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Gravitational collapse of type N spacetime, the naked singularity, and causality violation

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We present an axisymmetric type N gravitational collapse solution of the Einstein field equations. The spacetime is regular everywhere except on the symmetry axis, where it possesses a naked curvature singularity. The energy–momentum tensor is a combination of null dust (or pure radiation field) that has constant energy density and an anisotropic fluid satisfying the energy conditions for collapse on the symmetry axis. The spacetime satisfies neither the strong curvature condition nor the limiting focusing condition. The metric is radially geodesically complete, and gravitational lensing will be discussed. The physical interpretation of this solution, based on the study of the equation of the geodesic deviation, will be presented. It is demonstrated that this solution depends on the energy–momentum tensor terms consisting of a pure radiation field and an anisotropic fluid. Additionally, the spacetime admits closed timelike curves, which appear after a certain instant of time in a causally well behaved manner.

Subject Index E00, E01

1. Introduction

The gravitational collapse of cylindrically symmetric models forming a naked singularity has been discussed in Refs. [1,2]. Apostolatos and Thorne [3] investigated the collapse of a counter-rotating dust shell cylinder and showed that rotation, even if it is infinitesimally small, can halt the gravitational collapse of the cylinder. Echeverria [4] studied the evolution of a cylindrical dust shell analytically at late times and numerically for all times. Guttia et al. [5] studied the collapse of non-rotating infinite dust cylinders. Nakao et al. [6] studied the high-speed collapse of a cylindrically symmetric thick shell composed of dust and a perfect fluid with nonvanishing pressure [7]. Recent work describes the cylindrically symmetric collapse of counter-rotating dust shells [8–10], a self-similar scalar field [11,12], an axially symmetric vacuum solution [13], cylindrically symmetric collapse of an anisotropic fluid [14], and an axially symmetric null dust collapse [15]. Some other examples of nonspherical gravitational collapse are discussed in Refs. [16–25]. In addition, there are a number of examples of spherically symmetric gravitational collapse that form a naked curvature singularity (see, e.g., Refs. [26–32]). To counter the occurrence of naked singularities in a solution of the field equations, R. Penrose proposed a cosmic censorship conjecture [33–35]. According to the weak version, singularities have no effect on distant observers, i.e., they cannot communicate with faraway observers. Up to now, there have been no theorems or proofs (disproofs) that support (counter) this conjecture. In contrast, there is no reason that a naked singularity cannot exist in a solution of the field equations. Attempts have been made to provide the theoretical framework to devise a technique to distinguish between black holes and naked singularities from astrophysical data, mainly through gravitational lensing. The study of strong gravitational lensing in the

Janis–Newman–Winicour spacetime by Virbhadra and collaborators [36,37] and its rotating generalization [38], and notably the work in Refs. [39–42], has made some significant progress. Therefore, the study of naked singularities and spacetime with such objects is of considerable current interest.

In this paper, we construct a nonvacuum solution of the field equations with naked curvature singularity. The presence of naked singularities in a solution of the Einstein field equations may break down the causality condition and may form closed timelike curves (CTCs). The possibilities that a naked curvature singularity gives rise to a cosmic time machine has been discussed by Clarke and de Felice [43] (see also Refs. [44–46]). A cosmic time machine is a spacetime that is asymptotically flat and admits closed nonspacelike curves that extend to future infinity. Recent work on the cosmic time machine describes an axially symmetric vacuum solution [13], a cylindrically symmetric anisotropic fluid solution [14], and an axially symmetric null dust collapse [15]. For physical models, the weak energy condition (WEC) must hold along with the other energy conditions. The weak energy condition states that, for any physical (timelike) observer, the energy density is nonnegative, which is the case for all known types of (classical) matter fields. The matter–energy content of the solution discussed here satisfies the different energy conditions.

The Einstein field equations (taking a cosmological constant of $\Lambda = 0$) are given by

$$G_{\mu\nu} = T_{\mu\nu}, \quad \mu, \nu = 0, 1, 2, 3, \tag{1}$$

where the units are chosen such that $c = 1$ and $8\pi G = 1$. Here $G_{\mu\nu}$ is the Einstein tensor, and $T_{\mu\nu}$ is the energy–momentum tensor. For an anisotropic fluid and null dust, the energy momentum tensor is given by

$$\begin{aligned} T_{\mu\nu} &= (\rho + p_t)\, U_\mu\, U_\nu + p_t\, g_{\mu\nu} + (p_r - p_t)\, \zeta_\mu\, \zeta_\nu + \epsilon\, k_\mu\, k_\nu, \\ \text{and} \quad T &= T^\mu_\mu = 2\, p_t + p_r - \rho, \end{aligned} \tag{2}$$

where U^μ is a timelike unit four-velocity vector, ζ_μ is a spacelike unit vector along the radial direction r, and k_μ is the null vector. Here the different physical parameters ϵ, ρ, p_r, and p_t are the energy densities, the radial pressure, and the tangential pressure of the null dust and anisotropic fluid, respectively. The Ricci scalar from the field equations (1) using Eq. (2) is

$$-R = T = 2\, p_t + p_r - \rho. \tag{3}$$

2. Analysis of the type N spacetime

The line element of the axially symmetric metric in (t, r, ϕ, z) coordinates is

$$ds^2 = g_{rr}\, dr^2 + g_{\phi\phi}\, d\phi^2 + 2\, g_{t\phi}\, dt\, d\phi + 2\, g_{z\phi}\, dz\, d\phi + g_{zz}\, dz^2, \tag{4}$$

where the metric functions are given by

$$\begin{aligned} g_{rr} &= \sinh^2 2r\, \sinh^2 r, \\ g_{\phi\phi} &= -\sinh t\, \sinh^2 r, \\ g_{zz} &= \sinh^2 r, \\ g_{t\phi} &= g'_{\phi\phi}, \\ g_{z\phi} &= \beta^2\, z\, \sinh^2 r, \end{aligned} \tag{5}$$

where the prime denotes a derivative w.r.t. time t. Here we have assumed that the ϕ coordinate is periodic, with $\phi \in [0, 2\pi)$, and β is a real number. The ranges of the other coordinates are $-\infty < t < \infty$, $0 \leq r < \infty$, and $-\infty < z < \infty$. The coordinates are labeled as $x^0 = t$, $x^1 = r$, $x^2 = \phi$, and $x^3 = z$. The metric is Lorentzian with signature $(-, +, +, +)$ and the determinant of the corresponding metric tensor $g_{\mu\nu}$ is

$$\det g = -\sinh^2 2r \, \sinh^8 r \, \cosh^2 t. \tag{6}$$

The nonzero components of the Einstein tensor $G^{\mu\nu}$ are

$$G^t_t = -G^r_r = G^\phi_\phi = G^z_z = -\frac{3}{4}\,\mathrm{csch}^6 r, \quad G^t_\phi = -\frac{1}{2}\,\beta^2\,\mathrm{sech} t\,\mathrm{csch}^2 r. \tag{7}$$

The scalar curvature invariants are given by

$$\begin{aligned} R = R^\mu_\mu &= \frac{3}{2}\,\mathrm{csch}^6 r, \\ R_{\mu\nu}\,R^{\mu\nu} &= \frac{9}{4}\,\mathrm{csch}^9 r, \\ R_{\mu\nu\rho\sigma}\,R^{\mu\nu\rho\sigma} &= \frac{15}{4}\,\mathrm{csch}^{12} r, \\ R_{\mu\nu\rho\sigma;\tau}\,R^{\mu\nu\rho\sigma;\tau} &= \frac{81}{2}\,\mathrm{csch}^{18} r. \end{aligned} \tag{8}$$

The scalar curvature invariants and first-order differential invariant diverge (or blow up) at $r = 0$, a fact that indicates the existence of a naked curvature singularity. Since the naked singularity occurs without an event horizon (the presented solution does not represent a black-hole solution), the cosmic censorship conjecture has no physical interest here. These scalar curvature invariants vanish rapidly at $r \to \infty$, but the presented spacetime is not asymptotically flat [47].

For metric (4), we define the timelike unit four-velocity vector U^μ as

$$U^\mu = \frac{1}{\sqrt{2}}\,(f(r,t), 0, -1, 0), \quad U^\mu\,U_\mu = -1, \tag{9}$$

where

$$f(r,t) = -\mathrm{csch}^2 r\,\mathrm{sech} t + \frac{1}{2}\,\tanh t. \tag{10}$$

The spacelike unit vector ζ_μ along the radial direction r and the null vector k_μ are defined by

$$\begin{aligned} &\zeta_\mu = \sinh 2r\,\sinh r\,\delta^r_\mu, \quad \zeta_\mu\,\zeta^\mu = 1, \\ &k_\mu = \delta^\phi_\mu, \quad k_\mu\,k^\mu = 0. \end{aligned} \tag{11}$$

From the energy–momentum tensor (2) using Eqs. (9)–(11), we get

$$\begin{aligned} T^t_t &= \frac{1}{4}\,\{2\,(p_t - \rho) + (p_t + \rho)\,\sinh^2 r\,\sinh t\}, \\ T^t_\phi &= -\frac{1}{8}\,(p_t + \rho)\,\mathrm{sech} t\,\{4\,\mathrm{csch}^2 r - \sinh^2 r\,\sinh^2 t\} - \epsilon\,\mathrm{csch}^2 r\,\mathrm{sech} t, \\ T^t_z &= \frac{1}{4}\,\beta^2\,z\,(p_t + \rho)\,\mathrm{sech} t\,(2 - \sinh^2 r\,\sinh t), \end{aligned}$$

$$T^{\phi}_{t} = -\frac{1}{2}\,(p_t+\rho)\,\sinh^2 r\,\cosh t,$$
$$T^{\phi}_{\phi} = \frac{1}{4}\,\{2\,(p_t-\rho)-(p_t+\rho)\,\sinh^2 r\,\sinh t\},$$
$$T^{\phi}_{z} = \frac{1}{2}\,\beta^2\,z\,(p_t+\rho)\,\sinh^2 r,\quad T^{r}_{r}=p_r,\quad T^{z}_{z}=p_t. \tag{12}$$

From the field equations (1) using Eqs. (7), (12) and after simplification we get the following nonzero physical parameters:

$$\rho = p_r = -p_t = \frac{3}{4}\,\mathrm{csch}^6 r,\quad \epsilon = \frac{1}{2}\,\beta^2. \tag{13}$$

The matter-field distribution satisfies the following energy condition [48]:

$$\begin{aligned} \boldsymbol{WEC} &: \quad \rho > 0,\quad \epsilon > 0,\\ \boldsymbol{WEC}_t &: \quad \rho + p_t = 0,\\ \boldsymbol{WEC}_r &: \quad \rho + p_r > 0,\\ \boldsymbol{SEC} &: \quad \rho + p_r + 2\,p_t = 0,\\ \boldsymbol{DEC} &: \quad \rho = |p_i|,\quad i = t, r, z. \end{aligned} \tag{14}$$

The physical parameters (ρ, p_r, p_t) are singular on the symmetry axis $r = 0$. Therefore, the matter–energy field anisotropic fluid collapses on the symmetry axis, and a naked singularity forms.

2.1. Geodesic analysis: the radial geodesics

In order to understand the nature of events where the matter–energy density and/or curvature scalar diverge, one must analyze the behavior of the geodesics and the strength of such singular events. Here we focus on radial geodesics, which necessarily hit the singularity $r = 0$, i.e., on the symmetry axis [49].

The Lagrangian for metric (4) is given by

$$\begin{aligned} \pounds &= \frac{1}{2}\,g_{\mu\nu}\,\dot{x}^{\mu}\,\dot{x}^{\nu},\\ &= \frac{1}{2}\,\left[g_{rr}\,\dot{r}^2 + g_{zz}\,\dot{z}^2 + g_{\phi\phi}\,\dot{\phi}^2 + 2\,g_{t\phi}\,\dot{t}\,\dot{\phi} + 2\,g_{z\phi}\,\dot{z}\,\dot{\phi}\right], \end{aligned} \tag{15}$$

where the dot stands derivative w.r.t. an affine parameter. From Eqs. (4) and (15), it is clear that ϕ is a cyclic coordinate. There exist constants of motion corresponding to this cyclic coordinate, i.e., the azimuthal angular momentum p_ϕ, which is a constant given by

$$p_\phi = \mathrm{const} = g_{z\phi}\,\dot{z} + g_{\phi\phi}\,\dot{\phi} + g_{t\phi}\,\dot{t}. \tag{16}$$

For metric (4), the geodesic equations for the t, r coordinates in explicit form are

$$\ddot{t} = \mathrm{sech}t\,\left(-\frac{1}{2}\,\beta^4\,z^2\,\dot{\phi}^2 + \beta^2\,\dot{z}^2\right) - 2\,\mathrm{coth}r\,\dot{r}\,\dot{t} - \dot{t}\,\dot{\phi} - \tanh t\,\dot{t}^2 - \frac{1}{2}\,\tanh t\,\dot{\phi}^2, \tag{17}$$

$$\begin{aligned} \ddot{r} = -\frac{1}{8}\,\mathrm{csch}^3 r\,\mathrm{sech}r\,\{&(1-4\,\cosh 2\,r+3\,\cosh 4\,r)\,\dot{r}^2 + 2\,\sinh t\,\dot{\phi}^2 - 4\,\beta^2\,z\,\dot{\phi}\,\dot{z}\\ &- 2\,\dot{z}^2 + \cosh t\,\dot{\phi}\,\dot{t}\}. \end{aligned} \tag{18}$$

For the radial geodesics we have $\dot{z} = 0 = \dot{\phi}$. From Eqs. (17) and (18) we get

$$\ddot{t} = -\tanh t\, \dot{t}^2 - 2\coth r\, \dot{r}\, \dot{t},$$
$$\ddot{r} = -\frac{1}{8}\,\mathrm{csch}^3 r\,\mathrm{sech} r\,(1 - 4\,\cosh 2\,r + 3\,\cosh 4\,r)\,\dot{r}^2. \tag{19}$$

The first integral curves for t, r give

$$\dot{t}(s) = c_1\,\mathrm{sech} t\,\mathrm{csch}^2 r, \quad c_1 > 0,$$
$$\dot{r}(s) = \frac{c_2}{2\sqrt{2}}\,\mathrm{csch}^2 r\,\mathrm{sech} r, \quad c_2 > 0. \tag{20}$$

Solving Eqs. (20) yields

$$t(s) = \sinh^{-1}\left[\frac{c_1}{c_4}\,\{3\,(c_4\,s + c_3)\}^{1/3} + c_5\right],$$
$$r(s) = \sinh^{-1}[\{3\,(c_4\,s + c_3)\}^{1/3}], \quad c_4 = \frac{c_2}{2\sqrt{2}}, \tag{21}$$

where $c_i, i = 1, \ldots, 5$ are arbitrary constants. From Eq. (21) it is clear that the radial geodesics paths t, r are bounded for a finite value of the affine parameter s, and thus the presented spacetime is radially geodesically complete.

2.2. *Strengths of naked singularities*

To determine the strengths of naked singularities as strong or weak, we consider the criterion developed by Tipler [50] and Krolak [51], which provides insights into the magnitude of tidal forces experienced by an in-falling detector (or an observer) towards the singularity [52]. It is widely believed that the spacetime does not admit analytical extension through the singularity, if the strong curvature singularity holds.

A naked singularity is said to be strong if the collapsing objects get crushed to zero volume at the singularity, and weaker if they do not [26,53]. Following Clarke and Krolak [52], a sufficient condition for a singularity to be strong, in the sense of Tipler [50], is that along radial geodesic we must have

$$\lim_{s\to 0} s^2\,R_{\mu\nu}\,\frac{dx^\mu}{ds}\,\frac{dx^\nu}{ds} \neq 0 (> 0), \tag{22}$$

where $\frac{dx^\mu}{ds}$ is the tangent vector to the radial geodesics, and $R_{\mu\nu}$ is the Ricci tensor, while the weaker condition, which we called the *limiting focusing condition* [51] is defined by

$$\lim_{s\to 0} s\,R_{\mu\nu}\,\frac{dx^\mu}{ds}\,\frac{dx^\nu}{ds} \neq 0. \tag{23}$$

Hence, from Eq. (22) using Eq. (21), we have

$$\lim_{s\to 0} s^2\left[R_{rr}\left(\frac{dr}{ds}\right)^2 + R_{\phi\phi}\left(\frac{d\phi}{ds}\right)^2\right]$$
$$= \lim_{s\to 0} 6\,s^2\coth^2 r\left(\frac{dr}{ds}\right)^2, \quad \text{since} \quad \dot{\phi} = 0$$

$$
\begin{aligned}
&= \frac{c_2^2}{12} \lim_{s\to 0} \left[\frac{s^2}{(c_3 + c_4\, s)^2}\right] \\
&= 0, \quad \text{if} \quad c_3 \neq 0 \\
&= \text{const}, \quad \text{if} \quad c_3 = 0.
\end{aligned} \tag{24}
$$

Similarly, the spacetime does not satisfy the limiting focusing condition. Thus the naked singularity that is formed due to the curvature singularity and/or the matter–energy (anisotropic fluid) collapse does not satisfy the *strong curvature condition* provided we assume that $c_3 > 0$. Therefore, analytical extension of the spacetime through the singularity is possible.

2.3. *Equation of orbit and gravitational lensing*

We first calculate the equation of orbit for $r(\phi)$. Here we restrict our discussion to the orbital motion of the free test particle that moves on the $z = \text{const}$ plane. To get an equation for $r(\phi)$ we start with

$$
r'(\phi) = \frac{dr}{d\phi} = \frac{\dot{r}}{\dot{\phi}}. \tag{25}
$$

To work out the lens equation we have to calculate the null geodesics in the $z = \text{const}$ plane. For null geodesics $ds^2 = 0$, we get

$$
g_{rr}\, \dot{r}^2 + g_{\phi\phi}\, \dot{\phi}^2 + 2\, g_{t\phi}\, \dot{t}\, \dot{\phi} + 2\, g_{z\phi}\, \dot{z}\, \dot{\phi} = 0. \tag{26}
$$

Now we define the angular velocity Ω of a ZAMO (zero-angular-momentum particle as measured by an observer) in the constant z-plane. From Eq. (16) we have

$$
\Omega = \frac{\dot{\phi}}{\dot{t}} = -\frac{g_{t\phi}}{g_{\phi\phi}}, \tag{27}
$$

such that $p_\phi = 0$ for $z = 0$ (we have chosen the $z - \text{const}$ plane defined by $z = 0$). Therefore, from Eq. (26) using Eq. (27) one can obtain the following equation for the photon orbit:

$$
\begin{aligned}
& g_{rr}\, \dot{r}^2 - g_{\phi\phi}\, \dot{\phi}^2 = 0 \\
& \Rightarrow \left(\frac{\dot{r}}{\dot{\phi}}\right)^2 = \frac{g_{\phi\phi}}{g_{rr}} \\
& \Rightarrow \frac{\dot{r}}{\dot{\phi}} = \pm\sqrt{\frac{g_{\phi\phi}}{g_{rr}}} \\
& \Rightarrow \frac{dr}{d\phi} = \pm\sqrt{\frac{g_{\phi\phi}}{g_{rr}}} = \pm\sqrt{H(r,t)}.
\end{aligned} \tag{28}
$$

Substitution of $g_{\phi\phi}$ and g_{rr} and integration of Eq. (28) immediately gives the azimuthal angle ϕ as a function of r:

$$
\begin{aligned}
\phi &= \pm\frac{1}{2}\sqrt{-\text{csch} t}\; \cosh 2\, r \\
&= \pm\frac{1}{2}\sqrt{\text{csch} T}\; \cosh 2\, r, \quad t = -T < 0.
\end{aligned} \tag{29}
$$

Therefore, the Einstein deflection angle $\hat{\boldsymbol{\alpha}}$ is then calculated to give

$$\begin{aligned}\hat{\boldsymbol{\alpha}}(r) &= 2\int \frac{dr}{\sqrt{H(r,t)}} - \pi \\ &= 2\,|\phi(r,T)| - \pi \\ &= \sqrt{\text{csch} T}\,\cosh 2\,r - \pi .\end{aligned} \tag{30}$$

3. Classification of the rotating spacetime

For classification of the presented spacetime, one can construct a set of null tetrad vectors $(\mathbf{k}, \mathbf{l}, \mathbf{m}, \bar{\mathbf{m}})$. The metric tensor for metric (4) can be expressed as

$$g_{\mu\nu} = -k_\mu\, l_\nu - l_\mu\, k_\nu + m_\mu\, \bar{m}_\nu + \bar{m}_\mu\, m_\nu, \tag{31}$$

where the tetrad vectors are orthogonal except for $k^\mu\, l_\mu = -1$ and $m_\mu\, \bar{m}^\mu = 1$. Using this set of null tetrads, we calculate the five Weyl scalars, of which only

$$\Psi_4 = -\frac{\beta^2}{4} \tag{32}$$

is nonvanishing, while rest are $\Psi_0 = \Psi_1 = 0 = \Psi_2 = \Psi_3$. The metric is clearly of type N in the Petrov classification scheme.

An important note is that, if one takes $\beta = 0$, then the spacetime represented by Eq. (4) is conformally flat since the Weyl scalar vanishes; $C_{\mu\nu\rho\sigma} = 0$. The matter–energy content of this conformally flat spacetime is only anisotropic fluid with the energy density, the radial pressure, and the tangential pressure being given by Eq. (13).

We calculate the Ricci–Newmann scalars for metric (4); in terms of energy densities these are

$$\begin{aligned}\boldsymbol{\Phi}_{11} &= \frac{1}{4}\, R_{\mu\nu}\,(k^\mu\, l^\nu + m^\mu\, \bar{m}^\nu) = \frac{1}{4}\,\rho, \\ \boldsymbol{\Phi}_{22} &= \frac{1}{2}\, R_{\mu\nu}\, l^\mu\, l^\nu = \frac{1}{4}\,\beta^2, \\ \boldsymbol{\Phi}_{02} = \bar{\boldsymbol{\Phi}}_{20} &= \frac{1}{2}\, R_{\mu\nu}\, m^\mu\, m^\nu = \frac{1}{2}\,\rho,\end{aligned} \tag{33}$$

while the others are all vanishing, where the symbols are the same as in Ref. [54].

We set up an orthonormal frame $\mathbf{e}_{(a)} = \{\mathbf{e}_{(0)}, \mathbf{e}_{(1)}, \mathbf{e}_{(2)}, \mathbf{e}_{(3)}\}$, $\mathbf{e}_{(a)} \cdot \mathbf{e}_{(b)} \equiv e^\mu_{(a)}\, e^\nu_{(b)}\, g_{\mu\nu} = \eta_{ab} = \text{diag}(-1,+1,+1,+1)$, which consists of one timelike unit vector $\mathbf{e}_{(0)} = \mathbf{U}$ and three spacelike unit vectors $\mathbf{e}_{(i)}$, $i = 1,2,3$. The notations are such that the small Latin indices are raised and lowered with the Minkowski metric η^{ab}, η_{ab} and the Greek indices are raised and lowered with $g^{\mu\nu}$, $g_{\mu\nu}$. The dual basis is $\mathbf{e}^{(0)} = -\mathbf{e}_{(0)}$ and $\mathbf{e}^{(i)} = \mathbf{e}_{(i)}$. These frame components can conveniently be expressed in terms of null tetrad vectors:

$$\begin{aligned}\mathbf{k} &= \frac{1}{\sqrt{2}}\,(\mathbf{e}_{(0)} + \mathbf{e}_{(2)}), \quad \mathbf{l} = \frac{1}{\sqrt{2}}\,(\mathbf{e}_{(0)} - \mathbf{e}_{(2)}), \\ \mathbf{m} &= \frac{1}{\sqrt{2}}\,(\mathbf{e}_{(1)} + i\,\mathbf{e}_{(3)}), \quad \bar{\mathbf{m}} = \frac{1}{\sqrt{2}}\,(\mathbf{e}_{(1)} - i\,\mathbf{e}_{(3)}).\end{aligned} \tag{34}$$

3.1. The relative motion of free test particles in the spacetime

In order to analyze the effects of the gravitational field of the presented solution, it is natural to investigate the specific influence of various components of these fields on the relative motion of the free test particles. Such a local characterization of spacetimes, based on the equation of the geodesic deviation frame, was described by Pirani [55,56] and Szekeres [57,58] (see also Refs. [59,60]):

$$\frac{D^2 Z^\mu}{d\tau^2} = -R^\mu_{\ \nu\rho\sigma}\, U^\nu\, Z^\rho\, U^\sigma, \tag{35}$$

where $\mathbf{U}\cdot\mathbf{U} = -1$ is the timelike four-velocity of the free test particle (observer), and $Z(\tau)$ is the displacement vector, where τ is the proper time. By projecting Eq. (35) onto the orthonormal frame $\mathbf{e}_{(a)}$ given by Eq. (34), we get

$$\ddot{Z}^{(i)} = -R^{(i)}_{\ (0)(j)(0)}\, Z^{(j)}, \tag{36}$$

where $i, j = 1, 2, 3$, $\mathbf{e}_{(\mathbf{0})} = \mathbf{U}$, and

$$R^{(i)}_{\ (0)(j)(0)} = e^\mu_{(i)}\, U^\nu\, e^\rho_{(j)}\, U^\sigma\, R_{\mu\nu\rho\sigma}. \tag{37}$$

The frame components of the displacement vector $Z^{(j)} \equiv e^{(i)}_\mu\, Z^\mu$ directly determine the distance between close test particles. Their physical relative accelerations are given by

$$\ddot{Z}^{(i)} = e^{(i)}_\mu \left(\frac{D^2 Z^\mu}{d\tau^2}\right). \tag{38}$$

Equation (35) also implies

$$\ddot{Z}^{(0)} = -U_\mu \left(\frac{D^2 Z^\mu}{d\tau^2}\right) = R_{\mu\nu\rho\sigma}\, U^\mu\, U^\nu\, Z^\rho\, U^\sigma = 0, \tag{39}$$

so that $Z^{(0)} = a_0\,\tau + b_0$, a_0, b_0 are constants. Setting $Z^{(0)} = 0$, all test particles are synchronized by the proper time. From the standard decomposition of the curvature tensor we get

$$R_{(i)(0)(j)(0)} = C_{(i)(0)(j)(0)} + \frac{1}{6}\, R\, \delta_{ij} + \frac{1}{2}\, [\delta_{ij}\, R_{(0)(0)} - R_{(i)(j)}], \tag{40}$$

where $C_{(i)(0)(j)(0)} \equiv e^\mu_{(i)}\, U^\nu\, e^\rho_{(j)}\, U^\sigma\, C_{\mu\nu\rho\sigma}$ are the components of the Weyl tensor, and R and $R_{(i)(j)}$, respectively, denote the Ricci scalar and Ricci tensor components. From the field equations (1) using Eq. (2), we get

$$R_{(i)(0)(j)(0)} = C_{(i)(0)(j)(0)} - \frac{1}{2}\left[T_{(i)(j)} - \delta_{ij}\left\{T_{(0)(0)} + \frac{2}{3}\, T\right\}\right], \tag{41}$$

where $T = T^{(a)}_{(a)}$.

The only nonvanishing Weyl scalars are given by Eq. (32) so that

$$C_{(1)(0)(1)(0)} = \frac{1}{2}\, \Re e \Psi_4, \quad C_{(3)(0)(3)(0)} = -\frac{1}{2}\, \Re e \Psi_4. \tag{42}$$

Therefore, the equation of geodesic deviation (36) takes the form

$$\ddot{Z}^{(1)} = -R^{(1)}_{\ (0)(j)(0)}\, Z^{(j)}$$

$$
\begin{aligned}
&= -C_{(1)(0)(1)(0)}\, Z^{(1)} + \frac{1}{2}\left[T_{(1)(1)} - \{T_{(0)(0)} + \frac{2}{3}\,T\}\right] Z^{(1)} \\
&= -\frac{1}{2}\,\Re e\Psi_4\, Z^{(1)} + \frac{1}{2}\left[T_{(1)(1)} - T_{(0)(0)} - \frac{2}{3}\,T\right] Z^{(1)} \\
&= -\frac{1}{2}\,\Re e\Psi_4\, Z^{(1)} - \frac{1}{4}\,\epsilon\, Z^{(1)} + \frac{2}{3}\,\rho\, Z^{(1)} \\
&= \frac{2}{3}\,\rho\, Z^{(1)}, \qquad (43)\\
\ddot{Z}^{(2)} &= -R^{(2)}_{(0)(j)(0)}\, Z^{(j)} \\
&= -C_{(2)(0)(2)(0)}\, Z^{(2)} + \frac{1}{2}\left[T_{(2)(2)} - \left\{T_{(0)(0)} + \frac{2}{3}\,T\right\}\right] Z^{(2)} \\
&= -\frac{1}{3}\,\rho\, Z^{(2)}, \qquad (44)\\
\ddot{Z}^{(3)} &= -R^{(3)}_{(0)(j)(0)}\, Z^{(j)} \\
&= -C_{(3)(0)(3)(0)}\, Z^{(3)} + \frac{1}{2}\left[T_{(2)(2)} - \left\{T_{(0)(0)} + \frac{2}{3}\,T\right\}\right] Z^{(3)} \\
&= \frac{1}{2}\,\Re e\Psi_4\, Z^{(3)} + \frac{1}{2}\left[T_{(2)(2)} - T_{(0)(0)} - \frac{2}{3}\,T\right] Z^{(3)} \\
&= \frac{1}{2}\,\Re e\Psi_4\, Z^{(3)} - \frac{1}{4}\,\epsilon\, Z^{(3)} - \frac{1}{3}\,\rho\, Z^{(3)}, \\
&= -\left(\frac{\epsilon}{2} + \frac{\rho}{3}\right) Z^{(3)}, \qquad (45)
\end{aligned}
$$

where

$$
\begin{aligned}
&\Re e\Psi_4 = -\frac{\beta^2}{4} = -\frac{1}{2}\,\epsilon, \quad T_{(0)(0)} = \rho + \frac{1}{2}\,\epsilon, \\
&T_{(1)(1)} = p_r = \rho, \quad T_{(2)(2)} = p_t + \frac{1}{2}\,\epsilon = -\rho + \frac{1}{2}\,\epsilon, \\
&T_{(3)(3)} = p_t = -\rho, \quad T = -2\,\rho, \qquad (46)
\end{aligned}
$$

where $\Re e$ stands for real. Clearly, the relative motions of the nearby test particle depend on the energy–momentum tensor $T_{(a)(b)}$ terms, which consist of two components (null dust and anisotropic fluid) describing the interaction with matter content that affects the motion of the test particles.

The solutions of Eqs. (43)–(45) are

$$
\begin{aligned}
&Z^{(1)}(\tau) = A_1\, e^{\sqrt{\frac{2\rho}{3}}\,\tau} + B_1\, e^{-\sqrt{\frac{2\rho}{3}}\,\tau}, \\
&Z^{(2)}(\tau) = A_2\, \cos\left(\sqrt{\frac{\rho}{3}}\,\tau\right) + B_2\, \sin\left(\sqrt{\frac{\rho}{3}}\,\tau\right), \\
&Z^{(3)}(\tau) = A_3\, \cos\left(\sqrt{\frac{\epsilon}{2} + \frac{\rho}{3}}\,\tau\right) + B_3\, \sin\left(\sqrt{\frac{\epsilon}{2} + \frac{\rho}{3}}\,\tau\right). \qquad (47)
\end{aligned}
$$

If one takes $\beta = 0$, then the presented spacetime (4) is conformally flat and the matter–energy content anisotropic fluid satisfies the different energy condition. Therefore, for a conformally flat solution, the equation of the geodesic deviation takes the following form:

$$\ddot{Z}^{(1)} = \frac{2}{3}\,\rho\, Z^{(1)},$$
$$\ddot{Z}^{(2)} = -\frac{1}{3}\,\rho\, Z^{(2)},$$
$$\ddot{Z}^{(3)} = -\frac{1}{3}\,\rho\, Z^{(3)}. \tag{48}$$

The solutions of Eqs. (48) are

$$Z^{(1)}(\tau) = A_1\, e^{\sqrt{\frac{2\rho}{3}}\,\tau} + B_1\, e^{-\sqrt{\frac{2\rho}{3}}\,\tau},$$
$$Z^{(2)}(\tau) = A_2\, \cos\left(\sqrt{\frac{\rho}{3}}\,\tau\right) + B_2\, \sin\left(\sqrt{\frac{\rho}{3}}\,\tau\right),$$
$$Z^{(3)}(\tau) = A_4\, \cos\left(\sqrt{\frac{\rho}{3}}\,\tau\right) + B_4\, \sin\left(\sqrt{\frac{\rho}{3}}\,\tau\right), \tag{49}$$

where $A_i, B_i, i = 1, \ldots, 4$ are arbitrary constants.

4. Closed timelike curves of the spacetime

The presented spacetime admits closed timelike curves, which appear after a certain instant and thus violate the causality condition. There are many known solutions of the field equations that admit closed timelike curves (CTCs) and closed timelike geodesics (CTGs) (see, e.g., Refs. [61,62] and references therein).

Consider an azimuthal curve γ defined by $r = r_0$, $z = z_0$, $t = t_0$, where $r_0 > 0$, z_0, and t_0 are constants. From metric (4) we have

$$ds^2 = -\sinh t\, \sinh^2 r\, d\phi^2. \tag{50}$$

These curves are null at $t = t_0 = 0$, spacelike throughout for $t = t_0 < 0$, but become timelike for $t = t_0 > 0$, which indicates the presence of CTCs. Hence CTCs form at a definite instant of time satisfying $t = t_0 > 0$. The above analysis is valid provided that the CTCs evolve from an initial spacelike $t =$ const hypersurface. A hypersurface $t =$ const is spacelike provided that $g^{tt} < 0$ for $t < 0$, timelike provided that $g^{tt} > 0$ for $t > 0$, and a null hypersurface provided that $g^{tt} = 0$ for $t = 0$. From metric (4), we find that

$$g^{tt} = \text{csch}^2 r\, \text{sech}^2 t\, (\beta^4 z^2 + \sinh t). \tag{51}$$

The hypersurface $t = \text{const} = t_0 < 0$ is spacelike ($r \neq 0$) and can be chosen as the initial hypersurface over which the initial data may be specified. Here we have chosen constant z-planes defined by $z = z_0$, where z_0 is a constant equal to zero such that the above condition is satisfied, i.e., the $t =$ const < 0 hypersurface is spacelike. There is a Cauchy horizon at $t = t_0 = 0$ for any such initial spacelike hypersurface $t = t_0 < 0$. Hence, the spacetime evolves from an initial spacelike hypersurface in a causally well behaved manner, up to a moment, i.e., a null hypersurface $t = t_0 = 0$, and the formation of CTCs takes place from causally well behaved initial conditions in the $z =$ const planes. Thus the presented solution possesses a naked singularity (due to curvature scalars and matter–energy anisotropic fluid collapse) and admits closed timelike curves, which appear after a certain instant of time.

5. Conclusions

In this paper, we present gravitational collapse in an axisymmetric solution of the field equations that is regular everywhere except on the symmetry axis, where it possesses a naked curvature singularity. The matter–energy content of the spacetime is a combination of anisotropic fluid satisfying the energy condition collapse on the symmetry axis, and null dust (or pure radiation field) of constant energy density (ϵ). The physical parameters, the energy density (ρ), the radial pressure (p_r), and the tangential pressure (p_t) of anisotropic fluid diverge on the symmetry axis; therefore, a naked singularity forms. A physical interpretation of this solution, based on a study of the equation of the geodesic deviation, was presented. It was demonstrated that this solution depend on the terms describing the matter–energy content, which consists of two different components: the null dust field (ϵ) and the anisotropic fluid. A special case of the spacetime leads to a conformally flat metric with anisotropic fluid, as in the matter–energy content satisfying the energy conditions. Additionally, the rotating spacetime admits closed timelike curves (CTCs), which appear after a certain instant of time. These curves evolve from an initial spacelike hypersurface in a causally well behaved manner in $z = \text{const}$ planes.

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References

[1] K. S. Thorne, in *Magic without Magic: John Archibald Wheeler*, ed. J. R. Klauder (Freeman and Co., San Francisco, CA, 1972).
[2] S. A. Hayward, Classical Quantum Gravity **17**, 1749 (2000).
[3] T. A. Apostolatos and K. S. Thorne, Phys. Rev. D **46**, 2435 (1992).
[4] F. Echeverria, Phys. Rev. D **47**, 2271 (1993).
[5] S. Guttia, T. P. Singh, P. A. Sundararaj, and C. Vaz, arXiv:gr-qc/0212089v3.
[6] K. Nakao and Y. Morisawa, Classical Quantum Gravity **21**, 2101 (2004).
[7] K. Nakao and Y. Morisawa, Prog. Theor. Phys. **113**, 73 (2005).
[8] S. Goncalves and S. Jhingan, Int. J. Mod. Phys. D **11**, 1469 (2002).
[9] B. C. Nolan, Phys. Rev. D **65**, 104006 (2002).
[10] P. R. C. T. Periera and A. Wang, Phys. Rev. D **62**, 124001 (2000).
[11] A. Wang, Phys. Rev. D **68**, 064006 (2003).
[12] A. Wang, Phys. Rev. D **72**, 108501 (2005).
[13] D. Sarma, F. Ahmed, and M. Patgiri, Adv. High Energy Phys. **2016**, 2546186 (2016).
[14] F. Ahmed, Ann. Phys. **382**, 127 (2017).
[15] F. Ahmed, Adv. High Energy Phys. **2017**, 3587018 (2017).
[16] T. Chiba, Prog. Theor. Phys. **95**, 321 (1996).
[17] K. Nakao and Y. Morisawa, Phys. Rev. D **71**, 124007 (2005).
[18] T. Piran, Phys. Rev. Lett. **41**, 1085 (1978).
[19] J. M. M. Senovilla and R. Vera, Classical Quantum Gravity **17**, 2843 (2000).
[20] H. Bondi, Proc. R. Soc. Lond. A **427**, 259 (1990).
[21] K. S. Thorne, Phys. Rev. **138**, B251 (1965).
[22] T. A. Morgan, Gen. Relat. Gravit. **4**, 273 (1973).
[23] P. S. Letelier and A. Wang, Phys. Rev. D **49**, 5105 (1994).
[24] M. A. Melvin, Phys. Lett. **8**, 65 (1964).
[25] M. A. Melvin, Phys. Rev. **139**, B225 (1965).
[26] P. S. Joshi, *Global Aspects in Gravitation and Cosmology* (Clarendon, Oxford, UK, 1993).
[27] P. S. Joshi, *Singularities, Black Holes and Cosmic Censorship* (IUCAA, Pune, India, 1997).
[28] S. S. Deshingkar, P. S. Joshi, and I. H. Dwivedi, Phys. Rev. D **59**, 044018 (1999).
[29] S. Barve, T. P. Singh, C. Vaz, and L. Witten, Classical Quantum Gravity **16**, 1727 (1999).

[30] P. S. Joshi and I. H. Dwivedi, Classical Quantum Gravity **16**, 41 (1999).
[31] K. S. Govinder and M. Govender, Phys. Rev. D **68**, 024034 (2003).
[32] A. Krasinski, *Inhomogeneous Cosmological Models* (Cambridge University Press, Cambridge, 1997).
[33] R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969).
[34] R. Penrose, Singularities and time-asymmetry, in *General Relativity: An Einstein Centenary Survey*, eds. S. W. Hawking and W. Israel (Cambridge University Press, Cambridge, 1979), pp. 581–638.
[35] R. Penrose, The question of cosmic censorship, in *Black Holes and Relativistic Stars*, ed. R. M. Wald (Chicago University Press, Chicago, IL, 1994).
[36] K. S. Virbhadra, D. Narasimha, and S. M. Chitre, Astron. Astrophys. **337**, 1 (1998).
[37] K. S. Virbhadra and G. F. R. Ellis, Phys. Rev. D **65**, 103004 (2002).
[38] G. N. Gyulchev and S. S. Yazadjiev, Phys. Rev. D **78**, 083004 (2008).
[39] M. Werner and A. O. Petters, Phys. Rev. D **76**, 064024 (2007).
[40] C. Bambi and N. Yoshida, Classical Quantum Gravity **27**, 205006 (2010).
[41] C. Bambi and K. Freese, Phys. Rev. D **79**, 043002 (2009).
[42] K. Hioki and K.-i. Maeda, Phys. Rev. D **80**, 024042 (2009).
[43] C. J. S. Clarke and F. de Felice, Gen. Relat. Gravit. **16**, 139 (1984).
[44] F. de Felice, in *Birth of the Universe and Fundamental Physics*, ed. F. Occhionero (Springer-Verlag, Berlin Heidelberg, 1995), Lecture Notes in Physics Vol. 455, pp. 99–102.
[45] F. de Felice, Nuovo Cimento B **122**, 481 (2007).
[46] F. de Felice, EPJ Web Conf. **58**, 01001 (2013).
[47] J. C. N. de Araujo and A. Wang, Gen. Relat. Gravit. **32**, 1971 (2000) [arXiv:gr-qc/0004082].
[48] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973).
[49] Y. Kurita and K. Nakao, Phys. Rev. D **73**, 064022 (2006).
[50] F. J. Tipler, Phys. Lett. A **64**, 8 (1977).
[51] A. Krolak, J. Math. Phys. **28**, 138 (1987).
[52] C. J. S. Clarke and A. Krolak, J. Geom. Phys. **2**, 127 (1985).
[53] I. H. Dwivedi and P. S. Joshi, Classical Quantum Gravity **6**, 1599 (1989).
[54] H. Stephani, D. Kramer, M. MacCallum, C. Hoenselaers, and E. Herlt, *Exact Solutions to Einstein's Field Equations* (Cambridge University Press, Cambridge, 2003).
[55] F. A. E. Pirani, Acta Phys. Polon. **15**, 389 (1956).
[56] F. A. E. Pirani, Phys. Rev. **105**, 1089 (1957).
[57] P. Szekeres, J. Math. Phys. **6**, 1387 (1965).
[58] P. Szekeres, J. Math. Phys. **7**, 751 (1966).
[59] J. Bičák and J. Podolský, J. Math. Phys. **40**, 4506 (1999).
[60] J. Podolský and M. Ortaggio, Classical Quantum Gravity **20**, 1685 (2003).
[61] F. Ahmed, Prog. Theor. Exp. Phys. **2017**, 043E02 (2017).
[62] F. Ahmed, Ann. Phys. **382**, 127 (2017).