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The tetrahexahedric angular Calogero model

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Abstract. We investigate the spherical reduction of the rational A_{n-1} Calogero model. It defines a quantum superintegrable model of a particle trapped in one of $n!$ spherical $(n-2)$-simplices on S^{n-2} in a special S_n-symmetric potential. The energy levels (including degeneracy) and eigenstates are given, and the construction of conserved charges and Hamiltonian intertwiners outlined. We describe superintegrable complex $\mathcal{PT}$ deformations which remove the trapping walls and in some cases create a $\mathbb{Z}_2$-graded ("supersymmetric") extension of the spectrum. Details are worked out for the cases of n=3 and n=4.

1. Introduction – some history

The Calogero model [1] has been the workhorse for integrable systems for 45 years. It is therefore mildly surprising that new aspects of it can still be uncovered. This talk describes the superintegrable spherical reduction of the rational A_{n-1} quantum Calogero model and some of its complex $\mathcal{PT}$ deformations. The emphasis is on the energy spectrum including degeneracy and eigenstates, and on the conserved charges and intertwiners, in particular for a coupling strength $g(g-1)$ with $g \in \mathbb{Z}$. We discuss all features in some detail for the cases n=3 (the hexagonal or Pöschl-Teller model) and for n=4 (the tetrahexahedric model), the latter being the simplest non-separable case. We close with a number of open issues.

It is worth recalling the relevant part (for this talk) of the Calogero model's long history:

- 1971 Calogero [2]:
  Solution of the one-dimensional N-body problem with . . . inversely quadratic pair potentials
- 1981 Olshanetsky & Perelomov [3, 4]:
  Classical integrable finite-dimensional systems related to Lie algebras (1983: quantum)
- 1983 Wojciechowski [5]:
  Superintegrability of the Calogero–Moser system
- 1989 Dunkl [6]:
  Differential-difference operators associated to reflection groups
- 1990 Veselov & Chalykh [7]:
  Commutative rings of partial differential operators and Lie algebras, supercompleteness
- 1991 Heckman [8]:
  Elementary construction for commuting charges and intertwiners (shift operators)
- 2003 M. Feigin [9]:
  Intertwining relations for the spherical parts of generalized Calogero operators

- 2008 A. Fring, M. Znojil [10, 11]:
  $\mathcal{PT}$-symmetric deformations of Calogero models
- 2008 Hakobyan, Nersessian, Yeghikyan [12]:
  The cuboctahedric Higgs oscillator from the rational Calogero model (classical)
- 2013 M. Feigin, Lechtenfeld, Polychronakos [13]:
  The quantum angular Calogero–Moser model (spectra, eigenstates)
- 2013 Correa, Lechtenfeld, Plyushchay [14]:
  Nonlinear supersymmetry in the quantum Calogero model
- 2014 M. Feigin, Hakobyan [15]:
  On the algebra of Dunkl angular momentum operators
- 2015 Correa, Lechtenfeld [16]:
  The tetrahexahedric angular Calogero model

This talk is based on [13] and [16] but contains yet unpublished material concerning $\mathcal{PT}$ deformations.

2. The angular (relative) Calogero model

The Hamiltonian for the A_{n-1} rational Calogero system reads

$$H \;=\; \sum_{\mu<\nu}^{n}\Bigl\{\tfrac{1}{2n}(p_\mu-p_\nu)^2 \;+\; \frac{g(g-1)}{(x^\mu-x^\nu)^2}\Bigr\}\ , \tag{1}$$

where x^μ and p_μ with $\mu=1,2,\ldots,n$ denote position and momentum in $\mathbb{R}$ for the μth particle, and g is a constant. Even though $H^{(1-g)}=H^{(g)}$, it is useful to admit any real value for g. The quantum theory demands that $[\,x^\mu\,,\,p_\nu\,]\;=\;\mathrm{i}\,\delta^\mu_\nu$. Note that the center-of-mass degree of freedom has already been removed, so we introduce *radial* variables in the remaining $(n{-}1)$-dimensional coordinate and momentum spaces,

$$\tfrac{1}{n}\sum_{\mu<\nu}(x^\mu-x^\nu)^2 \;=\; r^2 \qquad\text{and}\qquad \tfrac{1}{n}\sum_{\mu<\nu}(p_\mu-p_\nu)^2 \;=\; p_r^2+\tfrac{1}{r^2}L^2+\tfrac{(n-2)(n-4)}{4\,r^2}\ . \tag{2}$$

Furthermore, it is convenient to pass to $n{-}1$ *relative* coordinates and momenta,

$$r^2 \;=\; \sum_{i=1}^{n-1}(y^i)^2 \quad,\quad p_i\equiv p_{y^i} \quad,\quad L_{ij} \;=\; -\mathrm{i}(y^ip_j-y^jp_i) \quad,\quad L^2 \;=\; -\sum_{i<j}L_{ij}^2\ , \tag{3}$$

and to define the corresponding angular momenta L_{ij}. The $sl(2,\mathbb{R})$ conformal algebra is generated by

$$H \;=\; \tfrac12 p_r^2 \;+\; \tfrac{(n-2)(n-4)}{8\,r^2} \;+\; \tfrac{1}{r^2}H_\Omega \quad,\qquad D \;=\; \tfrac12(r\,p_r+p_r r) \quad,\qquad K \;=\; \tfrac12 r^2\ , \tag{4}$$

where all the interactions are hiding in the angular Calogero Hamiltonian H_Ω (or the $sl(2,\mathbb{R})$ Casimir C),

$$H_\Omega \;=\; \tfrac12 L^2+U(\vec\theta) \;=\; C-\tfrac18(n{-}1)(n{-}5) \qquad\text{with}\qquad C \;=\; K\,H+H\,K-\tfrac12 D^2\ . \tag{5}$$

It defines a model for a single particle moving on the $(n{-}2)$-sphere (angular coordinates $\vec\theta$) under the influence of a special potential

$$U(\vec\theta) \;=\; r^2\sum_{\mu<\nu}\frac{g(g-1)}{(x^\mu-x^\nu)^2} \;=\; r^2\sum_{\alpha\in\mathcal{R}_+}\frac{g(g-1)}{(\alpha\cdot y)^2} \;=\; \tfrac{g(g-1)}{2}\sum_{\alpha\in\mathcal{R}_+}\cos^{-2}\theta_\alpha \tag{6}$$

which is a superposition of "Higgs oscillators" $\cos^{-2}\theta$ [17, 18] centered at the directions of the $\frac12 n(n{-}1)$ positive roots α for A_{n-1}. It is singular at the $(n{-}3)$-dimensional intersections of the Weyl chamber walls with the unit $(n{-}2)$-sphere, so the particle is trapped in one of $n!$ spherical $(n{-}2)$-simplices tessalating the sphere. This potential breaks the SO$(n{-}1)$ invariance of the free particle on S^{n-2} to the permutation group S_n, which is the Weyl group of A_{n-1}.

From now on, we pass to the position representation,

$$p_i \ \mapsto\ -\mathrm{i}\partial_i \qquad \Longrightarrow \qquad p_r \ \mapsto\ -\mathrm{i}\big(\partial_r + \tfrac{n-2}{2\,r}\big)\ , \tag{7}$$

$$\begin{aligned} H \ &\mapsto\ -\tfrac12\big(\partial_r^2 + \tfrac{n-2}{r}\partial_r\big) \ +\ \tfrac{1}{r^2}H_\Omega \ =\ w^{-1}\Big[-\tfrac12\big(\partial_r^2 - \tfrac{(n-2)(n-4)}{4\,r^2}\big)\ +\ \tfrac{1}{r^2}H_\Omega\Big]\,w\ , \\ H_\Omega \mapsto\ &-\tfrac12\sum_{i<j}\big(y^i\partial_j - y^j\partial_i\big)^2 \ +\ r^2\sum_{\alpha\in\mathcal{R}_+}\frac{g(g-1)}{(\alpha\cdot y)^2} \qquad \text{with}\quad w = r^{\frac{n-2}{2}}\ . \end{aligned} \tag{8}$$

The energy spectrum and radial eigenfunctions of the full Calogero model are well known,

$$\begin{aligned} H\,\Psi_{E,q} \ &=\ E\,\Psi_{E,q} \qquad \text{with} \qquad E\in\mathbb{R}_{\ge 0}\ , \\ \Psi_{E,q}(r,\vec\theta) \ &=\ r^{-\frac{n-3}{2}}\,J_{q+(n-3)/2}(\sqrt{2E}\,r)\,v_q(\vec\theta)\ , \end{aligned} \tag{9}$$

but the angular part of these wave functions is part of the spectral problem for the angular model,

$$\begin{aligned} H_\Omega\,v_q \ &=\ \varepsilon_q\,v_q \qquad \text{with} \qquad \varepsilon_q \ =\ \tfrac12\,q\,(q+n-3) \qquad \text{and} \\ q \ &=\ \tfrac12 n(n{-}1)\,g + \ell \qquad \text{where} \qquad \ell \ =\ 3\ell_3 + 4\ell_4 + \ldots + n\ell_n \ \in\mathbb{N}_0\ . \end{aligned} \tag{10}$$

Naturally, the spectrum $\{\varepsilon_q\}$ is discrete, for a fixed g governed by an integer-valued angular momentum quantum number ℓ. For a free particle on S^{n-2}, the degeneracy at level ℓ is $\binom{\ell+n-2}{n-2} - \binom{\ell+n-4}{n-2}$. With the potential U turned on, the $SO(n{-}1)$ representation content has to be decomposed with respect to the S_n subgroup. The full A_{n-1} Calogero system describes n *identical* particles, whose wave function picks up just a phase factor of $\mathrm{e}^{\mathrm{i}\pi g}$ under a particle exchange, so – apart from this – only S_n singlet representations are kept. Here we keep this heritage for the angular model and, therefore, admit only permutation invariant states. This greatly reduces the degeneracy at level ℓ to the number of all $(n{-}2)$-tuples $(\ell_3,\ell_4,\ldots,\ell_n)$ subject to (10). It can be expressed as

$$\deg_n(\varepsilon_q) \ =\ p_n(\ell) - p_n(\ell{-}1) - p_n(\ell{-}2) + p_n(\ell{-}3)\ , \tag{11}$$

where $p_n(\ell)$ is the number of partitions of ℓ into integers not larger than n, which can be generated via

$$p_n(t) \ :=\ \sum_{\ell=0}^{\infty} p_n(\ell)\,t^\ell \ =\ \prod_{m=1}^{n}\big(1-t^m\big)^{-1}\ . \tag{12}$$

For small values of n, there are explicit formulæ [19]:

$$\begin{aligned} \deg_3(\ell) \ &=\ \begin{cases} 0 & \text{for} \quad \ell = 1,2 \bmod 3 \\ 1 & \text{for} \quad \ell = 0 \bmod 3 \end{cases}\ , \\ \deg_4(\ell) \ &=\ \Big\lfloor\frac{\ell}{12}\Big\rfloor \ +\ \begin{cases} 0 & \text{for} \quad \ell = 1,2,5 \bmod 12 \\ 1 & \text{for} \quad \ell = \text{else} \bmod 12 \end{cases}\ , \\ \deg_5(\ell) \ &=\ \Big\lfloor\frac{6\ell^2+72\ell-89}{720}\Big\rfloor \ +\ \begin{cases} 0 & \text{for} \quad \ell = 2,22,26,46 \bmod 60 \\ 2 & \text{for} \quad \ell = 0,48 \bmod 60 \\ 1 & \text{for} \quad \ell = \text{else} \bmod 60 \end{cases}\ . \end{aligned} \tag{13}$$

The main task lies in the construction of the angular eigenfunctions, which are given by

$$v_q(\vec\theta)\;\equiv\;v_\ell^{(g)}(\vec\theta)\;\sim\;r^{n-3+q}\Big(\prod_{\mu=3}^{n}\sigma_\mu\big(\{\mathcal{D}_i\}\big)^{\ell_\mu}\Big)\,\Delta^g\,r^{3-n-n(n-1)g}\;. \tag{14}$$

The key ingredients are the Vandermonde factor and the Dunkl operators,

$$\Delta\;=\;\prod_{\alpha\in\mathcal{R}_+}\alpha\cdot y \qquad\text{and}\qquad \mathcal{D}_i\;=\;\partial_i\;-\;g\sum_{\alpha\in\mathcal{R}_+}\frac{\alpha_i}{\alpha\cdot y}\,s_\alpha\;, \tag{15}$$

where s_α denotes the reflection about the hyperplane orthogonal to the root α. The latter are combined into the elementary Weyl-symmetric polynomials $\sigma_\mu(y)$ of degree μ, with $\sigma_2 = \sum_i (y^i)^2$ being absent because we freeze the radial excitations. The vanishing locus of the Vandermonde Δ produces singularities for $g < 0$, rendering these states non-normalizable. Splitting off a universal factor, the angular wave functions become Dunkl-deformed Weyl-symmetric harmonic polynomials of degree ℓ,,

$$v_\ell^{(g)}(\vec\theta)\;=\;r^{-q}\,\Delta^g\,h_\ell^{(g)}\qquad\text{with}\qquad H\left(\Delta^g\,h_\ell^{(g)}\right)\;=\;0\;. \tag{16}$$

Just like 'Dunkl-deforming' the momenta, i.e. $\partial_i \to \mathcal{D}_i$, generates the full Calogero model from the free particle, the angular submodel can be obtained from free motion on S^{n-2} by an analogous deformation of the angular momenta,

$$L_{ij}\;\mapsto\;-(y^i\partial_j - y^j\partial_i)\qquad\Longrightarrow\qquad \mathcal{L}_{ij}\;=\;-(y^i\mathcal{D}_j - y^j\mathcal{D}_i)\;. \tag{17}$$

In both cases, their square yields a differential-difference operator ('pre-Hamiltonian') whose restriction to Weyl symmetric functions is basically the Hamiltonian,

$$\mathcal{H}\;=\;-\tfrac12\sum_i\mathcal{D}_i^2 \qquad\text{and}\qquad \mathcal{H}_\Omega\;=\;-\tfrac12\sum_{i<j}\mathcal{L}_{ij}^2\;+\;\tfrac12\,g{\textstyle\sum_\alpha} s_\alpha\,(g{\textstyle\sum_\alpha} s_\alpha + n{-}3)\;, \tag{18}$$

$$H\;=\;\mathrm{res}(\mathcal{H}) \qquad\text{and}\qquad H_\Omega\;=\;\mathrm{res}(\mathcal{H}_\Omega)\;=\;\tfrac12\mathrm{res}\big(\mathcal{L}^2\big)\;+\;\varepsilon_q(\ell{=}0)\;. \tag{19}$$

In fact, any Weyl-invariant polynomial in the $\mathcal{L}_{ij}$ of some degree t provides a conserved quantity,

$$\mathcal{C}_t(\mathcal{L}_{ij})\quad\text{Weyl-invariant}\qquad\Longrightarrow\qquad C_t=\mathrm{res}(\mathcal{C}_t)\quad\text{commutes with } H_\Omega\;. \tag{20}$$

In contrast to the full model, these are not in involution and hence do not constitute Liouville charges. The reason is that, while $[\mathcal{D}_i,\mathcal{D}_j]=0$, angular momenta (deformed or not) do not commute, rather [15]

$$[\mathcal{L}_{ij},\mathcal{L}_{k\ell}]\;=\;\mathcal{L}_{i\ell}\mathcal{S}_{jk}-\mathcal{L}_{ik}\mathcal{S}_{j\ell}-\mathcal{L}_{j\ell}\mathcal{S}_{ik}+\mathcal{L}_{jk}\mathcal{S}_{i\ell} \tag{21}$$

$$\text{with}\qquad \mathcal{S}_{ij}\;=\;\begin{cases}-g\,s_{ij} & \text{for}\quad i\neq j\\ 1+g\sum_{k(\neq i)}s_{ik} & \text{for}\quad i=j\end{cases}\;, \tag{22}$$

$$[\mathcal{S}_{ij},\mathcal{L}_{k\ell}]=0\quad,\qquad \{\mathcal{S}_{ij},\mathcal{L}_{ij}\}=0\quad,\qquad \mathcal{S}_{ij}\mathcal{L}_{ik}\;=\;\mathcal{L}_{jk}\mathcal{S}_{ij}\;. \tag{23}$$

This algebra is a 'Dunkl deformation' of $so(n{-}1)$, with H_Ω being the Casimir invariant. More technically, it is a subalgebra of the rational Cherednik algebra generated by $\{\mathcal{D}_i, y^j\}$ and the Weyl reflections.

It is equally interesting to consider Weyl-*anti*invariant polynomials $\mathcal{M}_s(\mathcal{L}_{ij})$ of some degree s, since

$$\mathcal{M}_s(\mathcal{L}_{ij})\quad\text{Weyl-antiinvariant}\qquad\Longrightarrow\qquad M_s=\mathrm{res}(\mathcal{M}_s)\quad\text{is an intertwiner for } H \text{ and } H_\Omega\;. \tag{24}$$

Indeed, since $[\mathcal{L}_{ij},\mathcal{H}]=0$ we have

$$\begin{aligned} [\mathcal{M}_s,\mathcal{H}]=0 \qquad &\Longrightarrow \qquad M_s^{(g)}H^{(g)} \;=\; H^{(-g)}M_s^{(g)} \;=\; H^{(g+1)}M_s^{(g)} \\ &\text{and} \qquad M_s^{(g)}\Psi_{E,q}^{(g)} \;\sim\; \Psi_{E,q}^{(g+1)}\;, \end{aligned} \tag{25}$$

$$\begin{aligned} [\mathcal{M}_s,\mathcal{H}_\Omega]=0 \qquad &\Longrightarrow \qquad M_s^{(g)}H_\Omega^{(g)} \;=\; H_\Omega^{(-g)}M_s^{(g)} \;=\; H_\Omega^{(g+1)}M_s^{(g)} \\ &\text{and} \qquad M_s^{(g)}v_\ell^{(g)} \;\sim\; v_{\ell-n(n-1)/2}^{(g+1)}\;. \end{aligned} \tag{26}$$

This allows us to connect the states for integer increments of the coupling g and allows one to generate the wave functions for $g\in\mathbb{N}$ directly from the free ones (g=0 or g=1). Therefore, we restrict ourselves to $g\in\mathbb{Z}$ from now on (it will prove beneficial to formally admit $g<0$), keeping only Bose (g even) and Fermi (g odd) statistics. The intertwiners M_s clearly exchange bosons with fermions. Since $H_\Omega^{(g)}=H_\Omega^{(1-g)}$, at each $g\in\mathbb{N}$ the spectrum of $H_\Omega^{(g)}$ formally contains a bosonic and a fermionic tower of states, but one of them is unphysical because it is constructed with a non-positive g-value and thus not normalizable. The only exception occurs for $g=1$ because the $g=0$ states are non-singular, but this is obvious since for vanishing potential the only restriction arises from Weyl invariance.

3. $\mathcal{PT}$ deformation

It has been known for a long time that hermiticity is not an esential feature of a Hamiltonian for its spectrum to be real. For instance, it suffices that the Hamiltonian commutes with an antilinear involution (called $\mathcal{PT}$) which also leaves the eigenfunctions invariant ("unbroken $\mathcal{PT}$ symmetry") [20]. Such a non-hermitian Hamiltonian is related to a hermitian one by a (non-unitary) similarity transformation, which may be impossibly complicated. Often, however, there exists a family H_ϵ of non-hermitian $\mathcal{PT}$-invariant Hamiltonians representing a smooth deformation of a hermitian H_0. In this case we speak of a "$\mathcal{PT}$ deformation", with the parameter ϵ measuring the deviation from hermiticity. For rational Calogero models, a particularly nice set of $\mathcal{PT}$ deformations can be generated by a specific *complex* orthogonal deformation of the coordinates x^μ (or y^i) in the expression for the Hamiltonian. If such a $\mathcal{PT}$ deformation is in accordance with the Weyl symmetry of the system, its integrability will be preserved.

For our case of the rational A_{n-1} Calogero model, the following SU(1,1)×SU(1,1)×$\cdots\subset$ SO($n{-}1,\mathbb{C}$) coordinate deformations $y^i\mapsto y^i_\epsilon$ with $\epsilon_a\in\mathbb{R}$ are admissible:

$$\begin{pmatrix} y^1\\ y^2\\ y^3\\ y^4\\ \vdots \end{pmatrix} \quad\longmapsto\quad \begin{pmatrix} \cosh\epsilon_1 & -\mathrm{i}\sinh\epsilon_1 & 0 & 0 & \cdots\\ \mathrm{i}\sinh\epsilon_1 & \cosh\epsilon_1 & 0 & 0 & \cdots\\ 0 & 0 & \cosh\epsilon_2 & -\mathrm{i}\sinh\epsilon_2 & \cdots\\ 0 & 0 & \mathrm{i}\sinh\epsilon_2 & \cosh\epsilon_2 & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} y^1\\ y^2\\ y^3\\ y^4\\ \vdots \end{pmatrix}\;. \tag{27}$$

This is equivalent to a complex deformations of angles, $\phi_a\mapsto\phi_a+\mathrm{i}\epsilon_a$, which parametrize mutually orthogonal planes. The complex coordinate deformation induces a non-unitary similarity transformation in the Hilbert space, sending

$$L^2 \;\mapsto\; L^2 \qquad\text{and}\qquad U(\vec\theta)\;\mapsto\; U(\vec\theta_\epsilon)=:U_\epsilon(\vec\theta) \tag{28}$$

thus leaving the energy spectrum unchanged. Moreover, the singular Higgs oscillator potential gets regularized,

$$U_\epsilon(\vec\theta) \;=\; r^2\sum_{\mu<\nu}\frac{g(g{-}1)}{(x^\mu_\epsilon{-}x^\nu_\epsilon)^2} \;=\; r^2\sum_{\alpha\in\mathcal{R}_+}\frac{g(g{-}1)}{(\alpha\cdot y_\epsilon)^2} \;=\; \tfrac{g(g-1)}{2}\sum_{\alpha\in\mathcal{R}_+}\cos^{-2}\theta_\alpha(\epsilon)\;, \tag{29}$$

and the Weyl-chamber wall singularities disappear due to

$$\frac{1}{\cos^2(\theta_\alpha+\mathrm{i}\eta_\alpha(\vec{\theta},\epsilon))} \;=\; \frac{\cosh^2\eta_\alpha\cos^2\theta_\alpha-\sinh^2\eta_\alpha\sin^2\theta_\alpha\;+\;\frac{\mathrm{i}}{2}\sinh 2\eta_\alpha\sin 2\theta_\alpha}{(\cosh^2\eta_\alpha\cos^2\theta_\alpha+\sinh^2\eta_\alpha\sin^2\theta_\alpha)^2}\;, \tag{30}$$

where the azimuthal angle for the Higgs oscillator pertaining to the root α is $\theta_\alpha(\epsilon)=\theta_\alpha+\mathrm{i}\eta_\alpha(\vec{\theta})$. The $\mathcal{PT}$ deformation frees the particle up to escape from its Weyl chamber and move anywhere on the $(n{-}2)$-sphere. Singularities of codimension two remain at the vanishing loci of η_α. We shall see later that such a change of boundary conditions for the spectral problem may bring back previously unphysical singular wave functions to the physical sprectrum, essentially doubling the degeneracy of the energy levels.

4. Warmup: the hexagonal or Pöschl-Teller model

The first interesting case is $n{=}3$, leading to a one-dimensional integrable system known as the Pöschl-Teller model for a particle on a circle. The Jacobi relative coordinates on the two-plane perpendicular to the center-of-mass X are

$$\begin{aligned} x^1 &= X+\tfrac{1}{\sqrt{2}}\,y^1+\tfrac{1}{\sqrt{6}}\,y^2\;, & \partial_{x^1} &= \tfrac13\partial_X+\tfrac{1}{\sqrt{2}}\,\partial_{y^1}+\tfrac{1}{\sqrt{6}}\,\partial_{y^2}\;,\\ x^2 &= X-\tfrac{1}{\sqrt{2}}\,y^1+\tfrac{1}{\sqrt{6}}\,y^2\;, & \partial_{x^2} &= \tfrac13\partial_X-\tfrac{1}{\sqrt{2}}\,\partial_{y^1}+\tfrac{1}{\sqrt{6}}\,\partial_{y^2}\;,\\ x^3 &= X-\tfrac{2}{\sqrt{6}}\,y^2\;, & \partial_{x^3} &= \tfrac13\partial_X-\tfrac{2}{\sqrt{6}}\,\partial_{y^2}\;, \end{aligned} \tag{31}$$

and we also use polar and complex coordinates in that plane,

$$y^1 \;=\; r\,\cos\phi \quad\text{and}\quad y^2\;=\;r\,\sin\phi \qquad\Longrightarrow\qquad w\;:=\;y^1+\mathrm{i}y^2\;=\;r\,\mathrm{e}^{\mathrm{i}\phi}\;. \tag{32}$$

The angular Hamiltonian reads

$$H_\Omega \;=\; \tfrac12\bigl(w\partial_w-\bar{w}\partial_{\bar{w}}\bigr)^2\;+\;g(g{-}1)\frac{18\,(w\bar{w})^3}{(w^3+\bar{w}^3)^2}\;, \tag{33}$$

with the potential coming from

$$U(\phi)\;=\;\tfrac{g(g-1)}{2}\sum_{k=0,1,2}\cos^{-2}(\phi{+}k\tfrac{2\pi}{3})\;=\;\tfrac92 g(g{-}1)\,\cos^{-2}(3\phi)\;. \tag{34}$$

Specializing our general formulæ for the energy eigenvalues and -functions from the previous section and denoting by (s_0,s_+,s_-) the three Coxeter reflections, we get $\ell=3\,\ell_3$ and

$$\varepsilon_q\;=\;\tfrac12 q^2 \qquad\text{with}\qquad q\;=\;3g+\ell\;=\;3(g+\ell_3) \qquad\text{and}\qquad \deg(\varepsilon_q)=1\;, \tag{35}$$

$$\Psi_{E,q}(r,\phi)\;=\;J_q(\sqrt{2E}\,r)\,v_q(\phi)\;, \tag{36}$$

$$v_q(\phi)\;\equiv\;v_\ell^{(g)}(\phi)\;\sim\;r^q\left(\mathcal{D}_w^3-\mathcal{D}_{\bar{w}}^3\right)^{\ell_3}\Delta^g\,r^{-6g}\;=\;r^{-q}\,\Delta^g\,h_\ell^{(g)}(w^3,\bar{w}^3)\;, \tag{37}$$

$$\Delta\;\sim\;w^3+\bar{w}^3\;\sim\;r^3\,\cos(3\phi) \qquad\text{vanishing at } \phi=\pm\tfrac{\pi}{6},\pm\tfrac{\pi}{2},\pm\tfrac{5\pi}{6}\;, \tag{38}$$

$$\mathcal{D}_w\;=\;\partial_w\;-\;g\left\{\frac{1}{w+\bar{w}}\,s_0+\frac{\rho}{\rho w+\bar{\rho}\bar{w}}\,s_+ +\frac{\bar{\rho}}{\bar{\rho}w+\rho\bar{w}}\,s_-\right\} \quad\text{with}\quad \rho=\mathrm{e}^{2\pi\mathrm{i}/3}\;, \tag{39}$$

$$h_\ell^{(g)}(w^3,\bar{w}^3)\;=\;\sum_{k=0}^{\ell_3}(-1)^k\,\tfrac{\Gamma(1+\ell_3)\,\Gamma(g+k)\,\Gamma(g+\ell_3-k)}{\Gamma(2g+\ell_3)\,\Gamma(g)\,\Gamma(1+k)\,\Gamma(1+\ell_3-k)}\;w^{\ell-3k}\bar{w}^{3k}\;, \tag{40}$$

where the final equation is the result of a nontrivial computation. We display the homogeneous polynomials $h_\ell^{(g)}$ defining the wave functions (un-normalized, see above) for $g=0,1,2$ and the lowest five energy levels, using the notation $(m\,\bar{m})\;:=\;w^{3m}\bar{w}^{3\bar{m}}\;=\;(y_1+\mathrm{i}y_2)^{3m}(y_1-\mathrm{i}y_2)^{3\bar{m}}$,

ℓ	$h_\ell^{(0)}$	$h_\ell^{(1)}$	$h_\ell^{(2)}$	$\cdots$
0	(00)	(00)	(00)	$\cdots$
3	$(10)-(01)$	$(10)-(01)$	$(10)-(01)$	$\cdots$
6	$(20)+(02)$	$(20)-(11)+(02)$	$3(20)-4(11)+3(02)$	$\cdots$
9	$(30)-(03)$	$(30)-(21)+(12)-(03)$	$4(30)-6(21)+6(12)-4(03)$	$\cdots$
12	$(40)+(04)$	$(40)-(31)+(22)-(13)+(04)$	$5(40)-8(31)+9(22)-8(13)+5(04)$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

These states can also be obtained by applying the angular intertwiner $M_1 = \text{res}(\mathcal{M}_1)$,

$$\begin{aligned} \mathcal{M}_1^{(g)} &\sim \mathrm{i}\big(w\mathcal{D}_w - \bar{w}\mathcal{D}_{\bar{w}}\big) \\ &\sim \mathrm{i}\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big) \;-\; \mathrm{i}\,g\left\{\frac{w-\bar{w}}{w+\bar{w}}\,s_0 + \frac{\rho w-\bar{\rho}\bar{w}}{\rho w+\bar{\rho}\bar{w}}\,s_+ + \frac{\bar{\rho}w-\rho\bar{w}}{\bar{\rho}w+\rho\bar{w}}\,s_-\right\}, \end{aligned} \tag{41}$$

$$M_1^{(g)} \sim \mathrm{i}\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big) \;-\; 3\mathrm{i}\,g\,\frac{w^3-\bar{w}^3}{w^3+\bar{w}^3} \;=\; \mathrm{i}\,\Delta^g\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big)\Delta^{-g} \;=\; \partial_\phi + 3\,g\,\tan 3\phi\,. \tag{42}$$

Since $M_1^{(g)} v_\ell^{(g)} \sim v_{\ell-3}^{(g+1)}$, we learn that

$$h_\ell^{(g+1)} \sim \mathrm{i}\,\Delta^{-1}\big(w\partial_w - \bar{w}\partial_{\bar{w}}\big)\,h_{\ell+3}^{(g)}\,, \tag{43}$$

which may be iterated to arrive at (40). This intertwiner does not yield additional conserved charges, because

$$\big(M_1^\dagger M_1\big)^{(g)} \;=\; -2\,H_\Omega^{(g)} + 9\,g^2 \;=\; -\text{res}(\mathcal{L}^2) \;=\; -C_2^{(g)}\,. \tag{44}$$

What are the possible $\mathcal{PT}$ deformations of the Pöschl-Teller system? Without loss of generality, we may fix $\mathcal{P}$ to be the Coxeter reflection about the line perpendicular to $\alpha \sim \binom{1}{0}$, hence

$$\mathcal{P}:\; s_0 \;=\; \left(\begin{smallmatrix}-1 & 0\\ 0 & 1\end{smallmatrix}\right) \qquad \text{and} \qquad \mathcal{T}:\; \text{complex conjugation} \quad (\text{but } w\mapsto w \text{ and } \bar{w}\mapsto\bar{w}\,!)\,. \tag{45}$$

The most general $\mathcal{PT}$ deformation compatible with this involution is an SU(1,1) $\subset$ SO(2,$\mathbb{C}$) boost,

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \;\longmapsto\; \begin{pmatrix} \cosh\epsilon & -\mathrm{i}\sinh\epsilon \\ \mathrm{i}\sinh\epsilon & \cosh\epsilon \end{pmatrix}\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} \;=\; r\begin{pmatrix} \cos(\phi{+}\mathrm{i}\epsilon) \\ \sin(\phi{+}\mathrm{i}\epsilon) \end{pmatrix}, \tag{46}$$

which in the original particle coordinates acquires the less transparent form

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix} \;\longmapsto\; \frac{1}{3}\begin{pmatrix} 1{+}2\cosh\epsilon & 1{-}\cosh\epsilon{-}\mathrm{i}\sqrt{3}\sinh\epsilon & 1{-}\cosh\epsilon{+}\mathrm{i}\sqrt{3}\sinh\epsilon \\ 1{-}\cosh\epsilon{+}\mathrm{i}\sqrt{3}\sinh\epsilon & 1{+}2\cosh\epsilon & 1{-}\cosh\epsilon{-}\mathrm{i}\sqrt{3}\sinh\epsilon \\ 1{-}\cosh\epsilon{-}\mathrm{i}\sqrt{3}\sinh\epsilon & 1{-}\cosh\epsilon{+}\mathrm{i}\sqrt{3}\sinh\epsilon & 1{+}2\cosh\epsilon \end{pmatrix}\begin{pmatrix} x^1 \\ x^2 \\ x^3 \end{pmatrix}. \tag{47}$$

In polar and complex coordinates, it is simply

$$\phi \;\mapsto\; \phi + \mathrm{i}\epsilon \qquad \Longleftrightarrow \qquad (w,\bar{w}) \;\mapsto\; (\mathrm{e}^{-\epsilon}w, \mathrm{e}^{\epsilon}\bar{w})\,, \tag{48}$$

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leading to the complex and completely regular potential

$$U_\epsilon(\phi) \;=\; U(\phi{+}\mathrm{i}\epsilon) \;=\; 9\,g(g{-}1)\,\frac{(1+\cosh 6\epsilon\cos 6\phi)+2\mathrm{i}\sinh 6\epsilon\sin 6\phi}{(\cosh 6\epsilon+\cos 6\phi)^2}\;. \tag{49}$$

The energy levels are independent of ϵ, but previously singular states for $g{<}0$ now have become normalizable and thus appear in the physical spectrum (but $q<0$ yields nothing new!),

$$\varepsilon_q \;=\; \tfrac12 q^2 \qquad\text{with}\qquad q \;=\; 3g+\ell \;=\; 3(g+\ell_3) \qquad\text{and}\qquad \ell_3\geq \max(-g,0)\;, \tag{50}$$

$$\Delta_\epsilon \;\sim\; \mathrm{e}^{-3\epsilon}w^3+\mathrm{e}^{3\epsilon}\bar{w}^3 \;\sim\; r^3\bigl(\cosh(3\epsilon)\cos(3\phi)-\mathrm{i}\sinh(3\epsilon)\sin(3\phi)\bigr)\;\neq 0\;, \tag{51}$$

$$h_\ell^{\epsilon(g)}(w^3,\bar{w}^3) \;=\; \sum_{k=0}^{\ell_3}(-1)^k\,\tfrac{\Gamma(1+\ell_3)\,\Gamma(g+k)\,\Gamma(g+\ell_3-k)}{\Gamma(2g+\ell_3)\,\Gamma(g)\,\Gamma(1+k)\,\Gamma(1+\ell_3-k)}\;(\mathrm{e}^{-\epsilon}w)^{\ell-3k}(\mathrm{e}^{\epsilon}\bar{w})^{3k}\;, \tag{52}$$

where the latter expression also holds for $g<0$ with proper $\frac{\infty}{\infty}$ regularization. Here is a list of the low-lying deformed polynomials including the $g{=}{-1}$ tower as a negative-g example, with the adapted notation $(m\,\bar{m}) \;:=\; \mathrm{e}^{-3(m-\bar{m})\epsilon}\,w^{3m}\bar{w}^{3\bar{m}}$:

ℓ	$h_\ell^{\epsilon(-1)}$	$h_\ell^{\epsilon(0)}$	$h_\ell^{\epsilon(1)}$	$h_\ell^{\epsilon(2)}$
0		(00)	(00)	(00)
3	$(10)-(01)$	$(10)-(01)$	$(10)-(01)$	$(10)-(01)$
6	(00)	$(20)+(02)$	$(20)-(11)+(02)$	$3(20)-4(11)+3(02)$
9	$(30)+3(21)-3(12)-(03)$	$(30)-(03)$	$(30)-(21)+(12)-(03)$	$4(30)-6(21)+6(12)-4(03)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Since the Hamiltonian depends only on the combination $g(g{-}1)$, we should restrict to $g\geq 1$ but join to the energy eigenstates of any positive g-value the new tower of states at coupling $1{-}g$. This enhances the level degeneracy to

$$\deg(\varepsilon_q) \;\stackrel{g\geq 0}{=}\; \begin{cases} 1 & \text{for}\;\; q<3g \\ 2 & \text{for}\;\; q\geq 3g \end{cases} \tag{53}$$

and yields a totally Weyl symmetric and a totally Weyl antisymmetric state at each but the lowest energy levels. These states are related by a new conserved charge,

$$Q^{(g)} \;=\; M_1^{(g-1)}M_1^{(g-2)}\cdots M_1^{(1-g)} \quad\Longrightarrow\quad Q^{(g)}H_\Omega^{(g)} \;=\; Q^{(g)}H_\Omega^{(1-g)} \;=\; H_\Omega^{(g)}Q^{(g)}\;. \tag{54}$$

It is algebraically independent of the Hamiltonian but squares to

$$\bigl(Q_\epsilon^{(g)}\bigr)^2 \;=\; \prod_{j=1-g}^{g-1}\bigl(-2\,H_\epsilon^{(g)}+9j^2\bigr)\;, \tag{55}$$

which provides a kind of “nonlinear supersymmetry” [21].

5. Tetrahexahedric model: the spectrum

The A_2 model is atypical since it is completely separable. It is much more exciting to study the next more complicated system, which is based on $A_3 \simeq D_3$ and has been termed the "tetrahexahedric model", due to the A_3 Coxeter system and its tetrahedral finite reflection group. For convenience, we rotate the Jacobi relative coordinates adapted to the A_3 root system to a D_3 basis, called Walsh-Hadamard coordinates:

$$\begin{aligned} x^1 &= X + \tfrac12(+x+y+z) \quad , \quad & \partial_{x^1} &= \tfrac14\partial_X + \tfrac12(+\partial_x+\partial_y+\partial_z)\ , \\ x^2 &= X + \tfrac12(+x-y-z) \quad , \quad & \partial_{x^2} &= \tfrac14\partial_X + \tfrac12(+\partial_x-\partial_y-\partial_z)\ , \\ x^3 &= X + \tfrac12(-x+y-z) \quad , \quad & \partial_{x^3} &= \tfrac14\partial_X + \tfrac12(-\partial_x+\partial_y-\partial_z)\ , \\ x^4 &= X + \tfrac12(-x-y+z) \quad , \quad & \partial_{x^4} &= \tfrac14\partial_X + \tfrac12(-\partial_x-\partial_y+\partial_z)\ , \end{aligned} \tag{56}$$

and introduce polar coordinates

$$x = r\,\sin\theta\,\cos\phi \quad , \qquad y = r\,\sin\theta\,\sin\phi \quad , \qquad z = r\,\cos\theta\ . \tag{57}$$

The angular momenta

$$L_x = -(y\partial_z - z\partial_y)\ , \qquad L_y = -(z\partial_x - x\partial_z)\ , \qquad L_z = -(x\partial_y - y\partial_x) \tag{58}$$

are the building blocks of the S^2 Laplacian

$$L^2 \ =\ -(L_x^2+L_y^2+L_z^2) \ =\ -\tfrac{1}{\sin\theta}\partial_\theta \sin\theta\,\partial_\theta - \tfrac{1}{\sin^2\theta}\partial_\phi^2\ , \tag{59}$$

and the full and angular Hamiltonians read

$$H \ =\ -\tfrac12(\partial_x^2+\partial_y^2+\partial_z^2) \ +\ 2\,g(g-1)\Big(\frac{x^2+y^2}{(x^2-y^2)^2} + \frac{y^2+z^2}{(y^2-z^2)^2} + \frac{z^2+x^2}{(z^2-x^2)^2}\Big) \tag{60}$$

and $H_\Omega = \frac12 L^2 + U(\vec\theta)$, respectively, with

$$U(\theta,\phi) \ =\ 2g(g-1)\Big\{\frac{1}{\sin^2\theta\,\cos^2 2\phi} + \frac{\cos^2\theta+\sin^2\theta\cos^2\phi}{(\cos^2\theta-\sin^2\theta\cos^2\phi)^2} + \frac{\cos^2\theta+\sin^2\theta\sin^2\phi}{(\cos^2\theta-\sin^2\theta\sin^2\phi)^2}\Big\}\ , \tag{61}$$

which blows up on 6 great circles of S^2. The potential is invariant under the S_4 Weyl group action, whose elementary reflections are realized as

$$\begin{aligned} s_{x+y}:\ (x,y,z) &\mapsto (-y,-x,+z) \quad , \qquad & s_{x-y}:\ (x,y,z) &\mapsto (+y,+x,+z)\ , \\ s_{y+z}:\ (x,y,z) &\mapsto (+x,-z,-y) \quad , \qquad & s_{y-z}:\ (x,y,z) &\mapsto (+x,+z,+y)\ , \\ s_{z+x}:\ (x,y,z) &\mapsto (-z,+y,-x) \quad , \qquad & s_{z-x}:\ (x,y,z) &\mapsto (+z,+y,+x)\ . \end{aligned} \tag{62}$$

Specializing our general formulæ to $n{=}4$ with $\sigma_3 = x\,y\,z$ and $\sigma_4 = x^4+y^4+z^4$, we get

$$\varepsilon_q \ =\ \tfrac12 q\,(q{+}1) \qquad \text{with} \qquad q \ =\ 6g+\ell \ =\ 6g + 3\ell_3{+}4\ell_4\ , \tag{63}$$

$$\Psi_{E,q}(r,\theta,\phi) \ =\ j_q(\sqrt{2E}\,r)\,v_q(\theta,\phi)\ , \tag{64}$$

$$v_\ell^{(g)}(\theta,\phi) \ \sim\ r^{q+1}\,\big(\mathcal{D}_x\mathcal{D}_y\mathcal{D}_z\big)^{\ell_3}\,\big(\mathcal{D}_x^4{+}\mathcal{D}_y^4{+}\mathcal{D}_z^4\big)^{\ell_4}\,\Delta^g\,r^{1-12g} \ =\ r^{-q}\,\Delta^g\,h_\ell^{(g)}(x,y,z)\ , \tag{65}$$

$$\Delta \ \sim\ (x^2-y^2)(y^2-z^2)(x^2-z^2)\ , \tag{66}$$

with the linear Dunkl operators

$$\begin{aligned}
\mathcal{D}_x &= \partial_x - \frac{g}{x+y}\,s_{x+y} - \frac{g}{x-y}\,s_{x-y} - \frac{g}{z+x}\,s_{x+z} - \frac{g}{x-z}\,s_{z-x}\,, \\
\mathcal{D}_y &= \partial_y - \frac{g}{y+x}\,s_{x+y} - \frac{g}{y-x}\,s_{x-y} - \frac{g}{y+z}\,s_{y+z} - \frac{g}{y-z}\,s_{y-z}\,, \\
\mathcal{D}_z &= \partial_z - \frac{g}{z+x}\,s_{z+x} - \frac{g}{z-x}\,s_{z-x} - \frac{g}{z+y}\,s_{y+z} - \frac{g}{z-y}\,s_{y-z}\,.
\end{aligned} \tag{67}$$

With the short-hand notation $\{rst\} := x^r y^s z^t + x^r y^t z^s + x^s y^t z^r + x^s y^r z^t + x^t y^r z^s + x^t y^s z^r$, we display the eigenstate polynomials for ten lowest energy levels at g=0 and g=1,

ℓ	ℓ_3	ℓ_4	$h_\ell^{(0)}$
0	0	0	$\{000\}$
3	1	0	$\{111\}$
4	0	1	$\{400\} - 3\{220\}$
6	2	0	$\{600\} - 15\{420\} + 30\{222\}$
7	1	1	$3\{511\} - 5\{331\}$
8	0	2	$\{800\} - 28\{620\} + 35\{440\}$
9	3	0	$9\{711\} - 63\{531\} + 70\{333\}$
10	2	1	$\{1000\} - 45\{820\} + 42\{640\} + 504\{622\} - 630\{442\}$
11	1	2	$5\{911\} - 60\{731\} + 63\{551\}$
12	4	0	$36\{1200\} - 2376\{1020\} + 2445\{840\} + 46125\{822\} + 4893\{660\} - 215250\{642\} + 179375\{444\}$
12	0	3	$101\{1200\} - 6666\{1020\} + 47100\{840\} + 8685\{822\} - 42609\{660\} - 40530\{642\} + 33775\{444\}$

ℓ	ℓ_3	ℓ_4	$h_\ell^{(1)}$
0	0	0	$\{000\}$
3	1	0	$\{111\}$
4	0	1	$3\{400\} - 11\{220\}$
6	2	0	$3\{600\} - 39\{420\} + 196\{222\}$
7	1	1	$5\{511\} - 13\{331\}$
8	0	2	$\{800\} - 20\{620\} + 23\{440\} + 12\{422\}$
9	3	0	$3\{711\} - 27\{531\} + 56\{333\}$
10	2	1	$15\{1000\} - 425\{820\} + 576\{640\} + 7568\{622\} - 14454\{442\}$
11	1	2	$35\{911\} - 476\{731\} + 477\{551\} + 204\{533\}$
12	4	0	$12\{1200\} - 456\{1020\} + 657\{840\} + 13581\{822\} + 1137\{660\} - 88842\{642\} + 114007\{444\}$
12	0	3	$813\{1200\}-30894\{1020\}+165652\{840\}+72131\{822\}-147943\{660\}-169702\{642\}+57527\{444\}$

The first degeneracy occurs at $\ell = 12 = 4\cdot 3 = 3\cdot 4$, and it increases linearly in steps of 12.

What kind of $\mathcal{PT}$ deformation is admissible in this model? Without loss of generality, we may take $\mathcal{P} = s_{x-y}$. Compatibility with this $\mathcal{PT}$ involution restricts the coset $\mathrm{SO}(3,\mathbb{C})/\mathrm{SO}(3,\mathbb{R})$ of complex orthogonal deformations to a two-parameter family, which contains e.g. the SU(1,1) boost in the xy plane,

$$\begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \quad\longmapsto\quad \begin{pmatrix} \cosh\epsilon & -\mathrm{i}\sinh\epsilon & 0 \\ \mathrm{i}\sinh\epsilon & \cosh\epsilon & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} y^1 \\ y^2 \\ y^3 \end{pmatrix} \;=\; r\begin{pmatrix} \sin\theta\cos(\phi+\mathrm{i}\epsilon) \\ \sin\theta\sin(\phi+\mathrm{i}\epsilon) \\ \cos\theta \end{pmatrix} \tag{68}$$

equivalent to

$$\begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \quad\longmapsto\quad \begin{pmatrix} 1+\cosh\epsilon & \mathrm{i}\sinh\epsilon & -\mathrm{i}\sinh\epsilon & 1-\cosh\epsilon \\ -\mathrm{i}\sinh\epsilon & 1+\cosh\epsilon & 1-\cosh\epsilon & \mathrm{i}\sinh\epsilon \\ \mathrm{i}\sinh\epsilon & 1-\cosh\epsilon & 1+\cosh\epsilon & -\mathrm{i}\sinh\epsilon \\ 1-\cosh\epsilon & -\mathrm{i}\sinh\epsilon & \mathrm{i}\sinh\epsilon & 1+\cosh\epsilon \end{pmatrix} \begin{pmatrix} x^1 \\ x^2 \\ x^3 \\ x^4 \end{pmatrix} \tag{69}$$

but more simply given by

$$\phi \;\mapsto\; \phi + \mathrm{i}\epsilon \qquad \text{or} \qquad \big(x\pm\mathrm{i}y,z\big) \;\mapsto\; \big(\mathrm{e}^{\mp\epsilon}(x\pm\mathrm{i}y),z\big)\ . \tag{70}$$

The ensuing complex potential

$$\frac{U_\epsilon(\theta,\phi)}{2g(g{-}1)} \;=\; \frac{1}{\sin^2\theta\,\cos^2 2(\phi{+}\mathrm{i}\epsilon)} + \frac{\cos^2\theta+\sin^2\theta\cos^2(\phi{+}\mathrm{i}\epsilon)}{(\cos^2\theta-\sin^2\theta\cos^2(\phi{+}\mathrm{i}\epsilon))^2} + \frac{\cos^2\theta+\sin^2\theta\sin^2(\phi{+}\mathrm{i}\epsilon)}{(\cos^2\theta-\sin^2\theta\sin^2(\phi{+}\mathrm{i}\epsilon))^2} \tag{71}$$

is free of singular lines but still blows up at 5 pairs of antipodal points. Because these singularities are also carried by

$$\Delta_\epsilon \;\sim\; \sin^2\theta\,\cos 2(\phi{+}\mathrm{i}\epsilon)\,(\cos^2\theta-\sin^2\theta\cos^2(\phi{+}\mathrm{i}\epsilon))\,(\cos^2\theta-\sin^2\theta\sin^2(\phi{+}\mathrm{i}\epsilon))\ , \tag{72}$$

the formal $g{<}0$ wave functions $v_\ell^{\epsilon(g)} \sim \Delta_\epsilon^g$, obtained by simply applying the deformation to the argument of $v_\ell^{(g)}$, remain non-normalizable. Hence, except for the free system ($g{=}1$), the degeneracy remains to be given by $\deg_4(\ell)$ for any $g\in\mathbb{N}$.

If we were able to invent a (nonlinear) $\mathcal{PT}$ deformation (for a suitable involution) which gets rid of all zeros of Δ (and hence of the potential), then the $g{<}0$ states would be resurrected, extending the eigenspace for any given $g{\ge}2$ by the new states with angular momenta $\tilde\ell$ for a coupling $\tilde g = 1{-}g \le -1$. For small values of $\tilde\ell$ this requires $q<0$, so the second branch of (63) would get occupied, and the $g{>}0$ spectrum would be modified to

$$\varepsilon_q \;=\; \tfrac12 q\,(q{+}1) \qquad\text{with}\qquad q \;=\; 6g+\ell \;=\; 6(1{-}g)+\tilde\ell \quad\text{or}\quad -1{-}q \;=\; 6(1{-}g)+\tilde\ell\,, \tag{73}$$

where we restricted $q\ge0$ for uniqueness, and both ℓ and $\tilde\ell$ are composed of 3's and 4's. This would enhance the degeneracy to

$$\begin{aligned}\deg(\varepsilon_q) \;&=\; \deg_4(\ell)+\deg_4(\tilde\ell) \;=\; \deg_4(q{-}6g)+\deg_4(q{+}6g{-}6)+\deg_4(-q{+}6g{-}7)\\ &=\;\begin{cases} g{-}1 \;+\; \begin{Bmatrix} 0 & \text{for} & q+6g=0,3,4,7,8,11 \bmod 12 \\ 1 & \text{for} & q+6g=1,2,5,6,9,10 \bmod 12\end{Bmatrix} & \text{if}\quad q<6g{-}6 \\[2ex] \big\lfloor \tfrac{q}{6}\big\rfloor \;+\; \begin{Bmatrix} 0 & \text{for} & q=1,2,5 \bmod 6 \\ 1 & \text{for} & q=0,3,4 \bmod 6\end{Bmatrix} & \text{if}\quad q\ge 6g{-}6\end{cases}\end{aligned} \tag{74}$$

which shows that its linear growth with q would become g-independent at high energy.

6. Tetrahexahedric model: intertwiner & integrability

As can already be glanced from the tetrahexahedric case, the angular Calogero models possess a rich structure of intertwiners. For their construction, we need the angular Dunkl operators:

$$\begin{aligned}\mathcal{L}_x \;&=\; L_x \;+\; g\big\{\tfrac{z}{x-y}s_{x-y} - \tfrac{z}{x+y}s_{x+y} - \tfrac{y}{x-z}s_{z-x} + \tfrac{y}{z+x}s_{z+x} - \tfrac{y+z}{y-z}s_{y-z} + \tfrac{y-z}{y+z}s_{y+z}\big\}\ ,\\ \mathcal{L}_y \;&=\; L_y \;+\; g\big\{\tfrac{x}{y-z}s_{y-z} - \tfrac{x}{y+z}s_{y+z} - \tfrac{z}{y-x}s_{x-y} + \tfrac{z}{y+x}s_{x+y} - \tfrac{z+x}{z-x}s_{z-x} + \tfrac{z-x}{z+x}s_{z+x}\big\}\ ,\\ \mathcal{L}_z \;&=\; L_z \;+\; g\big\{\tfrac{y}{z-x}s_{z-x} - \tfrac{y}{z+x}s_{z+x} - \tfrac{x}{z-y}s_{y-z} + \tfrac{x}{z+y}s_{y+z} - \tfrac{x+y}{x-y}s_{x-y} + \tfrac{x-y}{x+y}s_{x+y}\big\}\ .\end{aligned} \tag{75}$$

Although we cannot (yet) prove it, we have ample evidence that there exist exactly two algebraically independent angular intertwiners. The first one, of order three, derives from the Weyl antiinvariant

$$\mathcal{M}_3 \;\sim\; \tfrac16\big(\mathcal{L}_x\mathcal{L}_y\mathcal{L}_z+\mathcal{L}_x\mathcal{L}_z\mathcal{L}_y+\mathcal{L}_y\mathcal{L}_z\mathcal{L}_x+\mathcal{L}_y\mathcal{L}_x\mathcal{L}_z+\mathcal{L}_z\mathcal{L}_x\mathcal{L}_y+\mathcal{L}_z\mathcal{L}_y\mathcal{L}_x\big)\ , \tag{76}$$

whose restriction to symmetric functions yields

$$\begin{aligned} M_3 \ \sim\ & y^2z\partial_{zxx} - yz^2\partial_{xxy} + \tfrac12(y^2{-}z^2)\,\partial_{xx} +\ 4g\,\tfrac{yz}{y^2-z^2}\big(yz\partial_{xx} + x^2\partial_{yz} - zx\partial_{xy}\big) \\ & + g\,\Big[2g\,y^2z^2\big(\tfrac{8g}{(x^2-y^2)(z^2-x^2)} + \tfrac{16g}{(z^2-x^2)(y^2-z^2)} - \tfrac{2g-1}{(x^2-y^2)^2} + \tfrac{2g-1}{(z^2-x^2)^2}\big) \\ & \qquad -\tfrac{2x^2y^2}{(z^2-x^2)^2} + \tfrac{2x^2z^2}{(x^2-y^2)^2} - \tfrac{2y^2}{x^2-y^2} - \tfrac{2z^2}{z^2-x^2} - 2\tfrac{y^2+z^2}{y^2-z^2}\Big]\,x\partial_x \\ & + 2g(g{-}1)(g{+}2)\,x^2\,\Big[\tfrac{y^2+z^2}{(y^2-z^2)^2} + z\big(\tfrac{1}{(y-z)^3} - \tfrac{1}{(y+z)^3}\big)\Big] + g\,\big(2g^2{+}8g{-}1\big)\tfrac{y^2+z^2}{y^2-z^2} \\ & + 2g^2(8{+}9g)\tfrac{x^2y^2z^2}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} - \tfrac23 g^3\tfrac{x^6+y^6+z^6}{(x^2-y^2)(x^2-z^2)(y^2-z^2)} +\ \text{cyclic}(x\,y\,z)\ . \end{aligned} \tag{77}$$

It somewhat simplifies in the "potential-free frame",

$$\begin{aligned} \Delta^{-g}M_3\,\Delta^g\ \sim\ & y^2z\partial_{zxx} - yz^2\partial_{xxy} + \tfrac12(y^2{-}z^2)\,\partial_{xx} +\ 2g\,\tfrac{y^2z^2(y^2-z^2)}{(x^2-y^2)(x^2-z^2)}\,\partial_{xx} \\ & + 4g\,\tfrac{xy^2z}{x^2-z^2}\,\partial_{xz} + 2g\,x\,\Big[\tfrac{y^2(x^2+3z^2)}{(x^2-z^2)^2} - \tfrac{z^2(x^2+3y^2)}{(x^2-y^2)^2}\Big]\,\partial_x\ +\ \text{cyclic}(x\,y\,z)\ . \end{aligned} \tag{78}$$

The second one, of order six, emerges from

$$\mathcal{M}_6\ \sim\ \{\mathcal{L}_x^4,\mathcal{L}_y^2\} - \{\mathcal{L}_y^4,\mathcal{L}_x^2\} + \{\mathcal{L}_y^4,\mathcal{L}_z^2\} - \{\mathcal{L}_z^4,\mathcal{L}_y^2\} + \{\mathcal{L}_z^4,\mathcal{L}_x^2\} - \{\mathcal{L}_x^4,\mathcal{L}_z^2\}\ , \tag{79}$$

whose symmetric restriction $M_6\ =\ \text{res}(\mathcal{M}_6)$ is a rather lengthy expression not displayed here. (Hopefully $\Delta^{-g}M_6\,\Delta^g$ is a bit shorter.)

Higher angular intertwiners can be reduced to M_3 and M_6 in combination with the conserved charges

$$J_k\ :=\ \text{res}\big(\mathcal{L}_x^k + \mathcal{L}_y^k + \mathcal{L}_z^k\big) \qquad \text{for} \quad k = (0,)2,4,6 \tag{80}$$

$$\text{with} \qquad J_0\ =\ C_0\ =\ 1 \qquad \text{and} \qquad J_2\ =\ -C_2\ =\ -2\,H_\Omega\ +\ 6g(6g{+}1)\ . \tag{81}$$

In fact, any word in the letters $\{J_2, J_4, J_6\}$ is conserved, but they are not in involution because $[J_4, J_6]$ and $\{J_4, J_6\}$ both are nontrivial new words, and the center of the algebra of conserved charges is spanned by the Casimir J_2 (and J_0 of course). This is different from the situation in the full A_3 model, since we deal with a Dunkl-deformed nonabelian so(3) algebra. However, like in the full model, higher conserved charges are algebraically dependent, e.g.

$$\begin{aligned} 6J_8\ =\ & 8J_6J_2 + 3J_4J_4 - 6J_4J_2J_2 + J_2J_2J_2J_2 \\ & - 12(8{+}5g{+}12g^2)J_6 + 4(34{+}23g{+}30g^2)J_4J_2 - 8(5{+}3g{+}3g^2)J_2J_2J_2 \\ & + 24(13{+}15g{-}102g^2{-}72g^3)J_4 - 4(43{+}70g{-}252g^2{-}144g^3)J_2J_2 \\ & - 48(1{+}3g)(1{+}4g)(1{-}12g)J_2\ . \end{aligned} \tag{82}$$

The intertwining relations for the conserved charges beyond the Hamiltonian are also more involved than in the full model. In particular, they are no longer diagonal in either set. The basic relations for M_3 are

$$\begin{aligned} M_3^{(g)}J_2^{(g)}\ &=\ \big(J_2^{(g+1)} - 6(7{+}12g)\big)\,M_3^{(g)}\ , \\ M_3^{(g)}J_4^{(g)}\ &=\ \big(J_4^{(g+1)} - 4(11{+}12g)J_2^{(g+1)} + 48(26{+}73g{+}48g^2)\big)\,M_3^{(g)} \\ &\quad +\ 2\,M_6^{(g)}\ , \\ M_3^{(g)}J_6^{(g)}\ &=\ \big(J_6^{(g+1)} - (35{+}36g)J_4^{(g+1)} - 3(7{+}4g)J_2^{(g+1)}J_2^{(g+1)} \\ &\quad +\ 2(1111{+}2668g{+}1392g^2)J_2^{(g+1)} \\ &\quad +\ 96(457{+}1933g{+}2717g^2{+}1368g^3{+}144g^4)\big)M_3^{(g)} \\ &\quad +\ \big(3J_2^{(g+1)} - (115{+}200g{+}48g^2)\big)M_6^{(g)}\ , \end{aligned} \tag{83}$$

and those for M_6 look similar (but more lengthy).

Finally, for $g > 0$ the intertwining ladder $1-g \to 2-g \to \ldots \to g-2 \to g-1 \to g$ extends the algebra of conserved charges to a $\mathbb{Z}_2$-graded one by adjoining the new "odd" charges ($* = 3$ or 6)

$$Q^{(g)} = M_*^{(g-1)} M_*^{(g-2)} \cdots M_*^{(1-g)} \tag{84}$$

relating bosonic and fermionic states at any given energy level and positive integer coupling g. However, these operators are only well defined if a suitable $\mathcal{PT}$ deformation can fully regularize the potential (as is the case for the Pöschl-Teller model) and so allows one to double the state space by combining the states at $1-g$ and g. It is clear that such fermionic charges are independent of the J_k because they are of odd order, but they square to a polynomial in the J_k. We conjecture that the various above choices for Q (depending on the details of '$*$') are all related by multiplication with appropriate even charges, and so the $\mathbb{Z}_2$-graded nonlinear algebra would be generated by $\{Q, J_2, J_4, J_6\}$.

7. Summary and outlook

The angular rational A_{n-1} Calogero model for rank $n-1 \geq 3$ is superintegrable but not separable. It describes the Weyl-symmetric states of a particle on S^{n-2} subject to a particular S_n-symmetric potential. We have formulated the spectral problem and given a constructive solution, including a discussion of the conserved charges and the Hamiltonian intertwiners shifting the coupling $g \mapsto g+1$. A complex $\mathcal{PT}$ deformation reduces the dimension of the singular loci of the potential. If all singularities can be removed (as we demonstrate for n=3), then the degeneracy of the energy spectrum will essentially double, and for $g \in \mathbb{Z}$ the algebra of conserved charges gets enhanced to a $\mathbb{Z}_2$-graded one ("supersymmetrized") by the appearance of an additional, odd, charge. We have analyzed in some detail the cases of n=3 (the hexagonal or Pöschl-Teller model) and n=4 (the tetrahexahedric model).

Various issues remain unresolved. First, the question of Liouville charges (the first test appears at n=5) and, more generally, the algebra of the conserved charges depend on the structure of the deformed so($n-1$) algebra of Dunkl-deformed angular momenta $\mathcal{L}_{ij}$, about which little is known. Second, a classification of all independent Hamiltonian intertwiners, via Weyl antiinvariants built from the $\mathcal{L}_{ij}$, is open. Third, a systematic study of $\mathcal{PT}$ deformations compatible with the Weyl symmetry is warranted, in order to decide if and when the state space can be doubled and a nonlinear supersymmetry can be realized for integral values of g. Finally, there is always the potential generalization to trigonometric, hyperbolic or elliptic Calogero systems.

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