Is accretion alone enough to explain the mass spectrum of black holes?

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In memory of Edward George Houlden (1911-1994) and Vera May Houlden née Walsh (1919-1983)

Abstract. The discovery in recent years by gravity wave detectors of dozens of merging binary black holes with masses much greater than one solar mass, when taken with the known population of black holes raises the possibility that it may not be possible to explain their all their origins and growth by credible accretion mechanisms. This paper offers an alternative perspective; that black holes can increase their mass by a quantum process linked to the acceleration of distant matter. Also, the idea that dark matter is predominantly composed of black holes in strongly supported. A testable equation $\dot{M}/M = 2H$ for the fractional rate of production of dark matter inside a large region of space with radius r and mass $M(r)$, where H is the Hubble Constant, is presented. A mechanism to produce early supermassive black holes is proposed.

1. Introduction

Today's standard cosmology is based on General Relativity (GR) which incorporates a cosmological constant Λ (now the origin of Dark Energy) to explain the observed accelerated expansion of the Universe, includes the Hubble law and requires the presence of substantial amounts of cold (non-relativistic) dark matter which had an important dynamical effect in producing the large scale structures that are seen; this model is called the ΛCDM model. It is usual to discuss this model in terms of the Friedmann equation which can be written in terms of the redshift parameter z as

$$H^2(z) = H_0^2[\Omega_{ro}(1+z)^4 + \Omega_{mo}(1+z)^3 + \Omega_{ko}(1+z)^2 + \Omega_{\Lambda o}]$$

where the Ω parameters are numbers that have to found by fitting to data. The first term is the radiation term, the second the matter term (dark matter plus baryonic matter), the third arises from any spacetime curvature present and the last one is the normalised dark energy term

$$\Omega_{ko} = \frac{\Lambda}{3H_0^2}$$

Also in this equation the radiation and matter terms have been normalised to the current critical density,

$$\rho_{co} = \frac{3H_0^2}{8\pi G} \tag{1}$$

where H_0 is the Hubble parameter and G is the Newtonian Gravitation constant.

This expression can be used to construct a reference model as the basis to fit the Planck and WMAP data to obtain the values of the four Ω parameters, which in numerical order (ignoring errors) are ~9.29 × 10^{-5}, 0.315, –0.0042 and 0.685. Within errors the curvature term (the third one) is consistent with being zero. The Hubble parameter H_0 = 67.3 km / s / Mpc. [1].

There are several features of this result that should be noted. The first is that, in the matter term, normal baryonic matter forms only about 0.05 from the total of 0.315. This means that the majority of the matter in the Universe is completely unknown 'dark matter', that to date has not been produced or detected in laboratory experiments such as at the LHC. Whatever its nature, current cosmology makes the assumption that dark matter is some sort of particle that has been created in the Big Bang and has remained as a constant proportion of matter ever since.

Secondly the Dark Energy term which forms some 68.5% of the total, was completely unexpected, not predicted by any popular theories and remains totally mysterious. Thus the current situation we are confronted with is that some 95% of the energy balance of the Universe is of a totally unknown nature, a remarkable situation. Furthermore, the current view is that the cosmic acceleration term only 'switched on' in the recent past at red shift z ~0.55 [2]. The ΛCDM explanation for this is that in the early epochs the inward gravitational force from the constant amount of Dark Matter dominated the dynamics of the expansion. But as the proportion of Dark Energy continually increased its effects began to dominate, thus starting the observed acceleration.

The ΛCDM model also requires that in the very early Universe, when it was of order 10^{-35} s old there was a rapid, short-lived but vast expansion called 'inflation'. One effect of this was to produce a Euclidean space-time, which requires the density to equal the critical density. A clear prediction is that gravitational waves produced during inflation would result in there being a large scale, swirling, pattern of polarization imprinted on the CMB. This is difficult to observe as there is a similar large background made by the CMB photons scattering from dust in the Milky Way galaxy. To date there is no unambiguous observational evidence to support the inflation hypothesis.

2. Preliminary comments

In this paper GR will play an important but not exclusive role. Mass is not a useful concept in GR; energy E of all forms is the source of gravitation. Nevertheless it is convenient for us to use the mass m = E/c^2 to facilitate comparison with familiar objects, like the Sun. Another point is that mass is a positive quantity whereas gravitational potential energy due to a mass m is negative.

In 1934 Milne discussed what has been called the most striking paradox of twentieth century cosmology: the relativistic laws of the Universe could be derived in the simplest possible way from Newtonian theory [3]. Consider a large region of space *in the Matter Dominated era* with radius r containing matter with a density equal to the critical density. This will have mass $M(r)$. It is an observed feature of the Universe that galaxies on the periphery of this region will be moving radially outwards with a velocity given the Hubble Law, $v = Hr$ such that

$$\frac{GM(r)}{r} = ½v^2 \tag{2}$$

This means that the negative gravitational potential of M (r) is exactly balanced by the positive kinetic energy of the galaxy, so that the total energy is always zero. We will make use of this equation shortly.

3. The Nature of Dark Matter and Black Hole Formation and growth

As stated in the Introduction current thinking is that dark matter consists of some sort long-lived particle that was created in the early stages of the Big Bang and has been a constant proportion of the matter in the Universe. However to date there has been no supporting experimental evidence at all found in Earth-based searches.

In recent years there has been resurgence in an old idea; that black holes constitute the bulk of dark matter [3, 4]. These papers suggest there are still regions of the black hole parameter space that have not been ruled out by observation and that this idea still has a lot of life left in it. Frampton sets out his stall boldly "Our proposal is that the Milky Way contains between ten million and ten billion massive black holes each with between a hundred and a hundred thousand times the solar mass" [5].

Those fired up by these ideas and needing to brush up their knowledge of black hole formation have a handy new reference available [6]. The 14 chapters in this volume review all aspects of black hole production and growth and it provides a comprehensive list of useful references. Reading through these chapters reminds one how hard it is to make a black hole and let it grow by the accretion of in-falling matter or merging with other black holes. The idea that *all* black holes were made and grew by the mechanisms that are discussed seems to be extremely optimistic. There is one exception. In the very early Universe it seems primordial black holes could be produced in abundance. However most of these evaporated by Hawking radiation, and any that were massive enough to survive into the present era would have masses greater than 10^{15} g and "have been touted as partly or fully responsible for the dark matter in the Universe" [6]. The formidable problem of how to grow this surviving population of low mass black holes (should it exist) into the heavyweights demanded by Frampton's proposal still remains.

We speculate that if one could find an extra mechanism that could be added to accretion to increase black hole masses then that might solve the problem. But this would require finding a mechanism that was new to science.

4. Quantum mechanics, entanglement and the inside of black holes

There are available on the web a series of insightful and magisterial lectures by Prof. Leonard Susskind on General Relativity that every physicist should watch at least once in their lifetime. Two of the most relevant for present purpose are entitled "ER = EPR" or "What's Behind the Horizons of Black Holes?" [7] These lectures provide the framework of new ideas that are needed in this paper. These topics are still the subject of active research and have by no means "reached the stage of consensus". Those reading this paper for the first time are advised to watch these lectures before continuing further as the density of ideas presented is impossible to summarise adequately here.

Thinking within the corpus of ideas presented in these lectures it seems that it is perfectly allowable to make the following statement. The internal geometry and quantum system of a black hole can be entangled with another macroscopic system that can be an arbitrary distance away. Hence, if there is a change in a quantum state within the black hole there can be a change in its geometry (i.e. an energy / mass change) and at the same time a change in the quantum state of the remote entangled system. It must be appreciated that any change is a quantum step processes and not a continuous process. Put another way, a black hole can increase its mass step by step and the remote system it is entangled with can accelerate step by step.

Referring to equation (2) above this implies that if *M*(*r*) increases then v must increase to compensate. As (2) must be time independent it can be differentiated (see Appendix A) giving, in the Matter Dominated era when the curvature term

$$\Omega_{ko} = 0,$$

$$3H\Omega_v = \frac{\dot{M}}{M}, \tag{3}$$

which yields

$$2H = \frac{\dot{M}}{M}, \tag{4}$$

when $\Omega_v = 2/3$. See later in this paper why this precise value is postulated rather than the fitted Planck value of $\rho_{vac} / \rho_c = 0.685$.

In the present epoch

$$\frac{\dot{M}}{M} = 2H_0 \sim 4.36 \times 10^{-18}\ s^{-1} \tag{5}$$

since $H_0 = 67.3$ km/s/Mpc, 1 Mpc = 3.085×10^{19} m, hence $H_0 \sim 2.18 \times 10^{-18}$ s^{-1}.

5. A Toy Model Universe to illustrate some points

At this stage is it helpful to play with a toy model and plug in some numbers to get a better feel for things. Consider a volume of space with radius $R = 1$ Gpc, i.e. a radius of ~3.0×10^{25} m. This is smaller than the Hubble radius c/H_0. This volume will be filled with matter at the critical density and so will have $M(r) = 1.065 \times 10^{51}$ kg. Due to the Hubble law the radius will be expanding by $\delta r = 67.3 \times 10^6$ m every second. Thus in one second the expansion will generate a new volume of $4\pi R^2\ \delta r\ \sim 8 \times 10^{59}$ m^3. This will be filled with Dark Energy which has $\rho_{vac} = 5.96 \times 10^{-27}$ kg / m^3 so that in one second $M(r)$ will increase by ~4.8×10^{33} kg, about 2400 solar masses. If M(r) was spread out in solar mass scale objects then there would be ~5×10^{20} of them. Thus the Dark Energy contribution per second is a minute fraction of this.

The above calculation has been done in the framework of the ΛCDM model using the fitted results reported in [1], which found that $\rho_{vac} / \rho_c = 0.685$. One can now ask, what does equation (5) predict?

$$M \overset{\cdot}{=} 2H_0M = 4.6 \times 10^{33}\ \mathrm{kg\ s^{-1}}$$

This is very close to the 4.8×10^{33} kg s^{-1} calculated above. For future reference we note that if $\rho_{vac} / \rho_c = 0.6667 = 2/3$ exactly, then the differences between these two mass values is reduced to rounding errors and one gets essentially perfect agreement between the two.

This paper speculates that most, or all, dark matter consists of black holes that increase their mass with time by a new quantum process. The toy model puts this into perspective. If there are ~5×10^{20} solar mass objects in $M(r)$ then even if a fraction of these were black holes, say 1/5th of them, then the 2400 solar mass per second increase in $M(r)$ predicted by (5) could be shared between 10^{20} black holes in a way that would be totally undetectable in real time and require aeons to pass for any mass increase to be detectable.

6. The hidden history of an acceleration equation a = H²r.

In 1994 Peter Rowlands published a book with a largely historical slant [8]. He had been interested in seeing how far one could push the use of Newtonian ideas in Cosmology (as discussed in [3]) and what this approach had to say about Mach's principle, which links local masses to the rest of the Universe to explain the origin of inertia. In his original paper on GR, Einstein started with a discussion of Mach's principle and said how important it was. However, the field equations he derives at the end of the paper are incompatible with Mach's principle.

There were two principal outcomes of this small part of Rowlands' work. The first was that he concluded *static* gravity must be an instantaneous effect on all distance scales. The second was that an observer looking at the effects of instantaneous static gravity but working in a Lorentzian framework where the speed of light was the limit would see a 'fictitious' centrifugal force, that produces an acceleration "*a*" (similar in concept to fictitious forces like the Coriolis force), and on page 235 in [8] he published the equation

$$a = \frac{c^2 r}{r_H^2} = H_0^2 r$$

This equation can be integrated and that yields the Hubble law $v = Hr$. This implies that there is a deep relationship between these two equations; the acceleration is the *cause* and the Hubble law the *effect or consequence*.

Twenty five or more years ago, before 1998, the idea that static gravity really was an instantaneous effect had few takers, and the idea that one would observe an *accelerating* Universe would have astonished people. But today we can note that the concept of instantaneous static gravity over cosmological distances is consistent with the ideas outlined in Susskind's lectures [7].

In 1998 everything changed with the discovery of the acceleration of distant supernovas [9, 10]. To explain this, an old idea that had been abandoned for decades was resurrected; the concept of a cosmological constant, Λ. As is well known, Einstein introduced this to get a static solution to the field equations but then dropped it when the Hubble flow was discovered. The cosmological constant concept says that empty space has a non-zero energy density Λ which has a gravitational effect. This can be converted to an equivalent mass density (Λ/c^2 kg m^{-3}) which in section 5 is called ρ_{vac}. So as the Universe expands and new space gets created, that gets filled with energy which can, within GR generate acceleration. Taking this idea and including cold dark matter as a necessary concept gave rise to the ΛCDM model.

When the Planck collaboration published their results in 2013 they showed a spectacularly good fit between the data and the ΛCDM model [1]. On seeing these results Rowlands spotted something interesting in them. As stated above, the fit gave $\rho_{vac} / \rho_c = 0.685$. What he spotted was that 0.685 was remarkably close to 2/3, and he proposed that as more and more data gets gathered the fit will get closer and closer to $\rho_{vac} / \rho_c = 0.666667$, i.e. exactly 2/3. This prediction can be tested. The question now is why is this? This will be discussed in the next section.

7. Fitting the ΛCDM model

In reference [11] Rowlands discusses the Friedmann equations in the context of fitting the ΛCDM model to the data. It's useful to remember that in this context, Λ exerts a negative pressure. What Rowlands shows in [11] is that if $\rho_{vac} = (2/3)\,\rho_c$ *exactly* then this value for Λ gives an acceleration $a = H^2r$, which is what is required in the context discussed above. He then points out that because of this, in the Planck Data fitting process one could force the value 2/3 rather than fit it, so removing one degree of freedom, which might improve the fitted values and fitted errors of the other free parameters. In the toy model in section 5 it was seen that imposing 2/3 gave perfect agreement between the two mass values.

In passing we note that in [11] Rowlands uses Sciama's 1953 model of Mach's principle to show that it can also lead to the same acceleration equation, $a = H^2r$. What this exercise shows is that GR does not necessarily have a stranglehold in providing the only possible explanation of cosmic acceleration. Other ideas are possible.

8. Overview and discussion of the proposal

Before the proposal can be outlined a few more comments are needed. The first is that in Susskind's lectures [7] it is made clear that entanglement only occurs between pairs of particles or more massive states. This is clearly an important point as it means that in any modelling of entangled systems more complex webs of relationships are not required.

A second point is that entangled black holes or other massive system can be built up from pairs of entangled elementary particles. There must be few environments in the Universe where this can happen, but one of them could be in the very early Universe when there were a lot of interactions between fundamental particles.

So the first part of the proposal is that in the very early Universe entangled pairs of primordial black holes were produced. As they separated they could grow by the quantum process that could increase the mass of one black hole and accelerate the one it was entangled with. The fact that their masses could increase by other than accretion enabled them to become heavy enough to escape evaporation by Hawking radiation and survive to keep growing, to become dark matter. This population of black holes *is separate and different* from the populations produced by the accretion mechanisms etc. outlined in [6] and the two populations can be added. In early epochs (high red shift z values) the two populations can be treated as independent, but as time goes on they will merge somewhat. But this overlap could be small so to first approximation the evolution of the two populations can be treated separately.

The second proposal is that the rate of production of black holes/dark matter in the Matter Dominated era of a flat Universe with $k = 0$ is governed by the global rate-limiting equation (4). Although each pair of entangled black holes can be treated as independent the combined growth rate of all black holes taken together is controlled by (4). As the growth process is quantum mechanical this is no problem as the rate can be controlled by a phase space term. This point requires that one considers the Universe to be a totally entangled quantum system. This concept can be used resolve any issues about momentum conservation.

It is important to appreciate the difference between the two black hole populations. The primordial entangled population (PEBH) produced in the early Universe can grow in mass and *increase* M(r) in equation (2). The other population is manufactured by the mechanisms discussed in [6], all of which, basically, involve recycling already existing materials and do not produce entangled systems. These could be called Manufactured Black Holes (MANBH). They can only grow by merger and / or accretion of existing materials hence *cannot increase* $M(r)$ in equation (2).

A final point is that the growth in mass of a black hole by the proposed process is done quantum step by quantum step. Each step changes the geometry inside the black hole a little, which changes the associated energy hence the mass. It could be that one quantum step increases the black hole mass by one Planck mass, $\sim 2.176 \times 10^{-8}$ kg. If the mass of a one solar mass black hole doubles then that implies it has swept through $\sim 10^{38}$ quantum states, which is a lot. But Susskind teaches that there are a vast number of quantum states inside a black hole, of order e^S where S is the entropy of the black hole [7]. This is called quantum complexity. A solar mass black hole has entropy $S \sim 4 \times 10^{77}$ [12], so e^S is truly vast and much bigger than 10^{38}. There is indeed a lot of phase space inside a black hole.

9. Learning how to model the proposal

This section discusses some points that need to be thought about to in order to plan an attack on the problem before rushing off to write computer code.

In the first instance one should always try to build on existing work. The fitting of the ΛCDM model to the data is well tested and polished work [1]. Recall this model assumes that the proportion of dark matter has remained constant since the earliest times, whereas the proposal in this paper says the amount of dark matter can grow over time, and that there was very little in the early Universe. So a promising approach might be to use this proposal to model a mass vs time curve and import this into the ΛCDM fitting code and see the outcome.

One difficulty is that this proposal has a large parameter space, where presently there is little guidance about what values to use. Another is the question; does the rate of increase of mass of a black hole depend on its mass?

One way to tackle this could be to limit initial studies to using the proposed process to see if it can be tuned in some way to produce very early super massive black holes. The origin of these is a major problem [16]. Factoring out this problem has the advantage that it is restricted in time to the first 700 million or so years and the distribution of matter is smoother than at later dates. Lessons learnt in doing this work can be incorporated into the next step.

Another problem that could be studied is this one. Currently two incompatible measurements for the Hubble parameter are obtained. The one from ‘local’ data gives ~73.2 km/s/Mpc in the current epoch and data at high-z, Planck/WMAP which yields ~ 67.3 km/s/Mpc from the early epoch [13,14,1]. The value of 67.3 km/s/Mpc is the result of a fit to the sound horizon that occurs in the time before the recombination era. Much work is still needed to ensure this difference is a real effect and not the result of some systematic error.

But, if this is a real problem then our proposal offers a possible solution. Currently it is assumed the proportion of dark matter has always been the same. But if there was less dark matter at the time of the high-z data then there will be less gravitational retardation of the expansion, allowing the fitting process to increase the fitted value of the Hubble parameter from ~67 km/s/Mpc to the higher value. If this idea solves the problem then restricting studies and tuning of the model to this period has lots of advantages; short time period and lots of data. Working within these constraints should, hopefully, enable the details of the model to be tightened up.

In the very early Radiation Dominated era one cannot assume that (4) applies. Also in this era primordial black holes can increase their mass by absorbing radiation from the environment they are in. The rate of absorption of energy will depend on the area of their event horizon and the current temperature of the radiation field. Perhaps a safe initial idea is to assume that as all the primordial black holes are very similar when they are created and exist in the same radiation environment then they all grow initially in the same way. Then, at the end of the Radiation Dominated era, there will be a population with masses distributed around some average value, perhaps with a Gaussian spread.

Once this preliminary work has been done, in outline the procedure would go something like this:

(a) Set up some initial conditions at the start of the Matter Dominated era, with a population of entangled black holes in some volume with mass $M(r)$.
(b) Calculate the expected mass increase $\dot{M} = 2MH$
(c) Choose a time step Δt, then the change in mass $\Delta M = 2\ MH\Delta t$.
(d) Increase the mass of a random selection of black holes in this time interval, but limited by equation (4). As this is a quantum process there is no requirement to increase the mass of every black hole in this time interval. The way to do this is to choose a (large) random *subset N* of the entangled black holes and distribute the mass roughly equally over the i^{th} black hole, in amounts $\delta m_i \sim \Delta M/N$, bearing in mind all the lessons learnt earlier. So larger mass black holes might need to get larger values of δm_i. When all the N black holes have had their mass increased, cross-check that
$$\sum_{i=1}^{N} \delta m_i = \Delta M$$
(e) Update $M(r) \Rightarrow M(r) + \Delta M$
(f) Repeat the process for as long as is required.

This procedure will generate a (dark matter vs time profile) that can be included in the ΛCDM model fitting code. But it’s clear that to get reliable results will require formidable computer resources. Calculating many different profiles could be very expensive. One way to save time and cost could be to generate the profiles at the extremes and centres of the parameter space and then estimate profiles between these extremes by interpolation. So, users of the ΛCDM model fitting code could quickly dial up different, approximate, profiles to try out. When promising ones are identified, improved profiles could be calculated using the restricted parameter space.

10. Final thoughts

The proposed mechanism offers an explanation of the origin of binary pairs of massive black holes whose mergers are being detected today by gravity wave receivers [6]. When the primordial entangled black holes are formed there is nothing to stop two of them forming a binary pair (or more than two making multi-black hole systems). They are *not* entangled with each other but each is entangled with some remote partner. Entanglement does not have any effect on the orbital dynamics. As the Universe expands, all of the partners can independently increase their mass by the proposed process and live on, orbiting each other, until they merge.

For a binary pair with masses m_1 and m_2 with period P and semi-major axis a, Kepler’s third law is

$$P^2 = \frac{4\pi^2 a^2}{G(m_1 + m_2)}$$

This shows that if the masses increase then the period gets smaller. This is because the bodies will move towards each other and increase their orbital speed. However this probably could only happen if there was a 3rd body present to conserve angular momentum. For a binary pair of black holes this 3rd body could be another black hole or a cloud of gas. Earth's Moon is needed to conserve angular momentum to allow tidal friction to slow down the Earth's rotation rate [15].

As the binary pair of black holes spiral inwards then gravitational radiation is now known to be emitted, carrying off energy. So the two processes acting together will drive the pair towards merger.

The same could happen with a small cloud of black holes; each would increase in mass and the laws of motion and GR gravitational radiation could drive them rather quickly to merge, forming early super-massive black holes [16]?

An interesting question is; does entanglement survive mergers of black holes? Do supermassive black holes continue to gain mass to become monsters, like TON 618 which has a mass of 66 billion solar masses? [17]. Survival of entanglement after a merger of black holes would seem to be a good hypothesis.

Finally, as this paper was being finished a paper by Kevin S Croker *et al* [18] appeared. This proposes that the mass of an initial stellar remnant of mass m_0 that formed an ultrarelativistic compact object could grow according to the formula

$$m(a) = m_0(a/a_i)^k$$

where a_i is the scale factor when the stellar remnant formed, $a > a_i$ is the scale factor and k is a dimensionless constant constrained to the range $-3 < k < 3$. This paper illustrates the fact that *the idea that black holes can increase their mass by other than accretion or merger is already in the mainstream of current ideas about Cosmology.* The proposal in this paper suggests a possible mechanism for this process.

11. Conclusions

This paper offers an alternative to the ΛCDM model, which requires that there is a constant amount of baryonic matter, a constant amount of a dark matter particle and an increasing amount of Dark Energy, Λ, which is responsible for the observed acceleration of the expanding Universe. Rather, we propose that there is a constant amount of particulate matter (baryons and any as-yet undiscovered entities like axions or WIMPS) and a monotonically increasing amount of dark matter. We propose this dark matter is made from primordial black holes that increase their mass by a quantum process that accelerates distant matter that is entangled with the black hole. The combined fractional growth rate of all the dark matter black holes taken together obeys the relationship

$$\frac{\dot{M}}{M} = 2H$$

at all red shifts in the Matter Dominated era of a flat Universe. If this proposal has any validity models based on it must give a good fit to the current astrophysics datasets without the need of Λ.

Should a good fit to the data be found then this new model might offer an explanation of the 'two Hubble constants' problem, if that problem is found to be a genuine one and not just due to measurement errors.

It is possible that clusters of primordial black holes could form that would orbit one another, gaining mass by the proposed mechanism, and be driven to merger by orbital dynamics and energy loss by gravitational radiation. The early merger of such clusters of black holes is a possible origin of early supermassive black holes.

Should this proposal pass all the tests then we believe it offers strong supporting evidence for Rowland's idea that the observed cosmic acceleration is a fundamental property of the Universe and the Hubble law is a consequence of this.

Furthermore this would mean that the Hubble law is a consequence of a quantum process so should be written

$$\langle v \rangle = Hr,$$

where <*v*> is the expectation value of the velocity. This implies that the spread of values one sees around the v versus r Hubble line is due partly to the intrinsic spread of a quantum process and partly to the random and systematic measurement errors.

Appendix A

In the Matter Dominated era of a flat Universe where $k = 0$, $\Omega_v + \Omega_m = 1$. The deceleration parameter

$$q = \frac{\ddot{r}r}{\dot{r}^2}$$

Hence

$$\dot{H} = -H^2(1+q) \tag{A1}$$

Also in a flat Matter Dominated Universe [19],

$$q = \frac{3}{2}\Omega_m - 1. \tag{A2}$$

Find the time rate of change of (H^2) and divide the result by H^2, then using (A1) gives

$$2\frac{H\dot{H}}{H^2} = \frac{2\dot{H}}{H} = \frac{2\left(-H^2(1+q)\right)}{H} = -2H(1+q)$$

Using (A2)

$$2\frac{H\dot{H}}{H^2} = 3H\Omega_m$$

so

$$2H\dot{H} = -3H^3\Omega_m \tag{A3}$$

Equation (2) can be written ($M(r) \Rightarrow M$)

$$2GM = v^2r - H^2r^3$$

Differentiate

$$2G\dot{M} = H^2 3r^2\dot{r} + 2H\dot{H}r^3$$

Using (A3)

$$2G\dot{M} = 3H^3r^3 - 3H^3\Omega_m r^3$$

$$2G\dot{M} = 3H^3r^3(1-\Omega_m) = 3H^3r^3\Omega_v \tag{A4}$$

From

$$M = \frac{4}{3}\pi r^3\rho_c,$$

where ρ_c is the critical density,

$$2GM = H^2r^3$$

Hence

$$\frac{2G\dot{M}}{2GM} = \frac{\dot{M}}{M} = 3H\Omega_v$$

Historical note: the equation $\dot{M}/M = 2H_0$ describing the fractional rate of increase of dark matter was first presented by Houlden at an ANPA conference in Liverpool UK on 12 Aug 2019. However on that date its derivation was erroneous and the correct derivation shown in the Appendix A to this paper was first done by Steve D. Barrett in May 2021.

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