THE EUROPEAN
PHYSICAL JOURNAL C

Larmor rotation in galaxies

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Abstract   The role of the gravitomagnetic field in the rotation of galaxies is clarified. Larmor's theorem simplifies the analysis of different galactic equilibrium solutions. In particular, the self-consistent solution, which takes into account both the equivalent Larmor gravitomagnetic field and fluid convection, recovers previous results that reproduce galactic rotation curves without recourse to dark matter.

1 Introduction

Joseph Larmor proposed dynamical models for the electrons based on the canonical formulation of classical mechanics [1,2]. He was the first to establish the connection between the motion of electrons in a magnetic field and the angular velocity of rotating bodies in the form of Larmor's theorem [3,4]. This connection can be extended to the gravitoelectromagnetic formulation of gravity [5]. In the gravitoelectromagnetic context Larmor's theorem relates the Lorentz force in a stationary frame with the Coriolis acceleration in a rotating frame (cf. Appendix A). An equivalent Larmor gravitomagnetic field can be associated with the angular velocity, so that the transformation from an inertial to a non-inertial rotating frame corresponds to a transformation of the gravitomagnetic field. The laws of physics do not depend on the reference frame – the change to a rotation frame is just a convenient transformation that may or not simplify the calculations.

Concerning the rotation of galaxies, a recent paper demonstrated that the observed velocity curves of characteristic galaxies can be reasonably reproduced using a self-consistent equilibrium solution, within the gravitoelectromagnetic context and without introducing dark matter [6]. Earlier efforts on this problem used the general gravity approach, adding considerable physical insight on the subject [7–9]. The problem with the general gravity approach is the limited number of known exact solutions of Einstein's equation and the difficulty in developing self-consistent equilibria. The gravito-electromagnetic formulation alleviates these issues.

The objective of the present paper is to clarify the role of the gravitomagnetic field and illustrate some advantages of using a fluid (Eulerian) model instead of a particle (Lagrangian) approach. Of course, both formulations are equivalent [10]. Two solutions of the fluid-like galactic equilibrium are presented, based on different simplifying assumptions. The driven galactic equilibrium, derived in Sect. 2, focuses on a pure rotation model, completely neglecting the local mass currents in the non-inertial frame. In this case, the dust flow is described by a simple hydrostatic balance condition controlled by the mean Larmor gravitomagnetic field, which is equivalent to a pure rotation. This equilibrium solution corresponds to the classical circular rotation model, but is not self-consistent. It needs an external source and weak dissipative processes to be maintained, and can be clearly ruled out. On the other hand, the self-consistent galactic equilibrium, derived in Sect. 3, is maintained by the internal currents in a perfect fluid, as previously shown [6]. This solution is obtained if both the gravitomagnetic Lorentz acceleration and the convective terms are taken into account. In this second case the Lorentz acceleration is formed by the contributions of the pure rotation Larmor field and the local mass currents. Although the physical results do not depend on the reference frame, it is simpler in this case to remain in a local inertial frame, where the gravitomagnetic field in the Lorentz acceleration term includes all the mass current effects in this frame, namely, the sum of the pure rotation mean fluid velocity and the deviations from the mean.

A brief derivation of Larmor's theorem is presented, in a form appropriate for the fluid approach, in the Appendix A. This derivation shows how the gravitomagnetic field contributions change for a transformation from a local inertial frame to a rotating frame. The transformation to a non-inertial frame explicitly shows the contributions of the fictitious centrifugal and Coriolis accelerations, which are, of

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course, included in the full Lorentz acceleration in the inertial frame (in a truly inertial reference frame all the forces vanish, including gravity, and the motion is described by free-fall). The relation between rotation and the Lorentz acceleration on a moving mass constitutes the essence of the Larmor theorem, as pointed out by Brillouin [2]. The theorem was originally formulated having in mind the motion of electrons in an electromagnetic field, reaching wide acceptance with the quantum mechanical description of the electronic motion in the atom, but has the character of a more general law as conjectured by Larmor [1].

2 Driven galactic equilibrium

The equations of motion of a perfect fluid in rotary flow can be formulated in a frame rotating at the angular velocity $\boldsymbol{\Omega}$. In this case the motion is for the most part steady, but the dynamical equation explicitly includes the non-inertial centrifugal and Coriolis accelerations. The momentum conservation equation in the weak relativistic approximation in a rotating frame, as shown in the Appendix A, is given by

$$
\begin{aligned}
&\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{v} + \left(\frac{\partial \boldsymbol{\Omega}}{\partial t} + (\boldsymbol{v} \cdot \nabla)\,\boldsymbol{\Omega} \right) \times \boldsymbol{r} \\
&+ [\boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r})] \cdot \nabla\,\boldsymbol{\Omega} = -\frac{\nabla p}{\rho} + \boldsymbol{E} + \nabla \phi_c \\
&+ \boldsymbol{v} \times (\boldsymbol{B} - 2\boldsymbol{B}_L) .
\end{aligned}
\tag{2.1}
$$

Now consider the steady motion ($\partial/\partial t \equiv 0$) of a rotating dust distribution with vanishing pressure ($p \cong 0$). Furthermore, consider that the flowing dust is in pure rotation with vanishing local mass currents ($\boldsymbol{v} = 0$). This corresponds to the strong simplifying assumption referred to in the Introduction, when the convective and Lorentz acceleration effects due to the local mass currents are completely neglected in the rotating frame. In this case the flow is described by the hydrostatic balance condition

$$
[\boldsymbol{r} \times (\boldsymbol{\Omega} \times \boldsymbol{r})] \cdot \nabla\,\boldsymbol{\Omega} = -\nabla \phi + \nabla \phi_c ,
\tag{2.2}
$$

which is equivalent to (cf. Appendix A)

$$
\nabla \phi = -\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{B}_L \times \boldsymbol{u} ,
\tag{2.3}
$$

where ϕ is the Newtonian potential, $\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r}$ is the mean fluid velocity in the stationary frame, and $\boldsymbol{B}_L = -\boldsymbol{\Omega}$ is the equivalent Larmor field. Note that for an uniform angular velocity Eq. (2.2) reduces to a simple balance between the gravitational attraction and the centrifugal force, valid only for an infinitely long cylinder or for an infinitesimally thin disk. However, for non vanishing vorticity (rotational flow) Eq. (2.2) and the equivalent Eq. (2.3) show that the gravitational attraction can be balanced by the gravitomagnetic
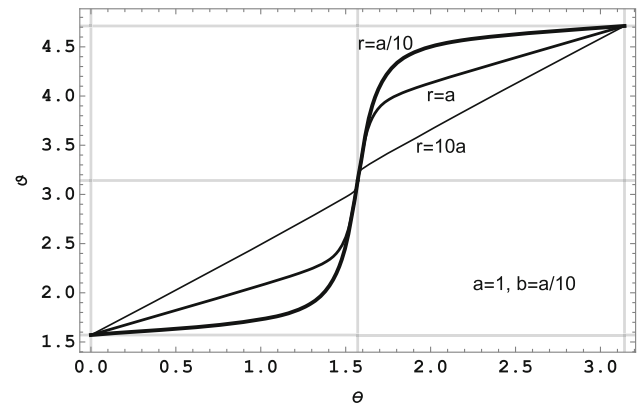
Fig. 1 Piecewise continuous representation of the tilt angle ϑ for an hypothetical thin disk approximately represented by the Kuzmin model described below

field everywhere. Note also that in both cases the convective acceleration term has been neglected.

Assuming azimuthal symmetry in the mean motion in spherical coordinates (r, θ, φ), the angular velocity $\boldsymbol{\Omega}$ can be represented in the form

$$
\boldsymbol{\Omega} = \{\cos (\theta - \vartheta) , -\sin (\theta - \vartheta) , 0\}\,\Omega\,(r, \theta) ,
\tag{2.4}
$$

where $\theta - \vartheta\,(r, \theta)$ is the angle formed by the axial vector $\boldsymbol{\Omega}$ with respect to the radial direction $\boldsymbol{r}$. This representation is in accordance with the differential rotation in galaxies demonstrated by Oort [11]. The mean axial velocity in the stationary frame is

$$
\boldsymbol{u} = \boldsymbol{\Omega} \times \boldsymbol{r} = \{0, 0, u_\varphi\,(r, \theta)\} ,
\tag{2.5}
$$

where $u_\varphi\,(r, \theta) = r \sin (\theta - \vartheta)\,\Omega\,(r, \theta)$. Note that individual particles undergoing a rotary motion in a plane normal to $\boldsymbol{\Omega}$ have velocity components along all directions. In this way, particles execute an up-and-down motion in the gravitational well, while rotating parallel to the galactic plane. In the fluid description the vertical motions between different particles cancel out and the mean velocity $\boldsymbol{u}$ has a single component in the azimuthal direction, considerably simplifying the analysis. Exactly on axis a single particle could exist executing a vertical oscillation in the gravitational potential well.

Using the above form for $\boldsymbol{\Omega}$, the equilibrium balance can be written as

$$
\begin{aligned}
\left\{ \frac{\partial \phi}{\partial r}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}, 0 \right\} &= \{\sin (\theta - \vartheta) , \cos (\theta - \vartheta) , 0\} \\
&\times \Omega\,(r, \theta)\,u\,(r, \theta) ,
\end{aligned}
\tag{2.6}
$$

It follows that

$$
\begin{cases}
\dfrac{\partial \phi}{\partial \theta} = \cot (\theta - \vartheta)\,r\,\dfrac{\partial \phi}{\partial r} \\
u\,(r, \theta) = \dfrac{\partial \phi / \partial r}{\sin (\theta - \vartheta)\,\Omega\,(r, \theta)}
\end{cases}
\tag{2.7}
$$

The tilt angle ϑ with respect to the vertical axis can be represented as a piecewise continuous function of θ by

$$\vartheta(r,\theta) = \theta - \text{arccotan}\left(\frac{\partial\phi/\partial\theta}{r\partial\phi/\partial r}\right) + \pi U\left(\theta - \frac{\pi}{2}\right), \quad (2.8)$$

where U is the unit step function. In this form $\vartheta = \pi$ at the equatorial plane $\theta = \pi/2$, where $\partial\phi/\partial\theta \to 0$ for vertically symmetric distributions as illustrated in Fig. 1 ($\boldsymbol{B}_L$ points downwards at $\theta = \pi/2$). Taking

$$\begin{cases} \sin(\theta - \vartheta) = \dfrac{1 - 2U(\theta - \pi/2)}{\dfrac{\partial\phi/\partial\theta}{r\partial\phi/\partial r}\sqrt{1 + \left(\dfrac{r\partial\phi/\partial r}{\partial\phi/\partial\theta}\right)^2}} \\[2em] \cos(\theta - \vartheta) = \dfrac{1 - 2U(\theta - \pi/2)}{\sqrt{1 + \left(\dfrac{r\partial\phi/\partial r}{\partial\phi/\partial\theta}\right)^2}} \end{cases} \quad (2.9)$$

the equilibrium equation gives (a positive sign is assumed for Ω, implying a negative sign for u_φ)

$$\begin{cases} \Omega(r,\theta) = \sqrt{\dfrac{r\partial\phi/\partial r}{r\sin^2(\theta - \vartheta)}} \\[1.5em] u_\varphi(r,\theta) = -\sqrt{r\dfrac{\partial\phi}{\partial r}} \end{cases} \quad (2.10)$$

Therefore, the equilibrium balance in its simplest form gives the circular velocity result used in most galactic rotation curve analyzes so far, eventually leading to the introduction of dark matter for compatibility with astronomical observations. However, as pointed out in the Introduction, this equilibrium is not self-consistent, needing an external source to drive the rotation as will be shown next. First, it is interesting to display the results of the above equilibrium solution for a simple dust distribution. In this respect, consider the exact Miyamoto-Nagai solution for Poisson's equation in spherical coordinates [12]

$$\begin{cases} \phi(r,\theta) \\ = -\dfrac{GM}{\left[r^2\sin^2\theta + \left(a + \sqrt{b^2 + r^2\cos^2\theta}\right)^2\right]^{1/2}} \\[1.5em] \rho(r,\theta) = \dfrac{Mb^2}{4\pi} \\[0.5em] \times \dfrac{ar^2\sin^2\theta + \left(a + 3\sqrt{b^2 + r^2\cos^2\theta}\right)\left(a + \sqrt{b^2 + r^2\cos^2\theta}\right)^2}{\left[r^2\sin^2\theta + \left(a + \sqrt{b^2 + r^2\cos^2\theta}\right)^2\right]^{5/2}\left(b^2 + r^2\cos^2\theta\right)^{3/2}} \end{cases} \quad (2.11)$$

where a and b are geometrical parameters that can be adjusted to represent the central bulge and the disk parts of a spheroidal mass distribution, and M is the total mass. In the limit $a \to 0$

this potential-density pair reduces to the spherical Plummer model [13]

$$\begin{cases} \phi(r,\theta) = -\dfrac{GM}{\sqrt{r^2 + b^2}} \\[1.5em] \rho(r,\theta) = \dfrac{3Mb^2}{4\pi\left(r^2 + b^2\right)^{5/2}} \end{cases} \quad (2.12)$$

and in the limit $b \ll a$ it approaches the thin disk Kuzmin model [13]

$$\begin{cases} \phi(r,\theta) \\ = -\dfrac{GM}{\sqrt{r^2 + a^2 + 2ar|\cos\theta|}} \\[1em] \rho(r,\theta) = \dfrac{Mb^2}{4\pi} \\[0.5em] \times \dfrac{a\left[(r^2 + a^2 + 6r^2\cos^2\theta) + r\left(5a^2 + 3r^2\cos^2\theta\right)|\cos\theta|\right]}{r^3\cos^2\theta\left[r^2 + a^2 + 2ar|\cos\theta|\right]^{5/2}|\cos\theta|} \end{cases} \quad (2.13)$$

Figures 2 and 3 show that the driven equilibrium solution approaches the Kepler profile at large distances, either for a nearly spherical configuration (Plummer model) or a thin disk configuration (Kuzmin model).

Now consider the Larmor field, responsible for the driven rotary motion of the dust distribution, whose poloidal components are (assuming $\partial\phi/\partial r \geq 0$)

$$\begin{aligned} \boldsymbol{B}_L &= -\boldsymbol{\Omega} = \{-\cos(\theta - \vartheta), \sin(\theta - \vartheta), 0\}\,\Omega(r,\theta) \\ &= \frac{[1 - 2U(\theta - \pi/2)]\,\text{sign}(\partial\phi/\partial\theta)}{r}\sqrt{r\frac{\partial\phi}{\partial r}} \\ &\quad \times \left\{-\frac{\partial\phi/\partial\theta}{r\partial\phi/\partial r}, 1, 0\right\}. \end{aligned} \quad (2.14)$$

The curl of $\boldsymbol{B}_L$ is

$$\begin{aligned} \nabla \times \boldsymbol{B}_L = \Bigg\{0, 0, \sin(\theta - \vartheta)&\left(\frac{\partial\Omega}{\partial r} + \frac{\Omega}{r}\frac{\partial\vartheta}{\partial\theta}\right) \\ + \cos(\theta - \vartheta)&\left(\frac{1}{r}\frac{\partial\Omega}{\partial\theta} - \Omega\frac{\partial\vartheta}{\partial r}\right)\Bigg\}, \end{aligned} \quad (2.15)$$

and the divergence of $\boldsymbol{B}_L$ is

$$\begin{aligned} \nabla \cdot \boldsymbol{B}_L = &-\cos(\theta - \vartheta)\left(\frac{\partial\Omega}{\partial r} + \frac{\Omega}{r}\frac{\partial\vartheta}{\partial\theta}\right) \\ &+ \sin(\theta - \vartheta)\left(\frac{1}{r}\frac{\partial\Omega}{\partial\theta} - \Omega\frac{\partial\vartheta}{\partial r}\right) \\ &- \frac{\sin(\theta - \vartheta)}{\sin\theta}\frac{\Omega}{r}. \end{aligned} \quad (2.16)$$
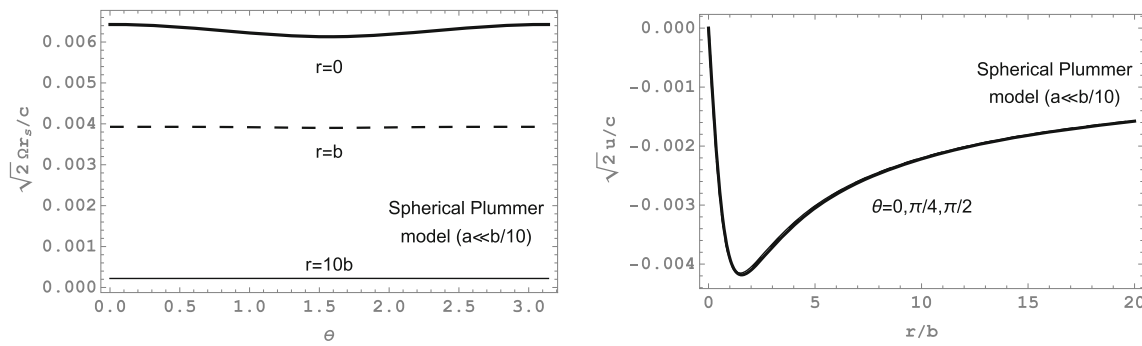
Fig. 2 Normalized angular velocity and rotation velocity profiles for an hypothetical dust distribution represented approximately by the spherical Plummer model. The right-hand panel shows that the rotation velocity profile hardly varies with the polar angle θ, corresponding to a nearly constant tangential velocity on a sphere. Note that the radial distance to the axis of rotation is $R = r \sin \theta$
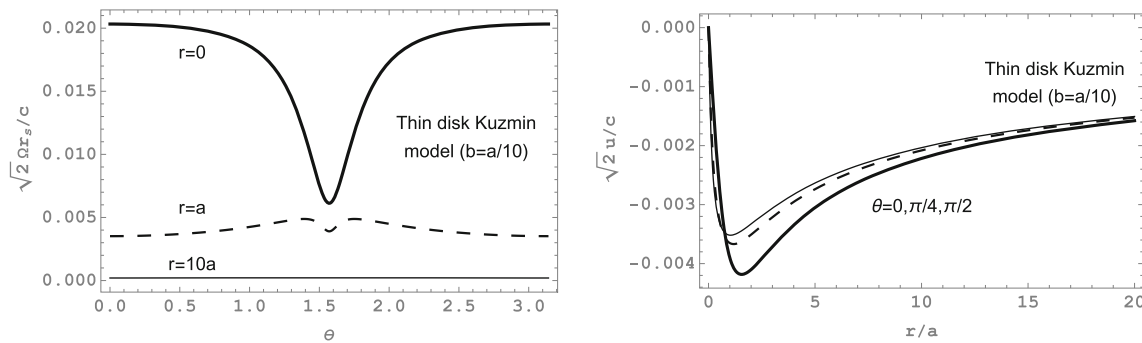

Fig. 3 Normalized angular velocity and rotation velocity profiles for an hypothetical dust distribution represented approximately by the thin disk Kuzmin model. Note that the radial distance to the axis of rotation is $R = r \sin \theta$

Assuming $\partial \phi / \partial r \geq 0$:

$$(\nabla \times \boldsymbol{B}_L)_\varphi = \frac{[1 - 2U(\theta - \pi/2)] \operatorname{sign}(\partial \phi / \partial \theta)}{2r (r \partial \phi / \partial r)^{3/2}}$$
$$\times \left[\frac{\partial \phi}{\partial r} \left(r^2 \frac{\partial^2 \phi}{\partial r^2} + r \frac{\partial \phi}{\partial r} + 2 \frac{\partial^2 \phi}{\partial \theta^2} \right) \right.$$
$$\left. - \frac{\partial \phi}{\partial \theta} \frac{\partial^2 \phi}{\partial r \partial \theta} \right], \tag{2.17}$$

and

$$\nabla \cdot \boldsymbol{B}_L = \frac{[1 - 2U(\theta - \pi/2)] \operatorname{sign}(\partial \phi / \partial \theta)}{2r (r \partial \phi / \partial r)^{3/2}}$$
$$\times \left[r \frac{\partial \phi}{\partial r} \left(2 \cot \theta \frac{\partial \phi}{\partial r} - \frac{\partial^2 \phi}{\partial r \partial \theta} \right) \right.$$
$$\left. + \frac{\partial \phi}{\partial \theta} \left(r \frac{\partial^2 \phi}{\partial r^2} - \frac{\partial \phi}{\partial r} \right) \right]. \tag{2.18}$$

The divergence is not null, implying the existence of an external nonconservative source driving the rotary flow. The Larmor field $\boldsymbol{B}_L$ in this case cannot be represented by the curl of a vector potential. In general, $\nabla \times \boldsymbol{B}_L$ is given by the sum of mass currents and time-varying gravitoelectric fields (these two terms were assumed to vanish in the above solution):

$$\nabla \times \boldsymbol{B}_L = -\frac{4\pi G}{c^2} \boldsymbol{j} + \frac{1}{c^2} \frac{\partial \boldsymbol{E}}{\partial t}. \tag{2.19}$$

According to Stokes' theorem

$$\iint_S (\nabla \times \boldsymbol{B}_L) \cdot d\boldsymbol{S} = \oint_\ell \boldsymbol{B}_L \cdot d\boldsymbol{\ell}, \tag{2.20}$$

where S is the surface of a poloidal cross-section and ℓ is the poloidal contour enclosing the surface. This implies that the boundary of the rotating system in the driven solution case (with $\nabla \times \boldsymbol{B}_L = 0$) must contain singularities injecting helicity in the system, either in the form of external mass currents $\boldsymbol{j}_{\text{ext}}$ or by a transformer action represented by $\partial \boldsymbol{E}_{\text{ext}} / \partial t$. These edge singularities correspond to the dark matter misconception. Furthermore, such driven equilibria can be maintained only in conjunction with a weak dissipative process.

The above discussion shows that the equilibrium described by the hydrostatic balance condition (2.3), which leads to the circular velocity solution, is not self-consistent. It requires an external source and some dissipative mechanism in order to be maintained. Ruling out this solution, and its erro-

neous simplifying assumptions, the self-consistent solution is examined in the next section.

3 Self-consistent galactic equilibrium

The momentum conservation equation in a stationary frame is (cf. Appendix A)

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\frac{\nabla p}{\rho} + \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}. \tag{3.1}$$

Consider again steady motion ($\partial/\partial t \equiv 0$) with vanishing pressure ($p \cong 0$):

$$(\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\nabla \phi + \boldsymbol{u} \times \boldsymbol{B}. \tag{3.2}$$

Note that the full convective and Lorentz acceleration terms have been maintained. The same representation used for $\boldsymbol{\Omega}$ in the previous section can be adopted for the gravitomagnetic field $\boldsymbol{B}$ in the present case

$$\boldsymbol{B} = \{\cos(\theta - \vartheta), -\sin(\theta - \vartheta), 0\} B(r, \theta). \tag{3.3}$$

The mean axial velocity in an axisymmetric equilibrium is $\boldsymbol{u} = \{0, 0, u_\varphi(r, \theta)\}$, so that

$$\boldsymbol{u} \times \boldsymbol{B} = \{\sin(\theta - \vartheta), \cos(\theta - \vartheta), 0\} u_\varphi(r, \theta) B(r, \theta), \tag{3.4}$$

and

$$(\boldsymbol{u} \cdot \nabla) \boldsymbol{u} = -\frac{u_\varphi^2(r, \theta)}{r} \{1, \cot \theta, 0\}. \tag{3.5}$$

The momentum balance equation gives

$$\begin{cases} \dfrac{\partial \phi}{\partial r} = \dfrac{u_\varphi^2(r, \theta)}{r} + \sin(\theta - \vartheta) u_\varphi(r, \theta) B(r, \theta) \\ \dfrac{1}{r} \dfrac{\partial \phi}{\partial \theta} = \dfrac{u_\varphi^2(r, \theta)}{r} \cot \theta + \cos(\theta - \vartheta) u_\varphi(r, \theta) B(r, \theta) \end{cases} \tag{3.6}$$

Hence

$$\begin{cases} \sin(\theta - \vartheta) = \dfrac{r\partial \phi/\partial r - u_\varphi^2(r, \theta)}{r u_\varphi(r, \theta) B(r, \theta)} \\ \cos(\theta - \vartheta) = \dfrac{\partial \phi/\partial \theta - u_\varphi^2(r, \theta) \cot \theta}{r u_\varphi(r, \theta) B(r, \theta)} \end{cases} \tag{3.7}$$

The tilt angle $\vartheta(r, \theta)$ of the gravitomagnetic field is given by the piecewise continuous function

$$\begin{aligned} \vartheta(r, \theta) &= \theta - \arctan\left(\frac{r\partial\phi/\partial r - u_\varphi^2(r, \theta)}{\partial\phi/\partial\theta - u_\varphi^2(r, \theta) \cot \theta}\right) \\ &\quad + \pi U\left(\theta - \frac{\pi}{2}\right), \end{aligned} \tag{3.8}$$

and the field magnitude is

$$B(r, \theta) = \frac{\partial\phi/\partial\theta - u_\varphi^2(r, \theta) \cot \theta}{\cos(\theta - \vartheta) r u_\varphi(r, \theta)}. \tag{3.9}$$

The solution at this point departs from the previous case (Sect. 2) in that $\boldsymbol{B}$ can be written in terms of a gravitomagnetic vector potential $\boldsymbol{A} = \{0, 0, A_\varphi(r, \theta)\}$, so that $\nabla \cdot \boldsymbol{B} = 0$. This is possible because the convective derivative of the flow velocity is included in the dynamical equation (cf. Eq. 3.2). The fluid model describes the collective effects of the gravitational Vlasov fields $\boldsymbol{E} = -\nabla\phi$ and $\boldsymbol{B}$ produced by the mass and mass current distributions, respectively. Introducing the gravitomagnetic flux function $\psi(r, \theta) = A_\varphi(r, \theta) r \sin \theta$, the field $\boldsymbol{B}$ is given by

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} = \frac{1}{r^2 \sin \theta}\left\{\frac{\partial\psi}{\partial\theta}, -r\frac{\partial\psi}{\partial r}, 0\right\}, \tag{3.10}$$

and

$$\begin{aligned} \nabla \times \boldsymbol{B} &= \nabla \times (\nabla \times \boldsymbol{A}) \\ &= \left\{0, 0, -\frac{1}{r^3 \sin \theta}\left[r^2\frac{\partial^2\psi}{\partial r^2} \right.\right. \\ &\quad \left.\left. + \sin\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right]\right\}. \end{aligned} \tag{3.11}$$

Ampère's law gives $\nabla \times \boldsymbol{B}$ in terms of the mass current $\rho\boldsymbol{u}$

$$\nabla \times \boldsymbol{B} = -\frac{4\pi G}{c^2}\rho\boldsymbol{u}, \tag{3.12}$$

which leads to the Grad-Shafranov equation for ψ

$$\frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = \frac{4\pi G}{c^2}\rho(r, \theta) u_\varphi(r, \theta) r \sin\theta. \tag{3.13}$$

The representation of $\boldsymbol{B}$ as a tilted axial vector combined with the momentum balance equation gives

$$\begin{cases} \dfrac{\partial\psi}{\partial r} = r B(r, \theta) \sin\theta \sin(\theta - \vartheta) \\ \quad = \dfrac{\sin\theta}{u_\varphi(r, \theta)}\left(r\dfrac{\partial\phi}{\partial r} - u_\varphi^2(r, \theta)\right) \\ \dfrac{\partial\psi}{\partial\theta} = r^2 B(r, \theta) \sin\theta \cos(\theta - \vartheta) \\ \quad = \dfrac{r\sin\theta}{u_\varphi(r, \theta)}\left(\dfrac{\partial\phi}{\partial\theta} - u_\varphi^2(r, \theta) \cot\theta\right) \end{cases} \tag{3.14}$$

Substitution of the above relations in the Grad-Shafranov equation gives a second-order nonlinear differential equation relating the mean fluid velocity $u_\varphi(r, \theta)$ to the Newtonian potential $\phi(r, \theta)$ and to the mass density $\rho(r, \theta)$

$$r \frac{\partial}{\partial r} \left[\frac{1}{u_\varphi(r, \theta)} \left(r \frac{\partial \phi}{\partial r} - u_\varphi^2(r, \theta) \right) \right]$$
$$+ \frac{\partial}{\partial \theta} \left[\frac{1}{u_\varphi(r, \theta)} \left(\frac{\partial \phi}{\partial \theta} - u_\varphi^2(r, \theta) \cot \theta \right) \right]$$
$$= \frac{4\pi G}{c^2} \rho(r, \theta) u_\varphi(r, \theta) r^2. \tag{3.15}$$

Furthermore, $\phi(r, \theta)$ and $\rho(r, \theta)$ are related by Poisson's equation $\nabla^2 \phi = 4\pi G \rho$ in spherical coordinates

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial \phi}{\partial \theta} \right) = 4\pi G \rho(r, \theta). \tag{3.16}$$

The two above equations can be combined in the form

$$\left(u_\varphi + r \frac{\partial u_\varphi}{\partial r} \right) r \frac{\partial \phi}{\partial r} + \left(u_\varphi \cot \theta + \frac{\partial u_\varphi}{\partial \theta} \right) \frac{\partial \phi}{\partial \theta}$$
$$= u_\varphi \left[\left(\frac{u_\varphi}{\sin^2 \theta} - r \frac{\partial u_\varphi}{\partial r} - \cot \theta \frac{\partial u_\varphi}{\partial \theta} \right) u_\varphi \right.$$
$$\left. + 4\pi G \rho r^2 \left(1 - \frac{u_\varphi^2}{c^2} \right) \right], \tag{3.17}$$

which does not explicitly depend on the second derivatives of ϕ. Introducing a normalization radius r_0 and a normalization mass density $\rho_0 = c^2 / (G r_0^2)$, the above nonlinear first order partial differential equation for $\beta = u_\varphi / c$ can be written in normalized form

$$\left(\beta + r \frac{\partial \beta}{\partial r} \right) r \frac{\partial(\phi/c^2)}{\partial r} + \left(\beta \cot \theta + \frac{\partial \beta}{\partial \theta} \right) \frac{\partial(\phi/c^2)}{\partial \theta}$$
$$= \beta \left[\left(\frac{\beta}{\sin^2 \theta} - r \frac{\partial \beta}{\partial r} - \cot \theta \frac{\partial \beta}{\partial \theta} \right) \beta + 4\pi \frac{\rho}{\rho_0} \frac{r^2}{r_0^2} \left(1 - \beta^2 \right) \right]. \tag{3.18}$$

The gravitational potential $\phi(r, \theta)$ is given in terms of the mass density $\rho(r, \theta)$ by the general integral in spherical coordinates

$$\frac{\phi(r, \theta)}{c^2} = -\frac{1}{r_0^2} \int_V \frac{\rho(r', \theta')}{\rho_0} \frac{d^3 r'}{|r' - r|}$$
$$= -\int_0^\infty \int_0^\pi \int_0^{2\pi} \frac{\rho(r', \theta')}{\rho_0}$$
$$\times \frac{r'^2 \sin \theta' dr' d\theta' d\varphi'}{r_0^2 \sqrt{r^2 + r'^2 - 2rr'(\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\varphi - \varphi'))}}$$
$$= -\int_0^\infty \int_0^\pi \frac{\rho(r', \theta')}{\rho_0} \frac{4K(m) r'^2 \sin \theta' dr' d\theta'}{r_0^2 \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}}, \tag{3.19}$$

where $m = m(r, r'; \theta, \theta')$ is the squared modulus of the elliptic integral of the first kind $K(m)$

$$m(r, r'; \theta, \theta') = -\frac{4rr' \sin \theta \sin \theta'}{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}, \tag{3.20}$$

and

$$r \frac{\partial}{\partial r} \left(\frac{\phi(r, \theta)}{c^2} \right) = \int_0^\infty \int_0^\pi \frac{\rho(r', \theta')}{\rho_0} \frac{r'^2 \sin \theta' dr' d\theta'}{r_0^2}$$
$$\times \frac{2}{\sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')}}$$
$$\times \left[K(m) + \frac{(r^2 - r'^2) E(m)}{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \right], \tag{3.21}$$

where $E(m)$ is the elliptic integral of the second kind. Assuming vertical symmetry and putting $\theta = \pi/2$ (with $\partial \phi / \partial \theta = 0$ at $\theta = \pi/2$), Eq. (3.18) can be simplified for calculating the rotation velocity $\beta(r, \pi/2)$ along the equatorial plane

$$\left(\beta + r \frac{\partial \beta}{\partial r} \right) r \frac{\partial}{\partial r} \left(\frac{\phi(r, \pi/2)}{c^2} \right)$$
$$= \beta \left[\left(\beta - r \frac{\partial \beta}{\partial r} \right) \beta + 4\pi \frac{\rho(r, \pi/2)}{\rho_0} \frac{r^2}{r_0^2} \left(1 - \beta^2 \right) \right], \tag{3.22}$$

where

$$r \frac{\partial}{\partial r} \left(\frac{\phi(r, \pi/2)}{c^2} \right) = \int_0^\infty \int_0^\pi \frac{\rho(r', \theta')}{\rho_0} \frac{r'^2 \sin \theta' dr' d\theta'}{r_0^2}$$
$$\times \frac{2}{\sqrt{r^2 + r'^2 - 2rr' \sin \theta'}} \left[K(m) + \frac{(r^2 - r'^2) E(m)}{r^2 + r'^2 - 2rr' \sin \theta'} \right], \tag{3.23}$$

and

$$m(r, r'; \theta') = -\frac{4rr' \sin \theta'}{r^2 + r'^2 - 2rr' \sin \theta'}. \tag{3.24}$$

Defining the two functions of the mass density and its integral

$$\begin{cases} f(r) = \dfrac{4\pi r^2}{r_0^2} \dfrac{\rho(r, \pi/2)}{\rho_0} \\ g(r) = r \dfrac{\partial}{\partial r} \left(\dfrac{\phi(r, \pi/2)}{c^2} \right) \end{cases} \tag{3.25}$$

the equation for $\beta^2(r, \pi/2)$ can be identified as an Abel equation of the second kind

$$\left[g(r) + \beta^2 \right] r \frac{d\beta^2}{dr} = 2 \left[f(r) - g(r) \right] \beta^2 + 2 \left(1 - f(r) \right) \beta^4, \tag{3.26}$$
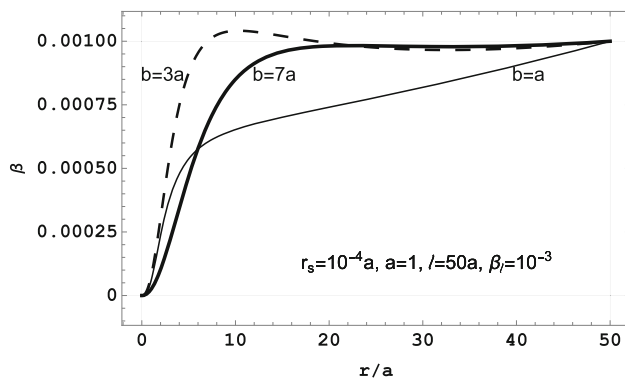
Fig. 4 Self consistent equilibrium of an hypothetical dust distribution described by the Miyamoto-Nagai potential-density pair

which is identical to the equation previously derived in cylindrical coordinates [6], taking into account that the radial direction r along the equatorial plane is the same.

Abel equation (3.26) for $\beta(r)$ can be solved numerically introducing a potential-density pair which is a solution of Poisson's equation $\nabla^2 \phi = 4\pi G \rho$. For example, using the simple Miyamoto-Nagai pair (2.11), the functions $f(r)$ and $g(r)$ are given by

$$\begin{cases} f(r) = \dfrac{r_s r^2 \left[(a+b)^2 (a+3b) + ar^2 \right]}{2b \left[(a+b)^2 + r^2 \right]^{5/2}} \\ g(r) = \dfrac{r_s r^2}{2 \left[(a+b)^2 + r^2 \right]^{3/2}} \end{cases} \tag{3.27}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius. The inward directed integration must be initiated with a rotation velocity value β_ℓ at a distant point ℓ, since the origin $r = 0$ is a singular point. Figure 4 shows the self-consistent equilibrium for an hypothetical dust distribution with normalized values $r_s/a = 10^{-4}$, $a = 1$ and $\beta_\ell = 10^{-3}$ at the distant point $\ell/a = 50$. The normalizing distance $a = 1$ can assume any convenient unit (in astronomical studies it usually takes the value 1 kpc). Three equilibrium solutions are displayed for $b/a = 1$, 3 and 7, showing that the nonlinear rotation velocity profiles vary in a somewhat unexpected way by the change of a single parameter b, but display the usually observed behavior of the galactic rotation curves. The Miyamoto-Nagai model has only two parameters, a and b, insufficient to represent a thin disk galactic distribution, which may include a long disk and localized ring currents. This solution is limited, in general, to compact, massive spheroidal objects. Furthermore, the mass density in this model extends to infinity, although with finite total mass. Unfortunately, one considerable difficulty is to obtain a free boundary solution (or reasonable approximation) for the galactic equilibrium. Nevertheless, since the Poisson equation is linear, the gravitational potential can be represented by

a superposition of mass density distributions, but the rotation velocity must be determined by a solution of the nonlinear self-consistent equilibrium in the presence of the mass currents.

The Abel equation of the second kind (Eq. 3.26) controls the rotation velocity in the equatorial plane. The coefficients in this equation depend on a localized function $f(r)$, which is proportional to the mass density $\rho(r)$, and on a cumulative function $g(r)$, which is given by the radial rate of change of the integrated mass density. These terms compete in a complicated way, making difficult an analysis of the transition between the asymptotic states near the singular origin and in the far away regions. One may conjecture that in the initial stages of formation of a galaxy the transition between asymptotic states is soft, but this could change rapidly at some point in the galactic evolution. An avalanche process may occur resulting in a hard transition. This is a phenomenon in which a small perturbation may eventually introduce a hard transition in an otherwise soft transition between stationary states. However, these assumptions must be tested with a study of the evolutionary and mass accretion processes in a galaxy, requiring much further work.

4 Comments and conclusions

Larmor theorem was implemented to describe the rotational flow of a dust distribution in gravitational confinement. The theorem was used to show the equivalence between rotation in a non-inertial frame and the gravitomagnetic Lorentz acceleration in a stationary frame. The physical results are obviously independent of the adopted frame, but the choice of a convenient frame can simplify the solution of a given problem, or alternatively lead to erroneous results depending on the simplifications introduced. This was exemplified by analyzing the galactic rotation curve problem. It was shown that, depending on the initial simplifications, either a driven or a self-consistent equilibrium solution can be found. The driven equilibrium is maintained by a balance between the gravitational potential and the centrifugal acceleration, which is produced by the differential rotation of the flow. According to Larmor's theorem, a gravitomagnetic field can be associated to the angular rotation vector. This leads to a Lorentz-like force term in the momentum balance equation. The driven equilibrium solution corresponds to the classical circular velocity model. However, an external source and weak dissipative processes are required to maintain the flow, which is not self-consistent. The external source required for gravitomagnetic helicity injection, through the boundaries of the rotating dust configuration, leads to the dark matter misconceptions.

Alternatively, the introduction of convective flow as the driving mechanism, together with the Lorentz acceleration,

leads to a self-consistent equilibrium solution, which can be formulated in any convenient frame. The Larmor equivalent gravitomagnetic field is linked to the gravitational potential by the momentum balance equation, which now includes Lorentz-like and convective terms. External sources are not needed to produce the gravitomagnetic field, which is maintained, according to Ampère's law, by the internal currents. On the other hand, the gravitational potential is produced by the mass distribution, according to Poisson's equation. The self-consistent link between the gravitational potential and the gravitomagnetic flux function leads to a nonlinear equation for the flow velocity in terms of both the mass density and the integrated mass density of the system. The rotation velocity in the equatorial plane is governed by a nonlinear Abel equation of the second kind, which is satisfactory in reproducing the astrophysical observations without introducing black matter [6].

In conclusion, the circular velocity solution (classical solution leading to the erroneous dark matter concept) corresponds to a driven equilibrium of the rotating dust configuration, when the convective forces are neglected. This solution is not self-consistent, depending on the application of external sources (presumably dark matter). In the alternative gravitoelectromagnetic approach the flowing dust equilibrium solution, which includes all the internal mass currents effects, is self-consistent. This corresponds to the self-consistent Larmor rotation of galactic dust. The difference between the two solutions presented in the paper results from the initial assumptions about the local mass currents, after subtracting the predominant rotation velocity.

The concepts introduced in this paper can be extended to other configurations. The transformation to a non-inertial frame may be advantageous in the study of perturbations about an equilibrium, like possible Rossby waves in a spiral galaxy. Furthermore, the velocity-dispersion in clusters can be analyzed taking into account the full contribution of the mean internal currents, including both the mean toroidal and poloidal mass currents. The toroidal component dominates the equilibrium in a galaxy, but the velocity-dispersion in clusters of galaxies is tentatively dominated by the poloidal component. Since the toroidal mass current cannot be neglected, due to the link with the gravitational potential in a spherical mass density distribution, the solution of the nonlinear three-dimensional motion is a complex problem. Possible equilibrium solutions, without recourse to dark matter, constitute work in progress.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors' comment: This is a theoretical paper for which no data has been generated.]

Appendix A Fluid motion in a rotating frame

The change in the position vector $\boldsymbol{r}$ for an infinitesimal rotation by an angle $d\theta$ about the direction $\boldsymbol{n}$ is

$$d\boldsymbol{r} = -\boldsymbol{r} \times \boldsymbol{n} \, d\theta, \tag{A1}$$

so that

$$\frac{d\boldsymbol{r}}{d\theta} = -\boldsymbol{r} \times \boldsymbol{n} = \boldsymbol{n} \times \boldsymbol{r}. \tag{A2}$$

Similarly

$$\frac{d\boldsymbol{r}}{dt} = -\boldsymbol{r} \times \boldsymbol{n} \frac{d\theta}{dt} = \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{A3}$$

This corresponds to an active transformation, giving the actual rotation of a vector, independent of the coordinate system. On the other hand, a passive transformation changes the coordinate system in which vector motion is described. In order for the vector to remain unchanged by the passive transformation, the coordinates of the vector must transform according to the inverse of the active transformation. In this way, the velocity vector $\boldsymbol{u}$ in a stationary frame of reference is related to the relative velocity $\boldsymbol{v}$ in a rotating frame of reference by

$$\boldsymbol{u} = \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{r}. \tag{A4}$$

For an arbitrary vector $\boldsymbol{x}$ in fluid motion:

$$\frac{d\boldsymbol{x}}{dt} = \left(\frac{d\boldsymbol{x}}{dt}\right)' + \boldsymbol{\Omega} \times \boldsymbol{x}, \tag{A5}$$

where $d/dt \equiv \partial/\partial t + \boldsymbol{v} \cdot \boldsymbol{\nabla}$ denotes the convective (hydrodynamic) derivative and $(d\boldsymbol{x}/dt)'$ refers to the rotating frame of reference. Accordingly, the differentiation of $\boldsymbol{u}$ gives

$$\frac{d\boldsymbol{u}}{dt} = \left(\frac{d\boldsymbol{u}}{dt}\right)' + \boldsymbol{\Omega} \times \boldsymbol{u}. \tag{A6}$$

Replacing u by $v + \mathbf{\Omega} \times r$ in the right-hand side leads to

$$
\begin{aligned}
\frac{du}{dt} &= \left(\frac{dv}{dt}\right)' + \left(\frac{d\mathbf{\Omega}}{dt}\right)' \times r + \mathbf{\Omega} \times \left(\frac{dr}{dt}\right)' \\
&\quad + \mathbf{\Omega} \times (v + \mathbf{\Omega} \times r) \\
&= \left(\frac{\partial v}{\partial t} + v \cdot \nabla v\right)' + \left(\frac{\partial \mathbf{\Omega}}{\partial t} + v \cdot \nabla \mathbf{\Omega}\right)' \times r \\
&\quad + \mathbf{\Omega} \times v + \mathbf{\Omega} \times (v + \mathbf{\Omega} \times r) \\
&= \left(\frac{dv}{dt}\right)' + \left(\frac{d\mathbf{\Omega}}{dt}\right)' \times r + 2\mathbf{\Omega} \times v + \mathbf{\Omega} \times (\mathbf{\Omega} \times r).
\end{aligned}
$$
(A7)

Now the equation of motion for an ideal fluid in a stationary frame in the gravitoelectromagnetic context is [10]

$$
\frac{d(\alpha u)}{dt} = \frac{\partial (\alpha u)}{\partial t} + (u \cdot \nabla)\,\alpha u = -\frac{\nabla p}{\rho} + E + u \times B, \quad \text{(A8)}
$$

where α is the relativistic thermo-inertial factor

$$
\alpha = \gamma \left(1 + \frac{\gamma_A}{\gamma_A - 1} \frac{\gamma k_B T}{c^2}\right). \tag{A9}
$$

Here $\gamma = 1/\sqrt{1 - u^2/c^2}$ is the relativistic Lorentz factor, γ_A is the adiabatic coefficient, T is the fluid temperature, $p = \rho k_B T/m$ is the fluid pressure, and ρ is the mass density of the fluid. The high-order relativistic contributions in the fluid motion inertia can be presently neglected taking $\alpha \cong 1$, since α introduces corrections of the order u^4/c^4 and higher in the convective term (these terms cannot by ignored in the calculation, for example, of the anomalous precession of the perihelion [14]). In the rotating frame of reference the equation of motion becomes (dropping the prime symbol)

$$
\begin{aligned}
\frac{dv}{dt} &+ \frac{d\mathbf{\Omega}}{dt} \times r + 2\mathbf{\Omega} \times v + \mathbf{\Omega} \times (\mathbf{\Omega} \times r) \\
&= -\frac{\nabla p}{\rho} + E + v \times B.
\end{aligned}
$$
(A10)

The double cross product in the left-hand side can be written as

$$
\mathbf{\Omega} \times (\mathbf{\Omega} \times r) = -\nabla \left[\frac{1}{2} (\mathbf{\Omega} \times r)^2\right] + [r \times (\mathbf{\Omega} \times r)] \cdot \nabla \mathbf{\Omega}.
$$
(A11)

Introducing the equivalent centrifugal potential $\phi_c = (\mathbf{\Omega} \times r)^2 /2$ and the equivalent Larmor gravitomagnetic field $B_L = -\mathbf{\Omega}$, the equation of motion in the rotating frame takes the final form

$$
\begin{aligned}
\frac{dv}{dt} &+ \frac{d\mathbf{\Omega}}{dt} \times r + [r \times (\mathbf{\Omega} \times r)] \cdot \nabla \mathbf{\Omega} \\
&= -\frac{\nabla p}{\rho} + E + \nabla \phi_c + v \times (B - 2B_L),
\end{aligned}
$$
(A12)

where the Coriolis acceleration $2\mathbf{\Omega} \times v = -2B_L \times v$ has been combined with the gravitomagnetic field in the Lorentz force term $v \times B$. If there are no local mass currents in the rotating frame (relative motion $v = 0$), the hydrostatic equilibrium in the rotating frame is described by the differential equation

$$
[r \times (\mathbf{\Omega} \times r)] \cdot \nabla \mathbf{\Omega} = -\frac{\nabla p}{\rho} - \nabla \phi + \nabla \phi_c, \tag{A13}
$$

which is equivalent to

$$
f = -\frac{\nabla p}{\rho} - \nabla \phi - \mathbf{\Omega} \times \underbrace{(\mathbf{\Omega} \times r)}_{u} = 0, \tag{A14}
$$

where ϕ is the gravitoelectric (Newtonian) potential, and f is the specific force density in the rotating frame of reference.

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