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ANALYTIC SOLUTION OF THE RELATIVISTIC COULOMB PROBLEM FOR A SPINLESS SALPETER EQUATION

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ABSTRACT

We construct an analytic solution to the spinless S-wave Salpeter equation for two quarks interacting via a Coulomb potential,

$$\left[2\sqrt{-\nabla^2+m^2}-M-\frac{\alpha}{r}\right]\psi(r)=0 \quad ,$$

by transforming the momentum space form of the equation into a mapping or boundary value problem for analytic functions. The principle part of the three dimensional wave function is identical to the solution of a one-dimensional Salpeter equation found by one of us and discussed here. The remainder of the wave function can be constructed by the iterative solution of an inhomogeneous singular integral equation. We show that the exact bound state eigenvalues for the Coulomb problem are

$$M_n = 2m\Big/\sqrt{1+\frac{\alpha^2}{4n^2}}\,, \quad n = 1,2,\ldots,$$

and that the wave function for the static interaction diverges for $r \to 0$ as $C(mr)^{-\nu}$ where $\nu = \frac{\alpha}{\pi}[1+\frac{\alpha}{\pi}+\ldots]$ is known exactly.

I. INTRODUCTION

In this paper, we present an analytic solution for the S-state wave functions of the spinless Salpeter-type equation for a static Coulomb potential,

$$\left[2\sqrt{-\nabla^2+m^2}-M-\frac{\alpha}{r}\right]\psi(r)=0. \tag{1}$$

This equation appears as a natural approximation to the relativistic Bethe-Salpeter-Schwinger equation[1] for two fermions of mass m and total energy M when the interaction kernel is approximated by the instantaneous Coulomb interaction, and spin-dependent effects and the coupling of the "large-large" and "small-small" components of the wave function are neglected. The solution--and the method used to construct it--should therefore be of fairly general interest. We find, for example, that we can determine the exact bound state eigenvalues of Eq. (1) without actually solving the equation. The result,

$$M_n = \frac{2m}{\sqrt{1+\frac{\alpha^2}{4n^2}}}\quad , \quad n = 1,2,\ldots, \tag{2}$$

can probably be generalized to orbital angular momenta $\ell > 0$, and the corresponding result may also be accessible in the spin-dependent problem.

We originally encountered Eq. (1) (with $\alpha = \frac{4}{3}\alpha_s$) in our study of short-range effects in bound quark-antiquark systems.[2] In that work (and later extensions[3]), it was important to know how $\psi(r)$ behaves for $mr \gtrsim 1$. By matching this (free) relativistic Coulomb function to the solution of Eq. (1) with an extra long range confining interaction and using the known short range gluonic radiative corrections to $\psi(r)$, we could estimate

$$\frac{dX}{dp}(p) + iB(p)\,X(p) = 0 \tag{7}$$

where

$$B(p) = \alpha/[2\sqrt{p^2+m^2} - M\,]. \tag{8}$$

The solution of Eq. (7) is straightforward,

$$\chi(p) = A\,\exp\left[-i\int^{p} B(p')dp'\right] \quad , \tag{9}$$

and

$$\tilde{\psi}(p) = -iB(p)\chi(p) = \frac{dX}{dp}(p)\ . \tag{10}$$

Explicit evaluation of the integral in Eq. (9) gives the rather complicated expression

$$\chi(p) = A\left[\frac{p}{\sqrt{p^2+m^2}+m} - \sqrt{\frac{M-2m}{M+2m}}\right]^{-i\eta}\left[\frac{p}{\sqrt{p^2+m^2}+m} + \sqrt{\frac{M-2m}{M+2m}}\right]^{i\eta}$$

$$\times\left[\frac{\sqrt{p^2+m^2}+m-p}{\sqrt{p^2+m^2}+m+p}\right]^{i\alpha/2}\ , \tag{11}$$

where $\eta = \alpha/2v$ and v is the velocity of a free quark with total energy $M/2$ and momentum p_o,

$$v = \sqrt{1-\frac{4m^2}{M^2}}\ , \quad p_o = Mv/2\ . \tag{12}$$

We can obtain equivalent results using the customary momentum-space form of the relativistic wave equation,

$$[2\sqrt{p^2+m^2} - M]\,\tilde{\psi}(p) - \frac{i\alpha}{2}\int_{-\infty}^{\infty} dk\ \varepsilon(p-k)\tilde{\psi}(k) = 0. \tag{13}$$

If we replace $\tilde{\psi}$ by $d\chi/dp$ as in Eq. (10), a partial integration reduces this integral equation to the differential equation for χ given in Eq. (7). This approach will be useful in the three-dimensional problem.

It is easily shown that the function $\chi(p)$ has branch points at $p = \pm p_o$ and at $\pm im$ but no other singularities in the (finite) p plane. We choose the branch cuts as shown in Fig. 1. To obtain a space wave function $\psi(x)$ which vanishes for $x = 0$, we choose the integration contour in Eq. (4) as the difference between contours which run from $-\infty$ to ∞ just above and below the real axis, that is, use the contour C_1 in Fig. 1. After a partial integration, we can collapse the contour and express $\psi(x)$ as a simple Fourier transform of $\chi(p)$,

$$\psi(x) = xA\int_{-p_o}^{p_o} dp\ e^{ipx}\left[\sqrt{\frac{M-2m}{M+2m}} - \frac{p}{\sqrt{p^2+m^2}+m}\right]^{-i\eta}$$

$$\times\left[\sqrt{\frac{M-2m}{M+2m}} + \frac{p}{\sqrt{p^2+m^2}+m}\right]^{i\eta}\left[\frac{\sqrt{p^2+m^2}+m-p}{\sqrt{p^2+m^2}+m+p}\right]^{i\alpha/2}\ , \tag{14}$$

where we have absorbed various constants into the overall normalization constant A. This expression reduces in the nonrelativistic limit ($m \to \infty$, p_o fixed) to a standard representation of the S-state Coulomb wave function,

$$\psi(x) = p_o x\ A\int_{-1}^{1} dt\ e^{ip_o xt}(1-t)^{-i\eta}(1+t)^{i\eta}\ , \quad t = p/p_o\ . \tag{15}$$

C. Evaluation of $\psi'(0)$

As we will show in Sec. III, the magnitude of the S-state Coulomb wave function near the origin in three dimensions is determined by the value of $\psi'(0)$, or equivalently, by the limiting value of $\psi(x)/x$ for $x \to 0$. (If we were dealing with an ordinary nonrelativistic Schrödinger equation, $\psi(x)$ could be identified with $u(r) = rR(r)$, and $\psi'(0) = R(0)$.) The value of $\psi'(0)$ is determined by Eq. (20). After dividing by x, we can set x equal to zero on the right hand side of this equation. A change of the variable of integration from p to

$$t = \frac{p}{\sqrt{p^2+m^2} + m} \tag{21}$$

then gives the expression

$$\psi'(0) = ma\frac{e^{\pi\eta/2}}{|\Gamma(1-i\eta)|}\int_{-1}^{1} dt \left(\frac{1+t}{1-t}\right)^{i\eta} \left(\frac{1-at}{1+at}\right)^{i\alpha/2} \frac{1+a^2t^2}{(1-a^2t^2)^2} \quad , \tag{22}$$

where

$$a = \sqrt{\frac{M-2m}{M+2m}} = \frac{2p_o}{M+2m} \ . \tag{23}$$

For particles which are not too relativistic, the parameter a is small, $a \sim p_o/2m \sim v/2$, and we can evaluate the integral in Eq. (22) approximately by expanding the last two factors in a power series in at,

$$\left(\frac{1-at}{1+at}\right)^{i\alpha/2} \frac{1+a^2t^2}{(1-a^2t^2)^2} = 1 - i\alpha a t + 3a^2t^2 + \dots \tag{24}$$

$$= 1 - \frac{1}{2} i\alpha a[(1+t)-(1-t)] + \frac{3}{2} a^2[(1+t)^2 + (1-t)^2-2]+\dots \ .$$

The integrals which appear can then be reduced to beta functions,

$$\int_{-1}^{1} dt(1+t)^x(1-t)^y = 2^{x+y+1}\frac{\Gamma(x+1)\Gamma(y+1)}{\Gamma(x+y+2)} \ , \tag{25}$$

and we find that

$$\psi'(0) = p_o e^{\pi\eta/2}|\Gamma(1-i\eta)| \frac{4m}{M+2m} [1 + 2\alpha\eta a + 2a^2(1-2\eta^2) + \dots] \tag{26}$$

$$= p_o e^{\pi\eta/2} |\Gamma(1-i\eta)| [1 + \frac{1}{4}\alpha^2 + \frac{1}{4}v^2 + O(\alpha^4,\alpha^2v^2,v^4)].$$

The leading factor is just the usual Coulomb factor with η calculated using the relativistic velocity of the quark,[5]

$$p_o e^{\pi\eta/2} |\Gamma(1-i\eta)| = p_o\left[\frac{2\pi\eta}{1-\exp(-2\pi\eta)}\right]^{1/2} , \quad \eta = \alpha/2v. \tag{27}$$

We have not found a simple expression for $\psi'(0)$ in the extreme relativistic case.

D. Comments on the one-dimensional and three-dimensional problems

The expression in Eq. (20) gives the exact solution to the one-dimensional equation

$$\left[2\sqrt{-\frac{d^2}{dx^2} + m^2} - M - \frac{\alpha}{x}\right]\psi(x) = 0 \tag{28}$$

which vanishes at $x = 0$. This solution reduces in the nonrelativistic limit to the $\ell=0$ solution of the radial Schrödinger equation for an attractive Coulomb potential, $\psi(x) \to u(r)$, $r = x > 0$, $E = M-2m$, differ slightly for $r \lesssim m^{-1}$ even though the solutions have the same non-relativistic limits and are essentially identical for $r >> m^{-1}$.

III. THE THREE-DIMENSIONAL COULOMB PROBLEM

A. The boundary value form of the problem

The three-dimensional Coulomb problem is defined by the integral equation given above,

$$[2\sqrt{p^2+m^2} - M]\phi(p) + \frac{\alpha}{2\pi}\int_{-\infty}^{\infty} dk\ \ln[(p-k)^2 + \varepsilon^2]\ \phi(k) = 0\ ,$$
$$\phi(p) = -\phi(-p), \qquad \varepsilon \to 0+\ . \tag{38}$$

To transform this equation into a more useful form, we make a redefinition suggested by our results in one dimension,

$$\phi(p) = \frac{d\chi}{dp}(p), \qquad \chi(-p) = \chi(p)\ , \tag{39}$$

and integrate once by parts. The result is the singular integral equation[6]

$$[2\sqrt{p^2+m^2} - M]\frac{d\chi}{dp} + \frac{\alpha}{\pi}\ P\int_{-\infty}^{\infty} dk\ \frac{\chi(k)}{k-p} = 0 \tag{40}$$

where P designates the principal value integral.

This integral equation can be transformed into a boundary value problem as follows. We represent $\chi(p)$, p real, as the difference of the boundary values of two functions $\Phi^+(p)$ and $\Phi^-(p)$ which are analytic respectively in the upper and lower halves of the complex p-plane and vanish at infinity,[7]

$$\chi(p) = \Phi^+(p+i\varepsilon) - \Phi^-(p-i\varepsilon)\ . \tag{41}$$

Use of the Plemelj relations for the principle value integral[8] (or direct calculation) then shows that

$$\frac{P}{\pi}\int_{-\infty}^{\infty} dk\ \frac{1}{k-p}\ [\Phi^+(k+i\varepsilon)-\Phi^-(k-i\varepsilon)] = i[\Phi^+(p+i\varepsilon)+\Phi^-(p-i\varepsilon)], \tag{42}$$

and we can rewrite Eq. (40) as

$$[\frac{d}{dp} + i\ B(p)]\Phi^-(p-i\varepsilon) = [\frac{d}{dp} - iB(p)]\Phi^+(p+i\varepsilon)\ . \tag{43}$$

Here B(p) is the function defined in Eq. (8),

$$B(p) = \alpha/[2\sqrt{p^2+m^2} - M]\ . \tag{44}$$

This function is symmetric in p, $B(-p) = B(p)$, and has poles at $p = \pm p_o$ (Eq. (12)) and branch points at $\pm im$. We will assume initially that $M > 2m$ so that p_o is real.

The radial wave function u(r) can be expressed in terms of the Φ's using Eqs. (34), (39), and (41). We first use the antisymmetry of $\phi(p) = d\chi/dp$ and Eq. (41) to rewrite Eq. (34) (with the factor $(2\pi)^{-2}$ absorbed in χ) as

$$u(r) = \int_{-\infty}^{\infty} dp\ \sin pr\ \frac{d\chi}{dp}(p)$$
$$= -\tfrac{1}{2}r\int_{-\infty}^{\infty} dp\ [e^{ipr} + e^{-ipr}][\Phi^+(p+i\varepsilon) - \Phi^-(p-i\varepsilon)]\ . \tag{45}$$

We then observe that the integrals which involve $e^{ipr}\ \Phi^+$ and $e^{-ipr}\ \Phi^-$ vanish for $r > 0$ (the functions are analytic in the upper and lower half planes respectively, and the integration contours can be pushed to $\pm i\infty$ where the integrands vanish exponentially). Thus

We therefore conclude that

$$\Phi^-(p) \propto (p_o \mp p)^{\mp i\eta}, \quad \Phi^+(p) \propto (p_o \mp p)^{\pm i\eta}, \quad p \to \pm p_o, \tag{52}$$

hence that the functions $\Phi^-(\Phi^+)$ have the same branch points at $\pm p_o$ as the function $\chi(p)$ encountered in the one-dimensional problem for an attractive (repulsive) Coulomb interaction. We will choose the cut in $\Phi^-(\Phi^+)$ to connect the branch points as shown in Fig. 3. We can then continue $\Phi^-(\Phi^+)$ around the cut into the upper (lower) half plane. We remark also that the expressions in Eq. (52) have the symmetry $\Phi^+(-p) = -\Phi^-(p)$, Eq. (49), for complex p if the constants of proportionality are chosen to be equal in magnitude and opposite in sign.

We next consider the continuation of Eq. (43) into the upper half plane starting on the real axis with $p > p_o$, that is, to the right of the cuts in $\Phi^{\pm}(p)$. The two sides of the equation are independently analytic near the real axis for $p > p_o$, so are equal as analytic functions and may be continued together. Since Φ^+ is analytic in the upper half plane by construction, the only singularity of the right hand side of Eq. (43) for Im p> 0 is the branch point of B(p) at p = im. The left hand side therefore has only this singularity, and we conclude that the only singularity of Φ^- in the upper half plane is a branch point at p = im. (Φ^- must have this branch point for a nontrivial solution to exist since $\Phi^-(p) \neq \Phi^+(p)$, see, e.g., Eq. (52).) We choose the branch cut in Φ^- to run from im to i∞ as shown in Fig. 3. The symmetry relation in Eq. (49) then implies that Φ^+ has a branch point at p =-im with a cut which runs from -im to -i∞ .

It will be convenient at this point to write Φ^- as a sum of two functions,

$$\Phi^-(p) = \Phi_1(p) + \Phi_2(p) \tag{53}$$

where $\Phi_1(p)$ has only the "short" cut on the real axis from $-p_o$ to p_o, and $\Phi_2(p)$ has only the "long" cut from im to i∞. Since $\Phi^-(p)$ vanishes at infinity, we can express the separate functions in terms of their discontinuities across the cuts using the Cauchy integral formula,

$$\begin{aligned}\Phi_1(p) &= \frac{1}{2\pi i}\int_{C_1} \frac{dp'}{p'-p}\,\Phi_1(p') \\ &= \frac{1}{2\pi i}\int_{-p_o}^{p_o} \frac{dp'}{p'-p}\,\text{disc}\,\Phi_1(p')\ ,\end{aligned} \tag{54}$$

$$\begin{aligned}\Phi_2(p) &= \frac{1}{2\pi i}\int_{C_2} \frac{dp'}{p'-p}\,\Phi_2(p') \\ &= \frac{1}{2\pi i}\int_{im}^{i\infty} \frac{dp'}{p'-p}\,\text{disc}\,\Phi_2(p')\ .\end{aligned} \tag{55}$$

In these expressions the point p lies outside the integration contours C_1 and C_2 shown in Fig. 3, and the discontinuities are defined by

$$\begin{aligned}\text{disc}\,\Phi_1(p) &= \Phi_1(p+i\epsilon) - \Phi_1(p-i\epsilon), \quad -p_o < p < p_o, \quad \epsilon \to 0+\ , \\ \text{disc}\,\Phi_2(p) &= \Phi_2(p-\epsilon) - \Phi_2(p+\epsilon), \quad m < -ip < \infty, \quad \epsilon \to 0+\ .\end{aligned} \tag{56}$$

The function disc $\Phi_1(p)$ is determined as follows. We note that the result obtained by continuing Eq. (43) from the region $p > p_o$ (where both sides are analytic) to the upper edge of the short cut must be consistent with the original equation for $-p_o < p < p_o$. Since Φ^- is given on the

$$p_o = i|p_o| = i\sqrt{m^2 - \frac{1}{4}M^2} \quad , \tag{65}$$

and the contour C_1 in Eq. (61) surrounds the segment $(-i|p_o|, i|p_o|)$ of the imaginary axis. The upper part of the contour gives an exponentially decreasing contribution to $u_1(r)$; the lower part of the contour gives an exponentially increasing contribution. To obtain an acceptable (normalizable) bound state wave function, we must be able to eliminate the lower part of the contour. This requires that there be no branch point or pole at $p = -i|p_o|$, hence from Eq. (64), that $i\eta$ be a positive integer, $i\eta = 1,2,...$ (the wave function vanishes for $i\eta = 0$). Using the relation $i\eta = \alpha M/4|p_o|$, we find that the exact bound state energies for $\ell = 0$ (S-states) are

$$M_n = 2m\Big/\sqrt{1 + \frac{\alpha^2}{4n^2}}$$
$$= 2m - \frac{\alpha^2 m}{4n^2} + \frac{3}{64}\frac{\alpha^4 m}{n^4} + \dots , \qquad n=1,2,\dots . \tag{66}$$

The α^4 term agrees with the correction to the Schrödinger energy $E_n = M_n - 2m$ obtained by expanding the square root in Eq. (30) to order ∇^4/m^3, and treating that term as a perturbation.

D. Integral equation for disc Φ_2

We showed in Secs. IIIA and IIIB that the solution of the three-dimensional relativistic Coulomb problem can be expressed in terms of a function $\Phi^-(p) = \Phi_1(p) + \Phi_2(p)$ which is analytic in the entire complex p

plane except for cuts from $-p_o$ to $+p_o$ (Φ_1) and from im to $i\infty$ (Φ_2). We found, furthermore, that disc Φ_1 (the discontinuity of Φ^- or Φ_1 across the "short" cut) satisfied the same differential equation as was encountered in the one-dimensional problem, and was given explicitly by disc $\Phi_1(p) = -\chi(p)$ where $\chi(p)$ is defined in Eqs. (11) and (19). In the present section, we will derive an integral equation for disc $\Phi_2(p)$. This equation relates disc Φ_2 to the known function disc Φ_1, and can be solved by iteration. The solution will be discussed in the following sections.

We begin by deriving an equation for $\Phi^-(p)$ which displays the analytic structure of that function. To do this, we note that the condition on disc Φ_1 in Eq. (58) allows us to write the boundary value equation, Eq. (43), as an analytic expression in the cut p plane,

$$\left[\frac{d}{dp} + iB(p)\right] \Phi^-(p) = \left[\frac{d}{dp} - iB(p)\right] \Phi^+(p) \, . \tag{67}$$

We multiply this equation by the Cauchy denominator and integrate on the contour C_2 shown in Fig. 1 to obtain a identity valid for general complex p,

$$\frac{1}{2\pi i}\int_{C_2} \frac{dp'}{p'-p}\left[\frac{d}{dp'} + iB(p')\right] \Phi^-(p') = \frac{1}{2\pi i}\int_{C_2} \frac{dp'}{p'-p}\left[\frac{d}{dp'} - iB(p')\right] \Phi^+(p') \, . \tag{68}$$

The integral of the right hand side is simple to evaluate: Φ^+ is analytic in the upper half plane, while B(p) has a cut from im to $i\infty$. Thus

$$\frac{1}{2\pi i}\int_{C_2} \frac{dp'}{p'-p}\left[\frac{d}{dp'} - iB(p')\right] \Phi^+(p')$$
$$= -\frac{1}{2\pi}\int_{im}^{i\infty} \frac{dp'}{p'-p}\, \Phi^+(p')\, \text{disc}\, B(p') \tag{69}$$

term" which involves the known function Φ_1. In the next section we will study the iterative solution of this equation.

E. Iterative solution for disc Φ_2

It will be convenient to make the change of variables $|p| = mx$, $1 \leq x < \infty$, in Eq. (76), and to write discΦ_2 as

$$\text{disc}\ \Phi_2(i|p|) = e^{\zeta(x)}\ \Psi(x) \quad , \tag{77}$$

$$\zeta(x) = \frac{\alpha}{2v} \tan^{-1} \frac{x_o}{x}, \quad x_o = p_o/m \quad . \tag{78}$$

This substitution eliminates the second term in Eq. (76), and leaves us with the equation

$$\frac{d\Psi}{dx}(x) - \frac{\alpha}{\pi} \frac{x\sqrt{x^2-1}}{x^2+x_o^2} e^{-\zeta(x)}\ P \int_1^\infty \frac{dx'}{x'^2-x^2}\ e^{\zeta(x')}\ \Psi(x')$$
$$= i\alpha \frac{\sqrt{x^2-1}}{x^2+x_o^2} e^{-\zeta(x)} [\Phi_1(imx) - \Phi_1(-imx)] \ . \tag{79}$$

This equation can be solved by iteration. We let

$$\frac{d\Psi^{(o)}}{dx} = i\alpha \frac{\sqrt{x^2-1}}{x^2+x_o^2}\ e^{-\zeta(x)}\ [\Phi_1(imx) - \Phi_1(-imx)] \ , \tag{80}$$

and define functions $\Psi_2^{(n)}$ recursively by

$$\frac{d\Psi^{(n)}}{dx} = \frac{\alpha}{\pi} \frac{x\sqrt{x^2-1}}{x^2+x_o^2} e^{-\zeta(x)}\ P \int_1^\infty \frac{dx'}{x'^2-x^2}\ e^{\zeta(x')}\ \Psi^{(n-1)}(x') \ , \tag{81}$$

$$\Psi_2^{(n)}(x) = - \int_x^\infty \frac{d\Psi^{(n)}}{dx'}\ dx' \quad . \tag{82}$$

Then

$$\Psi(x) = \sum_{n=0}^\infty \Psi^{(n)}(x) \ . \tag{83}$$

We note that each successive term in Eq. (83) involves an extra overall factor of α/π. However, Eq. (83) is not a simple power series in α because of the α-dependence introduced by the factors $e^{\pm\zeta}$ and by Φ_1. The second relevant parameter in Eq. (79) is x_o^2/x^2. This is of order v^2 for nonrelativistic quarks.

The function $[\Phi_1(imx) - \Phi_1(-imx)]$ is given from Eq. (54) by

$$\Phi_1(imx) - \Phi_1(-imx) = \frac{x}{\pi} \int_{-x_o}^{x_o} \frac{dx'}{x'^2+x^2}\ \text{disc}\ \Phi_1(mx') \ . \tag{84}$$

We will be primarily interested in the behavior of $\Psi(x)$ for $x \to \infty$. In this limit,

$$\Phi_1(imx) - \Phi_1(-imx) \xrightarrow[x>>1]{} \frac{1}{\pi x} \int_{-x_o}^{x_o} dx'\ \text{disc}\ \Phi_1(mx') \ , \tag{85}$$

where the corrections are of relative order x_o^2/x^2. The integral in Eq. (85) can be identified through Eqs. (61) and (64) with $-u_1'(0)/m = -\sqrt{4\pi}\ \psi_1(0)/m$, where

$$\psi(r) = [u_1(r) + u_2(r)]/\sqrt{4\pi}\ r \quad . \tag{86}$$

The same integral was encountered in Sec. IIIC in the evaluation of $\psi'(0)$, the derivative of the one-dimensional Coulomb wave function at the origin. To avoid confusion, we define $C = \sqrt{4\pi}\ \ \psi_1(0)$. Then

$$\Phi_1(imx) - \Phi_1(-imx) \xrightarrow[x>>1]{} - C/\pi mx \tag{87}$$

$$\nu = \nu^2 + \frac{\alpha}{2}\,\nu\cot\frac{\pi\nu}{2} \quad . \tag{96}$$

Expanding the cotangent for $\pi\nu/2$ small, we find that

$$\begin{aligned}\nu &= \frac{\alpha}{\pi} + \nu^2 - \frac{\alpha}{\pi}\,\frac{\pi^2\nu^2}{12} - \frac{\alpha}{\pi}\,\frac{\pi^4\nu^4}{720} + \ldots \\ &= \frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2} + \left(2 - \frac{\pi^2}{12}\right)\frac{\alpha^3}{\pi^3} + O\left(\frac{\alpha^4}{\pi^4}\right) \quad .\end{aligned} \tag{97}$$

We find similarly that

$$A = \frac{\alpha}{\pi}\,\frac{iC}{m}\,[1 + O(\alpha)] \quad , \tag{98}$$

and in a separate calculation in which we return higher order terms in x^{-1}, that the coefficient c_1 in Eq. (93) is

$$c_1 = -\frac{\alpha M}{8m}[1 + O(\alpha)] \; . \tag{99}$$

These results agree to the relevant order with the iterative results given in Eqs. (90) and (91),

$$\Psi(x) \underset{x>>1}{\sim} \frac{\alpha}{\pi}\,\frac{iC}{mx}\left[1 + \frac{\alpha}{\pi}\left(1 + \ln 2 - \frac{\pi}{4}\right) + \frac{\alpha}{\pi}\ln x - \frac{\alpha M}{8mx} + \ldots\right] \quad . \tag{100}$$

We see, in fact, that

$$A = \frac{\alpha}{\pi}\,\frac{iC}{m}\left[1 + \frac{\alpha}{\pi}\left(1 + \ln 2 - \frac{\pi}{4}\right) + O(\alpha^2, \nu^2)\right] \quad . \tag{101}$$

We conclude that the behavior of $\Psi(x)$ for $x \to \infty$ is given by the asymptotic expansion in Eq. (93) with ν, c_1, and A given by Eqs. (97), (99), and (101). The behavior of $\Psi(x)$ for $x \sim 1$ is given by the iterative series in Eq. (83). For α^2, $\nu^2 << 1$, the first two terms in the series, Eqs. (90) and (91), should give a satisfactory approximation to the exact result. The function disc Φ_2 is then given by Eq. (77).

F. Behavior of $\psi(r)$ for $r \to 0, \infty$.

We consider finally the behavior of the space wave function $\psi(r) = [u_1(r) + u_2(r)]/\sqrt{4\pi}\, r$ for $r \to 0$. The behavior of $u_1(r)/r$ is easily determined from Eq. (64),

$$u_1(r)/r = C[1 - \tfrac{1}{2}\alpha mr + \ldots] \tag{102}$$

where C is given in Eq. (88), and we have omitted corrections of orders α^2, ν^2 in the second term.

To determine the behavior of $u_2(r)/r$ for $r \to 0$, we use Eq. (62) and the results just obtained for disc Φ_2 for $|p| \to \infty$,

$$\begin{aligned} u_2(r)/r &= -i\int_m^\infty d|p|\; e^{-|p|r}\,\text{disc}\,\Phi_2(i|p|) \\ &= -im\int_1^\infty dx\; e^{-mrx}\, e^{\zeta(x)}\,\Psi(x) \\ &\underset{mr<<1}{\sim} \frac{\alpha}{\pi}\,C\int_1^\infty dx\; e^{-mrx}\,x^{\nu-1} + O(\alpha^2 C) \\ &= \frac{\alpha}{\pi}\,C\,\Gamma(\nu)(mr)^{-\nu} - \frac{\alpha}{\pi\nu}\,C + O(\alpha mr, \alpha^2)\,C \\ &\cong C[(mr)^{-\nu} - 1] + O(\alpha mr, \alpha/\pi)C \quad . \end{aligned} \tag{103}$$

In the last step, we have used the relation $\nu \cong \alpha/\pi$. Combining Eqs. (102) and (103), we find that

$$\psi(r) \underset{mr<<1}{\sim} C(mr)^{-\nu} + O(mr, \alpha/\pi)C. \tag{104}$$

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FOOTNOTES

1. E. E. Salpeter and H.A. Bethe, Phys. Rev. 84, 1232 (1951); J. Schwinger, Proc. Nat. Acad. Sci. U.S.A. 37, 452; 37, 455 (1951).
2. B. Durand and L. Durand, Phys. Rev. D 25, 2312 (1982); Phys. Lett. 113B, 338 (1982).
3. B. Durand and L. Durand, University of Wisconsin preprint MAD/TH-62, submitted to Phys. Rev. D.
4. This procedure is a generalization of Laplace's method for the solution of ordinary differential equations with coefficients linear in x; see, e.g., L.D. Landau and E.M. Lifschitz, Quantum Mechanics (Pergamon, New York, 1965), Appendix A, or E. Goursat, Differential Equations (Dover, New York, 1945), Sec. 46. BD has also used this

method to solve Eq. (3) for a one-dimensional linear potential (B. Durand, University of Wisconsin preprint MAD/TH-90).
5. See, e.g., M. Abramowitz and I. Stegun, Handbook of Mathematical Functions (Dover, New York, 1972), Sec. 14.1.
6. The general theory of singular integral equations is treated in N.I. Muskhelishvili, Singular Integral Equations (P. Noordhoff N.V., Groningen, Holland, 1953). See also W. Pogorzelski, Integral Equations and Their Applications (Pergamon Press, 1966). Unfortunately Eq. (40) is not of a standard type.
7. This representation is quite general. See Muskhelishvili, Ref. 6, Secs. 26, 27. For an extension to the representation of generalized functions (distributions), see H.J. Bremermann and L. Durand, J. Math. Phys. 2, 240 (1961).
8. Muskhelishvili, Ref. 6, Sec. 17.