Deformed Spaces, Symmetries and Gauge Theories*

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Abstract

This contribution is based on talks given by Frank Meyer (Section 1) and Marija Dimitrijević (Section 2). In the first section we review the basic concepts of deformed spaces and deformed symmetries. We discuss general features of differential calculi, introduce the star-product and star-product representations of differential operators. As examples we treat the canonically deformed space and the κ-deformed space. In the second section we study gauge theories on deformed spaces. Special attention is given to gauge theory the on κ-deformed space (which was introduced as an example in the first part). Nevertheless, the analysis is done in a rather general way such that it could also be applied to the other deformed spaces.

1. Deformed Spaces and Symmetries

1.1. Deformed Spaces

In gauge theories one usually considers differential space-time manifolds and fibers that admit a representation of a Lie-group. In the noncommutative realm, the notion of a point is no longer well-defined and we have to give up

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* The two talks given by the authors are based on common work with Larisa Jonke, Lutz Möller, Efrossini Tsouchnika, Julius Wess and Michael Wohlgenannt [1].

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the concept of differentiable manifolds. However, the space of functions on a manifold is an algebra. A generalization of this algebra can be considered in the noncommutative case. We take the algebra freely generated by the noncommutative coordinates $\hat{x}^\mu$, $\mu = 0 \dots n$, which respect commutation relations of the type

$$[\hat{x}^\mu, \hat{x}^\nu] = C^{\mu\nu}(\hat{x}) \neq 0\,. \tag{1}$$

Mathematically this means that we take the space of formal power series in the coordinates $\hat{x}^\mu$ and divide by the ideal generated by the above relations [2]:

$$\widehat{\mathcal{A}}_{\hat{x}} = \mathbb{C}\langle\langle \hat{x}^0, \dots, \hat{x}^n \rangle\rangle / ([\hat{x}^\mu, \hat{x}^\nu] - C^{\mu\nu}(\hat{x}))\,.$$

This we call a *deformed coordinate space.*

The function $C^{\mu\nu}(\hat{x})$ is unknown. It should be a function that vanishes at large distances where we experience the commutative world and may be determined by experiments. Nevertheless, one can consider a power-series expansion

$$C^{\mu\nu}(\hat{x}) = i\,\theta^{\mu\nu} + iC^{\mu\nu}_\rho \hat{x}^\rho + (q\hat{R}^{\mu\nu}_{\rho\sigma} - \delta^\nu_\rho \delta^\mu_\sigma)\,\hat{x}^\rho \hat{x}^\sigma + \dots,$$

where $\theta^{\mu\nu}$, $C^{\mu\nu}_\rho$ and $q\,\hat{R}^{\mu\nu}_{\rho\sigma}$ are constants, and study cases where the commutation relations are constant, linear or quadratic in the coordinates. At very short distances those cases provide a reasonable approximation for $C^{\mu\nu}(\hat{x})$ and lead to the following three structures which are of particular interest since they satisfy the so-called Poincare-Birkhoff-Witt property [1]

1. Canonical structure:
$$[\hat{x}^\mu, \hat{x}^\nu] = i\,\theta^{\mu\nu}. \tag{2}$$

2. Lie algebra structure:
$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\rho \hat{x}^\rho. \tag{3}$$

3. Quantum Space structure:
$$\hat{x}^\mu \hat{x}^\nu = q\,\hat{R}^{\mu\nu}_{\rho\sigma}\,\hat{x}^\rho \hat{x}^\sigma. \tag{4}$$

1.2. Symmetries on Deformed Spaces

In general the commutation relations (1) are not covariant with respect to undeformed symmetries. For example the canonical commutation relations (2) break Lorentz symmetry.

Then the question naturally arises whether we can *deform* the symmetry in such a way that it is consistent with the deformed space and that it

[1] The PBW-property states that the space of polynomials in noncommutative coordinates of a given degree is isomorphic to the space of polynomials in the commutative coordinates.

reduces to the undeformed symmetry in the commutative limit. The answer is yes: Lie groups can be deformed in the category of Hopf algebras [2]. The generated objects are called *Quantum Groups.* To make this more explicit we give two examples.

1.2.1. The Canonically Deformed Space

For a long time it was common belief that there does not exist a deformed symmetry for the canonically deformed space. However, recently a quantum group-symmetry was discovered [3] [3]. Let us state the result without deriving it:

$$\begin{aligned}
[\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0, \qquad\qquad [\hat{\delta}_\omega, \hat{\partial}_\rho] = \omega_\rho{}^\mu \hat{\partial}_\mu, \\
[\hat{\delta}_\omega, \hat{\delta}'_\omega] &= \hat{\delta}_{\omega\times\omega'}, \qquad (\omega\times\omega)'_\mu{}^\nu = -(\omega_\mu{}^\sigma \omega'_\sigma{}^\nu - \omega'_\mu{}^\sigma \omega_\sigma{}^\nu), \\
\Delta\hat{\partial}_\mu &= \hat{\partial}_\mu \otimes 1 + 1 \otimes \hat{\partial}_\mu, \\
\Delta\hat{\delta}_\omega &= \hat{\delta}_\omega \otimes 1 + 1 \otimes \hat{\delta}_\omega + \frac{i}{2}(\theta^{\mu\nu}\omega_\nu{}^\rho - \theta^{\rho\nu}\omega_\nu{}^\mu)\hat{\partial}_\rho \otimes \hat{\partial}_\mu.
\end{aligned} \tag{5}$$

Here the deformed generators of Lorentz-transformations are denoted by $\hat{\delta}_\omega$ with constant transformation parameters ω. Note that the algebra relations are undeformed and the deformation takes place exclusively in the co-sector of the Hopf-algebra. The coproduct $\Delta\hat{\delta}_\omega$ [4] of $\hat{\delta}_\omega$ contains $\theta^{\mu\nu}$-corrections. It is interesting that the coproduct of $\hat{\delta}_\omega$ closes only in the Poincare-algebra and not in the Lorentz-algebra. This may be the reason why this symmetry remained undiscovered for such a long time. The consequences of this new symmetry are part of future investigations by various groups.

1.2.2. κ-deformed Space-time

As an example for the Lie structure we introduce the κ-deformed space-time [5]:

$$[\hat{x}^\mu, \hat{x}^\nu] = iC^{\mu\nu}_\rho \hat{x}^\rho, \tag{6}$$

where $C^{\mu\nu}_\lambda = a\,(\eta^\mu_n \eta^\nu_\lambda - \eta^\nu_n \eta^\mu_\lambda)$ and where we use the signature $\eta^{\mu\nu} = \mathrm{diag}(1, -1, \ldots, -1)$. In the following Latin indices always run from 0 to $n-1$ whereas Greek indices run from 0 to n. The commutation relations (6) are covariant with respect to the κ-deformed Poincare algebra [6]. There is a basis where the Lorentz-algebra remains again undeformed

$$[M^{\mu\nu}, M^{\rho\sigma}] = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}, \tag{7}$$

[2] To be more precise the algebra of functions on a Lie group can be deformed. Since Lie groups themselves form a discrete set, a continuous deformation is not possible.

[3] Actually, a deformed symmetry which is just the dual to the one given here was already introduced some years ago in [4] but was basically unknown to the community of physicists working in that field.

[4] The coproduct is a structure map of a Hopf algebra. It tells us how to act on a product of functions.

[5] The κ-deformed space appears also naturally in the context of Doubly Special Relativity [5].

but the commutators of derivatives with the generators $M^{\mu\nu}$

$$\begin{aligned} [M^{ij}, \hat{\partial}_\mu] &= \eta^j_\mu \hat{\partial}^i - \eta^i_\mu \hat{\partial}^j, \\ [M^{in}, \hat{\partial}_n] &= \hat{\partial}^i, \\ [M^{in}, \hat{\partial}_j] &= \eta^i_j \, \frac{e^{2ia\hat{\partial}_n} - 1}{2\,i\,a} - \frac{i\,a}{2}\, \eta^i_j \, \hat{\partial}^l \hat{\partial}_l + i\,a\,\hat{\partial}^i \hat{\partial}_j, \\ [\hat{\partial}_\mu, \hat{\partial}_\nu] &= 0 \end{aligned} \tag{8}$$

and the co-algebra sector are deformed

$$\begin{aligned} \Delta M^{ij} &= M^{ij} \otimes 1 + 1 \otimes M^{ij}, \\ \Delta M^{in} &= M^{in} \otimes 1 + e^{ia\hat{\partial}_n} \otimes M^{in} + i\,a\,\hat{\partial}_k \otimes M^{ik}, \\ \Delta \hat{\partial}_i &= \hat{\partial}_i \otimes 1 + e^{i\,a\,\hat{\partial}_n} \otimes \hat{\partial}_i, \\ \Delta \hat{\partial}_n &= \hat{\partial}_n \otimes 1 + 1 \otimes \hat{\partial}_n \,. \end{aligned} \tag{9}$$

The generators $M^{\mu\nu}$ and $\hat{\partial}_\mu$ act as follows on the coordinates:

$$\begin{aligned} [M^{ij}, \hat{x}^\mu] &= \eta^{\mu j} \hat{x}^i - \eta^{\mu i} \hat{x}^j, \\ [M^{in}, \hat{x}^\mu] &= \eta^{\mu n} \hat{x}^i - \eta^{\mu i} \hat{x}^n + i\,a\,M^{i\mu}, \\ [\hat{\partial}_i, \hat{x}^\mu] &= \eta^\mu_i - i\,a\,\eta^{\mu n} \hat{\partial}_i, \qquad [\hat{\partial}_n, \hat{x}^\mu] = \eta^\mu_n \,. \end{aligned} \tag{10}$$

Note that all the commutation relations reduce the classical relations in the limit $a \to 0$.

1.3. Differential Calculus

Derivatives are maps on the deformed coordinate space [7]

$$\hat{\partial} : \widehat{\mathcal{A}}_{\hat{x}} \to \widehat{\mathcal{A}}_{\hat{x}} \,.$$

Such a map in particular has to map the ideal generated by the commutation relations (1) into itself. If this is the case we say that the map $\hat{\partial}$ respects the commutation relations (1) or is compatible with them.
To find a suitable map it is convenient to make a general ansatz for the commutator of a derivative and a coordinate:

$$[\hat{\partial}_\mu, \hat{x}^\nu] = \delta^\nu_\mu + \sum_j A^{\nu\rho_1 \ldots \rho_j}_\mu \hat{\partial}_{\rho_1} \ldots \hat{\partial}_{\rho_j} \,. \tag{11}$$

The coefficient functions $A^{\nu\rho_1 \ldots \rho_j}_\mu$ are of the order of the deformation parameter and vanish in the commutative limit. Requiring consistency of (11) with the commutation relations of the deformed space leads to conditions on the coefficients $A_\mu^{\nu\rho_1\ldots\rho_j}$. In general a solution for those conditions is not unique.

In the case of a canonically deformed space (1) one immediately verifies that actually the undeformed differential calculus

$$[\hat{\partial}_\mu, \hat{x}^\nu] := \delta_\mu^\nu \tag{12}$$

is compatible with the commutation relations (1).

For the κ-deformed space-time there exist several sets of differential calculi which are all equivalent. The derivatives obtained by requiring that the righthand side of (11) is at most linear in the derivatives are the ones introduced above in Section 1.2. as part of the generators of the κ-deformed Poincare algebra. Of special interest is the following set of derivatives which have a vector-like transformation property with respect to the κ-deformed Poincare symmetry. They will be used later on to establish a gauge theory on the κ-deformed space-time:

$$[M^{\mu\nu}, \hat{D}_\mu] = \eta_\rho^\nu \, \hat{D}^\mu - \eta_\rho^\mu \, \hat{D}^\nu, \tag{13}$$

where

$$\hat{D}_n = \frac{1}{a}\,\sin(a\hat{\partial}_n) - \frac{i\,a}{2}\,\hat{\partial}^l\hat{\partial}_l\,e^{-ia\hat{\partial}_n}, \qquad \hat{D}_i = \hat{\partial}_i e^{-ia\hat{\partial}_n}. \tag{14}$$

1.4. Towards a Physical Theory

So far we described how a deformed symmetry acts on a deformed space $\widehat{\mathcal{A}}_{\hat{x}}$ and how we construct differential calculi. To get a physical theory which makes predictions that can be checked by experiments we will express the noncommutative theory in terms of the known commutative variables. This means that the particle content does not change but the noncommutative theory predicts new interactions [8]. This can be achieved by the following two steps:

1. First we represent the abstract deformed space-time algebra $\widehat{\mathcal{A}}_{\hat{x}}$ on the common algebra of commutative functions $\mathcal{A}_x$ by a new product called *star-product ($\star$-product)* which is a deformation of the commutative product of functions.
2. Then we express all noncommutative fields in terms of their commutative counterparts by the *Seiberg-Witten map* (see Section 2.2.).

Using the results from the second step one can express the action of the noncommutative theory in terms of commutative fields and using the star-product from the first step we can expand this action in terms of the deformation parameter. The zeroth order gives back the commutative theory and one can study corrections of it in higher orders of the deformation parameter. Those two steps will be explained in a bit more detail in the following sections.

1.5. Star Product Approach

1.5.1. The Star Product

If the noncommutative algebra $\widehat{\mathcal{A}}_{\hat{x}}$ satisfies the PBW property (see the beginning of this section), the vector space of noncommutative functions is isomorphic (as a vector space) to the vector space of commutative functions [6]. Let

$$\begin{aligned}\rho : \mathbb{R}[[x^0, \ldots, x^n]] &\rightarrow \widehat{\mathcal{A}}_{\hat{x}} \\ f(x^\mu) &\mapsto \hat{f}(\hat{x}^\mu)\end{aligned}$$

be such an isomorphism of vector spaces.[7]

To render the vector space of commutative functions isomorphic as algebra to $\widehat{\mathcal{A}}_{\hat{x}}$ we just have to equip it with a new, noncommutative product. The isomorphism ρ tells us how to define this new product which we call *star-product* and which we denote with a $\star$:

$$f(x) \star g(x) := \rho^{-1}(\hat{f}(\hat{x}) \cdot \hat{g}(\hat{x}))\,. \tag{15}$$

Again we want to give explicit examples. For the canonically deformed space we have the well-known *Moyal-Weyl product*

$$\begin{aligned} f \star g &= \mu \circ e^{i\theta^{\mu\nu}\partial_\mu \otimes \partial_\nu}(f \otimes g) \\ &= fg + \frac{i}{2}\,\theta^{\mu\nu}(\partial_\mu f)(\partial_\nu g) + \ldots\,, \end{aligned} \tag{16}$$

where $\mu(f \otimes g) := fg$ is just the multiplication map. This star-product corresponds to the symmetric ordering prescription.

For the κ-deformed space-time we get the following more complicated expression from the symmetric ordering prescription:

$$\begin{aligned} f \star g(x) &= \lim_{\substack{z\to x\\ y\to x}} \exp\Bigg(x^j \partial_{z^j}\left(\frac{\partial_n}{\partial_{z^n}}\, e^{-ia\partial_{y^n}}\frac{1-e^{-ia\partial_{z^n}}}{1-e^{-ia\partial_n}} - 1\right) \\ &\qquad + x^j \partial_{y^j}\left(\frac{\partial_n}{\partial_{y^n}}\frac{1-e^{-ia\partial_{y^n}}}{1-e^{-ia\partial_n}} - 1\right)\Bigg) f(z)g(y) \\ &= f(x)\,g(x) + \frac{i}{2}\,C^{\mu\nu}_{\lambda}x^\lambda(\partial_\mu f)(\partial_\nu g) + \ldots\,. \end{aligned} \tag{17}$$

Both star-products start in zeroth order with the usual, commutative product and are deformations of it.

[6] It is obvious that they are not isomorphic as algebras since one is a commutative algebra and the other not.

[7] This isomorphism is not unique and every isomorphism describes an ordering prescription.

1.5.2. The Star-Representation of Differential Operators

An operator $\hat{O}$ acting on $\widehat{\mathcal{A}}_{\hat{x}}$ can be represented by a differential operator O^* acting on commutative functions:

$$\begin{array}{ccc} \hat{f}(\hat{x}) & \xrightarrow{\hat{O}} & \hat{O}(\hat{f}(\hat{x})) \\ \rho^{-1}\downarrow & & \downarrow\rho^{-1} \\ f(x) & \xrightarrow{O^*} & O^*(f(x)) \end{array}$$

The star-representation of the derivatives $\hat{\partial}_\mu$ for the canonically deformed space defined in (12) is quite easy: The differential calculus in this case is undeformed and we get

$$\partial^*_\mu = \partial_\mu. \tag{18}$$

In the case of κ-deformed spaces things are more complicated. For instance, the star-representation of the Dirac-derivatives introduced in (14) and their Leibnitz-rules read:

$$\begin{aligned} D^*_n f(x) &= (\frac{1}{a}\sin(a\partial_n) - \frac{\cos(a\partial_n)-1}{i\,a\partial_n^2}\,\partial_j\partial^j)\,f(x)\,, \\ D^*_i f(x) &= \frac{e^{-ia\partial_n}-1}{-i\,a\,\partial_n}\,\partial_i f(x)\,, \end{aligned} \tag{19}$$

$$\begin{aligned} D^*_n(f(x)\star g(x)) &= (D^*_n f(x))\star(e^{-ia\partial_n}g(x)) \\ &\quad + (e^{ia\partial_n}f(x))\star(D^*_n g(x)) \\ &\quad - i\,a\left(D^*_j e^{ia\partial_n}f(x)\right)\star(D^{j*}g(x))\,, \end{aligned} \tag{20}$$

$$\begin{aligned} D^*_i(f(x)\star g(x)) &= (D^*_i f(x))\star(e^{-ia\partial_n}g(x)) \\ &\quad + f(x)\star(D^*_i g(x)). \end{aligned} \tag{21}$$

We will see in the next section how the above star-representation of the Dirac-derivatives will be used to establish a gauge theory on κ-deformed space-time.

2. Gauge Theory on Deformed Spaces

Gauge theories are based on a gauge group. This is a compact Lie group with generators T^a

$$[T^a, T^b] = i\,f^{ab}_c\,T^c. \tag{22}$$

Infinitesimal transformation of the matter field ψ^0 is given by

$$\delta_\alpha\psi^0(x) = i\,\alpha(x)\,\psi^0(x), \tag{23}$$

where $\alpha(x) = \alpha^a(x)\,T^a$ is a Lie algebra-valued gauge parameter. Transformations (23) close in the algebra

$$\delta_\alpha\delta_\beta - \delta_\beta\delta_\alpha = \delta_{-i\,[\alpha,\beta]}. \tag{24}$$

In this section we will generalize this concept to deformed spaces as well. We choose to work in the $\star$-product representation and define noncommutative gauge transformations as

$$\delta_\alpha \psi = i\,\Lambda_\alpha \star \psi(x)\,, \tag{25}$$

where Λ_α is the noncommutative gauge parameter and ψ is the noncommutative matter field. Before proceeding to the standard construction of a covariant derivative one should check if this transformations close in the algebra (24). Explicit calculation gives

$$\begin{aligned}(\delta_\alpha\delta_\beta \,-\, \delta_\beta\delta_\alpha)\,\psi(x) &= (\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi \\ &= \frac{1}{2}\Big([\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta]\{T^a, T^b\} + \{\Lambda^a_\alpha \stackrel{\star}{,} \Lambda^b_\beta\}[T^a, T^b]\Big) \star \psi\,. \end{aligned} \tag{26}$$

If we take $\Lambda_\alpha = \Lambda^a_\alpha T^a$, that is a Lie algebra-valued gauge parameter, algebra (24) will not close because of the first term in the last line of (26) (anticommutator of two generators is no longer in the Lie algebra of generators). There are two ways of solving this problem. One is to consider only $U(N)$ gauge theories and that one we will not follow here. The other one is to go to the enveloping algebra [9] approach and we continue analysing this one.

2.1. Enveloping Algebra Approach

To start with, we define the basis in the enveloping algebra (we choose symmetric ordering)

$$\begin{aligned} :T^a: &= T^a\,, \\ :T^a T^b: &= \frac{1}{2}(T^a T^b + T^b T^a)\,, \\ :T^{a_1}\ldots T^{a_l}: &= \frac{1}{l!}\sum_{\sigma\in S_l}(T^{\sigma(a_1)}\ldots T^{\sigma(a_l)})\,. \end{aligned}$$

Gauge parameter Λ_α is said to be enveloping algebra-valued

$$\begin{aligned}\Lambda_\alpha(x) &= \sum_{l=1}^{\infty}\sum_{\text{basis}} \alpha_l^{a_1\ldots a_l}(x) :T^{a_1}\ldots T^{a_l} \\ &= \alpha^a(x) :T^a: + \;\alpha_2^{a_1 a_2}(x) :T^{a_1}T^{a_2}: + \ldots\,. \end{aligned} \tag{27}$$

In this case algebra (24) will close since we work in the enveloping algebra. Now one can proceed and define a covariant derivative $\mathcal{D}_\mu\psi(x) = \partial^*_\mu\psi(x) - iV_\mu \star \psi(x)$ by its transformation law

$$\delta_\alpha(\mathcal{D}_\mu\,\psi(x)) = i\,\Lambda_\alpha \star \mathcal{D}_\mu\psi(x)\,. \tag{28}$$

The choice of ∂_μ^* will depend on the choice of a deformed space on which we want to construct gauge theory. Since we are trying to keep the analysis as general as possible we do not specify (yet) what is ∂_μ^*. The noncommutative gauge field V_μ has to be enveloping algebra-valued as well

$$V_\mu = \sum_{l=1}^{\infty} \sum_{\text{basis}} V^l_{\mu\, a_1 \dots a_l} : T^{a_1} \dots T^{a_l} : .$$

From all this it looks like we have a theory with infinitely many degrees of freedom. This is an unphysical situation and the solution of the problem is given in terms of the Seiberg-Witten map [10].

2.2. Seiberg-Witten Map

The basic idea of this map is to suppose that the noncommutative gauge parameter (field) can be expressed in terms of the commutative gauge parameter and field, for example $\Lambda_\alpha = \Lambda_\alpha(x;\alpha,A^0_\mu)$. Then one uses (24) to calculate explicitly this dependance. Inserting $\Lambda_\alpha = \Lambda_\alpha(x;\alpha,A^0_\mu)$ in (24) gives [8]

$$(\Lambda_\alpha \star \Lambda_\beta - \Lambda_\beta \star \Lambda_\alpha) \star \psi + i\,(\delta_\alpha \Lambda_\beta - \delta_\beta \Lambda_\alpha) \star \psi = \delta_{-i\,[\alpha,\beta]}\,\psi\,. \tag{29}$$

What has been said up to now applies for a general deformed space since we have not yet specified the $\star$-product or the derivatives ∂_μ^*. But the equation (29) has to be solved perturbatively, therefore one has to expand the $\star$-product. Since we are mainly interested in the gauge theories on the κ-deformed space-time we use (17) and expand Λ_α as

$$\Lambda_\alpha = \alpha + a\,\Lambda^1_\alpha + \dots + a^k \Lambda^k_\alpha + \dots .$$

Up to first order in the deformation parameter a the solution of (29) is

$$\Lambda_\alpha = \alpha - \frac{1}{4}\, x^\lambda C_\lambda^{\mu\nu} \{A^0_\mu, \partial_\nu \alpha\}\,. \tag{30}$$

This solution is not unique, one can always add to it solutions of the homogeneous equation. Using (25) and solution for gauge parameter (30) one finds solution for the noncommutative matter field as well

$$\psi = \psi^0 - \frac{1}{2}\, x^\lambda C_\lambda^{\mu\nu} A^0_\mu\, \partial_\nu \psi^0 + \frac{i}{8}\, x^\lambda C_\lambda^{\mu\nu} [A^0_\mu, A^0_\nu]\, \psi^0, \tag{31}$$

where ψ^0 is the commutative matter field, $\delta_\alpha \psi^0 = i\,\alpha\,\psi^0$.

If one compares $\star$-products for the canonically deformed space (16) and for the κ-deformed space-time (17) one sees that up to first order in the

[8] One should notice that now $\delta_\alpha \Lambda_\beta \neq 0$ because Λ_β depends on the commutative gauge field A^0_μ as well and $\delta_\alpha A^0_\mu = \partial_\mu \alpha - i\,[A^0_\mu, \alpha]$.

deformation parameter they are of the same form (just replace $\theta^{\mu\nu}$ with $C_\lambda^{\mu\nu}x^\lambda$). Therefore it is not surprising that the solutions for Λ_α and ψ in the canonically deformed space can be obtained from (30) and (31) by replacing $C_\lambda^{\mu\nu}x^\lambda$ with $\theta^{\mu\nu}$ (and the other way around). However this analogy only applies in first order, in second order new terms will appear in the κ-deformed space-time compared to the canonically deformed space.

In order to solve the Seiberg-Witten map for the gauge field V_μ one first has to choose ∂_μ^* derivatives. In the canonically deformed space $\partial_\mu^* = \partial_\mu$ is the most natural choice. In the κ-deformed space-time there are more possibilities (see Section 1.3.). We choose D_μ^* derivatives because of their vector-like transformation law (13). From $\mathcal{D}_\mu\psi = D_\mu^*\psi - iV_\mu \star \psi$ and

$$\delta_\alpha(\mathcal{D}_\mu\psi) = i\,\Lambda_\alpha \star \mathcal{D}_\mu\psi$$

we get

$$\begin{aligned}(\delta_\alpha V_\mu)\star\psi &= D_\mu^*(\Lambda_\alpha\star\psi) - \Lambda_\alpha\star(D_\mu^*\psi) + i\,[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi \\ &\neq (D_\mu^*\Lambda_\alpha)\star\psi + i\,[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi\,.\end{aligned}$$

The last line follows from the nontrivial Leibnitz rules for D_μ^* derivatives (20,21). In order to continue we split between n and j indices.

First we have a look at the j index.

$$\begin{aligned}(\delta_\alpha V_j)\star\psi &= D_j^*(\Lambda_\alpha\star\psi) - \Lambda_\alpha\star(D_j^*\psi) + i\,[\Lambda_\alpha \stackrel{\star}{,} V_j]\star\psi \\ &= (D_j^*\Lambda_\alpha)\star e^{-ia\partial_n}\psi + i\,[\Lambda_\alpha \stackrel{\star}{,} V_\mu]\star\psi\,,\end{aligned} \tag{32}$$

where we have used (21). In order to solve this equation we have to allow for V_j to be derivative-valued, that is we make the following ansatz

$$V_j\star\psi = A_j\star(e^{-ia\partial_n}\psi)$$

and insert it into (32). After using $e^{-ia\partial_n}(f\star g) = (e^{-ia\partial_n}f)\star(e^{-ia\partial_n}g)$ and omitting $e^{-ia\partial_n}\psi$ we have

$$\delta_\alpha A_j = (D_j^*\Lambda_\alpha) + i\,\Lambda_\alpha\star A_j - i\,A_j\star(e^{-ia\partial_n}\Lambda_\alpha)\,. \tag{33}$$

This equation can be solved order by order in the deformation parameter. The solution up to first order in a is

$$\begin{aligned}V_j &= A_j^0 - i\,a\,A_j^0\,\partial_n - \frac{i\,a}{2}\,\partial_n A_j^0 - \frac{a}{4}\,\{A_n^0, A_j^0\} \\ &\quad + \frac{1}{4}\,x^\lambda C_\lambda^{\mu\nu}\Big(\{F_{\mu j}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_j^0\}\Big).\end{aligned} \tag{34}$$

For V_n one follows the same steps, using the Leibnitz rule for the D_n^* derivative (20) this time. The solution up to first order in a is

$$V_n = A_n^0 - i\,a\,A^{0j}\partial_j - \frac{i\,a}{2}\,\partial_j A^{0j} - \frac{a}{2}\,A_j^0\,A^{0j} + \frac{1}{4}\,x^\lambda C_\lambda^{\mu\nu}\Big(\{F_{\mu n}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_n^0\}\Big)\,. \qquad (35)$$

From (34) and (35) we see that besides being enveloping algebra-valued (consequence of noncommutativity, that is $\star$-product) the gauge field is also derivative-valued. This is the consequence of special properties of κ-deformed space-time, more concretely of nontrivial Leibnitz rules for D_μ^* derivatives.

For completeness we give here also the solution for V_μ in the canonically deformed space

$$V_\rho = A_\rho^0 + \frac{1}{4}\,\theta^{\mu\nu}\Big(\{F_{\mu\rho}^0, A_\nu^0\} - \{A_\mu^0, \partial_\nu A_\rho^0\}\Big)\,. \qquad (36)$$

This solution is not derivative valued since ∂_μ derivatives have undeformed Leibnitz rule.

Having solutions of the Seiberg-Witten map at hand, one calculates the field-strength tensor defined as

$$\mathcal{F}_{\mu\nu} = i\,[\mathcal{D}_\mu \stackrel{\star}{,} \mathcal{D}_\nu]\,. \qquad (37)$$

Since the gauge field V_μ is derivative-valued [9] it is not surprising that the field-strength tensor will also be derivative-valued. With a derivative-valued field-strength tensor we do not know how to write down the action for the gauge field. Therefore, we split the tensor $\mathcal{F}_{\mu\nu}$ into "curvature-like" and "torsion-like" terms, like one usually does in gravity theories

$$\mathcal{F}_{\mu\nu} = F_{\mu\nu} + T_{\mu\nu}^\rho \mathcal{D}_\rho + \ldots + T_{\mu\nu}^{\rho_1\ldots\rho_l} : \mathcal{D}_{\rho_1}\ldots\mathcal{D}_{\rho_l} : + \ldots \qquad (38)$$

For the action we will only use the "curvature-like" term $F_{\mu\nu}$ and ignore all "torsion-like" terms. With this we have all the ingredients to write Lagrangian densites up to the first order in a, see [1].

2.3. Integral and the Action

To come from the Lagrangian densities to the action for noncommutative gauge theory we need an integral. It should have the trace property

$$\int f \star g = \int g \star f\,. \qquad (39)$$

[9] The following does not apply to the canonically deformed space since V_μ in not derivative valued there.

This is required by gauge invariance of the action for the gauge field and can be used to formulate the variational principle. For the canonically deformed space (39) is automatically fulfilled and the following analysis is not needed there. Unfortunately, for κ-deformed space-time (39) is not fulfilled. The way to repair this is to introduce so-called measure function $\mu(x)$ such that

$$\int \mathrm{d}^{n+1}x\ \mu(x)\ (f \star g) = \int \mathrm{d}^{n+1}x\ \mu(x)\ (g \star f)\,. \tag{40}$$

From this request one gets conditions on $\mu(x)$

$$\partial_n \mu(x) = 0, \qquad x^j \partial_j \mu(x) = -n\,\mu(x)\,. \tag{41}$$

This equation can be solved, however the solution is not unique. But this is not the only problem. It turns out that the solution for $\mu(x)$ is a independent so it does not vanish in the limit $a \to 0$. This means that it will spoil the classical limit of the theory (equations of motion for example). Also, because of its explicit x-dependence [10] it will break the κ-Poincaré invariance of the integral.

On the other hand, one can construct an integral which is κ-Poincaré invariant using quantum trace [11]. The problem with the integral obtained that way is that it does not have the trace property, therefore it is not convenient for analysing gauge theories.

So far there has not been a completely satisfactory answer to the question of proper definition of the integral on κ-deformed space-time. It appears that one has to choose between having a gauge invariant theory or κ-Poincaré invariant theory. In the case of $U(1)$ gauge theory we have been able to write down the action using the first approach [12], but the analysis is still far from being complete.

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