all the spinors are found to be invariant, and the supersymmetry is enhanced to $\mathcal{N} = 8$. In the following, we will investigate what causes this supersymmetry enhancement in the ABJM model.

In the ABJM model, the classical global symmetry is $U(1)_b \times SU(4)_R$. The $SU(4)$ symmetry is realized by the 15 conserved traceless currents:

$$j^A_{\mu B} = i \operatorname{Tr}\left[Y^A \mathscr{D}_\mu Y^\dagger_B - (\mathscr{D}_\mu Y^A) Y^\dagger_B + i\psi^{\dagger A}\gamma_\mu \psi_B\right]. \tag{3.1.57}$$

The $U(1)_b$ transformation acts as:

$$Y^A \to e^{i\alpha} Y^A \ , \qquad \psi_B \to e^{i\alpha}\psi_B \ . \tag{3.1.58}$$

The corresponding current is related by the A^- equation of motion (in the $U(N) \times U(N)$ theory, $A^\pm \sim \operatorname{Tr} A \pm \operatorname{Tr} \hat{A}$) to the current $j_\mu \sim \epsilon_{\mu\nu\lambda} F^{+\nu\lambda}$, which is obviously conserved. Therefore, the $U(1)_b$ charge is carried only by the field configurations that have a flux of F^+ through the 2-sphere at infinity of $\mathbb{R}^3$. Such field configurations are created by the so-called "monopole operators" [112] that we discuss next. For some of their recent applications see [57, 113, 114, 115, 116, 117, 118, 119, 120].

In order for the supersymmetry to be enhanced to $\mathcal{N} = 8$, we need the global symmetry to be enhanced to $SO(8)_R$, which has 28 generators. Therefore, we need to find 12 conserved currents in addition to the 16 $U(1)_b \times SU(4)_R$ currents. Construction of these 12 currents is expected to involve the monopole operators [57]. Each of these operators creates a quantized flux in a $U(1)$ subgroup of the gauge group through a sphere surrounding the insertion point. In the ABJM theory, which has gauge group $U(N)_1 \times U(N)_2$, these monopole operators are labeled by the Cartan generators H and $\hat{H}$ of each of the two $U(N)$ factors of the gauge group:

$$H = \operatorname{diag}(q_1, q_2, \ldots, q_N), \qquad \hat{H} = \operatorname{diag}(\hat{q}_1, \hat{q}_2, \ldots, \hat{q}_N), \qquad q_i, \hat{q}_i \in \mathbb{N}, \tag{3.1.59}$$

where the condition that the entries of the generators have to be integers follows from the flux quantization condition around the S^2 surrounding the insertion point of the monopole. Also, for convenience, one can arrange the

q_i's and the $\hat{q}_i$'s such that:

$$q_1 \geq q_2 \geq \ldots \geq q_N, \qquad \hat{q}_1 \geq \hat{q}_2 \geq \ldots \geq \hat{q}_N. \tag{3.1.60}$$

We will restrict our attention to monopoles with $H = \hat{H}$. We also note that, for each monopole, there is an anti-monopole.

Generically, if a gauge group has Chern-Simons level k, the monopole operators transform in the representation of the gauge group given by a Young tableaux with kq_1 boxes in the first row, kq_2 boxes in the second row, etc[6]. It follows that in the ABJM theory, the monopoles are not gauge singlets and can combine with the chiral matter to form gauge invariant operators.

Consider, for example, the $k = 1$ theory. Here, the simplest, unit-charge monopole operator corresponds to $q_1 = \hat{q}_1 = 1$. The corresponding operator, $\mathcal{M}^a_{\hat{a}}$, transforms as a fundamental under $U(N)_1$ and anti-fundamental under $U(N)_2$. There is also the anti-monopole operator that transforms in the conjugate representation, $(\mathcal{M}^{-1})^{\hat{a}}_a$. Similarly, there are doubly charged monopole operators corresponding for example to $q_1 = \hat{q}_1 = 2$, that we denote $(\mathcal{M}^2)^{ab}_{\hat{a}\hat{b}}$, which transform in the symmetric tensor representation under $U(N)_1$, and in the conjugate representation under $U(N)_2$. We will denote their anti-monopole operators by $(\mathcal{M}^{-2})^{\hat{a}\hat{b}}_{ab}$. Another type of doubly charged BPS monopole operator has $q_1 = q_2 = \hat{q}_1 = \hat{q}_2 = 1$. Such an operator transforms as an anti-symmetric tensor under $U(N)_1$, and in the conjugate representation under $U(N)_2$. As we increase the monopole charge, the variety of different monopoles increases.

Now, let us consider the current operators:

$$j^{AB}_\mu = i\left[Y^A \mathscr{D}_\mu Y^B - \mathscr{D}_\mu Y^A Y^B + i\psi^{\dagger A}\gamma_\mu \psi^{\dagger B}\right]. \tag{3.1.61}$$

These operators are not gauge invariant, but they can be combined with the monopole $(\mathcal{M}^{-2})^{\hat{a}\hat{b}}_{ab}$, which has $kq_1 = 2, k\hat{q}_1 = 2$ and $q_i = \hat{q}_i = 0$ for $i \neq 1$, to form 6 invariant currents[7] that we shall denote as J^{AB}_μ. Combining their complex conjugates $j_{AB\mu}$ with the monopole $\mathcal{M}^2$, gives another six conserved currents which are also gauge invariant. To summarize, the

[6] Since the gauge groups we are dealing with are $U(N)$ rather than $SU(N)$, the columns of length N must be taken into account as well.

[7] The number 6 comes from the anti-symmetry of j^{AB}_μ under the exchange of A and B.

monopoles have provided us with 12 symmetry generators that we can add to the 16 obvious currents of $U(1)_b \times SU(4)_R$. With a total of 28 generators, the global symmetry is enhanced to $SO(8)_R$, as we expected. When $k \geq 3$, there is no way to construct the monopole operators $\mathcal{M}^{-2}$ and $\mathcal{M}^2$, which carry two indices under each $U(N)$; so the symmetry enhancement does not happen.

The importance of the monopoles goes beyond the supersymmetry enhancement for $k = 1, 2$. They are also necessary for matching the spectrum of the gauge theory with that of the dual gravity theory. In the ABJM theory with $k = 1$, the simplest gauge invariant operators have the form $Y^{\dagger\hat{a}}_{Ba} \mathcal{M}^a_{\hat{a}}$ and their conjugate. These operators, linear in the scalar fields, are expected to have scaling dimension $1/2$; they can be thought of as free fields dual to 'singleton' modes in AdS_4. The simplest scalar composite operators we can construct in the ABJM theory at level $k = 1$ can be written as:

$$\mathrm{Tr}\left[Y^\dagger_A Y^B - \frac{1}{4}\delta^B_A Y^\dagger_C Y^C \right] \ . \tag{3.1.62}$$

There are 15 such traceless operators of dimension 1. The AdS/CFT correspondence implies that the gauge invariant scalar operators should be in one-to-one correspondence with the Kaluza-Klein harmonics on S^7. There are 35 such harmonics that correspond to operators of dimension 1. Therefore, we need to find 20 more operators made of two scalar fields in order to match this part of spectrum with the gravity side. Once again, we proceed by forming non-gauge-invariant combinations of scalar fields and combining them with monopoles in order to make them gauge invariant. The 20 gauge invariant operators written as:

$$Y^\dagger_A Y^\dagger_B \mathcal{M}^2, \qquad Y^A Y^B \mathcal{M}^{-2}, \tag{3.1.63}$$

turn out to be what is needed to match the spectrum of the field theory with the gravity theory. These additional operators have non-vanishing $U(1)_b$ charge, which is dual to the momentum along the M-theory circle.[8]

[8]If we view the S^7 as a circle fibration over $\mathbb{CP}^3$, then the reduction to type IIA string theory produces an $AdS_4 \times \mathbb{CP}^3$ background.

3.2 The forward algorithm for M2-branes

Having reviewed the construction of the ABJM model, we now want to move to a more general level, trying to understand how it is possible to study the moduli space of vacua for a given $\mathcal{N} = 2$ theory. In particular, we will focus on a specific class of theories, namely those that can be represented by a brane tiling. In the following, we shall see first how the formalism of brane tilings can be extended to $(2+1)$-dimensional Chern-Simons theories and then how the forward algorithm can be employed to determine the Hilbert series of the mesonic moduli space. We shall begin by looking at the features of the moduli space of vacua for a generic $\mathcal{N} = 2$ supersymmetric Chern-Simons theory in $(2+1)$ dimensions.

3.2.1 The moduli space of $(2+1)$-dimensional supersymmetric Chern-Simons theories

In the following sections, we will be concerned with the moduli space of Chern-Simons (CS) theories in $(2+1)$ dimensions and, therefore, a more generic introduction of the subject is in order. We shall consider a generic theory with $\mathcal{N} = 2$ supersymmetry, i.e. 4 supercharges, a total number of E fields, a gauge group that can be written in the form $\prod_{a=1}^{G} U(N_a)$, with G being the number of factors and CS terms $(k_1, \ldots, k_G)$. In $\mathcal{N} = 2$ superspace notation, the Lagrangian for this generic theory can be written as:

$$\begin{aligned}\mathcal{L} \;=\; & -\int d^4\theta \left(\sum_{X_{ab}} X_{ab}^{\dagger} e^{-V_a} X_{ab} e^{V_b} - i \sum_{a=1}^{G} k_a \int_0^1 dt V_a \bar{\mathcal{D}}^{\alpha} (e^{tV_a} \mathcal{D}_{\alpha} e^{-tV_a}) \right) + \\ & + \int d^2\theta W(X_{ab}) + \text{c.c.}, \end{aligned} \tag{3.2.64}$$

where a indexes the factors in the gauge group, X_{ab} are the superfields accordingly charged, V_a are the vector multiplets, $\mathcal{D}$ is the superspace derivative, W is the superpotential and k_a are the CS levels which are integers; an overall trace is implicit since all the fields are matrix-valued.

Expanding the vector field in its components and carrying out the integration with respect to the fermionic coordinates, the relevant terms in the

Lagrangian can be written as:

$$\begin{aligned}\mathcal{L} &= -4\sum_{a=1}^{G} k_a \sigma_c D_a + \sum_{a=1}^{G} D_a \mu_a(X) - \sum_{X_{ab}} (\sigma_a X_{ab} - X_{ab}\sigma_b)(\sigma_a X_{ab} - X_{ab}\sigma_b)^{\dagger} + \\ &\quad - \sum_{Xab} |\partial_{X_{ab}} W|^2, \end{aligned} \tag{3.2.65}$$

where $\mu_a(X)$ is the moment map and can be defined as:

$$\mu_a(X) = \sum_{b \neq a} X_{ab} X_{ab}^{\dagger} - \sum_{c \neq a} X_{ca}^{\dagger} X_{ca} + [X_{aa}, X_{aa}^{\dagger}]. \tag{3.2.66}$$

Just as for the case of the ABJM model, by integrating out the auxiliary fields D_a the bosonic potential becomes a sum of squares. Thus, the vacua can be found by solving the following algebraic equations:

$$\begin{aligned} \partial_{X_{ab}} W &= 0 \ , \\ \mu_a(X) &= 4k_a \sigma_a \ , \\ \sigma_a X_{ab} - X_{ab}\sigma_b &= 0 \ . \end{aligned} \tag{3.2.67}$$

The first line gives the F-term constraints and, interestingly, it looks exactly the same as in the $(3+1)$-dimensional case, the second line resembles standard D-term equations with a set of effective FI terms $\zeta_a = 4k_a\sigma$ [92]. Strictly speaking, these cannot be considered the same as the FI parameters in $(3+1)$ dimensions because the latter are parameters of the Lagrangian, whereas σ_a's are auxiliary fields.

Finally, the last line is a complete novelty with respect to 4-dimensional supersymmetric gauge theories. In the following, we shall refer to the space of solutions to (3.2.67) as the **mesonic moduli space**, and denote it with $\mathcal{M}^{\text{mes}}$.

We will now proceed and show how these equations are solved in the abelian case. However, before we continue in this direction, two assumptions are in order:

- All gauge groups are $U(N)$ with N having the physical interpretation as the number of M2-branes in the stack on which the gauge theory is living;
- The superpotential W satisfies the **toric condition** [121]: Each chiral

multiplet appears precisely twice in W. Once with a positive sign and once with a negative sign. Under such assumptions, the moduli space is conjectured to receive no quantum corrections due to supersymmetry and due to conformal invariance in the IR.

The two conditions we have imposed guarantee that the mesonic moduli space is not only Calabi-Yau, but also toric manifold which will make it possible for the properly modified forward algorithm to be applied. Finally, we should make it clear that, as we did for 4-dimensional theories, we will be interested only in the classical moduli space of Chern-Simons $(2+1)$-dimensional theories and, therefore, we will treat their lagrangians as classical. We shall also assume that possible corrections to the Kähler potential do not affect our discussion.

We can now move to the abelian case, meaning that the gauge group of will generically be written as $U(1)^G$. In such situation, we can immediately appreciate that the last equation of (3.2.67) forces all the σ_a to be equal to a single field, say σ. Furthermore, since each quiver field appears twice in the sum of equation (3.2.67), once with a positive sign and once with a negative sign, we have the following:

$$\sum_a k_a \sigma_a = 0 \ . \tag{3.2.68}$$

The third equation of (3.2.67) sets all σ_a to a single field, say σ. From (3.2.68), we see that for $\sigma \neq 0$, we must impose the following constraints on the CS levels:

$$(k_1, \ldots, k_G) \neq 0 \ , \qquad \sum_{a=1}^{G} k_a = 0 \ . \tag{3.2.69}$$

Note that if the last equality is not satisfied, then σ is identically zero and (3.2.67) reduces to the usual vacuum equations for 3+1 dimensional gauge theories. For the time being, we will consider (3.2.69) as a necessary condition for the mesonic moduli space to be 4 dimensional, as we require. For simplicity, we will also take

$$\gcd(\{k_a\}) = 1 \tag{3.2.70}$$

so that we do not have to consider orbifold actions on the moduli space.

However, it is easy to generalise to the case of higher $\gcd(\{k_a\})$, and several explicit examples are given in [92, 91].

With what we have shown so far, the second set of equations in (3.2.67) can be written as:

$$\mu_a(X) = 4k_a\sigma, \tag{3.2.71}$$

with $a = 1, \ldots, G$. The constraint that all the CS terms sum to zero implies that only $G-1$ out of the original G equations are independent, thus making one redundant. Furthermore, we can see from the above equation that the vector containing the moment maps for each gauge group aligns along the direction singled out by the CS terms, which is one of the $(G-1)$ baryonic directions which are present in the $(3+1)$ dimensional theory. For the 3-dimensional theories we are examining, this direction remains mesonic and fibers over the Calabi-Yau 3-fold, which would be the space of vacua of the 4-dimensional theory, to give the Calabi-Yau 4-fold which constitutes the mesonic moduli space of the CS theory. We then proceed by pointing out that, by taking linear combinations, we can write $G-2$ equations in the form:

$$\mu_i(X) = 0, \qquad i = 1, \ldots, G-2. \tag{3.2.72}$$

Note that Eq. (3.2.72) says that we are imposing the vanishing of the D-terms for $G-2$ $U(1)$ gauge groups. As it is well-known for the case of 4-dimensional gauge theories, imposing the above equations and the gauge transformations amounts to modding out by the complexified gauge groups. As for the remaining two equations, one simply determines the value of σ without adding any further constraint, and the other one refers to the diagonal gauge field which simply leaves a discrete symmetry $\mathbb{Z}_k$, where $k = \gcd(\{k_a\})$.

Thus, in summary, there are $(G-2)$ baryonic charges *coming from the D-terms*. We emphasise a subtle point here: although there are indeed $G-2$ baryonic directions coming from the D-terms, this does not imply that all possible baryonic directions of the particular Calabi-Yau 4-fold are given by these $G-2$ directions. It only provides a lower bound. There are *at least* $G-2$ such baryonic directions and a different formulation may give more than this number. Below, we shall discuss how to count all baryonic charges using the toric diagram.

Note that the $G-2$ baryonic charges are in the null space of the matrix

$$C = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ k_1 & k_2 & k_3 & \dots & k_G \end{pmatrix} . \tag{3.2.73}$$

This can be seen as follows. If the charge vector $\mathbf{q} = (q_1, \dots, q_G)$ is in the null space of C, then $C \cdot \mathbf{q}^t = 0$, *i.e.* the G charges are subject to 2 relations:

$$\sum_{a=1}^{G} q_a = 0 \ , \qquad \sum_{a=1}^{G} k_a q_a = 0 \ . \tag{3.2.74}$$

The first equation, which fixes the total charge to be zero, implies that $\mathbf{q}$ is perpendicular to the vector $(1, \dots, 1)_{1\times G}$, and the second equation implies that $\mathbf{q}$ is perpendicular to the direction set by the CS integers $(k_1, \dots k_G)$. Since the vectors $(1, \dots, 1)_{1\times G}$ and $(k_1, \dots k_G)$ are orthogonal due to (3.2.69), the independent components of $\mathbf{q}$ are indeed the $G-2$ baryonic charges.

In the following sections, we will limit ourselves to the study of the mesonic moduli space of abelian theories. This is motivated by the fact that generically the moduli spaces in the non-abelian case is non-toric and, therefore, does not admit a description in terms of brane tilings. However, once the moduli space for the abelian case is understood, treating the non-abelian case should not be too difficult [104]. In fact, we can fix a gauge in which all the σ_a fields are diagonalised, and all with distinct eigenvalues. The last equation in (3.2.67), tells us that in this basis, the X_{ab} fields should generically be diagonal as well. Finally, the residual gauge symmetry is just the Weyl group of $SU(N)$, which permutes the diagonal entries of the chiral fields. This implies that the moduli space for non-abelian theories is expected to be the N^{th}-symmetric product of the abelian moduli space, N being the number of M2-branes probing the Calabi-Yau singularity.

3.2.2 Adapting the forward algorithm to M2-brane theories

Brane tilings. In the previous section we have required that the theories we will examine respect the toric condition, namely that each field appears exactly twice in the superpotential with opposite signs. As was the case for 4-dimensional theories arising from D3-branes, this condition naturally gives rise to a notion of a bipartite graph which, if drawn on a 2-torus, is precisely a brane tiling. Of course, a tiling does not require consideration of a torus, as it can be defined also on $\mathbb{R}^2$, as long as periodicity in both directions is preserved.

The main features of brane tilings for 3-dimensional CS theories are similar to the ones for the 4-dimensional theories we have considered in the previous chapter: each face of the tiling corresponds to a gauge group, each edge corresponds to a bi-fundamental field and each node corresponds to a superpotential term. Of course, the bipartiteness still naturally gives rise to a notion of orientation associated to each edge. With the convention that arrows 'circulates' clockwise around the white node and counterclockwise around the black nodes, we recall the definition of the incidence matrix d whose elements are:

$$d_{ai} = \begin{cases} +1 & \text{for an outgoing arrow from the face } a \ , \\ -1 & \text{for an incoming arrow to the face } a \ , \\ 0 & \text{if the edge } i \text{ is not a side of the face } a \ . \end{cases} \tag{3.2.75}$$

We call the $G \times E$ matrix d an *incidence matrix.* The real novelty, for the class of theories we are considering, is the assignment of integer numbers n_i to each edge i in the tiling. Together with the incidence matrix we just defined, these are used to associate Chern-Simons levels k_a to each gauge group a as shown below[9]:

$$k_a = \sum_i d_{ai} n_i \ . \tag{3.2.76}$$

Due to bipatiteness of the tiling, we see that the relation $\sum_a k_a = 0$ is satisfied as required. The superpotential can be written as

$$W = \sum_{\wp} \mathrm{sign}(\wp) \prod_{j_\wp} \Phi_{j_\wp} \ , \tag{3.2.77}$$

where the product is taken over the edges $j_\wp$ around the node $\wp$, and $\mathrm{sign}(\wp)$ is $+1$ if $\wp$ is a white node[10] and -1 if $\wp$ is a black node.

[9] This way of representing k_a is introduced in [92] and is also used in [122].

[10] The reader should note the similarity between white nodes and British roundabouts. They both have a positive effect and you go round them clockwise.

Brane realisation. As discussed in detail in [122], a brane tiling for the 2+1 dimensional CS theory can be regarded a D4-NS5 system in Type IIA theory on $\mathbb{R}^{1,7} \times T^2$. The NS5-brane fills an $\mathbb{R}^{1,3}$ subspace of $\mathbb{R}^{1,7}$ and is on a complex curve on $\mathbb{R}^2 \times T^2$ such that the NS5-brane forms a collection of tiles that wrap the T^2, with the NS brane forming the edges of the tiles. In the remaining two coordinates on $\mathbb{R}^{1,7}$ the brane system sits in a fixed position. The D4-branes span an $\mathbb{R}^{1,2}$ subspace of $\mathbb{R}^{1,3}$ and are wrapping the tiles in the T^2 directions, having boundaries that end on the NS5-brane. The gauge groups are realised on the D4-branes, giving rise to a $U(N)$ gauge group per N D4-branes that span the tile. The edges are separating two tiles and open strings stretched between them give rise to chiral multiplets in bi-fundamental representations. Let A be the gauge field on the D4-brane,

and let ϕ be the 0-form gauge field on the NS5-brane.

This 0-form gauge field couples to the field strength $\mathrm{d}A$ on the boundary of a D4-brane via the usual WZ coupling $\phi\ \mathrm{d}A \wedge \mathrm{d}A$. Integrating by parts, we may write down the boundary term in the D4-brane action as

$$S_{\text{boundary}} = \frac{1}{2\pi} \int_{\partial \mathrm{D4}} A \wedge \mathrm{d}A \wedge \mathrm{d}\phi\ . \tag{3.2.78}$$

This induces the CS coupling which is given by

$$k_a = \oint \mathrm{d}\phi\ , \tag{3.2.79}$$

where the integration is taken over the boundary of the face a (*i.e.,* along the boundary of the corresponding D4-brane). The one-form field strength $\mathrm{d}\phi$ along the edge i can be identified with the integer n_i. Being a field strength it is quantized and therefore k_a are integers. Linear combinations of the edge contributions n_i are integers and we therefore expect that each edge of the tiling gives an integer contribution with the orientation determining the sign. Thus, (3.2.79) is indeed equivalent to the relation (3.2.76).

Kasteleyn matrices. When we introduced the brane tilings in the previous chapter, we defined the Kasteleyn matrix, a weighted adjacency matrix with rows indexed by the black nodes and columns indexed by the white nodes of the fundamental domain. For the cases at hand, the entry of the Kasteleyn matrix corresponding to the black node $\wp$ and the white node ϱ

can be written as:

$$K_{\wp\varrho}(x, y, z) = \sum_{\{j_{\wp\varrho}\}} \Phi_{j_{\wp\varrho}} z^{n_{j_{\wp\varrho}}} w_{j_{\wp\varrho}}(x, y) \ , \tag{3.2.80}$$

where $j_{\wp\varrho}$ represent an edge connecting the black node $\wp$ to the white node ϱ, $\Phi_{j_{\wp\varrho}}$ is the field associated with this edge, and $w_{j_{\wp\varrho}}(x, y)$ is x or y (or x^{-1} or y^{-1}, depending on the orientation of the edge) if the edge $j_{\wp\varrho}$ crosses the fundamental domain [67, 68] and $w_{j_{\wp\varrho}}(x, y) = 1$ if it does not. Looking at (3.2.80), we can already see how the introduction of Chern-Simons levels have provided us with a new coordinate for the toric diagram that is needed to represent Calabi-Yau 4-folds.

Perfect matchings. The notion of perfect matching is still perfectly consistent with the modification we have made to the notion of brane tilings. In particular, perfect methinks can be generated by taking the permanent of the Kasteleyn matrix, which we can generically write as:

$$\text{perm } K = \sum_{\alpha=1}^{c} p_\alpha \ x^{u_\alpha} y^{v_\alpha} z^{w_\alpha} \ . \tag{3.2.81}$$

The coordinates $(u_\alpha, v_\alpha, w_\alpha)$, with $\alpha = 1, \ldots, c$, are points in a 3d *toric diagram* of the $(2+1)$ dimensional theory. From (3.2.80), we see that w_α is a linear combination of the integers n_i. Indeed, if we set $z = 1$, we then recover the 2d Newton polygon which gives a 2d toric diagram of the $(3+1)$ dimensional theory.

The perfect matching matrix. We collect the correspondence between the perfect matchings and the quiver fields in an $E \times c$ matrix (where E is the number of quiver fields and c is the number of perfect matchings) called the *perfect matching matrix* P. If $E = c$ (*i.e.* the fundamental domain contains precisely one pair of black and white nodes), we can relabel p_α so that P becomes an identity matrix. On the other hand, if $E \neq c$, then the null space of the matrix P is non-trivial, and there exists a $(c - G - 2) \times c$ matrix Q_F whose rows are basis vectors (which are taken to be orthogonal) of the nullspace of P:

$$Q_F = \ker(P) \ . \tag{3.2.82}$$

Therefore, by construction, we find the relation

$$P \cdot Q_F^t = 0 \ . \tag{3.2.83}$$

This matrix equation gives the relations between the perfect matchings p_α. Hence, the coherent component ${}^{\mathrm{Irr}}\mathcal{F}^\flat$ of the Master space can be viewed as the space $\mathbb{C}^c$ generated by the perfect matchings modded out by the relations encoded in Q_F:

$${}^{\mathrm{Irr}}\mathcal{F}^\flat = \mathbb{C}^c // Q_F \ . \tag{3.2.84}$$

The matrix Q_F can be regarded as the *charge matrix associated with the F-terms*. The coherent component ${}^{\mathrm{Irr}}\mathcal{F}^\flat$ is $c-(c-G-2) = G+2$ dimensional, as expected. Note that the sum of entries in each row of Q_F vanishes. This is equivalent to saying that $(1,1,\ldots,1)_{1\times G}$ is in the null space of Q_F, or in other words, is spanned by the row vectors of P^t (see (3.2.83)). It can be seen that the sum of all rows of P^t is proportional to $(1,\ldots,1)_{1\times G}$, and hence the statement in the previous sentence follows.

Baryonic charges of perfect matchings. Let us determine the baryonic charges of the perfect matchings. In order to do so, we remind the reader of the definition (3.2.75) of the incidence matrix d, which maps the fields into their quiver charges. Furthermore, we recall the definition of the perfect matching matrix P, which maps the perfect matchings to the fields. Let $\widetilde{Q}$ be a $G \times c$ matrix which maps the perfect matchings into their quiver charges. Then,

$$d_{G\times E} = \widetilde{Q}_{G\times c} \cdot (P^t)_{c\times E} \ , \tag{3.2.85}$$

where the subscripts denote the sizes of matrices. Recall that the $G-2$ baryonic charges are in the null space of C given by (3.2.73). We can define a $(G-2)\times G$ matrix $\ker(C)$ whose rows are orthogonal basis vectors of the null space of C. This matrix projects the space of quiver charges onto the null space of C. Hence, *the baryonic charges of the perfect matching* are given by the $(G-2)\times c$ matrix:

$$(Q_D)_{(G-2)\times c} = \ker(C)_{(G-2)\times G} \cdot \widetilde{Q}_{G\times c} \ . \tag{3.2.86}$$

In analogy to Q_F, the mesonic moduli space can be written as

$$\mathcal{M}^{\text{mes}} = {}^{\text{Irr}}\mathcal{F}^{\flat}//Q_D = (\mathbb{C}^c//Q_F)//Q_D \ . \tag{3.2.87}$$

The matrix Q_D can be regarded as the *charge matrix associated with the D-terms.* Note that the sum of entries in each row of Q_D vanishes, since $(1, 1, \ldots, 1)_{1\times G}$ is in the null space of $\ker(C)$ as discussed in the comment below (3.2.73). If the number of perfect matchings c is equal to the number of quiver fields E (*i.e.* there is precisely one pair of black and white nodes in the fundamental domain), then P can be arranged to be the identity matrix and hence

$$(Q_D)_{(G-2)\times c} = \ker(C)_{(G-2)\times G} \cdot d_{G\times E} \qquad \text{(for } c = E) \ . \tag{3.2.88}$$

The toric diagram. There are 2 methods of constructing the toric diagram:

- The first method was mentioned in the preceding paragraph. In particular, the coordinates $(u_\alpha, v_\alpha, w_\alpha)$ of the α-th point in the toric diagram are respectively given by the power of x, y, z in (3.2.81).
- The second method is to make use of the charge matrices Q_F and Q_D via (3.2.87). We construct a $(c-4)\times c$ matrix Q_t as follows:

$$(Q_t)_{(c-4)\times c} = \begin{pmatrix} (Q_D)_{(G-2)\times c} \\ (Q_F)_{(c-G-2)\times c} \end{pmatrix} \ . \tag{3.2.89}$$

Then, let us define a $4 \times c$ matrix

$$G_t = \ker(Q_t) \tag{3.2.90}$$

whose rows are basis vectors of the null space of Q_t. The matrix G_t projects the space of perfect matchings onto the null space of Q_t. Note that columns of length 4 of G_t signify a 4-fold. Since $(1, \ldots, 1)_{1\times G}$ lives in both the null spaces of Q_F and Q_D, it follows that we can always pick a row of G_t to be $(1, \ldots, 1)_{1\times G}$. This implies that the end points of these c 4-vectors lie in a 3 dimensional hyperplane. Therefore, we may remove the first row of G_t and obtain a $3\times c$ matrix G_t'. The columns of

G_t' give the coordinates of points in the toric diagram, which represent the toric 4-fold by an integer polytope in 3 dimensions.

We emphasise that the 3d toric diagram is defined up to a $GL(3, \mathbb{Z})$ transformation.

The mesonic symmetries. In the context of the AdS/CFT correspondence, $\mathcal{N} = 2$ superconformal gauge theories in 2+1 dimensions are dual to M-theory on $\mathrm{AdS}_4 \times \mathrm{SE}^7$ (where SE^7 denotes a Sasaki–Einstein 7-manifold). There are 4 global $U(1)$ symmetries which come from the metric and are isometries of the Sasaki–Einstein 7-manifold. The toric condition implies that the isometry group is $U(1)^4$ or an enhancement of $U(1)^4$ to a non-abelian group. This isometry group is called the **mesonic symmetry** and can be determined by the Q_t matrix. In particular, the existence of a non-abelian $SU(k)$ factor (with $k > 1$) in the mesonic symmetry is implied by the number k of repetitions of columns in the Q_t matrix. Since the mesonic symmetry has a total rank 4, we can classify all possible mesonic symmetries according to the partitions of 4 as follows:

- $SU(4) \times U(1)$,
- $SU(3) \times SU(2) \times U(1)$,
- $SU(3) \times U(1) \times U(1)$,
- $SU(2) \times SU(2) \times SU(2) \times U(1)$,
- $SU(2) \times SU(2) \times U(1) \times U(1)$,
- $SU(2) \times U(1) \times U(1) \times U(1)$,
- $U(1) \times U(1) \times U(1) \times U(1)$.

If it turns out that there is precisely one $U(1)$ factor in the mesonic symmetry, we can immediately identify it with the R-charge. Otherwise, there is a minimisation problem to be solved in order to determine which linear combination of these $U(1)$ charges gives the right R-charge in the IR [123]. In some simple cases, we can bypass this calculation using a symmetry argument.

The baryonic symmetries. Each external point in the toric diagram corresponds to a 5-cycle in the Sasaki–Einstein 7-manifold. Not all of these 5-cycles are homologically independent but one can choose a basis of homologically stable 5-cycles inside the Sasaki–Einstein 7-manifold. Every 5-cycle in this basis gives rise to a massless gauge field in AdS_4, coming from Kaluza–Klein reduction of the M-theory 6-form (dual to the 3-form) on the 5-cycle. These massless gauge fields are dual to the **baryonic** $U(1)$ symmetries in the gauge theory. The number of such homologically stable 5-cycles, which is thus equal to *the number of baryonic charges* $N(\mathscr{B})$, is equal to the number of external points $N(\mathscr{E})$ in the toric diagram minus 4:

$$N(\mathscr{B}) = N(\mathscr{E}) - 4 \ . \tag{3.2.91}$$

The global symmetry of the theory is a product of mesonic and baryonic symmetries.

Parity invariance of the Calabi–Yau 4-fold. We mention above that the vacuum equations and the mesonic moduli space are invariant under parity. This fact can also be seen from the toric diagram perspective as follows. Since under a parity transformation $k_a \to -k_a$, it follows from (3.2.76) that $n_i \to -n_i$ (the d_{ai} do not change sign as we are not dealing with charge conjugation). It follows from the discussion after (3.2.81) that, for each point in the toric diagram, the third coordinate $w_\alpha \to -w_\alpha$, whereas the first and second coordinates u_α, v_α remain unchanged. This is however a $GL(3,\mathbb{Z})$ action on the coordinates. We thus arrive at our conclusion.

A summary of the forward algorithm for M2-brane theories. We summarise the forward algorithm in the following diagram (as in [124]):

$$
\begin{array}{llll}
\boxed{\begin{array}{l}\text{INPUT 1:}\\ \text{Quiver}\end{array}} & \to\ d_{G\times E} & \to & (Q_D)_{(G-2)\times c} = \ker{(C)}_{(G-2)\times G}\cdot \widetilde{Q}_{G\times c} \\
 & & \nearrow & \\
\boxed{\begin{array}{l}\text{INPUT 2:}\\ \text{CS Levels}\end{array}} & \to\ C_{2\times G} & & \\
 & & \nearrow & \\
\boxed{\begin{array}{l}\text{INPUT 3:}\\ \text{Superpotential}\end{array}} & \to\ P_{E\times c} & \to & (Q_F)_{(c-G-2)\times c} = \ker(P) \\
 & & & \quad\downarrow \\
 & \multicolumn{3}{r}{(Q_t)_{(c-4)\times c} = \begin{pmatrix} (Q_D)_{(G-2)\times c} \\ (Q_F)_{(c-G-2)\times c} \end{pmatrix} \to \boxed{\begin{array}{l}\text{OUTPUT:}\\ (G_t)_{4\times c} = \ker(Q_t)\end{array}}}
\end{array}
\tag{3.2.92}
$$

Notation and nomenclature. We denote the i-th bi-fundamental field transforming in the fundamental (antifundamental) representation of the gauge group a (gauge group b) by X^i_{ab} and similarly ϕ^i_a denotes the i-th adjoint field in the gauge group a (when there is only a single arrow the i-index is dropped). We refer to gauge theories in subsequent sections by their mesonic moduli space (*e.g.,* the $\mathbb{C}^4$ theory), and in each subsection we name toric phases according to the features of their tilings (*e.g.,* Phase I of the $\mathbb{C}^4$ theory is called the 'chessboard model' as its tiling is similar to the chessboard). We use the shorthand notation listed in Table 3.3 for our nomenclature, *e.g.* the two double-bonded one-hexagon model is denoted by $\mathscr{D}_2\mathscr{H}_1$.

3.2.3 An example: the ABJM model

The chessboard model (which we shall refer to as $\mathscr{C}$) contains two gauge groups $U(N)_1 \times U(N)_2$ and bi-fundamental fields X^i_{12} and X^i_{21} (with $i = 1, 2$). The superpotential is given by

$$
W = \mathrm{Tr}(X^1_{12}X^1_{21}X^2_{12}X^2_{21} - X^1_{12}X^2_{21}X^2_{12}X^1_{21}) \ . \tag{3.2.93}
$$

Shorthand notation	Object referred to
$\mathscr{C}$	chessboard
$\mathscr{D}_n$	n double bonds
$\mathscr{H}_n$	n hexagons
$\mathscr{S}_n$	n squares
∂_n	n diagonals
$\mathscr{O}_n$	n octagons

Table 3.3: Shorthand notation for the nomenclature of the brane tilings used in paper.

According to (3.2.69), we take the Chern–Simons levels to be $k_1 = -k_2 = 1$. The quiver diagram and tiling are drawn in Figure 3.2. In 3+1 dimensions, the chessboard tiling actually gives rise to the conifold theory (which we shall refer to as $\mathcal{C}$); however, for the 2+1 dimensional theory, there is an additional structure, namely each edge in the tiling bears an integer n_i according to (3.2.76). In the following paragraph, we see that the mesonic moduli space of the 2+1 dimensional chessboard model indeed differs from the mesonic moduli space of the 3+1 dimensional conifold theory but still coincides with its master space.

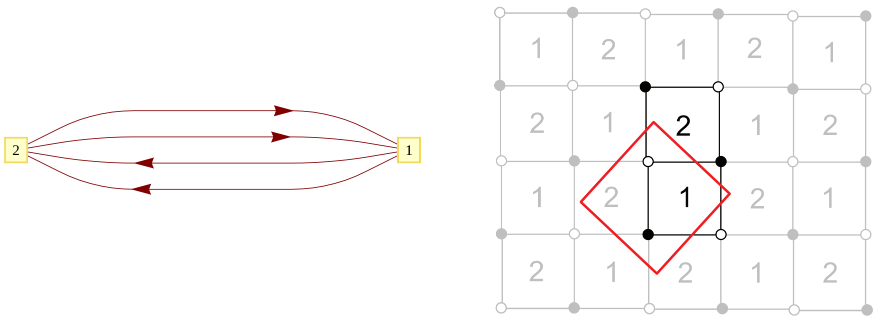


Figure 3.2: [Phase I of $\mathbb{C}^4$] (i) Quiver diagram for the $\mathscr{C}$ model. (ii) Tiling for the $\mathscr{C}$ model.

The toric diagram. We demonstrate two methods of constructing the toric diagram.

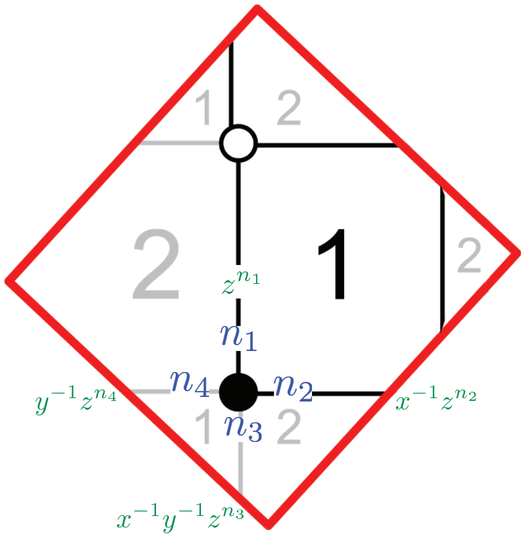


Figure 3.3: [Phase I of $\mathbb{C}^4$] The fundamental domain of the tiling for the $\mathscr{C}$ model: Assignments of the integers n_i to the edges are shown in blue and the weights for these edges are shown in green.

- **The Kasteleyn matrix.** We assign the integers n_i to the edges according to Figure 3.3. From (3.2.76), we find that

$$\begin{aligned} \text{Gauge group 1 :} \qquad k_1 &= 1 = n_1 - n_2 + n_3 - n_4 \ , \\ \text{Gauge group 2 :} \qquad k_2 &= -1 = -n_1 + n_2 - n_3 + n_4 \ . \end{aligned} \tag{3.2.94}$$

We choose

$$n_3 = 1, \quad n_1 = n_2 = n_4 = 0 \ . \tag{3.2.95}$$

We can now determine the Kasteleyn matrix for this phase of the theory. Since the fundamental domain contains only one white node and one black node, the Kasteleyn matrix is 1×1 and, therefore, coincides with its permanent:

$$\begin{aligned} K &= X_{12}^1 z^{n_1} + X_{21}^1 x^{-1} z^{n_2} + X_{12}^2 x^{-1} y^{-1} z^{n_3} + X_{21}^2 y^{-1} z^{n_4} \\ &= X_{12}^1 + X_{21}^1 x^{-1} + X_{12}^2 x^{-1} y^{-1} z + X_{21}^2 y^{-1} \\ &\quad (\text{for } n_3 = 1,\ n_1 = n_2 = n_4 = 0) \ . \end{aligned} \tag{3.2.96}$$

The powers of x, y, z in each term of (3.2.96) give the coordinates of each point in the toric diagram. We collect these points in the columns of the following G_K matrix:

$$G_K = \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \qquad (3.2.97)$$

- **The charge matrices.** From (3.2.96), the perfect matchings can therefore be taken as

$$p_1 = X^1_{12}, \quad p_2 = X^2_{12}, \quad p_3 = X^2_{21}, \quad p_4 = X^1_{21} . \qquad (3.2.98)$$

Since there is a one-to-one correspondence between the perfect matchings and the quiver fields, $Q_F = 0$. Since the number of gauge groups is $G = 2$, there is $G - 2 = 0$ baryonic charge from the D-terms and hence $Q_D = 0$. Thus, we have $Q_t = 0$. From (3.2.90), we find that $G_t = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. After removing the first row, the columns give the coordinates of points in the toric diagram:

$$G'_t = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \qquad (3.2.99)$$

We see that the toric diagram is merely 4 corners of a tetrahedron (Figure 3.4). This is in fact the toric diagram of $\mathbb{C}^4$ [123, 124].

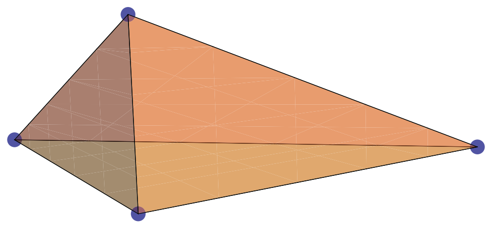


Figure 3.4: The toric diagram of the $\mathbb{C}^4$ theory.

The moduli space. For the abelian case, the fields are simply complex numbers and so the superpotential vanishes. Therefore, the Master space is $\mathcal{F}^{\flat}_{\mathscr{C}} = \mathbb{C}^4$. From Figure 3.4, there are 4 external points in the toric diagram. It follows that the number of baryonic charges is $4 - 4 = 0$, and hence the mesonic moduli space coincides with the Master space:

$$\mathcal{M}^{\text{mes}}_{\mathscr{C}} = \mathcal{F}^{\flat}_{\mathscr{C}} = \mathbb{C}^4 \ . \tag{3.2.100}$$

Since all four columns of the Q_t matrix are the same, the mesonic symmetry of this model is $SU(4) \times U(1)$. Note that this $U(1)$ is not the full R-symmetry, which is actually $Spin(8)$. However, since it assigns equal weight to all fields, it can be identified with the scaling dimension $1/2$. The four fields transform as the fundamental representation of the $SU(4)$. The Hilbert series is given by

$$\begin{aligned} g^{\text{mes}}_1(t, x_1, x_2, x_3; \mathscr{C}) &= \frac{1}{(1 - tx_1)\left(1 - \frac{tx_2}{x_1}\right)\left(1 - \frac{tx_3}{x_2}\right)\left(1 - \frac{t}{x_3}\right)} \\ &= \sum_{k=0}^{\infty} [k, 0, 0] t^k, \end{aligned} \tag{3.2.101}$$

where t is the fugacity counting scaling dimensions and x_1, x_2 and x_3 are fugacities for the $SU(4)$ weights. Let us compute the plethystic logarithm of the Hilbert series:

$$\mathrm{PL}[g^{\text{mes}}_1(t, x_1, x_2, x_3; \mathscr{C})] = t\left(x_1 + \frac{x_2}{x_1} + \frac{x_3}{x_2} + \frac{1}{x_3}\right) = [1, 0, 0]t. \tag{3.2.102}$$

The generators. We can see that the mesonic moduli space is generated by four operators:

$$X^1_{12}, \quad X^2_{12}, \quad X^1_{21}, \quad X^2_{21} \ .$$

We can represent these generators in a lattice (Figure 3.5) by plotting the powers of x_1, x_2, x_3 of the character in (3.2.102). Note that the lattice of generators is the dual of the toric diagram (nodes are dual to faces and edges are dual to edges). For the $\mathbb{C}^4$ theory, the toric diagram is a tetrahedron (4 nodes, 6 edges and 4 faces), which is a self-dual lattice. Therefore, the lattice of generators is the same as the toric diagram.

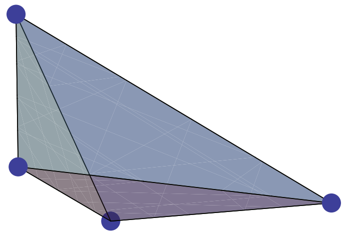


Figure 3.5: The lattice of generators of the $\mathbb{C}^4$ theory.

The 2+1 dimensional chessboard model $\mathscr{C}$ V.S. the 3+1 dimensional conifold theory $\mathcal{C}$. The Master space of the 3+1 dimensional conifold theory (see [90]) coincides with the Master space of the 2+1 dimensional chessboard model (see (3.2.100)):

$$\mathcal{F}^{\flat}_{\mathscr{C}} = \mathcal{F}^{\flat}_{\mathcal{C}} = \mathbb{C}^4 \ . \qquad (3.2.103)$$

However, the mesonic moduli spaces of these two theories are different. The space $\mathcal{M}^{\text{mes}}_{\mathcal{C}}$ of the conifold theory is a Calabi-Yau 3-fold whose affine coordinates are given by a hypersurface $\{xy - wz = 0\} \subset \mathbb{C}^4$. The Hilbert series is given by

$$g^{\text{mes}}_1(t;\mathcal{C}) \quad = \quad \frac{1-t^2}{(1-t)^4} = \frac{1+t}{(1-t)^3}. \qquad (3.2.104)$$

On the other hand, according to (3.2.100), the space $\mathcal{M}^{\text{mes}}_{\mathscr{C}}$ of the 2+1 dimensional chessboard theory is simply $\mathbb{C}^4$.

3.3 Toric duality

In the this section, we will introduce a particularly fascinating aspect of $(2+1)$-dimensional theories, called *toric duality*. This situation happens whenever one singular Calabi-Yau manifold is the mesonic moduli space of vacua of two or more gauge theories (henceforth referred to as *toric phases* or *models*).

Toric dualities have been studied in detail in the setup of D3-branes at singularities [125, 126, 121, 127, 128, 129, 130, 131, 132, 133] and for such cases it has been proven that the phenomenon coincides with Seiberg duality [126].

For M2-branes, toric duality is still a developing topic, but there has been seem progress in this direction, *e.g.* connections between models have been mentioned in [123, 135] and a number of models have been classified and systematically studied in [124]. In the following, we will support the argument of the existence of toric duality for M2-brane theories by presenting explicit examples. In order to check the duality we will compare the moduli space of vacua of different models, with particular reference to the toric diagram and the Hilbert series (and the information it contains).

3.3.1 The second phase of $\mathbb{C}^4$

In the previous section, we have introduced and discussed the so-called ABJM model. This was the first example of 3-dimensional CS theories living on the world-volume of M2-branes in flat space. However, at least at the classical level, this model is not the only one to have $\mathbb{C}^4$ as moduli space. In fact, let us consider the following model (which we shall refer to as $\mathscr{D}_1\mathscr{H}_1$), made of gauge group, $U(N)_1 \times U(N)_2$, 2 bi-fundamental fields X_{12} and X_{21} as well as 2 adjoint fields transforming in one of the two gauge groups. Without loss of generality, we take this gauge group to be $U(N)_1$ and denote the adjoint fields by ϕ_1^1 and ϕ_1^2. The superpotential is given by

$$W = \mathrm{Tr}(X_{21}[\phi_1^1, \phi_1^2]X_{12}) \ . \qquad (3.3.105)$$

According to (3.2.69), we take the Chern–Simons levels to be $k_1 = -k_2 = 1$. The quiver diagram and tiling[11] are drawn in Figure 3.6.

The toric diagram. We demonstrate two methods of constructing the toric diagram.

[11]The tiling for this theory was introduced in [123].

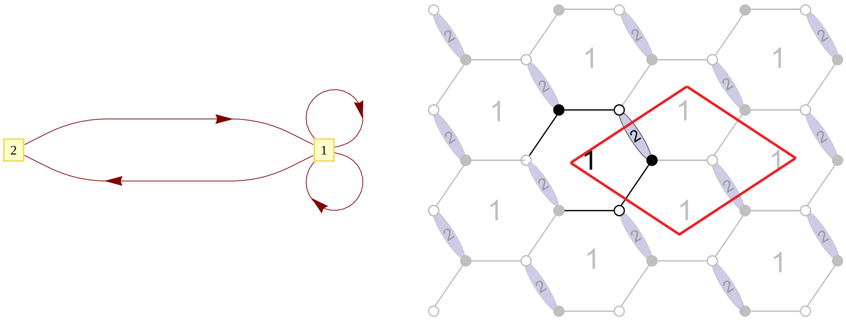


Figure 3.6: [Phase II of $\mathbb{C}^4$] (i) Quiver diagram for the $\mathscr{D}_1\mathscr{H}_1$ model. (ii) Tiling for the $\mathscr{D}_1\mathscr{H}_1$ model.

- **The Kasteleyn matrix.** We assign the integers n_i to the edges according to Figure 3.7. From (3.2.76), we find that

$$\begin{aligned} \text{Gauge group } 1: \quad k_1 &= 1 = -n_1 + n_2 \ , \\ \text{Gauge group } 2: \quad k_2 &= -1 = n_1 - n_2 \ . \end{aligned} \tag{3.3.106}$$

We choose

$$n_2 = 1, \quad n_1 = n_3 = n_4 = 0 \ . \tag{3.3.107}$$

We can now construct the Kasteleyn matrix for this model. Since the fundamental domain contains only one black node and one white node, the Kasteleyn matrix is a 1×1 matrix and, therefore, coincides with its permanent:

$$\begin{aligned} K &= \phi_1^1 z^{n_3} + \phi_1^2 y^{-1} z^{n_4} + X_{21} x z^{n_1} + X_{12} x z^{n_2} \\ &= \phi_1^1 + \phi_1^2 y^{-1} + X_{21} x + X_{12} x z \\ &\quad (\text{for } n_2 = 1, \ n_1 = n_3 = n_4 = 0) \ . \end{aligned} \tag{3.3.108}$$

The powers of x, y, z in each term of K give the coordinates of each point in the toric diagram. We collect these points in the columns of

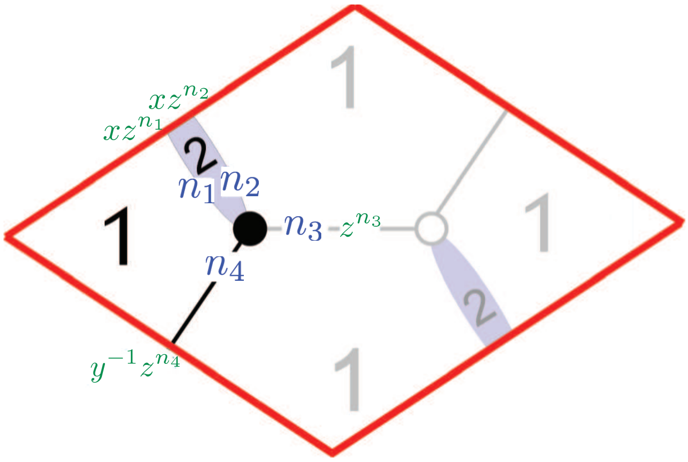


Figure 3.7: [Phase II of $\mathbb{C}^4$] The fundamental domain of tiling for the $\mathscr{D}_1\mathscr{H}_1$ model : Assignments of the integers n_i to the edges are shown in blue and the weights for these edges are shown in green.

the following G_K matrix:

$$G_K = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} . \qquad (3.3.109)$$

- **The charge matrices.** From (3.3.108), the perfect matchings can therefore be taken as

$$p_1 = X_{12}, \quad p_2 = \phi_1^2, \quad p_3 = X_{21}, \quad p_4 = \phi_1^1 . \qquad (3.3.110)$$

Since there is a one-to-one correspondence between the perfect matchings and the fields, $Q_F = 0$. Since the number of gauge groups is $G = 2$, there is $G - 2 = 0$ baryonic charge from the D-terms and hence $Q_D = 0$. Thus, we have $Q_t = 0$. Therefore, we have the same G'_t as in (3.2.99). The toric diagram is 4 corners of a tetrahedron as in Figure 3.4. Thus, we have shown that the toric diagram of phase II is indeed identical to that of phase I.

The moduli space. Since all four columns of the Q_t matrix are the same, the mesonic symmetry of this model is $SU(4)\times U(1)$. Note that this $U(1)$ is not the full R-symmetry, which is actually $Spin(8)$. However, since it assigns equal weight to all fields, it can be identified with the scaling dimension $1/2$. The four fields transform as the fundamental representation of the $SU(4)$. It follows that

$$\mathcal{M}^{\text{mes}}_{\mathscr{D}_1\mathscr{H}_1} = \mathcal{F}^{\flat}_{\mathscr{D}_1\mathscr{H}_1} = \mathbb{C}^4 \ , \tag{3.3.111}$$

with the Hilbert series given by (3.2.101). The plethystic logarithm, of course, coincides with that of the chessboard model and the generators are therefore

$$X_{12}, \quad X_{21}, \quad \phi_2^1, \quad \phi_2^2 \ . \tag{3.3.112}$$

Note that there is a one-to-one correspondence between the generators of this model and those of Phase I.

3.3.2 Another example: the phases of $Q^{1,1,1}/\mathbb{Z}_2$

We shall now introduce another interesting example of toric duality for theories on M2-branes, involving two different phases. These was generically introduced in [92, 123] as a modified $\mathbb{F}_0$ theory. In the following subsections, we examine these phases in detail, and compare their moduli spaces of vacua through their toric diagram and the information contained in their Hilbert series.

Phase I: The Four-Square Model This model (which we shall refer to as $\mathscr{S}_4$) has 4 gauge groups and bi-fundamental fields X^i_{12}, X^i_{23}, X^i_{34} and X^i_{41} (with $i = 1, 2$). The superpotential is given by

$$W = \epsilon_{ij}\epsilon_{pq} \operatorname{Tr}(X^i_{12}X^p_{23}X^j_{34}X^q_{41}) \ . \tag{3.3.113}$$

The quiver diagram and tiling are drawn in Figure 3.8. Note that in 3+1 dimensions, these correspond to Phase I of the $\mathbb{F}_0$ theory [133, 90, 88]. We choose the CS levels to be $k_1 = -k_2 = -k_3 = k_4 = 1$.

The Master space. A primary decomposition indicates that the Master space of this phase is a reducible variety and has 3 irreducible components

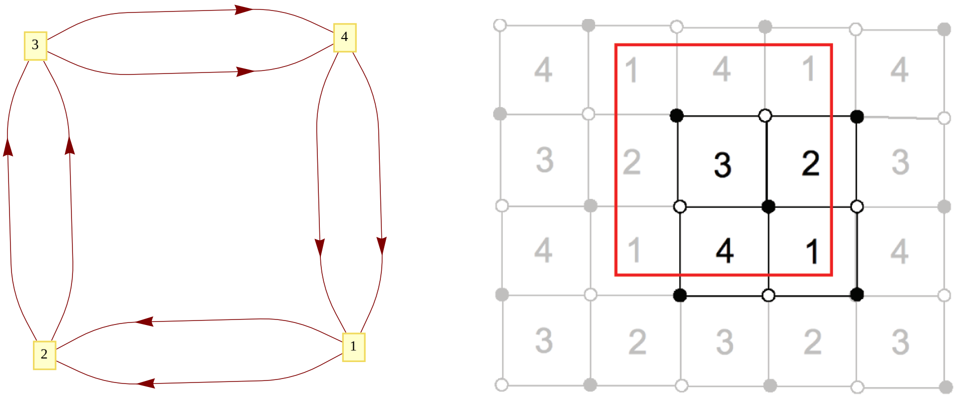


Figure 3.8: [Phase I of $Q^{1,1,1}/\mathbb{Z}_2$] (i) Quiver diagram for the $\mathscr{S}_4$ model. (ii) Tiling for the $\mathscr{S}_4$ model.

[90, 133]:

$$\mathcal{F}^\flat_{\mathscr{S}_4} = {}^{\text{Irr}}\mathcal{F}^\flat_{\mathscr{S}_4} \ \cup \ L^1_{\mathscr{S}_4} \ \cup \ L^2_{\mathscr{S}_4} \ , \tag{3.3.114}$$

where

$$\begin{aligned} {}^{\text{Irr}}\mathcal{F}^\flat_{\mathscr{S}_4} &= \mathbb{V}(X^1_{41}X^2_{23} - X^2_{41}X^1_{23}, X^1_{34}X^2_{12} - X^2_{34}X^1_{12}) \ , \\ L^1_{\mathscr{S}_4} &= \mathbb{V}(X^1_{23}, X^2_{23}, X^1_{41}, X^2_{41}) \ , \\ L^2_{\mathscr{S}_4} &= \mathbb{V}(X^1_{34}, X^2_{34}, X^1_{12}, X^2_{12}) \ . \end{aligned} \tag{3.3.115}$$

We see that the coherent component is the product of two conifolds:

$$ {}^{\text{Irr}}\mathcal{F}^\flat_{\mathscr{S}_4} = \mathcal{C} \times \mathcal{C} \ , \tag{3.3.116}$$

and the linear components are simply copies of $\mathbb{C}^4$:

$$L^i_{\mathscr{S}_4} = \mathbb{C}^4 \qquad (\text{for } i = 1, 2) \ . \tag{3.3.117}$$

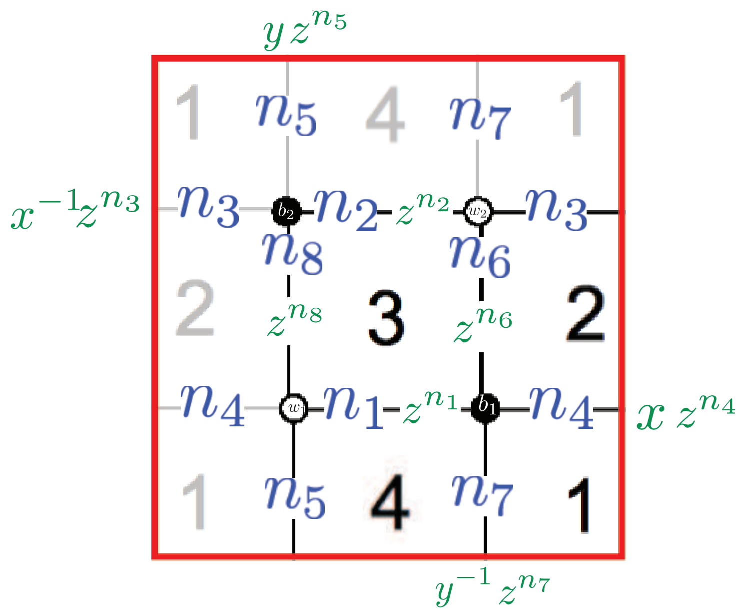


Figure 3.9: [Phase I of $Q^{1,1,1}/\mathbb{Z}_2$] The fundamental domain of tiling for the $\mathscr{S}_4$ model: Assignments of the integers n_i to the edges are shown in blue and the weights for these edges are shown in green.

The Kasteleyn matrix. We assign the integers n_i to the edges according to Figure 3.9. From (3.2.76), we find that

$$
\begin{aligned}
\text{Gauge group } 1: \quad & k_1 &=& \; 1 = n_3 + n_4 - n_5 - n_7 \;, \\
\text{Gauge group } 2: \quad & k_2 &=& \; -1 = n_6 + n_8 - n_3 - n_4 \;, \\
\text{Gauge group } 3: \quad & k_3 &=& \; -1 = n_1 + n_2 - n_6 - n_8 \;, \\
\text{Gauge group } 4: \quad & k_4 &=& \; 1 = -n_1 - n_2 + n_5 + n_7 \;.
\end{aligned}
\tag{3.3.118}
$$

We choose

$$
n_3 = -n_1 = 1, \quad n_i = 0 \text{ otherwise} \;. \tag{3.3.119}
$$

We can now construct the Kasteleyn matrix. The fundamental domain contains two black nodes and two white nodes and, therefore, the Kasteleyn matrix is a 2×2 matrix:

$$K = \left(\begin{array}{c|cc} & w_1 & w_2 \\ \hline b_1 & X_{34}^1 z^{n_1} + X_{12}^2 x z^{n_4} & X_{23}^1 z^{n_6} + X_{41}^2 y^{-1} z^{n_7} \\ b_2 & X_{23}^2 z^{n_8} + X_{41}^1 y z^{n_5} & X_{34}^2 z^{n_2} + X_{12}^1 x^{-1} z^{n_3} \end{array} \right) \ . \qquad (3.3.120)$$

The permanent of this matrix is given by

$$\begin{aligned} \text{perm } K &= X_{34}^1 X_{34}^2 z^{(n_1+n_2)} + X_{12}^1 X_{12}^2 z^{(n_3+n_4)} + X_{34}^1 X_{12}^1 x^{-1} z^{(n_1+n_3)} \\ &\quad + X_{34}^2 X_{12}^2 x z^{(n_2+n_4)} + X_{41}^1 X_{23}^1 y z^{(n_5+n_6)} + X_{41}^2 X_{23}^2 y^{-1} z^{(n_7+n_8)} \\ &\quad + X_{41}^1 X_{41}^2 z^{(n_5+n_7)} + X_{23}^1 X_{23}^2 z^{(n_6+n_8)} \\ &= X_{34}^1 X_{34}^2 z^{-1} + X_{12}^1 X_{12}^2 z + X_{34}^1 X_{12}^1 x^{-1} + X_{34}^2 X_{12}^2 x + X_{41}^1 X_{23}^1 y \\ &\quad + X_{41}^2 X_{23}^2 y^{-1} + X_{41}^1 X_{41}^2 + X_{23}^1 X_{23}^2 \\ &\quad (\text{for } n_3 = -n_1 = 1,\ n_i = 0 \text{ otherwise}) \ . \qquad (3.3.121) \end{aligned}$$

The perfect matchings. From (3.3.121), we write the perfect matchings as collections of fields as follows:

$$\begin{array}{l} p_1 = \{X_{34}^1, X_{34}^2\},\ \ p_2 = \{X_{12}^1, X_{12}^2\},\ \ q_1 = \{X_{34}^1, X_{12}^1\},\ \ q_2 = \{X_{34}^2, X_{12}^2\}, \\ r_1 = \{X_{41}^1, X_{23}^1\},\ \ r_2 = \{X_{41}^2, X_{23}^2\},\ \ s_1 = \{X_{41}^1, X_{41}^2\},\ \ s_2 = \{X_{23}^1, X_{23}^2\}. \end{array} \qquad (3.3.122)$$

From (3.3.121), we see that the perfect matchings p_i, q_i, r_i correspond to the external points in the toric diagram, whereas the perfect matchings s_i correspond to the internal point at the origin. In turn, we find the parameterisation of fields in terms of perfect matchings:

$$\begin{array}{llll} X_{34}^1 = p_1 q_1, & X_{34}^2 = p_1 q_2, & X_{12}^1 = p_2 q_1, & X_{12}^2 = p_2 q_2, \\ X_{41}^1 = r_1 s_1, & X_{23}^1 = r_1 s_2, & X_{41}^2 = r_2 s_1, & X_{23}^2 = r_2 s_2 \ . \end{array} \qquad (3.3.123)$$

This is summarised in the perfect matching matrix:

$$P = \left(\begin{array}{c|cccccccc} & p_1 & p_2 & q_1 & q_2 & r_1 & r_2 & s_1 & s_2 \\ \hline X_{34}^1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ X_{34}^2 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ X_{12}^1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ X_{12}^2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ X_{41}^1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ X_{23}^1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ X_{41}^2 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ X_{23}^2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right) . \qquad (3.3.124)$$

Basis vectors of the nullspace of P are given in the rows of the charge matrix:

$$Q_F = \begin{pmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix} . \qquad (3.3.125)$$

Hence, from (3.2.83), we see that the relations between the perfect matchings are given by

$$\begin{aligned} p_1 + p_2 - p_3 - p_4 &= 0 \ , \\ p_5 + p_6 - s_1 - s_2 &= 0 \ . \end{aligned} \qquad (3.3.126)$$

Since the coherent component ${}^{\mathrm{Irr}}\mathcal{F}^{\flat}_{\mathscr{S}_4}$ of the Master space is generated by the perfect matchings (subject to the relation (3.3.126)), it follows that

$${}^{\mathrm{Irr}}\mathcal{F}^{\flat}_{\mathscr{S}_4} = \mathbb{C}^8 // Q_F \ . \qquad (3.3.127)$$

The toric diagram. We demonstrate two methods of constructing the toric diagram.

- **The charge matrices.** Since the number of gauge groups is $G = 4$, there are $G - 2 = 2$ baryonic charges coming from the D-terms. We collect these charges of the perfect matchings in the Q_D matrix:

$$Q_D = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \end{pmatrix} . \qquad (3.3.128)$$

From (3.3.125) and (3.3.128), the total charge matrix is given by

$$Q_t = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{pmatrix} . \qquad (3.3.129)$$

We obtain the matrix G_t from (3.2.90), and after removing the first row, the columns give the coordinates of points in the toric diagram:

$$G'_t = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \qquad (3.3.130)$$

The toric diagram is drawn in Figure 3.10. Observe that there is an

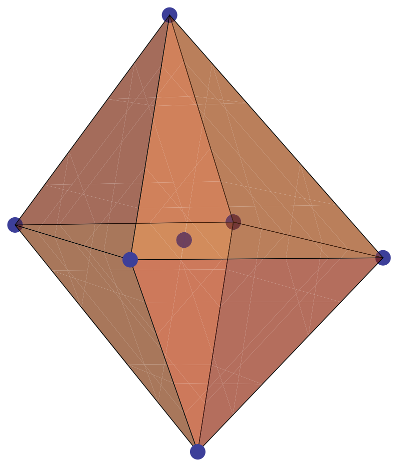


Figure 3.10: The toric diagram of the $Q^{1,1,1}/\mathbb{Z}_2$ theory.

internal point (with multiplicity 2) in the toric diagram for this theory, whereas the toric diagram for the $Q^{1,1,1}$ theory is simply 6 corners of an octahedron without an internal point (see Appendix A of [123]). Comparing Figure 3.10 with the 2d toric diagram of Phase I of $\mathbb{F}_0$ theory [90, 88], we see that the CS levels split two of the four points at the centre of the 2d toric diagram along the vertical axis into the two tips, and the rest remain at the centre of the octahedron.

- **The Kasteleyn matrix.** The powers of x, y, z in each term of (3.3.121) give the coordinates of each point in the toric diagram. We collect these points in the columns of the following G_K matrix:

$$G_K = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = G'_t \ . \qquad (3.3.131)$$

  Thus, the toric diagrams constructed from these two methods are indeed identical.

The baryonic charges. Since the toric diagram has 6 external points, this model has precisely $6 - 4 = 2$ baryonic charges which we shall denote by $U(1)_{B_1}, U(1)_{B_2}$. From the above discussion, we see that they arise from the D-terms. Therefore, the baryonic charges of the perfect matchings are given by the rows of the Q_D matrix.

The global symmetry. Since the Q_t matrix has 3 pairs of repeated columns, it follows that the mesonic symmetry of this model is $SU(2)^3 \times U(1)_R$. Since s_1 and s_2 are the perfect matchings corresponding to internal points in the toric diagram, we assign to each of them a zero R-charge. The remaining 6 external perfect matchings are completely symmetric and the requirement of R-charge 2 to the superpotential divides 2 equally among them, resulting in R-charge of 1/3 per each. The global symmetry of the theory is a product of mesonic and baryonic symmetries: $SU(2)^3 \times U(1)_R \times U(1)_{B_1} \times U(1)_{B_2}$. In Table 3.4, we present a consistent way of assigning charges to the perfect matchings under these global symmetries.

The Hilbert series. From (3.3.127), we compute the Hilbert series of the coherent component of the Master space by integrating the Hilbert series of

	$SU(2)_1$	$SU(2)_2$	$SU(2)_3$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	fugacity
p_1	1	0	0	1/3	1	0	tb_1x_1
p_2	-1	0	0	1/3	1	0	tb_1/x_1
q_1	0	1	0	1/3	0	0	tx_2
q_2	0	-1	0	1/3	0	0	t/x_2
r_1	0	0	1	1/3	-1	-1	$tx_3/(b_1b_2)$
r_2	0	0	-1	1/3	-1	-1	$t/(x_3b_1b_2)$
s_1	0	0	0	0	0	2	b_2^2
s_2	0	0	0	0	0	0	1
s_3	0	0	0	0	0	0	1

Table 3.4: Charges under the global symmetry of the $Q^{1,1,1}/\mathbb{Z}_2$ theory. Here t is the fugacity of R-charge, x_1, x_2, x_3 are weights of $SU(2)_1, SU(2)_2, SU(2)_3$, and b_1, b_2 are baryonic fugacities of $U(1)_{B_1}, U(1)_{B_2}$. Note that the perfect matching s_3 sdoes not exist in Phase I but exists in Phase II.

$\mathbb{C}^8$ over the fugacities z_1 and z_2 associated with the Q_F charges:

$$\begin{aligned}
& g_1^{\mathrm{Irr}\mathcal{F}^\flat}(t, x_1, x_2, x_3, b_1, b_2; \mathscr{S}_4) = \\
& \frac{1}{(2\pi i)^2} \oint\limits_{|z_1|=1} \frac{\mathrm{d}z_1}{z_1} \oint\limits_{|z_2|=1} \frac{\mathrm{d}z_2}{z_2} \frac{1}{(1 - tb_1z_1x_1)\left(1 - \frac{tb_1z_1}{x_1}\right)\left(1 - \frac{tx_2}{z_1}\right)} \times \\
& \times \frac{1}{\left(1 - \frac{t}{x_2z_1}\right)\left(1 - \frac{tx_3z_2}{b_1b_2}\right)\left(1 - \frac{tz_2}{b_1b_2x_3}\right)\left(1 - \frac{b_2^2}{z_2}\right)\left(1 - \frac{1}{z_2}\right)} \\
= \; & \frac{\left(1 - \frac{t^2}{b_1^2}\right)}{\left(1 - \frac{tb_2}{b_1x_3}\right)\left(1 - \frac{tb_2x_3}{b_1}\right)\left(1 - \frac{t}{b_1b_2x_3}\right)\left(1 - \frac{tx_3}{b_1b_2}\right)} \times \\
& \times \frac{\left(1 - t^4b_1^2\right)}{\left(1 - \frac{t^2b_1}{x_1x_2}\right)\left(1 - \frac{t^2b_1x_2}{x_1}\right)\left(1 - \frac{t^2b_1x_1}{x_2}\right)(1 - t^2b_1x_1x_2)} \; . \qquad (3.3.132)
\end{aligned}$$

The unrefined Hilbert series of the Master space can be written as:

$$g_1^{\mathrm{Irr}\mathcal{F}^\flat}(t, 1, 1, 1, 1, 1; \mathscr{S}_4) = \frac{1 - t^2}{(1-t)^4} \times \frac{1 - t^4}{(1-t^2)^4} \; . \qquad (3.3.133)$$

We see that this space is indeed the product of two conifolds. The Hilbert series of the mesonic moduli space can be obtained by integrating (3.3.132) over the two baryonic fugacities b_1 and b_2:

$$\begin{aligned} g_1^{\mathrm{mes}}(t,x_1,x_2,x_3;\mathscr{S}_4) &= \frac{1}{(2\pi i)^2}\oint\limits_{|b_1|=1}\frac{db_1}{b_1}\oint\limits_{|b_2|=1}\frac{db_2}{b_2}\, g_1^{\mathrm{Irr}\mathcal{F}^\flat}(t,x_1,x_2,x_3,b_1,b_2;\mathscr{S}_4) \\ &= \frac{P(t,x_1,x_2,x_3)}{\left(1-t^6x_1^2x_2^2x_3^2\right)\left(1-\frac{t^6x_1^2x_2^2}{x_3^2}\right)\left(1-\frac{t^6x_1^2x_3^2}{x_2^2}\right)\left(1-\frac{t^6x_2^2x_3^2}{x_1^2}\right)}\times \\ &\quad\times\frac{1}{\left(1-\frac{t^6x_1^2}{x_2^2x_3^2}\right)\left(1-\frac{t^6x_2^2}{x_1^2x_3^2}\right)\left(1-\frac{t^6x_3^2}{x_1^2x_2^2}\right)\left(1-\frac{t^6}{x_1^2x_2^2x_3^2}\right)} \\ &= \sum_{n=0}^{\infty}[2n;2n;2n]t^{6n}\ . \end{aligned} \tag{3.3.134}$$

where $P(t,x_1,x_2,x_3)$ is a polynomial of degree 42 in t which is too long to present here. The unrefined Hilbert series of the mesonic moduli space can be written as:

$$g_1^{\mathrm{mes}}(t,1,1,1;\mathscr{S}_4) = \frac{1+23t^6+23t^{12}+t^{18}}{(1-t^6)^4}\ . \tag{3.3.135}$$

This indicates that the mesonic moduli space is a Calabi–Yau 4-fold, as expected. The plethystic logarithm of the mesonic Hilbert series is given by

$$\begin{aligned} \mathrm{PL}[g_1^{\mathrm{mes}}(t,x_1,x_2,x_3;\mathscr{S}_4)] &= [2;2;2]t^6-([4;4;0]+[4;0;4]+[0;4;4]+[4;0;0]+[0;4;0]+ \\ &\quad +[0;0;4]+[4;2;2]+[2;4;2]+[2;2;4]+[2;2;0]+[2;0;2]+ \\ &\quad +[0;2;2]+1)t^{12}+O(t^{18})\ . \end{aligned} \tag{3.3.136}$$

The generators. Each of the generators can be written as a product of the perfect matchings:

$$p_i\ p_j\ q_k\ q_l\ r_m\ r_n\ s_1\ s_2\ , \tag{3.3.137}$$

where the indices i,j,k,l,m,n run from 1 to 2. Since, for example, p_ip_j has 3 independent components p_1p_1, p_1p_2, p_2p_2, it follows that there are indeed 27 independent generators. We can represent the generators in a lattice (Figure 3.11) by plotting the powers of the weights of the characters in (3.3.136). Note that the lattice of generators is the dual of the toric diagram (nodes are dual to faces and edges are dual to edges): The toric diagram has 6 nodes (external points), 12 edges and 8 faces, whereas the generators form a convex polytope that has 8 nodes (corners of the cube), 12 edges and 6 faces.

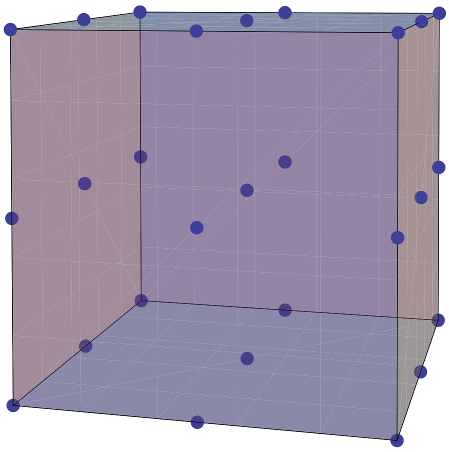


Figure 3.11: The lattice of generators of the $Q^{1,1,1}/\mathbb{Z}_2$ theory.

The $\mathbb{Z}_2$ orbifold action. It is interesting to compare the last equality of (3.3.134) to the Hilbert series of the $Q^{1,1,1}$ theory, which is given by (A.7) of [123]:

$$g_1^{\mathrm{mes}}(t, x_1, x_2, x_3; Q^{1,1,1}) \;=\; \sum_{n=0}^{\infty} [n; n; n] t^{3n} \; . \qquad (3.3.138)$$

This indicates that the $\mathscr{S}_4$ model is indeed the orbifold $Q^{1,1,1}/\mathbb{Z}_2$. The reason is as follows. As discussed in [91], under the $\mathbb{Z}_2$ orbifold action, $t \to -t$ and we need to sum over both sectors, with t and with $-t$. Therefore, starting from (3.3.138) and applying the $\mathbb{Z}_2$ action, we are left with the terms correponding to even j and hence (3.3.134).

Phase II: The Two-Square and Two-Octagon Model This model, first studied in [123], (which we shall denote as $\mathscr{S}_2\mathscr{O}_2$) has four gauge groups and bi-fundamental fields X_{12}^{ij}, X_{23}^{i}, $X_{23'}^{i}$, X_{31}^{i} and $X_{3'1}^{i}$ (with $i, j = 1, 2$). From the features of this quiver gauge theory, this phase is also known as a *three-block model* (see for example [66]). The superpotential is given by

$$W \;=\; \epsilon_{ij}\epsilon_{kl}\,\mathrm{Tr}(X_{12}^{ik} X_{23}^{l} X_{31}^{j}) - \epsilon_{ij}\epsilon_{kl}\,\mathrm{Tr}(X_{12}^{ki} X_{23'}^{l} X_{3'1}^{j}) \; . \qquad (3.3.139)$$

The quiver diagram and tiling of this phase of the theory are given in Figure 3.12. Note that in 3+1 dimensions, these quiver and tiling correspond to Phase II of the $\mathbb{F}_0$ theory [133, 90]. We choose the CS levels to be $k_1 = k_2 = -k_3 = -k_{3'} = 1$.

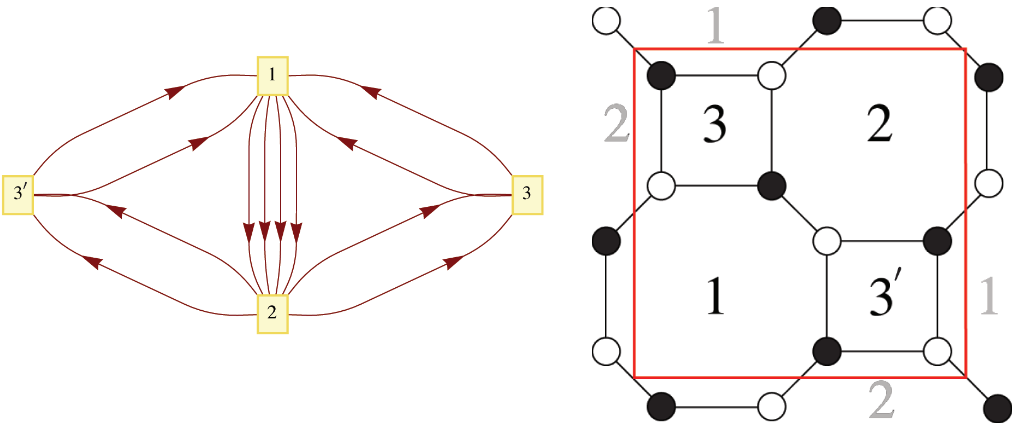


Figure 3.12: [Phase II of $Q^{1,1,1}/\mathbb{Z}_2$] (i) Quiver diagram for the $\mathscr{S}_2\mathscr{O}_2$ model. (ii) Tiling for the $\mathscr{S}_2\mathscr{O}_2$ model.

The Master space. A primary decomposition indicates that the Master space of this phase is a reducible variety and has 4 irreducible components [133, 90]:

$$\mathcal{F}^{\flat}_{\mathscr{S}_2\mathscr{O}_2} = {}^{\mathrm{Irr}}\mathcal{F}^{\flat}_{\mathscr{S}_2\mathscr{O}_2} \ \cup \ L^1_{\mathscr{S}_2\mathscr{O}_2} \ \cup \ L^2_{\mathscr{S}_2\mathscr{O}_2} \ \cup \ L^3_{\mathscr{S}_2\mathscr{O}_2} \ , \tag{3.3.140}$$

where

$$\begin{aligned}
{}^{\mathrm{Irr}}\mathcal{F}^{\flat}_{\mathscr{S}_1\mathscr{O}_1} \ &= \ \mathbb{V}(X^{12}_{12}X^{1}_{3'1} - X^{11}_{12}X^{2}_{3'1}, X^{2}_{31}X^{1}_{3'1} - X^{1}_{31}X^{2}_{3'1}, X^{22}_{12}X^{1}_{3'1} - X^{21}_{12}X^{2}_{3'1}, \\
&\quad X^{2}_{23}X^{1}_{23'} - X^{1}_{23}X^{2}_{23'}, X^{21}_{12}X^{1}_{23'} - X^{22}_{12}X^{2}_{23'}, X^{22}_{12}X^{1}_{23'} - X^{12}_{12}X^{2}_{23'}, \\
&\quad X^{21}_{12}X^{12}_{12} - X^{22}_{12}X^{22}_{12}, X^{1}_{31}X^{12}_{12} - X^{2}_{31}X^{11}_{12}, X^{2}_{31}X^{2}_{23} - X^{2}_{3'1}X^{2}_{23'}, \\
&\quad X^{1}_{31}X^{2}_{23} - X^{1}_{3'1}X^{2}_{23'}, X^{1}_{23}X^{21}_{12} - X^{2}_{23}X^{11}_{12}, X^{1}_{23}X^{2}_{31} - X^{1}_{23'}X^{2}_{3'1}, \\
&\quad X^{1}_{31}X^{22}_{12} - X^{2}_{31}X^{21}_{12}, X^{1}_{23}X^{22}_{12} - X^{2}_{23}X^{12}_{12}, X^{1}_{23}X^{1}_{31} - X^{1}_{23'}X^{1}_{3'1}) \ , \\
L^1_{\mathscr{S}_1\mathscr{O}_1} \ &= \ \mathbb{V}(X^{2}_{23'}, X^{2}_{3'1}, X^{1}_{3'1}, X^{1}_{23'}, X^{2}_{23}, X^{2}_{31}, X^{1}_{31}, X^{1}_{23}) \ , \\
L^2_{\mathscr{S}_1\mathscr{O}_1} \ &= \ \mathbb{V}(X^{2}_{3'1}, X^{1}_{3'1}, X^{11}_{12}, X^{12}_{12}, X^{21}_{12}, X^{2}_{31}, X^{22}_{12}, X^{1}_{31}) \ , \\
L^3_{\mathscr{S}_1\mathscr{O}_1} \ &= \ \mathbb{V}(X^{2}_{23'}, X^{1}_{23'}, X^{11}_{12}, X^{12}_{12}, X^{2}_{23}, X^{21}_{12}, X^{22}_{12}, X^{1}_{23}) \ .
\end{aligned} \tag{3.3.141}$$

We see that the linear components are simply copies of $\mathbb{C}^4$:

$$L^i_{\mathscr{S}_2\mathscr{O}_2} = \mathbb{C}^4 \qquad (\text{for } i = 1,2,3) \ . \tag{3.3.142}$$

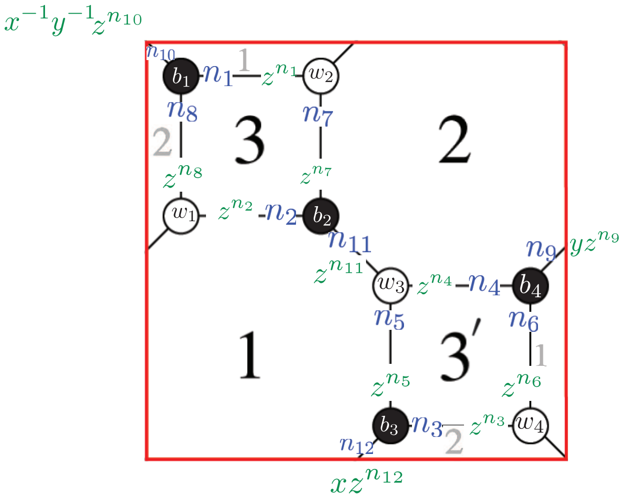


Figure 3.13: [Phase II of $Q^{1,1,1}/\mathbb{Z}_2$] The fundamental domain of tiling for the $\mathscr{S}_2\mathscr{O}_2$ model: Assignments of the integers n_i to the edges are shown in blue and the weights for these edges are shown in green.

The Kasteleyn matrix. We assign the integers n_i to the edges according to Figure 3.13. From (3.2.76), we find that

$$\begin{aligned}
\text{Gauge group 1}: \quad k_1 &= 1 = -n_1 - n_2 - n_5 - n_6 + n_9 + n_{10} + n_{11} + n_{12} \ , \\
\text{Gauge group 2}: \quad k_2 &= 1 = n_3 + n_4 + n_7 + n_8 - n_9 - n_{10} - n_{11} - n_{12} \ , \\
\text{Gauge group 3}: \quad k_3 &= -1 = n_1 + n_2 - n_7 - n_8 \ , \\
\text{Gauge group 4}: \quad k_{3'} &= -1 = -n_3 - n_4 + n_5 + n_6 \ .
\end{aligned} \tag{3.3.143}$$

We choose

$$n_2 = -1, \quad n_4 = 1, \quad n_i = 0 \text{ otherwise} \ . \tag{3.3.144}$$

We can now determine the Kasteleyn matrix. Since the fundamental domain contains 4 black nodes and 4 white nodes, the Kasteleyn matrix is a 4×4

matrix:

$$K = \left(\begin{array}{c|cccc} & w_1 & w_2 & w_3 & w_4 \\ \hline b_1 & X_{23}^2 z^{n_8} & X_{31}^1 z^{n_1} & 0 & X_{12}^{21} x^{-1} y^{-1} z^{n_{10}} \\ b_2 & X_{31}^2 z^{n_2} & X_{23}^1 z^{n_7} & X_{12}^{12} z^{n_{11}} & 0 \\ b_3 & 0 & X_{12}^{22} x z^{n_{12}} & X_{3'1}^1 z^{n_5} & X_{23'}^1 z^{n_3} \\ b_4 & X_{12}^{11} y z^{n_9} & 0 & X_{23'}^2 z^{n_4} & X_{3'1}^2 z^{n_6} \end{array}\right) . \tag{3.3.145}$$

The permanent of this matrix is given by

$$\begin{aligned} \text{perm } K &= X_{31}^1 X_{31}^2 X_{3'1}^1 X_{3'1}^2 z^{(n_1+n_2+n_5+n_6)} + X_{23'}^1 X_{23'}^2 X_{23}^2 X_{23}^1 z^{(n_3+n_4+n_7+n_8)} \\ &\quad + X_{3'1}^1 X_{23}^1 X_{12}^{11} X_{12}^{21} x^{-1} z^{(n_5+n_7+n_9+n_{10})} + X_{3'1}^2 X_{23}^2 X_{12}^{12} X_{12}^{22} x z^{(n_{11}+n_{12}+n_6+n_8)} \\ &\quad + X_{31}^1 X_{23'}^1 X_{12}^{11} X_{12}^{12} y z^{(n_1+n_3+n_9+n_{11})} + X_{31}^2 X_{23'}^2 X_{12}^{21} X_{12}^{22} y^{-1} z^{(n_2+n_4+n_{10}+n_{12})} \\ &\quad + X_{31}^1 X_{31}^2 X_{23'}^1 X_{23'}^2 z^{(n_1+n_2+n_3+n_4)} + X_{3'1}^1 X_{3'1}^2 X_{23}^2 X_{23}^1 z^{(n_5+n_6+n_7+n_8)} \\ &\quad + X_{12}^{11} X_{12}^{21} X_{12}^{12} X_{12}^{22} z^{(n_9+n_{10}+n_{11}+n_{12})} \\ &= X_{31}^1 X_{31}^2 X_{3'1}^1 X_{3'1}^2 z^{-1} + X_{23'}^1 X_{23'}^2 X_{23}^2 X_{23}^1 z + X_{3'1}^1 X_{23}^1 X_{12}^{11} X_{12}^{21} x^{-1} \\ &\quad + X_{3'1}^2 X_{23}^2 X_{12}^{12} X_{12}^{22} x + X_{31}^1 X_{23'}^1 X_{12}^{11} X_{12}^{12} y + X_{31}^2 X_{23'}^2 X_{12}^{21} X_{12}^{22} y^{-1} \\ &\quad + X_{31}^1 X_{31}^2 X_{23'}^1 X_{23'}^2 + X_{3'1}^1 X_{3'1}^2 X_{23}^2 X_{23}^1 + X_{12}^{11} X_{12}^{21} X_{12}^{12} X_{12}^{22} \\ &\quad (\text{for } n_2 = -1,\ n_4 = 1,\ n_i = 0 \text{ otherwise}) \ . \end{aligned} \tag{3.3.146}$$

The perfect matchings. We summarise the correspondence between the quiver fields and the perfect matchings in the P matrix as follows:

$$P = \left(\begin{array}{c|ccccccccc} & p_1 & p_2 & q_1 & q_2 & r_1 & r_2 & s_1 & s_2 & s_3 \\ \hline X_{31}^{1} & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ X_{31}^{2} & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ X_{23'}^{1} & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ X_{23'}^{2} & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ X_{3'1}^{1} & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ X_{3'1}^{2} & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ X_{23}^{1} & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ X_{23}^{2} & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ X_{12}^{11} & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ X_{12}^{21} & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ X_{12}^{12} & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ X_{12}^{22} & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \end{array} \right) . \qquad (3.3.147)$$

From (3.3.146), we see that the perfect matchings p_i, q_i, r_i correspond to the external points in the toric diagram, whereas the perfect matchings s_i correspond to the internal point at the origin. Basis vectors of the null space of P are given in the rows of the charge matrix:

$$Q_F = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{pmatrix} . \qquad (3.3.148)$$

Hence, from (3.2.83), we see that the relations between the perfect matchings are given by

$$\begin{aligned} p_1 + p_2 - s_1 - s_2 &= 0 , \\ q_1 + q_2 - s_2 - s_3 &= 0 , \\ r_1 + r_2 - s_1 - s_3 &= 0 . \end{aligned} \qquad (3.3.149)$$

Since the coherent component of the Master space is generated by the perfect matchings (subject to the relations (3.3.149)), it follows that

$$^{\mathrm{Irr}}\mathcal{F}^{\flat}_{\mathscr{S}_2\mathscr{O}_2} = \mathbb{C}^9 // Q_F . \qquad (3.3.150)$$

The toric diagram. We demonstrate two methods of constructing the toric diagram.

- **The charge matrices.** Since the number of gauge groups is $G = 4$, there are $G - 2 = 2$ baryonic charges coming from the D-terms. We collect these charges of the perfect matchings in the Q_D matrix:

$$Q_D = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \end{pmatrix} . \qquad (3.3.151)$$

  From (3.3.148) and (3.3.151), the total charge matrix is given by

$$Q_t = \begin{pmatrix} 1 & 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 & -1 \end{pmatrix} . \qquad (3.3.152)$$

  We obtain the matrix G_t from (3.2.90), and after removing the first row, the columns give the coordinates of points in the toric diagram:

$$G'_t = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \qquad (3.3.153)$$

  We see that the toric diagram is given by Figure 3.10, with three degenerate internal points at the centre. Comparing Figure 3.10 with the 2d toric diagram of Phase II of $\mathbb{F}_0$ theory [90, 88], we see that the CS levels split two of the five points at the centre of the 2d toric diagram along the vertical axis into the two tips, and the rest remain at the centre of the octahedron.

- **The Kasteleyn matrix.** The powers of x, y, z in each term of the permanent of the Kasteleyn matrix give the coordinates of each point in the toric diagram. We collect these points in the columns of the following G_K matrix:

$$G_K = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = G'_t \ . \quad (3.3.154)$$

Thus, the toric diagrams constructed from these two methods are indeed identical.

The baryonic charges. Since the toric diagram has 6 external points, this model has precisely $6 - 4 = 2$ baryonic charges which we shall denote by $U(1)_{B_1}, U(1)_{B_2}$. From the above discussion, we see that they arise from the D-terms. Therefore, the baryonic charges of the perfect matchings are given by the rows of the Q_D matrix.

The global symmetry. From the Q_t matrix, the charge assignment breaks the symmetry of the space of perfect matchings to $SU(2)^3 \times U(1)_R$. Since s_1, s_2, s_3 are the perfect matchings corresponding to internal points in the toric diagram, we assign to each of them a zero R-charge. The remaining 6 external perfect matchings are completely symmetric and the requirement of R-charge 2 to the superpotential divides 2 equally among them, resulting in R-charge of 1/3 per each. The global symmetry of the theory is a product of mesonic and baryonic symmetries: $SU(2)^3 \times U(1)_R \times U(1)_{B_1} \times U(1)_{B_2}$. In Table 3.4, we give a consistent charge assignment for the perfect matchings under the global symmetries.

The Hilbert series. From (3.3.150), we compute the Hilbert series of the coherent component of the Master space by integrating the Hilbert series of

$\mathbb{C}^9$ over the fugacities z_1, z_2, z_3 associated with the Q_F charges:

$$
\begin{aligned}
&g_1^{{}^{\mathrm{Irr}}\mathcal{F}^\flat}(t, x_1, x_2, x_3, b_1, b_2; \mathscr{S}_2\mathscr{O}_2) = \\
&\frac{1}{(2\pi i)^3} \oint\limits_{|z_1|=1} \frac{dz_1}{z_1} \oint\limits_{|z_2|=1} \frac{dz_2}{z_2} \oint\limits_{|z_3|=1} \frac{dz_3}{z_3} \frac{1}{(1-tb_1z_1x_1)\left(1-\frac{tb_1z_1}{x_1}\right)} \times \\
&\times \frac{1}{(1-tx_2z_2)\left(1-\frac{tz_2}{x_2}\right)\left(1-\frac{tx_3z_3}{b_1b_2}\right)\left(1-\frac{tz_3}{x_3b_1b_2}\right)\left(1-\frac{b_2^2}{z_1z_3}\right)} \times \\
&\times \frac{1}{\left(1-\frac{1}{z_1z_2}\right)\left(1-\frac{1}{z_2z_3}\right)} \ .
\end{aligned}
\tag{3.3.155}
$$

The unrefined Hilbert series of the Master space can be written as:

$$
g_1^{{}^{\mathrm{Irr}}\mathcal{F}^\flat}(t, 1, 1, 1, 1, 1; \mathscr{S}_2\mathscr{O}_2) = \frac{1+6t^2+6t^4+t^6}{(1-t^2)^6} \ . \tag{3.3.156}
$$

Integrating the Hilbert series of the Master space over the baryonic fugacities gives the Hilbert series of the mesonic moduli space:

$$
\begin{aligned}
g_1^{\mathrm{mes}}(t, x_1, x_2, x_3; \mathscr{S}_2\mathscr{O}_2) &= \frac{1}{(2\pi i)^2} \oint\limits_{|b_1|=1} \frac{db_1}{b_1} \oint\limits_{|b_2|=1} \frac{db_2}{b_2} g_1^{{}^{\mathrm{Irr}}\mathcal{F}^\flat}(t, x_1, x_2, x_3, b_1, b_2; \mathscr{S}_2\mathscr{O}_2) \\
&= \frac{P(t, x_1, x_2, x_3)}{(1-t^6x_1^2x_2^2x_3^2)\left(1-\frac{t^6x_1^2x_2^2}{x_3^2}\right)\left(1-\frac{t^6x_1^2x_3^2}{x_2^2}\right)\left(1-\frac{t^6x_2^2x_3^2}{x_1^2}\right)} \times \\
&\quad \times \frac{1}{\left(1-\frac{t^6x_1^2}{x_2^2x_3^2}\right)\left(1-\frac{t^6x_2^2}{x_1^2x_3^2}\right)\left(1-\frac{t^6x_3^2}{x_1^2x_2^2}\right)\left(1-\frac{t^6}{x_1^2x_2^2x_3^2}\right)} \\
&= \sum_{j=0}^{\infty} [2j; 2j; 2j] t^{6j} \ .
\end{aligned}
\tag{3.3.157}
$$

where $P(t, x_1, x_2, x_3)$ is a polynomial of order 42 in t mentioned in (3.3.134). This precisely identical to the Hilbert series (3.3.134) of the mesonic moduli space of Phase I .

The generators. Each of the generators can be written as a product of the perfect matchings:

$$
p_i \ p_j \ q_k \ q_l \ r_m \ r_n \ s_1 \ s_2 \ s_3 \ , \tag{3.3.158}
$$

where the indices i, j, k, l, m, n run from 1 to 2. Since, for example, $p_i p_j$ has 3 independent components $p_1 p_1$, $p_1 p_2$, $p_2 p_2$, it follows that there are indeed 27 independent generators. Note that the generators of this model are identical to those of Phase I, apart from a factor of the internal perfect matching s_3.

Discussion. The toric diagram and the Hilbert series (3.3.157) confirms that the mesonic moduli space of this model is indeed $Q^{1,1,1}/\mathbb{Z}_2$. However, from (3.3.133) and (3.3.156), we see that the Master spaces of the two phases are different. Since the mesonic and baryonic symmetries of the two phases are identical, it remains an open question why the Master spaces, which are expected to be the combined baryonic and mesonic moduli space, of the two phases are different. This situation was also encountered in [133], where two phases of the $\mathbb{F}_0$ were studied. There, it was found that the Hilbert series of the two phases are different unless the fugacities associated with the anomalous charges are set to 1.

3.4 The Higgs mechanism

One of the great advantages in using brane tilings to investigate supersymmetric theories, is that the Higgs mechanism for such theories is very easily studied. Interestingly, this mechanism allows for connections between different CS theories to be established as we shall see below.

The Higgs mechanism is realised by giving a vacuum expectation value (VEV) to one of the chiral matter fields in a certain M2-brane theory, call it X_{ab}, transforming in the bi-fundamental of $U(N)_a \times U(N)_b$. Flowing to an energy scale which is much below that introduced by the VEV, a new field theory is obtained where the two gauge groups under which the field transforms are broken to their diagonal subgroup and the X_{ab} is integrated out. From the point of view of the brane tiling, this simply amounts to deleting the edge corresponding to X_{ab} and merging the two faces that correspond to $U(N)_a$ and $U(N)_b$. Also, the CS levels of the higgsed gauge groups are added.

In what follows, we shall present the case study of all possible higgsings of the phases of $Q^{1,1,1}/\mathbb{Z}_2$ which we introduced in the previous section. Considering a specific example will allow us to appreciate the effects of the

Higgs mechanism on the toric diagram. Explicitly stated, we shall see that higgsing a field in the tiling amounts to removing one or more points in the toric diagram. This phenomenon is known as partial resolution and is very well-documented in the case of 4-dimensional toric gauge theories [67, 68, 125, 121, 126, 136, 137].

3.4.1 Higgsing Phase I of $Q^{1,1,1}/\mathbb{Z}_2$

Below, we shall discuss the Higgs mechanism on the first phase of $Q^{1,1,1}/\mathbb{Z}_2$ theory that we introduced in the preceding section. The origin of the names given to the theories obtained by higgsing can be found in [60].

Phase II of $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}^2$ **from giving a VEV to one of** X^i_{12}**,** X^i_{34}
By symmetry, turning on a VEV to one of the X^i_{12}, X^i_{34} fields yields the same result. For definiteness, let us give a VEV to X^1_{34}. This amounts to removing one of the edges that separate the squares corresponding to gauge groups 3 and 4. As a result, these gauge groups are combined into one gauge group, identified as 3. The quiver diagram and tiling of this model are presented in Figure 3.14. The superpotential is given by

$$W = \epsilon_{ij} \, \mathrm{Tr}(X^1_{12} X^i_{23} \phi_3 X^j_{31}) - \epsilon_{kl} \, \mathrm{Tr}(X^2_{12} X^k_{23} X^l_{31}) \ . \tag{3.4.159}$$

The CS levels associated with the higgsed gauge groups are added, and so the new CS levels are

$$k_1 = 1, \quad k_2 = -1, \quad k_3 = 0 \ . \tag{3.4.160}$$

We note that this model does not give rise to a consistent tiling in 3+1 dimensions and in fact is the simplest inconsistent model in the sense of [69]. It looks similar to the SPP theory but differs from it by being chiral, as opposed to the SPP quiver which is non-chiral.

The toric diagram is presented in Figure 3.15. We see that this is the toric diagram of a $\mathbb{Z}_2$ orbifold of $\mathbb{C}^4$. The discrete symmetry $\mathbb{Z}_2$ acts only on the perfect matchings r_1, r_2 (but not on p_1, p_2) and, as a result of this action, we gain a point on one of the edges (with multiplicity 2) corresponding to the perfect matchings s_1, s_2. Thus, the mesonic moduli space of this model

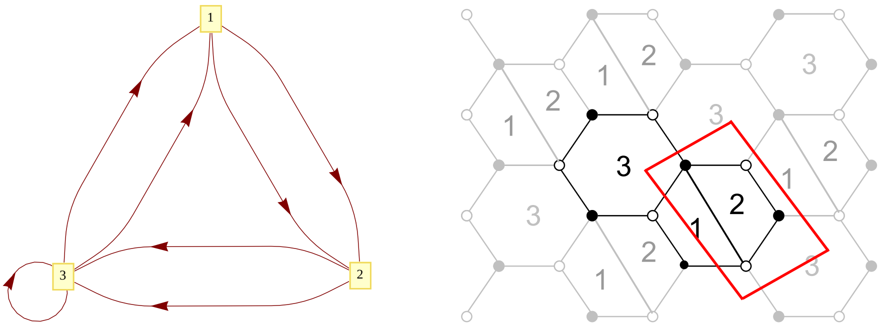


Figure 3.14: (i) Quiver diagram of Phase II of $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$. (ii) Tiling of Phase II of $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$.

is

$$\mathcal{M}^{\text{mes}} = \left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2 \ , \tag{3.4.161}$$

where the first $\mathbb{C}^2$ is parametrised by the perfect matchings r_1, r_2, and the second $\mathbb{C}^2$ is parametrised by the perfect matchings p_1, p_2. We refer to this model as **Phase II of** $(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}^2$.

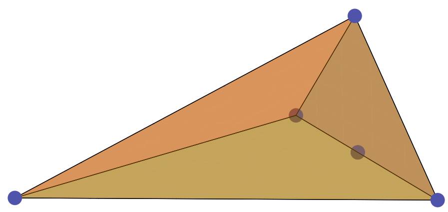


Figure 3.15: The toric diagram of the $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$ theory.

The global and baryonic charges of the perfect matchings are summarised in Table 3.5:

According to the charge assignment presented above, the mesonic Hilbert

	$SU(2)_1$	$SU(2)_2$	$U(1)_R$	$U(1)_q$	$U(1)_B$	fugacity
p_1	1	0	1/2	1	0	tx_1q
p_2	-1	0	1/2	1	0	$tq/(x_1)$
r_1	0	1	1/2	-1	0	tx_2/q
r_2	0	-1	1/2	-1	0	$t/(qx_2)$
s_1	0	0	0	0	-1	$1/b$
s_2	0	0	0	0	1	b

Table 3.5: Charges of the perfect matchings under the global symmetry of the $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$ theory. Here t is the fugacity of the R-charge (in the unit of 1/2), x_1, x_2 are the fugacities of the $SU(2)$ charge, q is the fugacity of the $U(1)$ symmetry and b is the fugacity of the $U(1)_B$ symmetry.

series can be written as:

$$g^{\text{mes}}(t, x_1, x_2, q) \quad = \quad \frac{1 + \frac{t^2}{q^2}}{\left(1 - \frac{tq}{x_1}\right)(1 - tqx_1)\left(1 - \frac{t^2}{q^2 x_2^2}\right)\left(1 - \frac{t^2 x_2^2}{q^2}\right)} \; . \tag{3.4.162}$$

The totally unrefined Hilbert series of the mesonic moduli space can be written as

$$g^{\text{mes}}(t, 1, 1, 1) = \frac{1 + t^2}{(1 - t)^2 (1 - t^2)^2} = \frac{1 + t^2}{(1 - t)^4 (1 + t)^2} \; . \tag{3.4.163}$$

The plethystic logarithm of (3.4.162) is given by

$$\text{PL}[g^{\text{mes}}(t, x_1, x_2, q)] \quad = \quad [1; 0]tq + [0; 2]\frac{t^2}{q^2} - \frac{t^4}{q^4} \; . \tag{3.4.164}$$

Therefore, the 5 generators of the mesonic moduli space can be written in terms of perfect matchings as

$$p_i, \quad r_i r_j s_1 s_2 \; , \tag{3.4.165}$$

where $i, j = 1, 2$. These generators can be represented in a lattice (Figure 3.16) by plotting the powers of each monomial in the characters of $SU(2) \times SU(2)$ and $U(1)_q$ in (3.4.164). Note that the lattice of generators is dual to the toric diagram (nodes are dual to faces and edges are dual to edges). For the $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$ theory, the lattice of generators is self-dual.

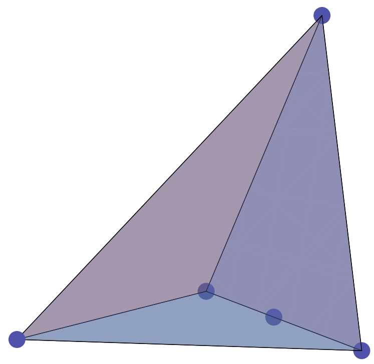


Figure 3.16: The lattice of generators of the $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$ theory.

***The* $\mathbb{F}_0 \times \mathbb{C}$ *theory* from giving a VEV to one of X^i_{23}, X^i_{41}** By symmetry, turning on a VEV to either X^i_{23} or X^i_{41} leads to the same theory. For definiteness, let us give a VEV to X^1_{23}. This means that we remove the edge that separates the gauge groups 2 and 3, and merge them into one gauge group, which is identified as 3. Let us relabel the gauge groups such that 4 becomes 2. Then, the quiver diagram and tiling of the resulting theory are given in Figure 3.14. The superpotential coincides with (3.4.159). The CS levels associated with the higgsed gauge groups are added, and so

$$k_1 = 1, \quad k_2 = 1, \quad k_3 = -2 \ . \tag{3.4.166}$$

The toric diagram is presented in Figure 3.17. We note that the perfect matching s corresponds to the internal point on the base, and the others correspond to external points at the corners. Therefore, the mesonic moduli space of this model is

$$\mathcal{M}^{\text{mes}} = \mathbb{F}_0 \times \mathbb{C} \ , \tag{3.4.167}$$

where $\mathbb{F}_0$, which is a $\mathbb{Z}_2$ orbifold of the conifold[12], is parametrised by

[12]Note that there is another $\mathbb{Z}_2$ orbifold of the conifold which is known as L^{222}. The toric diagram is drawn in Figure 4a of [138]. The Hilbert series of L^{222} is given by $\frac{1}{2}\left(\frac{1-t^2}{(1-t)^4} + \frac{1-t^2}{(1-t)^2(1+t)^2}\right) = \frac{1-t^4}{(1-t)^2(1-t^2)^2}$.

p_1, p_2, q_1, q_2, s (base of the pyramid in Figure 3.17), and $\mathbb{C}$ is parametrised by r (tip of the pyramid in Figure 3.17).

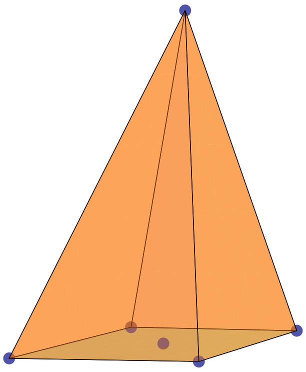


Figure 3.17: The toric diagram of the $\mathbb{F}_0 \times \mathbb{C}$ theory.

Table 3.5 collects all the global and baryonic charges of the perfect matchings.

	$SU(2)_1$	$SU(2)_2$	$U(1)_R$	$U(1)_q$	$U(1)_B$	fugacity
p_1	1	0	3/8	1	0	$t^3 x_1 q$
p_2	-1	0	3/8	1	0	$t^3 q/x_1$
q_1	0	1	3/8	1	-1	$t^3 x_2 q/b$
q_2	0	-1	3/8	1	-1	$t^3 q/(x_2 b)$
r	0	0	1/2	-4	0	t^4/q^4
s	0	0	0	0	2	b^2

Table 3.6: Charges of the perfect matchings under the global symmetry of the $\mathbb{F}_0 \times \mathbb{C}$ theory. Here t is the fugacity of the R-charge, x_1, x_2 are the fugacities of the $SU(2)$ charge, q is the fugacity of the $U(1)$ symmetry and b is the fugacity of the $U(1)_B$ symmetry.

The Hilbert series of the mesonic moduli space is given by:

$$g^{\mathrm{mes}}(t_1, t_2, x_1, x_2; \mathbb{F}_0 \times \mathbb{C}) = \frac{(1-t_1^4)\left[1+\left(2+\frac{1}{x_1^2}+x_1^2+\frac{1}{x_2^2}+x_2^2\right)t_1^4+t_1^8\right]}{(1-t_1^4 x_1^2 x_2^2)\left(1-\frac{t_1^4 x_1^2}{x_2^2}\right)\left(1-\frac{t_1^4 x_2^2}{x_1^2}\right)\left(1-\frac{t_1^4}{x_1^2 x_2^2}\right)(1-t_2)}$$
$$= \sum_{i=0}^{\infty} t_2^i \sum_{n=0}^{\infty} [2n; 2n] t_1^{4n} \ , \qquad (3.4.168)$$

where we note the first factor is the Hilbert series of $\mathbb{C}$ and the second factor is the Hilbert series of $\mathbb{F}_0$ [84]. This confirms that the mesonic moduli space of this model is $\mathbb{F}_0 \times \mathbb{C}$. The totally unrefined Hilbert series of the mesonic moduli space can be written as

$$g^{\mathrm{mes}}(t^3, t^4, 1, 1; \mathbb{F}_0 \times \mathbb{C}) \quad = \quad \frac{1 + 6t^{12} + t^{24}}{(1 - t^{12})^3} \times \frac{1}{(1 - t^4)} \ . \tag{3.4.169}$$

The plethystic logarithm of (3.4.168) is given by

$$\mathrm{PL}[g^{\mathrm{mes}}(t_1, t_2, x_1, x_2)] \quad = \quad [2; 2]t_1^4 + t_2 - O(t_1^8) \ . \tag{3.4.170}$$

Therefore, the 10 generators of the mesonic moduli space can be written in terms of perfect matchings as

$$r, \quad p_i p_j q_i q_j s \ , \tag{3.4.171}$$

where $i, j, l, k = 1, 2$. Note that the lattice of generators is the dual of the toric diagram (nodes are dual to faces and edges are dual to edges). For the $\mathbb{F}_0 \times \mathbb{C}$ theory, the lattice of generators is self-dual.

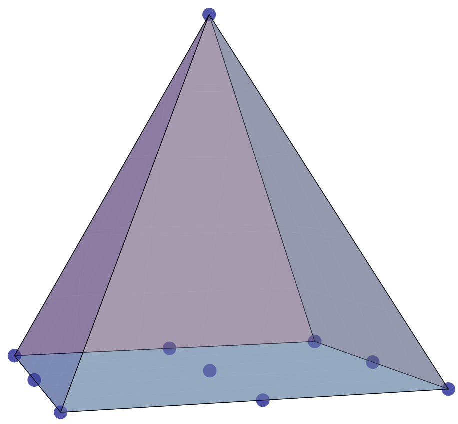


Figure 3.18: The lattice of generators of the $\mathbb{F}_0 \times \mathbb{C}$ theory.

The $\mathbb{Z}_2$ orbifold action. The mesonic Hilbert series of $\mathcal{C} \times \mathbb{C}$ is given by (4.14) of [59]:

$$g^{\text{mes}}(t_1, t_2, x_1, x_2; \mathcal{C} \times \mathbb{C}) = \sum_{i=0}^{\infty} t_2^i \sum_{n=0}^{\infty} [n; n] t_1^{2n} . \qquad (3.4.172)$$

As discussed in [91], under the $\mathbb{Z}_2$ orbifold action (on $\mathcal{C}$), $t_1^2 \to -t_1^2$, and we need to sum over both sectors, with t_1^2 and with $-t_1^2$. Therefore, starting from (3.4.172) and applying the $\mathbb{Z}_2$ action to t_1^2, we are left with the terms corresponding to even n, and hence we obtain (3.4.168).

Higgsing The $\mathbb{F}_0 \times \mathbb{C}$ Theory

From Figure 3.14, it can be seen that giving a VEV to X_{12}^1 of this model leads to the two-hexagon tiling with $k_1 = -k_2 = 2$. Hence, the mesonic moduli space is a $\mathbb{Z}_2$ orbifold of $\mathcal{C} \times \mathbb{C}$. This CS orbifold acts on the generators as one of the gauge groups [92]. Therefore, the mesonic moduli space of the resulting theory is $\mathbb{F}_0 \times \mathbb{C}$, with the fully refined Hilbert series given by (3.4.172)

3.4.2 Higgsing Phase II of $Q^{1,1,1}/\mathbb{Z}_2$

Having contemplated all the possible inequivalent higgsings for the first toric phase of the $Q^{1,1,1}/\mathbb{Z}_2$, we can now pass to the second.

Phase III of **$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}^2$ from giving a VEV to one of X_{12}^{ij}** By symmetry, we see that turning on a VEV to any of the X_{12}^{ij} fields yields the same result. For definiteness, let us consider the case of X_{12}^{12}. This amounts to removing one of the edges that separate the faces corresponding to gauge groups 1 and 2, and collapsing the two vertices adjacent to a bivalent vertex into a single vertex of higher valence. As a result, the gauge groups 1 and 2 are combined into one gauge group, identified as 1, and the edges corresponding to X_{23}^1, X_{31}^2 and $X_{23'}^2, X_{3'1}^1$ are removed. For convenience, let us relabel gauge group $3'$ as 2. The quiver diagram and tiling of this model are presented in Figure 3.19. Let us denote the adjoint fields by ϕ^i (with $i = 1, 2, 3$). The superpotential can be written as

$$W = \text{Tr}\left[(\phi^2 - \phi^3\phi^1)X_{12}X_{21} + (\phi^1\phi^3 - \phi^2)X_{13}X_{31}\right] . \qquad (3.4.173)$$

The CS levels associated with the higgsed gauge groups are added, and so

$$k_1 = 2, \quad k_2 = -1, \quad k_3 = -1 \ . \tag{3.4.174}$$

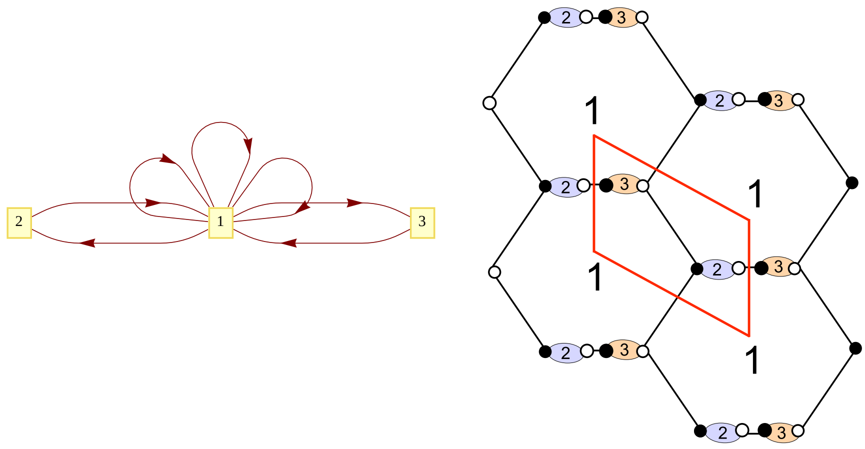


Figure 3.19: (i) Quiver diagram of Phase III of $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$. (ii) Tiling of Phase III of $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$.

Through the forward algorithm, it is possible to show that the toric diagram of this model is exactly that shown in Figure 3.15. Furthermore, we can adopt a consistent choice of global charge assignments to the perfect matchings which coincide with that presented in Table 3.5. Perhaps not too surprisingly, with the choices we made, one can show that the Hilbert series of the mesonic moduli space coincides exactly with Equation 3.4.162.

Phase II of **$(\mathbb{C}^2/\mathbb{Z}_2) \times \mathbb{C}^2$ from giving a VEV to one of X^i_{31}, X^i_{23}, $X^i_{3'1}$, $X^i_{23'}$**

By symmetry, giving a VEV to any of the $X^i_{13}, X^i_{23}, X^i_{13'}$ and $X^i_{23'}$ fields leads to the same theory. For definiteness, let us examine the case in which $X^2_{23'}$ acquires a VEV. This amounts to removing one of the edges separating the gauge groups 2 and $3'$ in Figure 3.12. Thus, the octagon corresponding to gauge group 2 combines with the square corresponding to gauge group $3'$ to form a decagon, which we shall label as 2. As a result of integrating out massive fields, the two vertices adjacent to a bivalent vertex collapse into a single vertex of higher valence. Therefore, decagons become hexagons and the remaining octagons become squares. It follows that the resulting tiling is simply two hexagons with one diagonal. The CS levels associated with the higgsed gauge groups are added, and so the new CS level is

$$k_1 = 1, \quad k_2 = 0, \quad k_3 = -1 \ . \tag{3.4.175}$$

Therefore, the resulting theory is Phase II of the $\left(\mathbb{C}^2/\mathbb{Z}_2\right) \times \mathbb{C}^2$ theory, with the quiver diagram and tiling presented in Figure 3.14. The toric diagram is drawn in Figure 3.15.

3.5 M2-brane theories and Fano 3-folds

Having discussed some interesting features of the M2-brane theories described by tilings, we now want to turn our attention to a classification problem that can be addressed using these techniques, namely that of Chern-Simons theories living on the world-volume of an M2-brane that probe Fano singularities.

Classification problems can be extremely useful in providing large classes of examples to explore their features in great detail. In fact, an analogous classification to the one we will review shortly, namely that for gauge theories on D3-branes probing smooth toric Fano 2-fold singularities, has played a central role in the study of quiver gauge theories, for example providing the first examples of toric duality [125, 121, 127, 128, 129, 130, 131, 133, 126]. Another interesting classification of the Fano surfaces can be found in [134]. The usefulness of these manifolds didn't certainly stop there, since those theories have been employed to construct a number of phenomenological models [139, 140, 141, 142, 143, 144, 145].

We chose to classify the CS theories corresponding to Fano 3-fold singularities inspired by the success that Fano 2-folds had in the past years and reassured by the fact that, for every dimension, there always exist a finite number of such manifolds [146, 147].

The Fano manifolds: a brief introduction From a strictly mathematical point of view, Fano manifolds are projective d-dimensional algebraic varieties whose anti-canonical sheaf is ample. One of the consequences of this definition which is of interest for us is that they are characterised by positive curvature, and can therefore be used to construct Calabi-Yau $(d+1)$-manifolds. A typical distinction is made between smooth and singular Fanos, and for our purposes we want to limit ourselves to the former case.

The simplest example one could consider is clearly in one complex dimension, and in such case the only smooth Fano manifold is the projective line $\mathbb{P}^1$. Things are still under control in two dimensions, where the only smooth Fano manifolds are the so called Hirzebruch zero$^{\text{th}}$ surface and the nine del Pezzo surfaces.

Classifying all Fano manifolds for a given dimension is a standard problem in Algebraic Geometry and many efforts to achieve this goal have been made over the years. Unfortunately, as the dimension grows, the number of manifolds increases significantly, thus making the problem a particularly hard one to tackle. However, it has been proved that in 3 dimensions there exists only 88 smooth Fanos [148, 149, 150]. Out of these, only 18 can be shown to be toric, and to them we should turn our attention, since this is a necessary condition for the corresponding gauge theories to be represented in terms of brane tilings.

A detailed presentation of these varieties and their main geometric features is given in Table 3.7.

The first column contains the 'name' of the Fano 3-fold, according to the nomenclature defined in Table 3.8. In particular, the Fano varieties can be divided into subclasses based on the number of external points in the toric diagram [147][13].

The second column of Table 3.7 encodes information about the symmetry of the CY 4-fold constructed by taking a complex cone over the Fano variety considered. In order to make the table compact, the following notation is used:

$$[3^{k_3}, 2^{k_2}, 1^{k_1}] = SU(3)^{k_3} \times SU(2)^{k_2} \times U(1)^{k_1}, \tag{3.5.176}$$

and, since the symmetry group of the CY must be of rank 4, we have:

$$2k_3 + k_2 + k_1 = 4. \tag{3.5.177}$$

[13] The reason why both $\mathcal{C}_i$ and $\mathcal{D}_i$ are used to denote varieties having 6 external points has to do with the structure of the toric diagram. The reader is referred to [147] for a detailed discussion of this point.

	Sym	Toric Data	Geometry	(b_2, g)
$\mathbb{P}^3$	$U(4)$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^3$	$(1, 33)$
$\mathcal{B}_4$	$[3, 2, 1]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^2 \times \mathbb{P}^1$	$(2, 28)$
$\mathcal{B}_1$	$[3, 1^2]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2 & -1 & 1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(2))$	$(2, 32)$
$\mathcal{B}_2$	$[3, 1^2]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$	$(2, 29)$
$\mathcal{C}_3$	$[2^3, 1]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	$(3, 25)$
$\mathcal{C}_4$	$[2^2, 1^2]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix}$	$dP_1 \times \mathbb{P}^1$	$(3, 25)$
$\mathcal{C}_5$	$[2^2, 1^2]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & -1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, -1))$	$(3, 23)$
$\mathcal{B}_3$	$[2^2, 1^2]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$	$(2, 28)$
$\mathcal{C}_1$	$[2^2, 1^2]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1, 1))$	$(3, 27)$
$\mathcal{C}_2$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 & 0 \end{pmatrix}$	$\mathbb{P}(\mathcal{O}_{dP_1} \oplus \mathcal{O}_{dP_1}(\ell)), \quad \ell^2\big\vert_{dP_1} = 1$	$(3, 26)$
$\mathcal{D}_1$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$	$\mathbb{P}^1$-blowup of $\mathcal{B}_2$	$(3, 26)$
$\mathcal{D}_2$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}$	$\mathbb{P}^1$-blowup of $\mathcal{B}_4$	$(3, 24)$
$\mathcal{E}_1$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$	dP_2 bundle over $\mathbb{P}^1$	$(4, 24)$
$\mathcal{E}_2$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & -1 & 1 & 0 \end{pmatrix}$	dP_2 bundle over $\mathbb{P}^1$	$(4, 23)$
$\mathcal{E}_3$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}$	$dP_2 \times \mathbb{P}^1$	$(4, 22)$
$\mathcal{E}_4$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & 0 \end{pmatrix}$	dP_2 bundle over $\mathbb{P}^1$	$(4, 21)$
$\mathcal{F}_2$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$	dP_3 bundle over $\mathbb{P}^1$	$(5, 19)$
$\mathcal{F}_1$	$[2, 1^3]$	$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 & 0 \end{pmatrix}$	$dP_3 \times \mathbb{P}^1$	$(5, 19)$

Table 3.7: The 18 smooth toric Fano 3-folds and some important geometric data.

Number of external points	4	5	6	6	7	8
Number of varieties	1	4	5	2	4	2
Nomenclature	$\mathbb{P}^3$	$\mathcal{B}_i$	$\mathcal{C}_i$	$\mathcal{D}_i$	$\mathcal{E}_i$	$\mathcal{F}_i$

Table 3.8: The number of smooth toric Fano three-folds for each number of external points in the toric diagram.

The order of the rows of Table 3.7 is determined by the amount of symmetry of the corresponding CY, with the rule that manifolds with the greatest number of non-abelian factors of highest rank come first.

The third column contains the G_t matrices that represent the toric diagram[14] of the CY 4-folds corresponding to the Fano varieties. In particular, the entries of each column of the matrices are the coordinates in a three-dimensional lattice of a point in the toric diagram of a specific CY.

Note that the point $(0,0,0)$ is always internal. This situation is similar to the cases where del Pezzo surfaces are considered: their toric diagram contains precisely one internal point. This property of the toric diagrams is not a mere coincidence, as it corresponds to the condition that the variety is Fano.

An interesting property of the G_t (and G_K) matrix of a model is that it is always possible to perform a series of elementary row operations such that the resulting matrix contains the simple roots of the non-abelian symmetries of the mesonic moduli space of the considered model. For example, the mesonic symmetry of the real cone over $M^{1,1,1}$ is $SU(3) \times SU(2) \times U(1)$ and the G_t matrix of this model can be written as:

$$G_t = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{pmatrix} . \qquad (3.5.178)$$

The first two rows of this matrix clearly contain the simple roots of $SU(3)$, whereas the third row contains the simple root of $SU(2)$. Therefore it can be seen that this example is consistent with the fact that the non-abelian mesonic symmetry of a given model is encoded in the coordinates of the toric diagram.

Finally, the last column of Table 3.7 contains two topological invariants that characterise the Fano 3-folds, the second Betti number and the genus. The former, which is denoted by b_2, can be related to the toric diagram by

$$b_2 = E - 3, \qquad (3.5.179)$$

where E is the number of external points.

The genus, which is denoted by g, is another important quantity used to

[14] Up to multiplicities.

characterise a manifold. It is of interest here because it can give information about the number of generators of the CS gauge theories living on an M2-brane probing the CY 4-folds considered in this paper. In particular, as a consequence of its defining property, a Fano variety can always be embedded in a projective space. It can be shown that this embedding is of degree $d = c_1(X)^3$ into $\mathbb{P}^{g+1}$, with $d = 2g - 2$. The $g + 2$ homogeneous coordinates of the ambient space are given precisely by the gauge invariant operators that generate the vacuum moduli space of the CS gauge theory.

A word of caution is necessary here. In the mathematical literature the usual approach is to embed a Fano variety in a projective space where all the coordinates have the same weight under multiplication by a scalar. This is known as the *canonical embedding*. If we think of the homogeneous coordinates of the ambient space as gauge invariant operators, then the standard embedding corresponds to the situation where all these operators have the same R-charge. This situation is equivalent to the UV limit of the gauge theory. The physically interesting properties are in the IR, of course, where R-charges and scaling dimensions vary according to the dynamics of the theory. This translates to having gauge invariant operators with different R-charges that corresponds to embedding the Fano variety in a weighted projective space.

It is important to observe that, although in principle one is free to choose how to embed the Fano, the requirement that the volume of the Sasaki-Einstein 7-fold (a *real* cone constructed over the Fano 3-fold) is minimised forces the choice of a specific embedding. In other words, the IR dynamics of the theory chooses a very specific embedding of the Fano 3-fold.

Since the genus of the Fano 3-fold is related to the number of generators of the mesonic moduli space, it is expected that the Hilbert series can be written with explicit g dependence [84]. In fact, by looking at all the examples reported in this paper, it is easy to see that the Hilbert series of the mesonic moduli space can be written as[15]:

$$
\begin{aligned}
g^{\text{mes}}(t;\ X) &= \frac{1+(g-2)t+(g-2)t^2+t^3}{(1-t)^4} \\
&= \sum_{n=0}^{\infty} \frac{t^n}{6}(2n+1)((g-1)n^2+(g-1)n+6),
\end{aligned}
\tag{3.5.180}
$$

where X is a Fano 3-fold of genus g [151].

3.5.1 A list of theories

In the following, for each Fano 3-fold we shall present the quiver, brane tiling and Hilbert series of the corresponding$(2+1)$-dimensional Chern Simons theories that arise when an M2-brane is placed at the singularity of the Calabi-Yau constructed using the Fano manifold.

$\mathcal{B}_4$: $\mathbb{P}^2 \times \mathbb{P}^1$ (The $M^{1,1,1}$ Theory)

The $M^{1,1,1}$ theory [92, 104, 60, 123, 59, 151, 152, 153, 154, 155] has 3 gauge groups and 9 chiral multiplets, which are denoted as $X^i_{12}, X^i_{23}, X^i_{31}$ (with $i = 1, 2, 3$). The quiver diagram and tiling are given in Figure 3.20. Note that in $3+1$ dimensions, this tiling corresponds to the gauge theory living on D3-branes probing the cone over the dP_0 surface. The superpotential is given by

$$
W = \text{Tr}\left(\epsilon_{ijk} X^i_{12} X^j_{23} X^k_{31}\right) \ . \tag{3.5.181}
$$

The CS levels are $\vec{k} = (1, -2, 1)$.

The charges of the perfect matchings under the global and baryonic symmetries are summarised in Table 3.9. The toric diagram of $\mathcal{C}_4$ is presented in Figure 3.20:

The Hilbert series of the mesonic moduli space of this theory can be

[15]For the sake of simplicity it is assumed that X is embedded in a projective space in a canonical way.

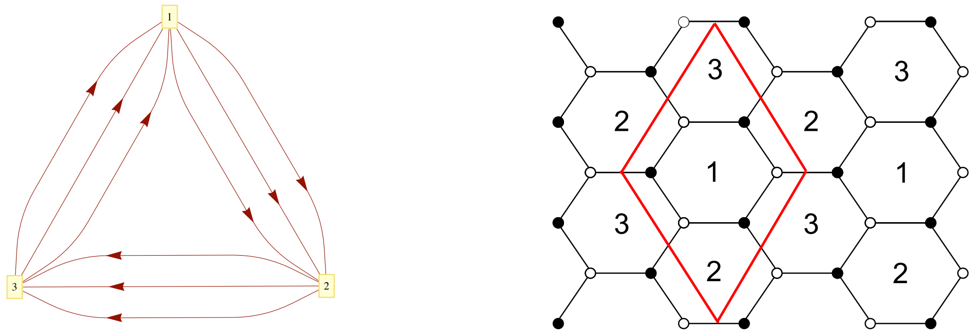


Figure 3.20: (i) Quiver diagram of the $M^{1,1,1}$ theory. (ii) Tiling of the $M^{1,1,1}$ theory.

	$SU(3)$	$SU(2)$	$U(1)_R$	$U(1)_B$	fugacity
p_1	$(1,0)$	0	$4/9$	0	$t^4 y_1$
p_2	$(-1,1)$	0	$4/9$	0	$t^4 y_2/y_1$
p_3	$(0,-1)$	0	$4/9$	0	t^4/y_2
r_1	$(0,0)$	1	$1/3$	1	$t^3 x b$
r_2	$(0,0)$	-1	$1/3$	1	$t^3 b/x$
v_1	$(0,0)$	0	0	-2	$1/b^2$

Table 3.9: Charges of the perfect matchings under the global symmetry of the $M^{1,1,1}$ theory. Here t is the fugacity of the R-charge (in units of $1/9$), y_1, y_2 are the weights of the $SU(3)$ symmetry, x is the weight of the $SU(2)$ symmetry and b is the fugacity of the $U(1)_B$ symmetry. The notation (a, b) is used to represent a weight of $SU(3)$.

written as:

$$\begin{aligned} g^{\mathrm{mes}}(t,x,y_1,y_2;\mathcal{B}_4) &= \frac{P(t,x,y_1,y_2;\mathcal{B}_4)}{\left(1-\frac{t^{18}y_1^3}{x^2}\right)\left(1-t^{18}x^2y_1^3\right)\left(1-\frac{t^{18}x^2}{y_2^3}\right)\left(1-\frac{t^{18}}{x^2y_2^3}\right)} \\ &\times \frac{1}{\left(1-\frac{t^{18}y_2^3}{x^2y_1^3}\right)\left(1-\frac{t^{18}x^2y_2^3}{y_1^3}\right)} \\ &= \sum_{n=0}^{\infty} [3n,0;2n]\, t^{18n} \ , \end{aligned} \tag{3.5.182}$$

where $P(t,x,y_1,y_2;\mathcal{B}_4)$ is a polynomial whose particular form is not relevant for the present discussion.

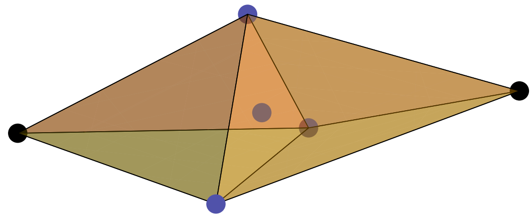


Figure 3.21: The toric diagram of the $M^{1,1,1}$ theory.

The generators of the mesonic moduli space can be determined from the plethystic logarithm of (3.5.182):

$$\mathrm{PL}[g^{\mathrm{mes}}(t,x,y_1,y_2,\mathcal{B}_4)] = [3,0;2]\,t^{18} - O(t^{36}) \ . \tag{3.5.183}$$

The 30 generators can be written in terms of perfect matchings as:

$$p_i \ p_j \ p_k \ r_l \ r_m \ v_1 \ , \tag{3.5.184}$$

where $i,j,k = 1,2,3$ and $l,m = 1,2$. As a check, let us note that $p_i p_j p_k$ has $\frac{3\times4\times5}{3!} = 10$ independent components and $r_l r_m$ has $\frac{2\times3}{2!} = 3$ independent components, which implies that there are indeed 30 generators.

$\mathcal{C}_3$: $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

This Fano 3-fold is precisely $Q^{1,1,1}/\mathbb{Z}_2$, whose 2 Phases we have discussed in Section 3.4.

$\mathcal{C}_4$: $dP_1 \times \mathbb{P}^1$

This model has 4 gauge groups and chiral fields X_{14}, X_{12}, X_{32}, X^i_{43}, X^j_{24} and X^j_{31} (with $i = 1,2,3$ and $j = 1,2$). The quiver diagram and the tiling are presented in Figure 3.22. Note that in $3+1$ dimensions this tiling corresponds to the gauge theory on D3-branes probing a cone over the dP_1 surface. The superpotential can be read off from the tiling and can be written as:

$$W = \mathrm{Tr}\left[\epsilon_{ij}\left(X_{14}X^i_{43}X^j_{31} + X_{32}X^i_{24}X^j_{43} - X_{12}X^i_{24}X^3_{43}X^j_{31}\right)\right] \ . \tag{3.5.185}$$

The CS levels are chosen to be $\vec{k} = (1, 1, -1, -1)$

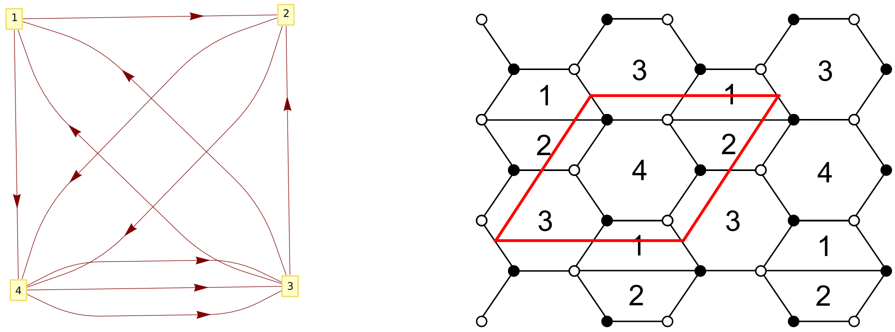


Figure 3.22: (i) Quiver diagram of the $dP_1 \times \mathbb{P}^1$ theory. (ii) Tiling of the $dP_1 \times \mathbb{P}^1$ theory.

The toric diagram of this theory is shown in Figure 3.23.

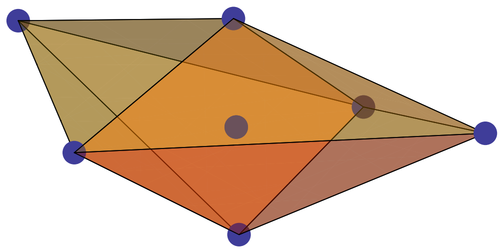


Figure 3.23: The toric diagram of $dP_1 \times \mathbb{P}^1$.

The charges of the perfect matchings under the global symmetries are presented in Table 3.10

	$SU(2)_1$	$SU(2)_2$	$U(1)_q$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	fugacity
p_1	1	0	1	0.335	1	0	$t_1 x_1 q b_1$
p_2	-1	0	1	0.335	1	0	$t_1 q b_1 / x_1$
q_1	0	1	1	0.353	0	0	$t_2 x_2 q$
q_2	0	-1	1	0.353	0	0	$t_2 q / x_2$
r_1	0	0	-2	0.241	0	0	t_3/q^2
r_2	0	0	-2	0.383	0	0	t_4/q^2
v_1	0	0	0	0	0	1	b_2
v_2	0	0	0	0	-2	-1	$1/(b_1^2 b_2)$

Table 3.10: Charges of the perfect matchings under the global symmetry of the $\mathcal{C}_4$ theory. Here t_i are the fugacities of the R-charges, x_1, x_2 are the weight of the $SU(2)$ symmetries, q, b_1 and b_2 are, respectively, the charges under the mesonic abelian symmetries $U(1)_q$ and under the two baryonic $U(1)_{B_1}$ and $U(1)_{B_2}$.

The Hilbert series of the mesonic moduli space of $\mathcal{C}_4$ can be written as:

$$\begin{aligned} g^{\mathrm{mes}}(t_\alpha, x_1, x_2, q; \mathcal{C}_4) &= \oint_{|b_1|=1} \frac{\mathrm{d}b_1}{2\pi i b_1} \oint_{|b_2|=1} \frac{\mathrm{d}z_2}{2\pi i b_2} g^{{}^{\mathrm{Irr}}\mathcal{F}^\flat}(t_\alpha, x_1, x_2, q, b_1, b_2; \mathcal{C}_4) \\ &= \frac{P(t_\alpha, x_1, x_2, q; \mathcal{C}_4)}{(1 - t_1^2 t_2^3 t_3^2 x_1^3 x_2^2 q)\left(1 - \frac{t_1^2 t_2^3 t_3^2 x_1^3 q}{x_2^2}\right)\left(1 - \frac{t_1^2 t_2^3 t_3^2 x_2^2 q}{x_1^3}\right)\left(1 - \frac{t_1^2 t_2^3 t_3^2 q}{x_1^3 x_2^2}\right)} \\ &\times \frac{1}{\left(1 - \frac{t_1^2 t_2 t_4^2 x_1 x_2^2}{q}\right)\left(1 - \frac{t_1^2 t_2 t_4^2 x_2^2}{x_1 q}\right)\left(1 - \frac{t_1^2 t_2 t_4^2 x_1}{x_2^2 q}\right)\left(1 - \frac{t_1^2 t_2 t_4^2}{x_1 x_2^2 q}\right)}, \end{aligned} \tag{3.5.186}$$

where $P(t_\alpha, x_1, x_2, q; \mathcal{C}_4)$ is a polynomial that is not reported here.

The plethystic logarithm of the mesonic Hilbert series is

$$\mathrm{PL}[g^{\mathrm{mes}}(t_\alpha, x_1, x_2, q; \mathcal{C}_4)] = [2;3] q t_1^4 t_2^6 t_3^4 + [2;2] t_1^4 t_2^4 t_3^2 t_4^2 + [2;1] \frac{t_1^4 t_2^2 t_4^4}{q} - O(t_1^{12} t_2^{12} t_3^6 t_4^6), \tag{3.5.187}$$

Therefore, the generators of the mesonic moduli space are

$$p_i p_j q_k q_l q_m r_1^2 v_1 v_2, \qquad p_i p_j q_k q_l r_1 r_2 v_1 v_2, \qquad p_i p_j q_k r_2^2 v_1 v_2 \tag{3.5.188}$$

$\mathcal{C}_5$**:** $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1)\right)$**, Phase I**

The quiver diagram and tiling of this model is given in Figure 3.8 and the CS levels are $\vec{k} = (1, -2, 1, 0)$. The toric diagram of this theory is given in Figure 3.24

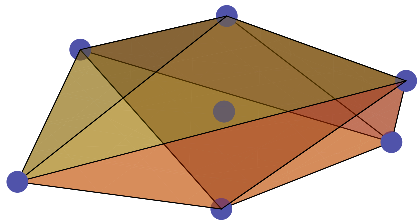


Figure 3.24: The toric diagram of the $\mathcal{C}_5$ theory.

The assignment of global and baryonic charges to the perfect matchings is given in Table 3.11.

	$SU(2)_1$	$SU(2)_2$	$U(1)_q$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	fugacity
p_1	1	0	1	4/11	0	0	$t^4 x_1 q$
p_2	-1	0	1	4/11	0	0	$t^4 q/x_1$
q_1	0	1	-1	4/11	0	0	$t^4 x_2/q$
q_2	0	-1	-1	4/11	0	0	$t^4/(x_2 q)$
r_1	0	0	0	3/11	1	0	$t^3 b_1$
r_2	0	0	0	3/11	1	0	$t^3 b_1$
v_1	0	0	0	0	0	1	b_2
v_2	0	0	0	0	-2	-1	$1/(b_1^2 b_2)$
v_3	0	0	0	0	0	0	1

Table 3.11: Charges of the perfect matchings under the global symmetry of both phases of the $\mathcal{C}_5$ theory. Here t is the fugacity of the R-charge (in units of 1/11), x_1, x_2 are the weights of the $SU(2)$ symmetry, q, b_1 and b_2 are, respectively, the charges under the mesonic abelian symmetry $U(1)_q$ and of the two baryonic symmetries $U(1)_{B_1}$ and $U(1)_{B_2}$. Note that the perfect matching v_3 (represented in blue) does not exist in Phase I but exists in Phase II.

Using this assignment, the Hilbert series of the mesonic moduli space of this model is given by:

$$
\begin{aligned}
g^{\mathrm{mes}}(t,x_1,x_2,q;\mathcal{C}_5^{(I)}) &= \frac{P(t,x_1,x_2,q;\mathcal{C}_5^{(I)})}{\left(1-\frac{t^{22}x_1^3q^2}{x_2}\right)\left(1-\frac{t^{22}x_2q^2}{x_1^3}\right)\left(1-\frac{t^{22}q^2}{x_1^3x_2}\right)\left(1-t^{22}x_1^3x_2q^2\right)} \\
&\times \frac{1}{\left(1-\frac{t^{22}x_1}{q^2x_2^3}\right)\left(1-\frac{t^{22}x_2^3}{x_1q^2}\right)\left(1-\frac{t^{22}}{x_2^3x_1q^2}\right)\left(1-\frac{t^{22}x_1x_2^3}{q^2}\right)} \ .
\end{aligned}
\tag{3.5.189}
$$

The unrefined Hilbert series can be written as:

$$
g^{\mathrm{mes}}(t,1,1,1;\mathcal{C}_5^{(I)}) = \frac{1+21t^{22}+21t^{44}+t^{66}}{(1-t^{22})^4} \ . \tag{3.5.190}
$$

The plethystic logarithm of this Hilbert series is given by:

$$
\mathrm{PL}[g^{\mathrm{mes}}(t,x_1,x_2,q;\mathcal{C}_5^{(I)})] = \left([3;1]q^2+[2;2]+\frac{1}{q^2}[1;3]\right)t^{22}-O(t^{44}) \tag{3.5.191}
$$

The generators of the mesonic moduli space are:

$$
p_ip_jp_kq_lr_1^2v_1v_2, \qquad p_ip_jq_kq_lr_1r_2v_1v_2, \qquad p_iq_jq_lq_kr_2^2v_1v_2 \ , \tag{3.5.192}
$$

where $i,j,k,l=1,2$

$\mathcal{C}_5$: $\mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,-1)\right)$, Phase II

The quiver diagram and tiling of this model are identical to those of Phase II of $Q^{1,1,1}/\mathbb{Z}_2$ (Figure 3.12). However, for this model the CS levels are chosen to be $\vec{k}=(0,0,1,-1)$.

For this theory, the perfect matchings are assigned charges according to Table 3.11. Correspondingly, the Hilbert series of the mesonic moduli space is given by:

$$
\begin{aligned}
g^{\mathrm{mes}}(t,x_1,x_2,q;\mathcal{C}_5^{(II)}) &= \frac{P(t,x_1,x_2,q;\mathcal{C}_5^{(II)})}{\left(1-\frac{t^{22}x_1^3q^2}{x_2}\right)\left(1-\frac{t^{22}x_2q^2}{x_1^3}\right)\left(1-\frac{t^{22}q^2}{x_1^3x_2}\right)\left(1-t^{22}x_1^3x_2q^2\right)} \\
&\times \frac{1}{\left(1-\frac{t^{22}x_1}{q^2x_2^3}\right)\left(1-\frac{t^{22}x_2^3}{x_1q^2}\right)\left(1-\frac{t^{22}}{x_2^3x_1q^2}\right)\left(1-\frac{t^{22}x_1x_2^3}{q^2}\right)} \ ,
\end{aligned}
\tag{3.5.193}
$$

where $P(t, x_1, x_2, q; \mathcal{C}_5^{(II)})$ is a polynomial which is not reported here. The unrefined mesonic Hilbert series of this equation is

$$g^{\text{mes}}(t, 1, 1, 1; \mathcal{C}_5^{(II)}) \quad = \quad \frac{1 + 21t^{22} + 21t^{44} + t^{66}}{(1 - t^{22})^4} \ . \qquad (3.5.194)$$

As was to be expected, this is identical to the mesonic Hilbert series of Phase I. The plethystic logarithm is given by

$$\text{PL}[g^{\text{mes}}(t, x_1, x_2, q; \mathcal{C}_5^{(II)})] = \left([3;1]q^2 + [2;2] + \frac{1}{q^2}[1;3] \right) t^{22} - O(t^{22}) \ . \qquad (3.5.195)$$

Therefore, the generators of the mesonic moduli space can be written in terms of perfect matchings as:

$$p_i p_j p_k q_l r_1^2 v_1 v_2 v_3, \qquad p_i p_j q_k q_l r_1 r_2 v_1 v_2 v_3, \qquad p_i q_j q_l q_k r_2^2 v_1 v_2 v_3 \ , \qquad (3.5.196)$$

with $i, j, k, l = 1, 2$.

$\mathcal{C}_1$**:** $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(1,1))$

This theory has 4 gauge groups and 12 chiral fields, which are denoted by X_{12}^{ij}, X_{23}^{i}, $X_{23'}^{i}$, X_{31}^{i} and $X_{3'1}^{i}$ (with $i, j = 1, 2$). The quiver diagram and tiling are given in Figure 3.12, with CS levels $\vec{k} = (2, 0, -1, -1)$.

The toric diagram of this theory is presented in Figure 3.25.

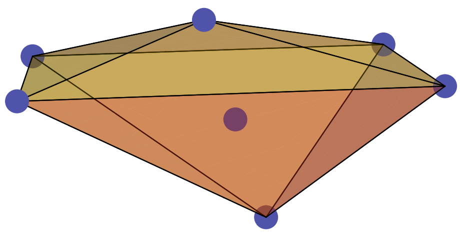


Figure 3.25: The toric diagram of $\mathcal{C}_1$.

Charges of the perfect matchings under global and baryonic symmetries are presented in Table 3.12.

	$SU(2)_1$	$SU(2)_2$	$U(1)_q$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	fugacity
p_1	1	0	0	0.344	0	0	$t_1 x_1$
p_2	-1	0	0	0.344	0	0	t_1/x_1
q_1	0	1	0	0.344	0	0	$t_1 x_2$
q_2	0	-1	0	0.344	0	0	t_1/x_2
r_1	0	0	1	0.447	0	0	$t_2 q$
r_2	0	0	-1	0.177	1	0	$t_3 b_1/q$
r_2'	0	0	0	0	-1	0	$1/b_1$
v_1	0	0	0	0	0	1	b_2
v_2	0	0	0	0	0	-1	$1/b_2$

Table 3.12: Charges of the perfect matchings under the global symmetry of the $\mathcal{C}_1$ model. Here t_i is the fugacity of the R-charge, x_1 and x_2 are the weights of the $SU(2)$ symmetries, q, b_1 and b_2 are, respectively, the charges under the mesonic abelian symmetry $U(1)_q$ and of the two baryonic $U(1)_{B_1}$ and $U(1)_{B_2}$.

Finally, the Hilbert series of the mesonic moduli space can be computed to be:

$$\begin{aligned} g^{\text{mes}}(t_\alpha, x_1, x_2, q; \mathcal{C}_1) &= \oint\limits_{|b_1|=1} \frac{\mathrm{d}b_1}{2\pi i b_1} \oint\limits_{|b_2|=1} \frac{\mathrm{d}b_2}{2\pi i b_2} g^{^{\mathrm{Irr}}\mathcal{F}^\flat}(t_\alpha, x_1, x_2, q, b_1, b_2; \mathcal{C}_1) \\ &= \frac{P(t_\alpha, x_1, x_2, q; \mathcal{C}_1)}{(1 - t_1^2 t_2^2 x_1 x_2 q^2)\left(1 - \frac{t_1^2 t_2^2 x_1 q^2}{x_2}\right)\left(1 - \frac{t_1^2 t_2^2 x_2 q^2}{x_1}\right)\left(1 - \frac{t_1^2 t_2^2 q^2}{x_1 x_2}\right)} \\ &\times \frac{1}{\left(1 - \frac{t_1^6 t_3^2 x_1^3 x_2^3}{q^2}\right)\left(1 - \frac{t_1^6 t_3^2 x_2^3}{x_1^3 q^2}\right)\left(1 - \frac{t_1^6 t_3^2 x_1^3}{x_2^3 q^2}\right)\left(1 - \frac{t_1^6 t_3^2}{x_1^3 x_2^3 q^2}\right)} . \end{aligned} \tag{3.5.197}$$

The plethystic logarithm can be written in terms of the fugacities t_1 and t_2 as

$$\mathrm{PL}[g^{\text{mes}}(t_1, t_2, t_3, x_1, x_2, q; \mathcal{C}_1)] = \frac{1}{q^2}[3;3] t_1^6 t_3^2 + [2;2] t_1^4 t_2 t_3 + q^2 [1;1] t_1^2 t_2^2 - O(t_1^4 t_2 t_3) \tag{3.5.198}$$

From this function, it can be deduced that the generators of the mesonic moduli space can be written in terms of perfect matchings as:

$$p_i p_j p_k q_l q_m q_n r_2^2 r_2'^2 v_1 v_2, \qquad p_i p_j q_k q_l r_1 r_2 r_2' v_1 v_2, \qquad p_i q_j r_1^2 v_1 v_2, \tag{3.5.199}$$

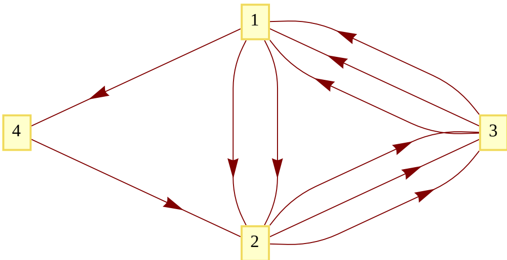
with $i, j, k, l, m, n = 1, 2$.

$\mathcal{C}_2$**:** $\mathbb{P}(\mathcal{O}_{dP_1} \oplus \mathcal{O}_{dP_1}(l)),\ l^2|_{dP_1} = 1$

This model has 4 gauge groups and chiral fields X^i_{23}, X^i_{31} (with $i = 1, 2, 3$), X^j_{12} (with $j = 1, 2$), X_{14} and X_{42}. The tiling and the quiver diagram are presented in Figure 3.26. Note that the former can be obtained by 'double bonding' the tiling of $\mathcal{B}_4$. The superpotential of this model can be written as

$$W = \epsilon_{ij}\,\mathrm{Tr}(X^i_{31}X^j_{12}X^3_{23}) + \epsilon_{ij}\,\mathrm{Tr}(X^i_{12}X^j_{23}X^3_{31}) + \epsilon_{ij}\,\mathrm{Tr}(X^i_{23}X^j_{31}X_{14}X_{42})\ . \tag{3.5.200}$$

The CS levels are $\vec{k} = (-1, 2, 0, -1)$.



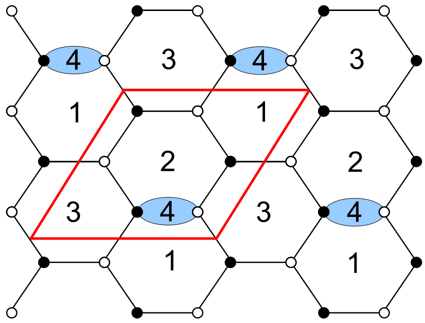


Figure 3.26: (i) Quiver diagram of the $\mathcal{C}_2$ model. (ii) Tiling of the $\mathcal{C}_2$ model.

The charges of the perfect matchings under the global and baryonic symmetries of the model are presented in Table 3.13

The toric diagram of the model is given in Figure 3.27 The Hilbert series

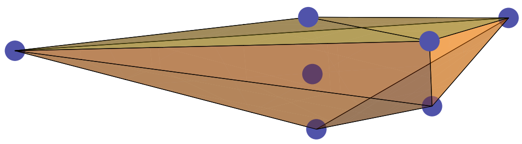


Figure 3.27: The toric diagram of $\mathcal{C}_2$.

	$SU(2)_1$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	fugacity
p_1	1	0	0	0.458	0	0	$t_1 x$
p_2	-1	0	0	0.458	0	0	t_1/x
q_1	0	1	0	0.291	0	0	$t_2 q_1$
q_2	0	-1	0	0.314	0	0	t_3/q_1
r_1	0	0	1	0.376	1	0	$t_4 q_2 b_1$
r_2	0	0	-1	0.103	1	1	$t_5 b_1 b_2/q_2$
r'_2	0	0	0	0	0	-1	$1/b_2$
v_1	0	0	0	0	-2	0	$1/b_1^2$

Table 3.13: Charges of the perfect matchings under the global symmetry of the $\mathcal{C}_2$ model. Here t_i are the fugacities of the R-charges, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1 and b_2 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1$, $U(1)_2$ and under the two baryonic $U(1)_{B_1}$ and $U(1)_{B_2}$. The fugacity t_6 is set to 1 as it corresponds to a perfect matching with zero R-charge.

of the mesonic moduli space of $\mathcal{C}_2$ is given by:

$$\begin{aligned} g^{\text{mes}}(t_\alpha, x, q_1, q_2; \mathcal{C}_2) &= \frac{P(t_\alpha, x, q_1, q_2; \mathcal{C}_2)}{\left(1 - \frac{t_1^4 t_3 t_5^2 x^4}{q_1 q_2^2}\right)\left(1 - \frac{t_1^4 t_3 t_5^2}{x^4 q_1 q_2^2}\right)\left(1 - t_1^3 t_4 t_5 x^3\right)\left(1 - \frac{t_1^3 t_4 t_5}{x^3}\right)} \\ &\times \frac{1}{\left(1 - t_1 t_2 t_4^2 x q_1 q_2^2\right)\left(1 - \frac{t_1 t_2 t_4^2 q_1 q_2^2}{x}\right)\left(1 - \frac{t_2^4 t_3^5 t_5^2}{q_1 q_2^2}\right)\left(1 - t_2^2 t_3 t_4^2 q_1 q_2^2\right)} , \end{aligned} \tag{3.5.201}$$

where $P(t_\alpha, x, q_1, q_2; \mathcal{C}_2)$ is a polynomial that is not reported here. The plethystic logarithm of (3.5.201) can be written as:

$$\begin{aligned} \text{PL}[g^{\text{mes}}(t_\alpha, x, q_1, q_2; \mathcal{C}_2)] &= [4]\frac{t_1^4 t_3 t_5^2}{q_1 q_2^2} + [3]\left(t_1^3 t_4 t_5 + \frac{t_1^3 t_2 t_3^2 t_5^2}{q_1 q_2^2}\right) + [2] t_1^2 t_2 t_3 t_4 t_5 \\ &+ [2]\frac{t_1^2 t_2^2 t_3^3 t_5^2}{q_1 q_2^2} + [1]\left(q_1 q_2^2 t_1 t_2 t_4^2 + \frac{t_1 t_2^3 t_3^4 t_5^2}{q_1 q_2^2} + t_1 t_2^2 t_3^2 t_4 t_5\right) \\ &+ q_1 q_2^2 t_2^2 t_3 t_4^2 + t_2^3 t_3^3 t_4 t_5 + \frac{t_2^4 t_3^5 t_5^2}{q_1 q_2^2} - O(t_1^4 t_2^3 t_3 t_4^5 t_5) \end{aligned} \tag{3.5.202}$$

The generators of the mesonic moduli space are

$$
\begin{array}{llll}
p_i p_j p_k p_l q_2 r_2^2 r_2'^2 v_1, & p_i p_j p_k r_1 r_2 r_2' v_1, & p_i p_j p_k q_1 q_2^2 r_2^2 r_2'^2 v_1, & p_i p_j q_1 q_2 r_1 r_2 r_2' v_1, \\
p_i p_j q_1^2 q_2^3 r_2^2 r_2'^2 v_1, & p_i q_1^2 q_2^2 r_1 r_2 r_2' v_1, & p_i q_1 r_1^2 v_1, & p_i q_1^3 q_2^4 r_2^2 r_2'^2 v_1, \\
q_1^2 q_2 r_1^2 v_1, & q_1^3 q_2^3 r_1 r_2 r_2' v_1, & q_1^4 q_2^5 r_2^2 r_2'^2 v_1 \ . &
\end{array}
$$

with $i, j, k, l = 1, 2$.

$\mathcal{D}_1$: $\mathbb{P}^1$-blowup of $\mathcal{B}_2$

This theory has 4 gauge groups and chiral fields X_{13}, X_{12}, X_{42}, X_{34}^i, X_{23}^j and X_{41}^j (with $i = 1, 2, 3$ and $j = 1, 2$). The tiling and the quiver diagram coincide with those presented in Figure 3.22, with CS levels $\vec{k} = (-1, -1, 0, 2)$. The superpotential coincide with that presented in (3.5.185).

The toric diagram of this model is represented in Figure 3.28

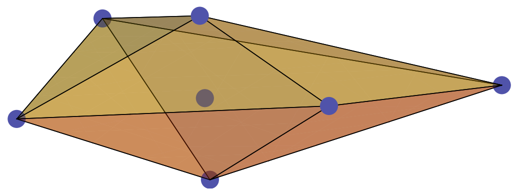


Figure 3.28: The toric diagram of $\mathcal{D}_1$.

In Table 3.14 we report the charges of the perfect matchings under the global and baryonic symmetries.

According to this charge assignment, the Hilbert series of the mesonic moduli space is given by:

$$
\begin{aligned}
g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{D}_1) &= \frac{P\left(t_\alpha, x, q_1, q_2; \mathcal{D}_1\right)}{\left(1 - \frac{t_1^4 t_2^3 t_5^2 q_1^3 x^4}{q_2^2}\right)\left(1 - \frac{t_1^4 t_2^3 t_5^2 q_1^3}{x^4 q_2^2}\right)\left(1 - t_1^2 t_2 t_4^2 x^2 q_1 q_2^2\right)\left(1 - \frac{t_1^2 t_2 t_4^2 q_1 q_2^2}{x^2}\right)} \\
&\times \frac{1}{\left(1 - \frac{t_1 t_3^3 t_5^2 x}{q_1^3 q_2^2}\right)\left(1 - \frac{t_1 t_3^3 t_5^2}{x q_1^3 q_2^2}\right)\left(1 - \frac{t_1 t_3 t_4^2 x q_2^2}{q_1}\right)\left(1 - \frac{t_1 t_3 t_4^2 q_2^2}{x q_1}\right)},
\end{aligned}
\tag{3.5.203}
$$

where $P\left(t_\alpha, x, q_1, q_2; \mathcal{D}_1\right)$ is a polynomial that is not reported here. The

	$SU(2)_1$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	fugacity
p_1	1	0	0	0.354	0	0	$t_1 x$
p_2	-1	0	0	0.354	0	0	t_1/x
q_1	0	1	0	0.255	0	0	$t_2 q_1$
q_2	0	-1	0	0.401	0	0	t_3/q_1
r_1	0	0	1	0.419	1	0	$t_4 q_2 b_1$
r_2	0	0	-1	0.217	1	1	$t_5 b_1 b_2/q_2$
r_2'	0	0	0	0	0	-1	$1/b_2$
v_1	0	0	0	0	-2	0	$1/b_1^2$

Table 3.14: Charges of the perfect matchings under the global symmetry of the $\mathcal{D}_1$ theory. Here t_i are the fugacities of the R-charges, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1 and b_2 are respectively the charges under the mesonic abelian symmetries $U(1)_1, U(1)_2$, and under the two baryonic $U(1)_{B_1}$ and $U(1)_{B_2}$. The perfect matching r_2' is found to have zero R-charge and, for this reason, its R-charge fugacity (s_6) is set to 1.

plethystic logarithm of the mesonic Hilbert series is given by

$$\begin{aligned}\mathrm{PL}[g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{D}_1)] \quad = \quad & [4]\frac{q_1^3 t_1^4 t_2^3 t_5^2}{q_2^2} + [3]\left(q_1^2 t_1^3 t_2^2 t_4 t_5 + \frac{q_1 t_1^3 t_2^2 t_3 t_5^2}{q_2^2}\right) \\ & +[2]\left(q_1 q_2^2 t_1^2 t_2 t_4^2 + t_1^2 t_2 t_3 t_4 t_5 + \frac{t_1^2 t_2 t_3^2 t_5^2}{q_1 q_2^2}\right) \\ & +[1]\left(\frac{q_2^2 t_1 t_3 t_4^2}{q_1} + \frac{t_1 t_3^2 t_4 t_5}{q_1^2} + \frac{t_1 t_3^3 t_5^2}{q_1^3 q_2^2}\right) - O(t_1^4 t_2 t_3^4 t_4^4 t_5^2)\end{aligned} \tag{3.5.204}$$

Therefore, the generators of the mesonic moduli space are

$$\begin{array}{lll} p_i p_j p_k p_l q_1^3 r_2^2 r_2'^2 v_1, & p_i p_j p_k q_1^2 r_1 r_2 r_2' v_1, & p_i p_j p_k q_1^2 q_2 r_2^2 r_2'^2 v_1 \ , \\ p_i p_j q_1 r_1^2 v_1, & p_i p_j q_1 q_2 r_1 r_2 r_2' v_1, & p_i p_j q_1 q_2^2 r_2^2 r_2'^2 v_1 \ , \\ p_i q_2 r_1^2 v_1, & p_i q_2^2 r_1 r_2 r_2' v_1, & p_i q_2^3 r_2^2 r_2'^2 v_1 \ . \end{array}$$

with $i, j, k, l = 1, 2$.

$\mathcal{D}_2$: $\mathbb{P}^1$-blowup of $\mathcal{B}_4$

This theory has 4 gauge groups and chiral fields X_{23}^i, X_{31}^i (with $i = 1, 2, 3$), X_{12}^j (with $j = 1, 2$), X_{14} and X_{42}. The tiling and the quiver diagram are presented in Figure 3.26. Note that they are the identical to those of the

$\mathcal{C}_2$ theory (*i.e.* the 'double bonding' of the $M^{1,1,1}$ tiling). However, the CS levels of this theory are $\vec{k} = (-1, 1, 1, -1)$. The superpotential is given by (3.5.200).

We show the toric diagram of the model in Figure 3.29

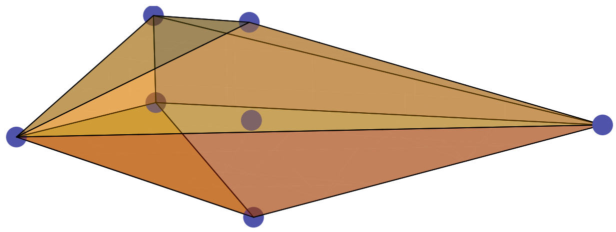


Figure 3.29: The toric diagram of $\mathcal{D}_2$.

Following the forward algorithm, the global and baryonic symmetries of this theory are computed and shown in Table 3.15.

	$SU(2)_1$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	fugacity
p_1	1	0	0	0.441	0	0	$t_1 x$
p_2	-1	0	0	0.441	0	0	t_1/x
q_1	0	1	0	0.295	0	0	$t_2 q_1$
q_2	0	0	1	0.301	0	0	$t_3 q_2$
r_1	0	-1	0	0.215	1	0	$t_4 b_1/q_1$
r_2	0	0	-1	0.306	1	0	$t_5 b_1/q_2$
v_1	0	0	0	0	0	1	b_2
v_2	0	0	0	0	-2	-1	$1/(b_1^2 b_2)$

Table 3.15: Charges of the perfect matchings under the global symmetry of the $\mathcal{D}_2$ model. Here t_i are the fugacities of the R-charges, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1 and b_2 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1$, $U(1)_2$ and under the two baryonic $U(1)_{B_1}$ and $U(1)_{B_2}$.

The Hilbert series of the mesonic moduli space is given by:

$$g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{D}_2) = \frac{P(t_\alpha, x, q_1, q_2; \mathcal{D}_2)}{\left(1 - q_1 q_2^4 t_2^3 t_3^4 t_4^2\right)\left(1 - q_1^3 t_2^3 t_3^2 t_5^2\right)\left(1 - \frac{q_2 t_1^3 t_3 t_4^2}{q_1^2 x^3}\right)\left(1 - \frac{q_2 t_1^3 t_3 t_4^2 x^3}{q_1^2}\right)}$$
$$\times \frac{1}{\left(1 - \frac{t_1^3 t_4 t_5}{q_1 q_2 x^3}\right)\left(1 - \frac{t_1^3 t_4 t_5 x^3}{q_1 q_2}\right)\left(1 - \frac{q_1 t_1^2 t_2 t_5^2}{q_2^2 x^2}\right)\left(1 - \frac{q_1 t_1^2 t_2 t_5^2 x^2}{q_2^2}\right)} \quad (3.5.205)$$

where $P(t_\alpha, x, q_1, q_2; \mathcal{D}_2)$ is a polynomial that is not reported here. The plethystic logarithm of (3.5.205) is

$$\begin{aligned} \mathrm{PL}[g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{D}_2)] &= [3]\left(\frac{t_1^3 t_4 t_5}{q_1 q_2} + \frac{q_2 t_1^3 t_3 t_4^2}{q_1^2}\right) + [2]\left(\frac{q_2^2 t_1^2 t_2 t_3^2 t_4^2}{q_1} + \frac{q_1 t_1^2 t_2 t_5^2}{q_2^2} +\right) \\ &+ [1]\left(q_2^3 t_1 t_2^2 t_3^3 t_4^2 + q_1 q_2 t_1 t_2^2 t_3^2 t_4 t_5 + \frac{q_1^2 t_1 t_2^2 t_3 t_5^2}{q_2}\right) + q_1 q_2^4 t_2^3 t_3^4 t_4^2 \\ &+ q_1^2 q_2^2 t_2^3 t_3^3 t_4 t_5 + q_1^3 t_2^3 t_3^2 t_5^2 - O(t_1^8 t_2 t_3 t_4^3 t_5^3) \end{aligned} \quad (3.5.206)$$

The generators of the mesonic moduli space are

$$\begin{array}{llll} p_i p_j p_k q_2 r_1^2 v_1 v_2, & p_i p_j p_k r_1 r_2 v_1 v_2, & p_i p_j q_1 q_2^2 r_1^2 v_1 v_2, & p_i p_j q_1 q_2 r_1 r_2 v_1 v_2, \\ p_i p_j q_1 r_2^2 v_1 v_2, & p_i q_1^2 q_2^3 r_1^2 v_1 v_2, & p_i q_1^2 q_2^2 r_1 r_2 v_1 v_2, & p_i q_1^2 q_2 r_2^2 v_1 v_2, \\ q_1^3 q_2^4 r_1^2 v_1 v_2, & q_1^3 q_2^3 r_1 r_2 v_1 v_2, & q_1^3 q_2^2 r_2^2 v_1 v_2\ . & \end{array}$$

with $i, j, k = 1, 2$.

$\mathcal{E}_1$: dP_2 bundle over $\mathbb{P}^1$

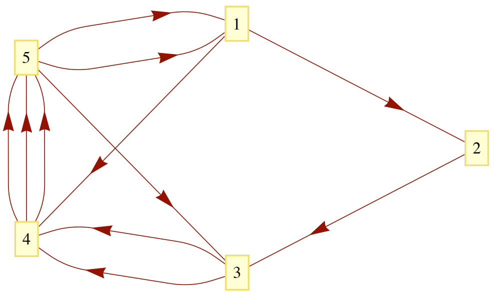
This theory has 5 gauge groups and chiral superfields X_{45}^i (with $i = 1, 2, 3$), X_{51}^j, X_{34}^j (with $j = 1, 2$), X_{14}, X_{12}, X_{53} and X_{23}. The tiling and quiver of this theory are shown in Figure 3.30. The superpotential can be read off from the tiling:

$$W = \mathrm{Tr}\left[\epsilon_{ij}\left(X_{51}^i X_{12} X_{23} X_{34}^j X_{45}^3 + X_{53} X_{34}^i X_{45}^j + X_{14} X_{45}^i X_{51}^j\right)\right] \quad (3.5.207)$$

Let us choose the CS levels to be $\vec{k} = (1, -1, 0, -1, 1)$

The toric diagram of this theory is shown in Figure 3.31

The global and baryonic charges to the perfect matchings are assigned as reported in Table 3.16.



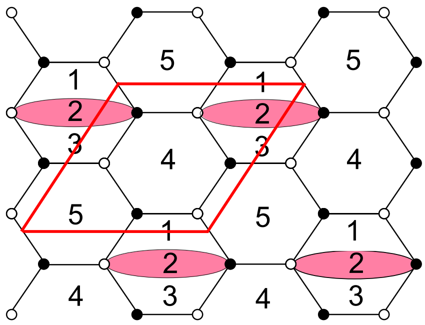


Figure 3.30: (i) Quiver diagram of the $\mathcal{E}_1$ model. (ii) Tiling of the $\mathcal{E}_1$ model.

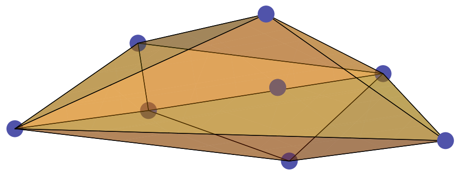


Figure 3.31: The toric diagram of $\mathcal{E}_1$.

	$SU(2)$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	fugacity
p_1	1	0	0	0.347	0	0	0	$t_1 x$
p_2	-1	0	0	0.347	0	0	0	t_1/x
q_1	0	1	0	0.201	0	0	0	$t_2 q_1$
q_2	0	-1	0	0.201	1	0	0	$t_2 b_1/q_1$
r_1	0	0	1	0.357	0	0	0	$t_3 q_2$
r_2	0	0	1	0.357	1	0	0	$t_3 b_1/q_2$
u_1	0	0	-2	0.189	0	0	0	t_4/q_2^2
v_1	0	0	0	0	0	1	0	b_2
v_2	0	0	0	0	0	0	1	b_3
v_3	0	0	0	0	-2	-1	-1	$1/(b_1^2 b_2 b_3)$

Table 3.16: Charges of the perfect matchings under the global symmetry of the $\mathcal{E}_1$ model. Here t_α is the fugacity of the R-charge, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1, b_2 and b_3 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1, U(1)_2$ and of the three baryonic $U(1)_{B_1}, U(1)_{B_2}$ and $U(1)_{B_3}$.

The Hilbert series of the mesonic moduli space can be written as:

$$\begin{aligned} g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{E}_1) &= \frac{P\left(t_\alpha, x, q_1, q_2; \mathcal{E}_1\right)}{\left(1-\frac{t_1 t_2 t_3^3 x q_1}{q_2^3}\right)\left(1-\frac{t_1 t_2 t_3^3 q_1}{x q_2^3}\right)\left(1-\frac{t_1 t_2 t_3^3 x q_2^3}{q_1}\right)\left(1-\frac{t_1 t_2 t_3^3 q_2^3}{x q_1}\right)} \\ &\times \frac{1}{(1-t_1^2 t_2^2 t_3^2 t_4 x^2 q_1^2)\left(1-\frac{t_1^2 t_2^2 t_3^2 t_4 q_1^2}{x^2}\right)\left(1-\frac{t_1^2 t_2^2 t_3^2 t_4 x^2}{q_1^2}\right)\left(1-\frac{t_1^2 t_2^2 t_3^2 t_4}{x^2 q_1^2}\right)} \\ &\times \frac{1}{\left(1-\frac{t_1^4 t_2^4 t_4^3 x^4}{q_2^6}\right)\left(1-\frac{t_1^4 t_2^4 t_4^2}{x^4 q_2^6}\right)}\ , \end{aligned} \tag{3.5.208}$$

where $P\left(t_\alpha, x, q_1, q_2; \mathcal{E}_1\right)$ is polynomial which is not reported here.

The plethystic logarithm of this function can be written as:

$$\begin{aligned} \mathrm{PL}[g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{E}_1)] &= [4]\frac{t_1^4 t_2^4 t_4^3}{q_2^6} + [3]\left(q_1 + \frac{1}{q_1}\right)\frac{t_1^3 t_2^3 t_3 t_4^2}{q_2^3} + [2]\left(q_1^2 + 1 + \frac{1}{q_1^2}\right) t_1^2 t_2^2 t_3^2 t_4 \\ &+[1]\left(q_1 + \frac{1}{q_1}\right) q_2^3 t_1 t_2 t_3^3 - O(t_1^3 t_2^3 t_3^5 t_4) \end{aligned} \tag{3.5.209}$$

From the plethystic logarithm it's clear that in the mesonic moduli space the abelian symmetry $U(1)_1$ is enhanced to $SU(2)$.

The generators of the mesonic moduli space are

$$\begin{aligned} &p_i p_j p_k p_l q_1^2 q_2^2 u_1^3 v_1 v_2 v_3, \quad && p_i p_j p_k q_m^2 q_n r_n u_1^2 v_1 v_2 v_3, \quad && p_i p_j q_m^2 r_n^2 u_1 v_1 v_2 v_3, \\ &p_i p_j q_1 q_2 r_1 r_2 u_1 v_1 v_2 v_3, && p_i q_m r_n^2 r_m v_1 v_2 v_3, && \end{aligned} \tag{3.5.210}$$

with $i, j, k, l, m, n = 1, 2$ and $m \neq n$.

$\mathcal{E}_2$: dP_2 bundle over $\mathbb{P}^1$

This model has 5 gauge groups and bi-fundamental fields X_{34}^i, X_{12}^i, X_{23}^i, X_{41}, X_{51}, X_{45} (with $i = 1, 2$). The quiver diagram and tiling are drawn in Figure 3.32. Note that the tiling of this model is a 'double bonding' of the tiling of Phase I of the $\mathbb{F}_0$ theory.

The superpotential is given by

$$W = \mathrm{Tr}\left[\epsilon_{ij}(X_{45}X_{51}X_{12}^i X_{23}^1 X_{34}^j - X_{41}X_{12}^i X_{23}^2 X_{34}^j)\right]\ . \tag{3.5.211}$$

The CS levels are chosen to be $\vec{k} = (1, 0, -1, -1, 1)$.

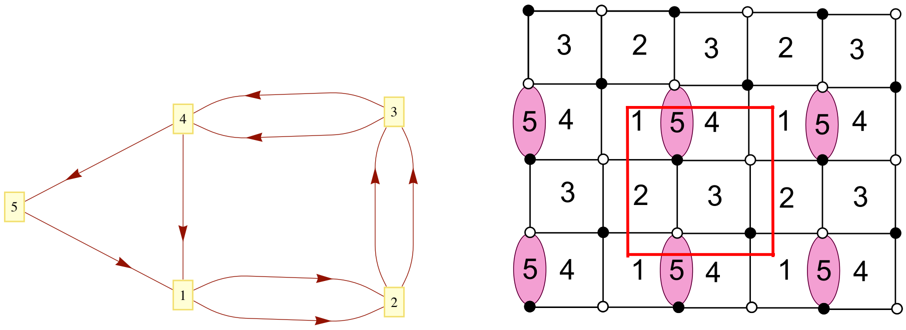


Figure 3.32: (i) Quiver of the $\mathcal{E}_2$ model. (ii) Tiling of the $\mathcal{E}_2$ model.

The toric diagram of this model is given in Figure 3.33.

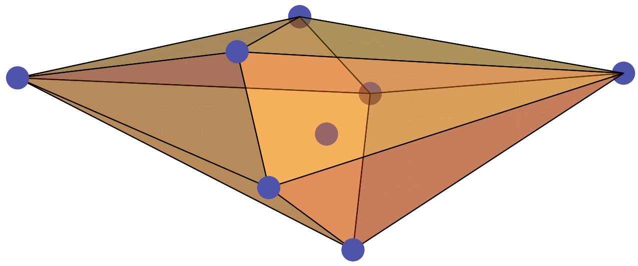


Figure 3.33: The toric diagram of the toric $\mathcal{E}_2$.

The charges of the perfect matchings under the global and baryonic symmetries are reported in 3.17

The Hilbert series of the mesonic moduli space of this model is given by:

$$\begin{aligned} g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{E}_2) &= \frac{P\left(t_\alpha, x, q_1, q_2; \mathcal{E}_2\right)}{(1 - t_1^3 t_2^2 t_4 t_6^2 x^3 q_1^2 q_2)\left(1 - \frac{t_1^3 t_2^2 t_4 t_6^2 q_1^2 q_2}{x^3}\right)\left(1 - \frac{t_1^3 t_3^2 t_4^3 t_6^2 x^3 q_2^3}{q_1^2}\right)\left(1 - \frac{t_1^3 t_3^2 t_4^3 t_6^2 q_2^3}{x^3 q_1^2}\right)} \\ &\times \frac{1}{\left(1 - \frac{t_1^2 t_2^2 t_5 t_6 x^2 q_1^2}{q_2}\right)\left(1 - \frac{t_1^2 t_2^2 t_5 t_6 q_1^2}{x^2 q_2}\right)\left(1 - \frac{t_1 t_3^2 t_4 t_5^2 x}{q_1^2 q_2}\right)\left(1 - \frac{t_1 t_3^2 t_4 t_5^2}{x q_1^2 q_2}\right)} \\ &\times \frac{1}{\left(1 - \frac{t_1 t_2 t_3 t_5^2 x}{q_2^2}\right)\left(1 - \frac{t_1 t_2 t_3 t_5^2}{x q_2^2}\right)}, \end{aligned} \tag{3.5.212}$$

where $P\left(t_\alpha, x, q_1, q_2; \mathcal{E}_2\right)$ is polynomial which is not reported here. The

	$SU(2)_1$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	fugacity
p_1	1	0	0	0.347	0	0	0	$t_1 x$
p_2	-1	0	0	0.347	0	0	0	t_1/x
r_1	0	1	0	0.304	0	0	0	$t_2 q_1$
r_2	0	-1	0	0.245	0	0	0	t_3/q_1
r_3	0	0	1	0.234	0	0	0	$t_4 q_2$
r_4	0	0	-1	0.359	1	0	0	$t_5 b_1/q_2$
r_5	0	0	0	0.164	1	1	0	$t_6 b_1 b_2$
r'_5	0	0	0	0	0	-1	0	$1/b_2$
v_1	0	0	0	0	-2	0	1	b_3/b_1^2
v_2	0	0	0	0	0	0	-1	$1/b_3$

Table 3.17: Charges of the perfect matchings under the global symmetry of the $\mathcal{E}_2$ model. Here t_α are the fugacities of the R-charges, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1, b_2 and b_3 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1$, $U(1)_2$ and under the three baryonic $U(1)_{B_1}, U(1)_{B_2}$ and $U(1)_{B_3}$.

plethystic logarithm of (3.5.212) can be written as:

$$\begin{aligned}
\mathrm{PL}[g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{E}_2)] \quad = \quad & [3]\left(q_1^2 q_2 t_1^3 t_2^2 t_4 t_6^2 + q_2^2 t_1^3 t_2 t_3 t_4^2 t_6^2 + \frac{q_2^3 t_1^3 t_3^2 t_4^3 t_6^2}{q_1^2}\right) \\
& +[2]\left(t_1^2 t_2 t_3 t_4 t_5 t_6 + \frac{q_1^2 t_1^2 t_2^2 t_5 t_6}{q_2} + \frac{q_2 t_1^2 t_3^2 t_4^2 t_5 t_6}{q_1^2}\right) \\
& +[1]\left(\frac{t_1 t_2 t_3 t_5^2}{q_2^2} + \frac{t_1 t_3^2 t_4 t_5^2}{q_1^2 q_2}\right) - O(t_1^4 t_2^3 t_3^3 t_4 t_5^5 t_6)
\end{aligned} \tag{3.5.213}$$

Therefore, the generators of the mesonic moduli space are:

$$\begin{array}{lll}
p_i p_j p_k r_1^2 r_3 r_5^2 r_5'^2 v_1 v_2, & p_i p_j p_k r_1 r_2 r_3^2 r_5^2 r_5'^2 v_1 v_2, & p_i p_j p_k r_2^2 r_3^3 r_5^2 r_5'^2 v_1 v_2, \\
p_i p_j r_1 r_2 r_3 r_4 r_5 r'_5 v_1 v_2, & p_i p_j r_1^2 r_4 r_5 r'_5 v_1 v_2, & p_i p_j r_2^2 r_3^2 r_4 r_5 r'_5 v_1 v_2, \\
p_i r_1 r_2 r_4^2 v_1 v_2, & p_i r_2^2 r_3 r_4^2 v_1 v_2\ . &
\end{array}$$

with $i, j, k = 1, 2$.

$\mathcal{E}_3$: $dP_2 \times \mathbb{P}^1$

This model has 5 gauge groups and bi-fundamental fields X^i_{34}, X^i_{12}, X^i_{23}, X_{41}, X_{51}, X_{45} (with $i = 1, 2$). The tiling and quiver of this model coincide with those in Figure 3.32. Of course, the superpotential is the same as (3.5.211). Let us choose that CS levels to be $\vec{k} = (1, 1, -1, 0, -1)$.

The toric diagram of this model is given in Figure 3.34

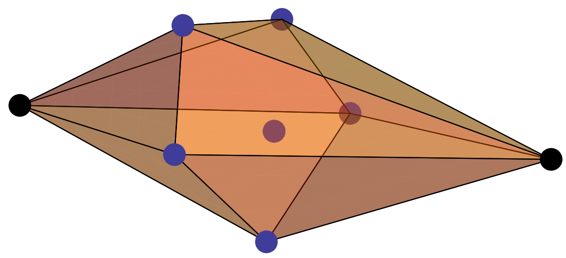


Figure 3.34: The toric diagram of the $\mathcal{E}_3$ model.

The charges of the perfect matchings under the global and baryonic symmetries are reported in 3.18

The Hilbert series of the mesonic moduli space of this model is given by:

$$
\begin{aligned}
g^{\text{mes}}(t_\alpha, x, q_1, q_2; \mathcal{E}_3) \quad = \quad & \frac{P\left(t_\alpha, x, q_1, q_2; \mathcal{E}_3\right)}{(1 - t_1^2 t_2^2 t_3^2 t_4 x^2 q_1^2)\left(1 - \frac{t_1^2 t_2^2 t_3^2 t_4 q_1^2}{x^2}\right)\left(1 - \frac{t_1^2 t_2^2 t_3^2 t_4 x^2}{q_1^2}\right)\left(1 - \frac{t_1^2 t_2^2 t_3^2 t_4}{x^2 q_1^2}\right)} \\
\times \quad & \frac{1}{(1 - t_1^2 t_2 t_3^3 x^2 q_1 q_2^3)\left(1 - \frac{t_1^2 t_2 t_3^3 q_1 q_2^3}{x^2}\right)\left(1 - \frac{t_1^2 t_2 t_3^3 q_2^3 x^2}{q_1}\right)\left(1 - \frac{t_1^2 t_2 t_3^3 q_2^3}{x^2 q_1}\right)} \\
\times \quad & \frac{1}{\left(1 - \frac{t_1^2 t_2^4 t_4^3 x^2}{q_2^6}\right)\left(1 - \frac{t_1^2 t_2^4 t_4^3}{x^2 q_2^6}\right)} \ ,
\end{aligned} \tag{3.5.214}
$$

where $P\left(t_\alpha, x, q_1, q_2; \mathcal{E}_3\right)$ is a polynomial which is not reported here. The

	$SU(2)_1$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	fugacity
p_1	1	0	0	0.334	0	0	0	$t_1 x$
p_2	-1	0	0	0.334	0	0	0	t_1/x
q_1	0	1	0	0.226	0	0	0	$t_2 q_1$
q_2	0	-1	0	0.226	1	0	0	$t_2 b_1/q_1$
r_1	0	0	1	0.310	0	0	0	$t_3 q_2$
r_2	0	0	1	0.310	1	0	0	$t_3 b_1 q_2$
u_1	0	0	-2	0.260	0	0	0	t_4/q_2^2
v_1	0	0	0	0	-2	1	0	b_2/b_1^2
v_2	0	0	0	0	0	0	1	b_3
v_3	0	0	0	0	0	-1	-1	$1/(b_2 b_3)$

Table 3.18: Charges of the perfect matchings under the global symmetry of the $\mathcal{E}_3$ model. Here t_α are the fugacities of the R-charges, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1, b_2 and b_3 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1$, $U(1)_2$ and under the three baryonic $U(1)_{B_1}, U(1)_{B_2}$ and $U(1)_{B_3}$.

plethystic logarithm of (3.5.214) can be written as:

$$\begin{aligned} \mathrm{PL}[g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{E}_3)] \;=\;& [2]\left(q_1 + \frac{1}{q_1}\right)\left(q_2^3 t_1^3 t_2 t_3^3 + \frac{t_1^2 t_2^3 t_3 t_4^2}{q_2^3}\right) \\ &+ [2]\left(q_1^2 + 1 + \frac{1}{q_1^2}\right) t_1^2 t_2^2 t_3^2 t_4 + [2]\frac{t_1^2 t_2^4 t_4^3}{q_2^6} \\ &- O(t_1^4 t_2^4 t_3^4 t_4^2)\ . \end{aligned} \tag{3.5.215}$$

From (3.5.215), it can be seen that the abelian symmetry $U(1)_1$ is enhanced to $SU(2)$ in the mesonic moduli space. The generators of the mesonic moduli space are

$$\begin{aligned} & p_i p_j q_k r_k r_l^2 v_1 v_2 v_3, \quad p_i p_j q_k^2 q_l r_l u_1^2 v_1 v_2 v_3, \quad p_i p_j q_k^2 r_l^2 u_1 v_1 v_2 v_3, \\ & p_i p_j q_1 q_2 r_1 r_2 u_1 v_1 v_2 v_3, \quad p_i p_j q_1^2 q_2^2 u_1^3 v_1 v_2 v_3. \end{aligned} \tag{3.5.216}$$

with $i, j, k, l = 1, 2$ and $k \neq l$.

$\mathcal{E}_4$: dP_2 bundle over $\mathbb{P}^1$

This theory has 9 chiral fields: X^i_{12}, X^i_{23}, X^i_{41} (with $i = 1, 2$), X_{35}, X_{54} and X_{34}. The quiver diagram and the tiling coincide with those of Figure 3.32. The superpotential can be read from (3.5.211). However, for this model let us choose the CS levels to be $\vec{k} = (1, -1, 0, -1, 1)$.

The toric diagram of this model is given in Figure 3.35.

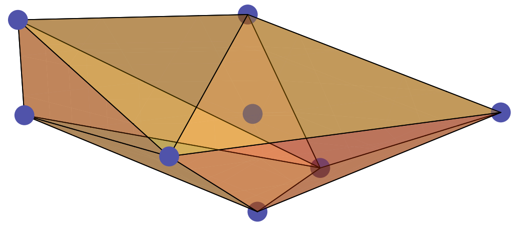


Figure 3.35: The toric diagram of $\mathcal{E}_4$.

The charges of the perfect matchings under the global and baryonic symmetries are reported in 3.19.

	$SU(2)$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	fugacity
p_1	1	0	0	0.357	0	0	0	$t_1 x$
p_2	-1	0	0	0.357	0	0	0	t_1/x
r_1	0	1	0	0.221	1	0	0	$t_2 q_1 b_1$
r_2	0	-1	0	0.258	1	0	0	$t_3 b_1/q_1$
r_3	0	0	1	0.282	0	0	0	$t_4 q_2$
r_4	0	0	-1	0.206	0	0	0	t_5/q_2
r_5	0	0	0	0.319	0	0	0	t_6
v_1	0	0	0	0	0	1	0	b_2
v_2	0	0	0	0	0	-1	1	b_3/b_2
v_3	0	0	0	0	-2	0	-1	$1/(b_1^2 b_3)$

Table 3.19: Charges of the perfect matchings under the global symmetry of the $\mathcal{E}_4$ model. Here t_α is the fugacity of the R-charge, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1, b_2 and b_3 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1, U(1)_2$ and of the three baryonic $U(1)_{B_1}, U(1)_{B_2}$ and $U(1)_{B_3}$.

The Hilbert series of the mesonic moduli space of this model is given by:

small

$$g^{\rm mes}(t_\alpha,x,q_1,q_2;\mathcal{E}_4) = \frac{P\left(t_\alpha,x,q_1,q_2;\mathcal{E}_4\right)}{\left(1-\frac{t_1^3t_2^2t_4t_5^2x^3q_1^2}{q_2}\right)\left(1-\frac{t_1^3t_2^2t_4t_5^2q_1^2}{x^3q_2}\right)\left(1-\frac{t_1^3t_2^2t_5t_6q_1^2x^3}{q_2}\right)\left(1-\frac{t_1^3t_2^2t_5t_6q_1^2}{x^3q_2}\right)}$$
$$\times\frac{1}{\left(1-t_1^2t_2t_3t_6^2x^2\right)\left(1-\frac{t_1^2t_2t_3t_6^2}{x^2}\right)\left(1-\frac{t_1t_3^2t_4^3t_5^2xq_2}{q_1^2}\right)\left(1-\frac{t_1t_3^2t_4^3t_5^2q_2}{xq_1^2}\right)}$$
$$\times\frac{1}{\left(1-\frac{t_1t_3^2t_4t_6^2xq_2}{q_1^2}\right)\left(1-\frac{t_1t_3^2t_4t_6^2q_2}{xq_1^2}\right)}\,, \qquad (3.5.217)$$

where $P\left(t_\alpha,x,q_1,q_2;\mathcal{E}_4\right)$ is a polynomial which is not reported here. The plethystic logarithm of the Hilbert series above can be written as:

$$\begin{aligned}\mathrm{PL}[g^{\rm mes}(t_\alpha,x,q_1,q_2;\mathcal{E}_4)] &= [3]\frac{q_1^2}{q_2}\left(t_1^3t_2^2t_4t_5^2+t_1^3t_2^2t_5t_6\right)\\ &+ [2]\left(t_1^2t_2t_3t_4^2t_5^2+t_1^2t_2t_3t_4t_5t_6+t_1^2t_2t_3t_6^2\right)\\ &+ [1]\frac{q_2}{q_1^2}\left(t_1t_3^2t_4^2t_5t_6+t_1t_3^2t_4^3t_5^2+t_1t_3^2t_4t_6^2\right)\\ &- O(t_1^3t_2t_3^3t_4^2t_5t_6^3).\end{aligned} \qquad (3.5.218)$$

Thus, the generators of the mesonic moduli space are

$$\begin{array}{lll} p_ip_jp_kr_1^2r_3r_4^2v_1v_2v_3, & p_ip_jp_kr_1^2r_4r_5v_1v_2v_3, & p_ip_jr_1r_2r_3^2r_4^2v_1v_2v_3,\\ p_ip_jr_1r_2r_3r_4r_5v_1v_2v_3, & p_ip_jr_1r_2r_5^2v_1v_2v_3, & p_ir_2^2r_3^2r_4r_5v_1v_2v_3,\\ p_ir_2^2r_3^3r_4^2v_1v_2v_3, & p_ir_2^2r_3r_5^2v_1v_2v_3\ . & \end{array}$$

with $i,j,k=1,2$.

$\mathcal{F}_2$: dP_3 bundle over $\mathbb{P}^1$

This theory has 6 gauge groups and chiral fields $X_{23}^i, X_{31}^i, X_{42}^i$ (with $i=1,2$), $X_{12}, X_{34}, X_{26}, X_{63}, X_{15}$ and X_{54}. The quiver diagram and the tiling of this model are presented in Figure 3.36. Note that this tiling is actually that of dP_1 with 2 double bonds. The superpotential of this model can be read off from the tiling as:

$$W=\mathrm{Tr}\left[\epsilon_{ij}\left(X_{12}X_{23}^iX_{31}^j+X_{34}X_{42}^iX_{23}^j+X_{26}X_{63}X_{31}^iX_{15}X_{54}X_{42}^j\right)\right]. \qquad (3.5.219)$$

The CS levels are $\vec{k} = (0, -1, 0, -1, 1, 1)$.

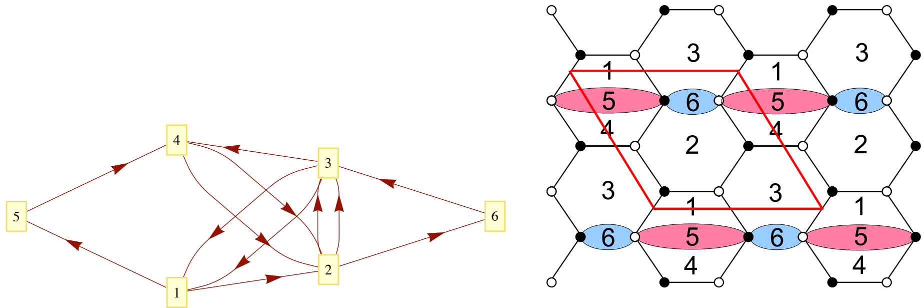


Figure 3.36: (i) Quiver of the $\mathcal{F}_2$ model. (ii) Tiling of the $\mathcal{F}_2$ model.

The toric diagram of this model is given in Figure 3.37.

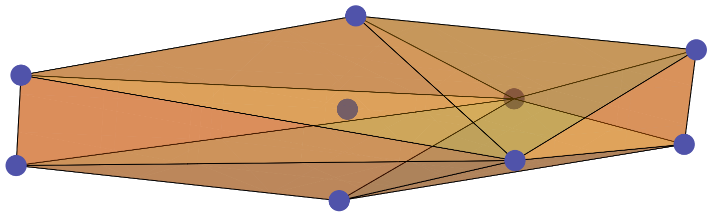


Figure 3.37: The toric diagram of $\mathcal{F}_2$.

The charges of the perfect matchings under the global and baryonic symmetries are reported in 3.20.

The Hilbert series of the mesonic moduli space of this model is given by:

$$
\begin{aligned}
g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{F}_2) \quad &= \quad \frac{P(t_\alpha, x, q_1, q_2; \mathcal{F}_2)}{\left(1 - \frac{t_1^3 t_2^3 t_3 t_4^2 x^3 q_1}{q_2}\right)\left(1 - \frac{t_1^3 t_2^3 t_3 t_4^2 q_1}{x^3 q_2}\right)\left(1 - \frac{t_1^3 t_2^3 t_3 t_4^2 x^3}{q_1 q_2}\right)\left(1 - \frac{t_1^3 t_2^3 t_3 t_4^2}{x^3 q_1 q_2}\right)} \\
&\times \quad \frac{1}{(1 - t_1^2 t_2^2 t_3^2 t_4 t_5 x^2 q_1^2)\left(1 - \frac{t_1^2 t_2^2 t_3^2 t_4 t_5 q_1^2}{x^2}\right)\left(1 - \frac{t_1^2 t_2^2 t_3^2 t_4 t_5 x^2}{q_1^2}\right)\left(1 - \frac{t_1^2 t_2^2 t_3^2 t_4 t_5}{x^2 q_1^2}\right)} \\
&\times \quad \frac{1}{(1 - t_1 t_2 t_3^3 t_5^2 x q_1 q_2)\left(1 - \frac{t_1 t_2 t_3^3 t_5^2 q_1 q_2}{x}\right)\left(1 - \frac{t_1 t_2 t_3^3 t_5^2 x q_2}{q_1}\right)\left(1 - \frac{t_1 t_2 t_3^3 t_5^2 q_2}{x q_1}\right)} \, .
\end{aligned}
\tag{3.5.220}
$$

where $P(t_\alpha, x, q_1, q_2; \mathcal{F}_2)$ is a polynomial that is not reported here. The

	$SU(2)$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	$U(1)_{B_4}$	fugacity
p_1	1	0	0	0.350	0	0	0	0	$t_1 x$
p_2	-1	0	0	0.350	0	0	0	0	t_1/x
q_1	0	1	0	0.199	0	0	0	0	$t_2 q_1$
q_2	0	-1	0	0.199	1	0	0	0	$t_2 b_1/q_1$
r_1	0	0	1	0.244	0	0	0	0	$t_3 q_2$
r_2	0	0	1	0.244	1	0	0	0	$t_3 b_1 q_2$
u_1	0	0	-1	0.160	0	0	0	0	t_4/q_2
u_2	0	0	-1	0.254	0	0	0	0	t_5/q_2
v_1	0	0	0	0	0	1	0	0	b_2
v_2	0	0	0	0	0	0	1	0	b_3
v_3	0	0	0	0	-2	0	0	1	b_4/b_1^2
v_4	0	0	0	0	0	-1	-1	-1	$1/(b_2 b_3 b_4)$

Table 3.20: Charges of the perfect matchings under the global symmetry of the $\mathcal{F}_2$ model. Here t_α is the fugacity of the R-charge, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1, b_2, b_3 and b_4 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1, U(1)_2$ and of the three baryonic $U(1)_{B_1}, U(1)_{B_2}, U(1)_{B_3}$ and $U(1)_{B_4}$.

plethystic logarithm of the Hilbert series of the mesonic moduli space is:

$$\begin{aligned} \mathrm{PL}[g^{\mathrm{mes}}(t_\alpha, x, q_1, q_2; \mathcal{F}_2)] &= [3]\left(q_1 + \frac{1}{q_1}\right)\frac{t_1^3 t_2^3 t_3 t_4^2}{q_2} + [2]\left(q_1^2 + 1 + \frac{1}{q_1^2}\right) t_1^2 t_2 t_3 t_4 t_5 \\ &+ [1]\left(q_1 + \frac{1}{q_1}\right) q_2 t_1 t_2 t_3^3 t_5^2 - O(t_1^4 t_2^2 t_3^2 t_4^2 t_5^2) \end{aligned} \tag{3.5.221}$$

The plethystic logarithm shows that in the mesonic moduli space the abelian symmetry $U(1)_1$ is enhanced to $SU(2)$. From the positive terms of the plethystic logarithm, it can be deduced that the generators of the mesonic moduli space are:

$$\begin{array}{ll} p_i p_j p_k q_l^2 q_m r_m u_1^2 v_1 v_2 v_3 v_4, & p_i p_j q_l^2 r_m^2 u_1 u_2 v_1 v_2 v_3 v_4, \\ p_i p_j q_1 q_2 r_1 r_2 u_1 u_2 v_1 v_2 v_3 v_4, & p_i q_l r_l r_m^2 u_2^2 v_1 v_2 v_3 v_4, \end{array} \tag{3.5.222}$$

with $i, j, k, l, m = 1, 2$ with $l \neq m$.

$\mathcal{F}_1$**:** $dP_3 \times \mathbb{P}^1$

The model has 6 gauge groups and 10 chiral fields: $X^i_{12}, X^i_{23}, X^i_{34}$ (with $i = 1,2$), X_{46}, X_{61}, X_{45} and X_{51}. The quiver diagram and tiling are presented in Figure 3.38. The superpotential can be read off from the tiling as

$$W = \mathrm{Tr}\left[\epsilon_{ij}\left(X^i_{12}X^1_{23}X^j_{34}X_{45}X_{51} - X^j_{12}X^2_{23}X^i_{34}X_{46}X_{61}\right)\right] \ . \quad (3.5.223)$$

Let us choose the CS levels to be $\vec{k} = (0,0,0,0,-1,1)$.

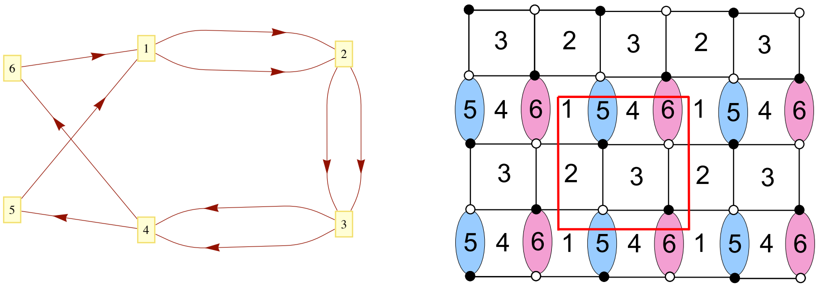


Figure 3.38: (i) Quiver of the $\mathcal{F}_1$ model. (ii) Tiling of the $\mathcal{F}_1$ model.

The toric diagram of this model is given in Figure 3.39

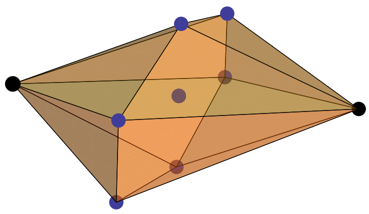


Figure 3.39: The toric diagram of $\mathcal{F}_1$.

The charges of the perfect matchings under the global and baryonic symmetries are reported in 3.21

	$SU(2)$	$U(1)_1$	$U(1)_2$	$U(1)_R$	$U(1)_{B_1}$	$U(1)_{B_2}$	$U(1)_{B_3}$	$U(1)_{B_4}$	fugacity
p_1	1	0	0	1/3	0	0	0	0	$t^3 x$
p_2	-1	0	0	1/3	0	0	0	0	t^3/x
q_1	0	1	0	2/9	0	0	0	0	$t^2 q_1$
q_2	0	-1	0	2/9	0	0	0	0	t^2/q_1
r_1	0	0	1	2/9	0	0	0	0	$t^2 q_2$
r_2	0	0	-1	2/9	0	0	0	0	t^2/q_2
u_1	0	0	0	2/9	0	0	0	0	t^2
u_2	0	0	0	2/9	0	0	0	0	t^2
v_1	0	0	0	0	1	0	0	0	b_1
v_2	0	0	0	0	0	1	0	0	b_2
v_3	0	0	0	0	-1	0	1	0	b_3/b_1
v_4	0	0	0	0	0	-1	0	1	b_4/b_2
v_5	0	0	0	0	0	0	-1	-1	$1/(b_3 b_4)$

Table 3.21: Charges of the perfect matchings under the global symmetry of the $\mathcal{F}_1$ model. Here t is the fugacity of the R-charge, x is the weight of the $SU(2)$ symmetry, q_1, q_2, b_1, b_2, b_3 and b_4 are, respectively, the charges under the mesonic abelian symmetries $U(1)_1, U(1)_2$ and of the three baryonic $U(1)_{B_1}, U(1)_{B_2}, U(1)_{B_3}$ and $U(1)_{B_4}$.

The Hilbert series of the mesonic moduli space of this model is given by:

$$\begin{aligned} g^{\text{mes}}(t,x,q_1,q_2;\mathcal{F}_1) &= \frac{P(t,x,q_1,q_2;\mathcal{F}_1)}{\left(1-t^{18}x^2q_1^2\right)\left(1-\frac{t^{18}q_1^2}{x^2}\right)\left(1-\frac{t^{18}x^2}{q_1^2}\right)\left(1-\frac{t^{18}}{x^2q_1^2}\right)} \\ &\quad\times\frac{1}{\left(1-t^{18}x^2q_2^2\right)\left(1-\frac{t^{18}q_2^2}{x^2}\right)\left(1-\frac{t^{18}x^2}{q_2^2}\right)\left(1-\frac{t^{18}}{x^2q_2^2}\right)} \\ &\quad\times\frac{1}{\left(1-t^{18}x^2q_1^2q_2^2\right)\left(1-\frac{t^{18}q_1^2q_2^2}{x^2}\right)\left(1-\frac{t^{18}x^2}{q_1^2q_2^2}\right)\left(1-\frac{t^{18}}{x^2q_1^2q_2^2}\right)} \end{aligned} \tag{3.5.224}$$

The fully unrefined version of the Hilbert series of the mesonic moduli space can be written as:

$$g^{\text{mes}}(t,1,1,1;\mathcal{F}_1) = \frac{1+17t^{18}+17t^{36}+t^{54}}{(1-t^{18})^4} \tag{3.5.225}$$

The plethystic logarithm of the Hilbert series of the mesonic moduli space is:

$$\mathrm{PL}[g^{\mathrm{mes}}(t,x,q_1,q_2;\mathcal{F}_1)] \;=\; [2]\left(q_1^2+q_2^2+q_1^2q_2^2+1+\frac{1}{q_1^2q_2^2}+\frac{1}{q_2^2}+\frac{1}{q_1^2}\right)t^{18}-O(t^{36}). \tag{3.5.226}$$

The generators of the mesonic moduli space are

$$\begin{array}{ll} p_ip_jq_k^2r_k^2u_1u_2v_1v_2v_3v_4v_5, & p_ip_jq_k^2r_1r_2u_k^2v_1v_2v_3v_4v_5, \\ p_ip_jq_1q_2r_k^2u_l^2v_1v_2v_3v_4v_5, & p_ip_jq_1q_2r_1r_2u_1u_2v_1v_2v_3v_4v_5, \end{array} \tag{3.5.227}$$

with $i,j,k,l=1,2$, and $k\neq l$.

3.5.2 The missing models

The catalogue of theories we have presented above strikingly shows the power of forward algorithm. Indeed, knowing the tiling of a (2+1)-dimensional CS theory is enough to determine its classical moduli space, its generators, the R-charges of the quiver fields and other interesting properties of the field theory. However, the limits of the algorithm are also clear from this chapter, especially since an extensive search has shed light only on the tilings of 14 out of the original 18 smooth toric Fano manifolds we introduced above. In Figure 3.40, we present the toric diagram corresponding to the missing models.

In contrast to the (3 + 1)-dimensional case, an inverse algorithm does not exist for theories on M2-branes, so that finding the correct tiling for a given theory requires a certain amount of guess work. This problem keeps us from making definite statements about these 4 mysterious theories. For example, it could be that the four manifolds we have been looking at do not admit a consistent gauge theory on M2-branes, or perhaps, it could be the case that they do indeed admit a CS gauge theory description, but this does not correspond to a consistent brane tiling. This would be an even more interesting case from our point of view since, at present, all theories arising from branes at toric singularities can be represented as brane tilings. Whatever the situation, though, we shall postpone any clear consideration after the invention of an inverse algorithm for M2-branes.

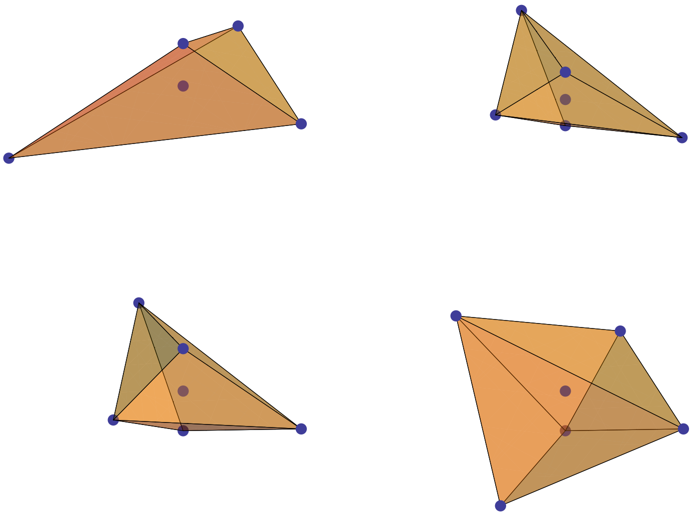


Figure 3.40: The toric diagrams of the missing Fano manifolds. From top left to bottom right, in order, we have $\mathbb{P}^3$, $\mathcal{B}_1$, $\mathcal{B}_2$, and $\mathcal{B}_3$

4 Using Hilbert series for SQCD theories

In the previous chapters we have seen how the Hilbert series can be used to study the mesonic moduli space of various supersymmetric theories lying on D3 and M2-branes. In such situations, brane tilings and the forward algorithm provided a natural framework to compute the partition functions of the gauge theories. Another fascinating application of Hilbert series is in the study of the moduli spaces of $\mathcal{N} = 1$ Supersymmetric Quantum Chromodynamics. The chiral ring and the moduli spaces of these theories has been the subject of a lot of research using the Plethystic Programme [83, 84, 85, 86, 87, 88, 89], Molien–Weyl formula [93, 94, 95, 96, 97, 98, 99], and character expansions [91, 92].

In the following, we shall discuss how Hilbert series can be used to approach the problem of counting gauge invariant operators concentrating on a class of theories known as adjoint SQCD, where the matter content includes $2N_f$ chiral multiplets (half transforming in the fundamental representation of the gauge group and the other half in the anti-fundamental), a chiral field transforming in the adjoint, a vanishing superpotential and a gauge group that we will take to be any of the classical Lie groups.

There have been a series of works [156, 157, 158, 159, 160, 161] on the $SU(N_c)$ adjoint SQCD, as well as [162, 163] on the $SO(N_c)$ and $Sp(N_c)$ theories with various classical superpotentials. It is known that the classical moduli space of the $SU(N_c)$ theory does not get quantum corrections [158, 159, 160]. However, due to technical difficulties, many aspects (e.g., Seiberg duality) of the zero classical superpotential theories have yet to be fully understood[1]. In the following we will examine the structure of the chiral rings of adjoint SQCD (with zero superpotential) through the generators of

[1]Regarding this, let us quote the authors of [157]: 'This interesting model has so far resisted all attempts at a detailed understanding.'

the gauge invariant operators (GIOs) and their relations.

4.1 Dimension of the Moduli Space

At a generic point of the moduli space, the gauge symmetry G is broken completely, and hence there are $d(G)$ broken generators. In the Higgs mechanism, a massless vector multiplet 'eats' an entire chiral multiplet to form a massive vector multiplet. Originally, we have $d(G)$ degrees of freedom coming from the chiral superfields in the adjoint representation (which is $d(G)$ dimensional), and $N_\chi d(\square)$ degrees of freedom coming from the N_χ chiral superfields in the fundamental (and antifundamental) representation (which is $d(\square)$ dimensional). Therefore, of the original $d(G)+N_\chi d(\square)$ chiral degrees of freedom, only $[d(G) + N_\chi d(\square)] - d(G) = N_\chi d(\square)$ singlets are left massless. Therefore, the dimension of the moduli space $\mathcal{M}$ is

$$\dim \mathcal{M} = N_\chi d(\square) \ . \tag{4.1.1}$$

For the $SU(N_c)$ adjoint SQCD with N_f chiral superfields in the fundamental representation and N_f chiral superfields in the anti-fundamental representation, we have $N_\chi = 2N_f$ and $d(\square) = N_c$. Therefore,

$$\dim \mathcal{M}_{(N_f, SU(N_c))} = 2N_f N_c \ . \tag{4.1.2}$$

For the $Sp(N_c)$ adjoint SQCD with $2N_f$ chiral superfields in the fundamental representation, we have $N_\chi = 2N_f$ and $d(\square) = 2N_c$. Therefore,

$$\dim \mathcal{M}_{(N_f, Sp(N_c))} = 4N_f N_c \ . \tag{4.1.3}$$

For the $SO(N_c)$ adjoint SQCD with N_f chiral superfields in the fundamental (vector) representation, we have $N_\chi = N_f$ and $d(\square) = N_c$. Therefore,

$$\dim \mathcal{M}_{(N_f, SO(N_c))} = N_f N_c \ . \tag{4.1.4}$$

4.2 The $SU(N_c)$ Gauge Groups

Let us consider the $SU(N_c)$ theory with N_f chiral superfields transforming in the fundamental representation, N_f chiral superfields transforming in the antifundamental representation (*i.e.* N_f flavours), and 1 chiral superfield transforming in the adjoint representation. The anomaly-free global symmetry of this theory [158] is $SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_{R_1} \times U(1)_{R_2}$.

4.2.1 Examples of Hilbert Series

Below we shall derive Hilbert series for various cases.

The $SU(2)$ Gauge Group

We start the analysis by the simplest case of the $SU(2)$ gauge theory with $2N_f$ chiral superfields transforming in the fundamental representation (N_f flavours) [2] and 1 chiral multiplet transforming in the adjoint representation. The Molien–Weyl formula can be written explicitly as:

$$
\begin{aligned}
g^{(N_f, SU(2))}(s,t) &= \frac{1}{2\pi i}\oint_{|z|=1} \frac{dz}{z}(1-z^2)\ \mathrm{PE}\ [2N_f[1]t + [2]s] \\
&= \frac{1}{2\pi i}\oint_{|z|=1} \frac{dz}{z} \frac{1-z^2}{(1-tz)^{2N_f}(1-\frac{t}{z})^{2N_f}(1-s)(1-sz^2)(1-\frac{s}{z^2})}\ ,
\end{aligned} \tag{4.2.1}
$$

where the Plethystic Exponential, PE, is a functional defined by:

$$
\mathrm{PE}[f(t_1,\ldots,t_n)] = \exp\left(\sum_{k=1}^{\infty} \frac{f(t_1^k,\ldots,t_n^k) - f(0,\ldots,0)}{k}\right). \tag{4.2.2}
$$

Noting that $0 < |t|, |s| < 1$, we use the residue theorem with the poles $z = t,\ \sqrt{s}, -\sqrt{s}$ and find that

$$
\begin{aligned}
g^{(1,SU(2))}(s,t) &= \frac{1+st^2}{(1-s^2)(1-t^2)(1-st^2)^2} \\
g^{(2,SU(2))}(s,t) &= \frac{1+t^2+6st^2-9st^4+s^2t^4+st^6-9s^2t^6+6s^2t^8+s^3t^8+s^3t^{10}}{(1-s^2)(1-t^2)^5(1-st^2)^4} \\
g^{(3,SU(2))}(s,t) &= 1+s^2+s^4+15t^2+21st^2+15s^2t^2+21s^3t^2+15s^4t^2+21s^5t^2+105t^4+ \\
&\quad 210st^4+231s^2t^4+210s^3t^4+231s^4t^4+210s^5t^4+O(s^6)O(t^6)\ , \\
g^{(4,SU(2))}(s,t) &= 1+s^2+s^4+28t^2+36st^2+28s^2t^2+36s^3t^2+28s^4t^2+36s^5t^2+336t^4+ \\
&\quad 630st^4+666s^2t^4+630s^3t^4+666s^4t^4+630s^5t^4+O(s^6)O(t^6)\ ,
\end{aligned} \tag{4.2.3}
$$

[2] Note that the number of fundamental chiral superfields must be even due to the global $\mathbb{Z}_2$ anomaly.

Looking at these generating functions, it is possible to predict the order of the numerator and the terms in the denominator of the generating function for a case with N_f fundamental quarks:

$$g^{(N_f,SU(2))} = \frac{P_{(2N_f-1),(8N_f-6)}(s,t)}{(1-s^2)(1-t^2)^{4N_f-3}(1-st^2)^{2N_f}}, \tag{4.2.4}$$

where $P_{a,b}(s,t)$ is a polynomial of degree a in s and of degree b in t. The calculation is simpler when we make a further identification $s=t$. In which case, we can write down a general form of the generating function:

$$g^{(N_f,SU(2))}(t) = \frac{P_{8N_f-6}(t)}{(1+t)^{2N_f-3}(1-t^2)^{2N_f}(1-t^3)^{2N_f}}, \tag{4.2.5}$$

where $P_{8N_f-6}(t)$ is a palindromic polynomial of degree $8N_f-6$ in t with $P_{8N_f-6}(1) \neq 0$ for all N_f. Observe that the order of the pole at $t=1$ of $g^{(N_f,SU(2))}(t)$ is $4N_f$. Therefore, the dimension of the moduli space $\mathcal{M}_{(N_f,SU(2))}$ is $4N_f$, in agreement with (4.1.2) and (4.1.3).

Character expansion. We can write down the generating function for an *arbitary* number of flavous N_f in terms of representations of the global symmetry $SU(2N_f)$ as follows:

$$\begin{aligned} g^{(N_f,SU(2))} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\sum_{m=0}^{\infty}[2n_1,n_2,0,\ldots,0]s^{n_1+2m}t^{2n_1+2n_2} \\ &= \frac{1}{1-s^2}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}[2n_1,n_2,0,\ldots,0]s^{n_1}t^{2n_1+2n_2}\,. \end{aligned} \tag{4.2.6}$$

We emphasise that $\frac{1}{1-s^2}$ does factor out from the character expansion.

Plethystic logarithms. We shall calculate the plethystic logarithms of the generating functions in (4.2.3):

$$\begin{aligned}
\mathrm{PL}[g^{(1,SU(2))}(s,t)] &= s^2+t^2+3st^2-s^2t^4 \ , \\
\mathrm{PL}[g^{(2,SU(2))}(s,t)] &= s^2+6t^2+10st^2-t^4-15st^4-20s^2t^4+O(s^6)O(t^6) \ , \\
\mathrm{PL}[g^{(3,SU(2))}(s,t)] &= s^2+15t^2+21st^2-15t^4-105st^4-105s^2t^4+O(s^6)O(t^6) \\
\mathrm{PL}[g^{(4,SU(2))}(s,t)] &= s^2+28t^2+36st^2-70t^4-378st^4-336s^2t^4+O(s^6)O(t^6) \\
\mathrm{PL}[g^{(5,SU(2))}(s,t)] &= s^2+45t^2+55st^2-210t^4-990st^4-825s^2t^4+O(s^6)O(t^6) \ .
\end{aligned} \tag{4.2.7}$$

Observe that only the plethystic logarithm of $g^{(1,SU(2))}$ is a polynomial. Therefore, the moduli space of the $SU(2)$ gauge theory with 1 flavour and 1 adjoint matter is a *complete intersection.*

Generators of the GIOs. According to (4.2.7), we see that there are only 3 types of generators of the GIOs in the $SU(2)$ theory, namely

$$\begin{array}{lllll}
\textbf{Casimir invariants}: & s^2 & \to & u \equiv \mathrm{Tr}(\phi^2) & : \ [0,\ldots,0] \ , \\
\textbf{Mesons}: & t^2 & \to & M^{ij} \equiv \epsilon^{ab} Q^i_a \, Q^j_b & : \ [0,1,\ldots,0] \ , \\
\textbf{Adjoint mesons}: & st^2 & \to & A^{ij} \equiv \epsilon^{ab}\epsilon^{cd} Q^i_a \phi_{bc} \, Q^j_d & : \ [2,0,\ldots,0] \ .
\end{array}$$

Note that the total number of generators is quadratic in N_f,

$$1+\binom{2N_f}{2}+N_f(2N_f+1)=4N_f^2+1 \ . \tag{4.2.8}$$

Relations between the generators. From plethystic logarithms (4.2.7), we see that there are 3 types of basic relations:

- **Order t^4:** The relations are known from the theory without adjoint:

  $$\mathrm{Pf}\ M = \epsilon_{i_1 \ldots i_{2N_f}} M^{i_1 i_2} M^{i_3 i_4} = 0 \ . \tag{4.2.9}$$

  They transform in the $SU(2N_f)$ representation $[0,0,0,1,0,\ldots,0]$. We note that this is contained in the decomposition of the symmetric square of the representation $[0,1,0,\ldots,0]$ at order t^2.

- **Order st^4:** The relations transform in the representation $[1,0,1,0,\ldots,0]$, which is contained in the decomposition of the antisymmetric product of the representation $[2,0,\ldots,0]$ at order st^2 and the representation $[0,1,0,\ldots,0]$ at order t^2.

- **Order s^2t^4:** The relations transform in the representation $[0,2,0,\ldots,0]$, which is contained in the decomposition of the symmetric square of the representation $[2,0,\ldots,0]$ at order st^2. In the case of 1 flavour, there is only 1 basic relation which can be written out explicitly as

$$A^{11}A^{22} - \left(A^{12}\right)^2 + \frac{1}{2}u\left(M^{12}\right)^2 = 0 \ . \tag{4.2.10}$$

In summary, for the $SU(2)$ theory, we have the basic relations which transform in the $SU(2N_f)$ representations $[0,0,0,1,0,\ldots,0]$, $[1,0,1,0,\ldots,0]$, and $[0,2,0,\ldots,0]$.

Therefore, we may write down a general expression of the plethystic logarithm in terms of $SU(2N_f)$ representations as

$$\begin{aligned} \mathrm{PL}[g^{(N_f,SU(2))}(s,t)] &= [0,\ldots,0]s^2 + [0,1,\ldots,0]t^2 + [2,0,\ldots,0]st^2 \\ &- [0,0,0,1,0,\ldots,0]t^4 - [1,0,1,0,\ldots,0]st^4 \\ &- [0,2,0,\ldots,0]s^2t^4 + O(s^6)O(t^6) \ . \end{aligned} \tag{4.2.11}$$

The $SU(3)$ Gauge Group

Now let us turn to the $SU(3)$ theory with N_f flavours and 1 adjoint matter. We have N_f chiral superfields transforming in the fundamental representation, N_f chiral superfields transforming in the antifundamental representation, and 1 chiral superfield transforming in the adjoint representation. Therefore, we can apply the Molien–Weyl formula to our theory as follows:

$$\begin{aligned} g^{(N_f,SU(3))} &= \oint d\mu_{SU(3)}\ \mathrm{PE}\left[N_f[1,0]t + N_f[0,1]\tilde{t} + [1,1]s\right] \\ &= \oint_{|z_1|=1} \frac{dz_1}{2\pi i z_1} \oint_{|z_2|=1} \frac{dz_2}{2\pi i z_2} \frac{(1-z_1z_2)\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_2^2}{z_1}\right)}{\left((1-tz_1)(1-t\frac{z_2}{z_1})(1-\frac{t}{z_2})\right)^{N_f}} \times \\ & \frac{1}{\left((1-\tilde{t}z_2)(1-\tilde{t}\frac{z_1}{z_2})(1-\frac{\tilde{t}}{z_1})\right)^{N_f}(1-sz_1z_2)(1-s\frac{z_1^2}{z_2})(1-s\frac{z_2^2}{z_1})(1-s\frac{z_1}{z_2^2})} \times \\ & \frac{1}{(1-s\frac{z_2}{z_1^2})(1-s\frac{1}{z_1z_2})(1-s)^2} \ . \end{aligned} \tag{4.2.12}$$

Applying the residue theorem, we have

$$
\begin{aligned}
g^{(1,SU(3))}(s,t,\tilde{t}) &= \frac{1-s^6t^3\tilde{t}^3}{(1-s^2)(1-s^3)\left(1-t\tilde{t}\right)\left(1-st\tilde{t}\right)\left(1-s^2t\tilde{t}\right)(1-s^3t^3)\left(1-s^3\tilde{t}^3\right)} \\
g^{(2,SU(3))}(s,t,\tilde{t}) &= 1+s^2+s^3+s^4+s^5+2s^6+2st^3+2s^2t^3+6s^3t^3+4s^4t^3+8s^5t^3+ \\
&\quad 8s^6t^3+4t\tilde{t}+4st\tilde{t}+8s^2t\tilde{t}+8s^3t\tilde{t}+12s^4t\tilde{t}+12s^5t\tilde{t}+16s^6t\tilde{t}+10t^2\tilde{t}^2+ \\
&\quad 16st^2\tilde{t}^2+35s^2t^2\tilde{t}^2+41s^3t^2\tilde{t}^2+60s^4t^2\tilde{t}^2+66s^5t^2\tilde{t}^2+85s^6t^2\tilde{t}^2+2s\tilde{t}^3+ \\
&\quad 2s^2\tilde{t}^3+6s^3\tilde{t}^3+4s^4\tilde{t}^3+8s^5\tilde{t}^3+8s^6\tilde{t}^3+20t^3\tilde{t}^3+40st^3\tilde{t}^3+96s^2t^3\tilde{t}^3+ \\
&\quad 136s^3t^3\tilde{t}^3+204s^4t^3\tilde{t}^3+244s^5t^3\tilde{t}^3+316s^6t^3\tilde{t}^3+O(s^7)O(t^4)O(\tilde{t}^4)\ , \\
g^{(3,SU(3))}(s,t,\tilde{t}) &= 1+s^2+s^3+s^4+s^5+2s^6+t^3+8st^3+9s^2t^3+19s^3t^3+17s^4t^3+ \\
&\quad 27s^5t^3+28s^6t^3+9t\tilde{t}+9st\tilde{t}+18s^2t\tilde{t}+18s^3t\tilde{t}+27s^4t\tilde{t}+27s^5t\tilde{t}+ \\
&\quad 36s^6t\tilde{t}+45t^2\tilde{t}^2+81st^2\tilde{t}^2+162s^2t^2\tilde{t}^2+198s^3t^2\tilde{t}^2+279s^4t^2\tilde{t}^2+ \\
&\quad 315s^5t^2\tilde{t}^2+396s^6t^2\tilde{t}^2+\tilde{t}^3+8s\tilde{t}^3+9s^2\tilde{t}^3+19s^3\tilde{t}^3+17s^4\tilde{t}^3+27s^5\tilde{t}^3+ \\
&\quad 28s^6\tilde{t}^3+165t^3\tilde{t}^3+404st^3\tilde{t}^3+893s^2t^3\tilde{t}^3+1301s^3t^3\tilde{t}^3+1881s^4t^3\tilde{t}^3+ \\
&\quad 2289s^5t^3\tilde{t}^3+2878s^6t^3\tilde{t}^3+O(s^7)O(t^4)O(\tilde{t}^4)\ , \\
g^{(4,SU(3))}(s,t,\tilde{t}) &= 1+s^2+s^3+s^4+s^5+2s^6+4t^3+20st^3+24s^2t^3+44s^3t^3+44s^4t^3+ \\
&\quad 64s^5t^3+68s^6t^3+16t\tilde{t}+16st\tilde{t}+32s^2t\tilde{t}+32s^3t\tilde{t}+48s^4t\tilde{t}+48s^5t\tilde{t}+ \\
&\quad 64s^6t\tilde{t}+136t^2\tilde{t}^2+256st^2\tilde{t}^2+492s^2t^2\tilde{t}^2+612s^3t^2\tilde{t}^2+848s^4t^2\tilde{t}^2+ \\
&\quad 968s^5t^2\tilde{t}^2+1204s^6t^2\tilde{t}^2+4\tilde{t}^3+20s\tilde{t}^3+24s^2\tilde{t}^3+44s^3\tilde{t}^3+44s^4\tilde{t}^3+64s^5\tilde{t}^3+ \\
&\quad 68s^6\tilde{t}^3+816t^3\tilde{t}^3+2160st^3\tilde{t}^3+4576s^2t^3\tilde{t}^3+6736s^3t^3\tilde{t}^3+9536s^4t^3\tilde{t}^3+ \\
&\quad 11696s^5t^3\tilde{t}^3+14512s^6t^3\tilde{t}^3+O(s^7)O(t^4)O(\tilde{t}^4)\ ,
\end{aligned}
\tag{4.2.13}
$$

We remark that, although these results seem to be rather lengthy, they contain information which proves to be extremely useful for analyses of the chiral ring. As we shall see from plethystic logarithms, $s^6t^3\tilde{t}^3$ is the minimum order up to which Hilbert series contain all necessary information about the generators and their basic relations.

The calculation is significantly simpler when we make an identification $s=t=\tilde{t}$. In which case, we can write down a general form of the generating function:

$$
g^{(N_f,SU(3))}(t)=\frac{P_{14N_f-10}(t)}{(1-t)^{N_f-1}(1-t^2)^{N_f+2}(1-t^3)^{N_f+1}(1-t^4)^{2N_f-2}(1-t^6)^{N_f}}\ , \tag{4.2.14}
$$

where $P_{14N_f-10}(t)$ is a palindromic polynomial of degree $14N_f - 10$ with $P_{14N_f-10}(1)$ being a non-zero number for any number of flavour. Observe that the order of the pole at $t = 1$ of $g^{(N_f,SU(3))}(t)$ is $6N_f$. Therefore, the dimension of the moduli space $\mathcal{M}_{(N_f,SU(3))}$ is $6N_f$ in agreement with (4.1.2).

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions:

$$\begin{aligned}
\mathrm{PL}[g^{(1,SU(3))}(s,t,\tilde{t})] &= s^2 + s^3 + t\tilde{t} + st\tilde{t} + s^2t\tilde{t} + s^3t^3 + s^3\tilde{t}^3 - s^6t^3\tilde{t}^3 \ , \\
\mathrm{PL}[g^{(2,SU(3))}(s,t,\tilde{t})] &= s^2 + s^3 + 2st^3 + 2s^2t^3 + 4s^3t^3 + 4t\tilde{t} + 4st\tilde{t} + 4s^2t\tilde{t} - s^2t^2\tilde{t}^2 - \\
&\quad s^3t^2\tilde{t}^2 - s^4t^2\tilde{t}^2 + 2s\tilde{t}^3 + 2s^2\tilde{t}^3 + 4s^3\tilde{t}^3 - 4s^2t^3\tilde{t}^3 - 8s^3t^3\tilde{t}^3 - \\
&\quad 20s^4t^3\tilde{t}^3 - 16s^5t^3\tilde{t}^3 - 16s^6t^3\tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4) \ , \\
\mathrm{PL}[g^{(3,SU(3))}(s,t,\tilde{t})] &= s^2 + s^3 + t^3 + 8st^3 + 8s^2t^3 + 10s^3t^3 + 9t\tilde{t} + 9st\tilde{t} + 9s^2t\tilde{t} - \\
&\quad 9s^2t^2\tilde{t}^2 - 9s^3t^2\tilde{t}^2 - 9s^4t^2\tilde{t}^2 + \tilde{t}^3 + 8s\tilde{t}^3 + 8s^2\tilde{t}^3 + 10s^3\tilde{t}^3 - \\
&\quad t^3\tilde{t}^3 - 17st^3\tilde{t}^3 - 81s^2t^3\tilde{t}^3 - 148s^3t^3\tilde{t}^3 - 207s^4t^3\tilde{t}^3 - 143s^5t^3\tilde{t}^3 - \\
&\quad 84s^6t^3\tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4) \ , \\
\mathrm{PL}[g^{(4,SU(3))}(s,t,\tilde{t})] &= s^2 + s^3 + 4t^3 + 20st^3 + 20s^2t^3 + 20s^3t^3 + 16t\tilde{t} + 16st\tilde{t} + 16s^2t\tilde{t} - \\
&\quad 36s^2t^2\tilde{t}^2 - 36s^3t^2\tilde{t}^2 - 36s^4t^2\tilde{t}^2 + 4\tilde{t}^3 + 20s\tilde{t}^3 + 20s^2\tilde{t}^3 + 20s^3\tilde{t}^3 - \\
&\quad 16t^3\tilde{t}^3 - 176st^3\tilde{t}^3 - 576s^2t^3\tilde{t}^3 - 960s^3t^3\tilde{t}^3 - 1024s^4t^3\tilde{t}^3 - 624s^5t^3\tilde{t}^3 - \\
&\quad 240s^6t^3\tilde{t}^3 + O(s^7)O(t^4)O(\tilde{t}^4) \ ,
\end{aligned} \tag{4.2.15}$$

As for the case of $N_c = 2$, the moduli space of the $SU(3)$ theory with 1 flavour and 1 adjoint matter is a complete intersection.

Generators of the GIOs. Armed with plethystic logarithms, we can write down the generators of the GIOs.

$$\begin{array}{llll}
\textbf{Casimir invariants}: & s^k & \to & u_k = \mathrm{Tr}(\phi^k),\ k=2,\ 3 \\
& & & [0,\dots,0;0,\dots,0] \quad 1 \text{ dimensional}\ , \\
\textbf{Mesons}: & t\tilde{t} & \to & M^i_j = Q^i_a \widetilde{Q}^a_j \\
& & & [1,0,\dots,0;0,\dots,0,1] \quad N_f^2 \text{ dimensional}\ , \\
\textbf{Adjoint mesons}: & s^l t\tilde{t} & \to & (A_l)^i_j = Q^i_a (\phi^l)^a_b \widetilde{Q}^b_j,\ l=1,\ 2 \\
& & & [1,0,\dots,0;0,\dots,0,1] \quad N_f^2 \text{ dimensional}\ , \\
\textbf{Baryons}: & t^3 & \to & B^{i_1 i_2 i_3} = \epsilon^{a_1 a_2 a_3} Q^{i_1}_{a_1} Q^{i_2}_{a_2} Q^{i_3}_{a_3} \\
& & & [0,0,1,0,\dots,0;0,\dots,0] \quad \binom{N_f}{3} \text{ dimensional}\ , \\
\textbf{Antibaryons}: & \tilde{t}^3 & \to & \widetilde{B}_{i_1 i_2 i_3} = \epsilon_{a_1 a_2 a_3} \widetilde{Q}^{a_1}_{i_1} \widetilde{Q}^{a_2}_{i_2} \widetilde{Q}^{a_3}_{i_3} \\
& & & [0,\dots,0;0,\dots,1,0,0] \quad \binom{N_f}{3} \text{ dimensional}\ .
\end{array}$$

In addition, we have **adjoint baryons**:

$$\begin{array}{lll}
st^3 & \to & \mathcal{B}^{i_1 i_2 j_1}_{0,0,1} = \epsilon^{a_1 a_2 b_1} Q^{i_1}_{a_1} Q^{i_2}_{a_2} (P_1)^{j_1}_{b_1} \\
& & [1,1,0,\dots,0;0,\dots,0]^{*} \qquad \frac{1}{3}(N_f-1)N_f(N_f+1) \text{ dimensional}\ , \\
s^2 t^3 & \to & \mathcal{B}^{i_1 j_1 j_2}_{0,1,1} = \epsilon^{a_1 b_1 b_2} Q^{i_1}_{a_1} (P_1)^{j_1}_{b_1} (P_1)^{j_2}_{b_2}\ ,\ \mathcal{B}^{i_1 i_2 j_1}_{0,0,2} = \epsilon^{a_1 a_2 b_1} Q^{i_1}_{a_1} Q^{i_2}_{a_2} (P_2)^{j_1}_{b_1} \\
& & [1,1,0,\dots,0;0,\dots,0]^{**} \qquad \frac{1}{3}(N_f-1)N_f(N_f+1) \text{ dimensional}\ , \\
s^3 t^3 & \to & \mathcal{B}^{ijk}_{1,1,1} = \epsilon^{abc} (P_1)^i_a (P_1)^j_b (P_1)^k_c\ ,\ \mathcal{B}^{i_1 j_1 k_1}_{0,1,2} = \epsilon^{a_1 b_1 c_1} Q^{i_1}_{a_1} (P_1)^{j_1}_{b_1} (P_2)^{k_2}_{c_2}\ , \\
& & \mathcal{B}^{i_1 i_2 j_1}_{0,0,3} = \epsilon^{a_1 a_2 b_1} Q^{i_1}_{a_1} Q^{i_2}_{a_2} (P_3)^{j_1}_{b_1}, \\
& & [3,0,\dots,0;0,\dots,0]^{***} \qquad \frac{1}{3!} N_f (N_f+1)(N_f+2) \text{ dimensional}\ ,
\end{array}$$

where $(P_m)^i_a = \phi^{b_1}_a \phi^{b_2}_{b_1} \dots \phi^{b_m}_{b_{m-1}} Q^i_{b_m}$, and the subscript of $\mathcal{B}$ indicates the partition of the power of s in the adjoint baryon. Moreover, in the same spirit as antibaryons, we also have **adjoint antibaryons** which transform in the conjugate representations of adjoint baryons.

***The generator at order st^3.** Note that the generator $\mathcal{B}^{i_1 i_2 j_1}_{0,0,1}$ is subject to a relation:

$$\mathcal{B}^{[i_1 i_2 j_1]}_{0,0,1} = 0\ , \tag{4.2.16}$$

where the square bracket denotes an antisymmetrisation without a normalisation factor. This means that the completely antisymmetric part, which transforms in the $SU(N_f)$ representation $[0,0,1,0,\dots,0]$, vanishes. Note that we can construct the generator by considering the following $SU(N_f)$ tensor product:

$$[0,1,0,\ldots,0] \times [1,0,\ldots,0] = [1,1,0,\ldots,0] + [0,0,1,0,\ldots,0] \ .$$

Therefore, after taking (4.2.16) into account, we conclude that $\mathcal{B}^{i_1 i_2 j_1}_{0,0,1}$ transforms in the $SU(N_f) \times SU(N_f)$ representation [1,1,0, …, 0; 0,…, 0], as stated in the list above.

Two generators at order s^2t^3. We can construct *each* of the generators $\mathcal{B}_{0,1,1}$ and $\mathcal{B}_{0,0,2}$ by considering the following $SU(N_f)$ tensor product:

$$[0,1,0,\ldots,0] \times [1,0,\ldots,0] = [1,1,0,\ldots,0] + [0,0,1,0,\ldots,0] \ .$$

Therefore, if there were no relations, we would say that the generators transform in $2[1,1,0,\ldots,0] + 2[0,0,1,0,\ldots,0]$. However, $\mathcal{B}_{0,1,1}$ and $\mathcal{B}_{0,0,2}$ are subject to the relations:

$$\mathcal{B}^{ijk}_{0,0,2} = -\mathcal{B}^{[ij]k}_{0,1,1} \ , \qquad \mathcal{B}^{[ijk]}_{0,0,2} = -2\mathcal{B}^{[ijk]}_{0,1,1} \ , \tag{4.2.17}$$

which transforms respectively in the $SU(N_f)$ representation $[1,1,0,\ldots,0]+$ $[0,0,1,0,\ldots,0]$ and $[0,0,1,0,\ldots,0]$. Therefore, we are left with the global $SU(N_f) \times SU(N_f)$ representation $[1,1,0,\ldots,0;0,\ldots,0]$, as stated in the above list.

***Three generators at order s^3t^3.** We can construct the generators $\mathcal{B}_{1,1,1}$, $\mathcal{B}_{0,1,2}$, $\mathcal{B}_{0,0,3}$ from the following $SU(N_f)$ tensor products:

$$\begin{aligned}
\Lambda^3[1,0,\ldots,0] &= [0,0,1,0,\ldots,0] \ , \\
[1,0,\ldots,0]^3 &= [3,0,\ldots,0] + 2[1,1,0,\ldots,0] + [0,0,1,0,\ldots,0] \ , \\
[0,1,0,\ldots,0] \times [1,0,\ldots,0] &= [1,1,0,\ldots,0] + [0,0,1,0,\ldots,0] \ .
\end{aligned}$$

Therefore, if there were no relations, we would say that the generators transform in

$$[3,0,\ldots,0] + 3[1,1,0,\ldots,0] + 3[0,0,1,0,\ldots,0] \ .$$

However, these generators are subject to the relations:

$$
\begin{aligned}
\mathcal{B}^{ijk}_{0,0,3} &= -\mathcal{B}^{[ij]k}_{0,1,2} \ , && (4.2.18)\\
\mathcal{B}^{ijk}_{1,1,1} &= -\mathcal{B}^{i[jk]}_{0,1,2} \ , && (4.2.19)\\
2\mathcal{B}^{ijk}_{1,1,1} + \mathcal{B}^{ikj}_{0,0,3} &= -\mathcal{B}^{[i|j|k]}_{0,1,2} \equiv -\left(\mathcal{B}^{ijk}_{0,1,2} - \mathcal{B}^{kji}_{0,1,2}\right) \ . && (4.2.20)
\end{aligned}
$$

These relations transform in the $SU(N_f)$ reperesentation $3[1,1,0,\ldots,0] + 3[0,0,1,0,\ldots,0]$. Therefore, we are left with the global $SU(N_f) \times SU(N_f)$ representation $[3,0,\ldots,0;0,\ldots,0]$, as stated in the above list.

Total number of generators. The total number of generators is cubic in N_f:

$$2 + 3N_f^2 + 2N_f^3 \ . \qquad (4.2.21)$$

Dimensions from plethystic logarithms: A trick. In the above, we computed dimensions of representations for relations using the following technique. For definiteness, let us consider the relations at order $s^2 t^3 \tilde{t}^3$. We know that the dimension $D(N_f)$ of the $SU(N_f) \times SU(N_f)$ representation $[0,0,1,0,\ldots,0;0,\ldots,0,1,0,0]+[1,1,0,\ldots,0;0,\ldots,0,1,0,0]+[0,0,1,0\ldots,0;0,\ldots,0,1,1]+[1,1,0,\ldots,0;0,\ldots,0,1,1]$ must be a polynomial of order 6 in N_f:

$$D(N_f) = \sum_{k=0}^{6} a_k N_f^k \ . \qquad (4.2.22)$$

Observe that we can determine the unknowns $a_0, \ldots, a_6$ from the 7 data points which come from the coefficients of $s^2 t^3 \tilde{t}^3$ in (4.2.15) for $N_f = 0, \ldots, 6$ (where the Hilbert series for $N_f = 0$ can be obtained by setting $t = \tilde{t} = 0$). Solving the following 7 equations simultaneously

$$
\begin{aligned}
D(0) &= 0, \quad D(1) = 0, \quad D(2) = 4, \quad D(3) = 81 \ ,\\
D(4) &= 576, \qquad D(5) = 2500, \qquad D(6) = 8100 \ , && (4.2.23)
\end{aligned}
$$

we find that

$$
\begin{aligned}
a_0 &= 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = 0,\\
a_4 &= \frac{1}{4}, \qquad a_5 = -\frac{1}{2}, \qquad a_6 = \frac{1}{4} \ . && (4.2.24)
\end{aligned}
$$

Substituting back to (4.2.22), we arrive at

$$D(N_f) = \frac{1}{4}N_f^4(N_f-1)^2 \ . \tag{4.2.25}$$

The $SU(4)$ Gauge Group

Let us examine the $SU(4)$ theory with N_f flavours and 1 adjoint matter. We have N_f chiral superfields transforming in the fundamental representation, N_f chiral superfields transforming in the antifundamental representation, and 1 chiral superfield transforming in the adjoint representation. Therefore, the Hilbert series can be written as:

$$\begin{aligned}
g^{(N_f,SU(4))} &= \oint d\mu_{SU(4)} \mathrm{PE}\left[N_f[1,0,0]t + N_f[0,0,1]\tilde{t} + [1,0,1]s\right] \\
&= \oint_{|z_1|=1} \frac{dz_1}{2\pi i z_1} \oint_{|z_2|=1} \frac{dz_2}{2\pi i z_2} \oint_{|z_3|=1} \frac{dz_3}{2\pi i z_3} \frac{\left(1-\frac{z_1^2}{z_2}\right)\left(1-\frac{z_1z_2}{z_3}\right)(1-z_1z_3)}{(1-s)^3\left(1-\frac{sz_1^2}{z_2}\right)\left(1-\frac{sz_2}{z_1^2}\right)\left(1-\frac{sz_2}{z_3^2}\right)} \times \\
&\quad \frac{\left(1-\frac{z_2^2}{z_1z_3}\right)\left(1-\frac{z_2z_3}{z_1}\right)\left(1-\frac{z_3^2}{z_2}\right)}{\left(1-\frac{s}{z_1z_3}\right)\left(1-\frac{sz_1}{z_2z_3}\right)\left(1-\frac{sz_1z_2}{z_3}\right)\left(1-\frac{sz_2^2}{z_1z_3}\right)(1-sz_1z_3)\left(1-\frac{sz_1z_3}{z_2^2}\right)} \times \\
&\quad \frac{1}{\left(1-\frac{sz_3}{z_1z_2}\right)\left(1-\frac{sz_2z_3}{z_1}\right)\left(1-\frac{sz_3^2}{z_2}\right)\left((1-tz_1)\left(1-t\frac{z_2}{z_1}\right)\left(1-t\frac{z_3}{z_2}\right)\left(1-\frac{t}{z_3}\right)\right)^{N_f}} \times \\
&\quad \frac{1}{\left(\left(1-\frac{\tilde{t}}{z_1}\right)\left(1-\tilde{t}\frac{z_1}{z_2}\right)\left(1-\tilde{t}\frac{z_2}{z_3}\right)(1-\tilde{t}z_3)\right)^{N_f}}
\end{aligned} \tag{4.2.26}$$

Applying the residue theorem, we find that

$$\begin{aligned}
g^{(1,SU(4))}(s,t,\tilde{t}) &= \frac{1-s^{12}t^4\tilde{t}^4}{(1-s^2)(1-s^3)(1-s^4)\left(1-t\tilde{t}\right)\left(1-st\tilde{t}\right)\left(1-s^2t\tilde{t}\right)\left(1-s^3t\tilde{t}\right)(1-s^6t^4)\left(1-s^6\tilde{t}^4\right)} \\
g^{(2,SU(4))}(s,t,\tilde{t}) &= 1+s^2+s^3+2s^4+s^5+3s^6+s^2t^4+3s^3t^4+5s^4t^4+7s^5t^4+ \\
&\quad 14s^6t^4+4t\tilde{t}+4st\tilde{t}+8s^2t\tilde{t}+12s^3t\tilde{t}+16s^4t\tilde{t}+20s^5t\tilde{t}+28s^6t\tilde{t}+ \\
&\quad 10t^2\tilde{t}^2+16st^2\tilde{t}^2+36s^2t^2\tilde{t}^2+57s^3t^2\tilde{t}^2+87s^4t^2\tilde{t}^2+114s^5t^2\tilde{t}^2+ \\
&\quad 163s^6t^2\tilde{t}^2+20t^3\tilde{t}^3+40st^3\tilde{t}^3+100s^2t^3\tilde{t}^3+180s^3t^3\tilde{t}^3+296s^4t^3\tilde{t}^3+ \\
&\quad 432s^5t^3\tilde{t}^3+624s^6t^3\tilde{t}^3+s^2\tilde{t}^4+3s^3\tilde{t}^4+5s^4\tilde{t}^4+7s^5\tilde{t}^4+14s^6\tilde{t}^4+O(s)O(t^4)O(\tilde{t}^4), \\
g^{(3,SU(4))}(s,t,\tilde{t}) &= 1+s^2+s^3+2s^4+s^5+3s^6+3st^4+9s^2t^4+21s^3t^4+33s^4t^4+ \\
&\quad 48s^5t^4+75s^6t^4+9t\tilde{t}+9st\tilde{t}+18s^2t\tilde{t}+27s^3t\tilde{t}+36s^4t\tilde{t}+45s^5t\tilde{t}+ \\
&\quad 63s^6t\tilde{t}+45t^2\tilde{t}^2+81st^2\tilde{t}^2+171s^2t^2\tilde{t}^2+279s^3t^2\tilde{t}^2+414s^4t^2\tilde{t}^2+ \\
&\quad 558s^5t^2\tilde{t}^2+774s^6t^2\tilde{t}^2+165t^3\tilde{t}^3+405st^3\tilde{t}^3+974s^2t^3\tilde{t}^3+1787s^3t^3\tilde{t}^3+ \\
&\quad 2920s^4t^3\tilde{t}^3+4297s^5t^3\tilde{t}^3+6103s^6t^3\tilde{t}^3+3s\tilde{t}^4+9s^2\tilde{t}^4+21s^3\tilde{t}^4+33s^4\tilde{t}^4+ \\
&\quad 48s^5\tilde{t}^4+75s^6\tilde{t}^4+O(s)O(t^4)O(\tilde{t}^4), \\
g^{(4,SU(4))}(s,t,\tilde{t}) &= 1+s^2+s^3+2s^4+s^5+3s^6+t^4+15st^4+36s^2t^4+76s^3t^4+117s^4t^4+ \\
&\quad 171s^5t^4+248s^6t^4+16t\tilde{t}+16st\tilde{t}+32s^2t\tilde{t}+48s^3t\tilde{t}+64s^4t\tilde{t}+80s^5t\tilde{t}+ \\
&\quad 112s^6t\tilde{t}+136t^2\tilde{t}^2+256st^2\tilde{t}^2+528s^2t^2\tilde{t}^2+868s^3t^2\tilde{t}^2+1276s^4t^2\tilde{t}^2+ \\
&\quad 1736s^5t^2\tilde{t}^2+2380s^6t^2\tilde{t}^2+816t^3\tilde{t}^3+2176st^3\tilde{t}^3+5152s^2t^3\tilde{t}^3+9488s^3t^3\tilde{t}^3+ \\
&\quad 15424s^4t^3\tilde{t}^3+22720s^5t^3\tilde{t}^3+32032s^6t^3\tilde{t}^3+\tilde{t}^4+15s\tilde{t}^4+36s^2\tilde{t}^4+76s^3\tilde{t}^4+ \\
&\quad 117s^4\tilde{t}^4+171s^5\tilde{t}^4+248s^6\tilde{t}^4+O(s)O(t^4)O(\tilde{t}^4).
\end{aligned} \tag{4.2.27}$$

We emphasise that these results with explicit expansion up to order $s^8t^4\tilde{t}^4$, albeit rather lengthy, turn out to be essential for analyses of the generators and their basic relations in the chiral ring.

Plethystic logarithms. We shall calculate plethystic logarithms of the generating functions:

$$
\begin{aligned}
\mathrm{PL}\left[g^{(1,SU(4))}(s,t,\tilde{t}\ )\right] &= s^2+s^3+s^4+t\tilde{t}+st\tilde{t}+s^2t\tilde{t}+s^3t\tilde{t}+s^6t^4+s^6\tilde{t}^4-s^{12}t^4\tilde{t}^4\ , \\
\mathrm{PL}\left[g^{(2,SU(4))}(s,t,\tilde{t}\ )\right] &= s^2+s^3+s^4+s^2t^4+3s^3t^4+4s^4t^4+3s^5t^4+5s^6t^4+ \\
&\quad 4t\tilde{t}+4st\tilde{t}+4s^2t\tilde{t}+4s^3t\tilde{t}+s^2\tilde{t}^4+3s^3\tilde{t}^4+4s^4\tilde{t}^4+ \\
&\quad 3s^5\tilde{t}^4+5s^6\tilde{t}^4-s^3t^2\tilde{t}^2-s^4t^2\tilde{t}^2-s^5t^2\tilde{t}^2-s^6t^2\tilde{t}^2- \\
&\quad s^4t^4\tilde{t}^4-6s^5t^4\tilde{t}^4-17s^6t^4\tilde{t}^4-O(s^7)O(t^5)O(\tilde{t}^5)\ , \\
\mathrm{PL}\left[g^{(3,SU(4))}(s,t,\tilde{t}\ )\right] &= s^2+s^3+s^4+3st^4+9s^2t^4+18s^3t^4+21s^4t^4+15s^5t^4+ \\
&\quad 15s^6t^4+9t\tilde{t}+9st\tilde{t}+9s^2t\tilde{t}+9s^3t\tilde{t}+17s^6t^3\tilde{t}^3+3s\tilde{t}^4+ \\
&\quad 9s^2\tilde{t}^4+18s^3\tilde{t}^4+21s^4\tilde{t}^4+15s^5\tilde{t}^4+15s^6\tilde{t}^4-9s^3t^2\tilde{t}^2- \\
&\quad 9s^4t^2\tilde{t}^2-9s^5t^2\tilde{t}^2-9s^6t^2\tilde{t}^2-s^2t^3\tilde{t}^3-s^3t^3\tilde{t}^3-s^4t^3\tilde{t}^3- \\
&\quad 9s^2t^4\tilde{t}^4-54s^3t^4\tilde{t}^4-189s^4t^4\tilde{t}^4-O(s^5)O(t^4)O(\tilde{t}^4)\ ,
\end{aligned}
$$

Generators of the GIOs. Armed with plethystic logarithms, we can write down the generators of the GIOs.

$$
\begin{array}{llcl}
\textbf{Casimir invariants}: & s^k & \to & u_k=\mathrm{Tr}(\phi^k),\ k=2,\ 3,\ 4 \\
 & & & [0,\ldots,0;0,\ldots,0]\quad 1 \text{ dimensional}\ , \\
\textbf{Mesons}: & t\tilde{t} & \to & M^i_j=Q^i_a\widetilde{Q}^a_j \\
 & & & [1,0,\ldots,0;0,\ldots,0,1]\quad N_f^2 \text{ dimensional}\ , \\
\textbf{Adjoint mesons}: & s^lt\tilde{t} & \to & (A_l)^i_j=Q^i_a\,(\phi^l)^a_b\,\widetilde{Q}^b_j,\ l=1,\ 2,\ 3 \\
 & & & [1,0,\ldots,0;0,\ldots,0,1]\quad N_f^2 \text{ dimensional}\ , \\
\textbf{Baryons}: & t^4 & \to & B^{i_1i_2i_3i_4}=\epsilon^{a_1a_2a_3a_4}Q^{i_1}_{a_1}Q^{i_2}_{a_2}Q^{i_3}_{a_3}Q^{i_4}_{a_4} \\
 & & & [0,0,0,1,0,\ldots,0;0,\ldots,0]\quad \binom{N_f}{4} \text{ dimensional}\ , \\
\textbf{Antibaryons}: & \tilde{t}^4 & \to & \text{similar expression to baryons with } Q\to\widetilde{Q} \\
 & & & [0,\ldots,0;0,\ldots,0,1,0,0,0]\quad \binom{N_f}{4} \text{ dimensional}\ .
\end{array}
$$

In addition, we have **adjoint baryons** (in a similar fashion to the case of $N_c = 3$):

$$
\begin{array}{lll}
st^4 & \to & \mathcal{B}_{0,0,0,1} = \epsilon QQQP_1 \\
 & & [1,0,1,0\ldots,0;0,\ldots,0]^{\dagger} \\
s^2t^4 & \to & \mathcal{B}_{0,0,0,2} = \epsilon QQQP_2 \ , \ \mathcal{B}_{0,0,1,1} = \epsilon QQP_1P_1 \\
 & & [0,2,0,\ldots,0;0,\ldots,0] + [1,0,1,0,\ldots,0;0,\ldots,0]^{\ddagger} \\
s^3t^4 & \to & \mathcal{B}_{0,0,0,3} = \epsilon QQQP_3 \ , \ \mathcal{B}_{0,0,1,2} = \epsilon QQP_1P_2 \ , \ \mathcal{B}_{0,1,1,1} = \epsilon QP_1P_1P_1 \\
 & & [2,1,0,\ldots,0;0,\ldots,0] + [1,0,1,0,\ldots,0;0,\ldots,0] \\
s^4t^4 & \to & \mathcal{B}_{0,0,0,4} = \epsilon QQQP_4 \ , \ \mathcal{B}_{0,0,1,3} = \epsilon QQP_1P_3 \ , \ \mathcal{B}_{0,0,2,2} = \epsilon QQP_2P_2 \\
 & & \mathcal{B}_{0,1,1,2} = \epsilon QP_1P_1P_2 \ , \ \mathcal{B}_{1,1,1,1} = \epsilon P_1P_1P_1P_1 \\
 & & [2,1,0,\ldots,0;0,\ldots,0] + [0,2,0,\ldots,0;0,\ldots,0] \\
s^5t^4 & \to & \mathcal{B}_{0,0,0,5} = \epsilon QQQP_5 \ , \ \mathcal{B}_{0,0,1,4} = \epsilon QQP_1P_4 \ , \ \mathcal{B}_{0,0,2,3} = \epsilon QQP_2P_2 \\
 & & \mathcal{B}_{0,1,1,3} = \epsilon QP_1P_1P_3 \ , \ \mathcal{B}_{0,1,2,2} = \epsilon QP_1P_2P_2 \ , \ \mathcal{B}_{1,1,1,2} = \epsilon P_1P_1P_1P_2 \\
 & & [2,1,0,\ldots,0;0,\ldots,0] \\
s^6t^4 & \to & \mathcal{B}_{0,0,0,6} = \epsilon QQQP_6 \ , \ \mathcal{B}_{0,0,1,5} = \epsilon QQP_1P_5 \ , \ \mathcal{B}_{0,0,3,3} = \epsilon QQP_3P_3 \\
 & & \mathcal{B}_{0,1,1,4} = \epsilon QP_1P_1P_4 \ , \ \mathcal{B}_{0,1,2,3} = \epsilon QP_1P_2P_3 \ , \ \mathcal{B}_{0,2,2,2} = \epsilon QP_2P_2P_2 \\
 & & \mathcal{B}_{1,1,1,3} = \epsilon P_1P_1P_1P_3 \ , \ \mathcal{B}_{1,1,2,2} = \epsilon P_1P_1P_2P_2 \ , \ \mathcal{B}_{0,0,2,4} = \epsilon QQP_2P_4 \\
 & & [4,0,\ldots,0;0,\ldots,0]
\end{array}
$$

where $(P_m)^i_a = \phi^{b_1}_a \phi^{b_2}_{b_1} \ldots \phi^{b_m}_{b_{m-1}} Q^i_{b_m}$, and the subscript of $\mathcal{B}$ indicates the partition of the power of s in the adjoint baryon. In the above, we suppressed the indices with the understanding that each epsilon tensor is contracted over all colour indices. Moreover, we have **adjoint antibaryons** which transform in the conjugate representations of adjoint baryons.

As for the case of $SU(3)$ gauge group, we emphasise that the representations written above are *not* the ones in which the generators transform; however, they are the ones in which the relations have already been taken into account. For example,

†**The generator at order st^4.** Note that the generator $\mathcal{B}^{i_1i_2i_3j_1}_{0,0,0,1} = \epsilon^{a_1a_2a_3b_1} Q^{i_1}_{a_1} Q^{i_2}_{a_2} Q^{i_3}_{a_3} (P_1)^{j_1}_{b_1}$ satisfies a relation:

$$\mathcal{B}^{[i_1i_2i_3j_1]}_{0,0,0,1} = 0 \ , \tag{4.2.28}$$

where the square bracket denotes an antisymmetrisation without a normalisation factor. This means that the completely antisymmetric part, which transforms in the $SU(N_f)$ representation [0,0,0,1,0,...,0], vanishes. Note that we can construct the generator by considering the following $SU(N_f)$ tensor product:

$$[0,0,1,0,\dots,0] \times [1,0,\dots,0] = [1,0,1,0,\dots,0] + [0,0,0,1,0,\dots,0] \ .$$

Therefore, after taking (4.2.28) into account, we conclude that $\mathcal{B}^{i_1 i_2 i_3 j_1}_{0,0,0,1}$ transforms in the $SU(N_f) \times SU(N_f)$ representation [1,0,1,0, ..., 0; 0,..., 0], as stated in the list above.

‡**Two generators at order $s^2 t^4$.** We can construct $\mathcal{B}_{0,0,1,1}, \mathcal{B}_{0,0,0,2}$ by considering the $SU(N_f)$ tensor products:

$$\begin{aligned} [0,1,0,\dots,0]^2 &= [0,2,0,\dots,0] + [1,0,1,0,\dots,0] + [0,0,0,1,0,\dots,0] \ , \\ [0,0,1,0,\dots,0] \times [1,0,\dots,0] &= [1,0,1,0,\dots,0] + [0,0,0,1,0,\dots,0] \ . \end{aligned}$$

They are however subject to the relations:

$$\mathcal{B}^{[ijkm]}_{0,0,0,2} = 6\left(\mathcal{B}^{i[jkm]}_{0,0,0,2} + \mathcal{B}^{i[jkm]}_{0,0,1,1}\right) \ , \qquad (4.2.29)$$

$$\mathcal{B}^{[ijkm]}_{0,0,0,2} = -3\mathcal{B}^{[ijkm]}_{0,0,1,1} \ , \qquad (4.2.30)$$

which transform respectively in the $SU(N_f)$ representations $[1,0,1,0,\dots,0]+[0,0,0,1,0,\dots,0]$, $[0,0,0,1,0,\dots,0]$. Therefore, we are left with the global $SU(N_f) \times SU(N_f)$ representation $[0,2,0,\dots,0;0,\dots,0]+[1,0,1,0,\dots,0;0,\dots,0]$, as stated in the above list.

Total number of generators. Using the trick mentioned in the previous subsection, we find that the total number of generators is

$$3 + 4N_f^2 + 2N_f^4 \ . \qquad (4.2.31)$$

From (4.2.8), (4.2.21) and (4.2.31), we establish the following observation[3]:

Observation 4.2.1. *The total number of generators in the $SU(N_c)$ theory with N_f fundamental chiral superfields and 1 adjoint chiral superfield is of*

[3]From now on, we use the word **Observation** to refer to a strong conjecture which can be deduced, in a consistent manner, from a number of non-trivial results presented earlier.

order $N_f^{N_c}$.

Note that this is substantially higher that the theory with no adjoints.

A comment on representations. From a number of examples in the cases of $SU(3)$ and $SU(4)$ gauge groups, we establish the following observations:

Observation 4.2.2. *Any adjoint baryon of the form* $\mathcal{B}_{\alpha_1,\ldots,\alpha_{N_c}}$ *(with* $0 \leq \alpha_1 \leq \ldots \leq \alpha_{N_c} \leq N_c$*) exists in the theory as a generator. Note that* $N_A \equiv \alpha_1 + \ldots + \alpha_{N_c}$ *is the total number of adjoint fields appearing in this particular adjoint baryon. It satisfies the bounds:* $0 \leq N_A \leq \frac{1}{2}N_c(N_c - 1)$.

Observation 4.2.3. *Whenever there is more than one way in partitioning adjoint fields into an adjoint baryon, there exists a relation between those options. The relation must transform in such a way that it cancels some representations associated with the generators, so that the leftover agrees with plethystic logarithms.*

4.2.2 Adjoint Baryons: A Combinatorial Problem of Partitions

From a combinatorial point of view, Observation 4.2.2 suggests that an adjoint baryon is simply a partition of the N_A objects into N_c slots (without distinction between the slots). This leads to an interesting problem: For given N_A and N_c, how many adjoint baryons can be constructed?

The partition function. This problem can be elegantly solved using a partition function (Hilbert series). Suppose that the number of slots N_c is held fixed. Let t be a fugacity conjugate to the number of adjoint fields N_A. The required partition function is

$$\mathcal{Z}_{N_c}(t) = \sum_{N_A=0}^{\infty} a_{N_A,N_c} t^{N_A} \ , \qquad (4.2.32)$$

where a_{N_A,N_c} is the number of adjoint baryons which can be constructed for given N_A and N_c. We can write the partition $\mathcal{Z}$ in another way as follows. Let n_i be the number of slots which contain i adjoint fields. It is then easy to see that $n_1 + 2n_2 + \ldots + N_c n_{N_c}$ is the total number of adjoint fields. We can therefore write

$$\mathcal{Z}_{N_c}(t) \quad = \quad \sum_{\{n_i\}} t^{n_1+2n_2+\ldots+N_c n_{N_c}} = \frac{1}{(1-t)(1-t^2)\ldots(1-t^{N_c})} = \prod_{j=1}^{N_c} \frac{1}{1-t^j} \; . \tag{4.2.33}$$

This formula is also known as a partition function of N_c bosonic one-dimensional harmonic oscillators [83]. Equating (4.2.32) and (4.2.33), we find that

$$\sum_{N_A=0}^{\infty} a_{N_A,N_c} t^{N_A} = \prod_{j=1}^{N_c} \frac{1}{1-t^j} \; . \tag{4.2.34}$$

Thus, the number of adjoint baryons a_{N_A,N_c} is given by the coefficient of t^{N_A} in the power series of the product $\prod_{j=1}^{N_c} \frac{1}{1-t^j}$. In other words,

$$a_{N_A,N_c} = \frac{1}{2\pi i} \oint_{|t|=1} \frac{\mathrm{d}t}{t^{N_A+1}} \prod_{j=1}^{N_c} \frac{1}{1-t^j} \; . \tag{4.2.35}$$

We note that this result is correct for any $N_A \geq 0$ but we are particularly interested in the case of $0 \leq N_A \leq \binom{Nc}{2}$.

Example: $N_c = 3$**.** The power series of the last expression in (4.2.34) is given by

$$\prod_{j=1}^{3} \frac{1}{1-t^j} = 1 + t + 2t^2 + 3t^3 + \ldots \; . \tag{4.2.36}$$

Therefore, for 0, 1, 2 and 3 adjoint fields, we can construct 1, 1, 2 and 3 adjoint baryons, respectively. This agrees with the earlier results above Equation (4.2.16) for the $SU(3)$ gauge group.

Example: $N_c = 4$**.** The power series of the last expression in (4.2.34) is given by

$$\prod_{j=1}^{4} \frac{1}{1-t^j} = 1 + t + 2t^2 + 3t^3 + 5t^4 + 6t^5 + 9t^6 + \ldots \; . \tag{4.2.37}$$

Therefore, for 0, 1, 2, 3, 4, 5 and 6 adjoint fields, we can construct 1, 1, 2, 3, 5, 6 and 9 adjoint baryons, respectively. This agrees with the earlier results above Equation (4.2.28) for the $SU(4)$ gauge group.

4.2.3 The Canonical Free Energy

An immediate consequence of (4.1.1) is a general form of the unrefined generating function:

$$g^{(N_f,SU(N_c))}(t) = \frac{P(t)}{\prod_i (1-t^{n_i})^{d_i}} \ , \tag{4.2.38}$$

where $P(t)$ is a palindromic polynomial with $P(1) \neq 0$, and the order of the pole $t = 1$ of $g^{(N_f,SU(N_c))}(t)$ is

$$\sum_i d_i = \dim \mathcal{M}_{(N_f,SU(N_c))} = 2N_f N_c \ . \tag{4.2.39}$$

We can define the **canonical free energy** of the system as

$$F(t) = -\log g^{(N_f,SU(N_c))}(t) \ . \tag{4.2.40}$$

It is easy to see from (4.2.38), (4.2.39) and (4.2.40) that in the large and N_c limit

$$F(t) \sim f(t) N_f N_c \ , \tag{4.2.41}$$

where $f(t)$ is some function of order 1. In other words, in this limit, the canonical free energy scales linearly with the dimension of the moduli space, which in turn is linear in both the number of colours and the number of flavours.

4.2.4 Complete Intersection Moduli Space

Having seen from a number of examples in preceding subsections that the moduli space of the theories with 1 flavour and 1 adjoint chiral multiplet is a complete intersection, we shall conjecture that this statement is true for any $SU(N_c)$ gauge group.

Generators and relations. The generators of the theories with 1 flavour and 1 adjoint matter are

$$\begin{array}{lllll}
\textbf{Casimir invariants}: & s^k & \to & u_k = \mathrm{Tr}(\phi^k),\ k = 2, \ldots,\ N_c \\
 & & & \text{1 operator for each } k\ , \\
\textbf{Meson}: & t\tilde{t} & \to & M = Q_a \widetilde{Q}^a \\
 & & & \text{1 operator}\ , \\
\textbf{Adjoint mesons}: & s^l t\tilde{t} & \to & (A_l) = Q_a\ (\phi^l)^a_b\ \widetilde{Q}^b,\ l = 1, \ldots,\ N_c - 1 \\
 & & & \text{1 operator for each } l\ , \\
\textbf{Adjoint baryon}: & s^{(N_c-1)N_c/2} t^{N_c} & \to & \mathcal{B}_{0,1,2,\ldots,N_c-1} = \epsilon^{abc\ldots d} Q_a (P_1)_b (P_2)_c \ldots (P_{N_c-1})_d \\
 & & & \text{1 operator}\ , \\
\textbf{Adjoint antibaryon}: & s^{(N_c-1)N_c/2} \tilde{t}^{N_c} & \to & \text{adjoint baryon with } Q \to \widetilde{Q} \\
 & & & \text{1 operator}\ .
\end{array}$$

We see that there are altogether $2N_c + 1$ generators. Note that in all examples we checked the number of relations is 1. We therefore assume that there is precisely one basic relation at order $s^{(N_c-1)N_c} t^{N_c} \tilde{t}^{N_c}$.

Since the dimension of the moduli space (which is $2N_c$ from (4.1.1)) is equal to the number of generators (which is $2N_c + 1$) minus the number of basic relations (which is assumed to be 1), it gives a strong indication that the moduli space is a **complete intersection**.

General formula. As a consequence, we can write down a fully refined generating function for an arbitrary N_c as

$$g^{(1,SU(N_c))}(s,t,\tilde{t}) = \frac{1 - s^{(N_c-1)N_c} t^{N_c} \tilde{t}^{N_c}}{(1 - s^{(N_c-1)N_c/2} t^{N_c})(1 - s^{(N_c-1)N_c/2} \tilde{t}^{N_c}) \prod_{k=2}^{N_c} (1 - s^k) \prod_{l=0}^{N_c-1} (1 - s^l t\tilde{t})}\ . \tag{4.2.42}$$

A General Expression for The Relation

In §4.2.1, the relation for the case of $N_c = 2, N_f = 1$ is written explicitly in (4.2.10). It is interesting to find a general expression of the relation for any N_c (with $N_f = 1$).[4] In this and only this subsection, we include the factor of $1/k$ into the Casimir invariant u_k, namely $u_k = \frac{1}{k}\mathrm{Tr}(\phi^k)$.

[4]Special thanks to Nathan Seiberg, Kenneth Intriligator and Michael Douglas for discussions.

The case of $N_c = 3$. Let us introduce the operators A_3 and A_4 in the usual way, *i.e.* $A_l = Q_a \, \phi^l \, \widetilde{Q}^a$. Note that they can be written in terms of basic generators as

$$A_3 = u_3 A_0 + u_2 A_1 \ , \qquad A_4 = u_3 A_1 + u_2 A_2 \ , \tag{4.2.43}$$

where A_0 denotes the meson M. Then, the relation can be written as

$$\mathcal{B}\widetilde{\mathcal{B}} = A_0 A_2 A_4 - A_0 A_3^2 + 2A_1 A_2 A_3 - A_1^2 A_4 - A_2^3 \ , \tag{4.2.44}$$

where $\mathcal{B}$ and $\widetilde{\mathcal{B}}$ respectively denote the baryon and antibaryon. The moduli space is $\mathbb{C}^9/\mathcal{I}_3$, where the ideal $\mathcal{I}_3$ is given by the 3 relations: (4.2.43) and (4.2.44).

The case of $N_c = 4$. We introduce the operators $A_4, \ldots, A_6$, which can be written in terms of basic generators as

$$\begin{aligned}
A_4 &= u_4 A_0 - \frac{1}{2} u_2^2 A_0 + u_3 A_1 + u_2 A_2 \ , \\
A_5 &= u_4 A_1 - \frac{1}{2} u_2^2 A_1 + u_3 A_2 + u_2 A_3 \ , \\
A_6 &= u_2 u_4 A_0 - \frac{1}{2} u_2^3 A_0 + u_2 u_3 A_1 + \frac{1}{2} u_2^2 A_2 + u_4 A_2 + u_3 A_3 \ .
\end{aligned} \tag{4.2.45}$$

Then, the relation can be written as

$$\begin{aligned}
\mathcal{B}\widetilde{\mathcal{B}} = \; & A_0 A_2 A_4 A_6 - A_0 A_2 A_5^2 - A_0 A_3^2 A_6 + 2A_0 A_3 A_4 A_5 - A_0 A_4^3 - A_1^2 A_4 A_6 + A_1^2 A_5^2 \\
& + 2A_1 A_2 A_3 A_6 - 2A_1 A_2 A_4 A_5 - 2A_1 A_3^2 A_5 + 2A_1 A_3 A_4^2 - A_2^3 A_6 + 2A_2^2 A_3 A_5 \\
& + A_2^2 A_4^2 - 3A_2 A_3^2 A_4 + A_3^4 \ .
\end{aligned} \tag{4.2.46}$$

The moduli space is $\mathbb{C}^{12}/\mathcal{I}_4$, where the ideal $\mathcal{I}_4$ is given by the 4 relations: (4.2.45) and (4.2.46).

A general expression. We can generalise (4.2.44) and (4.2.46) to any number of colours. The relation can be written compactly as[5]

$$\mathcal{B}\widetilde{\mathcal{B}} \;=\; \det \mathcal{A} \ , \tag{4.2.47}$$

[5] We thank Michael Douglas for pointing out this elegant expression.

where $\mathcal{A}_{ij} = A_{i+j}$ and $0 \leq i, j \leq N_c - 1$.

It is interesting to examine this formula in the spacial case of $N_c = 2$. The adjoint baryon is given by $\mathcal{B} = \epsilon^{ab} Q^1_a (P_1)^1_b = A^{11}$, whereas the adjoint antibaryon is given by $\widetilde{\mathcal{B}} = \epsilon_{ab} Q^{a2} (P_1)^{b2} = \epsilon_{ab} \epsilon^{ad} Q^2_d \epsilon^{bc} (P_1)^2_c = -\delta^d_b Q^2_d \epsilon^{bc} (P_1)^2_c = -\epsilon^{bc} Q^2_b (P_1)^2_c = -A^{22}$ (note the minus sign). Similarly, it is easy to see that $\det \mathcal{A} = A_0 A_2 - A_1^2 = M^{12} \left(\frac{1}{2} u M^{12} \right) - (A^{12})^2$. We thus correctly recover the formula (4.2.10).

4.3 The $Sp(N_c)$ Gauge Groups

Let us turn to the $Sp(N_c)$ gauge theory[6] with $2N_f$ chiral superfields transforming in the fundamental representation (N_f flavours)[7] and 1 chiral superfield transforming in the adjoint representation. The anomaly-free global symmetry of this theory [163] is $SU(2N_f) \times U(1) \times U(1)_R$.

4.3.1 Examples of Hilbert Series

Below we shall derive Hilbert series for various cases.

The $Sp(2)$ Gauge Group

Let us now examine the $Sp(2)$ gauge theory with $2N_f$ chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint

[6] We shall use the notation where the rank of $Sp(n)$ is n and $Sp(1)$ is isomorphic to $SU(2)$.

[7] Note that the number of fundamental chiral multiplets must be even due to the global $\mathbb{Z}_2$ anomaly.

representation. The Hilbert series for this theory is

$$\begin{aligned} g^{(N_f,Sp(2))} &= \int_{Sp(2)} d\mu_{Sp(2)}(z_1,z_2) PE\left[2N_f[1,0]t\ +[2,0]s\right] \\ &= \oint_{|z_1|=1} \frac{dz_1}{2\pi i z_1} \oint_{|z_2|=1} \frac{dz_2}{2\pi i z_2} \frac{(1-z_1^2)(1-z_2)(1-\frac{z_1^2}{z_2})(1-\frac{z_2^2}{z_1^2})}{\left((1-tz_1)(1-t\frac{z_2}{z_1})(1-t\frac{z_1}{z_2})(1-t\frac{1}{z_1})\right)^{2N_f}} \times \\ & \quad \frac{1}{(1-s)^2(1-sz_1^2)(1-sz_2)(1-s\frac{z_1^2}{z_2})(1-s\frac{z_2^2}{z_1^2})(1-s\frac{z_1^2}{z_2^2})(1-s\frac{z_2}{z_1^2})} \times \\ & \quad \frac{1}{(1-s\frac{1}{z_2})(1-s\frac{1}{z_1^2})} \end{aligned} \tag{4.3.48}$$

Applying the residue theorem, we can compute Hilbert series for various N_f:

$$\begin{aligned} g^{(1,Sp(2))}(s,t) &= \frac{(1-s^6t^4)(1-s^4t^4)}{(1-s^2)(1-s^4)(1-t^2)(1-st^2)^3(1-s^2t^2)(1-s^3t^2)^3} , \\ g^{(2,Sp(2))}(s,t) &= 1+s^2+2s^4+2s^6+6t^2+10st^2+12s^2t^2+20s^3t^3+18s^4t^2+30s^5t^2+ \\ & \quad 24s^6t^2+21t^4+60st^4+111s^2t^4+165s^3t^4+232s^4t^4+270s^5t^4+357s^6t^4+ \\ & \quad 56t^6+210st^6+500s^2t^6+890s^3t^6+1330s^4t^6+1836s^5t^6+O(s^6)O(t^6) , \\ g^{(3,Sp(2))}(s,t) &= 1+s^2+2s^4+2s^6+15t^2+21st^2+30s^2t^2+42s^3t^2+45s^4t^2+63s^5t^2+ \\ & \quad 60s^6t^2+120t^4+315st^4+561s^2t^4+840s^3t^4+1122s^4t^4+1365s^5t^4+1689s^6t^4+ \\ & \quad 679t^6+2485st^6+5530s^2t^6+9436s^3t^6+13895s^4t^6+18571s^5t^6+O(s^6)O(t^6) , \\ g^{(4,Sp(2))}(s,t) &= 1+s^2+2s^4+2s^6+28t^2+36st^2+56s^2t^2+72s^3t^2+84s^4t^2+108s^5t^2+ \\ & \quad 112s^6t^2+406t^4+1008st^4+1786s^2t^4+2646s^3t^4+3488s^4t^4+4284s^5t^4+ \\ & \quad 5198s^6t^4+4032t^6+14196st^6+30576s^2t^6+51192s^3t^6+74424s^4t^6+ \\ & \quad 98352s^5t^6+O(s^6)(t^6) . \end{aligned} \tag{4.3.49}$$

A general form of the generating function when we set $s=t=\tilde{t}$ is

$$g^{(N_f,Sp(2))}(t) = \frac{P_{24N_f-16}(t)}{(1+t)^{6N_f-7}(1-t^2)^{2N_f+1}(1+t^2)(1-t^3)^{4N_f-1}(1-t^5)^{2N_f}} . \tag{4.3.50}$$

Plethystic Logarithms. We shall calculate the plethystic logarithms of the generating functions:

$$\begin{aligned}
\mathrm{PL}\left[g^{(1,Sp(2))}(s,t)\right] &= s^2+s^4+t^2+3st^2+s^2t^2+3s^3t^2-s^4t^4-s^6t^4 \ , \\
\mathrm{PL}\left[g^{(2,Sp(2))}(s,t)\right] &= s^2+s^4+6t^2+10st^2+6s^2t^2+10s^3t^2-s^2t^4-15s^3t^4-21s^4t^4- \\
&\quad 15s^5t^4-20s^6t^4-6s^2t^6-10s^3t^6+16s^5t^6+O(s^6)O(t^6) \ , \\
\mathrm{PL}\left[g^{(3,Sp(2))}(s,t)\right] &= s^2+s^4+15t^2+21st^2+15s^2t^2+21s^3t^2-15s^2t^4-105s^3t^4- \\
&\quad 120s^4t^4-105s^5t^4-105s^6t^4-t^6-35st^6-189s^2t^6-175s^3t^6+ \\
&\quad 36s^4t^6+539s^5t^6+O(s^6)O(t^6) \ , \\
\mathrm{PL}\left[g^{(4,Sp(2))}(s,t)\right] &= s^2+s^4+28t^2+36st^2+28s^2t^2+36s^3t^2-70s^2t^4-378s^3t^4- \\
&\quad 406s^4t^4-378s^5t^4-336s^6t^4-28t^6-420st^6-1512s^2t^6- \\
&\quad 1176s^3t^6+448s^4t^6+4452s^5t^6+O(s^6)O(t^6) \ . \qquad (4.3.51)
\end{aligned}$$

The $Sp(3)$ Gauge Group

We now move to examining the generating funtions and their plethystic logarithms for the $Sp(3)$ gauge group with $2N_f$ chiral fields transforming in the fundamental representation and one in the adjoint representation of the group. The Hilbert series for this theory is

$$\begin{aligned}
g^{(N_f,Sp(3))} &= \int_{Sp(3)} d\mu_{Sp(3)}(z_1,z_2,z_3)\mathrm{PE}\left[2N_f[1,0,0]t\ +[2,0,0]s\right] \\
&= \left(\prod_{k=1}^{3}\oint_{|z_k|=1}\frac{dz_k}{2\pi i z_k}\right)\frac{(1-z_1^2)(1-\frac{z_1^2}{z_2})(1-z_2)(1-\frac{z_2^2}{z_1^2})(1-\frac{z_1z_2}{z_3})}{(1-s\frac{1}{z_1^2})(1-sz_1^2)(1-s\frac{z_1^2}{z_2^2})(1-s\frac{1}{z_2})(1-s\frac{z_1^2}{z_2})}\times \\
&\quad \frac{(1-\frac{z_2^2}{z_1z_3})(1-\frac{z_3}{z_1})(1-\frac{z_1z_3}{z_2})(1-\frac{z_3^2}{z_2^2})}{(1-s\frac{z_2^2}{z_1^2})(1-s\frac{z_2^2}{z_3^2})(1-s\frac{z_1}{z_3})(1-s\frac{z_2}{z_1z_3})(1-s\frac{z_1z_2}{z_3})(1-s\frac{z^2}{z_1z_3})(1-s\frac{z_3}{z_1})}\times \\
&\quad \frac{1}{(1-s\frac{z_3}{z_1z_2})(1-s\frac{z_2}{z_1^2})(1-s\frac{z_1z_3}{z_2})(1-s\frac{z_3^2}{z_2^2})(1-sz_2)(1-s\frac{z_1z_3}{z_2^2})(1-s)^3}\times \\
&\quad \frac{1}{((1-t\frac{1}{z_1})(1-tz_1)(1-t\frac{z_1}{z_2})(1-t\frac{z_2}{z_1})(1-t\frac{z_2}{z_3})(1-t\frac{z_3}{z_2}))^{2N_f}}
\end{aligned} \qquad (4.3.1)$$

Applying the residue theorem we can compute Hilbert series for various N_f:

$$
\begin{aligned}
g^{(1,Sp(3))}(s,t) &= \frac{(1-s^6t^4)(1-s^8t^4)(1-s^{10}t^4)}{(1-s^2)(1-s^4)(1-s^6)(1-t^2)(1-st^2)^3(1-s^2t^2)(1-s^3t^2)^3(1-s^4t^2)(1-s^5t^2)^3}\ , \\
g^{(2,Sp(3))}(s,t) &= 1+s^2+2s^4+3s^6+6t^2+10st^2+12s^2t^2+20s^3t^2+24s^4t^2+40s^5t^2+ \\
&\quad 36s^6t^2+60s^7t^2+54s^8t^2+21t^4+60st^4+112s^2t^4+180s^3t^4+289s^4t^4+ \\
&\quad 405s^5t^4+571s^6t^4+56t^6+210st^6+512s^2t^6+1000s^3t^6+1738s^4t^6+ \\
&\quad 2790s^5t^6+O(s^6)O(t^6)\ , \\
g^{(3,Sp(3))}(s,t) &= 1+s^2+2s^4+3s^6+15t^2+21st^2+30s^2t^2+42s^3t^2+60s^4t^2+84s^5t^2+ \\
&\quad 90s^6t^2+126s^7t^2+135s^8t^2+120t^4+315st^4+576s^2t^4+945s^3t^4+1467s^4t^4+ \\
&\quad 2100s^5t^4+2820s^6t^4+680t^6+2520st^6+5944s^2t^6+11501s^3t^6+19889s^4t^6+ \\
&\quad 31262s^5t^6+O(s^6)O(t^6)\ , \\
g^{(4,Sp(3))}(s,t) &= 1+s^2+2s^4+3s^6+28t^2+36st^2+56s^2t^2+72s^3t^2+112s^4t^2+144s^5t^2+ \\
&\quad 168s^6t^2+216s^7t^2+252s^8t^2+406t^4+1008st^4+1856s^2t^4+3024s^3t^4+ \\
&\quad 4678s^4t^4+6678s^5t^4+8874s^6t^4+4060t^6+14616st^6+34048s^2t^6+ \\
&\quad 65472s^3t^6+112280s^4t^6+175044s^5t^6+O(s^6)(t^6)\ .
\end{aligned}
\tag{4.3.2}
$$

Plethystic Logarithms. We shall calculate the plethystic logarithms of the generating functions:

$$
\begin{aligned}
\mathrm{PL}\left[g^{(1,Sp(3))}(s,t)\right] &= s^2+s^4+s^6+t^2+3st^2+s^2t^2+3s^3t^2+s^4t^2+3s^5t^2-s^6t^4- \\
&\quad s^8t^4-s^{10}t^4\ , \\
\mathrm{PL}\left[g^{(2,Sp(3))}(s,t)\right] &= s^2+s^4+s^6+6t^2+10st^2+6s^2t^2+10s^3t^2+6s^4t^2+10s^5t^2- \\
&\quad s^4t^4-15s^5t^4-21s^6t^4-15s^7t^4-21s^8t^4-6s^4t^6-10s^5t^6- \\
&\quad O(s^6)O(t^6)\ , \\
\mathrm{PL}\left[g^{(3,Sp(3))}(s,t)\right] &= s^2+s^4+s^6+15t^2+21st^2+15s^2t^2+21s^3t^2+15s^4t^2+21s^5t^2- \\
&\quad 15s^4t^4-105s^5t^4-120s^6t^4-s^2t^6-35s^3t^6-190s^4t^6-210s^5t^6- \\
&\quad O(s^6)O(t^6)\ , \\
\mathrm{PL}\left[g^{(4,Sp(3))}(s,t)\right] &= s^2+s^4+s^6+28t^2+36st^2+28s^2t^2+36s^3t^2+28s^4t^2+36s^5t^2- \\
&\quad 70s^4t^4-378s^5t^4-406s^6t^4-28s^2t^6-420s^3t^6-1540s^4t^6- \\
&\quad 1596s^5t^6-O(s^9)O(t^9)\ .
\end{aligned}
\tag{4.3.3}
$$

4.3.2 Generators of the Chiral Ring

Below we summarise the generators of $Sp(N_c)$ adjoint SQCD [162, 163] and representations of $SU(2N_f) \times SU(2N_f)$ in which they transform.

$$
\begin{array}{llll}
\textbf{Casimir invariants}: & s^{2k} & \rightarrow & u_{2k} = \mathrm{Tr}(\phi^{2k}) \qquad (k = 1,\, 2,\, \ldots, N_c) \\
 & & & [0,\ldots,0] \quad 1 \text{ dimensional }, \\
\textbf{Mesons}: & t^2 & \rightarrow & M^{ij} = J^{ab} Q^i_a Q^j_b \qquad (a, b = 1,\, 2,\, \ldots, 2N_c) \\
 & & & [0,1,0,\ldots,0] \quad N_f(2N_f - 1) \text{ dimensional }, \\
\textbf{Even adjoint mesons}: & s^{2l} t^2 & \rightarrow & (A_{2l})^{ij} = J^{a_1 b_1} \ldots J^{c_{2l-1} b_{2l}} J^{c_{2l} a_{2l}} Q^i_{a_1} \phi_{b_1 c_1} \ldots \phi_{b_{2l} c_{2l}} Q^j_{a_{2l}} \\
 & & & [0,1,0,\ldots,0] \quad N_f(2N_f - 1) \text{ dimensional}, \\
\textbf{Odd adjoint mesons}: & s^{2k-1} t^2 & \rightarrow & (A_{2k-1})^{ij} = J^{a_1 b_1} \ldots J^{c_{2k-1} a_{2k-1}} Q^i_{a_1} \phi_{b_1 c_1} \ldots \phi_{b_{2k-1} c_{2k-1}} Q^j_{a_{2k-1}} \\
 & & & [2,0,\ldots,0] \quad N_f(2N_f + 1) \text{ dimensional },
\end{array}
$$

where $l = 1, \ldots, N_c - 1$ and $k = 1, \ldots, N_c$.

The total number of generators is

$$N_c(1 + 4N_f^2) \ . \tag{4.3.4}$$

We note that for $N_c = 1$, $Sp(1)$ is isomorphic to $SU(2)$. In which case, we recover the $SU(2)$ adjoint SQCD and hence (4.3.4) reduces to (4.2.8).

4.3.3 Complete Intersection Moduli Space

We claim that the moduli space of the $Sp(N_c)$ adjoint SQCD with 2 fundamental chiral multiplets ($N_f = 1$) and 1 adjoint chiral multiplet is a complete intersection. A general expression of the phethystic logarithm for the complete intersection case can be written as

$$\mathrm{PL}[g^{(1,Sp(N_c))}(s,t)] = \sum_{k=1}^{N_c} s^{2k} + \sum_{k=1}^{N_c} (3s^{2k-1} + s^{2(k-1)}) t^2 - \sum_{k=1}^{N_c} s^{2(N_c+k-1)} t^4 \ . \tag{4.3.5}$$

Note that the number of relations is equal to the rank of the gauge group and not to 1 as might naively be expected from the case of $SU(N_c)$ gauge group.

4.4 The $SO(N_c)$ Gauge Groups

Let us turn to the $SO(N_c)$ gauge theory with N_f chiral superfields transforming in the fundamental (vector) representation (N_f flavours) and 1 chiral superfield transforming in the adjoint representation. The global symmetry of this theory [162] is $SU(N_f) \times U(1)_R$.

4.4.1 Examples of Hilbert Series

We shall derive Hilbert series for various cases. Note that the following subsections are *not* merely a collection of results. They will turn out to be essential for analysing the generators of the chiral ring.

The $SO(3)$ Gauge Group

Let us now examine the $SO(3)$ gauge theory with N_f chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint representation. The Hilbert series for this theory is

$$
\begin{aligned}
g^{(N_f,SO(3))} &= \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}(1-z)\ \mathrm{PE}\ \left[N_f[1]t+[1]s\right] \\
&= \frac{1}{2\pi i}\oint_{|z|=1}\frac{dz}{z}\frac{1-z}{(1-t)^{N_f}(1-tz)^{N_f}(1-\frac{t}{z})^{N_f}(1-s)(1-sz)(1-\frac{s}{z})}\ .
\end{aligned}
\tag{4.4.1}
$$

Applying the residue theorem, we find that

$$
\begin{aligned}
g^{(1,SO(3))}(s,t) &= \frac{1}{(1-s^2)\,(1-st)\,(1-t^2)}\ , \\
g^{(2,SO(3))}(s,t) &= \frac{1-s^2t^4}{(1-s^2)\,(1-st)^2(1-st^2)\,(1-t^2)^3}\ , \\
g^{(3,SO(3))}(s,t) &= \frac{1-t+t^2+3st^2-3st^3-s^2t^3+s^2t^4-s^2t^5}{(1-s^2)(1-t)^6(1+t)^5(1-st)^3}\ ,
\end{aligned}
\tag{4.4.2}
$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions:

$$
\begin{aligned}
\mathrm{PL}\left[g^{(1,SO(3))}(s,t)\right] &= s^2+st+t^2 \ , \\
\mathrm{PL}\left[g^{(2,SO(3))}(s,t)\right] &= s^2+2st+3t^2+st^2-s^2t^4 \ , \\
\mathrm{PL}\left[g^{(3,SO(3))}(s,t)\right] &= s^2+3st+6t^2+3st^2+t^3-s^2t^3-3st^4-6s^2t^4+O(s^5)O(t^5) \ ,
\end{aligned}
\tag{4.4.3}
$$

The $N_f = 1$ moduli space is freely generated, *i.e.* there is no relation between the generators. Since the plethystic logarithm for $N_f = 2$ is a polynomial (not an infinite series), it follows that the $N_f = 2$ moduli space is a complete intersection.

The $SO(4)$ Gauge Group

Let us now examine the $SO(4)$ gauge theory with N_f chiral multiplets in the fundamental representation and 1 chiral superfield in the adjoint representation. The Hilbert series for this theory is

$$
g^{(N_f,SO(4))} = \oint d\mu_{SO(4)} \mathrm{PE}\ [N_f[1,0]t+[1,1]s] \ . \tag{4.4.4}
$$

Applying the residue theorem, we find that small

$$
\begin{aligned}
g^{(1,SO(4))}(s,t) &= \frac{1}{(1-s^2)^2(1-t^2)(1-s^2t^2)} \ , \\
g^{(2,SO(4))}(s,t) &= \frac{(1-s^2t^4)(1-s^4t^4)}{(1-s^2)^2(1-t^2)^3(1-st^2)^2(1-s^2t^2)^3} \ , \\
g^{(3,SO(4))}(s,t) &= \frac{1}{(1-s^2)^2(1-st)^3(1+st)^3(1-t^2)^6(1-st^2)^3} \times (1+3st^2+ \\
&\quad 3s^2t^2+3s^3t^4-2s^2t^6-8s^3t^6-8s^4t^6-2s^5t^6+3s^4t^8+3s^5t^{10}+ \\
&\quad 3s^6t^{10}+s^7t^{12}) \ , \\
g^{(4,SO(4))}(s,t) &= 1+2s^2+3s^4+10t^2+12st^2+30s^2t^2+24s^3t^2+50s^4t^2+ \\
&\quad 56t^4+120st^4+267s^2t^4+330s^3t^4+513s^4t^4+O(s^5)O(t^5) \ ,
\end{aligned}
\tag{4.4.5}
$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions:

$$
\begin{aligned}
\mathrm{PL}\left[g^{(1,SO(4))}(s,t)\right] &= 2s^2+t^2+s^2t^2 \ , \\
\mathrm{PL}\left[g^{(2,SO(4))}(s,t)\right] &= 2s^2+3t^2+2st^2+3s^2t^2-s^2t^4-s^4t^4 \ , \\
\mathrm{PL}\left[g^{(3,SO(4))}(s,t)\right] &= 2s^2+6t^2+6st^2+6s^2t^2-6s^2t^4-6s^3t^4- \\
&\quad 6s^4t^4+O(s^5)O(t^5) \ , \\
\mathrm{PL}\left[g^{(4,SO(4))}(s,t)\right] &= 2s^2+10t^2+12st^2+10s^2t^2+t^4-23s^2t^4- \\
&\quad 30s^3t^4-20s^4t^4+O(s^5)O(t^5) \ , \qquad (4.4.6)
\end{aligned}
$$

The $N_f = 1$ moduli space is freely generated, *i.e.* there is no relation between the generators. Since the plethystic logarithm for $N_f = 2$ is a polynomial (not an infinite series), it follows that the $N_f = 2$ moduli space is a complete intersection.

The $SO(5)$ Gauge Group

Let us now examine the $SO(5)$ gauge theory with N_f chiral multiplets in the fundamental representation $[1,0]$ and 1 chiral superfield in the adjoint representation $[0,2]$. The Hilbert series for this theory is

$$
g^{(N_f,SO(5))} = \oint d\mu_{SO(5)} \times \mathrm{PE}\ [N_f[1,0]t+[0,2]s] \ . \qquad (4.4.7)
$$

Applying the residue theorem, we find that

$$
\begin{aligned}
g^{(1,SO(5))}(s,t) &= \frac{1}{(1-s^2)(1-s^4)(1-s^2t)(1-t^2)(1-s^2t^2)} \ , \\
g^{(2,SO(5))}(s,t) &= \frac{(1-s^4t^4)(1-s^6t^4)}{(1-s^2)(1-s^4)(1-s^2t)^2(1-t^2)^3(1-st^2)(1-s^2t^2)^3(1-s^3t^2)} \ , \\
g^{(3,SO(5))}(s,t) &= 1+s^2+2s^4+3s^2t+3s^4t+6t^2+3st^2+12s^2t^2+6s^3t^2+ \\
&\quad 24s^4t^2st^3+18s^2t^3+10s^3t^3+36s^4t^3+21t^4+18st^4+63s^2t^4+ \\
&\quad 54s^3t^4+O(s^4)O(t^4) \ .
\end{aligned}
\qquad (4.4.8)
$$

Plethystic logarithms. We shall calculate plethystic logarithms of generating functions:

$$\begin{aligned}
\mathrm{PL}\left[g^{(1,SO(5))}(s,t)\right] &= s^2+s^4+s^2t+t^2+s^2t^2 \ , \\
\mathrm{PL}\left[g^{(2,SO(5))}(s,t)\right] &= s^2+s^4+2s^2t+3t^2+st^2+3s^2t^2+s^3t^2-s^4t^4-s^6t^4 \ , \\
\mathrm{PL}\left[g^{(3,SO(5))}(s,t)\right] &= s^2+s^4+3s^2t+6t^2+3st^2+6s^2t^2+3s^3t^2+st^3-s^5t^3- \\
&\quad 3s^3t^4-6s^4t^4-3s^5t^4-6s^6t^4-3s^4t^5-s^2t^6-s^4t^6+O(s^6)O(t^6) \ ,
\end{aligned} \tag{4.4.9}$$

The $N_f = 1$ moduli space is freely generated, *i.e.* there is no relation between the generators. Since the plethystic logarithm for $N_f = 2$ is a polynomial (not an infinite series), it follows that the $N_f = 2$ moduli space is a complete intersection.

4.4.2 Generators of the Chiral Ring

Using plethystic logarithms computed in preceding subsections, we can write down the generators of $SO(N_c)$ adjoint SQCD and representations of $SU(N_f)$ in which they transform. We will make a distinction between $SO(2m)$ and $SO(2m+1)$ gauge groups.

The $SO(2m)$ Gauge Groups

The generators of the chiral ring in the case of $SO(2m)$ are as follows:

$$\begin{array}{llll}
\textbf{Casimir invariants}: & s^{2k} & \to & u_{2k}=\mathrm{Tr}(\phi^{2k}) \quad k=1,\dots,m-1 \\
 & & & [0,\dots,0] \quad 1 \text{ dimensional} \ , \\
\textbf{Even adjoint mesons}: & s^{2k}t^2 & \to & (A_{2k})^{ij}=Q^i_{a_1}(\phi^{2k})_{a_1a_2}Q^j_{a_2} \quad k=0,\dots,m-1 \\
 & & & [2,0,\dots,0] \quad \frac{1}{2}N_f(N_f+1) \text{ dimensional} \ , \\
\textbf{Odd adjoint mesons}: & s^{2k+1}t^2 & \to & (A_{2k+1})^{ij}=Q^i_{a_1}(\phi^{2k+1})_{a_1a_2}Q^j_{a_2} \quad k=0,\dots,m-2 \\
 & & & [0,1,0,\dots,0] \quad \frac{1}{2}N_f(N_f-1) \text{ dimensional} \ , \\
\textbf{Adjoint baryons}: & s^kt^l & \to & \mathcal{B}_k^{i_1\dots i_l}=\epsilon^{a_1\dots a_{2k+l}}\phi_{a_1a_2}\dots\phi_{a_{2k-1}a_{2k}}Q^{i_1}_{a_{2k+1}}\dots Q^{i_l}_{a_{2k+l}} \\
 & & & \text{with } 2k+l=2m, \ k=0,\dots,m \\
 & & & [0,\dots,0,1_{l;L},0,\dots,0] \quad \binom{N_f}{2m-2k} \text{ dimensional} \ .
\end{array}$$

The total number of generators is

$$m(1+N_f^2) - \frac{N_f(N_f-1)}{2} - 1 + \sum_{k=0}^{m} \binom{N_f}{2m-2k} , \qquad (4.4.10)$$

which behaves as $N_f^{N_c}/N_c!$ for large values of N_c.

The $SO(2m+1)$ Gauge Groups

The generators of the chiral ring in the case of $SO(2m+1)$ are as follows:

$$\begin{array}{llll}
\textbf{Casimir invariants}: & s^{2k} & \to & u_{2k} = \mathrm{Tr}(\phi^{2k}) \quad k=1,\ldots,m \\
 & & & [0,\ldots,0] \quad 1 \text{ dimensional} , \\
\textbf{Even adjoint mesons}: & s^{2k}t^2 & \to & (A_{2k})^{ij} = Q^i_{a_1}(\phi^{2k})_{a_1a_2}Q^j_{a_2} \quad k=0,\ldots,m-1 \\
 & & & [2,0,\ldots,0] \quad \frac{1}{2}N_f(N_f+1) \text{ dimensional} , \\
\textbf{Odd adjoint mesons}: & s^{2k+1}t^2 & \to & (A_{2k+1})^{ij} = Q^i_{a_1}(\phi^{2k+1})_{a_1a_2}Q^j_{a_2} \quad k=0,\ldots,m-1 \\
 & & & [0,1,0,\ldots,0] \quad \frac{1}{2}N_f(N_f-1) \text{ dimensional} , \\
\textbf{Adjoint baryons}: & s^k t^l & \to & \mathcal{B}^{i_1\ldots i_l}_k = \epsilon^{a_1\ldots a_{2k+l}}\phi_{a_1a_2}\ldots\phi_{a_{2k-1}a_{2k}}Q^{i_1}_{a_{2k+1}}\ldots Q^{i_l}_{a_{2k+l}} \\
 & & & \text{with } 2k+l=2m+1,\ k=0,\ldots,m \\
 & & & [0,\ldots,0,1_{l;L},0,\ldots,0] \quad \binom{N_f}{2m+1-2k} \text{ dimensional}
\end{array}$$

The total number of generators is

$$m\left(1+N_f^2\right) + \sum_{k=0}^{m} \binom{N_f}{2m+1-2k}, \qquad (4.4.11)$$

which behaves as $N_f^{N_c}/N_c!$ for large values of N_c. It is to be noted that, among the adjoint mesons of $SO(N_c)$ gauge theories we have just listed, the generator A_0 is what we referred to as meson in the preceding sections. We have listed it among the adjoint mesons only for simplicity.

4.4.3 Complete Intersection Moduli Space

We have seen from several examples in the preceding sections that

- The moduli space of the $SO(N_c)$ gauge theories with 1 fundamental chiral superfield and 1 adjoint chiral superfield is freely generated,

- The moduli space of the $SO(N_c)$ gauge theories with 2 fundamental chiral superfields and 1 adjoint chiral superfield is a complete intersection.

Generalising these examples, we write down general expressions for the fully refined plethystic logarithms in the case of 2 fundamental chiral superfields as

$$\begin{aligned}
\mathrm{PL}[g^{(2,SO(2m+1))}(s,t_1,t_2)] &= \sum_{k=1}^{m} s^{2k} + s^m t_1 + s^m t_2 + \sum_{k=1}^{m} \left[s^{2(k-1)}(t_1^2 + t_1 t_2 + t_2^2) + s^{2k-1} t_1 t_2 \right] \\
&\quad - \sum_{k=1}^{m} s^{2(m+k-1)} (t_1 t_2)^2 \ , \\
\mathrm{PL}[g^{(2,SO(2m))}(s,t_1,t_2)] &= \sum_{k=1}^{m-1} s^{2k} + s^m + s^{m-1} t_1 t_2 + \sum_{k=1}^{m} s^{2(k-1)}(t_1^2 + t_1 t_2 + t_2^2) \\
&\quad + \sum_{k=1}^{m-1} s^{2k-1} t_1 t_2 - \sum_{k=1}^{m} s^{2(m+k-2)} (t_1 t_2)^2 \ .
\end{aligned} \tag{4.4.12}$$

Note that the number of relations is equal to the rank of the gauge group and not to 1 as might naively be expected from the case of $SU(N_c)$ gauge group. The plethystic logarithms in the case of 1 fundamental chiral superfields can be easily obtained by setting $t_1 = t,\ t_2 = t$:

$$\begin{aligned}
\mathrm{PL}[g^{(1,SO(2m+1))}(s,t)] &= \sum_{k=1}^{m} s^{2k} + s^m t + \sum_{k=1}^{m} s^{2(k-1)} t^2 \ , \\
\mathrm{PL}[g^{(1,SO(2m))}(s,t)] &= \sum_{k=1}^{m-1} s^{2k} + s^m + \sum_{k=1}^{m} s^{2(k-1)} t^2 \ .
\end{aligned} \tag{4.4.13}$$

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