

Exact Finite-Difference Calculus: Beyond Set of Entire Functions

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Abstract: In this paper, a short review of the calculus of exact finite-differences of integer order is proposed. The finite-difference operators are called the exact finite-differences of integer orders, if these operators satisfy the same characteristic algebraic relations as standard differential operators of the same order on some function space. In this paper, we prove theorem that this property of the exact finite-differences is satisfied for the space of simple entire functions on the real axis (i.e., functions that can be expanded into power series on the real axis). In addition, new results that describe the exact finite-differences beyond the set of entire functions are proposed. A generalized expression of exact finite-differences for non-entire functions is suggested. As an example, the exact finite-differences of the square root function is considered. The use of exact finite-differences for numerical and computer simulations is not discussed in this paper. Exact finite-differences are considered as an algebraic analog of standard derivatives of integer order.

Keywords: difference calculus; non-standard finite-difference; exact finite-difference; fractional derivative

MSC: 39A70; 47B39; 26A33; 65L12; 39A12



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1. Introduction

Equations with derivatives and integrals of integer and non-integer orders are a very important tool for describing various processes in mechanics, physics, chemistry, biology, economics, and other sciences [1–7]. As another important tool for describing such processes, equations with finite-differences of integer and non-integer orders are also often used. However, the well-known finite-differences of integer and the non-integer orders cannot be considered as exact discrete analogs of derivatives of the same orders. This is due to the fact that the finite-differences have characteristic algebraic properties, which do not coincide with the properties of differential operators. This is clearly seen in the finite-differences and derivatives of the integer order. It is well-known that the standard finite-differences (S-FDs) do not have the same algebraic properties as the standard derivatives of the integer order [8–10].

Exact finite-differences can be called as such finite-difference operators that satisfy the same characteristic algebraic relations as the corresponding standard differential operators of the same order for some function sets. Therefore, one can state that the standard finite-differences of the integer order cannot be considered as the exact finite-differences of the same order. In 1982, the problem of constructing the exact finite-differences of integer orders was first formulated in the form of a problem of the exact discretization of differential equations of integer orders by Potts in the papers [11,12], and then by Mickens in articles [13–15] in 1988. In these works, a concept of non-standard finite-differences (NS-FDs) was proposed. Mickens proved that for differential equations there is a “locally exact” finite-difference discretization, where the local truncation errors are zero. These NS-FDs were used for modeling various processes and systems that described by

differential equations of integer orders in the books and edited volumes [16–19], Mickens papers [20–28], and papers by various scientists [29–63].

A main disadvantage of the non-standard finite-differences is that these difference operators strongly depend on the form of the considered differential equation and the parameters of these equations. In addition, the proposed non-standard finite-differences do not have the same algebraic properties as derivative operators of integer orders. For example, the standard Leibniz rule is a characteristic property of the derivative operators of integer orders [64–66] is not satisfied for the NS-FDs of the integer order. Moreover, these non-standard finite-differences do not form a calculus. For example, the sequential action of the NS-FDs of the first order does not coincide with the action of the NS-FDs of the second order.

Equations with derivatives and integrals of the non-integer orders [1–7] are actively used in various sciences. In addition to the standard finite-differences of the integer order, there are finite-differences of the non-integer order. Such difference operators of the non-integer orders have been first proposed by Grunwald [67] and Letnikov [68] in 1867 and 1868, respectively, (see also [1,4]). It should be noted that the Grunwald–Letnikov differences, and almost all other types of finite-differences of the non-integer order [1,4], cannot be considered as exact finite-differences. This is easy to see by the fact that the discrete Fourier transform of these difference operators does not coincide with the Fourier transform of derivatives of the non-integer order [69]. In addition, for the integer values of orders, these fractional differences do not have the same algebraic properties as the standard derivatives of the same integer-orders.

In 2011, a generalization of NS-FDs for the exact discretization of fractional differential equations of the non-integer order was first proposed by Moaddy, Momani, and Hashim in papers [70,71]. In these papers, the non-standard discretization scheme, which is proposed by Mickens, has been applied to the partial differential equations with fractional derivatives. The Grunwald–Letnikov fractional derivatives have been used in numerical analysis to discretize the fractional differential equations. From this year, the non-standard discretization scheme began to be actively studied and applied to fractional differential equations of the non-integer orders in various works [70–84]. However, the disadvantages of this scheme after generalization for operators of the non-integer order remained almost the same. The proposed difference operators significantly depend on the form of the fractional differential equation and parameters of this equation. In addition, these NS-FDs of the non-integer order do not form a fractional calculus.

Let us briefly describe some basic steps from discrete systems with long-range interactions to the creation of a fractional calculus on physical lattices, which then led to the construction of exact finite-differences of the integer and non-integer order.

Step 1: Power-Law Long-Range Interaction.

In the 2006 papers [85–89], we proposed discrete systems with power-law long-range interactions, which, in a continuous limit, are described by fractional differential equations of non-integer-orders. Equations of these discrete systems can be considered as fractional finite-difference equations with power-law kernels. In this work, it was first shown that these discrete equations of motions with power-law long-range interactions give fractional differential equations of non-integer order in the continuous limit.

Step 2: Long-Range Interaction of Power-Law Type (Alpha Interaction):

In 2006 works [88,89] (see also Section 8 in [90], pp. 335–353), a whole family of long-range interactions was proposed that are called alpha-interactions, for which the equations of motions led to fractional differential equations of non-integer orders in a continuous limit. The set of alpha-interactions, which can also be called long-range interactions of power-law type, contains the power-law long-range interactions as a particular case. In 2006 works [88,89], it was proved that equations with derivatives of non-integer orders can be directly connected with discrete models of lattice systems with long-range interactions of power-law type. In the next few years, some applications of the proposed approach were also considered (for example, see [90–97]).

In most works, equations of discrete systems with such long-range interactions gave equations with a fractional derivative in the continued limit. However, the algebraic properties of discrete and continuous operators, which, in fact, are used in the discrete equations and associated with them are equations with fractional derivatives, which were different and did not coincide. This property is manifested, for example, in the fact that the properties of the Fourier images of discrete operators of a non-integer order and fractional derivatives did not coincide.

Step 3: Alpha Interaction described by the Lommel function.

It should be emphasized that, in two articles [88,89], the finite-difference operators, which, in fact, define the exact finite-difference, were proposed. These finite-differences describe a special form of the long-range interactions of power-law type (alpha-interaction).

The finite-difference operators with this kernel have the Fourier image the exactly coincides with the Fourier image of the fractional differential operator of non-integer order. In the 2006 papers [88,89] and book [90], we proposed a kernel of the difference (lattice) operator, which describe the long-range interactions, which has a discrete Fourier transform coinciding with the Fourier transform of the fractional operator of a non-integer order. The kernel of the difference operator is expressed through the Lommel function (see Equation (41) in [89], p. 092901-7; see Table on page 14900 in [88], and Appendix on page 14908 in [88]; see Definition 8.2 and Example 1 in [90], pp. 166–167, and Table on page 170 of [90]). Note that the Lommel function can be expressed through the generalized hypergeometric function ${}_1F_2(a; b, c; x)$ that is used in the exact finite-difference of arbitrary order, which was proposed in [69], by equation that is given in [98], p. 372, [99], p. 428, and [100], p. 682.

In fact, the proposed discrete (difference) equations of physical lattices and chains with long-range interactions with these kernels, which were proposed in the 2006 papers [88,89], can be considered as equations with exact finite-differences. Unfortunately, the concept of exact discretization of the fractional derivative and the calculus of exact finite-differences was not clearly formulated. This was performed only 10 years later.

Step 4: Lattice Fractional Calculus.

In the articles of 2014–2016 [101–103], a finite-difference operator, which exactly corresponded to the fractional differential operators of the Riesz type, was actually rediscovered. The kernel of these operators are represented through a generalized hypergeometric function ${}_1F_2(a; b, c; x)$. Using these operators, a discrete fractional calculus on physical lattices was formulated and it was called the lattice fractional calculus. In fact, these finite-difference operators, which were called as lattice fractional derivatives, are exact finite-differences, which can be considered as exact discrete analogs of the well-known fractional operators of the Riesz type.

The interconnection between the equations of these discrete systems (lattices) and the fractional differential equations is proved by the special transform operator that includes the Fourier series transform and Fourier integral transform [88,89,101–103]. The suggested differences are derived by the discretization of the Riesz-type differentiation and the integration of non-integer and integer orders.

As a result, in the 2014–2016 papers [101–103], we propose a lattice fractional calculus of finite-difference operators of non-integer orders that have the same characteristic properties of the derivatives of integer and non-integer orders. This approach has been applied to the discrete and lattice models of non-local continua and fields (for example, see [104–108]).

Step 5: Calculus of Exact Finite-Differences.

In the 2015–2017 papers [69,109–111], the exact finite-differences of integer and non-integer orders have been suggested. The proposed new finite-difference operators, which can be considered as an exact discrete analog of derivatives of integer (and non-integer) orders, form a calculus.

These finite-difference operators of integer orders satisfy the same characteristic algebraic relations as the corresponding standard differential operators of the same order on the space of simple entire functions $E(\mathbb{R})$ (i.e., functions that are expandable into power

series in \mathbb{R}). These finite-differences do not depend on the form of differential equations and the parameters of these equations. Using the E-FDs of integer orders, we can obtain the exact discrete analogs of the differential equation of integer orders. The suggested E-FDs allow us to obtain difference equations that exactly correspond to the differential equations, where the exact correspondence exists not only between the equations, but also between their solutions. The discrete analogs of the exact solutions of differential equations are solutions to the corresponding equations with exact finite-differences.

Note that, in contrast to the standard derivatives of a integer order, for which Leibniz's rule is a characteristic property, for the derivatives of a non-integer order, the standard Leibniz rule is violated [64–66]. Because of this, the comparison of the algebraic properties of the fractional finite differences and fractional derivatives is complicated. For the operators of non-integer orders, the most important indicator of the coincidence or difference in properties is the comparison of their Fourier transforms [69].

In paper [69], we proposed an approach to the exact discretization that is based on the principle of universality and the algebraic correspondence principle. The universality principle means that exact finite-differences should not depend on the form of differential equations and the parameters of these equations. An algebraic correspondence means that the exact finite-differences must satisfy the same algebraic relations as the differential operators. Therefore, the proposed exact finite-differences satisfy the exact discrete analogs of algebraic properties.

As the main characteristic algebraic properties of the derivatives of integer orders, one can consider the Leibniz rule (the product rule), the chain rule, the semi-group property, the rule for the action of an integer-order derivative on a power function [64–66,112] (see also [1,113–116]). Therefore, these algebraic properties should be the characteristic algebraic property of exact discrete analogs of the derivative, i.e., the properties of the E-FDs of integer orders. For example, the semi-group property means that the E-FD of the second order should be equal to the sequential action of the E-FDs of the first order. The exact finite-difference analog of the rule for the action of an integer-order derivative on a power function means that the action of the E-FD on power-law functions should give the same expression as an action of derivatives. For E-FDs of integer orders, these properties is important for the space of simple entire functions $E(\mathbb{R})$.

The calculus of exact finite-differences and lattice derivatives are applied to continuum mechanics [97,104,105,117], statistical mechanics [106,107], economics [118–120], quantum mechanics [121–123], and quantum field theory [108,124,125]. The E-FDs, the results and methods proposed in these works were developed in an article on quantum theory [126–129].

Step 6: Exact Finite-Differences Beyond Entire Functions.

Due to the fact that exact finite differences were considered only on the space of simple entire functions $E(\mathbb{R})$, a natural question arises about the properties of these finite-differences outside this space. This is exactly what this 2024 article is dedicated to. It will be shown that the equation that defines the exact finite-difference on the space $E(\mathbb{R})$ of simple entire functions cannot define the exact finite-difference for a wider class of functions. For a wider class of the function equation, which should define E-FDs, must be generalized to preserve its characteristic properties of standard derivatives of integer orders. A generalized equation of the exact finite-differences is proposed to use these differences for a wider class of functions. The equation of E-FDs, which is used for simple entire functions, is a special case of the new proposed generalized equation that defines the E-FDs.

In this paper, a short review of the basic properties of exact finite-differences is proposed. The sixth step is performed in this article, which gives a generalized definition of exact finite-differences beyond the space of simple entire functions, is suggested in this paper. The fact that the equation, which defines the generalized form of the exact finite-differences, gives an exact algebraic analog of the standard derivatives of a integer order is proven. For simple entire functions, this generalized equation takes the previously proposed (not generalized) form. As an example, the exact finite-difference of the square

root function is described. Note that the use of exact finite-differences for numerical and computer simulations is not discussed in this paper.

Let us briefly describe the contents of the paper. In Section 2, it is demonstrated that standard and well-known non-standard finite-differences cannot be considered as exact finite-differences. In Section 3, a short review of exact finite-differences and its properties for the space of entire functions is suggested. In Section 4, a generalized definition and basic properties of exact finite-differences for a set of non-entire functions are proposed. A short conclusion is given in Section 5.

2. Standard and Non-Standard Finite-Differences

In this section, we demonstrate that standard finite-differences (S-FD) and non-standard finite-differences (NS-FD) cannot be considered as candidates for exact finite-differences (E-FD). In order for finite-difference operators to be exact finite-differences of integer orders, these operators must satisfy the same characteristic algebraic relations as the corresponding standard differential operators of the same order on some function space [69]. To demonstrate the fact that S-FD and NS-FD cannot be E-FD, we will list the well-known properties of the standard and non-standard finite-differences of integer orders that do not coincide with the corresponding properties of the standard differential operators of the same orders.

2.1. Standard Finite-Differences

Standard finite-differences Δ^k and Δ_h^k can be defined in various forms. Let us give some examples of the well-known standard finite-differences.

(1) The forward difference

$${}^f\Delta^1 f[n] := f[n+1] - f[n], \quad (1)$$

$${}^f\Delta_h^1 f(x) := f(x+h) - f(x), \quad (2)$$

$${}^f\Delta_h^m f(x) := \sum_{k=0}^m (-1)^{m-k} \binom{m}{k} f(x+kh). \quad (3)$$

(2) The backward difference

$${}^b\Delta^1 f[n] := f[n] - f[n-1], \quad (4)$$

$${}^b\Delta_h^1 f(x) := f(x) - f(x-h), \quad (5)$$

$${}^b\Delta_h^m f(x) := \sum_{k=0}^m (-1)^k \binom{m}{k} f(x-kh). \quad (6)$$

(3) The central difference

$${}^c\Delta_h^m f(x) := \sum_{k=0}^m (-1)^k \binom{m}{k} f\left(x + \left(\frac{m}{2} - k\right)h\right). \quad (7)$$

Here $n \in \mathbb{Z}$, $m \in \mathbb{N}$, $k \in \mathbb{N}$, $x \in \mathbb{R}$.

The standard finite-differences Δ^k of the positive integer order $m \in \mathbb{N}$, do not preserve the main characteristic property of the derivatives d^m/dx^m (the standard Leibniz rule) in general.

Let us note some properties of the S-FDs, which do not coincide with the standard properties of integer-order derivatives. To simplify the consideration, one can consider the forward S-FDs.

(1) The standard Leibniz rule is violated as follows:

$${}^f\Delta^1 (f[n]g[n]) \neq ({}^f\Delta^1 f[n])g[n] + f[n]({}^f\Delta^1 g[n]) \quad (n \in \mathbb{Z}), \quad (8)$$

since the forward finite-difference satisfies the equality

$${}^f\Delta^1 (f[n]g[n]) = ({}^f\Delta^1 f[n])g[n] + f[n]({}^f\Delta^1 g[n]) + ({}^f\Delta^1 f[n])({}^f\Delta^1 g[n]) \quad (n \in \mathbb{Z}), \quad (9)$$

where $({}^f\Delta^1 f[n])({}^f\Delta^1 g[n]) \neq 0$, in the general case.

For an algebra A over a ring or a field R , a R -linear map $D : A \rightarrow A$ is called the derivation if D satisfies the standard Leibniz rule $D(fg) = D(f)g + fD(g)$ for all $f, g \in A$. Using (8), one can see that the S-FDs cannot be considered as a derivation on an algebra. In mathematics, a derivation is a map on an algebra, which generalizes the derivatives of the first order. As a result, the S-FDs cannot be considered as an exact analog of derivatives of integer orders since the S-FDs violate the standard Leibniz rule.

- (2) An action of the S-FD ${}^f\Delta^1$ on a function $f(n)$, $n \in \mathbb{Z}$, cannot give the same result as an action of the derivative d/dx on $f(x)$, $x \in \mathbb{R}$:

$${}^f\Delta^1 f(n) \neq \left(\frac{df(x)}{dx} \right)_{x=n}. \quad (10)$$

For example, inequality (10) can be demonstrated for the elementary functions x^m , $\sin(kx)$ and $\exp(kx)$ as follows:

- (a) The power-law function

$${}^f\Delta n^m \neq m n^{m-1} \quad (n \in \mathbb{Z}, \quad m \in \mathbb{N}), \quad (11)$$

$${}^f\Delta^1 n^m = (n+1)^m - n^m = \sum_{s=1}^m \binom{m}{s} n^{m-s} \neq \binom{m}{1} n^{m-1} \quad (n \in \mathbb{Z}). \quad (12)$$

- (b) The sine and cosine function

$${}^f\Delta^1 \sin(kn) = k \cos(kn) \cdot \frac{\sin(k)}{k} + \sin(kn) \cdot (\cos(k) - 1) \neq k \cos(kn). \quad (13)$$

- (c) The exponential function

$${}^f\Delta^1 \exp(kn) = \exp(kn) \cdot (\exp(k) - 1) \neq k \exp(kn). \quad (14)$$

Here, $n \in \mathbb{N}$, $k \in \mathbb{R}$.

- (3) The solutions of equations with S-FDs do not coincide with the solutions of the corresponding differential equation with integer-order derivatives. For example, one can compare the first-order differential equation with its solution as follows:

$$\frac{df(x)}{dx} = -\lambda f(x) \iff f(x) = f(0) e^{-\lambda x}, \quad (15)$$

and the equation with forward S-FD with its solution

$${}^f\Delta^1 f[n] = -\lambda f[n] \iff f[n] = f[0] (1 - \lambda)^n \neq f[0] e^{-\lambda n}. \quad (16)$$

It can be seen that these solutions do not coincide.

In addition to the mentioned properties, the Fourier integral transform \mathcal{F} of the integer-order derivatives is

$$\mathcal{F}\left(\frac{d^m f(x)}{dx^m}\right)(k) = (ik)^m \mathcal{F}(f(x))(k) \quad m \in \mathbb{N}. \quad (17)$$

The Fourier series transform \mathcal{F}_Δ of the standard backward difference of order is

$$\mathcal{F}_\Delta \left({}^b\Delta^m f(x) \right) (k) = (1 - \exp(ik))^m \mathcal{F}_\Delta(f(x))(k) \quad m \in \mathbb{N}. \quad (18)$$

One can see that these Fourier transforms are not similar.

As a result, the S-FDs and standard integer-order derivatives have different algebraic properties and, therefore, the S-FDs cannot be considered as exact finite-difference analogs of derivatives of an integer order.

2.2. Non-Standard Finite-Differences

The concept of non-standard finite-differences (NS-FDs) was first proposed in Potts papers [11,12], Mickens articles [13–15], and the books [16,19]. The NS-FDs have the following important property: the solutions of equations with NS-FDs coincide with the solutions of the corresponding differential equation with integer-order derivatives. Unfortunately, for each differential equation this approach associates its own NS-FD operator. Therefore, there are no universal NS-FDs that can be used for all types of differential equations.

Let us give some examples of equations with the NS-FDs and the expressions of the corresponding NS-FDs. In these examples, we emphasize some disadvantages of the Mickens approach from the point of view of a construction of exact analogs finite-differences.

Example 1. For the following differential equation:

$$\frac{df(x)}{dx} + \lambda f(x) = 0 \quad (x \in \mathbb{R}) \quad (19)$$

with $\lambda \in \mathbb{R}$ and its solution $f(x) = f(0) \exp(-\lambda x)$, the difference equation with HS-FDs is

$${}_1^M\Delta_{h,\lambda}^1 f[n] + \lambda f[n] = 0, \quad f[0] = C \quad (n \in \mathbb{Z}), \quad (20)$$

where

$${}_1^M\Delta_{h,\lambda}^1 f[n] := \frac{f[n+1] - f[n]}{(1 - \exp(-\lambda h))/\lambda}. \quad (21)$$

One can see that NS-FD (21) depends on the parameter λ of Equation (19).

Example 2. Equation

$$\frac{d^2 f(x)}{dx^2} + \lambda^2 f(x) = 0 \quad (x \in \mathbb{R}), \quad (22)$$

where λ is a real constant, can be represented in the following form:

$$\frac{df_1(x)}{dx} - f_2(x) = 0, \quad \frac{df_2(x)}{dx} + \lambda^2 f_1(x) = 0 \quad (x \in \mathbb{R}). \quad (23)$$

The system of equations with NS-FD is

$${}_2^M\Delta_{h,\lambda}^1 f_1[n] - f_2[n] = 0, \quad {}_2^M\Delta_{h,\lambda}^1 f_2[n] + \lambda^2 f_1[n] = 0 \quad (n \in \mathbb{Z}), \quad (24)$$

where

$${}_2^M\Delta_{h,\lambda}^1 f[n] := \frac{f[n+1] - \cos(\lambda h) f[n]}{\sin(\lambda h)/\lambda}. \quad (25)$$

One can see that in Examples 1 and 2 the NS-FDs of the first order are not identical since the form of NS-FDs depends on the differential equations and its parameters.

Example 3. For the differential equation

$$\frac{d^2 f(x)}{dx^2} + \lambda^2 f(x) = 0 \quad (x \in \mathbb{R}), \quad (26)$$

where λ is a real constant, the equation with NS-FDs is

$${}^M\Delta_{h,\lambda}^2 f[n] + \lambda^2 f[n] = 0 \quad (n \in \mathbb{Z}), \quad (27)$$

where

$${}^M\Delta_{h,\lambda}^2 f[n] := \frac{f[n+1] - 2f[n] + f[n-1]}{(4/\lambda^2) \sin^2(\lambda h/2)}. \quad (28)$$

Using these examples of equations with NS-FDs and its solutions, one can see the following disadvantages of the NS-FDs.

1. Using Equations (21) and (25), one see that the NS-FDs of the same order are not identical, i.e.,

$${}^M_1\Delta_{h,\lambda}^1 \neq {}^M_2\Delta_{h,\lambda}^1. \quad (29)$$

2. Using Equations (21) and (25), and Equation (28), one can see that the second-order NS-FD (28) cannot be represented as a sequential action of the first-order NS-FDs (21) or (25). Therefore the semi-group property of NS-FDs is violated;

$${}^M\Delta_{h,\lambda}^2 \neq {}^M\Delta_{h,\lambda}^1 {}^M\Delta_{h,\lambda}^1. \quad (30)$$

3. The integer-order NS-FDs strongly depends on the form of the differential equation and the parameters in it. Therefore, the set of the NS-FDs cannot form a calculus of NS-FDs since the set of integer-order NS-FDs cannot be known completely;
4. If the solution of differential equation is known, then the NS-FDs can be constructed. For a general differential equation, for which the solution is not known, NS-FDs cannot be proposed in a general form. If we do not know in advance a solution for the differential equation, then the corresponding NS-FDs cannot be constructed;
5. The standard Leibniz rule is violated;

$${}^M\Delta^1 (f[n] g[n]) \neq ({}^M\Delta^1 f[n]) g[n] + f[n] ({}^M\Delta^1 g[n]) \quad (n \in \mathbb{Z}). \quad (31)$$

It can easy to see from the following representation:

$${}^M\Delta_{h,\lambda}^1 f[n] := \frac{f[n+1] - f[n]}{(1 - \exp(-\lambda h))/\lambda} = \frac{\lambda}{(1 - \exp(-\lambda h))} \cdot {}^f\Delta_h^1 f[n] \quad (32)$$

and Equation (8).

Using (31), one can see that the NS-FDs cannot be considered as a derivation on an algebra. As a result, the NS-FDs cannot be considered as an exact analog of the integer-order derivatives, since the NS-FDs violate the standard Leibniz rule.

3. Exact Finite-Differences for Entire Functions

A set of finite-difference operators ${}^{\mathcal{T}}\Delta_h^m$ and ${}^{\mathcal{T}}\Delta_h^m$ will be called the exact finite-differences (E-FDs) of integer orders $m \in \mathbb{N}$, if all operators of this set satisfy the same characteristic algebraic relations as the corresponding differential operators on some function space. The correspondence between the calculus of E-FDs and the calculus of the standard differential operators of integer order lies not so much in the limiting condition, when the step $h \rightarrow 0$. This correspondence lies in the fact that operators of these two theories should obey, in many cases, the same algebraic rules. This statement is called the “Algebraic Correspondence Principle” [69].

3.1. Set of Entire Functions

Let us consider exact finite-differences of simple entire functions in \mathbb{R} , i.e., functions which are expandable into a power series in \mathbb{R} . The space of such functions will be denoted as $E(\mathbb{R})$.

Let us note some properties of the entire functions $f(x)$ with $x \in \mathbb{R}$.

1. Every real-valued entire function $f(x)$ with $x \in \mathbb{R}$, can be represented as a power series

$$f(x) = \sum_{k=0}^{\infty} f_k x^k \quad (|x| < \infty), \quad (33)$$

which converges everywhere in the real line.

2. Any series

$$\sum_{k=0}^{\infty} f_k x^k \quad (|x| < \infty), \quad (34)$$

for which the Cauchy–Hadamard condition

$$\lim_{k \rightarrow \infty} \sqrt[k]{|f_k|} = 0 \quad (35)$$

is satisfied, represents an entire function $f(x)$ from $E(\mathbb{R})$.

3. Let us give examples of the following entire functions:
 - (A) The polynomials $P_n(x)$; exponential function: $e^x = \exp(x)$; sine and cosine functions $\sin(x)$, $\cos(x)$; hyperbolic sine and cosine functions $\sinh(x)$, $\cosh(x)$;
 - (B) The Mittag–Leffler function $E_\alpha(x)$; Gamma function reciprocal $1/\Gamma(x)$; Wright function $\phi(\rho, \beta; x)$; generalized hypergeometric function ${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; x)$;
 - (C) Bessel function of the first and second kinds $J_n(x)$, $Y_n(x)$; sine integral $\text{Si}(x)$; error function $\text{erf}(x)$; Airy functions $\text{Ai}(x)$, $\text{Bi}(x)$; Jacobi theta functions $\theta_n(x, q)$;

3.2. Generalized (Cesaro and Poisson–Abel) Summation

In the definitions of exact finite-differences, we use generalized convergences (generalized summations) that assign values to some infinite sums that are not convergent in the usual sense. For example, one can consider the Cesaro, Poisson–Abel, and Mittag–Leffler summations [130–133].

Let us define the Cesaro summation.

Definition 1. Let $g[m]$ be a discrete function (sequence) with $m \in \mathbb{Z}$ and let

$$A[k] = \sum_{m=1}^k g[m] \quad (36)$$

be the partial sum of the series

$$\sum_{m=1}^{\infty} g[m]. \quad (37)$$

The series (37) and the sequence $g[m]$ are called Cesaro summable, with the sum $A \in \mathbb{R}$, if the average (mean) value of its partial sums $A[k]$ tends to A :

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=1}^N A[k] = A. \quad (38)$$

The Cesaro sum is a limit of the arithmetic mean (average) of first N partial sums of the series, as N goes to infinity. It is well-known that any convergent series is Cesaro summable, and the sum of the series is equal to its Cesaro sum. There are divergent series that are Cesaro summable.

Let us define the Poisson–Abel summation.

Definition 2. Let $g[m]$ be a discrete function (sequence) with $m \in \mathbb{Z}$ and let

$$B(x) := \sum_{m=0}^{\infty} g[m] x^m = \sum_{m=0}^{\infty} g[m] \exp(-m t) \quad (39)$$

be the corresponding power series. If this series is convergent for $0 < x = e^{-t} < 1$, (if $t > 0$, then $0 < e^{-t} < 1$) and its sum $B(x)$ has the limit B at $x \rightarrow 1-0$ ($t \rightarrow 0+$),

$$\lim_{x \rightarrow 1-0} B(x) = B, \quad (40)$$

then the series (39) (and the sequence $g[m]$) is called Poisson–Abel summable and A is a Poisson–Abel sum.

Note that if the series is Cesaro summable with the sum A then this series is Poisson–Abel summable with sum $B = A$ (for example, see Chapters 11 and 12 of [130], including Section 449).

3.3. Definition and Examples of Exact Finite-Differences of Integer Orders

Let us define the exact finite-differences (E-FDs) of the first order.

Definition 3. Let $f(x)$ belong to the set $E(\mathbb{R})$ of the simple entire functions.

The exact finite-difference of the first order with the step $h > 0$ for the function $f(x)$ is defined as

$$\mathcal{T}\Delta_h^1 f(x) := \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} \left(f(x - h m) - f(x + h m) \right), \quad (41)$$

where the sum means the Poisson–Abel (or Cesaro) summation.

The exact finite-difference of a positive integer order $m \in \mathbb{N}$ is

$$\mathcal{T}\Delta_h^m f(x) := (\mathcal{T}\Delta_h^1)^m f(x), \quad (42)$$

where $f(x) \in E(\mathbb{R})$.

Definition 4. Let $f[n]$ belong to the set $E(\mathbb{Z})$ of the simple entire functions.

Then, the first-order exact finite-difference is defined as

$$\mathcal{T}\Delta^1 f[n] := \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{(-1)^m}{m} f[n - m] = \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} \left(f(n - m) - f(n + m) \right), \quad (43)$$

where the sum means the Poisson–Abel (or Cesaro) summation.

The exact finite-difference of the positive integer order $m \in \mathbb{N}$ is

$$\mathcal{T}\Delta^m f[n] := (\mathcal{T}\Delta^1)^m f[n], \quad (44)$$

where $f[n] \in E(\mathbb{Z})$.

Remark 1. Using the following function:

$$g[m] := \frac{(-1)^m}{m} \left(f(n - m) - f(n + m) \right), \quad (45)$$

the Poisson–Abel summation of (48) means that the following limits exist:

$$\mathcal{T}\Delta^1 f[n] := \lim_{x \rightarrow 1-0} \sum_{m=1}^{\infty} \frac{(-x)^m}{m} \left(f[n - m] - f[n + m] \right), \quad (46)$$

or

$$\mathcal{T}\Delta^1 f[n] := \lim_{t \rightarrow 0+} \sum_{m=1}^{\infty} \frac{(-1)^m e^{-mt}}{m} (f[n-m] - f[n+m]). \quad (47)$$

Let us give examples of the exact finite-differences of orders equal to 1, 2, 3, and 4. The first-order exact finite-difference is

$$\mathcal{T}\Delta^1 f[n] := \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{(-1)^m}{m} f[n-m]. \quad (48)$$

The second-order exact finite-difference is

$$\mathcal{T}\Delta^2 f[n] := (\mathcal{T}\Delta^1)^2 f[n] = - \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{2(-1)^m}{m^2} f[n-m] - \frac{\pi^2}{3} f[n]. \quad (49)$$

The third-order exact finite-difference is

$$\mathcal{T}\Delta^3 f[n] := (\mathcal{T}\Delta^1)^3 f[n] = - \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \left(\frac{(-1)^m \pi^2}{m} - \frac{6(-1)^m}{m^3} \right) f[n-m]. \quad (50)$$

The fourth-order exact finite-difference is

$$\mathcal{T}\Delta^4 f[n] := (\mathcal{T}\Delta^1)^4 f[n] = \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \left(\frac{4\pi^2(-1)^m}{m^2} - \frac{24(-1)^m}{m^4} \right) f[n-m] + \frac{\pi^4}{5} f[n]. \quad (51)$$

3.4. Exact Finite-Difference and Derivative of Integer Orders

Let us prove one of the most important theorems, which directly shows the relationship between the exact finite-difference and the standard derivative of an integer order.

Theorem 1. Let $f(x)$ belong to the set $E(\mathbb{R})$ of the simple entire functions.

For the exact finite-difference of the first order with the step $h > 0$ of the following form:

$$\mathcal{T}\Delta_h^1 f(x) := \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} (f(x-hm) - f(x+hm)), \quad (52)$$

where the sum means the Poisson–Abel (or Cesaro) summation, the following equality:

$$\frac{1}{h} \mathcal{T}\Delta_h^1 f(x) = f^{(1)}(x) \quad (53)$$

holds for all $x \in \mathbb{R}$, and all $h \in (0, \infty)$, where $f^{(1)}(x) = df(x)/dx$ is the standard first-order derivative.

Proof. The exact finite-difference of the first order with the step $h > 0$ is

$$\mathcal{T}\Delta_h^1 f(x) := \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} (f(x-hm) - f(x+hm)). \quad (54)$$

The Taylor series of a real-valued entire function $f(z)$, where $z = x + \Delta x$, and x is a real number, can be written as the following power series:

$$f(x \pm \Delta x) = f(x) \pm f^{(1)}(x) \cdot \Delta x + \frac{1}{2} f^{(2)}(x) \cdot (\Delta x)^2 \pm R_{3,\pm}(x, \Delta x), \quad (55)$$

where $R_{3,\pm}(x, \Delta x)$ is a remainder term of the Taylor series in the Lagrange's form

$$R_{3,\pm}(x, \Delta x) := \frac{1}{3!} f^{(3)}(c_{\pm}) (\Delta x)^3 \quad (0 < |c_{\pm}| < |\Delta x|). \quad (56)$$

Using $\Delta x = h m$, we have the following:

$$f(x - h m) - f(x + h m) = -2 m h f^{(1)}(x) - \frac{1}{3!} (f^{(3)}(c_-) + f^{(3)}(c_+)) (h m)^3. \quad (57)$$

Substituting (57) into (54) gives the following:

$$\frac{1}{h} {}^{\mathcal{T}}\Delta_h^1 f(x) = -2 f^{(1)}(x) \sum_{m=1}^{+\infty} (-1)^m - \frac{1}{3!} (f^{(3)}(c_-) + f^{(3)}(c_+)) h^2 \sum_{m=1}^{+\infty} (-1)^m m^2. \quad (58)$$

For the Poisson–Abel summation (or Cesaro summation), we have the following:

$$\sum_{m=1}^{+\infty} (-1)^m = -\frac{1}{2} \quad (\text{the Grandi's series}), \quad \sum_{m=1}^{+\infty} (-1)^m m^{2j} = 0 \quad (j \in \mathbb{N}). \quad (59)$$

As the result, we obtain the following:

$$\frac{1}{h} {}^{\mathcal{T}}\Delta_h^1 f(x) = f^{(1)}(x) \quad (\text{the Poisson–Abel summation}), \quad (60)$$

where $x \in \mathbb{R}$ and $h \in \mathbb{R}_+$. \square

Remark 2. Another proof of Theorem 1 is given in paper [69] (see Theorem 8).

Remark 3. It should be emphasized that Equation (53) holds without passing to the limit $h \rightarrow 0$.

The following statements directly follows from Theorem 1.

Corollary 1. Let $f(x)$ belong to the set $E(\mathbb{R})$ of the simple entire functions.
For the exact finite-difference of the order $k \in \mathbb{N}$ with the step $h > 0$ of the form

$${}^{\mathcal{T}}\Delta_h^k f(x) := \left({}^{\mathcal{T}}\Delta_h^1 \right)^k f(x), \quad (61)$$

the equality

$$\frac{1}{h^k} {}^{\mathcal{T}}\Delta_h^k f(x) = f^{(k)}(x) \quad (62)$$

holds for all $x \in \mathbb{R}$, and all $h \in (0, \infty)$, where $f^{(k)}(x) = d^k f(x) / dx^k$ is the standard derivative of the order $k \in \mathbb{N}$.

Corollary 2. Let $f[n]$ belong to the set $E(\mathbb{Z})$ of the simple entire functions.
For the exact finite-difference of the first order

$${}^{\mathcal{T}}\Delta^1 f[n] := \left({}^{\mathcal{T}}\Delta_{h=1}^1 f(x) \right)_{x=n} = \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} (f(n-m) - f(n+m)), \quad (63)$$

where the sum means the Poisson–Abel (or Cesaro) summation, the equality

$${}^{\mathcal{T}}\Delta^1 f[n] = \left(\frac{df(x)}{dx} \right)_{x=n} \quad (64)$$

holds for all $n \in \mathbb{Z}$, where $df(x) / dx$ is the standard first-order derivative.

Corollary 3. Let $f(x)$ belong to the set $E(\mathbb{R})$ of the simple entire functions.

For the exact finite-difference of the first order

$$\mathcal{T}\Delta_h^1 f(x) := \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} \left(f(x - hm) - f(x + hm) \right), \quad (65)$$

where the sum means the Poisson–Abel (or Cesaro) summation, the equality

$$\lim_{h \rightarrow 0+} \frac{1}{h} \mathcal{T}\Delta_h^1 f(x) = f^{(1)}(x) \quad (66)$$

holds for $x \in \mathbb{R}$, where $f^{(1)}(x)$ is the standard first-order derivative.

Proof. Using $\sum_{m=1}^{+\infty} (-1)^m = -1/2$ and passing to the limit $h \rightarrow 0+$ in (58), we obtain the following:

$$\lim_{h \rightarrow 0+} \frac{1}{h} \mathcal{T}\Delta_h^1 f(x) = f^{(1)}(x) \implies \frac{df(x)}{dx} = \lim_{h \rightarrow 0+} \frac{\mathcal{T}\Delta_h^1 f(x)}{h}. \quad (67)$$

□

Therefore, the derivatives of integer order can be defined through the exact finite-differences of the same order by Equation (66).

Corollary 4. For derivative of the integer order $k \in \mathbb{N}$ and the exact finite-difference of the order $k \in \mathbb{N}$ the following equality is satisfied:

$$\frac{d^k f(x)}{dx^k} = \lim_{h \rightarrow 0+} \frac{\mathcal{T}\Delta_h^k f(x)}{h^k} \quad (k \in \mathbb{N}), \quad (68)$$

where $\mathcal{T}\Delta_h^k := (\mathcal{T}\Delta_h^1)^k, k \in \mathbb{N}$.

3.5. Properties of Exact Finite-Differences of Integer Orders

Let us prove the main characteristic algebraic property of the exact finite-differences $\mathcal{T}\Delta^1$ of the first order on the space of entire functions $E(\mathbb{Z})$.

Theorem 2 (Leibniz rule). Let $f[n]$ and $g[n]$ belong to the set $E(\mathbb{Z})$ of the simple entire functions. For the exact finite-difference of the first order

$$\mathcal{T}\Delta^1 f[n] := \mathcal{T}\Delta_{h=1}^1 f(x=n) := \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} \left(f(n-m) - f(n+m) \right), \quad (69)$$

where the sum means the Poisson–Abel (or Cesaro) summation, the standard product rule (the Leibniz rule) of the form

$$\mathcal{T}\Delta^1 \left(f[n] g[n] \right) = f[n] \left(\mathcal{T}\Delta^1 g[n] \right) + \left(\mathcal{T}\Delta^1 f[n] \right) g[n] \quad (70)$$

holds for all $n \in \mathbb{Z}$.

Proof. Using Equation (64) and the standard product rule for the first-order derivative, we obtain the following:

$$\begin{aligned} \mathcal{T}\Delta^1 \left(f[n] g[n] \right) &= \left(\frac{d}{dx} (f(x) g(x)) \right)_{x=n} = \\ &= \left(f^{(1)}(x) g(x) + f(x) g^{(1)}(x) \right)_{x=n} = \left(f^{(1)}(x) \right)_{x=n} g[n] + f[n] \left(g^{(1)}(x) \right)_{x=n} = \end{aligned}$$

$$f[n] \left({}^{\mathcal{T}}\Delta^1 g[n] \right) + \left({}^{\mathcal{T}}\Delta^1 f[n] \right) g[n], \quad (71)$$

where we use the property $f[n] g[n] \in E(\mathbb{Z})$, if $f[n] \in E(\mathbb{Z})$ and $g[n] \in E(\mathbb{Z})$. \square

Remark 4. Another proof of Theorem 2 is given in paper [69], p. 49, (see Theorem 7).

For exact finite-differences ${}^{\mathcal{T}}\Delta_h^1$ on the space of entire functions $E(\mathbb{R})$, the chain rule can be proved by using Theorem 1 and the standard chain rule for a derivative of the first order.

Theorem 3 (Chain rule). Let $f(x)$, $y(x)$ and $g(x) := f(y(x))$ belong to the set $E(\mathbb{R})$ of the simple entire functions.

For the exact finite-difference of the first order with the step $h > 0$ of the form

$${}^{\mathcal{T}}\Delta_h^1 f(x) := \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} \left(f(x - hm) - f(x + hm) \right), \quad (72)$$

where the sum means the Poisson–Abel (or Cesaro) summation, the equality

$${}^{\mathcal{T}}\Delta_h^1 f(y(x)) = \frac{1}{h} \left({}^{\mathcal{T}}\Delta_h^1 f(y) \right)_{y=y(x)} {}^{\mathcal{T}}\Delta_h^1 y(x) \quad (73)$$

holds for all $x \in \mathbb{R}$, and all $h \in (0, \infty)$, where $f^{(1)}(x) = df(x)/dx$ is the standard first-order derivative.

Proof. Using equality (53) for $g(x) := f(y(x)) \in E(\mathbb{R})$ and the standard chain rule for $g(x)$, one can obtain the following:

$$\begin{aligned} \frac{1}{h} {}^{\mathcal{T}}\Delta_h^1 g(x) &= g^{(1)}(x) = \frac{d}{dx} f(y(x)) = \\ &= \left(\frac{df(y)}{dy} \right)_{y=y(x)} \frac{dy(x)}{dx} = \left(\frac{1}{h} {}^{\mathcal{T}}\Delta_h^1 f(y) \right)_{y=y(x)} \frac{1}{h} {}^{\mathcal{T}}\Delta_h^1 y(x), \end{aligned} \quad (74)$$

where

$${}^{\mathcal{T}}\Delta_h^1 f(y) := \sum_{m=1}^{+\infty} \frac{(-1)^m}{m} \left(f(y - hm) - f(y + hm) \right). \quad (75)$$

and $f(y) \in E(\mathbb{R})$ and $y(x) \in E(\mathbb{R})$. \square

Remark 5. The chain rule for the E-FD ${}^{\mathcal{T}}\Delta^1$ of the first order on the space $E(\mathbb{Z})$ is given as follows:

$${}^{\mathcal{T}}\Delta^1 f(y[n]) = \left({}^{\mathcal{T}}\Delta^1 f(y) \right)_{y=y(n)} {}^{\mathcal{T}}\Delta^1 y[n] \quad (76)$$

holds for all $n \in \mathbb{Z}$. Equality (76) is a corollary of Theorem 3 and the equation

$${}^{\mathcal{T}}\Delta^1 f[n] = \left({}^{\mathcal{T}}\Delta_{h=1}^1 f(x) \right)_{x=n}. \quad (77)$$

The exact finite-differences ${}^{\mathcal{T}}\Delta^k$ of positive integer order $k \in \mathbb{N}$ preserve the following characteristic property of the derivatives d^k/dx^k on the space of entire functions $f(x), g(x) \in E(\mathbb{R})$, which are proved in [69,103,109].

(1) The Leibniz rule

$${}^{\mathcal{T}}\Delta^1 \left(f[n] g[n] \right) = \left({}^{\mathcal{T}}\Delta^1 f[n] \right) g[n] + f[n] \left({}^{\mathcal{T}}\Delta^1 g[n] \right) \quad (n \in \mathbb{Z}), \quad (78)$$

$${}^{\mathcal{T}}\Delta^k (f[n]g[n]) = \sum_{j=0}^k \binom{k}{j} ({}^{\mathcal{T}}\Delta^{k-j} f[n]) ({}^{\mathcal{T}}\Delta^j g[n]) \quad (n \in \mathbb{Z}, k, j \in \mathbb{N}). \quad (79)$$

(2) The chain rule

$${}^{\mathcal{T}}\Delta^1 f(g[n]) = ({}^{\mathcal{T}}\Delta^1 f(g))_{g=g(n)} {}^{\mathcal{T}}\Delta^1 g[n] \quad (n \in \mathbb{Z}). \quad (80)$$

(3) The semi-group property

$${}^{\mathcal{T}}\Delta^1 ({}^{\mathcal{T}}\Delta^1 f[n]) = {}^{\mathcal{T}}\Delta^2 f[n] \quad (n \in \mathbb{Z}), \quad (81)$$

$${}^{\mathcal{T}}\Delta^k ({}^{\mathcal{T}}\Delta^j f[n]) = {}^{\mathcal{T}}\Delta^{k+j} f[n] \quad (n \in \mathbb{Z}, k, j \in \mathbb{N}). \quad (82)$$

(4) The equations for power-law entire functions

$${}^{\mathcal{T}}\Delta^1 n^j = j n^{j-1} \quad (n \in \mathbb{Z}, j \in \mathbb{N} \quad j \geq 1), \quad (83)$$

$${}^{\mathcal{T}}\Delta^k n^j = \frac{j!}{(j-k)!} n^{j-k} \quad (n \in \mathbb{Z}, k, j \in \mathbb{N} \quad j \geq k). \quad (84)$$

(5) Let $f(x)$ and $g(x)$ be entire functions ($f(x), g(x) \in E(\mathbb{R})$). If these functions satisfy the following equation:

$$\frac{d^k f(x)}{dx^k} = g(x) \quad (85)$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{N}$, then the following equation:

$${}^{\mathcal{T}}\Delta^k f[n] = g[n] \quad (86)$$

holds for all $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

(6) The Fourier series transform \mathcal{F}_{Δ} of the exact finite-difference of order $m \in \mathbb{N}$ is

$$\mathcal{F}_{\Delta} ({}^{\mathcal{T}}\Delta^m f[n]) (k) = (ik)^m (\mathcal{F}_{\Delta} f[n]) (k), \quad (87)$$

which has the same form as the Fourier integral transform \mathcal{F} of the integer-order derivatives

$$\mathcal{F} \left(\frac{d^m f(x)}{dx^m} \right) (k) = (ik)^m \mathcal{F}(f(x)) (k), \quad (88)$$

where $n \in \mathbb{N}$.

(7) If $p \geq 1$, then we can define a norm on the l^p -space by the following equation:

$$\|f(m)\|_p := \left(\sum_{n=-\infty}^{+\infty} |f(m)|^p \right)^{1/p}, \quad (89)$$

where $m \in \mathbb{Z}$. The sequence space l^p with $p > 0$ is a complete metric space with respect to this norm (89) and, therefore, it is the Banach space. Using that $l^q \subset l^p$ ($1 \leq p < q$), then $l^q \subset l^2$ if $q > 2$, and $f[m] \in l^q$ with $q \geq 2$.

Theorem 4. Let ${}^{\mathcal{T}}\Delta^n$ be an exact finite-difference that is defined by convolutions of its kernel $K_n(m) \in l^p$ ($p > 1$) and function $f[m] \in l^q$ ($q \geq 2$) in the form

$${}^{\mathcal{T}}\Delta^n f[m] := \sum_{k=-\infty}^{\infty} K_n(k) f[m-k]. \quad (90)$$

Then, the operator ${}^{\mathcal{T}}\Delta^n$ maps the discrete function $f[m] \in l^q$ ($q \geq 2$) into functions $g[m] \in l^r$ ($r \geq 2$) such that

$${}^{\mathcal{T}}\Delta^n f[m] = g[m] \in l^r, \quad (91)$$

where $m \in \mathbb{Z}$, and

$$\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}. \quad (92)$$

The proof of this Theorem is given in [69,109,121,122] by using the Young's inequality for convolution [134–136].

Corollary 5. The exact finite-differences ${}^{\mathcal{T}}\Delta^{2s}$ of even orders $2s$ of a function $f[n] \in l^2$ belongs to the Hilbert space l^2 of square-summable sequences, i.e.,

$${}^{\mathcal{T}}\Delta^{2s} f[n] = g[n] \in l^2 \quad (f[n] \in l^2). \quad (93)$$

Corollary 5 means that the exact finite-differences ${}^{\mathcal{T}}\Delta^{2s}$ is an operator on the Hilbert space l^2 of square-summable sequences:

$${}^{\mathcal{T}}\Delta^{2s} : l^2 \longrightarrow l^2, \quad (94)$$

where $s \in \mathbb{N}$.

The proof of Corollary 5 is given in [69,109,121,122]. This corollary is important to application in quantum mechanics [8,121–123,126,127,129].

3.6. Exact Finite-Differences of Arbitrary Integer Orders

Let us give equations that represent the exact finite-differences of arbitrary positive integer orders [69].

Theorem 5. Let $f[n]$ belong to the set $E(\mathbb{Z})$ of the simple entire functions.

Then, the exact finite-differences of integer orders $2s - 1$ and $2s$ with $s \in \mathbb{N}$ can be represented as follows:

$${}^{\mathcal{T}}\Delta^{2s-1} f[n] := \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} K_{2s-1}(m) f[n-m], \quad (s \in \mathbb{N}), \quad (95)$$

$${}^{\mathcal{T}}\Delta^{2s} f[n] := \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} K_{2s}(m) f[n-m] + K_{2s}(0) f[n], \quad (s \in \mathbb{N}), \quad (96)$$

where the sum means the Poisson–Abel (or Cesaro) summation, and the kernels

$$K_{2s-1}(m) = \sum_{k=0}^{s-1} \frac{(-1)^{m+k+s+1} (2s-1)! \pi^{2s-2k-2}}{(2s-2k-1)!} \frac{1}{m^{2k+1}} \quad (m \in \mathbb{Z}, \quad m \neq 0), \quad (97)$$

$$K_{2s}(m) = \sum_{k=0}^{s-1} \frac{(-1)^{m+k+s} (2s)! \pi^{2s-2k-2}}{(2s-2k-1)!} \frac{1}{m^{2k+2}} \quad (m \in \mathbb{Z}, \quad m \neq 0), \quad (98)$$

and

$$K_{2s-1}(0) = 0, \quad K_{2s}(0) = \frac{(-1)^s \pi^{2s}}{2s+1} \quad (99)$$

with $s \in \mathbb{N}$.

The proof is given in [69,103].

Remark 6. Using the even and odd properties of the kernels

$$K_{2s}(-m) = +K_{2s}(m), \quad (100)$$

$$K_{2s-1}(-m) = -K_{2s-1}(m), \quad (101)$$

and replacing

$$\sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \Rightarrow \sum_{m=1}^{+\infty}, \quad (102)$$

the exact finite-differences (103) and (104) can be represented as

$${}^{\mathcal{T}}\Delta^{2s-1} f[n] := \sum_{m=1}^{+\infty} K_{2s-1}(m) (f[n-m] - f[n+m]) \quad (s \in \mathbb{N}), \quad (103)$$

$${}^{\mathcal{T}}\Delta^{2s} f[n] := \sum_{m=1}^{+\infty} K_{2s}(m) (f[n-m] + f[n+m]) + K_{2s}(0) f[n] \quad (s \in \mathbb{N}), \quad (104)$$

that simplify the direct calculations for some cases.

Corollary 6. Let $f(x)$ belong to the set $E(\mathbb{R})$ of the simple entire functions.

Then, two Equations (103) and (104) can be written as one equation

$${}^{\mathcal{T}}\Delta_h^k f(x) := \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} K_k(m) f(x - m h) + K_k(0) f(x), \quad (k \in \mathbb{N}), \quad (105)$$

where the kernel $K_k(m)$ with $m \in \mathbb{Z}$, and $m \neq 0$, is defined by

$$K_k(m) = \sum_{j=0}^{[(k+1)/2]+1} \frac{(-1)^{m+j} k! \pi^{k-2j-2}}{(k-2j)! m^{2j+2}} \left((k-2j) \cos\left(\frac{\pi k}{2}\right) + \pi j \sin\left(\frac{\pi k}{2}\right) \right), \quad (106)$$

and for $m = 0$ it is

$$K_k(0) = \frac{\pi^k}{k+1} \cos\left(\frac{\pi k}{2}\right) \quad (k \in \mathbb{N}). \quad (107)$$

Examples of the E-FDs of the orders 1, 2, 3, and 4 are presented by Equations (48)–(51).

3.7. Exact Finite-Difference of Arbitrary Positive Order

The differential and integral operators of arbitrary positive orders form fractional calculus [1–7]. The study of such operators has a long history [137–142]. Another important tool is the calculus of finite-differences of integer and non-integer orders.

In the papers [69,103], the exact finite-differences ${}^{\mathcal{T}}\Delta^\alpha$ of arbitrary order $\alpha \in (-1, \infty)$, which are called fractional exact finite-differences, have been proposed. Note that these proposed differences can be considered for the orders $\alpha > -2$; that is, for orders $\alpha \in (-2, -1]$.

Definition 5. The fractional exact finite-difference ${}^{\mathcal{T}}\Delta^\alpha$ of the order $\alpha > -2$ is defined by the equation

$${}^{\mathcal{T}}\Delta^\alpha f[n] := \sum_{m=-\infty}^{+\infty} K_\alpha(m) f[n-m], \quad (108)$$

where the function $f[n]$ and the kernel $K_\alpha(m)$ are real-valued functions of the variable $m \in \mathbb{Z}$, and

$$K_\alpha(m) = \cos\left(\frac{\pi \alpha}{2}\right) K_\alpha^+(m) + \sin\left(\frac{\pi \alpha}{2}\right) K_\alpha^-(m), \quad \alpha > -1, \quad (109)$$

and

$$K_{\alpha}(m) = \sin\left(\frac{\pi\alpha}{2}\right) K_{\alpha}^{-}(m), \quad \alpha \in (-2, -1], \quad (110)$$

where the kernels $K_{\alpha}^{\pm}(n-m)$ are defined as

$$K_{\alpha}^{+}(n-m) = \frac{\pi^{\alpha}}{\alpha+1} {}_1F_2\left(\frac{\alpha+1}{2}; \frac{1}{2}, \frac{\alpha+3}{2}; -\frac{\pi^2(n-m)^2}{4}\right), \quad \alpha > -1, \quad (111)$$

$$K_{\alpha}^{-}(n-m) = -\frac{\pi^{\alpha+1}(n-m)}{\alpha+2} {}_1F_2\left(\frac{\alpha+2}{2}; \frac{3}{2}, \frac{\alpha+4}{2}; -\frac{\pi^2(n-m)^2}{4}\right), \quad \alpha > -2, \quad (112)$$

where ${}_1F_2$ is the generalized hypergeometric function.

The generalized hypergeometric function is defined as

$${}_pF_q(a_1, \dots, a_p; b_1, \dots, b_q; z) := \sum_{k=0}^{\infty} \frac{(a_1)_k \dots (a_p)_k}{(b_1)_k \dots (b_q)_k} \frac{z^k}{k!}, \quad (113)$$

where $a_l \in \mathbb{C}$ ($l = 1, \dots, p$), $b_j \in \mathbb{C}$ and $b_j \neq 0, -1, -2, \dots$ ($j = 1, \dots, q$), and $(a)_k$ is the Pochhammer symbol (rising factorial) that is defined by

$$(a)_0 = 1, \quad (a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = a(a+1)(a+2)\dots(a+k-1) \quad (k \in \mathbb{N}). \quad (114)$$

The function ${}_1F_2(a; b, c; z)$ is the special case of (113) for $p = 1$ and $q = 2$,

$${}_1F_2(a; b, c; z) := \sum_{k=0}^{\infty} \frac{(a)_k}{(b)_k (c)_k} \frac{z^k}{k!}, \quad (115)$$

where ${}_1F_2(a; b, c; z) = {}_1F_2(a; c, b; z)$.

Remark 7. The exact finite-differences ${}^{\mathcal{J}}\Delta^{\alpha}$ of the arbitrary positive order $\alpha > 0$ has the following characteristic property of the Liouville fractional derivative of the arbitrary positive order $\alpha > 0$. The exact finite-differences ${}^{\mathcal{J}}\Delta^{\alpha}$ of the arbitrary positive order $\alpha \in (-2, 0)$ has the property of the Liouville fractional integral of the arbitrary positive order $\alpha \in (0, 2)$.

The Fourier series transform \mathcal{F}_{Δ} of the exact finite-difference of the arbitrary positive order $\alpha > -2$

$$\mathcal{F}_{\Delta}\left({}^{\mathcal{J}}\Delta^{\alpha} f[n]\right)(k) = (-ik)^{\alpha} (\mathcal{F}_{\Delta} f[n])(k), \quad (116)$$

which has the same form [4], p. 90, as the Fourier integral transform \mathcal{F} of the Liouville fractional derivative of the arbitrary positive order $\alpha > 0$

$$\mathcal{F}((D_{+}^{\alpha} f)(x))(k) = (-ik)^{\alpha} \mathcal{F}(f(x))(k), \quad (117)$$

and the Liouville fractional integral of the arbitrary positive order $\alpha \in (0, 2)$.

$$\mathcal{F}((I_{+}^{\alpha} f)(x))(k) = (-ik)^{-\alpha} \mathcal{F}(f(x))(k), \quad (118)$$

where

$$(-ik)^{\alpha} = |k|^{\alpha} \exp(i\pi\alpha \operatorname{sgn}(k)). \quad (119)$$

The Liouville fractional integral is defined as

$$(I_{+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_{-\infty}^x (x-u)^{\alpha-1} f(u) du, \quad (120)$$

and the Liouville fractional derivative is defined as

$$(D_+^\alpha f)(x) := \frac{d^n}{dx^n} \frac{1}{\Gamma(n-\alpha)} \int_{-\infty}^x (x-u)^{n-\alpha-1} f(u) du \quad (121)$$

with $n-1 < \alpha \leq n$ (see Section 2.3 [4], pp. 87–90). For integer values of the order $\alpha = n \in \mathbb{N}$, the Liouville fractional derivatives (121) are equal to the standard derivatives of integer orders

$$(D_+^n f)(x) = \frac{d^n f(x)}{dx^n}, \quad (122)$$

where $n \in \mathbb{N}$ (see Equation (2.3.5) in [4]).

Remark 8. It should be noted that Equation (108) of Definition 5 for the order $\alpha = k \in \mathbb{N}$ gives Equation (105) of Corollary 6 that defines the E-FD of the integer orders $k \in \mathbb{N}$. This fact can be written by the equation

$$\mathcal{T} \Delta^{\alpha=k} f[n] = \mathcal{T} \Delta^k f[n] \quad \text{if } \alpha = k \in \mathbb{N}. \quad (123)$$

As a result, for the integer values of the order $\alpha > 0$, the fractional E-FDs gives the E-FDs of the integer order. Therefore, the E-FDs of integer orders are special cases of the fractional E-FDs given by Definition 5.

Remark 9. It is possible to make a mathematical hypothesis (assumption) in the form of the equality

$$\mathcal{T} \Delta^\alpha f[n] = ((D_+^\alpha f)(x))_{x=n} \quad (124)$$

for all positive orders $\alpha > 0$. Equation can be considered as a generalization of Equation (64). This mathematical hypothesis is based on the following statements, which have already been proven earlier in previous works.

The first statement says that equality (124) holds for all positive integer values of the order $\alpha = k \in \mathbb{N}$ as

$$\mathcal{T} \Delta^{\alpha=k} f[n] = ((D_+^{\alpha=k} f)(x))_{x=n} = \left(\frac{d^k f(x)}{dx^k} \right)_{x=n}, \quad (125)$$

which is given as Corollary 2.

The second statement says that the Fourier series transform \mathcal{F}_Δ of the exact finite-difference of the order $\alpha > 0$ coincides with the Fourier integral transform \mathcal{F} of the Liouville fractional derivative of this order

$$\mathcal{F}_\Delta(\mathcal{T} \Delta^\alpha f[n])(k) = \mathcal{F}((D_+^\alpha f)(x))(k) \quad (126)$$

for all positive orders $\alpha > 0$.

However, an accurate and consistent proof of this mathematical hypothesis for some space of functions (for example, for the space of simple entire functions) has not yet been obtained.

Note that this proof allows for us to consider examples of calculations of E-FDs of non-integer orders that are similar to equations for the Liouville fractional derivative of the arbitrary positive order given in Table 9.2 in [1], p. 174, and Section 2.3 in [4], pp. 87–90.

Remark 10. Note that the kernel of the fractional exact finite-difference that is expressed through the Lommel function has been proposed in the 2006 papers [88,89], and in the 2010 book [90]. It has been given by Equation (41) in [89], p. 092901-7; in the table on page 14900 in [88], and the appendix on page 14908 in [88]; in Definition 8.2 and Example 1 in [90], pp. 166–167, and the table on page 170 of [90]. Note that the Lommel function can be expressed through the generalized hypergeometric function ${}_1F_2(a; b, c; x)$, which is used in the exact finite-difference of arbitrary order [69], and described by equations in [98], p. 372, [99], p. 428, and [100], p. 682.

Remark 11. For the integer values of $\alpha = 2s$ and $\alpha = 2s - 1$ with $s \in \mathbb{N}$, Equation (108) gives the exact finite-differences of the integer orders (95) and (96) with kernels (97) and (98).

Remark 12. Note that the fractional exact finite-differences

$$({}^{\mathcal{T}}\Delta^{\alpha,+}f)[n] := \sum_{m=-\infty}^{+\infty} K_{\alpha}^{+}(n-m)f[m], \quad (127)$$

where $\alpha > -1$ are the exact discrete analog of the Riesz fractional derivatives and integrals.

The fractional exact finite-differences

$$({}^{\mathcal{T}}\Delta^{\alpha,-}f)[n] := \sum_{m=-\infty}^{+\infty} K_{\alpha}^{-}(n-m)f[m] = \sum_{m=-\infty}^{+\infty} K_{\alpha}^{-}(m)f[n-m], \quad (128)$$

where $\alpha > -2$ are exact discrete analog of the conjugated Riesz fractional derivatives and integrals [102].

Note that exact discrete (finite-difference) analog of the fractional Laplacian of the Riesz form for N -dimensional space is proposed in paper [110].

3.8. Exact Finite-Difference of Negative Order

Using Definition 5, it is clear that the order α can be negative $\alpha \in (-2, 0)$. Therefore, one can consider the exact finite-difference of the integer negative order $\alpha = -1$. The exact finite-difference of the order $\alpha = -1$ has been proposed in the papers [69,103,109]. Note that a possibility to use the finite-differences for negative orders is not unique. For example, the Grünwald–Letnikov derivatives can be used for orders $\alpha < 0$ (see Section 20 in [1] and Section 2.2 in [3]), if the functions $f(x)$ satisfies the condition $|f(x)| < c(1 + |x|)^{-\mu}$, where $\mu > |\alpha|$.

Let us consider the exact finite-difference for $\alpha = -1$. Using Equation (108) of the E-FD

$${}^{\mathcal{T}}\Delta^{-1}f[n] := \sum_{m=-\infty}^{+\infty} K_{-1}(m)f[n-m] \quad (129)$$

with kernel (110) in the form

$$K_{-1}(m) = -K_{-1}^{-}(m) = m {}_1F_2\left(\frac{1}{2}; \frac{3}{2}, \frac{3}{2}; -\frac{\pi^2 m^2}{4}\right) = \frac{1}{\pi} \int_0^{\pi} k^{-1} \sin(mk) dk = \frac{1}{\pi} \text{Si}(\pi m), \quad (130)$$

and $\text{Si}(z)$ is the sine integral. Note that the function ${}_1F_2(1/2; 3/2, 3/2; -\pi^2 x^2/4)$ is the anti-derivative of the cardinal sine. As a result, one can define the following exact finite-difference.

Definition 6. Let the function $f(x)$ belong to space $f(x) \in L_1(\mathbb{R})$ and satisfy the condition $f[n] = f(n) \in E(\mathbb{Z})$.

The exact finite-difference of first negative order is defined as

$${}^{\mathcal{T}}\Delta^{-1}f[n] := \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{\text{Si}(\pi m)}{\pi} f[n-m], \quad (131)$$

where $\text{Si}(z)$ is the sine integral.

The finite-difference can be interpreted as anti-derivative [69,103,109].

Theorem 6 (Fundamental theorem of E-FD calculus). Let function $f(x)$ belong to space $f(x) \in L_1(\mathbb{R})$ and satisfy the condition ${}^{\mathcal{T}}\Delta^1 f[n] \in E(\mathbb{Z})$, where $f[n] = f(x = n)$ and $n \in \mathbb{Z}$.

Then, the following equality is satisfied:

$$\mathcal{T} \Delta^1 \mathcal{T} \Delta^{-1} f[n] = \sum_{m=-\infty}^{+\infty} \delta_{m,0} f[n-m] = f[n], \quad (132)$$

where

$$\delta_{n,0} := \begin{cases} 0, & n \neq 0, \\ 1, & n = 0, \end{cases} \quad (133)$$

Proof. Let us define the convolution

$$(f * g)(n) := \sum_{m=-\infty}^{+\infty} f[m] g[n-m] = \sum_{m=-\infty}^{+\infty} g[m] f[n-m] = (g * f)(n). \quad (134)$$

Then, Equation (108) can be written as

$$\mathcal{T} \Delta^\alpha f[n] = (K_\alpha * f)(n). \quad (135)$$

Therefore

$$\begin{aligned} \mathcal{T} \Delta^{-1} \mathcal{T} \Delta^1 f[n] &= \\ (K_1 * (K_{-1} * f))(n) &= ((K_1 * K_{-1}) * f)(n) = \sum_{m=-\infty}^{+\infty} (K_1 * K_{-1})(m) f[n-m], \end{aligned} \quad (136)$$

where

$$(K_1 * K_{-1})(n) = \sum_{m=-\infty}^{+\infty} K_1(m) * K_{-1}(n-m) = \sum_{\substack{m=-\infty \\ m \neq 0, m \neq n}}^{+\infty} K_1(m) K_{-1}(n-m), \quad (137)$$

since $K_{\pm 1}(0) = 0$.

The substitution of expression of the kernels $K_{\pm}(x)$ gives

$$\begin{aligned} (K_1 * K_{-1})(n) &= \sum_{\substack{m=-\infty \\ m \neq 0, m \neq n}}^{+\infty} K_{-1}(m) K_1(n-m) = \\ \sum_{\substack{m=-\infty \\ m \neq 0, m \neq n}}^{+\infty} \frac{1}{\pi} \operatorname{Si}(\pi m) \frac{(-1)^{n-m}}{n-m} &= \frac{1}{\pi} \sum_{\substack{m=-\infty \\ m \neq 0, m \neq n}}^{+\infty} \frac{(-1)^{n-m}}{n-m} \int_0^{\pi m} \frac{\sin(z)}{z} dz. \end{aligned} \quad (138)$$

Using the variable $t = z/m$ ($z = mt, z = mt, dz = m dt$), we obtain the following:

$$\begin{aligned} (K_1 * K_{-1})(n) &= \frac{1}{\pi} \int_0^\pi \frac{dt}{t} \left(\sum_{\substack{m=-\infty \\ m \neq 0, m \neq n}}^{+\infty} \frac{(-1)^{n-m} \sin(mt)}{n-m} \right) = \\ \frac{1}{\pi} \int_0^\pi \frac{dt}{t} \left(\sum_{\substack{m=-\infty \\ m \neq n}}^{+\infty} \frac{(-1)^{n-m} \sin(mt)}{n-m} - \frac{(-1)^n \sin(0t)}{n} \right), \end{aligned} \quad (139)$$

where the second term is equal to zero. Making the change in variable $k = m - n$, and then using that $(-1)^{-k-1} = (-1)^{k-1}$ for $k \in \mathbb{Z}$ and $\sin(-x) = -\sin(x)$, we obtain

$$\begin{aligned} (K_1 * K_{-1})(n) &= \frac{1}{\pi} \int_0^\pi \frac{dt}{t} \left(\sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{(-1)^{k-1} \sin(kt + nt)}{k} \right) = \\ &= \frac{1}{\pi} \int_0^\pi \frac{dt}{t} \left(\sum_{k=1}^{+\infty} \frac{(-1)^{k-1} \sin(kt + nt)}{k} + \sum_{k=-1}^{-\infty} \frac{(-1)^{k-1} \sin(kt + nt)}{k} \right) = \\ &= \frac{1}{\pi} \int_0^\pi \frac{dt}{t} \left(\sum_{k=1}^{+\infty} \frac{(-1)^{k-1} \sin(kt + nt)}{k} + \sum_{k=1}^{+\infty} \frac{(-1)^{k-1} \sin(kt - nt)}{k} \right). \end{aligned} \quad (140)$$

Using Equation (5.4.2.10) of [143], where we replace $a = nt$ and $x = t$, in the form

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k-1} \sin(kt \pm nt)}{k} = \frac{t}{2} \cos(nt) \pm \ln(2 \cos(t/2)) \sin(nt), \quad (141)$$

we obtain

$$(K_1 * K_{-1})(n) = \frac{1}{\pi} \int_0^\pi \frac{dt}{t} (t \cos(nt)) = \frac{1}{\pi} \int_0^\pi dt \cos(nt), \quad (142)$$

where terms with logarithms are canceled. Using $\sin(\pi n) = 0$ and $\cos(0) = 1$, we obtain

$$\begin{aligned} (K_1 * K_{-1})(n) &= \frac{1}{\pi} \int_0^\pi dt \cos(nt) = \\ &= \begin{cases} \frac{\sin(\pi n)}{\pi n} = 0, & n \neq 0, \\ \frac{1}{\pi} \int_0^\pi dt = 1, & n = 0, \end{cases} = \delta_{n,0} \end{aligned} \quad (143)$$

Therefore, we obtain

$$(K_1 * K_{-1})(n) = \delta_{n,0}. \quad (144)$$

As a result, we prove the following equation:

$$\mathcal{T} \Delta^1 \mathcal{T} \Delta^{-1} f[n] = \sum_{m=-\infty}^{+\infty} \delta_{m,0} f[n-m] = f[n]. \quad (145)$$

□

Remark 13. Theorem can be considered as an exact finite-difference analog of Lemma 2.20 [4], p. 89, for $\alpha = 1$.

It can be assumed that an exact finite-difference analog of Lemma 2.20 [4], p. 89 is satisfied for the arbitrary values of the order $\alpha \in (0, 2)$ in the form of the following equation:

$$\mathcal{T} \Delta^\alpha \mathcal{T} \Delta^{-\alpha} f[n] = \sum_{m=-\infty}^{+\infty} \delta_{m,0} f[n-m] = f[n], \quad (146)$$

where $-\alpha \in (-2, 0)$.

Remark 14. The Fourier series transform \mathcal{F}_Δ of this difference is

$$\mathcal{F}_\Delta \left(\mathcal{T} \Delta^{-1} f[n] \right) (k) = (-ik)^{-1} \mathcal{F}_\Delta(f[n])(k), \quad (147)$$

which has the same form [4], p. 90, as the Fourier integral transform \mathcal{F} of the Liouville fractional derivative of the arbitrary positive the Liouville fractional integral of the order $\alpha = 1$ (see Property 2.15 in [4], p. 90).

3.9. Exact Finite-Difference Laplacian of Arbitrary Positive Order

The exact finite-difference form of the fractional Laplacian is proposed in [110].

Definition 7. The exact finite-difference fractional Laplacian of the Riesz form ${}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N,\alpha}$ is defined by the equation

$${}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N,\alpha} f[\mathbf{n}] := \sum_{\mathbf{m}} {}^{\mathcal{T}}K_{N,\alpha}(\mathbf{m}) f[\mathbf{n} - \mathbf{m}], \quad (148)$$

where $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^N$, $\alpha > -N$, the kernel ${}^{\mathcal{T}}K_{N,\alpha,h}(\mathbf{m})$ is

$${}^{\mathcal{T}}K_{N,\alpha,h}(\mathbf{m}) = \frac{1}{h^\alpha} \frac{\pi^{\alpha+N/2}}{(\alpha+N) 2^{N-1} \Gamma(N/2)} {}_1F_2\left(\frac{\alpha+N}{2}; \frac{N}{2}, \frac{\alpha+N+2}{2}; -\frac{\pi^2 |\mathbf{m}|^2}{4}\right), \quad (149)$$

and ${}_1F_2[a; b, c; z]$ is the generalized hypergeometric function. Here we assume that $h_j = h$ for all $j = 1, \dots, N$.

The important property of the exact finite-difference fractional Laplacian (148) is equation of the Fourier series transform

$$\mathcal{F}_{h,\Delta}\left({}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N,\alpha} f[\mathbf{n}]\right) = (|\mathbf{k}|^\alpha) \mathcal{F}_{h,\Delta}(f[\mathbf{n}]). \quad (150)$$

The Fourier series transform $\mathcal{F}_{h,\Delta}\{f[n]\}$ is defined as

$$\mathcal{F}_{h,\Delta}\{f[n]\} = \sum_{n=-\infty}^{+\infty} f[n] e^{-ikx(n)}. \quad (151)$$

Here $x(n) = nh$, and $h > 0$ is step of differences. In the general case, $f[n] \neq f(n)$ and $f[n] \neq f(x(n))$. The functions $f[n]$ and $f(x)$ are connected by the relation $f[n] = hf(hn)$. For $h = 1$, we have $f[n] = f(n)$.

Remark 15. The exact finite-difference fractional Laplacian (148) satisfies the property

$${}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N,2} f[\mathbf{n}] = - \sum_{j=1}^N {}^{\mathcal{T}}\Delta_j^2 f[\mathbf{n}], \quad (152)$$

where ${}^{\mathcal{T}}\Delta_j^2$ is the exact E-FDs of the second order with respect to the variable n_j that is represented by the equation

$${}^{\mathcal{T}}\Delta_j^2 f[n_j] := - \sum_{\substack{m_j=-\infty \\ m_j \neq 0}}^{+\infty} \frac{2(-1)^{m_j}}{m_j^2} f[n_j - m_j] - \frac{\pi^2}{3} f[n_j]. \quad (153)$$

The suggested exact finite-difference fractional Laplacian allows us to consider and solve various equations with such operators.

Let us give particular solutions of some linear equations with the exact finite-difference fractional Laplacians [110].

Theorem 7. The linear nonhomogeneous equation with the exact finite-difference fractional Laplacians

$$\sum_{j=1}^m a_j {}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N,\alpha_j} f[\mathbf{n}] + a_0 f[\mathbf{n}] = g[\mathbf{n}], \quad (154)$$

where $\alpha_n > (N-1)/2$, $\alpha_m > \dots > \alpha_1 > 0$, $a_j \in \mathbb{R}$ ($j = 0, 1, 2, \dots, m$), $a_0 \neq 0$, $a_m \neq 0$, has the particular solution

$$f[\mathbf{n}] = \sum_{\mathbf{m}} G_{\alpha}[\mathbf{n} - \mathbf{m}] g[\mathbf{m}], \quad (155)$$

where

$$G_{\alpha}[\mathbf{n}] = \frac{1}{(2\pi)^{N/2} |\mathbf{r}(\mathbf{n})|^{(N-2)/2}} \int_0^{\pi/|\mathbf{h}|} \left(\sum_{j=1}^m a_j \lambda^{\alpha_j} + a_0 \right)^{-1} \lambda^{N/2} J_{N/2-1}(\lambda |\mathbf{r}(\mathbf{n})|) d\lambda, \quad (156)$$

and $J_a(z)$ is the Bessel function of the first kind.

The proof of Theorem 7 is given in [110] as proof of Theorem 4.

Theorem 8. The linear nonhomogeneous equation with the exact finite-difference fractional Laplacians

$$\sum_{j=1}^m a_j {}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N, \alpha_j} f[\mathbf{n}] = g[\mathbf{n}] \quad (m \geq 2), \quad (157)$$

where $\alpha_n > (N-1)/2$, $a_1 < N$, $\alpha_m > \dots > \alpha_1 > 0$, $a_j \in \mathbb{R}$ ($j = 1, 2, \dots, m$), $a_1 \neq 0$, $a_m \neq 0$, has the particular solution of the form

$$f[\mathbf{n}] = \sum_{\mathbf{m}} G_{\alpha}[\mathbf{n} - \mathbf{m}] g[\mathbf{m}], \quad (158)$$

where

$$G_{\alpha}[\mathbf{n}] := \frac{1}{(2\pi)^{N/2} |\mathbf{r}(\mathbf{n})|^{(N-2)/2}} \int_0^{\pi/|\mathbf{h}|} \left(\sum_{j=1}^m a_j \lambda^{\alpha_j} \right)^{-1} \lambda^{N/2} J_{N/2-1}(\lambda |\mathbf{r}(\mathbf{n})|) d\lambda, \quad (159)$$

where $J_a(z)$ is the Bessel function of the first kind.

The proof of Theorem 8 is similar to proof of Theorem 7 that is given in [110].

Example 4. If $\alpha > (N-1)/2$, $\mathbf{n} \in \mathbb{Z}^N$, $\omega \in \mathbb{R}$ and $\omega^2 \neq 0$, the difference equation

$${}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N, \alpha} f[\mathbf{n}] - \omega^2 f[\mathbf{n}] = g[\mathbf{n}] \quad (160)$$

is solvable, and its particular solution has the form

$$f[\mathbf{n}] = \sum_{\mathbf{m}} G_{\alpha}[\mathbf{n} - \mathbf{m}] g[\mathbf{m}], \quad (161)$$

where

$$G_{\alpha}[\mathbf{n}] := \frac{1}{(2\pi)^{N/2} |\mathbf{r}(\mathbf{n})|^{(N-2)/2}} \int_0^{\pi/|\mathbf{h}|} \left(\lambda^{\alpha} - \omega^2 \right)^{-1} \lambda^{N/2} J_{N/2-1}(\lambda |\mathbf{r}(\mathbf{n})|) d\lambda. \quad (162)$$

For the case $\alpha = 2$, Equation (160) can be considered as a fractional Helmholtz equation with E-FDs in N -dimensional space.

Example 5. If $\alpha > (N-1)/2$, $\alpha > \beta > 0$, $\beta < N$, $\mathbf{n} \in \mathbb{Z}^N$, $\omega \neq 0$, then the exact finite-difference equation

$${}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N, \alpha} f[\mathbf{n}] + \omega {}^{\mathcal{T}}\Delta_{\mathbf{h}}^{N, \beta} f[\mathbf{n}] = g[\mathbf{n}] \quad (163)$$

has the particular solution

$$f[\mathbf{n}] = \sum_{\mathbf{m}} G_{\alpha, \beta}[\mathbf{n} - \mathbf{m}] g[\mathbf{m}], \quad (164)$$

where

$$G_{\alpha,\beta}[\mathbf{n}] := \frac{1}{(2\pi)^{N/2} |\mathbf{r}(\mathbf{n})|^{(N-2)/2}} \int_0^{\pi/|\mathbf{h}|} \left(\lambda^\alpha + \omega \lambda^\beta \right)^{-1} \lambda^{N/2} J_{N/2-1}(\lambda |\mathbf{r}(\mathbf{n})|) d\lambda. \quad (165)$$

Example 6. If $N = 1$, $\alpha > \beta > 0$, $\beta < 1$, $\omega \neq 0$, then the difference equation

$$\mathcal{T} \Delta_{\mathbf{h}}^{N,\alpha} f[n] + \omega \mathcal{T} \Delta_{\mathbf{h}}^{N,\beta} f[n] = g[n], \quad (166)$$

has the particular solution

$$f[n] = \sum_{m=-\infty}^{\infty} G_{\alpha,\beta}[n-m] g[m], \quad (167)$$

where

$$G_{\alpha,\beta}[n] := \frac{1}{\pi} \int_0^{\pi/h} \frac{\cos(\lambda h |n|)}{\lambda^\alpha + \omega \lambda^\beta} d\lambda. \quad (168)$$

Example 7. If $N = 1$, $\alpha > 0$, and $\omega \neq 0$, then the difference equation

$$\mathcal{T} \Delta_{\mathbf{h}}^{N,\alpha} f[n] + \omega f[n] = g[n], \quad (169)$$

has the particular solution

$$f[n] = \sum_{m=-\infty}^{\infty} G_{\alpha}[n-m] g[m], \quad (170)$$

with the Green's function

$$G_{\alpha}[n] := \frac{1}{\pi} \int_0^{\pi/|h|} \frac{\cos(\lambda h |n|)}{\lambda^\alpha + \omega} d\lambda, \quad (171)$$

where we use

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x) \quad (172)$$

for a one-dimensional case ($N = 1$).

4. Exact Finite-Differences for Non-Entire Functions

Theorem 1, which describes the connection between the E-FDs and the standard derivatives of integer orders, holds for functions $f[n]$ that belong to the set $E(\mathbb{Z})$ of the simple entire functions. In the general case, when the functions do not belong to the set of simple entire functions, we can obtain the following inequality:

$$\mathcal{T} \Delta^1 f[n] \neq \left(\frac{df(x)}{dx} \right)_{x=n}. \quad (173)$$

This fact will lead to the need to generalize the equation, defining the exact finite-differences to the sets of functions outside the space $E(\mathbb{Z})$.

In this section, a generalization of equation, which defines the exact finite-difference of the first order for simple, entire functions, is proposed.

Theorem 9. Let C be a contour enclosing the all points $z = \pm n$, where $n \in \mathbb{N}$ (except the point $z = 0$).

Let $f(z)$ be analytic inside and on C , except possibly for a number of poles a_1, \dots, a_N none of which coincide with $z \in \mathbb{Z}$.

Then, the following equation is satisfied:

$$\sum_{k=-\infty, k \neq 0}^{\infty} \frac{(-1)^k}{k} f[n-k] = \left(\frac{\partial f(z)}{\partial z} \right)_{z=n} + \mathcal{D}(f, n), \quad (174)$$

where

$$\mathcal{D}(f, n) = \frac{1}{2\pi i} \int_C \frac{f(n-z)}{z} \frac{\pi dz}{\sin \pi z} - \sum_{j=1}^N \operatorname{Res}_{z=a_j} \left(\frac{\pi f(n-z)}{z \sin(\pi z)} \right), \quad (175)$$

where $\operatorname{Res}_{z=a_j} g(z)$ is the residue of $g(z)$ at $z = a_j$.

Proof. Let us use the residues for the summation of the series (for example, see Section 7.7-4. of [144]). Given a contour C enclosing the points $z = m, z = m+1, z = m+2, \dots, z = n$, where m is an integer, let $f(z)$ be analytic inside and on C , except possibly for a number of poles a_1, \dots, a_N none of which coincide with $z = m, z = m+1, z = m+2, \dots, z = n$. Then

$$\sum_{k=m}^n (-1)^k f[k] = \frac{1}{2\pi i} \int_C f(z) \frac{\pi dz}{\sin \pi z} - \sum_{j=1}^N \operatorname{Res}_{z=a_j} \left(\frac{\pi f(z)}{\sin(\pi z)} \right), \quad (176)$$

where $\operatorname{Res}_{z=a_j} g(z)$ is the residue of $g(z)$ at $z = a_j$.

For the calculation of the sum of the series $\sum_{k=-\infty}^{\infty} (-1)^k f[k]$ one can use the Cauchy formula (see Section 4.5 of [145]). Let us first replace the series by a contour integral. For this aim, we need a function that has simple poles at the points $z = n$, and it is bounded on the infinity beyond the real axis. As such a function of this type we use $\pi / \sin(\pi z)$, which has simple poles at $z = 0, \pm 1, \pm 2, \dots$. \square

As a corollary of Theorem 9, we proposed the following generalization of the first-order exact finite-difference for non-entire functions.

Definition 8. Let C be a contour enclosing the all points $z = \pm n$, where $n \in \mathbb{N}$ (except the point $z = 0$). Let $f(z)$ be analytic inside and on C , except possibly for a number of poles a_1, \dots, a_N none of which coincide with $z \in \mathbb{Z}$.

Then, the generalized exact finite-difference (GE-FD) of the first order is defined as

$${}^{\mathcal{G}}\Delta^1 f[n] := {}^{\mathcal{T}}\Delta^1 f[n] - \mathcal{D}(f, n), \quad (177)$$

and

$${}^{\mathcal{G}}\Delta^{m+1} f[n] := {}^{\mathcal{G}}\Delta^1 {}^{\mathcal{G}}\Delta^m f[n], \quad (m \in \mathbb{N}), \quad (178)$$

where $\mathcal{D}(f, n)$ is defined by Equation (175).

Remark 16. Note that for functions $f[n]$ that belong to the set $E(\mathbb{Z})$ of simple entire function, we have $\mathcal{D}(f, n) = 0$.

As a corollary of Theorem 9, we have a generalization of Theorem 1 from a exact finite-difference to a generalized exact finite-difference.

Theorem 10. Let C be a contour enclosing the all points $z = \pm n$, where $n \in \mathbb{N}$ (except the point $z = 0$).

Let $f(z)$ be analytic inside and on C , except possibly for a number of poles a_1, \dots, a_N none of which coincide with $z \in \mathbb{Z}$.

Then, the generalized first-order exact finite-difference satisfies the equality

$${}^{\mathcal{G}}\Delta^1 f[n] := \left(\frac{\partial f(z)}{\partial z} \right)_{z=n} \quad (179)$$

for all $n \in \mathbb{Z}$ and all $h > 0$, where $\partial f(z) / \partial z$ is the standard first-order derivative.

Using Theorem 10, the following properties of the generalized exact finite-differences ${}^{\mathcal{G}}\Delta^k$ of the positive integer order $k \in \mathbb{N}$ can be proved similar to the proofs of the properties

of the exact finite differences ${}^{\mathcal{T}}\Delta^k$. Instead of simple entire functions, we can consider functions that are described in Theorem 10.

(1) The Leibniz rule

$${}^{\mathcal{S}\mathcal{T}}\Delta^1(f[n]g[n]) = ({}^{\mathcal{S}\mathcal{T}}\Delta^1 f[n])g[n] + f[n]({}^{\mathcal{S}\mathcal{T}}\Delta^1 g[n]) \quad (n \in \mathbb{Z}), \quad (180)$$

$${}^{\mathcal{S}\mathcal{T}}\Delta^k(f[n]g[n]) = \sum_{j=0}^k \binom{k}{j} ({}^{\mathcal{S}\mathcal{T}}\Delta^{k-j} f[n]) ({}^{\mathcal{S}\mathcal{T}}\Delta^j g[n]) \quad (n \in \mathbb{Z}, k, j \in \mathbb{N}). \quad (181)$$

(2) The chain rule

$${}^{\mathcal{S}\mathcal{T}}\Delta^1 f(g[n]) = ({}^{\mathcal{S}\mathcal{T}}\Delta^1 f(g))_{g=g(n)} {}^{\mathcal{S}\mathcal{T}}\Delta^1 g[n] \quad (n \in \mathbb{Z}). \quad (182)$$

(3) The semi-group property

$${}^{\mathcal{S}\mathcal{T}}\Delta^1({}^{\mathcal{S}\mathcal{T}}\Delta^1 f[n]) = {}^{\mathcal{S}\mathcal{T}}\Delta^2 f[n] \quad (n \in \mathbb{Z}), \quad (183)$$

$${}^{\mathcal{S}\mathcal{T}}\Delta^k({}^{\mathcal{T}}\Delta^j f[n]) = {}^{\mathcal{S}\mathcal{T}}\Delta^{k+j} f[n] \quad (n \in \mathbb{Z}, k, j \in \mathbb{N}). \quad (184)$$

(4) The equations for power-law functions

$${}^{\mathcal{S}\mathcal{T}}\Delta^1 n^j = j n^{j-1} \quad (n \in \mathbb{Z}, j \in \mathbb{N} \quad j \geq 1), \quad (185)$$

$${}^{\mathcal{S}\mathcal{T}}\Delta^k n^j = \frac{j!}{(j-k)!} n^{j-k} \quad (n \in \mathbb{Z}, k, j \in \mathbb{N} \quad j \geq k). \quad (186)$$

(5) If functions $f(z)$ and $g(z)$ satisfy the equation

$$\frac{d^k f(z)}{dz^k} = g(z), \quad (187)$$

where $k \in \mathbb{N}$, then the equation

$${}^{\mathcal{S}\mathcal{T}}\Delta^k f(n) = g(n) \quad (188)$$

holds for $n \in \mathbb{Z}$ and $k \in \mathbb{N}$.

Let us consider an example of the calculation of generalized exact finite-differences without using Theorem 10. As such, an example of non-entire function, one can consider the square root function. It should be emphasized that there is no need to carry out calculations in the way described below, since Theorem 10 strictly proves the connection between the generalized exact finite-difference and the standard first-order derivative.

The standard partial derivative of the square root $f(x, y) = \sqrt{x^2 + y^2}$ of the form

$$\frac{\partial}{\partial x} \sqrt{x^2 + y^2} = \frac{x}{\sqrt{x^2 + y^2}}, \quad (189)$$

where $x, y \neq 0$ and $x, y \in \mathbb{R}$.

Let us consider discretization $f(x, y) \rightarrow f[n, m]$, when the partial derivative is replaced by the exact finite-difference. Then,

$$\frac{\partial}{\partial x} \sqrt{x^2 + y^2} \rightarrow {}^{\mathcal{T}}\Delta_x^1 \sqrt{n^2 + m^2}, \quad (190)$$

where $n \neq 0$ and $n, m \in \mathbb{Z}$. Using the equation of the exact finite-difference in the form

$${}^T\Delta_x^1 f[n] = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} (f[k-n] - f[-k-n]), \quad (191)$$

we obtain

$${}^T\Delta_x^1 \sqrt{n^2 + m^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\sqrt{(k-n)^2 + m^2} - \sqrt{(k+n)^2 + m^2} \right). \quad (192)$$

To calculate (192), we can use the complex analysis. Let us consider the following integral:

$$\mathcal{D}_N(f, n) := \frac{1}{2\pi i} \int_{C_N} \left(\sqrt{(z-n)^2 + m^2} - \sqrt{(z+n)^2 + m^2} \right) \frac{\pi dz}{z \sin(\pi z)}, \quad (193)$$

where C_N is the square contour $|x| < N + 1/2$, $|y| < N + 1/2$ with vertical cuts from points $\pm n + im$ upwards and from points $\pm n - im$ downwards. The branches of the roots are chosen so that the roots were positive for real z . Calculating this integral through the residues, we obtain

$$\mathcal{D}_N(f, n) = -\frac{2n}{\sqrt{n^2 + m^2}} + 2 \sum_{k=1}^N \frac{(-1)^k}{k} \left(\sqrt{(k-n)^2 + m^2} - \sqrt{(k+n)^2 + m^2} \right). \quad (194)$$

On the other hand, using the oddness of the integrand and the fact that the conjugate points it takes the conjugated values, we obtain

$$\mathcal{D}_N(f, n) = 4 \operatorname{Re} \frac{1}{2\pi i} \int_{C'_N} \left(\sqrt{(z-n)^2 + m^2} - \sqrt{(z+n)^2 + m^2} \right) \frac{\pi dz}{z \sin \pi z}, \quad (195)$$

where C'_N is a part of the contour C_N that lies in the first quarter.

Then, it is necessary to show that the integrals over the horizontal and vertical sides of C'_N tend to zero with a growth of N . For the horizontal side, it is quite simply because there is a sine that increases exponentially. For the vertical side, it is enough that the difference roots in the brackets will be bounded, and this can be seen since their arguments are very close at large N .

Thus, when $N \rightarrow \infty$ we have only the integral along the section from the point $n + im$ upwards. The second root takes the same values on both edges of the cut, so it does not give the contribution to the integral since these boundaries are traversed in opposite directions. The values of the first root on the boundary of cuts differ in sign, so it will give a double contribution to the integral. It remains to substitute $z = n + it$, where $t \in [m, \infty)$ and one considers that roots is equal to $-i\sqrt{t^2 - m^2}$ on the left side of the cut, and to use the reduction formula for sine and select the real part.

As a result, we obtain

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{(-1)^k}{k} \left(\sqrt{(k-n)^2 + m^2} - \sqrt{(k+n)^2 + m^2} \right) = \\ \frac{n}{\sqrt{n^2 + m^2}} + 2n(-1)^{n+1} \int_m^{\infty} \frac{\sqrt{t^2 - m^2}}{(t^2 + n^2) \sinh(\pi t)} dt, \end{aligned} \quad (196)$$

where $\sinh(z)$ is the hyperbolic sine.

Therefore, we have

$${}^{\mathcal{ST}}\Delta^1 \sqrt{n^2 + m^2} = \frac{n}{\sqrt{n^2 + m^2}}. \quad (197)$$

The absolute values of difference between the results, which is given by right side of Equation (196), and the initial sum, which is given by the left part of Equation (196), have been calculated for n, m from 1 to 10. These absolute values are presented by the following two tables:

$$\left(\begin{array}{c|ccccc} & m=1 & m=2 & m=3 & m=4 & m=5 \\ \hline n=1 & 2.343 \cdot 10^{-14} & 5.551 \cdot 10^{-16} & 3.009 \cdot 10^{-14} & 1.565 \cdot 10^{-14} & 2.473 \cdot 10^{-13} \\ n=2 & 4.108 \cdot 10^{-14} & 4.663 \cdot 10^{-15} & 2.220 \cdot 10^{-16} & 3.014 \cdot 10^{-13} & 7.772 \cdot 10^{-16} \\ n=3 & 6.473 \cdot 10^{-14} & 4.319 \cdot 10^{-14} & 1.366 \cdot 10^{-14} & 2.428 \cdot 10^{-11} & 5.299 \cdot 10^{-13} \\ n=4 & 8.027 \cdot 10^{-14} & 6.950 \cdot 10^{-14} & 7.452 \cdot 10^{-13} & 1.499 \cdot 10^{-13} & 5.833 \cdot 10^{-13} \\ n=5 & 3.675 \cdot 10^{-14} & 3.886 \cdot 10^{-14} & 3.191 \cdot 10^{-11} & 7.413 \cdot 10^{-12} & 5.503 \cdot 10^{-13} \\ n=6 & 2.529 \cdot 10^{-13} & 2.928 \cdot 10^{-12} & 7.151 \cdot 10^{-13} & 1.567 \cdot 10^{-12} & 3.457 \cdot 10^{-11} \\ n=7 & 1.116 \cdot 10^{-13} & 7.317 \cdot 10^{-11} & 5.211 \cdot 10^{-12} & 1.222 \cdot 10^{-11} & 9.166 \cdot 10^{-9} \\ n=8 & 8.162 \cdot 10^{-13} & 3.802 \cdot 10^{-12} & 1.859 \cdot 10^{-12} & 3.386 \cdot 10^{-9} & 7.526 \cdot 10^{-10} \\ n=9 & 2.145 \cdot 10^{-10} & 8.797 \cdot 10^{-12} & 2.037 \cdot 10^{-10} & 5.725 \cdot 10^{-8} & 5.067 \cdot 10^{-10} \\ n=10 & 1.695 \cdot 10^{-11} & 1.545 \cdot 10^{-11} & 9.187 \cdot 10^{-8} & 6.296 \cdot 10^{-10} & 5.311 \cdot 10^{-11} \end{array} \right), \quad (198)$$

$$\left(\begin{array}{c|ccccc} & m=6 & m=7 & m=8 & m=9 & m=10 \\ \hline n=1 & 1.541 \cdot 10^{-13} & 1.837 \cdot 10^{-13} & 4.337 \cdot 10^{-9} & 3.850 \cdot 10^{-12} & 4.633 \cdot 10^{-12} \\ n=2 & 1.280 \cdot 10^{-13} & 1.384 \cdot 10^{-12} & 4.989 \cdot 10^{-9} & 9.842 \cdot 10^{-13} & 1.454 \cdot 10^{-12} \\ n=3 & 3.519 \cdot 10^{-14} & 2.228 \cdot 10^{-11} & 4.562 \cdot 10^{-9} & 2.020 \cdot 10^{-12} & 6.582 \cdot 10^{-13} \\ n=4 & 2.633 \cdot 10^{-12} & 8.793 \cdot 10^{-9} & 2.069 \cdot 10^{-11} & 1.101 \cdot 10^{-10} & 3.110 \cdot 10^{-13} \\ n=5 & 1.147 \cdot 10^{-10} & 1.359 \cdot 10^{-10} & 6.487 \cdot 10^{-12} & 1.762 \cdot 10^{-12} & 1.619 \cdot 10^{-11} \\ n=6 & 4.815 \cdot 10^{-9} & 1.377 \cdot 10^{-12} & 1.048 \cdot 10^{-11} & 4.489 \cdot 10^{-12} & 1.925 \cdot 10^{-10} \\ n=7 & 6.341 \cdot 10^{-11} & 7.431 \cdot 10^{-11} & 3.444 \cdot 10^{-12} & 3.630 \cdot 10^{-10} & 9.514 \cdot 10^{-12} \\ n=8 & 2.506 \cdot 10^{-9} & 5.160 \cdot 10^{-12} & 5.195 \cdot 10^{-10} & 2.632 \cdot 10^{-11} & 2.025 \cdot 10^{-11} \\ n=9 & 1.518 \cdot 10^{-11} & 5.442 \cdot 10^{-9} & 2.659 \cdot 10^{-11} & 6.951 \cdot 10^{-11} & 1.703 \cdot 10^{-11} \\ n=10 & 1.302 \cdot 10^{-9} & 4.240 \cdot 10^{-12} & 5.527 \cdot 10^{-10} & 2.449 \cdot 10^{-11} & 2.996 \cdot 10^{-13} \end{array} \right). \quad (199)$$

In Tables (198) and (199), the absolute values of difference represent the errors of summation and integration (196).

Note that there is no need to consider calculations by the method described above, since Theorem 10 strictly proves the connection between the generalized exact finite-difference (GE-FDs) and the standard first-order derivative. Therefore, using Theorem 10 and the properties of the GE-FDs described above, one can give the other examples of calculations of the GE-FDs.

(a) First example of the action of the GE-FD

$${}^{\mathcal{G}\mathcal{T}}\Delta^1 \left(n^2 + m^2 \right)^{s/k} = \frac{2sn}{k} \left(n^2 + m^2 \right)^{-(k-s)/k}, \quad (200)$$

where $n, m \in \mathbb{Z}$ and $k, s \in \mathbb{N}$.

(b) Second example of the action of the GE-FD

$${}^{\mathcal{G}\mathcal{T}}\Delta^1 \left(n^2 + m^2 \right)^{s/k} e^{\lambda n^2} = \frac{2sn}{k} \left(n^2 + m^2 \right)^{-(k-s)/k} e^{\lambda n^2} + 2\lambda n \left(n^2 + m^2 \right)^{s/k} e^{\lambda n^2}, \quad (201)$$

where $n, m \in \mathbb{Z}$ and $k, s \in \mathbb{N}$, $\lambda \in \mathbb{R}$.

(c) Third example of the action of the GE-FD

$${}^{\mathcal{G}\mathcal{T}}\Delta^k E_{\alpha, \beta}[n] = k! E_{\alpha, \beta+\alpha k}^{k+1}[n], \quad (202)$$

where $\alpha, \beta > 0$, $n, m \in \mathbb{Z}$, $k \in \mathbb{N}$, $\lambda \in \mathbb{R}$, and $E_{\alpha, \beta}[z]$ is two-parameter Mittag-Leffler function and $E_{\alpha, \beta}^k[z]$ is generalized two-parameter Mittag-Leffler function [4,146,147].

(d) The linear equation with GE-FD in the form

$$n {}^{\mathcal{G}\mathcal{T}}\Delta^2 f[n] + (c - n) {}^{\mathcal{G}\mathcal{T}}\Delta^1 f[n] - a f[n] = 0 \quad (203)$$

has the solution

$$f[n] = {}_1F_1(a, c; n) = \sum_{k=0}^{\infty} \frac{(a)_k}{(c)_k} \frac{n^k}{k!}, \quad (204)$$

where $n \in \mathbb{Z}$, $k \in \mathbb{N}$, and $(z)_k = \Gamma(z+k)/\Gamma(z)$ is the Pochhammer symbol. Note that

$${}^{\mathcal{G}\mathcal{T}}\Delta^k {}_1F_1(a, c; n) = \frac{(a)_k}{(c)_k} {}_1F_1(a, c; n), \quad (205)$$

where ${}_1F_1(a, c; z)$ is the confluent hypergeometric Kummer function [4,148].

(e) The linear equation with GE-FD in the form

$$n^2 {}^{\mathcal{G}\mathcal{T}}\Delta^2 f[n] + n {}^{\mathcal{G}\mathcal{T}}\Delta^1 f[n] + (n^2 - m^2) f[n] = 0 \quad (206)$$

has the solution

$$f[n] = J_m[n] = \sum_{j=0}^{\infty} \frac{(-1)^j}{2^{j+m} j! \Gamma(m+j+1)} n^{2j+m}, \quad (207)$$

where $n \in \mathbb{Z}$, $m \in \mathbb{N}$ and $J_m[z]$ is the Bessel function of the first kind [4,149].

Remark 17. It should be emphasized that the proposed Tables (198) and (199) are not intended to approximately prove Equations (197) and (179) or Theorem 10, since these equation is exact and it has been proven analytically using well-known formulas for example, see Section 7.7-4 of [144] and Section 4.5 of [145]).

In fact, Theorems 1 and 10, prove that exact finite-differences and generalized exact finite-differences of an integer order are algebraic analogs of the derivatives of an integer order, in contrast to the standard finite-differences [8–10] and non-standard finite-differences proposed in works [11–16,19].

5. Examples of Difference and Differential Equations with Its Solutions

In this section, we give examples of equations with the exact finite-differences (E-FDs) (Tables 1–3). These equations are compared with the equations that contains the standard integer-order derivatives (Der), standard finite-differences (S-FDs), and non-standard finite-differences (NS-FDs) in the form of tables [69].

Table 1. Examples of linear equations of the first order.

Operator Type	Equation and Operator	Solution
Der	$D^1 f(x) = -\lambda f(x)$ $D^1 f(x) := \frac{df(x)}{dx}$	$f(x) = f(0) e^{-\lambda x}$
S-FD	${}^f\Delta^1 f[n] = -\lambda f[n]$ ${}^f\Delta^1 f[n] := f[n+1] - f[n]$	$f[n] = f[0] (1 - \lambda)^n$ $f[n] \neq f[0] e^{-\lambda n}$
NS-FD	${}^M\Delta_{\lambda}^1 f[n] = -\lambda f[n]$ ${}^M\Delta_{\lambda}^1 f[n] := \frac{f[n+1] - f[n]}{(1 - e^{\lambda n})/\lambda}$	$f[n] = f[0] e^{-\lambda n}$
E-FD	${}^{\mathcal{T}}\Delta^1 f[n] = -\lambda f[n]$ ${}^{\mathcal{T}}\Delta^1 f[n] := \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{(-1)^m}{m} f[n-m]$	$f[n] = f[0] e^{-\lambda n}$

Table 2. Examples of linear equations of second order.

Operator Type	Equation, Operator, and Solution
Der	$D^2 f(x) + \lambda^2 f(x) = 0$ $D^2 f(x) := \frac{d^2 f(x)}{dx^2}$ $f(x) = A \cos(\lambda x + \phi)$
S-FD	${}^f \Delta^2 f[n] + \lambda^2 f[n] = 0$ ${}^f \Delta^1 f[n] := f[n+1] - 2f[n] + f[n-1]$ $f[n] \neq A \cos(\lambda n + \phi)$
NS-FD	${}^M \Delta_\lambda^2 f[n] + \lambda^2 f[n] = 0$ ${}^M \Delta_\lambda^2 f[n] := \frac{f[n+1] - 2f[n] + f[n-1]}{4 \sin(\lambda/2)/\lambda^2}$ $f[n] = A \cos(\lambda n + \phi)$
E-FD	${}^T \Delta^2 f[n] + \lambda^2 f[n] = 0$ ${}^T \Delta^2 f[n] := - \sum_{\substack{m=-\infty \\ m \neq 0}}^{+\infty} \frac{2(-1)^m}{m^2} f[n-m] - \frac{\pi^2}{3} f[n]$ $f[n] = A \cos(\lambda n + \phi)$

Table 3. Examples of nonlinear equations of the first order.

	Differential Equation and Solution	Equation with E-FD and Solution
1.	$\frac{df(x)}{dx} - a^2 (f(x))^{k/(k+1)} = 0$ $f(x) = \left(\frac{a^2}{k+1} x + C \right)^{k+1}$	${}^T \Delta^1 f[n] - a^2 (f[n])^{k/(k+1)} = 0$ $f[n] = \left(\frac{a^2}{k+1} n + C \right)^{k+1}$
2.	$\frac{df(x)}{dx} + \lambda f(x) \ln(f(x)) = 0$ $f(x) = \exp(\exp(-\lambda x + C))$	${}^T \Delta^1 f[n] + \lambda f[n] \ln(f[n]) = 0$ $f[n] = \exp(\exp(-\lambda n + C))$
3.	$\frac{df(x)}{dx} - \lambda \exp(x) f^{k/(k+1)}(x) = 0$ $f(x) = \left(\frac{\lambda}{k+1} \exp(x) + C \right)^{k+1}$	${}^T \Delta^1 f[n] - \lambda \exp(n) f^{k/(k+1)}[n] = 0$ $f[n] = \left(\frac{\lambda}{k+1} \exp(n) + C \right)^{k+1}$
4.	$\frac{df(x)}{dx} - \lambda x^k f^{(k-1)/k}(x) = 0$ $f(x) = \left(\frac{\lambda}{k(k+1)} x^{k+1} + C \right)^k$	${}^T \Delta^1 f[n] - \lambda n^k f^{(k-1)/k}[n] = 0$ $f[n] = \left(\frac{\lambda}{k(k+1)} n^{k+1} + C \right)^k$
5.	$\frac{df(x)}{dx} - \lambda \cos(x/k) f^{(k-1)/k}(x) = 0$ $f(x) = (\lambda \sin(x/k) + C)^k$	${}^T \Delta^1 f[n] - \lambda \cos(n/k) f^{(k-1)/k}[n] = 0$ $f[n] = (\lambda \sin(n/k) + C)^k$

6. Conclusions

In this article, a short review of exact finite-differences calculus of integer orders is proposed. The standard finite-differences [8–10] and non-standard finite-differences [11–15] cannot be considered as exact discrete (difference) analogs of the standard derivatives of integer orders. The proposed exact finite-differences are exact algebraic analogs of derivatives; that is, they satisfy the same characteristic relationships as the standard derivatives in the space of the simple entire functions. In this regard, the question arises of the behavior of these final differences outside this space. As an answer to this question, a generalized equation that defines the exact finite-differences for non-entire functions is proposed. As an example of calculation, the square root function is considered.

It should be noted that a fractional generalization of the exact finite-differences of integer orders to non-integer orders has been proposed in the papers [69,110]. In preliminary form in [101–103], where the kernel of the exact finite-difference operators, which is expressed through the Lommel function was proposed in the 2006 papers by Equation (41)

and the Table in [89], in the Appendix of [88] and by Definition 8.2 and Example 1 in [90]. One can emphasize that there are many interesting open mathematical questions about the properties of such fractional operators that are exact fractional finite-difference of non-integer orders.

It should be noted that there are many interesting applications of the exact finite-differences of integer orders in continuum mechanics [97,104,105,117], statistical mechanics [106,107], economics [118–120], quantum mechanics [121–123], and quantum field theory [108,124,125]. The results and methods proposed in these works were developed in article on quantum theory [126–129]. We assume that these applications and others possible applications [90,150–161] can be generalized for fractional exact finite-difference to describe processes and systems with non-locality in space and time.

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