Boundary condition for D-brane from Wilson loop at the AdS boundary[1]

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Abstract

We study the supersymmetric Wilson loops in the four-dimensional $\mathcal{N}=4$ super Yang-Mills theory in the context of the AdS/CFT correspondence. In the gauge theory side, it is known that the expectation value of the Wilson loops of circular shape with winding number k, $W_k(C)$, is calculable by using a Gaussian matrix model. In the gravity side, the expectation value of the loop is conjectured to be given by the classical value of the action $S_{\rm D3}$ for a probe D3-brane with k electric fluxes as $\langle W_k(C)\rangle = {\rm e}^{-S_{\rm D3}}$. However, according to the spirit of the AdS/CFT correspondence, in principle we have to perform the path integral for the D3-brane action in the $\rm AdS_5\times S^5$ under appropriate boundary conditions which should be given in terms of data of the Wilson loop at the AdS boundary. We clarify what kind of boundary conditions are imposed on the D3-brane from the Wilson loop. As an application, our boundary conditions provide a natural interpretation of a position of an eigenvalue in the Gaussian matrix model as an integrated flux on the D3-brane.

1 Introduction

Much progress in string theory for the last ten years suggests that quantum gravity will be formulated as the large-N limit of gauge theories or matrix models [2–4]. The key in this approach is that all the information on gravity or geometry in the bulk is encoded into gauge theory degrees of freedom on the boundary. This idea called holography is realized in the AdS/CFT correspondence. In particular, we consider a circular Wilson loop in the gauge theory in the context of the AdS/CFT correspondence as a nice and concrete realization of holography idea. In our analysis, we emphasize importance of boundary conditions the gauge theory imposes on the geometry. We hope this kind of study has some implications to the brane world scenario, cosmology, or other quantum gravity formulations.

2 Circular Wilson loop in $\mathcal{N}=4$ $U(N)$ SYM

In this section we review the circular Wilson loop in the four-dimensional $\mathcal{N}=4$ $U(N)$ supersymmetric Yang-Mills (SYM) theory in the large-N limit. Bosonic fields in this theory are the $U(N)$ gauge field A_μ ($\mu = 0\sim 3$), the scalar fields Φ_i ($i=4\sim 9$) in the adjoint representation of $U(N)$. In terms of these fields, the circular Wilson loop is defined as

$$W_k(C) = \frac{1}{N}{\rm tr}P\exp\left(\int_C ds(iA_\mu \dot{x}^\mu(s) + \Phi_i \dot{y}^i(s))\right), \tag{1}$$

where C is a circle parametrized by s, k is the winding number of the Wilson loop, and $x^\mu(s)$, $y^i(s)$ represents the shape of the loop in the four-dimensional, and six-dimensional space, respectively. In particular, when we are interested in the circular Wilson loop, $x^\mu(s)$ describes the circle C, e.g. $x^\mu(s) = (\cos s, \sin s, 0, 0)$. In order to take advantage of the AdS/CFT correspondence, eventually we have to take $N\to\infty$ limit.

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In this setup, let us summarize the main result in the gauge theory side [5–7]. We are interested in the vacuum expectation value of the circular Wilson loop in the large-N limit

$$\langle W_k(C)\rangle \equiv \frac{1}{Z_{\text{gauge}}}\int \mathcal{D}(\text{all fields}) W_k(C) e^{-S_{\text{gauge}}}, \tag{2}$$

where S_{gauge} is the action of the gauge theory. The standard way to compute this is by the perturbation theory in terms of the gauge theory coupling constant g_{YM}. We choose two points on the Wilson loop s_1, s_2, and connect fields on these points by their propagators as $\langle A_\mu(x(s_1))A_\nu(x(s_2))\rangle$, $\langle \Phi_i(x(s_1))\Phi_j(x(s_2))\rangle$. However, since these fields also carry $\dot{x}^\mu$ and $\dot{y}^i$, the net contribution of the propagators becomes

$$-\langle A_\mu(x(s_1))A_\nu(x(s_2))\rangle\, \dot{x}^\mu(s_1)\dot{x}^\nu(s_2) + \langle \Phi_i(x(s_1))\Phi_j(x(s_2))\rangle\, \dot{y}^i(s_1)\dot{y}^j(s_2). \tag{3}$$

The crucial property of the circular Wilson loop is that this becomes constant, namely independent of the space-time points $x^\mu(s_1)$ and $x^\mu(s_2)$ provided that $\dot{x}^2 - \dot{y}^2 = 0$. More precisely, when this condition is satisfied, the above quantity becomes $g_{\text{YM}}^2/8\pi^2$. Thus the combined propagator loses space-time dependence. Due to this property, the computation of $\langle W_k(C)\rangle$ is greatly simplified. Therefore, in the following let us concentrate on the circular Wilson loop (1) with $\dot{x}^2 - \dot{y}^2 = 0$ satisfied. In fact, this condition is known as the one under which the Wilson loop preserves the half of supersymmetries. Furthermore, it is known that diagrams with internal vertices vanish because of the supersymmetry. Thus the computation is reduced to the sum over all planar diagrams with the constant propagator, which is just a combinatorics problem. Actually, the calculation boils down to the one-matrix model

$$\langle W_k(C)\rangle = \left\langle \frac{1}{N}\operatorname{tr} e^{kM}\right\rangle_{\text{MM}} \equiv \frac{1}{Z_{\text{MM}}}\int dM \frac{1}{N}\operatorname{tr} e^{kM} e^{-S_{\text{MM}}}, \tag{4}$$

$$S_{\text{MM}} = \frac{2N}{\lambda}\operatorname{tr}M^2, \qquad \lambda \equiv g_{YM}^2 N, \tag{5}$$

where M is an $N\times N$ Hermitian matrix. From the observations above, it is easy to see that this matrix model reproduces the calculation of $\langle W_k(C)\rangle$ in the large-N limit, because it generates all planar diagrams with the constant propagator proportional to $\lambda/N = g_{\text{YM}}^2$. Note that the operator $\frac{1}{N}\operatorname{tr} e^{kM}$ is a remnant of the Wilson loop $W_k(C)$, where the winding number k appears in the exponent. We can calculate (4) by the standard technique [8]: performing integration over angular variables, (4) can be written as integration over N eigenvalues of M

$$\langle W_k(C)\rangle = \frac{1}{Z}\int \prod_i dm_i \exp(-NV_{\text{eff}}),$$
$$V_{\text{eff}} = \sum_i \frac{2}{\lambda}m_i^2 - \sum_{i,j}\log(m_i - m_j)^2 - \frac{k}{N}m_N. \tag{6}$$

This implies that the system now becomes that of N particles in the Gaussian potential with strong repulsive logarithmic force between them. Moreover, in the presence of the Wilson loop, the last eigenvalue m_N feels extra linear potential proportional to k. In particular, when k is of order N, it survives in the large-N limit. This situation is quite interesting, so henceforth let us discuss the case where k is of order N.

In the large-N limit, these eigenvalues are expected to form a continuous distribution, and the distribution function can be derived from the saddle point method in the large-N limit. The result is [1]

$$\rho(m) = \frac{2}{\pi\lambda}\sqrt{\lambda - m^2} + \frac{1}{N}\delta(m - m_*), \tag{7}$$

which is displayed in Figure 1. Here $m_* = \sqrt{\lambda(1+\kappa^2)}$ and $\kappa \equiv k\sqrt{\lambda}/4N$ which is $\mathcal{O}(1)$ when k is $\mathcal{O}(N)$. The isolated eigenvalue distribution at m_* originates from the last eigenvalue m_N. Given (7), it is easy to calculate $\langle W_k(C)\rangle$ as [1, 9]

$$\langle W_k(C)\rangle = \exp\left[N\left(2\kappa\sqrt{1+\kappa^2} + 2\operatorname{arcsinh}\kappa\right)\right]. \tag{8}$$

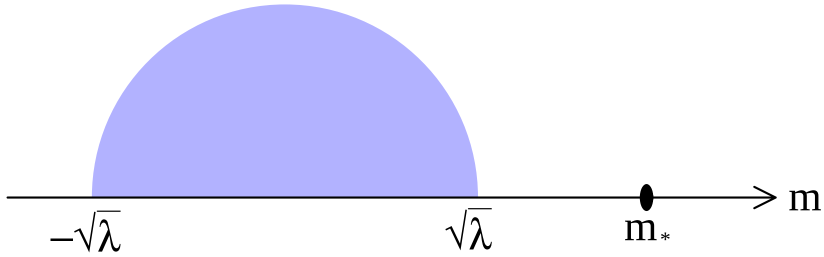

Figure 1: The eigenvalue distribution with k of $\mathcal{O}(N)$.

Conversely, detailed analysis [1] of the matrix model tells us that on general grounds, we can deduce from $\langle W_k(C)\rangle$ the position of the isolated eigenvalue as $\langle W_k(C)\rangle = \exp(-V_{\text{eff}}(k)) \to m_* = -V'_{\text{eff}}(k)$. In the gravity side, by a totally different method we can calculate $\langle W_k(C)\rangle$, from which we can read off a bulk interpretation of m_* by using this relation.

3 AdS/CFT for Wilson loop

In this section we review main results of the Wilson loop in the AdS/CFT correspondence. First we consider the case of the Wilson loop with winding number $k = 1$, then we turn to k of $\mathcal{O}(N)$ case.

3.1 $k = 1$ case

The statement of AdS/CFT for the Wilson loop with $k = 1$ is [10, 11]

$$\langle W_{k=1}(C)\rangle = \int_{\text{b.c.}} e^{-(S_{\text{NG}}+S_b)}, \tag{9}$$

where S_{NG} and S_b is the Nambu-Goto action and a possible boundary term, respectively. In the right hand side the path integral should be over all fields on a string world sheet in $\text{AdS}_5\times\text{S}^5$ attached to the loop C at the AdS boundary under appropriate boundary conditions. Here we stress importance of the boundary conditions in the right hand side in (9). First of all, from the theoretical point of view, in the spirit of the AdS/CFT correspondence a bulk or gravity quantity should be completely fixed by the boundary or gauge theory data, which should enter in the right hand side through boundary conditions. On the other hand, from the practical point of view, usually the relation (9) is applied in the case of $\lambda \gg 1$, where the path integral in the right hand side can be replaced by $e^{-S_{\text{cl}}}$ with S_{cl} the action evaluated for the classical solution. However, in order to fix the classical solution, we have to specify appropriate boundary conditions at, for example, the AdS boundary corresponding to the presence of the Wilson loop. Thus an important issue in (9) is what kind of boundary conditions and/or boundary terms the Wilson loop impose on the bulk or geometry.

A nice argument on boundary conditions based on the T-duality is given in [12]: let us start from the 10-dimensional gauge theory, namely D9-brane world volume theory. If a string is attached to a Wilson loop in this theory, string coordinates in all directions X^μ ($\mu = 0 \sim 9$) should have the Dirichlet boundary condition because their boundary values are all fixed by the position of the Wilson loop. Then applying the T-duality in $i = 4 \sim 9$ directions, we find that X^μ ($\mu = 0 \sim 3$) still have the Dirichlet boundary condition, while X^i ($i = 4 \sim 9$) should have the Neumann boundary condition. Namely, the string coordinates have the Dirichlet boundary condition for the D3-brane world volume directions, and the Neumann boundary condition for the orthogonal directions.

The above T-duality argument implies that the right hand side in (9) will be a function of X^μ ($\mu = 0 \sim 3$) and P_i ($i = 4 \sim 9$). This requires boundary terms for the Neumann directions $i = 4 \sim 9$. In order to see this, let us parametrize a string world sheet attached to the Wilson loop by σ^1, σ^2 as follows: the world sheet boundary exists at $\sigma^2 = 0$, while the boundary itself is parametrized by σ^1. Then variation of the classical world sheet action reads

$$\delta S\Big|_{\rm cl} = \int d^2\sigma \left[\frac{\delta S}{\delta X}\delta X + \frac{\delta S}{\delta\partial X}\delta\partial X\right]\Bigg|_{\rm cl} = -\oint_{\sigma^2=0} d\sigma^1 P_i \delta X^i, \tag{10}$$

where $|_{\rm cl}$ represents evaluation via a classical solution and $P_i = \delta\mathcal{L}/\delta\partial_2 X^i$ is the boundary momentum. This equation suggests that the classical action is a function of X^i. The standard way to flip the boundary condition, namely to change a function of X^i to that of P_i is the Legendre transformation. We add the boundary term as

$$\tilde{S} = S + \oint_{\sigma^2=0} d\sigma^1 P_i X^i \quad\rightarrow\quad \delta\tilde{S}\Big|_{\rm cl} = \oint_{\sigma^2=0} d\sigma^1 X^i \delta P_i, \tag{11}$$

which shows that $\tilde{S}\Big|_{\rm cl}$ is a function of P_i as expected. Thus we conclude that we have to add the boundary term $S_b = \oint_{\sigma^2=0} d\sigma^1 P_i X^i$ for the Neumann direction.

In order to give a concrete form of the above boundary conditions, let us choose the $\mathrm{AdS}_5\times\mathrm{S}^5$ metric as

$$\begin{aligned} ds^2 &= \frac{L^2}{Y^2}\left((dX^\mu)^2 + (dY^i)^2\right) \\ &= L^2\left(\left(\frac{2\pi\alpha' U}{L^2}\right)^2 (dX^\mu)^2 + \frac{(dU^i)^2}{U^2}\right), \quad (U^i)^2 = U = \frac{L^2}{2\pi\alpha' Y}, \quad U^i = U\theta^i, \end{aligned} \tag{12}$$

where $L = \lambda^{\frac{1}{4}}\sqrt{\alpha'}$, θ^i is a coordinate of the unit S^5, and the gauge theory lives in the four-dimensional space-time X^μ ($\mu = 0 \sim 3$). Then the Wilson loop (1) provides following boundary conditions [12]:

- Dirichlet: $X^\mu(\sigma^1, \sigma^2 = 0) = x^\mu(\sigma^1)$,
- Neumann: $P_i(\sigma^1, \sigma^2 = 0) = \dot{y}_i(\sigma^1)$,

where P_i is the conjugate momentum of U^i. The latter equation is nontrivial, but there are some arguments supporting it based on symmetries and constraints [12]. In the next section we derive a similar boundary condition in the case of k of $\mathcal{O}(N)$. It is worth noticing that these boundary conditions are along the spirit of AdS/CFT, namely the Wilson loop data provides the boundary conditions for the fields in the bulk.

3.2 k of $\mathcal{O}(N)$ case

Now let us turn to the case where k is of $\mathcal{O}(N)$. A crucial observation is that in this case we have to consider k world sheets, because the Wilson loop with winding number k originates from k fundamental open strings connecting N D3-branes and a probe D3-brane, which should correspond to k world sheets attached to the loop in the gravity side. However, since k is now of $\mathcal{O}(N)$, we are considering $N \sim 1/g_s$ world sheets, which means that we can no longer neglect string interactions, namely we need nonperturbative description. For this system, an interesting proposal was made in [13] that when k is of $\mathcal{O}(N)$, the string world sheet attached to the loop should be replaced by a D3-brane world volume. Namely, the basic relation for the Wilson loop in the AdS/CFT correspondence given in (9) now becomes

$$\langle W_k(C)\rangle = \int_{\rm b.c.} e^{-(S_{\rm D3}+S_b)}, \tag{13}$$

where $S_{\rm D3}$ is the Dirac-Born-Infeld action for a D3-brane including the Wess-Zumino term. Their proposal is based on the fact that a fundamental string can be regarded as a BPS configuration from the point of view of the D3-brane world volume theory [14]. In fact, as we will see later, using the relation (13), we can compute $\langle W_k(C)\rangle$ in the gravity side and it agrees exactly with the gauge theory result given in (8).

In this approach, there are apparently different points from the string world sheet ($k = 1$) case: first of all, the world volume is now four-dimensional and hence attachment to the one-dimensional loop at the

AdS boundary is somewhat nontrivial. In the following σ^a ($a = 1 \sim 4$) denote coordinates on the world volume. Secondly, apart from the scalar fields $X^\mu(\sigma^a)$, $U^i(\sigma^a)$ ($\mu = 0 \sim 3$, $i = 4 \sim 9$) which describe the position of the D3-brane in $\mathrm{AdS}_5 \times \mathrm{S}^5$ in terms of the coordinates given in (12), we have a $U(1)$ gauge field $A_a(\sigma^a)$. Thus we have to specify a boundary condition and a possible boundary term even for this gauge field. $A_a(\sigma^a)$ on a D3-brane world volume should not be confused with the original $U(N)$ gauge field in the gauge theory side.

For the purpose of examining boundary conditions for a D3-brane, let us parametrize the D3-brane world volume in such a way that the world volume boundary is again given by $\sigma^2 = 0$, and the one-dimensional boundary itself is parametrized by σ^1. Other directions are parametrized by σ^3, σ^4. The same argument as before yields apparent boundary conditions

1. X^μ: Dirichlet, $X^\mu(\sigma^1, \sigma^2 = 0) = x^\mu(\sigma^1)$,
   (Here by using reparametrization invariance of the Wilson loop, we make an identification $s = \sigma^1$.)
2. U^i: Neumann,
3. $\Pi^{a=1}(\sigma^1, \sigma^2 = 0) = -ik$ for each σ^1.

Several notices are in order. As for 2, it should be emphasized that we do not yet know explicitly how to specify P_{U^i} at the boundary. It is true that in the string case $P_{U^i} = \dot{y}_i$, but it is not guaranteed that this is also the case with the D3-brane. The boundary condition 3 reflects that the fact that the end point of a string attached to a D-brane can be regarded as an electric charge from the D3 world volume viewpoint and therefore if it moves along the circle to form the Wilson loop, it induces flux of $U(1)$ gauge field in the σ^1 direction. Recalling the discussion in (11), we have to add the boundary terms for fields with the Neumann boundary conditions, namely the boundary term for the transverse scalar fields $\oint_{\sigma^2=0} d\sigma^1 d\sigma^3 d\sigma^4 P_{U^i} U^i$ and that for the gauge field $\oint_{\sigma^2=0} d\sigma^1 d\sigma^3 d\sigma^4 \Pi^1 A_1$.

Now we make a short review of the explicit form of the D3-brane solution given in [13]. If we take the AdS_5 metric as

$$ds^2_{\mathrm{AdS}_5} = L^2 \left(\left(\frac{2\pi\alpha' U}{L^2} \right)^2 \left(dr_1^2 + r_1^2 d\psi^2 + dr_2^2 + r_2^2 d\phi^2 \right) + \frac{dU^2}{U^2} \right), \tag{14}$$

then the loop can be assumed to be located at $r_1 = R$, $r_2 = 0$, which corresponds to the world volume parametrization $\sigma^1 = \psi$, $\sigma^2 = r_2$ and σ^3, σ^4 parametrize S^2 which shrinks at the boundary $\sigma^2 = r_2 = 0$. In this case, the half BPS nature is strong enough to fix the form of the classical solution uniquely once we take account of only the gauge field boundary condition $\Pi^{a=1} = -ik$. Essentially the solution takes the same form as in [14] and near the boundary $r_2 \ll 1$, it looks like

$$U \sim \frac{\kappa\sqrt{\lambda}}{2\pi r_2}, \quad A_1 \sim \frac{-iR\kappa\sqrt{\lambda}}{2\pi r_2}, \tag{15}$$

where κ is defined below (7). Plugging the solution into the action, we have

$$S_{\mathrm{D3}} + S_b = -N \left(2\kappa\sqrt{1+\kappa^2} + 2\mathrm{arcsinh}\,\kappa \right). \tag{16}$$

Then according to (13) we can calculate $\langle W_k(C) \rangle$ as

$$\begin{aligned} \langle W_k(C) \rangle &= \int e^{-(S_{\text{D3-brane}} + S_b)} \\ &= \exp\left(-(S_{\text{D3-brane}} + S_b) \right)\big|_{\mathrm{cl}} \\ &= \exp\left[N \left(2\kappa\sqrt{1+\kappa^2} + 2\mathrm{arcsinh}\,\kappa \right) \right], \end{aligned} \tag{17}$$

where in the second equality we have used $\lambda \gg 1$. This indeed agrees with the gauge theory result (8). Since $\langle W_k(C) \rangle$ is quite a complicated function, this agreement strongly supports the claim (13), at least in strong coupling regime $\lambda \gg 1$.

4 Boundary condition for D-brane from Wilson loop

The agreement (17) shown in [13] looks quite nice, but we emphasize here that the derivation in [13] is not completely along the spirit of AdS/CFT. Namely, in the basic relation (13), the right hand side will be a function of X^μ, Π^a and P_{U^i} and their boundary conditions should be provided in terms of the Wilson loop. Then the path integral, or the evaluation of the classical action should be done under these boundary conditions. In contrast to this, in [13] they do not take account of the boundary condition for P_{U^i} and rather they evaluate the action from the explicit form of the classical solution which can be uniquely fixed due to many supersymmetries the configuration preserves. It should be noticed that in the spirit of the AdS/CFT correspondence, the boundary term and the boundary condition should be specified in (13) in terms of the Wilson loop without referring to the equation of motion. Moreover, a prescription itself of giving boundary conditions from a Wilson loop would be generic and independent of its shape. Thus our aim is to deduce D3-brane boundary conditions for generic shape of the Wilson loop without using the equation of motion. Our problem is quite unique, because usually a D-brane specifies boundary conditions for an open string attached to it, while in the present case we are considering boundary conditions for a D-brane itself imposed by a Wilson loop.

4.1 Derivation of boundary conditions

For our purpose, let us take the (Wick rotated) $\mathrm{AdS}_5\times\mathrm{S}^5$ metric as

$$ds^2 = \left(\frac{2\pi\alpha' U}{L}\right)^2 (dt^2 + d\rho^2 + \rho^2(d\theta^2 + \sin^2\theta d\phi^2)) + L^2\frac{(dU^i)^2}{U^2}, \tag{18}$$

where the gauge theory lives in the four-dimensional space-time parametrized by t, ρ, θ and ϕ, and the AdS boundary exists at $U=\infty$. We choose t, ρ in such a way that a loop of generic shape is located at $\rho = 0$, and that it is extended into the t-direction in the four-dimensional space-time. A D3-brane is in the AdS_5, and is attached to the loop $\rho = 0$ at the AdS boundary. We take σ^a $(a = 1 \sim 4)$ as the D3-brane world volume coordinates, hence world volume fields are embedding coordinates $t = t(\sigma^a)$, $\rho = \rho(\sigma^a)$, $\cdots$, $U^i = U^i(\sigma^a)$, and the $U(1)$ gauge field $A_a(\sigma^a)$. As before, we take σ^a in such a way that the world volume boundary is at $\sigma^2 = 0$, and there the boundary itself is parametrized by σ^1. Thus $\rho \to 0$ as $\sigma^2 \to 0$. Note that at the beginning the world volume boundary has nothing to do with the AdS boundary $U = \infty$. Rather, we impose a condition later that the world volume boundary is located at the AdS boundary. As for other world volume coordinates, it is natural to set $\sigma^3 = \theta$, $\sigma^4 = \phi$. Near the world volume boundary $\sigma^2 \sim 0$, $\rho \sim 0$, then the S^2 parametrized by σ^3 and σ^4 shrinks, hence all fields become independent of them. Namely, at least near the world volume boundary, t, ρ and U^i are fields only of σ^1, σ^2: $t = t(\sigma^1, \sigma^2)$, $\rho = \rho(\sigma^1, \sigma^2)$ $U^i = U^i(\sigma^1, \sigma^2)$ for $\sigma^2 \sim 0$. As for the gauge field, at the world volume boundary it is along the σ^1-direction as we discussed above (14): $A_{a=1} = A_{a=1}(\sigma^2)$.

Using these coordinate choice, let us consider the D3-brane action. Near the world volume boundary, S^2-part can be integrated trivially and we obtain

$$S_{\mathrm{D3}}|_{\sigma^2\sim 0} = \int_{\sigma^2\sim 0} d\sigma^1 d\sigma^2 L_{\mathrm{D3}} = \int_{\sigma^2\sim 0} d\sigma^1 d\sigma^2 \sqrt{\det_{\sigma^1,\sigma^2}(g_{ab} + 2\pi\alpha' F_{ab})}. \tag{19}$$

From the effective Lagrangian L_{D3} we define conjugate momenta near the world volume boundary as

$$P_t = \frac{\partial L_{\mathrm{D3}}}{\partial(\partial_2 t)}, \quad P_\rho = \frac{\partial L_{\mathrm{D3}}}{\partial(\partial_2 \rho)}, \quad P_{U^i} = \frac{\partial L_{\mathrm{D3}}}{\partial(\partial_2 U^i)}, \quad \Pi^{a=1} = \frac{\partial L_{\mathrm{D3}}}{\partial(\partial_2 A_1)}. \tag{20}$$

The diffeomorphism invariance of the action implies the Hamiltonian constraint among these conjugate momenta

$$\begin{aligned} 0 = {} & (P_{U^i})^2 + (\Pi^1)^2\big((\partial_1 t)^2 + (\partial_1\rho)^2\big) - \frac{16\pi}{\lambda} N\big(P_t\partial_1\rho - P_\rho\partial_1 t\big)\rho^2 \\ & - \left(64\pi^2\lambda^{-2}N^2(U\rho)^4 - \frac{\lambda}{4\pi^2}(\Pi^1)^2\right)\left(\frac{\partial_1 U^i}{U^2}\right)^2 + \frac{\lambda}{4\pi^2}(P_t^2 + P_\rho^2)\frac{1}{U^4}. \end{aligned} \tag{21}$$

Let us examine what this equation means at the world volume boundary, i.e. $\rho \to 0$ ($\sigma^2 \to 0$)[3]. Here we make a crucial requirement that when $\sigma^2 \to 0$, $U \to \infty$. Namely, the world volume boundary lies at the AdS boundary. Then the constraint is greatly simplified to yield

$$P_{U^i}^2 + (\Pi^1)^2(\partial_1 t)^2 = 0, \quad \text{at } \sigma^2 = 0. \tag{22}$$

As discussed in subsection 3.2, a natural boundary condition for the gauge field is $\Pi^1(\sigma^1, \sigma^2 = 0) = -ik$ for each σ^1. Plugging this into the above, we get $P_{U^i} = -k\,|\partial_1 t|\,\theta_i$ with $\theta_i = U_i/U$ being S^5 coordinate, where we have used $P_{U^i} = (U^i/U)P_U$ and chosen the minus sign in order to make it consistent with the equation of motion. This is only the point we refer to information of the equation of motion. However, at $\sigma^2 = 0$, the loop is assumed to be extended along the t-direction and the D3-brane world volume is attached to it. Thus we have $|\partial_1 t| = |\dot{X}^\mu(\sigma^1)|$ at least in a local patch around a point on the loop. By using the Dirichlet boundary condition for X^μ: $X^\mu(\sigma^1, \sigma^2 = 0) = x^\mu(\sigma^1)$, we find that $|\partial_1 t|$ can be identified with $|\dot{x}^\mu|$ at least in a local patch. Thus we see that the Hamiltonian constraint (21) at the world volume boundary $\sigma^2 = 0$ under the requirement $U(\sigma^1, \sigma^2 = 0) = \infty$ implies the following boundary conditions:

$$X^\mu(\sigma^1, \sigma^2 = 0) = x^\mu(\sigma^1), \tag{23}$$

$$\Pi^1(\sigma^1, \sigma^2 = 0) = -ik, \qquad P_{U^i}(\sigma^1, \sigma^2 = 0) = -k\left|\dot{x}(\sigma^1)\right|\theta_i(\sigma^1), \tag{24}$$

where we have again identified s with σ^1 by using the reparametrization of the loop. By use of the embedding coordinates X^μ, we can convert the world volume indices of Π^a into the space-time ones like $\Pi^\mu \equiv \partial_a X^\mu \Pi^a$. Since Π^a has the only non-vanishing component for $a = 1$, Π^μ satisfies the following boundary condition

$$\Pi^\mu = \partial_1 X^\mu \Pi^1 = -ik\dot{x}^\mu\,, \qquad \text{at } \sigma^2 = 0, \tag{25}$$

where we have used (23). Using the second boundary condition in (24), this gives the following relation

$$(\Pi^\mu)^2 + (P_{U^i})^2 = 0\,. \tag{26}$$

It is worth noting that the boundary condition (24) thus corresponds to the BPS condition in [14], i.e., force balance between the electric charge Π^1 and the deformation of the D3-brane which is characterized by P_{U^i}, in the case of the spike solution in the flat space. Since the spike solution presented in [14] is the half BPS, it is natural that the equation (26) also implies a local BPS condition for the Wilson loop. In fact, in the gauge theory side, the local BPS condition for the Wilson loop (1) is given by

$$\dot{x}^2 = \dot{y}^2, \tag{27}$$

as commented in section 2. For the Wilson loop satisfying this condition, $\dot{y}_i = |\dot{x}|\theta_i$. Therefore, by using this relation in (24), we deduce boundary conditions in a general case as

$$\Pi^\mu(\sigma^1, \sigma^2 = 0) = -ik\dot{x}^\mu(\sigma^1), \qquad P_{U^i}(\sigma^1, \sigma^2 = 0) = -k\dot{y}_i(\sigma^1). \tag{28}$$

We find our boundary condition quite natural because once it is assumed, the local BPS conditions in both sides become equivalent. From the point of view of our general boundary conditions (28), (24) are those in a local patch along the loop where the loop can be regarded as the straight line with $\dot{x}^2 = \dot{y}^2$. In summary, the Wilson loop with winding number k of $\mathcal{O}(N)$ given in (1) provides following boundary conditions on a D3-brane in the AdS/CFT correspondence:

$$\text{Dirichlet for } X^\mu: \qquad X^\mu(\sigma^1, \sigma^2 = 0) = x^\mu(\sigma^1), \tag{29}$$

$$\text{Neumann for } A_a: \qquad \Pi^\mu(\sigma^1, \sigma^2 = 0) = -ik\dot{x}^\mu(\sigma^1), \qquad \left(\Pi^\mu = \frac{\partial X^\mu}{\partial \sigma^1}\Pi^1\right) \tag{30}$$

$$\text{Neumann for } U^i: \qquad P_{U^i}(\sigma^1, \sigma^2 = 0) = -k\dot{y}_i(\sigma^1). \tag{31}$$

[3] In $k = 1$ (string world sheet) case, relation between the Hamiltonian constraint and a boundary condition was discussed in [12].

Note that these are in accordance with the spirit of the AdS/CFT correspondence: the boundary conditions are given in terms of the data of the Wilson loop $x^\mu(s)$, $y^i(s)$ and k, namely the shape of the loop and the winding number.

There are several evidences supporting our boundary conditions. For example, they are consistent with the boundary conditions for the string world sheet ($k=1$) case. Moreover, we can explicitly check that the D3-brane solution obtained in [13] which we briefly review in subsection 3.2 actually satisfies our boundary conditions. Here we again emphasize that in our derivation we do not refer to the equation of motion, hence this fact is an evidence for validity of our boundary conditions. Another interesting aspect of our boundary condition is that under them the Gauss' law constraint $\Pi^{a=2}=0$ implies $\partial\sigma^2/\partial\sigma^1=0$, namely orthogonality relation between the tangential and perpendicular directions of the world volume boundary.

5 Bulk interpretation of the position of eigenvalue

In this section as an application of our boundary conditions (29)∼(31), we examine what happens if they are applied to the basic relation (13). As mentioned in (17), when $\lambda \gg 1$, the path integral in the right hand side in (13) can be replaced by $e^{-S^{\rm cl}_{\rm D3}-S^{\rm cl}_b}$ where the action is evaluated by its classical value under the boundary conditions (29)∼(31). Therefore, for $\lambda \gg 1$, we have

$$\langle W_k(C)\rangle = e^{-\left(S^{\rm cl}_{\rm D3}+S^{\rm cl}_b\right)\big|_{\rm b.c.}}, \tag{32}$$

where $|_{\rm b.c.}$ denotes the evaluation under (29)∼(31). Thus the exponent of $\langle W_k(C)\rangle$, $V_{\rm eff}(k)$ defined below (8) can be read from this equation as $V_{\rm eff}(k) = \left(S^{\rm cl}_{\rm D3}+S^{\rm cl}_b\right)\big|_{\rm b.c.}$. Then by using the fact we mentioned at the end of section 2, we can make a connection between the position of the isolated eigenvalue m_* and the bulk quantity as

$$m_* = -V'_{\rm eff}(k) = -\frac{\partial}{\partial k}\left(S^{\rm cl}_{\rm D3}+S^{\rm cl}_b\right)\big|_{\rm b.c.}. \tag{33}$$

In order to calculate this, it is instructive to notice that k is a part of the boundary conditions and as such variation of k gives rise to that of the classical action. However, the variation of the classical action only comes from the boundary action due to the equation of motion. More precisely, for the Dirichlet direction the variation is given as (10), while that for the Neumann direction is as (11). From the boundary conditions (29)∼(31), the variation with respect to k does not affect the Dirichlet direction X^μ. Thus as in (11) the variation comes only from the boundary term

$$S^{\rm cl}_b = \oint_{\sigma^2=0} d\sigma^1 d\sigma^3 d\sigma^4 (\Pi^1 A_1 + P_{U^i}U^i). \tag{34}$$

Now from (30) and (31), we obtain

$$S^{\rm cl}_b\big|_{\rm b.c.} = -k\oint_{\sigma^2=0} d\sigma^1\left(iA_1\dot{x}^1 + U^i\dot{y}_i\right). \tag{35}$$

Notice here that in (29)∼(31), the integration over S^2-part (namely σ^3 and σ^4) is done as in (19) and (20). Therefore we finally find that

$$m_* = -\frac{\partial\, S^{\rm cl}_b\big|_{\rm b.c.}}{\partial k} = \oint_{\sigma^2=0} d\sigma^1\left(iA_1\dot{x}^1 + U^i\dot{y}_i\right). \tag{36}$$

Thus we obtain a clear interpretation of the isolated eigenvalue as a bulk quantity, namely as flux of the gauge field, more precisely, an integration over the $U(1)$ gauge field plus the scalar field along the loop.

As for this result, several notes are in order: first of all, the above derivation is exactly in accordance with the spirit of the AdS/CFT correspondence in contrast to the result in [13] where they use the explicit form of the solution to the equation of motion as in (15) to get (16). On the other hand, in our derivation we do not use the explicit form of the solution. Rather, plugging it into (36), we obtain

$$\int_{\sigma^2=0} d\sigma^1\left(iA_1\dot{x}^1 + U^i\dot{y}_i\right) = \sqrt{\lambda(1+\kappa^2)}, \tag{37}$$

which indeed reproduces the gauge theory result below (7). This is a nontrivial consistency check of our boundary conditions. It is also important to recognize that the boundary condition for U^i in (31) plays an essential role in deriving (37), which is missed in [13]. Finally we note that we again have (the exponent of) the $U(1)$ Wilson loop. It would be quite an interesting aspect of our result, although the $U(1)$ gauge field on the D3-brane in the bulk of course should not be confused with the original $U(N)$ gauge field on the N D3-brane in the gauge theory side.

6 Conclusions

We have analyzed the circular Wilson loop with winding number k of $\mathcal{O}(N)$ in the gauge theory by using a D3-brane carrying k units of charge in the context of the AdS/CFT correspondence. It is known that the calculation of the expectation value of this Wilson loop in the gauge theory side amounts to considering a Gaussian matrix model with an exponential operator insertion due to its symmetry. In this calculation an isolated eigenvalue plays an essential role. After emphasizing importance of boundary conditions the Wilson loop imposes on the gravity side, we deduce them based on the Hamiltonian constraint. We have checked that our boundary conditions pass several nontrivial tests. As an application, we have taken account of them in the AdS/CFT correspondence for the Wilson loop and succeeded in giving an interesting interpretation in terms of fields in the gravity side to the position of the eigenvalue in the matrix model.

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