Reconstructing the metric of the local Universe from number counts observations

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Number counts observations available with new surveys such as the Euclid mission will be an important source of information about the metric of the Universe. We compute the low red-shift expansion for the density contrast using an exact spherically symmetric solution in presence of a cosmological constant. At low red-shift the expansion is more precise than linear perturbation theory prediction. We then use the local expansion to reconstruct the metric from the monopole of the density contrast. We test the inversion method using numerical calculations and find a good agreement within the regime of validity of the red-shift expansion. The method could be applied to observational data to reconstruct the metric of the local Universe with a level of precision higher than the one achievable using perturbation theory.

Keywords: Number counts, density contrast, redshift, metric, local Universe.

1. Introduction

The standard cosmological model is based on the assumption that the Universe is homogeneous and isotropic on sufficiently larges scales, and is confirmed by different observations such as for example the cosmic microwave background (CMB) radiation[1] or of galaxy catalogues. However the presence of structure at smaller scales can affect local observations as it was shown in Ref. 2, and it is therefore important to understand its consequences. The effects of inhomogeneities on cosmological observables have been studied in different cases, such as dark energy, the luminosity distance[3–5] or the expansion scalar.[6] These effects are due to the fact that spatial inhomogeneities change the energy of photons, modifying the cosmological red-shift due to the Universe expansion. As a consequence some errors are produced in the estimation of parameters based on homogeneous cosmological models.

One important source of information about the Universe are galaxy catalogues since they allow to map the local density field. Since we can only measure the red-shift of astrophysical objects for which other distance measurement methods such as stellar parallax cannot be applied, it is important to take into account the effects of these inhomogeneities on the metric in order to compute self-consistently the density in red-shift space. This is particularly important when trying to determine the metric of the Universe.

2. Modeling the local Universe

In order to model the monopole component of the local structure we use the LTB solution[7–11]

$$ds^2 = -dt^2 + \frac{R'(t,r)^2}{1+2\,E(r)}dr^2 + R(t,r)^2 d\Omega^2\,, \tag{1}$$

where $E(r)$ is an arbitrary function of r, and $R'(t,r) = \partial_r R(t,r)$. The analytical solution of Einstein's equations can be derived[12,13] if we introduce a new function $k(r)$, and a new coordinate $\eta = \eta(t,r)$ given by

$$\frac{\partial \eta}{\partial t}|_r = \frac{r}{R} = \frac{1}{a}\,, \qquad k(r) = -\frac{2E(r)}{r^2}\,. \tag{2}$$

The Einstein equations imply

$$\left(\frac{\partial a}{\partial \eta}\right)^2 = -k(r)a^2 + \frac{\rho_0(r)}{3}a + \frac{\Lambda}{3}a^4\,, \tag{3}$$

where $\rho_0(r)$ is an arbitrary function of r, and we adopt a system of units in which $c = 8\pi G = 1$. Without any loss of generality we adopt the coordinate system in which $\rho_0(r)$ is a constant, which is known as the FLRW gauge. The solution to the above equation can be written in the form[14]

$$a(\eta,r) = \frac{\rho_0}{k(r) + 3\wp(\frac{\eta}{2}; g_2(r), g_3(r))}\,. \tag{4}$$

where $\wp(x; g_2, g_3)$ is the Weierstrass elliptic function and

$$g_2(r) = \frac{4}{3}k(r)^2\,, \quad g_3(r) = \frac{4}{27}\left(2k(r)^3 - \Lambda\rho_0^2\right). \tag{5}$$

The relation between t and η can be found by integrating Eq. (2) and is given by[6]

$$t(\eta,r) = \frac{2\rho_0}{3\wp'\left(\wp^{-1}\left(-\frac{k(r)}{3}\right)\right)}\left[\ln\left(\frac{\sigma\left(\frac{\eta}{2} - \wp^{-1}\left(-\frac{k(r)}{3}\right)\right)}{\sigma\left(\frac{\eta}{2} + \wp^{-1}\left(-\frac{k(r)}{3}\right)\right)}\right) + \eta\zeta\left(\wp^{-1}\left(-\frac{k(r)}{3}\right)\right)\right].$$

The radial null geodesic equations take the form[15]

$$\frac{d\eta}{dz} = -\frac{\partial_r t(\eta,r) + G(\eta,r)}{(1+z)\partial_\eta G(\eta,r)}\,, \qquad \frac{dr}{dz} = \frac{a(\eta,r)}{(1+z)\partial_\eta G(\eta,r)}\,, \tag{6}$$

where

$$G(\eta,r) \equiv \frac{R'(t(\eta,r),r)}{\sqrt{1-k(r)r^2}}\,.$$

The density profile is given by

$$\rho(\eta,r) = \rho(t(\eta,r),r) = \frac{\rho_0}{a(\eta,r)^2 R'(t(\eta,r),r)}\,. \tag{7}$$

The background density of the universe is given by the sub-horizon volume average of ρ on constant time slices

$$\overline{\rho}(t) = \frac{\int_V \rho(t,r)dV}{\int_V dV} = \frac{\int_0^{r_{Hor}(t)} \rho(t,r)\frac{R(t,r)^2 R'(t,r)}{\sqrt{1-k(r)r^2}}dr}{\int_0^{r_{Hor}(t)} \frac{R(t,r)^2 R'(t,r)}{\sqrt{1-k(r)r^2}}dr}\,, \tag{8}$$

where the upper limit of the integrals $r_{Hor}(t)$ is the comoving horizon as a function of time, and determines the region of space causally connected with the central observer at time t. We can then evaluate $\overline{\rho}(t)$ at the time $t(z)$ corresponding to a given redshift z, i.e. the time along null radial geodesics, and define the background value of ρ at redshift z as $\overline{\rho}(z) \equiv \overline{\rho}(t(z))$. We can then define the density contrast

$$\delta(z) = \frac{\rho(z)}{\overline{\rho}(z)} - 1\,. \tag{9}$$

If the size of the local inhomogeneity is sufficiently smaller than the volume over which the integral in Eq. (8) is performed then $\overline{\rho}$ will get most of its contribution from the asymptotically homogeneous region and the average density will be well approximated by the asymptotic density

$$\overline{\rho}(z) = 3(H_0^b)^2\Omega_M^b(1+z)^3\,; \quad H_0^b = \overline{H}(0)\,, \quad \Omega_M^b = \frac{\overline{\rho}(0)}{3(H_0^b)^2}\,, \tag{10}$$

where the upper-script ^{b} stands for background and H is the expansion scalar $H(t,r)$[6].

3. Reconstruction of the local metric

In order to reconstruct the metric of the local universe from the density contrast we expand the curvature function $k(r)$ as

$$k(r) = k_0 + k_1 r + k_2 r^2 + \cdots\,. \tag{11}$$

We also expand the solution of the geodesic equations according to

$$r(z) = r_1 z + r_2 z^2 + r_3 z^3 + \cdots\,, \quad \eta(z) = \eta_0 + \eta_1 z + \eta_2 z^2 + \cdots\,. \tag{12}$$

We expand $t(\eta, r)$ as

$$t(\eta,r) = t_0(r) + a(\eta_0,r)(\eta-\eta_0) + \frac{1}{2}\partial_\eta a(\eta_0,r)(\eta-\eta_0)^2 + \cdots\,, \tag{13}$$

where $t_0(r) \equiv t(\eta_0, r)$. We now define the dimensionless parameters $K_n \equiv k_n(a_0H_0)^{-(n+2)}$ and applying the chain rule to $t_0'(0)$ and $t_0''(0)$ we find

$$t_0'(0) = \frac{\partial t_0(r)}{\partial k}\frac{\partial k}{\partial r}|_{r=0} = a_0\alpha K_1\,, \quad t_0''(0) = a_0(a_0H_0)(\beta K_1^2 + 2\alpha K_2)\,, \tag{14}$$

where α and β are dimensionless parameters.

We will consider the case in which $k_0 = 0$, which is enough to understand qualitatively the effects of the inhomogeneity, since this term corresponds to the homogeneous component of the curvature function $k(r)$. Expanding the density contrast up to second order we find

$$\delta(z) = \delta_0 + \delta_1 z + \delta_2 z^2 \,, \tag{15}$$

$$\delta_0 = \left(\frac{H_0}{H_0^b}\right)^2 \left(\frac{\Omega_M}{\Omega_M^b} - 1\right) , \tag{16}$$

$$\delta_1 = \left(\frac{H_0}{H_0^b}\right)^2 \frac{4K_1\left(3\alpha\Omega_M + 1\right)}{3\Omega_M^b} , \tag{17}$$

$$\begin{aligned}\delta_2 = -\left(\frac{H_0}{H_0^b}\right)^2 \frac{1}{36\Omega_\Lambda\Omega_M\Omega_M^b}\Bigg[&18K_1\Omega_\Lambda\Omega_M^2\left(3\alpha\Omega_M + 2\right) \\ &+ K_1^2\Big\{\Omega_\Lambda\Big(-18\left(25\alpha^2 - 4\alpha + 5\beta\right)\Omega_M^2 + 81\alpha^2\Omega_M^3 - 300\alpha\Omega_M - 40\Big) \\ &+ 20\left(\Omega_M - \zeta_0\right)\Big\} - 60K_2\Omega_\Lambda\Omega_M\left(3\alpha\Omega_M + 1\right)\Bigg] .\end{aligned} \tag{18}$$

From these equations we can finally obtain

$$k(r) \approx K_1(a_0H_0)^3 r + K_2(a_0H_0)^4 r^2 \,, \tag{19}$$

$$K_1 = \left(\frac{H_0^b}{H_0}\right)^2 \frac{3\Omega_M^b\delta_1}{4\left(3\alpha\Omega_M + 1\right)} , \tag{20}$$

$$\begin{aligned}K_2 = \left(\frac{H_0^b}{H_0}\right)^4 \frac{3\Omega_M^b}{320\Omega_\Lambda\Omega_M\left(3\alpha\Omega_M + 1\right)^3}\Bigg[&8\Omega_\Lambda\Omega_M\left(3\alpha\Omega_M + 1\right)\Big\{9\alpha\delta_1\Omega_M^2 \\ &+ 6\left(4\alpha\delta_2 + \delta_1\right)\Omega_M + 8\delta_2\Big\} + \delta_1^2\Omega_M^b\left(\frac{H_0}{H_0^b}\right)^2\Big\{\Omega_\Lambda\Big(-18\left(25\alpha^2 - 4\alpha\right. \\ &\left. + 5\beta\right)\Omega_M^2 + 81\alpha^2\Omega_M^3 - 300\alpha\Omega_M - 40\Big) + 20\left(\Omega_M - \zeta_0\right)\Big\}\Bigg] .\end{aligned} \tag{21}$$

In the above equations we have introduced the parameters a_0, H_0, Ω_M, Ω_Λ, T_0 and ζ_0 according to their corresponding definitions given in Refs. 14, 6.

4. Testing the accuracy of the method

In order to test the formulae we have derived we chose models defined by the spatial curvature function $k(r)$ according to

$$k(r) = \pm(a_0H_0)^2\frac{a_0H_0r}{5}\Big[1 - \tanh\left(2a_0H_0r\right)\Big] . \tag{22}$$

We also compare our formula for the density contrast with the linear perturbation theory prediction that around a flat homogeneous background[16]

$$\delta(z) \approx -3\delta H(z)(\Omega_M^b)^{-0.55} . \tag{23}$$

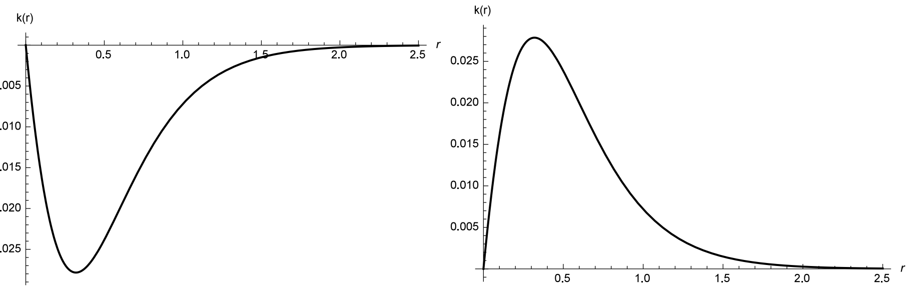

Fig. 1. The function $k(r)$ is plotted in units of H_0^2 as a function of the radial coordinate in units of H_0^{-1}.

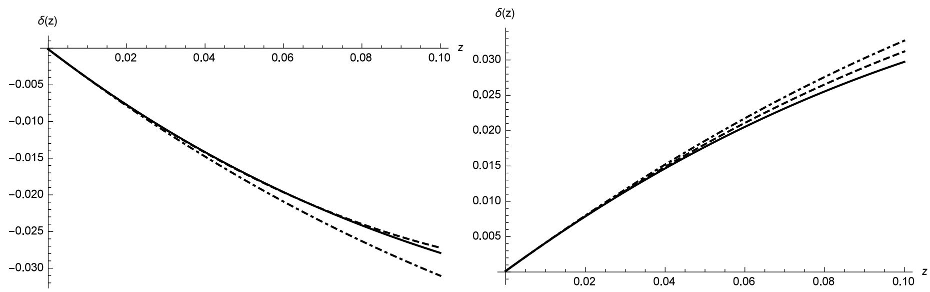

Fig. 2. The density contrast is plotted as a function of redshift. The left and right plots are for the inhomogeneities corresponding to Fig. 1. The solid lines correspond to the numerical solution, the dashed lines to the analytical formula we derived and the dot-dashed lines to the perturbation theory result.

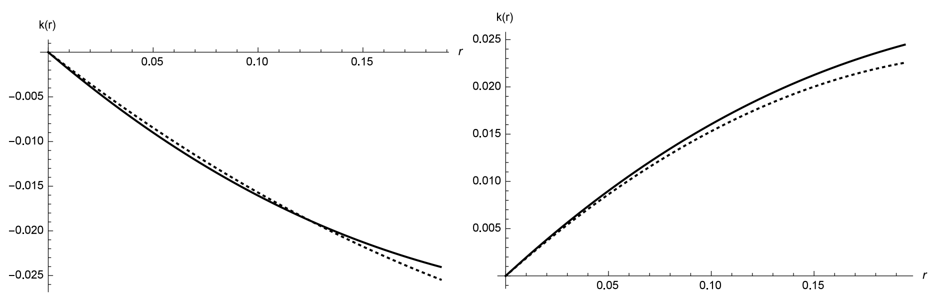

Fig. 3. The reconstructed metric function $k(r)$ is plotted in units of H_0^2 as a function of the radial coordinate in units of H_0^{-1} for the inhomogeneities corresponding to Fig 1. The black solid line corresponds to the original $k(r)$ function and the black dotted line to the reconstructed one.

5. Conclusions

We have derived the low-redshift expansion for the monopole of the density contrast. At low red-shift the formula is in good agreement with numerical solutions and is more accurate than the linear perturbation theory approximation. Using this formula we have then developed a new analytical inversion method to reconstruct the metric from the monopole of the density contrast. The inversion method could be applied to low red-shift observational data to determine the metric with a level of precision higher than the one achievable using perturbation theory.

For a full reconstruction of the metric beyond the monopole contribution other solutions of the Einstein equations could be used for the analytical approach, in order to accommodate more complex geometries. For a general numerical inversion able to reconstruct any type of metric more sophisticated methods in numerical relativity will be required.

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