

## The fuzzy disc: a review

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**Abstract.** In this talk we present a matrix approximation to the algebra of functions on the disc and define a fuzzy approximation to the Laplacian with Dirichlet boundary conditions. A basis for the NC algebra is introduced in terms of the eigenfunctions of the fuzzy Laplacian, which are seen to converge to the eigenfunctions of the ordinary Laplacian on the disc, that is, the Bessel functions.

### A. Introduction

Fuzzy spaces are an approximation of the abelian algebra of functions on an ordinary space with a sequence of finite rank matrix algebras, which preserve the symmetries of the original space, at the price of noncommutativity.

The idea was introduced by Madore [1] with the fuzzy sphere (see also [2, 3]): a sequence of nonabelian algebras, generated by three “noncommutative coordinates” which satisfy  $x_i x_i = 1$ ,  $[x_i, x_j] = \kappa \varepsilon_{ijk} x_k$  with  $\kappa$  depending on the dimension of representations of  $SU(2)$ . Such an algebra is seen to correctly reproduce the algebra of functions on the ordinary sphere in an appropriate limit [4]. (For applications to field theory a partial list of references is [5].) The main ingredient here being the existence of a compact group of which the sphere is an orbit, the same idea has been further applied to spaces with similar features [6, 7, 8, 9].

Here we present a generalization to a space, the disc, which cannot be seen as the coadjoint orbit of a compact group. First we briefly review the fuzzy sphere from a perspective which can be partly generalised to the disc case. Then introduce the fuzzy disc starting from the noncommutative plane and implementing the constraint  $x^2 + y^2 \leq R^2$  [10, 11, 12, 13]. As for the fuzzy sphere, the geometry is introduced through a fuzzy Laplacian, whose eigenfunctions furnish a basis for the matrix algebra approximating the algebra of functions on the disc. In the commutative limit they tend to the ordinary Bessel functions (from which the name of fuzzy Bessel functions) and the spectrum of the Laplacian recovers the correct spectrum.

### B. The fuzzy sphere

In this section we present the fuzzy sphere in the general context of the Weyl-Wigner formalism [7, 8, 14], which establishes invertible maps between operators and functions on a given manifold.

<sup>1</sup> Talk presented by this author

Since the sphere is an orbit of  $SU(2)$ , we may use coherent states of  $SU(2)$  to establish the map [15]. These are defined starting from unitary irreducible representations of the group. On each finite dimensional Hilbert space  $\mathbf{C}^N$ , with  $N = 2L + 1$  -here ( $L = 0, 1/2, 1, 3/2, \dots$ )-, a basis is given by vectors that, in ket notation, are represented as  $|L, M\rangle$  -with  $M = (-L, -L + 1, \dots, L - 1, L)$ . To each group element we associate an operator

$$u \in SU(2) \xrightarrow{\hat{R}^{(L)}} \mathcal{B}(\mathbf{C}^N) \quad (\text{B.1})$$

whose matrix elements are the Wigner functions

$$\langle L, M | \hat{R}^{(L)}(u) | L, M' \rangle = D_{MM'}^L(u) . \quad (\text{B.2})$$

The second step is to fix a fiducial state. We choose the highest weight in the representation:  $|\psi_0\rangle = |L, L\rangle$ . If the group manifold is parametrised by Euler angles, then  $u$  represents a point whose “coordinates” range through  $\alpha \in [0, 4\pi)$ ,  $\beta \in [0, \pi)$ ,  $\gamma \in [0, 2\pi)$ . Fixed the fiducial vector, its stability subgroup  $H_{\psi_0}$  by the  $\hat{R}^{(L)}$  representation is made by elements for which  $\beta = 0$  (this condition is seen to be valid independently of  $N$ ). Two elements  $u$  and  $u'$  are equivalent if  $u^\dagger u' \in H_{\psi_0}$ . It is possible to prove that:

$$SU(2)/H_{\psi_0} \approx S^2 \quad (\text{B.3})$$

identifying  $\theta = \beta$  and  $\varphi = \alpha \bmod 2\pi$ . To each equivalence class of the quotient we associate a coherent state. Therefore, varying  $\tilde{u}$ , a representative element for each equivalence class, the set of coherent states is defined as:

$$|\theta, \varphi, N\rangle = \hat{R}^{(L)}(\tilde{u}) |L, L\rangle . \quad (\text{B.4})$$

This set of states is nonorthogonal, and overcomplete ( $d\Omega = d\varphi \sin \theta d\theta$ ):

$$\begin{aligned} \langle \theta', \varphi', N | \theta, \varphi, N \rangle &= e^{-iL(\varphi' - \varphi)} \left[ e^{i(\varphi' - \varphi)} \cos \theta/2 \cos \theta'/2 + \sin \theta/2 \sin \theta'/2 \right]^{2L} , \\ \mathbb{1} &= \frac{2L+1}{4\pi} \int_{S^2} d\Omega |\theta, \varphi, N\rangle \langle \theta, \varphi, N| . \end{aligned} \quad (\text{B.5})$$

Using this set of vectors it is possible to map operators on a finite dimensional Hilbert space (finite rank matrices) to functions on the sphere  $S^2$  (Berezin symbols [16]):

$$\begin{aligned} \hat{A}^{(N)} \in \mathcal{B}(\mathbf{C}^N) &\approx \mathbf{M}_N \quad \mapsto \quad A^{(N)} \in \mathcal{F}(S^2) , \\ A^{(N)}(\theta, \varphi) &= \langle \theta, \varphi, N | \hat{A}^{(N)} | \theta, \varphi, N \rangle . \end{aligned} \quad (\text{B.6})$$

Among these operators, there are some special ones,  $\hat{Y}_{JM}^{(N)}$  whose symbols are the spherical harmonics, up to order  $2L$  (here  $J = 0, 1, \dots, 2L$  and  $M = -J, \dots, +J$ ):

$$\langle \theta, \varphi, N | \hat{Y}_{JM}^{(N)} | \theta, \varphi, N \rangle = Y_{JM}(\theta, \varphi) , \quad (\text{B.7})$$

these operators are called *fuzzy harmonics*. What is special to these operators? Till now we have just associated functions with finite dimensional representations of  $SU(2)$ . The geometry of the sphere is introduced through a Laplacian defined on each finite rank matrix algebra  $\mathbf{M}_N$ , whose eigenmatrices are precisely the fuzzy harmonics

$$\begin{aligned} \nabla^2 &: \mathbf{M}_N \mapsto \mathbf{M}_N , \\ \nabla^2 \hat{A}^{(N)} &= [\hat{L}_s^{(N)}, [\hat{L}_s^{(N)}, \hat{A}^{(N)}]] \end{aligned} \quad (\text{B.8})$$

where  $\hat{L}_a^{(N)}$  are the generators of the Lie algebra of This operator is called the *fuzzy Laplacian*. Its spectrum is given by the eigenvalues  $\mathcal{L}_j = j(j+1)$ , where  $j = 0, \dots, 2L$ , and every eigenvalue has a multiplicity of  $2j+1$ . The spectrum of the fuzzy Laplacian thus coincides, up to order  $2L$ , with the one of its continuum counterpart acting on the space of functions on a sphere. The cut-off of this spectrum is of course related to the dimension of the rank of the matrix algebra under analysis. The fuzzy harmonics, besides being the eigenstates of the fuzzy Laplacian, furnish a basis in each space of matrices  $\mathbf{M}_N$ . An element  $\hat{F}^{(N)}$  belonging to  $\mathbf{M}_N$  can be expanded as:

$$\hat{F}^{(N)} = \sum_{J=0}^{2L} \sum_{M=-J}^J F_{JM}^{(N)} \hat{Y}_{JM}^{(N)}, \quad (\text{B.9})$$

with coefficients

$$F_{JM}^{(N)} = \frac{\text{Tr} [\hat{Y}_{JM}^{(N)\dagger} \hat{F}^{(N)}]}{\text{Tr} \hat{Y}_{JM}^{(N)\dagger} \hat{Y}_{JM}^{(N)}}. \quad (\text{B.10})$$

Then a Weyl-Wigner map can be defined simply mapping spherical harmonics into fuzzy harmonics:

$$\hat{Y}_{JM}^{(N)} \Leftrightarrow Y_{JM}(\theta, \varphi). \quad (\text{B.11})$$

This map clearly depends on the dimension  $N$  of the space on which fuzzy harmonics are realized. It can be linearly extended by:

$$\hat{F}^{(N)} = \sum_{J=0}^{2L} \sum_{M=-J}^J F_{JM}^{(N)} \hat{Y}_{JM}^{(N)} \Leftrightarrow F^{(N)}(\theta, \varphi) = \sum_{J=0}^{2L} \sum_{M=-J}^{+J} F_{JM}^{(N)} Y_{JM}(\theta, \varphi). \quad (\text{B.12})$$

This is a Weyl-Wigner isomorphism and it can be used to define a fuzzy sphere. Given a function on a sphere, if it is square integrable with respect to the standard measure  $d\Omega$ , then it can be expanded in the basis of spherical harmonics:

$$f(\theta, \varphi) = \sum_{J=0}^{\infty} \sum_{M=-J}^J f_{JM} Y_{JM}(\theta, \varphi). \quad (\text{B.13})$$

Now consider the set of “truncated” functions:

$$f^{(N)}(\theta, \varphi) = \sum_{J=0}^{2L} \sum_{M=-J}^J f_{JM} Y_{JM}(\theta, \varphi). \quad (\text{B.14})$$

this is made into an algebra, isomorphic to the matrix algebra  $\mathbf{M}_N$ , if we define a new, noncommutative product,

$$(f^{(N)} * g^{(N)})(\theta, \varphi) = \langle \theta, \varphi, N | \hat{f}^{(N)} \hat{g}^{(N)} | \theta, \varphi, N \rangle. \quad (\text{B.15})$$

The Weyl-Wigner map (B.12) has been used to make each set of truncated functions a non abelian algebra  $\mathcal{A}^{(N)}(S^2, *)$ , isomorphic to  $\mathbf{M}_N$ .

To make contact with our initial definition of fuzzy sphere, let us observe that these algebras can be seen as formally generated by matrices which are the images of the norm 1 vectors in  $\mathbf{R}^3$ , i.e. points on the sphere. They are mapped into multiples of the generators  $\hat{L}_a^{(N)}$  of the Lie algebra:

$$\frac{x_a}{\| \vec{x} \|} \mapsto \hat{x}_a^{(N)} \quad [\hat{x}_a^{(N)}, \hat{x}_b^{(N)}] = \frac{2i\varepsilon_{abc}}{\sqrt{N^2 - 1}} \hat{x}_c^{(N)}. \quad (\text{B.16})$$

The commutation rules satisfied by generators of the algebras in the sequence  $\mathcal{A}^{(N)}(S^2, *)$  make it intuitively clear that in the limit for  $N \rightarrow \infty$  we recover the abelian algebra of functions on  $S^2$ .

### C. The fuzzy disc

In the previous section we have defined the fuzzy sphere as a sequence of finite-dimensional matrix algebras. The finite rank was connected to the dimension of UIRR's of the group  $SU(2)$ . The Weyl-Wigner map associating operators to functions was realised by means of coherent states of  $SU(2)$ , which are in one to one correspondence with points on the sphere  $S^2$ . Moreover, a basis in the set  $\mathbf{M}_N$  has been found in terms of eigenmatrices of the fuzzy Laplacian. This procedure works for all coadjoint orbits of compact groups, therefore, not for the disc. What to do then? Despite the evident differences, the procedure followed for the sphere can still be applied, with some modifications. In fact, in [10, 11] we propose the following approach. Considering the disc as a subspace of the plane  $\mathbf{R}^2$ , we first regard the plane as an orbit of the Heisenberg-Weyl group, consider coherent states for such a group and establish a Weyl-Wigner map between functions on the plane and operators which, being the HW group non-compact, are morally infinite dimensional matrices. At this point we have to implement the constraint  $x^2 + y^2 \leq R^2$ , where  $x, y$  are coordinates on the plane and  $R$  is the radius of the disc. This is achieved introducing a sequence of projections converging to the characteristic function of the disc. The projections identify a sequence of finite rank matrix algebras converging to the commutative algebra of functions on the disc. This algebra can be endowed with an additional structure, a fuzzy Laplacian, identifying the underlying geometry. In the commutative limit  $N \rightarrow \infty$ , with  $N\theta = R^2$  this geometry is seen to converge to a disc of radius  $R$ . It is this sequence which we call the fuzzy disc (some points of contact with our approach may be found in [17, 18]). Pursuing our analogy with the fuzzy sphere, we can go a step further and solve the eigenvalue problem for the fuzzy Laplacian. It is known that, in complete analogy with spherical harmonics and their role for the sphere, Bessel functions of integer order are a basis for the algebra of functions on the ordinary disc. Moreover, they are eigenfunctions of the Laplacian with Dirichlet boundary conditions on the disc. What happens in the fuzzy case? Here, the eigenmatrices are finite rank operators. Their symbols are seen to tend to Bessel functions of integer order, therefore deserving the name of fuzzy Bessel. As we will see in detail below, these finite rank operators, the fuzzy Bessel functions, furnish a basis in each finite dimensional matrix algebra of the sequence, completing the picture.

A noncommutative plane can be defined by means of a Weyl-Wigner map, following again the general procedure of Berezin: so the first step is the definition of a set of coherent states for the Heisenberg-Weyl group, since they are labelled by points of the plane. For details on the derivation, which follows the same steps we have seen for the sphere, we refer to [11]. However, coherent states for the HW group are very well known: they are the eigenstates of the annihilation operator  $\hat{a} = \theta \frac{\partial}{\partial z}$  where  $z = x + iy$  are complex coordinates for the plane. The commutation relations of  $\hat{a}$  with creation operator  $\hat{a}^\dagger$ ,  $(\hat{a}^\dagger f)(z) = zf(z)$  read

$$[\hat{a}, \hat{a}^\dagger] = \theta \mathbb{1} \quad (\text{C.17})$$

with the unconventional presence of  $\theta$ . In the basis of the number operator  $\hat{N} = \hat{a}^\dagger \hat{a}$  coherent states read

$$|z\rangle = \sum_{n=0}^{\infty} e^{-\bar{z}z/2\theta} \frac{z^n}{\sqrt{n!\theta^n}} |\psi_n\rangle. \quad (\text{C.18})$$

Thus a Weyl-Wigner correspondence can be established. To operators  $\hat{f}$  acting on the Hilbert space of complex analytic functions of complex variable  $z$  we associate

$$f(\bar{z}, z) = \langle z | \hat{f} | z \rangle. \quad (\text{C.19})$$

This relation can be inverted to:

$$\hat{f} = \int \frac{d^2\xi}{\pi\theta} \int \frac{d^2z}{\pi\theta} f(z, \bar{z}) e^{-(\bar{z}\xi - \bar{\xi}z)/\theta} e^{\xi\hat{a}^\dagger/\theta} e^{-\bar{\xi}\hat{a}/\theta}. \quad (\text{C.20})$$

Instead than the integral form (C.20) we look for a series expansion, which is more manageable to be adapted to the disc. To start with, we can restrict to functions which can be written as Taylor series in  $\bar{z}, z$ :

$$f(\bar{z}, z) = \sum_{m,n=0}^{\infty} f_{mn}^{Tay} \bar{z}^m z^n. \quad (\text{C.21})$$

An easy calculation shows that this  $f$  is the symbol of the operator:

$$\hat{f} = \sum_{m,n=0}^{\infty} f_{mn}^{Tay} \hat{a}^{\dagger m} \hat{a}^n. \quad (\text{C.22})$$

More generally we can consider operators written in a density matrix notation:

$$\hat{f} = \sum_{m,n=0}^{\infty} f_{mn} |\psi_m\rangle \langle \psi_n|. \quad (\text{C.23})$$

The Berezin symbol of this operator is the function:

$$f(\bar{z}, z) = e^{-|z|^2/\theta} \sum_{m,n=0}^{\infty} f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}}, \quad (\text{C.24})$$

where the relation between the Taylor coefficients  $f_{mn}^{Tay}$  and the  $f_{mn}$  is

$$f_{lk} = \sum_{q=0}^{\min(l,k)} f_{l-q k-q}^{Tay} \frac{\sqrt{k!l!\theta^{l+k}}}{q!\theta^q}, \quad (\text{C.25})$$

while the inverse relation is given by:

$$f_{mn}^{Tay} = \sum_{p=0}^{\min(m,n)} \frac{(-1)^p}{p! \sqrt{(m-p)!(n-p)!\theta^{m+n}}} f_{m-p n-p}. \quad (\text{C.26})$$

Equation (C.22) shows that the quantization of a monomial in the variables  $z, \bar{z}$  is an operator in  $\hat{a}, \hat{a}^{\dagger}$ , formally a monomial in these two noncommuting variables, with all terms in  $\hat{a}^{\dagger}$  acting at the left side with respect to terms in  $\hat{a}$ .

The invertibility of the Weyl map (on a suitable domain of functions on the plane) enables to define a noncommutative product in the space of functions, known as Voros product [19, 20], a variant of the more popular Grönewold-Moyal product [21, 22]:

$$(f * g)(\bar{z}, z) = \langle z | \hat{f} \hat{g} | z \rangle. \quad (\text{C.27})$$

It is a non local product:

$$(f * g)(\bar{z}, z) = e^{-\bar{z}z/\theta} \int \frac{d^2\xi}{\pi\theta} f(\bar{z}, \xi) g(\bar{\xi}, z) e^{-\bar{\xi}\xi/\theta} e^{\bar{\xi}z/\theta} e^{\bar{z}\xi/\theta}. \quad (\text{C.28})$$

Its asymptotic expansion, on a suitable domain, acquires the form:

$$(f * g)(\bar{z}, z) = f e^{\theta \overleftarrow{\partial}_{\bar{z}} \overrightarrow{\partial}_z} g, \quad (\text{C.29})$$

and makes it clear that it is a deformation in  $\theta$  of the pointwise commutative product. Since it is the translation, in the space of functions, of the product in the space of operators, if symbols are expressed in the form (C.24), then the product acquires a matrix form:

$$(f * g)_{mn} = \sum_{k=0}^{\infty} f_{mk} g_{kn} . \quad (\text{C.30})$$

The space of functions on the plane, with the standard definition of sum, and the product given by the Voros product (C.27), is a nonabelian algebra, a noncommutative plane. This algebra  $\mathcal{A}_\theta = (\mathcal{F}(\mathbf{R}^2), *)$  is isomorphic to an algebra of operators, or, as equation (C.30) suggests, to an algebra of infinite dimensional matrices.

As anticipated above, to obtain finite rank matrices we introduce a set of projectors

$$\hat{P}_\theta^{(N)} = \sum_{n=0}^N |\psi_n\rangle\langle\psi_n| \quad (\text{C.31})$$

in the space of operators. Their symbols are projectors in the algebra  $\mathcal{A}_\theta$  of the noncommutative plane, in the sense that they are idempotent functions of order 2 with respect to the Voros product (here  $z = re^{i\varphi}$ ):

$$\begin{aligned} P_\theta^{(N)}(r, \varphi) &= \sum_{n=0}^N \langle z | \psi_n \rangle \langle \psi_n | z \rangle = e^{-r^2/\theta} \sum_{n=0}^N \frac{r^{2n}}{n! \theta^n} \\ P_\theta^{(N)} * P_\theta^{(N)} &= P_\theta^{(N)} . \end{aligned} \quad (\text{C.32})$$

This finite sum can be performed yielding a rotationally symmetric function which is the ratio of an incomplete gamma function by a gamma function:

$$P_\theta^{(N)}(r, \varphi) = \frac{\Gamma(N+1, r^2/\theta)}{\Gamma(N+1)} . \quad (\text{C.33})$$

Let us analyse the limit  $N \rightarrow \infty$ . If  $\theta$  is kept fixed, and nonzero, in the limit for  $N \rightarrow \infty$  the symbol  $P_\theta^{(N)}(r, \varphi)$  converges, pointwise, to the constant function  $P_\theta^{(\infty)}(r, \varphi) = 1$ , which can be formally considered as the symbol of the identity operator: in this limit one recovers the whole noncommutative plane.

This situation changes if the limit for  $N \rightarrow \infty$  is performed keeping the product  $N\theta$  equals to a constant, say  $R^2$ . In a pointwise convergence, chosen  $R^2 = 1$ :

$$P_\theta^{(N)} \rightarrow \begin{bmatrix} 1 & r < 1 \\ 1/2 & r = 1 \\ 0 & r > 1 \end{bmatrix} = Id(r) . \quad (\text{C.34})$$

This sequence of projectors converges to a step function in the radial coordinate  $r$ , the characteristic function of a disc on the plane. Thus a sequence of subalgebras  $\mathcal{A}_\theta^{(N)}$  can be defined by:

$$\mathcal{A}_\theta^{(N)} = P_\theta^{(N)} * \mathcal{A}_\theta * P_\theta^{(N)} . \quad (\text{C.35})$$

As it has been said, the full algebra  $\mathcal{A}_\theta$  is isomorphic to an algebra of operators. What the previous relation says is that  $\mathcal{A}_\theta^{(N)}$  is isomorphic to  $\mathbf{M}_{N+1}$ , the algebra of  $(N+1)$  rank matrices. The effect of this projection on a generic function is:

$$\Pi_\theta^{(N)}(f) = f_\theta^{(N)} = P_\theta^{(N)} * f * P_\theta^{(N)} = e^{-|z|^2/\theta} \sum_{m,n=0}^N f_{mn} \frac{\bar{z}^m z^n}{\sqrt{m! n! \theta^{m+n}}} \quad (\text{C.36})$$

which, comparing with (C.24) is nothing but a truncation of the series expansion. On every subalgebra  $\mathcal{A}_\theta^{(N)}$ , the symbol  $P_\theta^{(N)}(r, \varphi)$  is then the identity, because it is the symbol of the projector  $\hat{P}^{(N)} = \sum_{n=0}^N |\psi_n\rangle\langle\psi_n|$ , which is the identity operator in  $\mathcal{A}_\theta^{(N)}$ , or, equivalently, the identity matrix in every  $\mathbf{M}_{N+1}$ .

Note that the rotation group on the plane,  $SO(2)$ , acts in a natural way on these subalgebras. Its generator is the truncated number operator  $\hat{N}^{(N)} = \sum_{n=0}^N n\theta|\psi_n\rangle\langle\psi_n|$ . Cutting at a finite  $N$  the expansion provides an infrared cutoff. This cutoff is “fuzzy” in the sense that functions in the subalgebra are still defined outside the disc of radius  $R$ , but are exponentially damped. In general functions are close to their projected version  $f_\theta^{(N)}$  if they are mostly supported on a disc of radius  $R = \sqrt{N\theta}$ , otherwise they are exponentially cut, provided they do not present oscillations of too small wavelength (compared to  $\theta$ ). In this case the projected function becomes very large on the boundary of the disc. More details and examples are in [10, 12].

The procedure outlined so far allows a fuzzy approximation to functions mostly supported on the disc. What is missing, however, is a direct association between functions on the disc and finite-dimensional matrices: in other words, we need a basis in each  $\mathcal{A}_\theta^N$  which is mapped one to one into a basis for  $(\mathcal{F}(D), \cdot)$ . We know that a basis for  $(\mathcal{F}(D), \cdot)$  is represented by the Bessel functions of integer order, which are in turn the eigenfunctions of the Laplacian on the disc with Dirichlet boundary conditions. Therefore, mimicking the procedure outlined for the fuzzy sphere, we consider a properly defined fuzzy Laplacian, we show that it is a good approximation because its spectrum reproduces the standard spectrum in the noncommutative limit and we study the eigenmatrices.

The first step is to define derivatives in  $\mathcal{A}_\theta^N$ . In  $\mathcal{A}_\theta$  we have

$$\begin{aligned}\partial_z f &= \frac{1}{\theta} \langle z | [\hat{f}, \hat{a}^\dagger] | z \rangle, \\ \partial_{\bar{z}} f &= \frac{1}{\theta} \langle z | [\hat{a}, \hat{f}] | z \rangle.\end{aligned}\tag{C.37}$$

Therefore we define [11]

$$\begin{aligned}\partial_z f_\theta^{(N)} &\equiv \frac{1}{\theta} \langle z | \hat{P}_\theta^{(N)} [\hat{P}_\theta^{(N)} \hat{f} \hat{P}_\theta^{(N)}, \hat{a}^\dagger] \hat{P}_\theta^{(N)} | z \rangle \\ \partial_{\bar{z}} f_\theta^{(N)} &\equiv -\frac{1}{\theta} \langle z | \hat{P}_\theta^{(N)} [\hat{P}_\theta^{(N)} \hat{f} \hat{P}_\theta^{(N)}, \hat{a}] \hat{P}_\theta^{(N)} | z \rangle\end{aligned}\tag{C.38}$$

which can be checked to be true derivations on each  $\mathcal{A}_\theta^{(N)}$ , that is a linear operation from  $\mathcal{A}_\theta^{(N)}$  to itself, satisfying the Leibnitz rule.

Let us come to the definition of the Laplacian operator. In the spirit of noncommutative geometry it is this additional structure which carries the information about the geometry of the space underlying  $\mathcal{A}_\theta^{(N)}$ .

Starting from the exact expression on  $\mathcal{A}_\theta$ :

$$\nabla^2 f(\bar{z}, z) = 4\partial_{\bar{z}}\partial_z f = \frac{4}{\theta^2} \langle z | [\hat{a}, [\hat{f}, \hat{a}^\dagger]] | z \rangle\tag{C.39}$$

it is possible to define, in each  $\mathcal{A}_\theta^{(N)}$ :

$$\nabla_{(N)}^2 \hat{f}_\theta^{(N)} \equiv \frac{4}{\theta^2} \hat{P}_\theta^{(N)} [\hat{a}, [\hat{P}_\theta^{(N)} \hat{f} \hat{P}_\theta^{(N)}, \hat{a}^\dagger]] \hat{P}_\theta^{(N)}.\tag{C.40}$$

The image of an element of the truncated algebra:

$$\hat{f}_\theta^{(N)} = \sum_{a,b=0}^N f_{ab} |\psi_a\rangle\langle\psi_b|$$

is then:

$$\begin{aligned}
\nabla_{(N)}^2 f_\theta^{(N)} = & 4N \left[ \sum_{s=0}^{N-1} \sum_{b=0}^{N-1} f_{s+1,b+1} \sqrt{(s+1)(b+1)} |\psi_s\rangle\langle\psi_b| + \right. \\
& - \sum_{s=0}^N \sum_{b=0}^N f_{sb} (s+1) |\psi_s\rangle\langle\psi_b| - \sum_{s=0}^{N-1} f_{0,s+1} (s+1) |\psi_0\rangle\langle\psi_{s+1}| + \\
& + \sum_{s=0}^{N-1} \sum_{b=0}^{N-1} f_{sb} \sqrt{(s+1)(b+1)} |\psi_{s+1}\rangle\langle\psi_{b+1}| + \\
& \left. - \sum_{s=0}^{N-1} \sum_{b=0}^{N-1} f_{s+1,b+1} (b+1) |\psi_{s+1}\rangle\langle\psi_{b+1}| \right]. \tag{C.41}
\end{aligned}$$

The eigenvalues of this Laplacian have been numerically calculated [10, 11]. They are seen to converge to the spectrum of the standard Laplacian defined on a disc, with boundary conditions on the edge of the disc of Dirichlet homogeneous kind. We recall that in the standard case all eigenvalues are negative, and their modules  $\lambda$  solve the implicit equation:

$$J_n(\sqrt{\lambda}) = 0, \tag{C.42}$$

where  $n$  is the order of the Bessel functions. In particular, those related to  $J_0$  are simply degenerate, the others are doubly degenerate: so there is a sequence of eigenvalues labelled by  $\lambda_{n,k}$  where  $n$  is the order of the Bessel function and  $k$  indicates that it is the  $k^{\text{th}}$  zero of the function. Moreover, the eigenfunctions of the standard Laplacian are:

$$\Phi_{n,k} = e^{in\varphi} \left( \frac{\sqrt{\lambda_{|n|,k}} r}{2} \right)^{|n|} \sum_{s=0}^{\infty} \frac{(-\lambda_{|n|,k})^s}{s! (|n|+s)!} \left( \frac{r}{2} \right)^{2s} = e^{in\varphi} J_{|n|} \left( \sqrt{\lambda_{|n|,k}} r \right). \tag{C.43}$$

with  $n$  integer number and  $|n|$  its absolute value. This is a way to write the eigenfunctions in a compact form, taking into account the degeneracy of eigenvalues for  $|n| \geq 1$ .

The spectrum of the fuzzy Laplacian is in good agreement with the spectrum of the continuum case, even for low values  $N$  of the dimension of truncation, as can be seen in figure 1. It is interesting to note that with this definition of Laplacian (C.40) we correctly reproduce the degeneracy pattern of the eigenvalues.

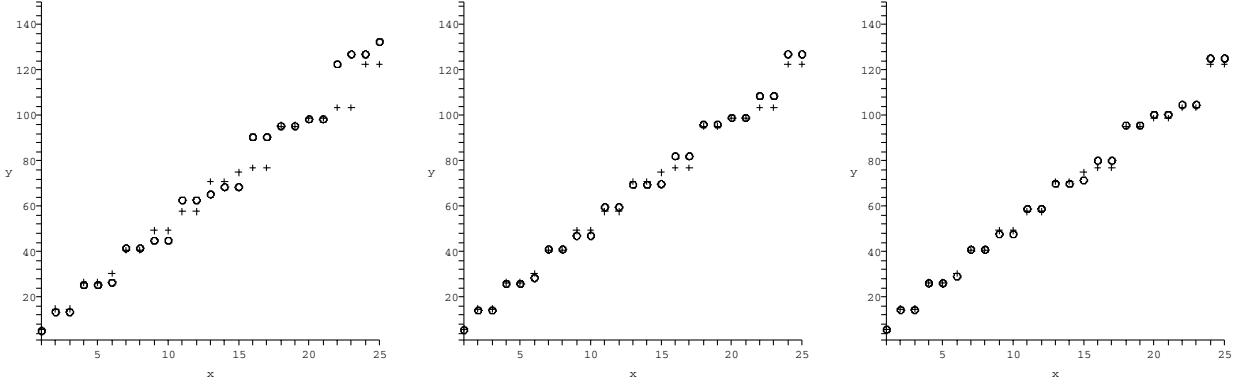
Despite many points of contact, there is an important difference with respect to the fuzzy sphere. There, the spectrum is truncated (we get the first  $2L$  eigenvalues, with  $2L = N - 1$ ) but exact up to the truncation. For the fuzzy disc the spectrum is not only cut but also approximated.

Let us consider now the eigenmatrices of the fuzzy Laplacian (C.40). As anticipated, their symbols will furnish a basis in  $\mathcal{A}_\theta^N$ . It can be shown after lengthy calculations [11] that the eigenmatrices are of the form

$$\hat{\Phi}_{n,k}^{(N)} = \left( \frac{\lambda_{n,k}}{4N} \right)^{n/2} \sum_{a=0}^{N-n} \sqrt{a!(a+n)!} \left[ \sum_{s=0}^a \left( -\frac{\lambda_{n,k}}{4N} \right)^s \frac{1}{s!(s+n)!(a-s)!} \right] |\psi_a\rangle\langle\psi_{a+n}|. \tag{C.44}$$

To understand what these operators are, let us consider the eigenfunctions of the standard Laplacian,  $\Phi_{n,k}$  represented by (C.43) and map them into operators via the Weyl map. We obtain for a fixed nonnegative  $n$ , such that  $0 \leq n \leq N$ ,

$$\hat{\Phi}_{n,k} = \left( \frac{\lambda_{n,k}}{4} \right)^{n/2} \sum_{s=0}^{\infty} \left( -\frac{\lambda_{n,k}}{4} \right)^s \frac{1}{s!(s+n)!} \hat{a}^{\dagger s} \hat{a}^{s+n}. \tag{C.45}$$



**Figure 1.** Comparison of the first eigenvalues of the fuzzy Laplacian (circles) with those of the continuum Laplacian (crosses) on the domain of functions with Dirichlet homogeneous boundary conditions. The orders of truncation are  $N = 10, 20, 30$ .

In the density matrix notation, it acquires the form:

$$\hat{\Phi}_{n,k} = \left( \frac{\lambda_{n,k}}{4} \right)^{n/2} \sum_{j=n}^{\infty} \sum_{s=0}^{j-n} \left( -\frac{\theta \lambda_{n,k}}{4} \right)^s \frac{\theta^{n/2}}{s! (s+n)!} \frac{\sqrt{j! (j-n)!}}{(j-s-n)!} |\psi_{j-n}\rangle \langle \psi_j| . \quad (\text{C.46})$$

Comparing this expression with the eigenmatrices of the fuzzy Laplacian (C.44) we can easily verify that (C.44) is a truncation of (C.46) with the constrain  $\theta N = 1$ . Therefore the eigenmatrices of the fuzzy Laplacian coincide with the eigenmatrices of the exact Laplacian (which are represented by an infinite series), up to the order of the approximation. We refer for details to [11].

As a final result, we now compare the behaviour of the symbols of the fuzzy Bessel with their ordinary counterparts.

From (C.44) the symbol of a fuzzy Bessel is:

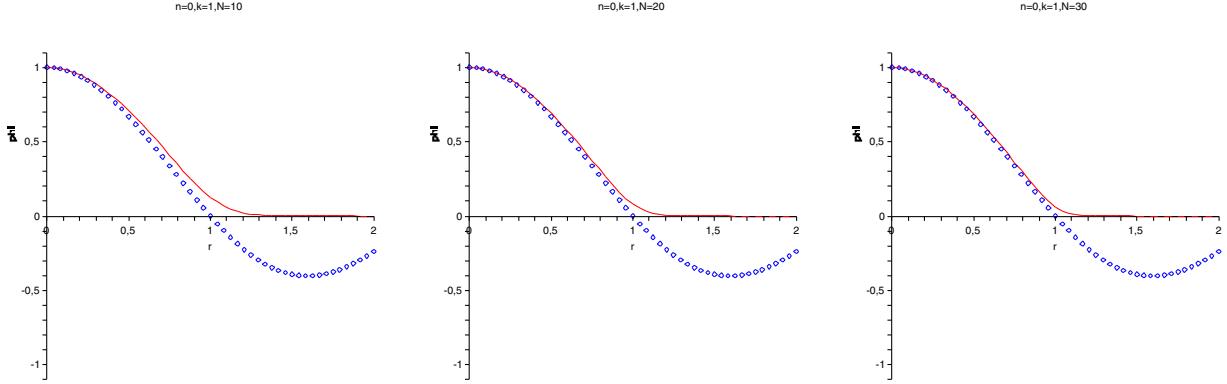
$$\Phi_{n,k}^{(N)} = \left( \frac{\lambda_{n,k}^{(N)}}{4} \right)^{n/2} r^n e^{in\varphi} e^{-Nr^2} \sum_{a=0}^{N-n} r^{2a} N^a \left[ \sum_{s=0}^a \frac{1}{s! (s+n)! (a-s)!} \left( -\frac{\lambda_{n,k}^{(N)}}{4N} \right)^s \right] . \quad (\text{C.47})$$

The integer  $n$  appears as a phase modulating factor for the variable  $\varphi$ . This would be the expansion of the corresponding Bessel function, where it not for the truncation in the sum, and the fact that the parameter  $\lambda_{0,k}$  has become the eigenvalue of the fuzzy Laplacian. For  $n = 0$  the expression can be simplified:

$$\begin{aligned} \Phi_{0,k}^{(N)}(r) &= \sum_{a=0}^N \left( \sum_{s=0}^a \left( -\frac{\lambda_{0,k}^{(N)}}{4N} \right)^s \frac{1}{s!} \binom{a}{s} \right) e^{-Nr^2} r^{2a} \frac{N^a}{a!} \\ &= e^{-Nr^2} \sum_{a=0}^N \frac{N^a}{a!} r^{2a} \mathcal{L}_{(a)} \left( \frac{\lambda_{0,k}^{(N)}}{4N} \right) . \end{aligned} \quad (\text{C.48})$$

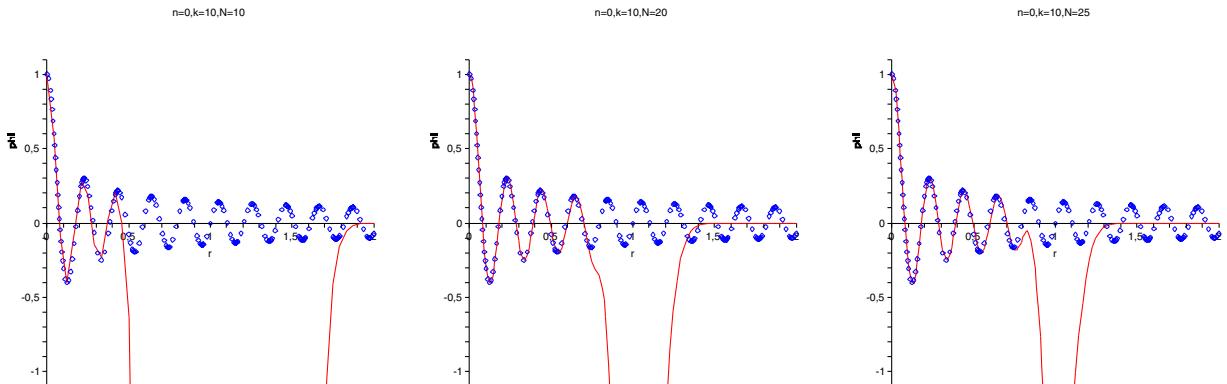
Where  $\mathcal{L}_{(a)} \left( \frac{\lambda_{0,k}^{(N)}}{4N} \right)$  is the  $a^{th}$  Laguerre polynomial in the variable  $\left( \frac{\lambda_{0,k}^{(N)}}{4N} \right)$ . We can plot the diagonal fuzzy elements. Fig. 2 shows that the zero order fuzzy Bessel state converges to the

continuum eigenfunctions  $\Phi_{0,1}(r, \varphi)$  for values of  $r$  inside the disc of radius 1, while it converges to zero outside the disc. This behaviour is seen to be valid also for eigenstates of different



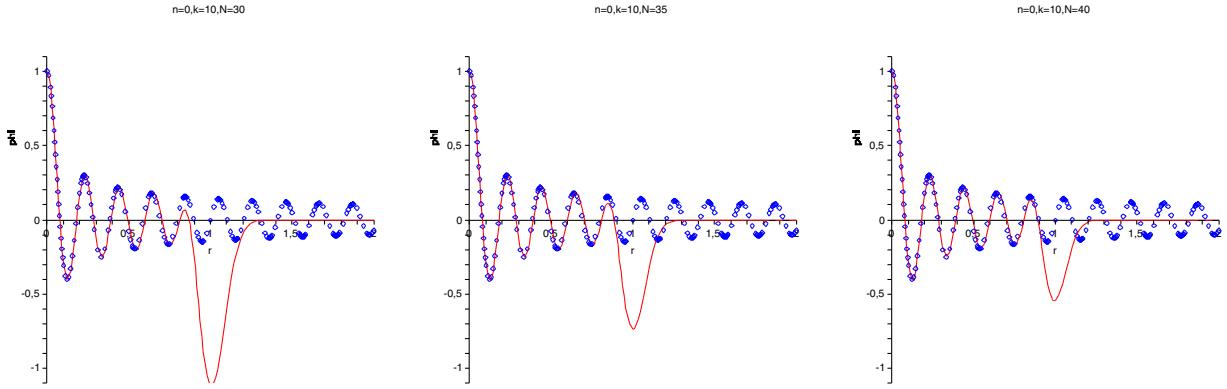
**Figure 2.** Comparison of the radial shape for the symbol  $\Phi_{0,1}^{(N)}(r, \varphi)$  (continuum line), the symbol of the eigenmatrix of the fuzzy Laplacian for  $N = 10, 20, 30$ , with  $\Phi_{0,1}(r, \varphi)$ .

eigenvalues. It is interesting to analyse the fuzzification of  $\Phi_{0,10}(r)$ . The fuzzy symbol is  $\Phi_{0,10}^{(N)}$ . For  $N = 10, 20, 25$  it is plotted in figure (3). It is evident that the fuzzy eigenfunction reproduces



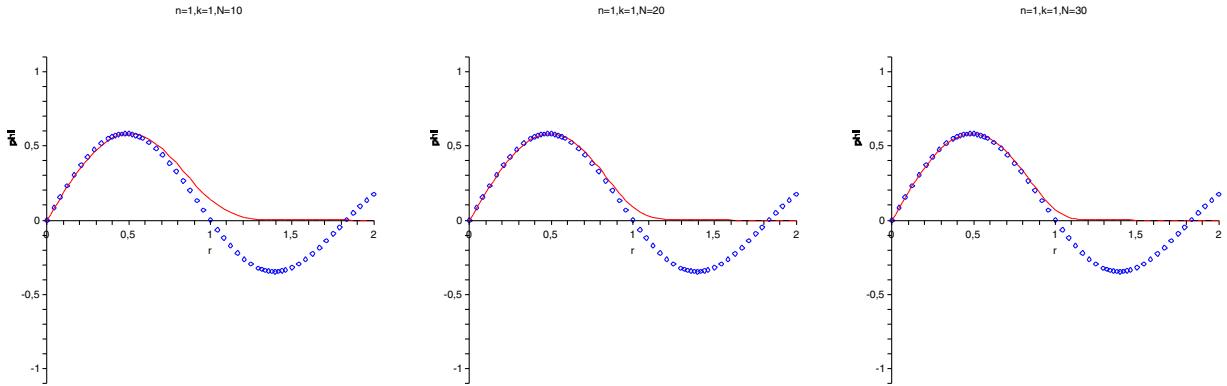
**Figure 3.** Comparison of the radial shape for the  $\Phi_{0,10}^{(N)}$  symbol (continuum line) for  $N = 10, 20, 25$ . The bump goes out of the picture. As  $N$  increases, it becomes narrower and narrower.

the continuum eigenfunction for values of  $r$  close to the centre of the disc, but not on the edge, where a huge bump appears. In [10] it has been explained that the presence of the bump on the edge of the disc, in the fuzzification of a function defined on the plane, is related to the fact that this function has oscillations of too small wavelength compared to  $\theta$ . This is a manifestation of the infrared-ultraviolet mixing characteristic of noncommutative theories. In the case of  $\Phi_{0,10}(r) = J_0(\sqrt{\lambda_{0,10}}r)$ , one can immediately see that the oscillation wavelength of the continuum eigenfunction is given by  $\rho_\lambda \sim 1/k \sim 1/10$ . In these plots, it is assumed  $\theta = 1/N$  (the fuzzy disc truncation), so  $\theta$  and  $\rho_\lambda$  are of compatible magnitude. In the fuzzy disc limit,  $N \rightarrow \infty$ , so  $\theta$  is infinitesimal. The bump disappears, as it is shown in the other plots (figure 4).



**Figure 4.** Comparison of the radial shape for the  $\Phi_{0,10}^{(N)}$  symbol for  $N = 30, 35, 40$ .

The non radial functions follow a similar pattern. Their phases are exactly as the ones of their continuum counterparts, while the radial parts are similar. A first plot is in Fig. 5, a second one in Fig. 6, where the fuzzification procedure gives again a bump, for small values of  $N$ . This bump is seen to disappear in the fuzzy disc limit.



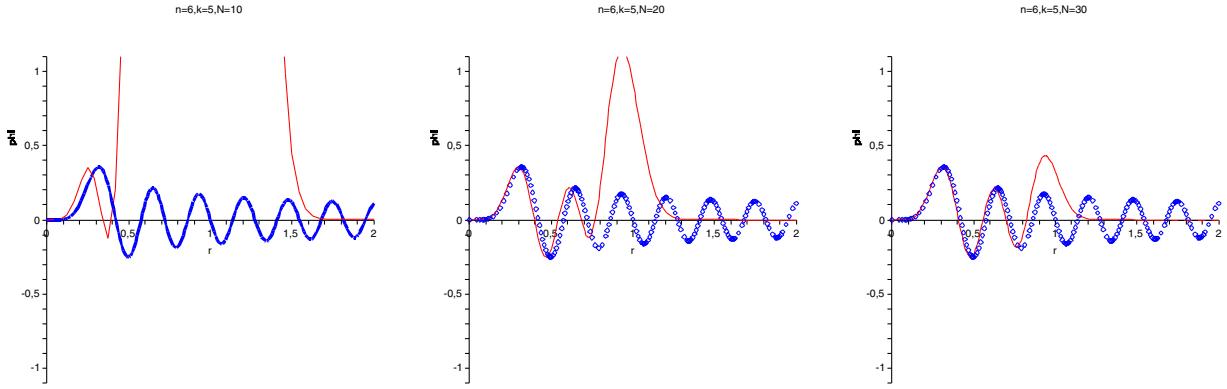
**Figure 5.** Comparison of the radial shape of the  $\Phi_{1,1}^{(N)}(r, \varphi)$  symbol (continuum line) for  $N = 10, 20, 30$ .

Since the fuzzy Bessels play a role similar to fuzzy harmonics for the fuzzy sphere algebra, we can now make the process of approximating the algebra of functions on a disc with matrices more precise. In complete analogy with (B.13) and (B.14), if  $f$  is square integrable with respect to the standard measure on the disc  $d\Omega = r dr d\varphi$ , it can be expanded in terms of Bessel functions:

$$f(r, \varphi) = \sum_{n=-\infty}^{+\infty} \sum_{k=1}^{\infty} f_{nk} e^{in\varphi} J_{|n|} \left( \sqrt{\lambda_{|n|,k}} r \right) \quad (\text{C.49})$$

and it is possible to truncate:

$$f^{(N)}(r, \varphi) = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk} e^{in\varphi} J_{|n|} \left( \sqrt{\lambda_{|n|,k}} r \right) = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk} \Phi_{n,k}(r, \varphi) . \quad (\text{C.50})$$



**Figure 6.** Comparison of the radial shape of the  $\Phi_{6,5}^{(N)}(r, \varphi)$  symbol (continuum line) for  $N = 10, 20, 30$ .

This set of functions is a vector space. It is made into a nonabelian algebra if we induce the product

$$f^{(N)} * g^{(N)} = \langle z | \hat{f}^{(N)} \hat{g}^{(N)} | z \rangle \quad (\text{C.51})$$

with

$$\hat{f}_\theta^{(N)} = \sum_{n=-N}^{+N} \sum_{k=1}^{N+1-|n|} f_{nk} \hat{\Phi}_{n,k}^{(N)}. \quad (\text{C.52})$$

The formal limit  $N \rightarrow \infty$  with the constraint  $N\theta = 1$  is the abelian algebra of functions on the disc. The sequence of nonabelian algebras  $\mathcal{A}_\theta^{(N)}$  is what we call the *fuzzy disc*.

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