

Finite 3D de-Sitter quantum Gravity

Rudranil Basu¹, Samir Ka Paul²

^{1,2} S.N. Bose National Centre for Basic Sciences, Sector-III, Block - JD, Salt Lake, Kolkata - 700 098

E-mail: ¹ rudranil@bose.res.in, ² smr@bose.res.in

Abstract. Non-perturbative quantization of 3d gravity is studied in the first order formulation with positive cosmological constant on the background topology of lens space, which is a three sphere modulo a discrete group. We concentrate on the Chern Simons formulation of 3d gravity for the purpose of quantization. However, instead of taking the standard theory we alter it by adding another topological term to the action, which keeps the equations of motion same. This additional term makes the quantization consistent. More importantly the introduction of a new parameter renders the theory finite.

In this short report, which is based on our paper [1], we present how a finite and consistent theory of pure 3D quantum gravity can be developed for the case when the cosmological constant is taken to be positive. Although most of the non-trivial results in 3D gravity are in the context of negative cosmological constant, our work can best be put in the recent upsurge in this arena of de-Sitter [2, 3]. In [2] one-loop partition function of the theory was computed in the functional integral approach. After summing contributions from all possible topologies, the result found was divergent and hence physically meaningless. It was also verified using the Chern Simons formulation too [4]. But in [3], through the introduction of a massive mode in pure gravity through topological massive gravity (TMG) theory, the divergence was shown to be regulated, which comes a bit contrary to one's intuition.

On the other hand people have been looking into the problem of an alternative theory of 3D gravity [6] and its viability for quite some time [7, 8, 10]. Standard theory of Euclidean 3D gravity ($\Lambda > 0$) can be described as a difference between two $SU(2)$ Chern Simons actions (in the units $16\pi G = 1 = c$):

$$S = l \left(I[A^{(+)}] - I[A^{(-)}] \right)$$

$$\text{where } I[A] = \int \left(A^I \wedge dA_I + \frac{1}{3} \epsilon_{IJK} A^I \wedge A^J \wedge A^K \right), \quad A^{(\pm)} = \omega \pm e/l. \quad (1)$$

Here $l = \frac{1}{\sqrt{\Lambda}}$ and e, ω are the triad and connection variables of gravity dynamics. The present author himself and others have worked many features of the alternative theory:

$$\tilde{S}[A^{(+)}, A^{(-)}] = \frac{k^{(+)}}{2\pi} I[A^{(+)}] + \frac{k^{(-)}}{2\pi} I[A^{(-)}] \quad (2)$$

$$\text{where } k_{(\pm)} = \frac{l(1/\gamma \pm 1)}{8G}, \quad \text{reinstating } G$$

Hereby we have introduced the dimensionless parameter γ . It is a feasible theory because equations of motion coming from (2) are same as those coming from (1). The only difference is that the canonical structure of this augmented theory is different from that of the standard theory. In absence of boundaries of the base manifold, the pre-symplectic structure looks as:

$$\Omega(\delta_1, \delta_2) = \frac{k^{(+)}}{\pi} \int_{\Sigma} \delta_1 A^{(+)} \wedge \delta_2 A^{(+)} + \frac{k^{(-)}}{\pi} \int_{\Sigma} \delta_1 A^{(-)} \wedge \delta_2 A^{(-)}. \quad (3)$$

In order to proceed for quantization of the theory, we note that, topology of the background manifold is the sole deciding factor about the quantum theory, since this is a topological field theory. Along with it, we remind ourselves that the vacuum solution of $\Lambda > 0$ gravity is de-Sitter space. When Euclideanized, this is $dS \rightarrow S^3$ and all solutions are locally isomorphic to S^3 , geometrically. We take for viable topology S^3/Γ , where Γ is a discrete group. For example, if we choose $\Gamma \sim \mathbb{Z}_p$ and specify its action on S^3 in a particular way (involving another natural number $q \leq p$ and $(q, p) = 1$), we get Lens spaces, $L(p, q)$. There are other choices for Γ , whose contributions are trivial.

One may now be optimist enough to look forward to the possibility of quantizing the theory directly (geometric quantization [5]). But this is the point where the background come in the way. Gauge invariant physical phase space of Chern Simons theory is given by $:\text{hom}(\pi_1(M) \rightarrow G)/\sim$. Here M is the base space and G is the gauge group. \sim here denotes gauge equivalence. In this case we can easily see that the gage invariant phase space consists of a total of p^2 points, hence cannot qualify for symplectic manifold. Obviously then we take a less direct route and follow [4].

In this later approach one contemplates $L(p, q)$ to be constructed by gluing together 2 solid tori at their surfaces through an element of the mapping class group $PSL(2, \mathbb{Z})$ of the toric surface. In the defining representation, an element U of the group, which generates $L(p, q)$ is

$$U = \begin{pmatrix} q & b \\ p & d \end{pmatrix} \in SL(2, \mathbb{Z}). \quad (4)$$

This U can be generated by the well-known S and T modular-transformation operators as

$$U = S \prod_{s=1}^{t-1} (T^{m_s} S), \quad m_s \in \mathbb{Z}^+.$$

The following restrictions of m_s are to be followed:

$$p/q = -m_{t-1} + \frac{1}{m_{t-2} - \frac{1}{\dots - \frac{1}{m_1}}}. \quad (5)$$

Meanwhile, one has to keep the quantization ready on the toric surfaces ready. This is easy and can be carried out in the framework of geometric quantization. Without going into the detail, we state that after performing the quantization we end up with a Hilbert space associated with each toric surface for each Chern Simons theory with label k . The dimensionality of the Hilbert space over the field \mathbb{C} is $k+1$. A convenient basis, for which the inner product structure is easily spelled out is given in terms of the $SU(2)$ theta functions:

$$\psi_{j,k}(z, \tau) = \frac{\vartheta_{j+1,k+1}(z, \tau) - \vartheta_{-j-1,k+2}(z, \tau)}{\vartheta_{1,2}(z, \tau) - \vartheta_{-1,2}(z, \tau)} \quad j = 0, \dots, k \quad (6)$$

Here $z = \frac{1}{\pi}(\theta + \tau\phi)$ coordinatizes the phase space with $\theta, \phi \in [0, 2\pi)$, τ being the modular parameter on the phase space, which is a Kähler manifold and also a Jacobian variety. This Hilbert space on the other hand makes sense if and only if:

$$a := \frac{l}{8l_p} = s/2 \quad s \in \mathbb{N} \quad \text{and} \quad \gamma = \frac{a}{(a-1)+t} \quad t \in \mathbb{N}. \quad (7)$$

$l_p = G\hbar/c^3$ is the Planck length. The axiomatics of topological field theory [9] now dictates that the partition function of the glued space (for each Chern Simons theory), ie the Lens space $L(p, q)$ must be given by

$$Z(r)_{L(p,q)} = \langle \psi_{0,k} | U | \psi_{0,k} \rangle.$$

In order to evaluate this, one keeps in mind the actions of the S and T transformations on the space of the theta functions. A non-trivial calculation in this direction gives us [4]:

$$Z(r)_{L(p,q)} = -\frac{i}{\sqrt{2rp}} \exp(6\pi is(q, p)/r) \sum_{\pm} \sum_{n=1}^p \exp\left(\frac{2\pi i q r n^2}{p} + \frac{2\pi i n(q \pm 1)}{p} \pm \frac{\pi i}{rp}\right), \quad (8)$$

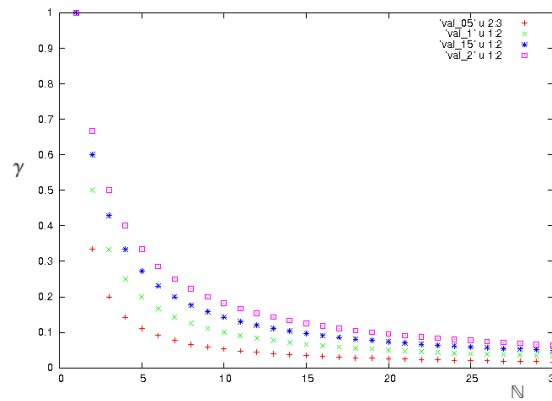


Figure 1. Some of the first allowed values of γ for $l/l_p = 4, 8, 12, 16$

denoting $r = k + 2$. Here $s(q, p) = \sum_{l=1}^{p-1} \frac{l}{p} \left(\frac{lq}{p} - \left[\frac{lq}{p} \right] - \frac{1}{2} \right)$ is the Dedekind sum. $[]$ is the standard floor function.

Returning to our original problem of gravity, we have the total partition function, summing over all topologies:

$$Z^{\text{tot}} = \sum_{p=1}^{\infty} \sum_{\substack{q(\text{mod } p) \\ (q,p)=1}} = Z(r_{(+)})_{L(p,q)} Z(r_{(-)})_{L(p,q)} \tag{9}$$

For the purpose of illuminating the convergence of our theory we choose a pair of particular saddles in the sum (8), $n = \frac{q \pm 1}{2}$ for the '+' and the '-' type Chern Simons theory respectively. Moreover we relax the above restrictions (7) a little bit by demanding $a/\gamma \in 2\mathbb{N}$. Provided these conditions are obeyed, we re-write (9):

$$Z^{\text{tot}} = -\frac{1}{2\sqrt{r_{(+)r_{(-)}}}} \sum_{p=1}^{\infty} \frac{1}{p} \sum_{\substack{q(\text{mod } p) \\ (q,p)=1}} \exp(6\pi i s(q, p)/R_+) \exp\left(\frac{\pi i}{p} (2a + (q + q^*) (a/\gamma + 2))\right) \times \left[e^{\frac{\pi i}{pR_+} + \frac{2\pi i}{p}(q+1)} + e^{-\frac{\pi i}{pR_+} + \frac{2\pi i}{p}(q-1)} - e^{\frac{\pi i}{pR_-} + \frac{4\pi i}{p}} - e^{\frac{-\pi i}{pR_-}} \right] \tag{10}$$

where $\frac{1}{R_{\pm}} = \frac{1}{r_{(+)}} \pm \frac{1}{r_{(-)}}$

This is the point where one encounters the divergence found in [2] through $\sum_{\substack{q(\text{mod } p) \\ (q,p)=1}} 1 = \phi(p)$, the Euler totient function. But in our case, in the above expression, there is no q independent term. This is the reason we can avoid the dangerous totient function. For the sake of investigation of convergence we further go to the limit: $0 < \gamma \ll 1$. This limit is consistent with the restrictions (7). In this limit,

$$Z^{\text{tot}} = -\frac{\gamma}{a} \left(1 - \frac{2\gamma}{a} \right) \sum_{p=1}^{\infty} \frac{1}{p} e^{\frac{2\pi i a}{p}} \cos(2\pi/p) \left[S\left(\frac{a}{2\gamma} + 2, \frac{a}{2\gamma} + 1; p\right) - e^{\frac{2\pi i}{p}} S\left(\frac{a}{2\gamma} + 1, \frac{a}{2\gamma} + 1; p\right) \right] \tag{11}$$

$$S(\alpha, \beta; p) = \sum_{\substack{q(\text{mod } p) \\ (q,p)=1}} \exp(2\pi i(\alpha q + \beta q^*)/p).$$

When one expands the cosine and the exponential above in powers of π/p , Kloostermann zeta functions:

$$L(m, n; s) = \sum_{p=1}^{\infty} p^{-2s} S(m, n; p)$$

are evident. These functions are analytic in $\Re s > 1/2$. In the small γ region, the above summand of (11) becomes:

$$\sum_{m,n,r=0}^{\infty} \frac{(2\pi i)^{r+n+2m+1}}{r+1} \frac{a^n}{(2m)!n!r!} L\left(\frac{a}{2\gamma}, \frac{a}{2\gamma}; \frac{r+n+2m}{2} + 1\right) \quad (12)$$

As we learnt above, all the above terms are analytic. Hence the partition function is free from any divergence. This is our main result.

Now we go on to present a brief comparative study between our theory and TMG. Denoting by E, the contributions coming from pure Einstein Hilbert theory with cosmological constant and by MG, the ones coming from massive graviton modes, we have the one-loop(in contrast to our non-perturbative result) result [3]:

$$\sum_{p=1}^{\infty} \sum_{\substack{q(\bmod p) \\ (q,p)=1}} Z_E^{(0)} Z_{MG}^{(0)} Z_E^{(1)} \sim \sum_{r=0}^{\infty} \frac{(2\pi a)^r}{r!} L\left(\frac{a}{2\gamma}, \frac{a}{2\gamma}; \frac{r}{2} + \frac{1}{2}\right) + \text{trivially analytic terms.} \quad (13)$$

Comparing this with (12), the interesting fact that crops up is that the term corresponding to $r = 0$ in the sum of the RHS is the source of divergence since it corresponds to Kloosterman zeta function with $s = 1/2$. But in [3] it is showed that when one includes $Z_{MG}^{(1)}$ as the product and then performs the sum over p , the divergence goes away. Up to one loop calculation they have

$$Z = \sum_{p=1}^{\infty} \sum_{\substack{q(\bmod p) \\ (q,p)=1}} Z_E^{(0)} Z_{MG}^{(0)} Z_E^{(1)} Z_{MG}^{(1)}.$$

The expression of $Z_{MG}^{(1)}$ as given in [3] is far too complicated for the above expression to be analytically simplified and compared with (12). The way the divergence in the above expression is controlled by $Z_{MG}^{(1)}$ is very like the way we showed (12) to be finite. We conclude that although the finiteness of TMG could be ascribed to its propagating graviton modes, our theory (2), being devoid of massive gravitons still yield a reasonably similar convergent partition function.

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