Parity codes in space and time

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***Abstract*—We present a single framework to describe parity codes in exclusively spatial dimensions as quantum error correction codes, as well as parity codes with a temporal dimension, as quantum circuits. This framework builds on the stabilizer formalism to introduce label tracking of parity information for illustrating and constructing quantum circuits and quantum error correction codes. Additionally, this framework proposes the necessary transformations and conditions to translate between the spatial and temporal versions of a given parity code. This technique is then applied to find the temporal variant of the spatially extended LHZ-code [1] using CNOT gates on a linear nearest-neighbor (LNN) qubit architecture. The resulting quantum circuit provides an efficient implementation of quantum algorithms on LNN architectures, that require the interactions found in all-to-all connected transverse-field Ising models. Two widely used quantum algorithms that require such interactions are the Quantum Approximate Optimization Algorithm (QAOA) [2] and the quantum Fourier transform (QFT). For the QAOA, this is the first proposal with a linear circuit depth while not requiring more than n^2 CNOT gates per iteration. Our approach for QFT, requiring only LNN connectivity, surpasses previous QFT implementations even on devices with high qubit connectivity in terms of circuit depth while not increasing the gate count in leading order.**

***Index Terms*—parity code, quantum circuit, QFT, QAOA**

I. Introduction

Parity codes [3] are computational methods which encode information of multiple logical qubits onto single physical qubits in order to manipulate it locally. If the multi-qubit information is given along the same axis, e.g., ZZZ or XX, it is referred to as the parity of the qubits involved. Typically, parity codes focus on localizing parities of a single basis, since they all commute with one another and can therefore always be localized at the same time. Following this convention, the physical qubits holding multi-qubit Z-information are called *parity qubits*, while physical qubits trivially encoding single-qubit Z-information are referred to as *base qubits*.

II. Parity label tracking

We demonstrate how to keep track of such parity information and show why this is a helpful tool to design and optimize quantum algorithms. To this end, we assign a label $P_a^{(k)}$ to every physical qubit a at time τ_k and define each qubit label to be $P_a^{(0)} = \{a\}$ at the starting point of the qubit tracking at time τ_0. The label of a qubit t can be changed by applying the gate $\mathrm{CNOT}_{\mathrm{c}\to\mathrm{t}}$ during time step k, which is defined as the time period between time τ_{k-1} and time τ_k. The labels can be tracked accordingly as

$$P_c^{(k)} = P_c^{(k-1)}, \tag{1}$$
$$P_t^{(k)} = P_c^{(k-1)} \triangle P_t^{(k-1)}, \tag{2}$$

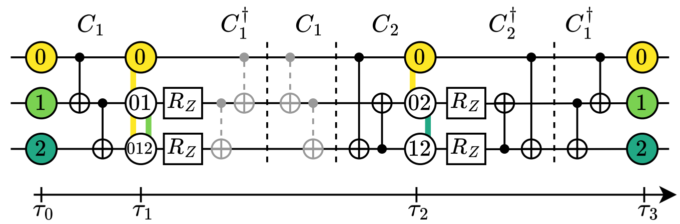

Fig. 1. Labeling of qubits demonstrating the effects of CNOT gates on the qubit encoding. The initial labels $P_j^{(0)}$ directly correspond to individual logical qubit information $\bar{Z}_j$ labeled $0, 1$ and 2. Furthermore, the labels change at the targets of the CNOTs according to (1). The colored lines between the qubit vertices represent logical lines, indicating how the corresponding logical X-operator is spread out onto physical operators.

where $\triangle$ denotes the symmetric difference. A set of CNOT gates in time step k is described by the unitary operator C_k, such that a set of CNOT gates between time τ_0 and τ_k can be written as a sequence of operators $C_k \cdots C_0$ giving rise to the time dependent labels as shown in Fig. 1. Auxiliary qubits can be added to the circuit which are initialized in the $+1$ Z eigenstate $|0\rangle$ such that applying a Z gate to these qubits has a trivial effect and therefore these qubits have no initial label. These qubits will be referred to as *empty qubits*, since their label represents the empty set, as opposed to *active qubits*, which have a parity label that is not empty. Using labeling, the expression for a logical Z rotation $\bar{U}_k$ using physical qubit a at time τ_k is

$$\begin{aligned}\bar{U}_k^Z &= e^{i\theta C_1^\dagger \cdots C_k^\dagger Z_a C_k \cdots C_1} \\ &= e^{i\theta (\prod_{j\in P_a^{(k)}} \bar{Z}_j)}\end{aligned} \tag{3}$$

and a dual expression for the logical X rotation using the labels is

$$\begin{aligned}\bar{U}_k^X &= e^{i\theta C_1^\dagger \cdots C_k^\dagger (\prod_{j\in \mathcal{L}_l} X_j) C_k \cdots C_1} \\ &= e^{i\theta \bar{X}_l}.\end{aligned} \tag{4}$$

where $\mathcal{L}_l$ holds the indices of all physical qubits which contain l in their k^{th} label. The labels allow any unitary $\bar{U}$ to be

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expressed as k alternating applications of logical Z and logical X rotations as

$$\bar{U}|\psi\rangle = \bar{U}_k^X \bar{U}_k^Z \cdots \bar{U}_1^X \bar{U}_1^Z |\psi\rangle. \tag{5}$$

III. Translating spatial codes to temporal codes

The labels assigned to the qubits are directly related to the Clifford tableau [4] for tracking Pauli operators under conjugation of Clifford gates. They specifically relate to the inverse of the block matrix T^Z, spanned by the rows and columns of the Clifford tableau corresponding to the Z-basis information, also known as a parity map [5]. The changing of the labels according to (1) by a sequence of CNOTs $C_k \cdots C_1$, then corresponds to applying row operations as matrix multiplications from the left acting on the parity map as $T_k^Z = D_k \cdots D_1 T_0^Z$, where $(D_k)_{ij} \in \{0, 1\}$, meaning that if $(D_k)_{ij} = 1$, the gate $\text{CNOT}_{j\to i}$ applied in round k is tracked. Given an active and empty qubit pair in one of the k time steps, the parity code can effectively be expanded or contracted by respectively adding or removing a CNOT from the active qubit to the empty qubit and additionally adapting the other CNOTs accordingly [6]. A condition on the qubit connectivity and a condition on the other CNOTs present in that time step need to be satisfied in order to apply such a transformation.

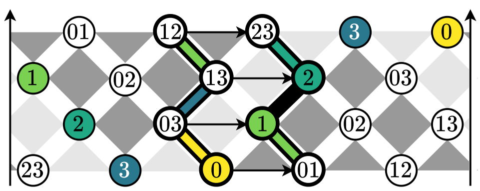

Fig. 2. A unit cell of the extended LHZ-code for 4 logical qubits, where the gray polygons indicate Z-stabilizers and the vertical arrows indicate the periodic boundaries. The spanning lines are emphasized by black edges and reach from the top boundary to the bottom boundary. The horizontal arrows indicate how the spanning line effectively moves from left to right by applying the corresponding CNOT gates.

IV. Application to the LHZ-code

The known triangular shape of the LHZ-encoding [1] can be extended with additional (redundant) qubits via localized constraints to effectively prolong the qubit lines generating the logical X information as shown in Fig. 2. An essential property of the extended layout is that each line of qubits spanning from the top boundary to the bottom boundary forms a *spanning line* of the original problem graph containing each logical qubit index at least once and at least one data qubit such that all the logical information is contained in the qubits along this line. Since the LHZ-code contains all possible $\bar{Z}_i \bar{Z}_j$ pairs, each logical two-qubit parity operator occurs at some point in time as a physical single-qubit operator, such that all two-qubit interactions can be implemented as single-qubit rotations around the Z-axis, whilst the logical X information always remains on a single or two neighboring qubits [6]. This technique results in lower resource requirements for implementing algorithms that include interactions similar to those appearing in the Hamiltonian of a transverse field Ising model

$$H_{\text{Ising}} = \sum_{j=1}^{n} \sum_{i<j} J_{ij} \bar{Z}_i \bar{Z}_j + \sum_{i=1}^{n} h_i \bar{Z}_i + \sum_{i=1}^{n} g_i \bar{X}_i. \tag{6}$$

Two of these algorithms are the Quantum Approximate Optimization Algorithm (QAOA) and the Quantum Fourier Transform (QFT). To apply one round of QAOA for a logical system size of n, our approach requires $n^2 - 1$ CNOT gates with circuit depth $2n + 2$, compared to $\frac{3}{2}n^2 - \frac{5}{2}n + 1$ CNOT gates with circuit depth $3n - 2$ for the typical approach in [7]. The improved resource efficiency for the QFT with our approach compared to other approaches is shown in table I.

TABLE I
Resource comparison for the QFT on n qubits. The gate set used consists of CNOT gates as well as single-qubit (SQ) H, R_x and R_z gates.

Resource	Ours	Ref. [8]	Ref. [9]	all-to-all
#CNOTs	$n^2 - 1$	$n^2 + n - 4$	$\frac{3}{2}n^2 - \frac{3}{2}n$	$n^2 - n$
#SQ gates	$\frac{1}{2}n^2 + \frac{5}{2}n - 2$	$\frac{1}{2}n^2 + \frac{5}{2}n - 2$	n^2	n^2
total gates	$\frac{3}{2}n^2 + \frac{5}{2}n - 3$	$\frac{3}{2}n^2 + \frac{7}{2}n - 6$	$\frac{5}{2}n^2 - \frac{3}{2}n$	$2n^2 - n$
CNOT depth	$4n - 4$	$n^2 + n - 4$	$6n - 9$	$4n - 6$
SQ depth	$n + 1$	$2n + 1$	$4n - 4$	$4n - 4$
total depth	$5n - 3$	$n^2 + 3n - 3$	$10n - 13$	$8n - 10$

V. Conclusion

This work describes parity codes with the corresponding label tracking and additionally introduces a method to translate between spatial and temporal parity codes. This method is then applied to a spatially extended LHZ-code to find implementations of the QFT and QAOA on linear nearest-neighbor architectures requiring fewer hardware resources than currently used methods.

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