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Black hole thermodynamics via Tsallis statistical mechanics

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Abstract   An investigation of black hole thermodynamics based on Tsallis statistical mechanics is explored through the study of the thermodynamics of a gas system located near the horizon of a black hole. In spite of the difficulty in exploring black hole thermodynamics through statistical mechanics, the entropy of the nearby gas system is found to be proportional to the black hole's horizon area using Gibbs–Boltzmann statistical mechanics. This allows us to study black hole thermodynamics by using statistical mechanics through the thermodynamic behaviors of the gas system. Since the entropy of the black hole is proportional to the horizon area, it is more suitable to use non-extensive statistical mechanics instead of the usual Gibbs–Boltzmann ones. In this work, the black hole entropy is derived based on Tsallis statistical mechanics, one of well-known non-extensive statistical mechanics. It is found that the black hole entropy gets a modification due to non-extensivity. By using such an entropy, the black hole can be stabilized due to the non-extensivity, and the bound on the non-extensive parameter is also determined.

1 Introduction

One of the important properties of black holes is the laws of black hole mechanics [1], such as Hawking's area theorem, etc. Based on these laws, their formulae are similar to those of thermodynamic laws and then they inspire physicists to study black holes in terms of thermodynamics. Bekenstein had considered a black hole as just a thermodynamic system and found that the black hole entropy must be proportional to the black hole's surface area [2,3]. By considering the quantum effect, a black hole can behave like a black body

emitting radiation with a certain temperature, T_H, known as Hawking temperature [4]. Based on the first law of black hole mechanics, the corresponding entropy is also obtained as S_{BH}, called the Bekenstein–Hawking entropy. As a result, it is possible to interpret the black hole as a thermodynamic system carrying certain entropy and temperature. The study of thermodynamic properties of black holes has been intensively investigated in order to explore characteristic phenomena and the quantum nature of spacetime.

One of the important issues in the study of black hole thermodynamics is the physical interpretation of the black hole entropy or what the black hole entropy actually measures. By considering the same manner of the interpretation of matter entropy which is quantified by using statistical mechanics, it might be worthwhile to interpret the black hole entropy from a description of statistical mechanics. However, one of the obstructions to this manner is the lack of knowledge of the gravitational entropy [5–7]. Moreover, for the thermodynamic system in curved spacetime, it is unclear how the usual thermodynamic entropy of matter precisely relates to the gravitational entropy. Note that for a specific example in string theory such that a collection of branes turns into an extremal black hole, the Bekenstein–Hawking entropy can be derived by counting the degeneracy of BPS soliton bound states [8]. Remarkably, the entropy of a system of gas near the black hole's horizon can be calculated by using statistical mechanics and found to be proportional to the horizon area [9–13]. In this work, we investigate the possible way to interpret such matter entropy as the black hole entropy [14,15].

The black hole entropy is proportional to its area; $S_{BH} \propto A_h$ rather than its volume; $S_{BH} \propto V$. Accordingly, the black hole entropy is non-extensive and non-additive. The effect of non-extensivity and non-additivity on entropy can be characterized by the emergence of long-range interaction. Since Gibbs–Boltzmann (GB) statistical mechanics does not rely on long-range interaction, the black hole entropy should be

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evaluated by using other statistical mechanics associated with long-range interaction.

One of the non-extensive entropies is Tsallis entropy [16], compatible with thermodynamic systems including gravitational interaction. Tsallis entropy obeys the pseudo-additive composition rule; $S_{q(1+2)} = S_{q(1)} + S_{q(2)} + \left[(1 - q)/k_B\right]S_{q(1)}S_{q(2)}$ where $q > 0$ is a non-extensive parameter. With the pseudo-additivity, the empirical temperature obtained from Tsallis entropy is not compatible with the zeroth law of thermodynamics. This implies that a system described by Tsallis statistical mechanics must belong to one with hydrostatic equilibrium instead of thermal equilibrium. Consequently, Tsallis statistical mechanics has been widely applied to investigate physical systems with long-range interactions, including self-gravitating systems [17–22], and cosmology [23–28]. Moreover, the applications of Tsallis entropy in black hole physics have been intensively investigated. By treating Bekenstein–Hawking entropy as Tsallis entropy, the black holes' thermodynamic properties have been explored [29–37]. Black hole thermodynamics with Tsallis entropy including quantum effect have been investigated via the generalized uncertainty principle [38,39]. Using Tsallis entropy (instead of Bekenstein-Hawking entropy) to characterize the black hole's entropy may lead to a thermodynamic inconsistency. It is based on the argument that a black hole emits Hawking radiation while maintaining a Hawking temperature [40]. However, in the framework of generalized statistical mechanics, the temperature does not necessarily correspond to the Hawking temperature [41]. The black hole's temperature associated with the generalized entropy should be defined in such a way that the algebraic relation among geometric quantities (Smarr formula) and the first law of black hole thermodynamics can be satisfied [42]. Besides the Tsallis entropy, other generalized entropies have been extensively investigated [43–51].

As suggested in Refs. [14,15], the entropy of the black hole may be investigated by using matter entropy near the black hole's horizon. This allows us to explore the black hole entropy via statistical mechanics. In fact, the matter entropy near the horizon has been investigated by using GB statistical mechanics [9–13], while there has not been any investigation of non-extensive statistical mechanics in such a context. Among the various types of generalized entropies, e.g., Rényi [52], Barrow [53], and Sharma-Mittal entropies [54], Tsallis entropy is a suitable candidate for non-extensive entropy since it is directly compatible with non-extensive statistical mechanics. Note that the Barrow entropy is relevant to quantum correction, rather than derived from a statistically mechanical point of view. For Rényi entropy, it is a formal logarithm map of Tsallis one [55], and then they are in fact based on the same statistical mechanics. The Sharma-Mittal entropy is more complicated than Tsallis and Rényi

ones since it is the two-parameter extension of the Gibbs–Boltzmann entropy.

Therefore, in this work, we restrict our consideration to the one-parameter extension of Gibbs–Boltzmann statistical mechanics and aim to explore the black hole entropy through the matter entropy near the black hole's horizon by using Tsallis statistical mechanics. Note that, by considering a system of the interacting bosonic gas near the horizon via Tsallis statistical mechanics, the black hole entropy still be in the form of the Bekenstein-Hawking entropy [56]. This suggests that for quantum gas, the non-extensivity does not alter the form of black hole entropy. In our case, we found that the black hole entropy will get the correction due to the non-extensivity.

For GB statistical mechanics, the entropy of the gas system is claimed to be the Bekenstein–Hawking entropy of the black hole by requiring that the number of particles corresponds to the area at the black hole's horizon in the unit of the Planck area and the gas should also be sufficiently closed to the horizon. We discuss this issue in Sect. 2. In Sect. 3, we investigate the thermodynamic properties of classical gas by using Tsallis statistical mechanics. To achieve this, we have to propose a suitable relation between the internal energies for in the GB and Tsallis gas. The black hole's q-entropy can be obtained by the similar manner as the GB case. In Sect. 4, the black hole thermodynamic phase space is constructed based on Tsallis statistical mechanics. The thermodynamic stability is also investigated for various processes. We find that the black hole can be stable in a certain situation. It may imply that there exists some classical correlation of the gas near the horizon in order to maintain the black hole to be thermodynamically stable. In Sect. 5, we summarize important results and discuss the effect of the non-extensivity on black hole thermodynamics. The units used in this work are set as the Boltzmann constant, the universal constant of gravitation, the reduced Planck constant, and the speed of light being unity, $k_B = G = \hbar = c = 1$, only in Sect. 4.

2 Black hole entropy based on Gibbs–Boltzmann statistical mechanics

In this section, we briefly review and devote to summarize the important results of the thermodynamic properties of the gas system described by usual GB statistical mechanics in curved spacetime [9]. Let us begin by assuming that there is a fancy isolated system with two subsystems, namely, a gas system and a black hole, undergoing the quasi-static process. In addition, they are in thermal contact with each other so there is heat transfer between them. This implies that the considered isolated system belongs to the canonical ensemble, in which all possible states of the system are in thermodynamic equilibrium. To demonstrate the aforementioned system, we

impose that N-indistinguishable classical gas particles with a temperature $T = 1/(k_B\beta)$ form as a thin spherical shell with a thickness of H. It is located in a static and spherically symmetric spacetime characterized by the interval:

$$ds^2 = -g(r)c^2 dt^2 + g^{-1}(r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$
$$(1)$$

where $g(r)$ is a horizon function. Note that the interval can be taken in the form of Eq. (1) (i.e., the (00) and (11) components of metric tensor are dependent) when the energy-momentum tensor T^μ_ν satisfies a condition: $T^0_0 = T^1_1$.

A phase-space volume element for a single particle of the gas with mass m and energy E in the curved spacetime can be expressed as [57]

$$dV(E) = \frac{(4\pi)^2}{3h^3}\left[\frac{r^2}{\sqrt{g(r)}}\left(\frac{E^2}{c^2 g(r)} - m^2 c^2\right)^{3/2}\right]dr, \quad (2)$$

where h is the Planck constant. It is noted that in the above expression, the volume elements for both 3-position and 3-momentum spaces obey the spherical symmetry so that the angular sectors can be integrated out. The leftover part of the position space is the radial one described by the coordinate r. For the (leftover) radial sector of the momentum space, the volume element is equivalent to that of the spherical hypersurface with a certain radius associated with the 4-momentum constraint: $p^2 = E^2/(c^2 g(r)) - m^2 c^2$ where p^2 is the square norm of the spatial components of the 4-momentum. In addition, this volume element is invariant under the general coordinate transformation.

We are interested in the scenario that the shell of the gas with a thickness of H is located close to the the black hole's horizon. The distance between the black hole's horizon and the inner radius of the shell is L which is assumed to be very small compared to the thickness of the shell H and horizon radius r_{bh}. Very close to the horizon; $r \to r_{bh}$, the horizon function $g(r)$ can be approximated:

$$g(r \to r_{bh}) \approx \left.\frac{dg(r)}{dr}\right|_{r_{bh}}(r - r_{bh}) = \frac{2\kappa}{c^2}(r - r_{bh}), \quad (3)$$

where κ is the surface gravity of the static and spherically symmetric black hole [58]. As a result, the phase-space volume of the gas particle with the energy E near the black hole's horizon can be approximately computed as

$$V(E) \approx \frac{4\pi^2 c}{3h^3\kappa^2}\int_{r_{bh}+L}^{r_{bh}+L+H} dr \frac{r^2}{\sqrt{r - r_{bh}}}$$
$$\times \left(\frac{E^2}{r - r_{bh}} - 2m^2 c^2\kappa\right)^{3/2} \approx \frac{\pi c E^3 A_h}{3h^3\kappa^2 L}, \quad (4)$$

where $A_h = 4\pi r_{bh}^2$ is the black hole's surface area at its horizon. The last approximation is obtained from the series expansion of the result with the small distance L.

Based on GB statistical mechanics, the partition function of the N-indistinguishable particles associated with the total energy of the system, denoted by E_{tot} can be written as

$$Z = \frac{1}{N!}\int dV \exp(-\beta E_{tot})$$
$$= \frac{1}{N!}\prod_{i=1}^{N}\left[\int dV_i \exp(-\beta E_i)\right]$$
$$= \frac{1}{N!}(Z^{(1)})^N, \quad (5)$$

where the quantities with a subscript i denote those belonging to the i-th particle and $Z^{(1)}$ is the one-particle partition function defined by

$$Z^{(1)} = \int dV \exp(-\beta E). \quad (6)$$

It is important to emphasize that the partition function and the one-particle partition function are also invariant under general coordinate transformation since the quantities βE and βE_{tot} are invariant. According to the expression of the phase-space volume in Eq. (4), the integration over phase-space can be done for all possible states of energy. Thus, the one-particle partition function is further expressed as

$$Z^{(1)} = \frac{\pi c A_h}{h^3\kappa^2 L}\int_0^\infty dE E^2 \exp(-\beta E)$$
$$= \frac{\pi c}{h^3}\frac{A_h}{\beta^3\kappa^2 L}\Gamma(3) \quad (7)$$

where $\Gamma(z)$ is the gamma function. It is also noticed that we keep the above expression in terms of the gamma function for the sake of considering generalized statistical mechanics as will be discussed in the next section.

In the rest frame of the gas particle, the distance between the inner shell and the horizon L can be measured as the particle's proper length:

$$l_{loc} = \int_{r_{bh}}^{r_{bh}+L}\frac{dr}{\sqrt{g(r)}} \approx \sqrt{\frac{2c^2 L}{\kappa}}. \quad (8)$$

The approximation for the horizon function (3) is employed in the above result. Another quantity that depends on the frame of reference is the gas temperature. The relation between the temperature measured by a local observer in the gravitational field T_{loc} and the temperature measured by an observer located far from the source of the gravitational field T is described by Tolman's law [59]: $T_{loc} = T_{loc}(r) = T/\sqrt{-g_{00}(r)}$. In other words, at the inner shell of the gas system, the position-dependent (local) inverse temperature of static fluid β_{loc} can be expressed in terms of the position-independent one β as

$$\beta_{loc}(r_{bh} + L) = \beta\sqrt{g(r_{bh} + L)} \approx \beta\sqrt{\frac{2\kappa L}{c^2}}. \quad (9)$$

Again, Eq. (3) is used in the last approximation. Using all of the approximations in Eqs. (7)–(9), the partition function in Eq. (5) is then written in terms of the local quantities as

$$Z = \frac{1}{N!} \left(\frac{2\pi}{h^3 c^3} \frac{A_h l_{loc}}{\beta_{loc}^3} \Gamma(3) \right)^N. \tag{10}$$

According to the fact that the partition function is the same for all observers, we can choose to consider it as the quantity measured by the local observer. In the thermodynamic limit, i.e., the number of the gas particles N is extremely large, the logarithmic function of Z can be obtained by using Stirling's approximation as

$$\ln Z \approx N \ln \left(\frac{2\pi}{h^3 c^3} \frac{A_h l_{loc}}{\beta_{loc}^3 N} \Gamma(3) \right) + N. \tag{11}$$

Following the framework for the canonical ensemble, $\ln Z$ is thought of as a function of the inverse temperature β_{loc} and a volume V_{loc}; $\ln Z = \ln Z(\beta_{loc}, V_{loc})$. To obtain the standard result for an ultra-relativistic gas [60], the volume V_{loc} must be taken in the form of $V_{loc} = A_h l_{loc}/2$. One can notice that the volume V_{loc} is a very thin shell with a thickness of $l_{loc}/2$, and the inner and outer radii are approximately the black hole's horizon radius. It can be interpreted that when the gas particles closely approach the black hole's horizon, their volume becomes very small as V_{loc} instead of the shell one $4\pi [(r_{bh} + L + H)^3 - (r_{bh} + L)^3]/3$. This feature arises due to the gravitational field near the horizon, affecting how to count the number of microstates. As a result, the partition in Eq. (11) can be rewritten as

$$\ln Z(\beta_{loc}, V_{loc}) = N \ln \left(\frac{4\pi}{h^3 c^3} \frac{V_{loc}}{\beta_{loc}^3 N} \Gamma(3) \right) + N. \tag{12}$$

The thermodynamic mean energy, mean pressure, and entropy are, respectively, given by

$$U_{loc} = -\frac{\partial \ln Z}{\partial \beta_{loc}} = \frac{3N}{\beta_{loc}}, \tag{13}$$

$$P_{loc} = \frac{1}{\beta_{loc}} \frac{\partial \ln Z}{\partial V_{loc}} = \frac{N}{V_{loc}\beta_{loc}}, \tag{14}$$

$$S_{loc} = k_B \left(1 - \beta_{loc} \frac{\partial}{\partial \beta_{loc}} \right) \ln Z$$
$$= k_B N \left[\ln \left(\frac{4\pi}{h^3 c^3} \frac{V_{loc}}{\beta_{loc}^3 N} \Gamma(3) \right) + 4 \right]. \tag{15}$$

It is very important to note that the above U_{loc} and P_{loc} depend on the frame of references while the entropy S_{loc} does not. This is reasonable because the entropy in GB statistical mechanics is proportional to the number of microstates which should be the same for all observers. These quantities obey the first law of thermodynamics:

$$dU_{loc} = \frac{1}{k_B \beta_{loc}} dS_{loc} - P_{loc} dV_{loc}. \tag{16}$$

In the frame of the observer at the asymptotically far region, the gas system also satisfies the first law of the quantities measured by that observer.

From the fact that the asymptotically far observer cannot know whether the gas crosses the black hole's horizon. Consequently, due to the very small (proper) distance between the inner radius of the gas' shell and the event horizon (supposed to be in the order of Planck length), it is not possible to distinguish them [9,12]. This may imply that the gas particles of the gas system can be thought of as sitting on the black hole's surface area. However, the number of the gas particles is constrained by the argument that each particle can occupy the smallest area of the Planck area l_P^2. The maximum number of gas particles is then counted as the surface area of the black hole in the unit of the Planck area:

$$N = \frac{A_h}{l_P^2}. \tag{17}$$

Using this argument, we can establish a link between the black hole and its surroundings (i.e., the gas shell). Via the thermal contact, these two systems can be in thermal equilibrium so that the temperature of the gas equals the Hawking temperature $T_H = h\kappa/(4\pi^2 c k_B)$ of the black hole [4], or equivalently,

$$\beta = \frac{4\pi^2 c}{h\kappa}. \tag{18}$$

Note that the Hawking temperature is measured by the observer at an asymptotically far region. As a result, the entropy of the gas in equilibrium $S_{loc}^{(0)}$ is expressed as

$$S_{loc}^{(0)} = \frac{k_B A_h}{l_P^2} \left[\ln \left(\frac{l_P^2 \Gamma(3)}{32\pi^5 l_{loc}^2} \right) + 4 \right]. \tag{19}$$

As the entropy of the gas shell is identical to that of the system located at the horizon, it is then equal to the black hole entropy $S_{bh}^{(0)}$ by setting the proper length being extremely small as

$$l_{loc} = \frac{e^{15/8} \sqrt{\Gamma(3)}}{4\sqrt{2}\pi^{5/2}} l_P = \frac{e^{15/8}}{4\pi^{5/2}} l_P \sim 10^{-1} l_P. \tag{20}$$

Note that this length scale can be promoted as a cut-off scale, at which the quantum effect is significant so that quantum description of gravitation has to be required [61]. In other words, the entropy in Eq. (19) becomes the Bekenstein-Hawking entropy S_{BH} [2,4];

$$S_{BH} \equiv S_{bh}^{(0)} = k_B \frac{A_h}{4l_P^2}. \tag{21}$$

It has been seen that the Bekenstein–Hawking entropy can be derived as the statistical entropy of the black hole based on GB statistical mechanics. In a standard manner, the GB entropy at equilibrium is written as $S_{bh}^{(0)} = k_B \ln \Omega$, where Ω is the number of microstates. The number of microstates

is straightforwardly obtained as

$$\Omega_{bh} = \exp\left(\frac{A_h}{4l_P^2}\right). \tag{22}$$

Remarkably, one can only derive the entropy via the thermal equilibrium condition. Based on this point of view, the other thermodynamic quantities for black holes cannot be defined as those belonging to the gas system. This may imply that to obtain the complete set of thermodynamic variables, the geometric relation of the black hole needs to be realized. For example, it is not possible to define the thermodynamic volume and pressure for the Schwarzschild black hole with GB statistical mechanics [1]. However, the two aforementioned quantities can be constructed for the black holes with the cosmological constant [62]. Consequently, after defining the entropy of black holes, the thermodynamic phase-space can be constructed by adopting the relations among the black holes' geometric quantities (it will be discussed in Sect. 4).

From Eq. (21), it is worthwhile to point out that the significant feature of the black hole entropy is scaled by its area instead of its volume. This property, however, contradicts the extensive (and also volume-scaling) entropy in conventional GB statistical mechanics. Therefore, the black hole entropy is not suitable to be described by the conventional GB entropy. A framework of statistical mechanics based on generalized entropy may be more appropriate to study the thermodynamic properties of black holes instead of using GB statistical mechanics. Since the derivation of the black hole entropy is general and can be applied to other types of statistical mechanics, the black hole entropy will be derived by following the procedure discussed in this section. The purpose of the next section is to achieve this objective.

3 Generalized entropy of black holes based on Tsallis statistical mechanics

In this section, by using a similar manner to the previous section, we will determine the black hole entropy in the aspect of statistical mechanics associated with one of the generalized entropies namely, Tsallis entropy (q-entropy). The q-entropy with non-extensive parameter, $q > 0$, can be written in terms of one-particle distribution function f as [16]

$$S_q = -k_B \int d\mathcal{V} f^q \ln_q f, \tag{23}$$

where $\ln_q f$ is the q-logarithmic function of f. In fact, the q-entropy is generalized from GB entropy by replacing normal logarithm with q-logarithm and promoting f to be f^q. The q-logarithm and its inverse function (i.e., the q-exponential)

are, respectively, expressed as [63,64]

$$\ln_q Y = \frac{1}{1-q}(Y^{1-q} - 1),$$

$$\exp_q Y = \left[1 + (1-q)Y\right]^{\frac{1}{1-q}}. \tag{24}$$

The Tsallis entropy obeys the non-additive composition rule, for example, $S_{q(1+2)} = S_{q(1)} + S_{q(2)} + \left[(1-q)/k_B\right]S_{q(1)}S_{q(2)}$ where the subscript "$(1+2)''$ refers to the combined system of two subsystems, while subscript "$(1)''$ and "$(2)''$ refer to each subsystem. Moreover, at the limit $q \to 1$, the Tsallis entropy is reduced to the GB entropy, where the additive composition rule: $S_{(1+2)} = S_{(1)} + S_{(2)}$ is restored.

For GB statistical mechanics, the distribution function can be obtained by maximizing the entropy, subjecting to the normalization condition: $\int d\mathcal{V} f = 1$, and then thermodynamic quantities, for example, the internal energy can be obtained by using the expectation value. For Tsallis statistical mechanics, the distribution function can be calculated by using the same manner as performed in the GB case. However, it is not unique to evaluate the expectation value in Tsallis statistical mechanics. There are various choices for the expectation value of internal energy [16,65–83]. However, there exists a transformation to link between such the various definitions [84]. In this work, we choose the choice, for convenience, by following [65] as

$$\int d\mathcal{V} f = 1, \qquad \int d\mathcal{V} f^q E = U_q. \tag{25}$$

To obtain the q-distribution function, one can extremize the Tsallis entropy, $\delta S_q = 0$, subjecting to the above constraints by using the Lagrange multiplier, a_1 and a_2 as follows

$$\delta\left[\frac{k_B}{1-q}\int d\mathcal{V}(f^q - f) - a_1\left(\int d\mathcal{V} f - 1\right)\right.$$
$$\left. - a_2\left(\int d\mathcal{V} f^q E - U_q\right)\right] = 0, \tag{26}$$

$$\int d\mathcal{V}\left[\frac{k_B}{1-q}(qf^{q-1} - 1) - a_1 - a_2 q f^{q-1} E\right]\delta f = 0. \tag{27}$$

Since Eq. (27) does not depend on the choice of δf, one obtains

$$\frac{k_B}{1-q}(qf^{q-1} - 1) - a_1 - a_2 q f^{q-1} E = 0, \tag{28}$$

and then the q-distribution function can be expressed as

$$f = B_q\left(1 - \frac{(1-q)a_2}{k_B}E\right)^{\frac{1}{1-q}}, \tag{29}$$

where B_q is a constant determined from the normalization condition found in Eq. (25). As a result, the constant B_q can

be obtained in terms of partition function, Z_q, as

$$B_q = \frac{1}{\int d\mathcal{V} \left(1 - \frac{(1-q)a_2}{k_B} E\right)^{\frac{1}{1-q}}} = \frac{1}{Z_q}, \qquad (30)$$

where the q-partition function plays the role of a constant for tuning the probability as in the appropriate range $[0, 1]$ defined explicitly by

$$Z_q \equiv \int d\mathcal{V} \left(1 - \frac{(1-q)a_2}{k_B} E\right)^{\frac{1}{1-q}} = \int d\mathcal{V} \exp_q(-\beta_q E). \qquad (31)$$

Note that we have set $a_2 = 1/T_q$ in order to recover the GB partition function at limit $q \to 1$.

Using Eq. (30), Eq. (29), the q-distribution function can be rewritten as

$$f = \frac{1}{Z_q}(1 - (1-q)\beta_q E)^{\frac{1}{1-q}} = \frac{1}{Z_q} \exp_q(-\beta_q E), \qquad (32)$$

where $\beta_q = 1/(k_B T_q)$ and T_q is q-temperature. For the N-indistinguishable particles with the total energy E_{tot}, the q-partition function has to be scaled by $N!$:

$$\begin{aligned} Z_q &= \frac{1}{N!} \int d\mathcal{V} \exp_q(-\beta_q E_{tot}) \\ &= \frac{1}{N!} \prod_{i=1}^{N} \left[\int d\mathcal{V}_i \exp_q(\beta_q E_i)\right] \\ &= \frac{1}{N!}(Z_q^{(1)})^N, \end{aligned} \qquad (33)$$

where $Z_q^{(1)}$ is the one-particle partition function defined by

$$Z_q^{(1)} = \int d\mathcal{V}(1 - (1-q)\beta_q E)^{\frac{1}{1-q}}. \qquad (34)$$

Note that at the limit $q \to 1$, the q-partition function is reduced to the conventional GB partition function. It is important to emphasize that in order to write the partition function to be factorized form shown in Eq. (33), the total energy must be non-additive expressed as [85]

$$E_{tot} = \sum_{i=1}^{N} E_i + \sum_{j=2}^{N} [(q-1)\beta_q]^{j-1} \sum_{n_1 < n_2 < ... < n_j} \prod_{l=1}^{j} E_l. \qquad (35)$$

Note that the additive form of the total energy can be obtained by using the formal logarithm map given by [55,86]

$$\begin{aligned} E_{add} &= -\frac{1}{(1-q)\beta_q} \ln|1 - (1-q)\beta_q E_{tot}| \\ &= -\frac{1}{(1-q)\beta_q} \sum_{i=1}^{N} \ln|1 - (1-q)\beta_q E_i|. \end{aligned} \qquad (36)$$

From the phase-space volume in Eq. (7), one can see that the contribution from the non-extensive effect does not alter the phase-space volume. In fact, the phase-space volume depends only on spacetime geometry, i.e., $g(r)$. In addition, the partition function is invariant under the general coordinate transformation, let us choose to consider it in the local frame as done for the GB case in the previous section. Consequently, using Eqs. (4) and (8), Eq. (34) can be rewritten as

$$Z_q^{(1)} = \frac{2\pi c^3 A_h}{h^3 \kappa^3 l_{loc}^2} \int_0^\infty dE E^2 (1 - (1-q)\beta_q E)^{\frac{1}{1-q}}. \qquad (37)$$

Actually, the integration on the right-hand side of Eq. (37) is the q-Laplace transform [87]:

$$\mathcal{L}_q(f(E)) = \int_0^\infty dE f(E)(1 - (1-q)\beta_q E)^{\frac{1}{1-q}}, \qquad (38)$$

where $E > 0$ and $\beta_q \in \mathcal{C}$ with $\text{Re}(\beta_q) > 0$. For the algebraic function $f(E) = E^{\alpha-1}$, the q-Laplace transform is given by $\mathcal{L}(E^{\alpha-1}) = \Gamma_q(\alpha)/\beta_q^\alpha$ where $\alpha, \beta_q \in \mathcal{C}$, $\text{Re}(\beta_q) > 0$ and $q \neq 1$, and $\Gamma_q(\alpha)$ is the q-gamma function given by [87]

$$\Gamma_q(\alpha) = \begin{cases} \dfrac{1}{(1-q)^\alpha} \dfrac{\Gamma(\alpha)\Gamma\left(\dfrac{1}{1-q}+1\right)}{\Gamma\left(\dfrac{1}{1-q}+\alpha+1\right)}, & \text{for } q < 1 \\ \Gamma(\alpha), & \text{for } q = 1 \\ \dfrac{1}{(q-1)^\alpha} \dfrac{\Gamma(\alpha)\Gamma\left(\dfrac{1}{q-1}-\alpha\right)}{\Gamma\left(\dfrac{1}{q-1}\right)}, & \text{for } q > 1. \end{cases} \qquad (39)$$

As a result, Eq. (37) can be written as

$$Z_q^{(1)} = \frac{4\pi c^3}{h^3} \frac{V_{loc}}{(\beta_q \kappa l_{loc})^3} \Gamma_q(3), \qquad (40)$$

where an explicit form of $\Gamma_q(3)$ in terms of the non-extensive parameter is given by

$$\Gamma_q(3) = \frac{2}{(2-q)(3-2q)(4-3q)}. \qquad (41)$$

Note that the above expression is valid for both $q < 1$ and $q > 1$. Generally, we have $\beta_q \neq \beta$ for $q \neq 1$, hence it is worthwhile to introduce a mathematical relation as

$$\beta_q = \mathcal{J}(q)\beta, \qquad (42)$$

where $\mathcal{J}(q)$ is a positive-valued function satisfied $\lim_{q\to 1} \mathcal{J}(q) = 1$. Using this relation together with Tolman's law for q-thermodynamics $\beta_{q,loc} = \mathcal{J}(q)\beta_{loc}$ as performed in the GB case, the q-partition function for N particles can be

expressed as

$$Z_q \approx \frac{1}{N!} \left[\frac{4\pi}{h^3 c^3} \frac{V_{loc}}{\beta_{q,loc}^3} \Gamma_q(3) \right]^N. \tag{43}$$

At the thermodynamic limit: $N \gg 1$, using Stirling's approximation: $N! \approx (N/e)^N$, the q-logarithmic function of Z_q can be obtained as

$$\ln_q Z_q = \frac{1}{1-q} \left(Z_q^{1-q} - 1 \right)$$
$$\approx \frac{1}{1-q} \left[\left(\frac{4\pi e}{h^3 c^3} \frac{V_{loc}}{\beta_{q,loc}^3 N} \Gamma_q(3) \right)^{N(1-q)} - 1 \right]. \tag{44}$$

Again, one can check that the above results reduce to that of the GB case in the limit $q \to 1$. To obtain the thermodynamic quantities of the gas system, by using Eq. (44), the mean energy and the mean pressure of the gas system can be computed as follows (see 1, for a detailed derivation)

$$U_{q,loc} = -\frac{\partial}{\partial \beta_{q,loc}} \ln_q Z_q$$
$$= \frac{3N}{\beta_{q,loc}} \left(\frac{4\pi e}{h^3 c^3} \frac{V_{loc}}{\beta_{q,loc}^3 N} \Gamma_q(3) \right)^{N(1-q)}, \tag{45}$$

and

$$P_{q,loc} = \frac{1}{\beta_{q,loc}} \frac{\partial}{\partial V_{loc}} \ln_q Z_q$$
$$= \frac{N}{V_{loc} \beta_{q,loc}} \left(\frac{4\pi e}{h^3 c^3} \frac{V_{loc}}{\beta_{q,loc}^3 N} \Gamma_q(3) \right)^{N(1-q)}. \tag{46}$$

Furthermore, the gas q-entropy can be obtained as follows

$$S_{q,loc} = k_B \left(1 - \beta_{q,loc} \frac{\partial}{\partial \beta_{q,loc}} \right) \ln_q Z,$$
$$= \frac{k_B}{1-q} \left[\left\{ 1 + 3N(1-q) \right\} \right.$$
$$\left. \times \left(\frac{4\pi e}{h^3 c^3} \frac{V_{loc}}{\beta_{q,loc}^3 N} \Gamma_q(3) \right)^{N(1-q)} - 1 \right]. \tag{47}$$

The q-thermodynamic quantities, namely, $P_{q,loc}$, $U_{q,loc}$ and $S_{q,loc}$, satisfy the first law of q-thermodynamics:

$$dU_{q,loc} = \frac{1}{k_B \beta_{q,loc}} dS_{q,loc} - P_{q,loc} dV_{loc}. \tag{48}$$

As seen in Eq. (42), we have introduced the auxiliary function $\mathcal{J}(q)$ characterizing how β_q differs from β. It is not unique to choose this function. For example, one can choose the function $\mathcal{J}(q)$ satisfying a condition of identical internal energy $U_{q,loc} = U_{loc}$ and one can also find that the function $\mathcal{J}$ can be solved exactly. Even though this choice is simple, it provides reasonable results since $\mathcal{J}(q) \to 1$ at the limit

$q \to 1$ and $U_{q,loc}$ taken as a linear function of U_{loc} is compatible with the description of Tsallis statistics. It should be noted that the function $\mathcal{J}(q)$ will, despite initially appearing to be arbitrary, actually reflect the thermodynamic properties of the black hole. For example, for the simple choice, $U_{q,loc} = U_{loc}$, the black hole entropy cannot be written in terms of a homogeneous function, and then the Smarr formula or the first law of black hole thermodynamics cannot be properly defined via a scaling law. Therefore, the form of the function $\mathcal{J}$ is also constrained by the condition that the consequent black hole's entropy must be a homogeneous function. In order to obtain a proper form of $\mathcal{J}(q)$ and the consistent thermodynamic properties of the black hole, let us consider a more general form as

$$U_{q,loc} = u_1 \Gamma_q(3)^{u_2} U_{loc}, \tag{49}$$

where u_1 and u_2 are arbitrary constants. As will be discussed in Sect. 4, by choosing $u_2 = -1/3$, $\Gamma_q(3)$ is eliminated in the q-entropy and then the q-entropy can be treated as a homogeneous function. As a result, black hole thermodynamics can be properly investigated. Note also that the q-internal energy is additive. As a result, using Eqs. (13) and (45), the function $\mathcal{J}(q)$ can be obtained as follows:

$$\mathcal{J}(q) = \left[\frac{\Gamma_q(3)^{1/3}}{u_1} \left(\frac{4\pi e}{h^3 c^3} \frac{V_{loc}}{\beta_{loc}^3 N} \Gamma_q(3) \right)^{N(1-q)} \right]^{\frac{1}{1+3N(1-q)}}. \tag{50}$$

To satisfy the condition: $\lim_{q \to 1} \mathcal{J}(q) = 1$, the constant is set as $u_1 = 2^{1/3}$. Due to the fact that the factor $u_1 \Gamma_q(3)^{u_2}$ in Eq. (49) is always positive. It leads to the constraint on the non-extensive parameter as follows:

$$0 < q < \frac{4}{3}, \qquad \frac{3}{2} < q < 2. \tag{51}$$

Moreover, since the mean pressure in Eq. (46) can be written in terms of the q-internal energy in Eq. (45), the condition (49) also leads to a relation between the mean pressures for both GB and Tsallis statistical mechanics as

$$P_{q,loc} = \frac{U_{q,loc}}{3V_{loc}} = \frac{\left(\frac{2}{\Gamma_q(3)} \right)^{1/3} U_{loc}}{3V_{loc}} = \left(\frac{2}{\Gamma_q(3)} \right)^{1/3} P_{loc}. \tag{52}$$

According to Eq. (41), one can check that for $q > 1$ (in fact, the range is $1 < q < 4/3$, and $3/2 < q < 2$, but let us keep this regime as $q > 1$ for simplicity), the pressure of the non-extensive gas system is less than the pressure of the ideal gas one: $P_{q,loc} < P_{loc}$. We may argue that a gas molecule attracts other molecules due to its non-extensivity. This causes the gas molecules to effectively exhibit a lower force hitting the

boundary of the system (corresponding to a lower pressure) compared to the GB case. In contrast, for $q < 1$, the pressure of the non-extensive gas system is less than that of the GB case, i.e., $P_{q,loc} > P_{loc}$ due to the gas molecules repelling each other in this regime of non-extensivity. As a consequence of the first law (48), the heat term obeys

$$\delta Q_{q,loc} = T_{q,loc} dS_{q,loc} = \left(\frac{2}{\Gamma_q(3)}\right)^{1/3} T_{loc} dS_{loc}$$

$$= \left(\frac{2}{\Gamma_q(3)}\right)^{1/3} \delta Q_{loc}. \tag{53}$$

Again, the heat transfer also gets a modification due to the non-extensivity. The efficiency in transferring the heat by the molecules of the non-extensive gas system is less (more) than that of the traditional extensive gas system caused by the attractive (repulsive) behavior of the non-extensive gas molecules with $q > 1$ ($q < 1$).

Let us turn our attention back to the q-entropy. This entropy can be expressed in terms of β_{loc} by substituting Eq. (50) into Eq. (47) as

$$S_{q,loc} = \frac{k_B}{1-q}\left[\left\{1 + 3N(1-q)\right\}\right.$$

$$\left. \times \left(\frac{8\pi e}{h^3 c^3} \frac{V_{loc}}{\beta_{loc}^3 N}\right)^{\frac{N(1-q)}{1+3N(1-q)}} - 1\right]. \tag{54}$$

As expected, this resulting entropy is independent of $\Gamma_q(3)$. The behavior of the gas entropy can be illustrated in Fig. 1. It can be seen that the behavior of the entropy depends on the parameter q and the base of the exponential function $\frac{8\pi e}{h^3 c^3} \frac{V_{loc}}{\beta_{loc}^3 N}$. For $q < 1$, the entropy grows smoothly as the number of particles increases (see the dashed lines in Fig. 1). However, divergent and local extremum points can emerge when $q > 1$. In the regime $q > 1$, the key feature of the entropy is affected by the value of the base X/ρ where $X \equiv 8\pi e/(h^3 c^3 \beta_{loc}^3)$ and $\rho \equiv N/V_{loc}$ describe how hot and dense the gas is, respectively. The density ρ of the gas is assumed to be a constant in this consideration (this assumption is applicable to the q-entropy of a black hole as will be discussed soon). It can be split into three regimes:

(i) Sufficiently cold or dense gas ($X/\rho < 1$): as increasing N, $S_{q,loc}$ first reaches its local maximum and then drops to minus infinity (see the solid blue line in Fig. 1).
(ii) Sufficiently hot or loose gas ($X/\rho > 1$): as increasing N, $S_{q,loc}$ diverges first to infinity, and then drops to its local minimum (see the solid red line in Fig. 1).
(iii) Critical point ($X/\rho = 1$): the entropy does not depend on q, but linearly depends on N as $S_{q,loc} = 3k_B N$.

The existence of the extremum and divergence of the entropy could be argued as a result of taking the self-
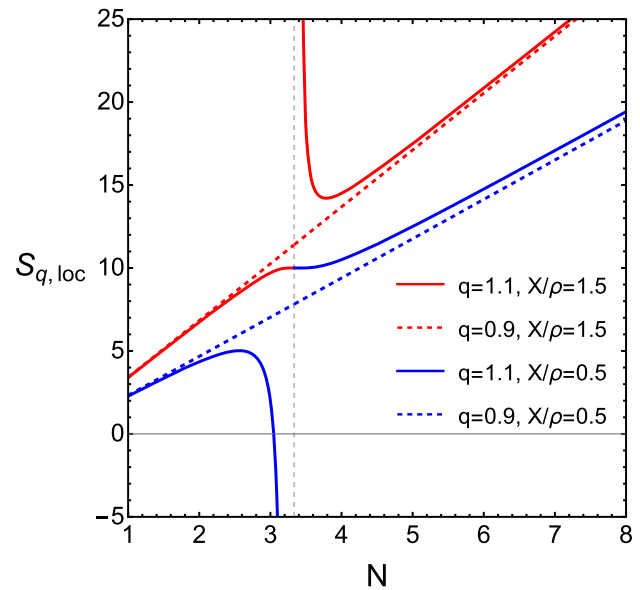

Fig. 1 The profiles of the gas entropy versus N for various values of the non-extensive parameter and the base X/ρ

gravitating effect (long-range interaction) into account, and this effect becomes significantly stronger. In fact, these results are in agreement with the ones of applying the Tsallis entropy in astrophysics (see, e.g., [68,88]). With GB statistical mechanics, the thermodynamic description of the self-gravitating system leads to a conclusion that the system is isothermal with negative heat capacity, referring to thermodynamic instability. Therefore, GB statistical mechanics may not be properly described in such a system. One needs statistical mechanics compatible with a non-isothermal system in hydrostatic equilibrium instead of thermal equilibrium, for example, Tsallis statistical mechanics [89]. We will see in the next section that a thermal system associated with a black hole is not well-defined when the number N is too large. This is because the system's temperature is diverging and becomes negative. Therefore, it is worthwhile to consider the system with a sufficiently small number of particles in order to restrict our consideration to the thermal system characterized by well-behaved temperatures.

Following the argument mentioned in the previous section, the number of particles, the inverse temperature and volume of the gas shell are set as $N = A_h/l_P^2$, $\beta_{loc} = 4\pi^2 l_{loc}/hc$, $V_h = A l_{loc}/2$, respectively. In addition, the proper distance between the inner shell of the gas and the black hole's horizon is also fixed as $l_{loc} = l_P \exp(15/8)/4\pi^{5/2}$. The generalized Bekenstein–Hawking entropy corresponding to the black hole entropy in q-thermodynamics can be obtained as

$$S_{q,BH} \equiv S_{q,loc(bh)}^{(0)}$$

$$= \frac{k_B}{1-q}\left[\left\{1 + 3(1-q)\frac{A_h}{l_P^2}\right\}\right.$$

$$\times \exp\left\{-\frac{11(1-q)\frac{A_h}{l_P^2}}{4\left(1+3(1-q)\right)\frac{A_h}{l_P^2}}\right\}\right] - 1\right]. \qquad (55)$$

One can straightforwardly check that the base X/ρ is less than unity so that the behavior of the black hole entropy is similar to the entropy represented as the solid and dashed blue lines Fig. 1. The form of the black hole entropy expressed in Eq. (55) is complicated, and not easy to capture its physical meaning. By keeping q closed to 1, the generalized Bekenstein–Hawking entropy can be approximated as the power series around $q = 1$:

$$S_{q,BH} \approx k_B \frac{A_h}{4l_P^2} + k_B \frac{121A_h^2}{32l_P^4}(1-q) + \mathcal{O}[(1-q)^2]. \quad (56)$$

From Eq. (56), it is seen that the black hole entropy is written as the usual Bekenstein–Hawking entropy with the correction term due to the non-extensivity. Furthermore, the Tsallis entropy at q-thermodynamic equilibrium is expressed as $S_q^{(0)} = k_B \ln_q \Omega$. Using Eq. (55), we can calculate the number of microstates of the black hole as

$$\Omega_{bh} = \left[\left\{1+3(1-q)\frac{A_h}{l_P^2}\right\}\exp\left\{-\frac{11(1-q)\frac{A_h}{l_P^2}}{4\left(1+3(1-q)\right)\frac{A_h}{l_P^2}}\right\}\right]^{\frac{1}{1-q}}. \qquad (57)$$

Expanding Eq. (57) around $q = 1$, one obtains

$$\Omega_{bh} \approx \exp\left(\frac{A_h}{4l_P^2}\right) + \frac{15A_h^2}{4l_P^4}\exp\left(\frac{A_h}{4l_P^2}\right)(1-q)$$
$$+ \mathcal{O}[(1-q)^2]. \qquad (58)$$

From this expression, one can see that for $q > 1$, the number of microstates is decreased due to the non-extensive nature of the system. Therefore, if black hole thermodynamics is supposed to rely on the non-extensive system, available states in which particles can occupy are less than one for the usual case. In this case, we can imagine that the Planck area units are partially overlapped and then there are over-counting states. As a result, the actual number of microstates is less than one counted from GB statistical mechanics. Oppositely, for $q < 1$, the number of microstates will be increased due to the Planck area units moving away from each other.

It is important to note that there are various applications of Tsallis entropy which take the form $S_\delta = \alpha_\delta A_h^\delta$ [90] where A_h denotes the horizon area and, α_δ and δ are constants. However, it is not perfectly clear what the form of the Tsallis entropy for the black hole is. This argument is still debatable [91–94]. Our result suggests that the form of Tsallis entropy does not satisfy the power-law form as argued in the literature. This may provide a hint to naturally include the non-extensive effect to study black hole thermodynamics as well as in a cosmological context.

4 Thermodynamic properties of black hole

In this section, we try to construct the thermodynamic quantities of the black hole by replacing the Bekenstein–Hawking entropy with the generalized Bekenstein–Hawking entropy, as well as investigate the thermodynamic properties of the black hole, e.g., the stability. In this section, we will work in the unit of $k_B = G = \hbar = c = 1$ (the Planck length l_P is also unity in this unit) for convenience. The first law of thermodynamics can be obtained by treating the black hole mass as a homogeneous function. In addition, by employing Euler's theorem, one hence obtains the algebraic relation known as Smarr formula [95]. Let us introduce the aforementioned procedure by starting with the definition of the homogeneous function. An arbitrary function $\tilde{f}(x_i)$, where $x_i \in R$, is said to be a homogeneous function of degree $n \in R$, if and only if it satisfies that $\tilde{f}(Ix_i) = I^n \tilde{f}(x_i)$, with $I \in R$. According to Euler's theorem for the homogeneous function, the function $\tilde{f}(x_i)$ satisfies the following relation:

$$n\tilde{f}(x_i) = x_j \frac{\partial \tilde{f}(x_i)}{\partial x_j}. \qquad (59)$$

For the simple case in consideration, we choose the static and spherically symmetric black hole as the Schwarzschild (Sch) black hole. The interval is taken in the form found in Eq. (1) with the horizon function given by

$$g(r) = 1 - \frac{2M}{r}, \qquad (60)$$

where M is the black hole mass. The black hole mass can be obtained by solving $g(r_{bh}) = 0$ as

$$M = \frac{r_{bh}}{2}. \qquad (61)$$

Furthermore, the Bekenstein–Hawking entropy can be written as

$$S_{BH(\text{Sch})} = \frac{A_h}{4} = \pi r_{bh}^2. \qquad (62)$$

By substituting Eq. (62) into Eq. (61), the black hole mass can be rewritten as

$$M = \frac{1}{2}\sqrt{\frac{S_{BH(\text{Sch})}}{\pi}}. \qquad (63)$$

The Smarr formula for the black hole mass written in the form of a homogeneous function of degree 1/2, i.e., $M(I S_{BH(\text{Sch})}) = I^{1/2}M(S_{BH(\text{Sch})})$, can be obtained by employing the Euler's theorem as

$$M = 2S_{BH(\text{Sch})}T_{H(\text{Sch})}, \qquad (64)$$

where the conjugate quantity of $S_{BH(\text{Sch})}$ is given by

$$T_{H(\text{Sch})} = \left(\frac{\partial M}{\partial S_{BH(\text{Sch})}}\right) = \frac{1}{4\pi r_{bh}}. \qquad (65)$$

In fact, the above equation is the Hawking temperature. Furthermore, the first law of thermodynamics can be written as

$$dM = T_{H(\text{Sch})} dS_{BH(\text{Sch})}. \tag{66}$$

Here, we have employed Euler's theorem to the black hole mass regarded as a homogeneous function of degree 1/2. By identifying the suitable expressions of thermodynamic quantities of the black hole, we obtain the thermodynamic relations that are equivalent to those obtained from the geometric approach. Moreover, the mentioned relations are obtained by considering the black hole entropy as the Bekenstein–Hawking entropy. In other words, the thermodynamic relations are rooted in GB statistical mechanics. However, as we have known, GB statistical mechanics is not suitable to describe the thermodynamic properties of the black hole due to the non-extensivity. In this work, in order to solve the aforementioned issue, the generalized Bekenstein–Hawking entropy is chosen to construct the thermodynamic relations instead of the usual one.

The Smarr formula with the generalized Bekenstein–Hawking entropy can be constructed by following the same manner done in the GB framework. Unfortunately, the expression of r_{bh} in terms of the generalized Bekenstein–Hawking entropy is not easy to display. We may write the generalized Bekenstein–Hawking entropy in terms of the black hole mass instead. As mentioned, by substituting Eq. (61) into Eq. (55), we can treat the generalized Bekenstein–Hawking entropy, with a re-defined non-extensive parameter: $\eta = (1 - q)$:

$$S_{\eta, BH(\text{Sch})} = \frac{1}{\eta} \left[(1 + 3\eta A_h) \exp\left\{ -\frac{11\eta A_h}{4(1 + 3\eta A_h)} \right\} - 1 \right],$$
$$= \frac{1}{\eta} \left[(1 + 48\pi M^2 \eta) \exp\left\{ -\frac{44\pi M^2 \eta}{1 + 48\pi M^2 \eta} \right\} - 1 \right], \tag{67}$$

as a homogeneous function of degree 1, i.e., $S_{\eta, BH(\text{Sch})}(IM^2, I\eta^{-1}) = IS_{\eta, BH(\text{Sch})}(M^2, \eta^{-1})$. As mentioned in the previous section, this is the reason for choosing the entropy to be independent of $\Gamma_q(3)$. If the term $\Gamma_q(3)$ appears in the black hole entropy, it is not possible to treat such an entropy as the homogeneous function. Furthermore, by using Euler's theorem, the generalized Bekenstein–Hawking entropy can be expressed as

$$S_{\eta, BH(\text{Sch})} = \frac{1}{2} M \left(\frac{\partial S_{\eta, BH(\text{Sch})}}{\partial M} \right)_\eta - \eta \left(\frac{\partial S_{\eta, BH(\text{Sch})}}{\partial \eta} \right)_M. \tag{68}$$

The generalized Smarr formula can be obtained by rearranging the above equation as

$$M = 2 S_{\eta, BH(\text{Sch})} \left(\frac{\partial M}{\partial S_{\eta, BH(\text{Sch})}} \right)_\eta$$
$$+ 2\eta \left(\frac{\partial S_{\eta, BH(\text{Sch})}}{\partial \eta} \right)_M \left(\frac{\partial M}{\partial S_{\eta, BH(\text{Sch})}} \right)_\eta. \tag{69}$$

The last term on the right-hand side of Eq. (69) can be rewritten by using the cyclical relation, and hence the generalized Smarr formula reads

$$M = 2 S_{\eta, BH(\text{Sch})} T_{\eta, H(\text{Sch})} - 2\eta \Phi_{\eta(\text{Sch})}, \tag{70}$$

where the conjugate thermodynamic quantities are properly defined by

$$T_{\eta, H(\text{Sch})} = \left(\frac{\partial M}{\partial S_{\eta, BH(\text{Sch})}} \right)_\eta,$$
$$\Phi_{\eta(\text{Sch})} = \left(\frac{\partial M}{\partial \eta} \right)_{S_{\eta, BH(\text{Sch})}}. \tag{71}$$

Note that the black hole mass in Eq. (70) is the same as the one in Eq. (64). Furthermore, the first law of thermodynamics can be written as follows

$$dM = \left(\frac{\partial M}{\partial S_{\eta, BH(\text{Sch})}} \right)_\eta dS_{\eta, BH(\text{Sch})} + \left(\frac{\partial M}{\partial \eta} \right)_{S_{\eta, BH(\text{Sch})}} d\eta,$$
$$= T_{\eta, H(\text{Sch})} dS_{\eta, BH(\text{Sch})} + \Phi_{\eta(\text{Sch})} d\eta. \tag{72}$$

Note that at the limit $\eta \to 0$, the generalized Smarr formula (70) and the first law of thermodynamics (72) can be, respectively, reduced to Eqs. (64) and (66) as expected. It is also important to note that the generalized Smarr formula found in Eq. (70) is derived from the fact that black hole entropy is thought of as the homogeneous function instead of considering the black hole mass. However, it is still written in the appropriate form and reduced to one in the GB case at limit $\eta \to 0$. Furthermore, the thermodynamic relations obtained from the same procedure with the Rényi entropy for the various kinds of black holes have been shown in Ref. [42].

In this part of our work, let us first consider the Sch black hole undergoing a process of fixing η. The first law of thermodynamics is, therefore, reduced to

$$dM = T_{\eta, H(\text{Sch})} dS_{\eta, BH(\text{Sch})}. \tag{73}$$

The behaviors of the black hole entropy in Eq. (67) for various values of η can be illustrated in Fig. 2. It is very important to point out that the entropy with negative η has an extremum point, while there is no such a point for one with $\eta \geq 0$. In addition, the entropy with negative η is unfortunately not well-behaved for the whole range of horizon radius, e.g. it becomes at a certain horizon radius as seen in the top panel of Fig. 2. It is also noticed that the greater the negative value of the non-extensive parameter, the smaller the area at which the entropy drops to zero. On the other hand, the entropy with positive η is a monotonically increasing function of the hori-
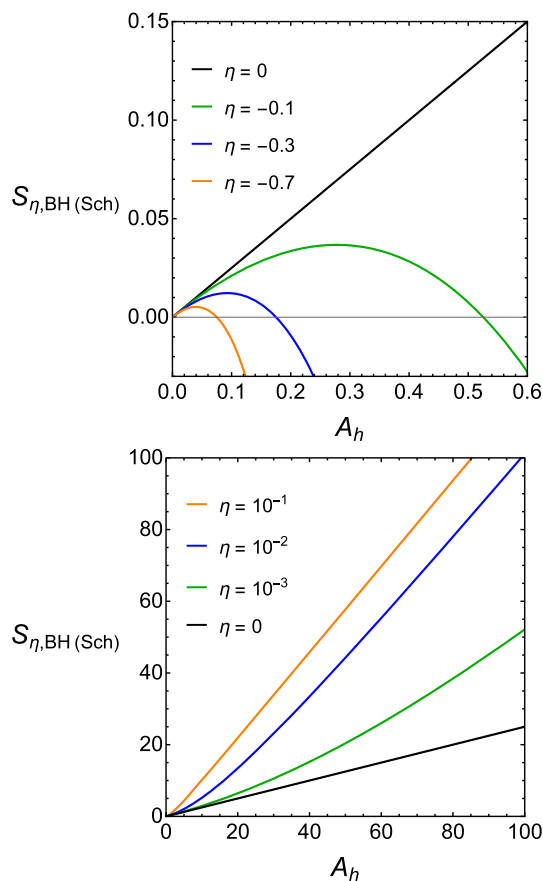
Fig. 2 The black hole entropy versus A_h with negative η (top) and positive η (bottom)

zon area. The contribution due to the non-extensivity makes the q-entropy grow faster than the GB one.

For the temperature of the black hole, it can be straightforwardly computed by using the first relation in Eq. (71) as

$$T_{\eta, H(\text{Sch})} = \frac{(1 + 3\eta A_h) \exp\left[\dfrac{11\eta A_h}{4(1 + 3\eta A_h)}\right]}{2\sqrt{\pi A_h}(1 + 36\eta A_h)}. \tag{74}$$

Again, it is easy to check that at limit $\eta \to 0$, Eq. (74) can be reduced to the Hawking temperature: $\lim_{\eta \to 0} T_{\eta, H(\text{Sch})} = 1/\left(2\sqrt{\pi A_h}\right)$. According to the definition in the first relation of Eq. (71), the temperature with negative η diverges when the entropy reaches its local maximum. It implies that the black hole's temperature in this regime of η cannot be defined for a sufficiently large size. This issue is relevant to the argument mentioned in the previous section that for large enough N, the temperature associated with Tsallis entropy is not easily defined for the self-gravitating system in general since the system is no longer in thermodynamic equilibrium. In order to restrict our attention to the thermodynamic equilibrium system, we will set the upper bound limit of N at

the local maximum of the entropy. Such an upper bound of N corresponds to the largest surface area of the mentioned maximum point denoted as $A_{\max}$, in which the temperature in the range $0 < A_h < A_{\max}$ is well-behaved.

The local maximum of the entropy in Eq. (67) is simply obtained from the condition $dS_{\eta, BH(\text{Sch})}/dA_h|_{A_{\max}} = 0$. As a result, the area at the local maximum is obtained as

$$A_{\max} = -\frac{1}{36\eta}. \tag{75}$$

The profiles of the temperature are illustrated in Fig. 3. According to the first relation in Eq. (71) together with the fact that the derivative of the black hole's mass with respect to A_h is always positive, the trend of the temperature depends only on the change in entropy. For the asymptotic values of the temperature, one can also check that $dM/dA_h \propto 1/\sqrt{A_h}$. Therefore, the temperature approaches infinity as $A_h \to 0$. For negative η, the temperature drops to its minimum at a certain area denoted as A_{C_η}, then grows to infinity again due to the maximum of the entropy. The area A_{C_η} is indeed the point at which the heat capacity diverges as will be discussed soon. By using the minimum condition: $dT_{\eta, BH(\text{Sch})}/dA_h = 0$, one can solve for an explicit form of A_{C_η} as follows:

$$A_{C_\eta} = \frac{1}{108\eta}\left[11^{2/3}\left(3\sqrt{7869} + 236\right)^{1/3}\right.$$
$$\left. - 55\left(\frac{11}{3\sqrt{7869} + 236}\right)^{1/3} - 25\right],$$
$$\approx 0.359 A_{\max}. \tag{76}$$

It is interestingly found that this behavior of the temperature with negative η does not depend on the magnitude of η. Therefore, as seen in the top panel of Fig. 3, the temperature for negative η profiles can be plotted without identifying the value of η. For the positive η regime, the temperature monotonically decreases from infinity to zero as the horizon area increases (see the bottom panel of Fig. 3) straightforwardly caused by the behavior of the entropy. In addition, since the area A_h is indeed in the unit of the Planck area, the non-zero area should be at least one (the black hole's horizon area is equal to the Planck area). It yields a bound to the magnitude of the negative η. As a result, the constraint (51) can be further restricted as

$$-\frac{1}{36} < \eta < 1. \tag{77}$$

Next, let us investigate the thermodynamic stability of the Sch black hole based on the Tsallis statistical mechanics. A thermal system is said to be locally stable if its heat capacity is positive. In addition, the phrase "local" in this context means that the black hole can exist by itself. When the black hole is locally unstable (characterized by negative heat capacity), it will evaporate due to thermal radiation and eventually disappear. For simplicity, we consider the process with fixing η,
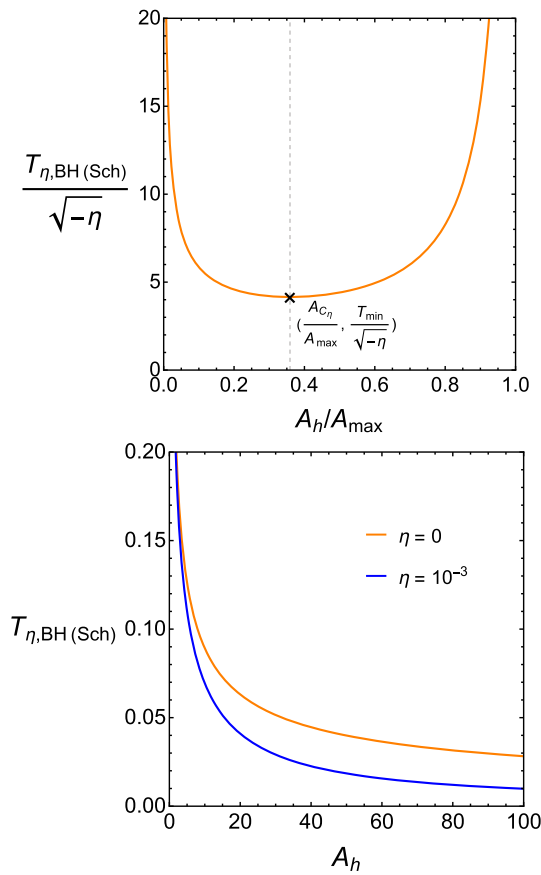
Fig. 3 Profiles of the temperature of the black hole with negative η (top) and positive η (bottom)

Fig. 4 Profiles of heat capacity for negative η (top) and positive η (bottom)

the heat capacity of the black hole undergoing this process is defined by

$$
\begin{aligned}
C_{\eta(\mathrm{Sch})} &= \left(\frac{\partial M}{\partial T_{\eta, H(\mathrm{Sch})}} \right)_\eta \\
&= -\frac{A_h \exp\left[\dfrac{11\eta A_h}{4(1 + 3\eta A_h)} \right](1 + 3\eta A_h)(1 + 36\eta A_h)^2}{2 + \eta A_h \left[205 + 18\eta A_h(25 + 36\eta A_h) \right]}.
\end{aligned}
$$

(78)

The behavior of the heat capacity is illustrated in Fig. 4. It is found that the heat capacity in the GB limit is always negative, $\lim_{\eta \to 0} C_{\eta(\mathrm{Sch})} = -A_h/2$, implying that the Sch black hole is locally unstable based on GB statistical mechanics. Due to the fact that the change under the process of fixing η is that with respect to the change of A_h. As previously mentioned, the change of the black hole mass is always positive so that the sign of the heat capacity is identified by the sign of the slope of the temperature $dT_{\eta, H(\mathrm{Sch})}/dA_h$ (see the first equality in Eq. (78)). According to the analysis of the behavior of the temperature, the black hole with positive η is always locally

unstable. On the other hand, the black hole with negative η can be both locally unstable and stable for sufficiently small size and large size, respectively. The transition between these two phases is at A_{C_η}. One can conclude that the non-extensive parameter is constrained by the local stability condition as

$$
-\frac{1}{36} < \eta < 0.
$$

(79)

The black hole with the above negative η is locally stable in the region: $A_{C_\eta} < A_h < A_{\max}$ (large-sized black hole phase).

Apart from the local stability, the global stability can be determined by analyzing the free energy. The phrase "global" refers to comparing the black hole with other possible states. A black hole is preferred to exist when its free energy must be less than the free energy of a state without a black hole, i.e., radiation or hot gas. A black hole with free energy greater than radiation's free energy cannot form. Suppose that the radiation has zero free energy, the global stability condition for the black hole is thus the phase with negative free energy. With η kept constant, the Gibbs free energy is defined by

$$\mathcal{G}_{(\text{Sch})} = M - T_{\eta, H(\text{Sch})} S_{\eta, BH(\text{Sch})},$$

$$= \frac{-2 - 11\eta A_h + 18\eta^2 A_h^2 + 2(1 + 3\eta A_h) \exp\left[\dfrac{11\eta A_h}{4(1 + 3\eta A_h)}\right]}{4\eta \sqrt{\pi A_h}(1 + 36\eta A_h)}. \tag{80}$$

The behavior of the free energy is illustrated on the top panel of Fig. 5. Remarkably, the free energy in Eq. (80) can be reduced to one in the GB limit as $\lim_{\eta \to 0} \mathcal{G}_{(\text{Sch})} = \sqrt{A_h}/(8\sqrt{\pi})$, which is always positive implying that the Sch black hole is globally unstable based on GB statistical mechanics. For the black hole described by Tsallis statistical mechanics (with negative η), the free energy can be negative due to the existence of non-extensivity. A lower bound on the area denoted as $A_{\mathcal{G}}$ of the globally stable black hole is straightforwardly obtained from the vanishing of the numerator of the second line of Eq. (80). Unfortunately, it is very complicated to analytically solve for $A_{\mathcal{G}}$ because of dealing with the exponential functions with different bases and exponents. Instead, we solve numerically and then obtain as

$$A_{\mathcal{G}} \approx 0.695 A_{\max}. \tag{81}$$

Comparing to the bound on A_h in Eq. (76), there always exists the phase of both locally and globally stable black holes with the area: $(A_{C_\eta} <) A_{\mathcal{G}} < A_h < A_{\max}$.

Furthermore, the behavior of the free energy versus the temperature of the black hole can be illustrated in the bottom panel of Fig. 5. From this figure, there exists a cusp corresponding to the area A_C at which the temperature is minimized, the heat capacity diverges and the Gibbs free energy is maximized. The red and blue dashed lines represent the locally unstable and locally stable but globally unstable phases of the black hole, respectively. The radiation or non-black hole and stable black hole phases are represented as the green line of zero free energy and the solid blue line, respectively. In addition, there exists a temperature, called the Hawking-Page temperature, denoted by T_{HP}, which can be obtained from the condition $\mathcal{G}_{(\text{Sch})} = M - T_{HP} S_{\eta, BH(\text{Sch})} = 0$ as

$$T_{HP} \approx 5.935\sqrt{-\eta}. \tag{82}$$

At this temperature, the first-order phase transition from the thermal radiation to the stable large-sized black hole emerges. It is well-known as the Hawking-Page phase transition [96]. Note that the aforementioned phase transitions occur due to the existence of the non-extensivity.

Another possible process, in which the heat transfer exists, is the process of fixing $\Phi_{\eta(\text{Sch})}$. The conjugate variable $\Phi_{\eta(\text{Sch})}$ can be computed using the definition in the second relation of Eq. (71) or the Smarr formula (70) as
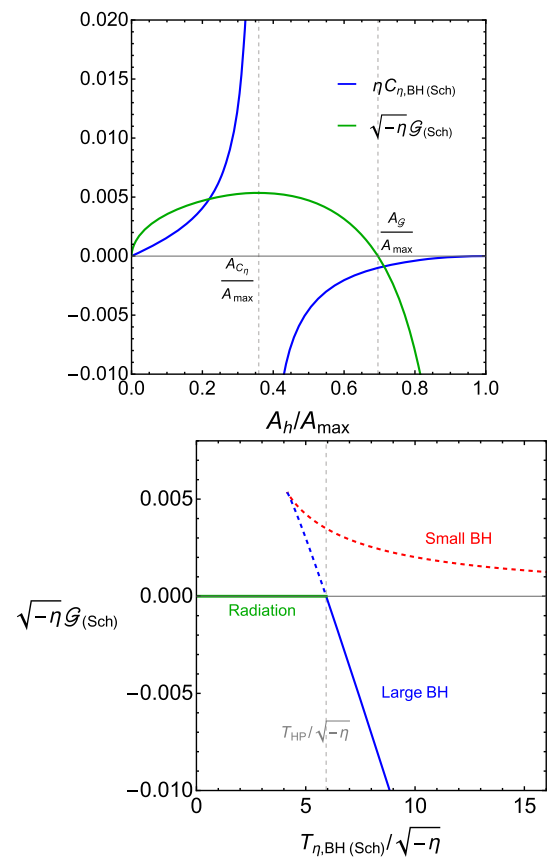

Fig. 5 The top panel shows the profiles of the heat capacity and Gibbs free energy for negative η. The bottom panel shows the Gibbs free energy versus the temperature of the black hole with negative η

$$\Phi_{\eta(\text{Sch})} = \frac{4 + 23\eta A_h - 4(1 + 3\eta A_h) \exp\left[\dfrac{11\eta A_h}{4(1 + 3\eta A_h)}\right]}{8\eta^2 \sqrt{\pi A_h}(1 + 36\eta A_h)}. \tag{83}$$

The behavior of $\Phi_{\eta(\text{Sch})}$ is illustrated in Fig. 6. It is noticed that $\Phi_{\eta(\text{Sch})}$ always has negative valued for all possible values of η and A_h.

According to the fact that $\Phi_{\eta(\text{Sch})}$ depends on both η and A_h, in principle, the behavior of a quantity under the process of fixing $\Phi_{\eta(\text{Sch})}$ should be expressed in the terms of $\Phi_{\eta(\text{Sch})}$ and η (or A_h) obeying the constraint (83). Practically, it is very complicated to solve the above constraint analytically, so a numerical technique may be required to achieve our purpose. For any change under this process, the derivative $dA_h/d\eta$ must be fixed as
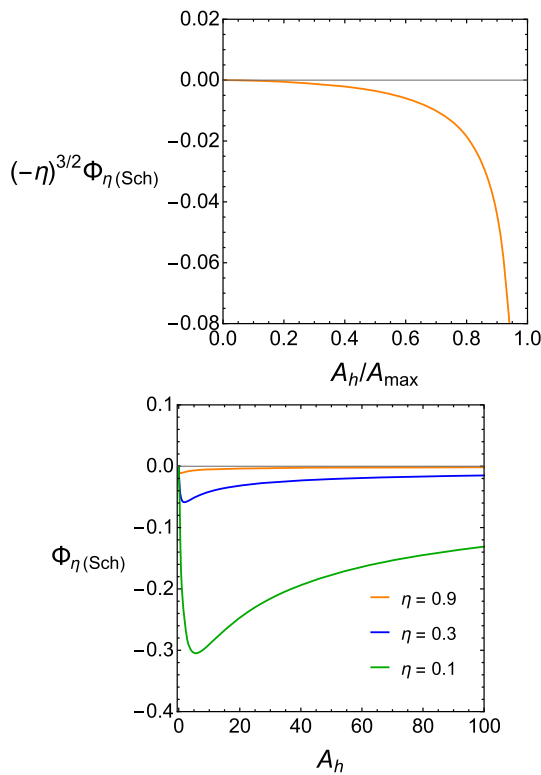
Fig. 6 The profiles of $\Phi_{\eta(\mathrm{Sch})}$ with negative η (top) and positive η (bottom

where

$$
c_1 = 119232\eta^4 A_h^4 + 55116\eta^3 A_h^3
$$
$$
+14205\eta^2 A_h^2 + 1744\eta A_h + 16,
$$
$$
c_2 = 648\eta^3 A_h^3 + 450\eta^2 A_h^2 + 205\eta A_h + 2,
$$
$$
c_3 = 7776\eta^4 A_h^4 + 5508\eta^3 A_h^3 + 2205\eta^2 A_h^2 + 423\eta A_h + 4,
$$
$$
c_4 = 59616\eta^4 A_h^4 + 56124\eta^3 A_h^3
$$
$$
+26181\eta^2 A_h^2 + 3464\eta A_h + 32,
$$
$$
c_5 = 1944\eta^4 A_h^4 + 1998\eta^3 A_h^3 + 1065\eta^2 A_h^2 + 211\eta A_h + 2.
$$

The corresponding Helmholtz free energy is given by

$$
\mathcal{F}_{(\mathrm{Sch})} = M - \eta\Phi_{\eta(\mathrm{Sch})} - T_{\eta,BH(\mathrm{Sch})}S_{\eta,BH(\mathrm{Sch})},
$$
$$
= -\frac{8 + 9\eta A_h(5 - 4\eta A_h) - 8(1 + 3\eta A_h)\exp\left[\dfrac{11\eta A_h}{4(1 + 3\eta A_h)}\right]}{8\eta\sqrt{\pi A_h}(1 + 36\eta A_h)}.
$$
(87)

The behaviors of the thermodynamic quantities are illustrated in the top panels of Fig. 7. As mentioned, we employ the numerical technique to find the pairs of (η, A_h) with fixing $\Phi_{\eta(\mathrm{Sch})}$. Then, substituting these pairs into Eqs. (74), (78) and (87), one can obtain the temperature, heat capacity, and free energy in terms of a suitable variable along the fixed-$\Phi_{\eta(\mathrm{Sch})}$ process, respectively. According to the examples in the top panels of Fig. 7, the black hole cannot be both locally

---

$$
\frac{dA_h}{d\eta} = 2A_h\frac{(3\eta A_h + 1)(\eta A_h b_1 + 8) - e^{\frac{11\eta A_h}{4(1+3\eta A_h)}}\left[\eta A_h\left(72\eta A_h b_2 + 457\right) + 8\right]}{2\eta e^{\frac{11\eta A_h}{4(1+3\eta A_h)}}\left[\eta A_h\left(18\eta A_h b_2 + 205\right) + 2\right] - \eta(3\eta A_h + 1)\left(\eta A_h b_3 + 4\right)}.
$$
(84)

---

where $b_1 = 1656\eta A_h + 455$, $b_2 = 36\eta A_h + 25$ and $b_3 = 828\eta A_h + 409$. In this part of this work, we will use these manners to deal with computing the relevant thermodynamic quantities.

To investigate the thermodynamic stability, let us introduce a new thermodynamic potential $M - \eta\,\Phi_{\eta(\mathrm{Sch})}$ in which its change corresponds to

$$
d(M - \eta\,\Phi_{\eta(\mathrm{Sch})}) = T_{\eta,BH(\mathrm{Sch})}dS_{\eta,BH(\mathrm{Sch})} - \eta d\Phi_{\eta(\mathrm{Sch})}.
$$
(85)

Therefore, the heat capacity under the process of fixing $\Phi_{\eta(\mathrm{Sch})}$) can be computed as follows

$$
C_{\Phi(\mathrm{Sch})} = \left(\frac{d(M - \eta\,\Phi_{\eta(\mathrm{Sch})})}{dT_{\eta,BH(\mathrm{Sch})}}\right)_{\Phi_{\eta(\mathrm{Sch})}},
$$
$$
= -\frac{(1 + 3\eta A_\eta)\,e^{-\frac{11\eta A_h}{4(1+3\eta A_h)}}}{\eta}
$$
$$
\times\left[\frac{c_1 + 8\,c_2\,e^{\frac{11\eta A_h}{2(1+3\eta A_h)}} - 8\,c_3\,e^{\frac{11\eta A_h}{4(1+3\eta A_h)}}}{c_4 - 16\,c_5\,e^{\frac{11\eta A_h}{4(1+3\eta A_h)}}}\right],
$$
(86)

and globally stable in the process of fixing $\Phi_{\eta(\mathrm{Sch})}$. These thermodynamic instabilities are confirmed by the plot in the bottom panel of Fig. 7. It is obviously seen that there is no overlapped region at which all the valid $\Phi_{\eta(\mathrm{Sch})}$, locally stable black hole phase ($C_{\Phi(\mathrm{Sch})} > 0$) and globally stable black hole phase ($\mathcal{F}_{(\mathrm{Sch})} < 0$) exist.

Before ending this section, let us move our attention to $\Phi_{\eta(\mathrm{Sch})}$. At the small η regime, the quantity Φ_η in Eq. (83) can be expanded around $\eta = 0$ as
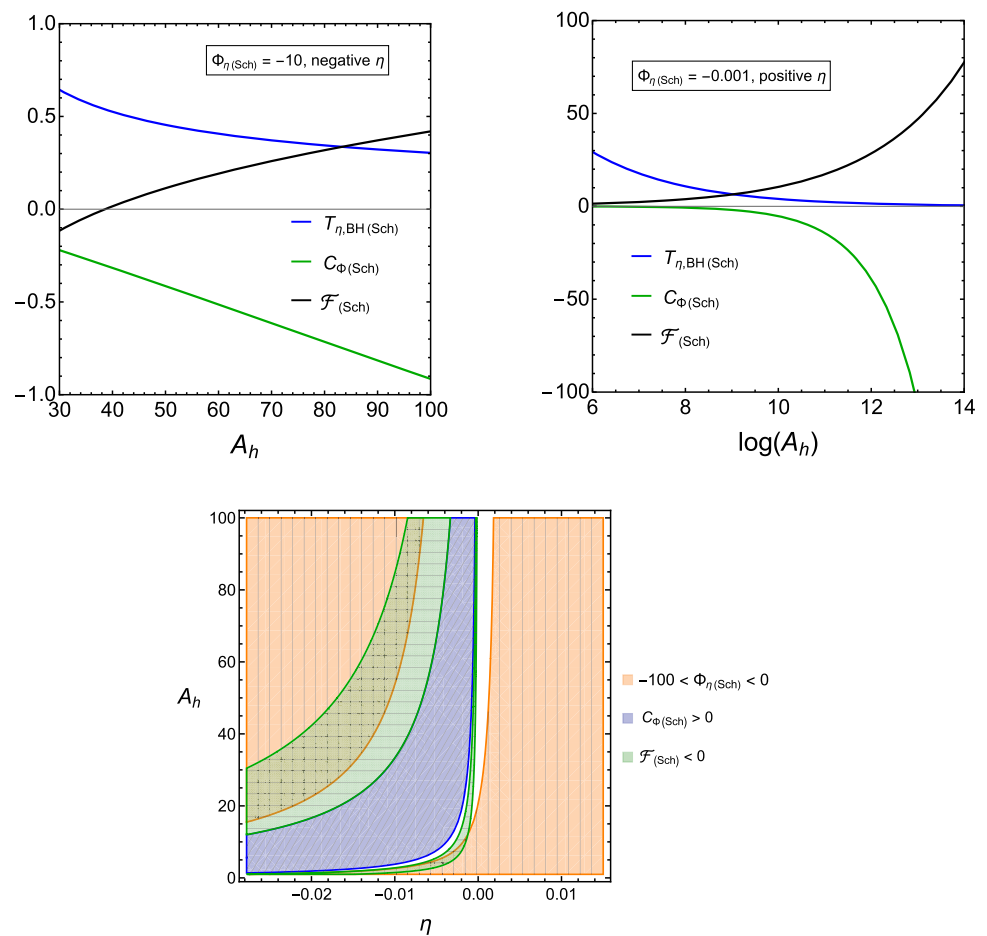
$$
\Phi_{\eta(\mathrm{Sch})} \approx -\frac{121}{8}\pi r_{bh}^3 + \frac{55297}{24}\pi^2 r_{bh}^2\eta + \mathcal{O}(\eta^2).
$$
(88)

The leading term of the above expression is proportional to the three-dimensional volume. Accordingly, it is worthwhile to introduce, respectively, the thermodynamic pressure and volume as

$$
P_{\eta(\mathrm{Sch})} = -\frac{363}{32}\eta, \qquad V_{\eta(\mathrm{Sch})} = -\frac{32}{363}\Phi_{\eta(\mathrm{Sch})}.
$$
(89)

At the limit $\eta \to 0$, one obtains $\lim_{\eta\to 0} V_{\eta(\mathrm{Sch})} = \frac{4}{3}\pi r_{bh}^3$. Note that $V_{\eta(\mathrm{Sch})}$ can behave properly as a positive-valued

Fig. 7 The top panels show the profiles of the black hole thermodynamic quantities under $\Phi_{\eta(\mathrm{Sch})}$-fixed process with negative η (left) and positive η (right). The bottom panel shows the regions of existence of $\Phi_{\eta(\mathrm{Sch})}$, locally stable and globally stable black hole phases

volume since $\Phi_{\eta(\mathrm{Sch})}$ is always negative. Regarding the non-extensive parameter as a state variable, the thermodynamic $P_{\eta(\mathrm{Sch})}$ behaves as a proper pressure for negative η and as a tension for positive η.

Remarkably, the thermodynamic pressure $P_{\eta(\mathrm{Sch})}$ is different from the pressure of gas found in Eq. (46). It is because the pressure $P_{\eta(\mathrm{Sch})} = 0$, while $P_{q,loc} \neq 0$, at limit $\eta \to 0$. Furthermore, according to $P_{\eta(\mathrm{Sch})}$ and $V_{\eta(\mathrm{Sch})}$ expressed in Eq. (89), the Smarr formula and the first law of thermodynamics of the Sch black hole can be rewritten, respectively, as follows

$$M = 2T_{\eta,H(\mathrm{Sch})}S_{\eta,BH(\mathrm{Sch})} - 2P_{\eta(\mathrm{Sch})}V_{\eta(\mathrm{Sch})}, \tag{90}$$

$$dM = T_{\eta,H(\mathrm{Sch})}dS_{\eta,BH(\mathrm{Sch})} + V_{\eta(\mathrm{Sch})}dP_{\eta(\mathrm{Sch})}. \tag{91}$$

The above first law is reminiscent of the change of the chemical enthalpy. Hence, one can interpret the black hole's mass as the enthalpy, and $M - P_{\eta(\mathrm{Sch})}V_{\eta(\mathrm{Sch})} = M - \eta\Phi_{\eta(\mathrm{Sch})}$ as the internal energy. In addition, η and $\Phi_{\eta(\mathrm{Sch})}$ can also be thought of as the chemical potential and number of particles, respectively. No matter how η and its conjugate are interpreted, the stability analysis done previously will still be reliable.

5 Conclusion and discussion

In the first part of this work, we proposed the black hole entropy based on q-statistical mechanics. Based on traditional GB statistical mechanics, the Bekenstein–Hawking entropy can be derived by treating the system near a black hole as the shell of the indistinguishable classical gas with the maximum number of gas particles, i.e., $N = A_h/l_P^2$. The significant conditions are imposed as follows: (i) the gas system is in thermal equilibrium with the black hole, i.e. the gas temperature is equal to the Hawking one, (ii) it approaches very closely the black hole's horizon with a distance of one-tenth of the Planck length. For Tsallis statistical mechanics, it can be thought of as the one-parameter extension of GB statistical mechanics which is characterized by the non-extensive entropy. With Tsallis statistical mechanics, the gas q-entropy was derived as found in Eq. (47). Moreover, we proposed that the internal energy is written as $U_{q,loc} = 2^{1/3}\Gamma_q(3)^{-1/3}U_{loc}$ in order to obtain the unknown function $\mathcal{J}(q)$, from which the term $\Gamma_q(3)$ in the gas q-entropy can be eliminated. According to the $\Gamma_q(3)$-independent entropy of gas, it is possible to formulate the black hole entropy as a homogeneous function. By following a similar manner in deriving the black hole entropy as done for the GB case, the resulting black hole q-

entropy has been shown in Eq. (55). Note that in our derivation, we also employ the formal logarithm map for the sake of factorization of the gas distribution function. With Taylor's series around the GB limit ($q \rightarrow 1$), the leading order term of the q-entropy is the standard Bekenstein–Hawking one. The non-extensivity of non-extensive statistical mechanics affects the number of microstates of black holes. According to the principle of the maximization of the entropy, the non-extensive parameter q must be positive, and then the non-extensivity can both reduce and increase the number of black hole's microstates (see Eq. (58)).

Thermodynamic stabilities of the Sch black hole have been investigated in the second part of this work. The thermodynamic phase-space associated with q-statistical mechanics can be constructed by adopting Euler's theorem for the q-entropy which is the homogeneous function of degree 1. As a result, the Smarr formula and the first law of thermodynamics were derived in Eqs. (70) and (72), respectively. It is emphasized that the non-extensive parameter $\eta = (1 - q)$ automatically plays a role as the thermodynamic variable. To realize the thermodynamic stability, we firstly focused on the scenario that the thermal system associated with Sch black hole undergoes the process of fixing η. We found that, unlike the consideration based on the GB entropy, the non-extensivity can stabilize the black hole. In particular, the system can be both locally and globally stable when the non-extensive parameter lies in the range $-1/36 < \eta < 0$. Furthermore, we considered another possible process that $\Phi_{\eta(\text{Sch})}$ is fixed. It is found that for such a process, the black hole cannot be both locally and globally stable confirmed by the plot in the bottom panel of Fig. 7.

According to Eq. (58), the black hole with negative η has a number of microstates less than the system described by GB statistical mechanics. In this case, we can imagine that the Planck area units are partially overlapped and then there are over-counting states. As a result, the actual number of microstates is less than the one counted from GB statistical mechanics. Furthermore, in the viewpoint of the total energy, the non-extensive system has non-additive total energy, for example, for $N = 2$, $E = E_1 + E_2 - (1 - q)\beta_q E_1 E_2$. The non-additive term, i.e., $-(1 - q)\beta_q E_1 E_2$, can be interpreted as the classical correlation between particles due to long-range interaction. This correlation might correspond to the overlapping region of the Planck area units as we have argued previously. The aforementioned argument may provide insight into the thermodynamic nature of black holes via statistical mechanics.

One of the key results of the present work is the outcome of black hole entropy, q-entropy, found in Eq. (67). Up to our knowledge, it is not expressed in the well-known form as found in the literature. In fact, this entropy is quite unique due to its complicated form as the exponential function of the horizon area. In order to compare our result to the other,

it is worthwhile to use an approximation where $\eta \rightarrow 0$ and then analyze the resulting correction by comparing it to one of the quantum corrections. One of the usual quantum corrections of the area law entropy is the logarithmic correction given by $\alpha \ln(A/4)$ [97,98] where α is a dimensionless constant which is usually $\alpha = -1/2$ or $\alpha = -3/2$ [99,100]. One can see that the quantum correction is always negative which is consistent with our result where the correction can be written as $\frac{\eta}{2}(11A/4)^2$. As a result, it may be inferred that quantum correction influences the number of microstates in the same way as non-extensivity does. This may shed light on the connection between the quantum nature of the black hole and the non-extensivity behavior of the black hole.

In the manner presented in this work, the black hole entropy is derived independently of the type of black hole solution and different types of gas near the black hole. Therefore, it is worthwhile to apply this manner to investigate the thermodynamic implications based on non-extensive statistical mechanics for other black holes. This may pave the way to exploring the effect of non-extensivity on black hole thermodynamics. Moreover, the thermodynamic stability analysis in our consideration corresponds to the system undergoing the processes of fixing η and its conjugate, Φ_{η}.

It is important to note that our investigation is mainly involved in the development of black hole thermodynamics which seems to be interested only in theoretical aspects. In order to capture such properties of the black hole, it is natural to ask whether it is possible to find some observational signatures to identify the black hole's thermodynamic properties. In fact, it is possible to characterize the behavior of the quasinormal modes (QNMs) by analyzing the signature in the gravitational waves emitted from a black hole during the ringdown stage [101–104]. Moreover, some intrinsic properties of the black hole can be identified by using null geodesics of test particles moving near the black hole [105–109]. Unfortunately, such direct signatures of black hole thermodynamics are still difficult to detect from recent observations. However, recent investigation into the geodesic instability of the test particles, specifically the Lyapunov exponents of both massless and massive particles, serves as an order parameter to investigate the thermodynamic phase transition of the black hole [110–115]. This might be a possible way to connect the observational signature to the properties of black hole thermodynamics. The further investigation of the thermodynamic phase transition of the black hole with Tsallis entropy considered in this work in order to link to the Lyapunov exponents may shed light on the interplay between the non-extensive nature of the black hole and the observational signatures. This is an interesting issue to investigate and we will leave this investigation for further work.

Appendix A: The first law of thermodynamics and consistent thermodynamic quantities

The first law of thermodynamics relating the thermodynamic quantities, e.g., the internal energy, can be constructed in the following manner. Let us begin with calculating the total differential of U_q:

$$dU_q = q \int d\mathcal{V} f^{q-1} E \delta f + \int d\mathcal{V} f^q \delta E. \tag{A.1}$$

Moreover, the total differential of S_q can be written as

$$dS_q = \frac{k_B}{1-q} \int d\mathcal{V}(q f^{q-1} - 1)\delta f$$
$$= \frac{k_B q}{1-q} \int d\mathcal{V} f^{q-1} \delta f. \tag{A.2}$$

The leftover term on the right-hand side in Eq. (A.2) is obtained by using the variation of the normalization condition: $\delta \int d\mathcal{V} f = 0 = \int d\mathcal{V} \delta f$. From Eq. (32), it can be rearranged as

$$E = \frac{1}{(1-q)\beta_q}(1 - f^{1-q} Z_q^{1-q}). \tag{A.3}$$

According to Eq. (A.3), the first term on the right-hand side of Eq. (A.1) can be expressed as

$$q \int d\mathcal{V} f^{q-1} E \delta f = \frac{q}{(1-q)\beta_q} \int d\mathcal{V} f^{q-1}(1 - f^{1-q} Z_q^{1-q})\delta f$$

$$= \frac{q}{(1-q)\beta_q} \int d\mathcal{V} f^{q-1}\delta f. \tag{A.4}$$

Substituting Eq. (A.2) in Eq. (A.4), it reads

$$q \int d\mathcal{V} f^{q-1} E \delta f = T_q dS_q. \tag{A.5}$$

Obviously, the first term on the right-hand side of Eq. (A.1) is a heat-term, i.e., $\delta Q = T_q dS_q$. Immediately, the second (leftover) term can be interpreted as a work term:

$$\int d\mathcal{V} f^q \delta E = \delta W_q. \tag{A.6}$$

Substituting Eqs. (A.5) and (A.6) in Eq. (A.1), the first law of thermodynamics can be expressed as [66]

$$dU_q = T_q dS_q + \delta W_q, \tag{A.7}$$

Specially, for the hydrostatic system described by (P_q, V, T_q), the first law of thermodynamics can be expressed as

$$dU_q = T_q dS_q - P_q dV, \tag{A.8}$$

with setting the work term as $\delta W_q = -P_q dV$ where P_q is the mean q-pressure. Furthermore, the consistent thermodynamic quantities are defined by

$$T_q = \left(\frac{\partial U_q}{\partial S_q}\right)_V, \qquad P_q = -\left(\frac{\partial U_q}{\partial V}\right)_{S_q}. \tag{A.9}$$

Instead of expressing the thermodynamic quantities in terms of the distribution function, e.g., $U_q = \int d\mathcal{V} f^q E$, those can be written in terms of the partition function as follows. Let us start with considering the internal energy:

$$U_q = \int d\mathcal{V} f^q E. \tag{A.10}$$

Using Eq. (31), the derivative of Z_q with respect to β_q can be obtained as

$$\frac{\partial Z_q}{\partial \beta_q} = -\int d\mathcal{V} E(1 - (1-q)\beta_q E)^{\frac{q}{1-q}}. \tag{A.11}$$

Substituting Eq. (A.11) in Eq. (A.10), the internal energy reads

$$U_q = -\frac{1}{Z_q^q}\frac{\partial Z_q}{\partial \beta_q} = -\frac{\partial}{\partial \beta_q} \ln_q Z_q. \tag{A.12}$$

Note that one obtains $\lim_{q \to 1} U_q = -\partial \ln_1 Z/\partial \beta$ as expected.

Furthermore, the mean q-pressure is defined by

$$P_q = \int d\mathcal{V} f^q P_q'. \tag{A.13}$$

By following the same manner done for the internal energy, we have

$$\int d\mathcal{V} f^q P_q' = \frac{1}{\beta_q}\frac{\partial}{\partial V} \ln_q Z_q. \tag{A.14}$$

Note that we have employed the thermodynamic relation $P'_q = -\partial E/\partial V$ in Eq. (A.14). Substituting Eq. (A.14) in Eq. (A.13), one hence obtains

$$P_q = \frac{1}{\beta_q} \frac{\partial}{\partial V} \ln_q Z_q. \tag{A.15}$$

Again, we can check that the mean q-pressure can be reduced to one in the GB case at limit $q \to 1$.

For the q-entropy related to those quantities expressed in Eqs. (A.12) and (A.15) via the first law of thermodynamics, it can be constructed as follows. Let us consider the partition function depending on β_q and V, i.e., $\ln_q Z_q = \ln_q Z_q(\beta_q, V)$, and hence the total differential of $\ln_q Z_q$ can be written as follows

$$\begin{aligned} d \ln_q Z_q &= \left(\frac{\partial}{\partial \beta_q} \ln_q Z_q \right)_V d\beta_q + \left(\frac{\partial}{\partial V} \ln_q Z_q \right)_{\beta_q} dV, \\ &= d \left(\beta_q \frac{\partial}{\partial \beta_q} \ln_q Z_q \right)_V - \beta_q d \left(\frac{\partial}{\partial \beta_q} \ln_q Z_q \right)_V \\ &\quad + \left(\frac{\partial}{\partial V} \ln_q Z_q \right)_{\beta_q} dV, \\ &\quad \times d \left[\left(1 - \beta_q \frac{\partial}{\partial \beta_q} \right) \ln_q Z_q \right] \\ &= \beta_q d \left(-\frac{\partial}{\partial \beta_q} \ln_q Z_q \right) + \left(\frac{\partial}{\partial V} \ln_q Z_q \right)_{\beta_q} dV. \end{aligned} \tag{A.16}$$

The total differential of the internal energy is obtained as

$$dU_q = T_q d \left[k_B \left(1 - \beta_q \frac{\partial}{\partial \beta_q} \right) \ln_q Z_q \right] - P_q dV. \tag{A.17}$$

From Eq. (A.17), it is obviously seen that the q-entropy can be properly defined by

$$S_q = k_B \left(1 - \beta_q \frac{\partial}{\partial \beta_q} \right) \ln_q Z_q. \tag{A.18}$$

Note that, the q-entropy can reduce to one in GB statistics:

$$\lim_{q \to 1} S_q = k_B \left(1 - \beta \frac{\partial}{\partial \beta} \right) \ln Z. \tag{A.19}$$

Besides the internal energy, the other thermodynamic potentials, e.g., Gibbs free energy, can be constructed by using the Legendre transform. As seen from Eq. (A.7), the quantities S_q and T_q are the Legendre pair of one another. Accordingly, the Helmholtz free energy can be defined by using the Legendre transform of the internal energy with the Legendre pair S_q and T_q as [66]

$$\mathcal{F}_q = U_q - T_q S_q. \tag{A.20}$$

Remarkably, the thermodynamic Legendre structure remains valid for all values of q. By employing the same manner as done for the those thermodynamic quantities, the Helmholtz free energy can be written in terms of the partition function as

$$\mathcal{F}_q = -k_B T_q \ln_q Z_q. \tag{A.21}$$

The first law of thermodynamics with the Helmholtz free energy can be expressed as follows

$$d\mathcal{F}_q = dU_q - T_q dS_q - S_q dT_q = -S_q dT_q - P_q dV. \tag{A.22}$$

In addition, the corresponding thermodynamic quantities can be defined by

$$S_q = -\left(\frac{\partial \mathcal{F}_q}{\partial T_q} \right)_V, \qquad P_q = -\left(\frac{\partial \mathcal{F}_q}{\partial V} \right)_{T_q}. \tag{A.23}$$

Apart from the Legendre pair S_q and T_q, the leftover pair are V and P_q. The Gibbs free energy can be constructed by adopting the Legendre transform of U_q together with the Legendre pair S_q and T_q, and V and P_q as

$$G_q = U_q - T_q S_q + P_q V = H_q - T_q S_q, \tag{A.24}$$

where H_q is a chemical enthalpy defined by

$$H_q = U_q + P_q V. \tag{A.25}$$

In addition, the first law of thermodynamics with the Gibbs free energy and the chemical enthalpy can be, respectively, written as follows

$$dG_q = -S_q dT_q + V dP_q, \tag{A.26}$$

and

$$dH_q = T_q dS_q + V dP_q. \tag{A.27}$$

The thermodynamic quantities corresponding to the Gibbs free energy and the chemical enthalpy are, respectively, defined by

$$S_q = -\left(\frac{\partial G_q}{\partial T_q} \right)_{P_q}, \qquad V = \left(\frac{\partial G_q}{\partial P_q} \right)_{T_q}, \tag{A.28}$$

and

$$T_q = \left(\frac{\partial H_q}{\partial S_q} \right)_{P_q}, \qquad V = \left(\frac{\partial H_q}{\partial P_q} \right)_{S_q}. \tag{A.29}$$

References

1. J.M. Bardeen, B. Carter, S.W. Hawking, The Four laws of black hole mechanics. Commun. Math. Phys. **31**, 161 (1973). https://doi.org/10.1007/BF01645742
2. J.D. Bekenstein, Black holes and entropy. Phys. Rev. D **7**, 2333 (1973). https://doi.org/10.1103/PhysRevD.7.2333
3. J.D. Bekenstein, Generalized second law of thermodynamics in black hole physics. Phys. Rev. D **9**, 3292 (1974). https://doi.org/10.1103/PhysRevD.9.3292

4. S.W. Hawking, Particle creation by black holes. Commun. Math. Phys. **43**, 199 (1975). https://doi.org/10.1007/BF02345020 (**Erratum: Commun. Math. Phys. 46, 206 (1976)**)

5. D. Wallace, Gravity, entropy, and cosmology: in search of clarity. Br. J. Philos. Sci. **61**, 513 (2010). https://doi.org/10.1093/bjps/axp048

6. T. Padmanabhan, Lessons from classical gravity about the quantum structure of spacetime. J. Phys. Conf. Ser. **306**, 012001 (2011). https://doi.org/10.1088/1742-6596/306/1/012001

7. D. Gregoris, Y.C. Ong, Understanding gravitational entropy of black holes: a new proposal via curvature invariants. Phys. Rev. D **105**(10), 104017 (2022). https://doi.org/10.1103/PhysRevD.105.104017

8. A. Strominger, C. Vafa, Microscopic origin of the Bekenstein–Hawking entropy. Phys. Lett. B **379**, 99 (1996). https://doi.org/10.1016/0370-2693(96)00345-0

9. S. Kolekar, T. Padmanabhan, Ideal gas in a strong gravitational field: area dependence of entropy. Phys. Rev. D **83**, 064034 (2011). https://doi.org/10.1103/PhysRevD.83.064034

10. B. Mirza, H. Mohammadzadeh, Z. Raissi, Condensation of an ideal gas with intermediate statistics on the horizon. Eur. Phys. J. C **72**, 2152 (2012). https://doi.org/10.1140/epjc/s10052-012-2152-5

11. S. Bhattacharya, S. Chakraborty, T. Padmanabhan, Entropy of a box of gas in an external gravitational field-revisited. Phys. Rev. D **96**(8), 084030 (2017). https://doi.org/10.1103/PhysRevD.96.084030

12. D. Li, B. Wu, Z.M. Xu, W.L. Yang, A shell of bosons in spherically symmetric spacetimes. Phys. Lett. B **820**, 136588 (2021). https://doi.org/10.1016/j.physletb.2021.136588

13. E. Sourtzinou, C. Anastopoulos, Quantum statistical mechanics near a black hole horizon. Phys. Rev. D **107**(8), 085006 (2023). https://doi.org/10.1103/PhysRevD.107.085006

14. F. Pretorius, D. Vollick, W. Israel, An operational approach to black hole entropy. Phys. Rev. D **57**, 6311 (1998). https://doi.org/10.1103/PhysRevD.57.6311

15. J. Oppenheim, Area scaling entropies for gravitating systems. Phys. Rev. D **65**, 024020 (2002). https://doi.org/10.1103/PhysRevD.65.024020

16. C. Tsallis, Possible generalization of Boltzmann–Gibbs statistics. J. Stat. Phys. **52**, 479 (1988). https://doi.org/10.1007/BF01016429

17. D.J. Louis-Martinez, Classical relativistic ideal gas in thermodynamic equilibrium in a uniformly accelerated reference frame. Class. Quantum Gravity **28**, 035004 (2011). https://doi.org/10.1088/0264-9381/28/3/035004

18. D. Jiulin, The nonextensive parameter and Tsallis distribution for self-gravitating systems. EPL **67**, 893 (2004). https://doi.org/10.1209/epl/i2004-10145-2

19. R. Silva, J.A.S. Lima, Relativity, nonextensivity, and extended power law distributions. Phys. Rev. E **72**, 057101 (2005). https://doi.org/10.1103/PhysRevE.72.057101

20. D. Jiulin, Nonextensive power law distributions and the q-kinetic theory for the systems with self-gravitating long-range interactions. Astrophys. Space Sci. **312**, 47 (2007). https://doi.org/10.1007/s10509-007-9611-8

21. L. Zhipeng, D. Jiulin, G. Lina, Nonextensivity and q-distribution of a relativistic gas under an external electromagnetic field. Chin. Sci. Bull. **56**, 3689 (2011). https://doi.org/10.1007/s11434-011-4750-2

22. Y. Zheng, J. Du, F. Liang, The limit behavior of the evolution of the Tsallis entropy in self-gravitating systems. EPL **118**(5), 50007 (2017). https://doi.org/10.1209/0295-5075/118/50007

23. A. Sheykhi, Modified Friedmann equations from Tsallis entropy. Phys. Lett. B **785**, 118 (2018). https://doi.org/10.1016/j.physletb.2018.08.036

24. E.N. Saridakis, K. Bamba, R. Myrzakulov, F.K. Anagnostopoulos, Holographic dark energy through Tsallis entropy. JCAP **2018**(012) (2018). https://doi.org/10.1088/1475-7516/2018/12/012

25. A. Lymperis, E.N. Saridakis, Modified cosmology through nonextensive horizon thermodynamics. Eur. Phys. J. C **78**(12), 993 (2018). https://doi.org/10.1140/epjc/s10052-018-6480-y

26. P. Jizba, G. Lambiase, Constraints on Tsallis cosmology from big bang nucleosynthesis and the relic abundance of cold dark matter particles. Entropy **25**(11), 1495 (2023). https://doi.org/10.3390/e25111495

27. A. Sheykhi, New explanation for accelerated expansion and flat galactic rotation curves. Eur. Phys. J. C **80**(1) (2020). https://doi.org/10.1140/epjc/s10052-019-7599-1

28. M. Dehpour, Thermal leptogenesis in nonextensive cosmology. Eur. Phys. J. C **84**(3), 340 (2024). https://doi.org/10.1140/epjc/s10052-024-12697-7

29. T.S. Biró, V.G. Czinner, A q-parameter bound for particle spectra based on black hole thermodynamics with Rényi entropy. Phys. Lett. B **726**, 861 (2013). https://doi.org/10.1016/j.physletb.2013.09.032

30. V.G. Czinner, H. Iguchi, Rényi entropy and the thermodynamic stability of black holes. Phys. Lett. B **752**, 306 (2016). https://doi.org/10.1016/j.physletb.2015.11.061

31. V.G. Czinner, H. Iguchi, Thermodynamics, stability and Hawking-Page transition of Kerr black holes from Rényi statistics. Eur. Phys. J. C **77**(12), 892 (2017). https://doi.org/10.1140/epjc/s10052-017-5453-x

32. L. Tannukij, P. Wongjun, E. Hirunsirisawat, T. Deesuwan, C. Promsiri, Thermodynamics and phase transition of spherically symmetric black hole in de Sitter space from Rényi statistics. Eur. Phys. J. Plus **135**(6), 500 (2020). https://doi.org/10.1140/epjp/s13360-020-00517-2

33. C. Promsiri, E. Hirunsirisawat, W. Liewrian, Thermodynamics and Van der Waals phase transition of charged black holes in flat spacetime via Rényi statistics. Phys. Rev. D **102**(6), 064014 (2020). https://doi.org/10.1103/PhysRevD.102.064014

34. C. Promsiri, E. Hirunsirisawat, W. Liewrian, Solid-liquid phase transition and heat engine in an asymptotically flat Schwarzschild black hole via the Rényi extended phase space approach. Phys. Rev. D **104**(6), 064004 (2021). https://doi.org/10.1103/PhysRevD.104.064004

35. R. Nakarachinda, E. Hirunsirisawat, L. Tannukij, P. Wongjun, Effective thermodynamical system of Schwarzschild-de Sitter black holes from Rényi statistics. Phys. Rev. D **104**(6), 064003 (2021). https://doi.org/10.1103/PhysRevD.104.064003

36. E. Hirunsirisawat, R. Nakarachinda, C. Promsiri, Emergent phase, thermodynamic geometry, and criticality of charged black holes from Rényi statistics. Phys. Rev. D **105**(12), 124049 (2022). https://doi.org/10.1103/PhysRevD.105.124049

37. P. Chunaksorn, E. Hirunsirisawat, R. Nakarachinda, L. Tannukij, P. Wongjun, Thermodynamics of asymptotically de Sitter black hole in dRGT massive gravity from Rényi entropy. Eur. Phys. J. C **82**(12), 1174 (2022). https://doi.org/10.1140/epjc/s10052-022-11110-5

38. A. Alonso-Serrano, M.P. Dąbrowski, H. Gohar, Nonextensive black hole entropy and quantum gravity effects at the last stages of evaporation. Phys. Rev. D **103**(2), 026021 (2021). https://doi.org/10.1103/PhysRevD.103.026021

39. I. Çimidiker, M.P. Dąbrowski, H. Gohar, Generalized uncertainty principle impact on nonextensive black hole thermodynamics. Class. Quantum Gravity **40**(14), 145001 (2023). https://doi.org/10.1088/1361-6382/acdb40

40. S. Nojiri, S.D. Odintsov, V. Faraoni, Area-law versus Rényi and Tsallis black hole entropies. Phys. Rev. D **104**(8), 084030 (2021). https://doi.org/10.1103/PhysRevD.104.084030

41. I. Çimdiker, M.P. Dąbrowski, H. Gohar, Equilibrium temperature for black holes with nonextensive entropy. Eur. Phys. J. C **83**(2) (2023). https://doi.org/10.1140/epjc/s10052-023-11317-0

42. R. Nakarachinda, C. Promsiri, L. Tannukij, P. Wongjun, Thermodynamics of black holes with Rényi entropy from classical gravity. Nucl. Phys. B **1011**, 116796 (2025). https://doi.org/10.1016/j.nuclphysb.2025.116796

43. S. Nojiri, S.D. Odintsov, V. Faraoni, From nonextensive statistics and black hole entropy to the holographic dark universe. Phys. Rev. D **105**(4), 044042 (2022). https://doi.org/10.1103/PhysRevD.105.044042

44. S. Nojiri, S.D. Odintsov, T. Paul, Early and late universe holographic cosmology from a new generalized entropy. Phys. Lett. B **831**, 137189 (2022). https://doi.org/10.1016/j.physletb.2022.137189

45. S. Nojiri, S.D. Odintsov, V. Faraoni, Alternative entropies and consistent black hole thermodynamics. Int. J. Geom. Methods Mod. Phys. **19**(13), 2250210 (2022). https://doi.org/10.1142/S0219887822502103

46. S. Nojiri, S.D. Odintsov, V. Faraoni, New entropies, black holes, and holographic dark energy. Astrophysics **65**(4), 534 (2022). https://doi.org/10.1007/s10511-023-09759-1

47. S.D. Odintsov, T. Paul, A non-singular generalized entropy and its implications on bounce cosmology. Phys. Dark Univ. **39**, 101159 (2023). https://doi.org/10.1016/j.dark.2022.101159

48. R. D'Agostino, G.G. Luciano, Lagrangian formulation of the Tsallis entropy. Phys. Lett. B **857**, 138987 (2024). https://doi.org/10.1016/j.physletb.2024.138987

49. G.E. Volovik. Tsallis-Cirto entropy of black hole and black hole atom (2024). arXiv:2409.15362

50. M.T. Manoharan, N. Shaji, Reconciling fractional entropy and black hole entropy compositions. Eur. Phys. J. C **85**(4), 373 (2025). https://doi.org/10.1140/epjc/s10052-025-14107-y

51. H. Lu, S. Di Gennaro, Y.C. Ong, Generalized entropy implies varying-G: horizon area dependent field equations and black hole-cosmology coupling. Ann. Phys. **474**, 169914 (2025). https://doi.org/10.1016/j.aop.2024.169914

52. A. Rényi, On the dimension and entropy of probability distributions. Acta Math. Acad. Sci. Hung. **10**, 193 (1959)

53. J.D. Barrow, The area of a rough black hole. Phys. Lett. B **808**, 135643 (2020). https://doi.org/10.1016/j.physletb.2020.135643

54. B.D. Sharma, D.P. Mittal, New non-additive measures of entropy for discrete probability distributions. J. Math. Sci. **10**(75), 28 (1975)

55. T.S. Biró, P. Ván, Zeroth law compatibility of nonadditive thermodynamics. Phys. Rev. E **83**(6), 061147 (2011). https://doi.org/10.1103/PhysRevE.83.061147

56. M. Maleki, Z. Ebadi, H. Mohammadzadeh, Nonextensive statistics and the entropy on the horizon. Int. J. Mod. Phys. A **39**(04), 2450019 (2024). https://doi.org/10.1142/S0217751X24500192

57. T. Padmanabhan, Phase volume occupied by a test particle around an incipient black hole. Phys. Lett. A **136**, 203 (1989). https://doi.org/10.1016/0375-9601(89)90562-8

58. E. Poisson, *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics* (Cambridge University Press, Cambridge, 2009). https://doi.org/10.1017/CBO9780511606601

59. R.C. Tolman, P. Ehrenfest, Temperature equilibrium in a static gravitational field. Phys. Rev. **36**, 1791 (1930). https://doi.org/10.1103/PhysRev.36.1791

60. R. Pathria, P. Beale, *Statistical Mechanics* (Elsevier Science, 2021). https://books.google.co.th/books?id=esr2DwAAQBAJ

61. X. Calmet, M. Graesser, S.D.H. Hsu, Minimum length from quantum mechanics and general relativity. Phys. Rev. Lett. **93**, 211101 (2004). https://doi.org/10.1103/PhysRevLett.93.211101

62. D. Kubiznak, R.B. Mann, M. Teo, Black hole chemistry: thermodynamics with Lambda. Class. Quantum Gravity **34**(6), 063001 (2017). https://doi.org/10.1088/1361-6382/aa5c69

63. C. Tsallis, F. Baldovin, R. Cerbino, P. Pierobon, Introduction to nonextensive statistical mechanics and thermodynamics (2003). arXiv e-prints arXiv:cond-mat/0309093

64. M. Gell-Mann, C. Tsallis, *Nonextensive Entropy: Interdisciplinary Applications* (Oxford University Press, Oxford, 2004)

65. E.M.F. Curado, C. Tsallis, Generalized statistical mechanics: connection with thermodynamics. J. Phys. A **24**, L69 (1991) (**Erratum: J.Phys.A 25, 1019 (1992)**)

66. C. Tsallis, R.S. Mendes, A.R. Plastino, The Role of constraints within generalized nonextensive statistics. Phys. A **261**, 534 (1998). https://doi.org/10.1016/S0378-4371(98)00437-3

67. A. Taruya, Ma. Sakagami, Gravothermal catastrophe and generalized entropy of selfgravitating systems. Phys. A **307**, 185 (2002). https://doi.org/10.1016/S0378-4371(01)00622-7

68. A. Taruya, M. Sakagami, Gravothermal catastrophe and Tsallis' generalized entropy of selfgravitating systems. 2. Thermodynamic properties of stellar polytrope. Phys. A **318**, 387 (2003). https://doi.org/10.1016/S0378-4371(02)01342-0

69. A. Taruya, Ma. Sakagami, Gravothermal catastrophe and Tsallis' generalized entropy of selfgravitating systems. 3. Equilibrium structure using normalized q values. Phys. A **322**, 285 (2003). https://doi.org/10.1016/S0378-4371(03)00039-6

70. Ma. Sakagami, A. Taruya, Selfgravitating stellar systems and nonextensive thermostatistics. Contin. Mech. Thermodyn. **16**, 279 (2004). https://doi.org/10.1007/s00161-003-0168-7

71. J. Zavala, D. Nunez, R.A. Sussman, L.G. Cabral-Rosetti, T. Matos, Stellar polytropes and NFW halo model: Stellar polytropes and Navarro-Frenk-White halo models: comparison with observations. JCAP **2006**(008) (2006). https://doi.org/10.1088/1475-7516/2006/06/008

72. M. Coraddu, G. Kaniadakis, A. Lavagno, M. Lissia, G. Mezzorani, P. Quarati, Thermal distributions in stellar plasmas, nuclear reactions and solar neutrinos. Braz. J. Phys. **29**, 153 (1999). https://doi.org/10.1590/S0103-97331999000100014

73. R. Chakrabarti, R. Chandrashekar, S. Naina Mohammed, Nonextensive statistics of the classical relativistic ideal gas. Phys. A Stat. Mech. Appl. **389**(8), 1571–1584 (2010). https://doi.org/10.1016/j.physa.2009.12.040

74. R. Chandrashekar, S.S. Naina Mohammed, A class of energy-based ensembles in tsallis statistics. J. Stat. Mech. Theory Exp. **2011**(05), P05018 (2011). https://doi.org/10.1088/1742-5468/2011/05/p05018

75. C. Tsallis, Entropic nonextensivity: a possible measure of complexity. Chaos Solitons Fractals **13**, 371 (2002). https://doi.org/10.1016/S0960-0779(01)00019-4

76. G. Wilk, Z. Wlodarczyk, Nonextensive information entropy for stochastic networks. Acta Phys. Pol. B **35**, 871 (2004)

77. G. Wilk, Z. Włodarczyk, Quasi-power law ensembles. Acta Phys. Pol. B **46**(6), 1103 (2015). https://doi.org/10.5506/APhysPolB.46.1103

78. D. Jiulin, Nonextensive in nonequilibrium plasma systems with Coulombian long-range interactions. Phys. Lett. A **329**, 262 (2004). https://doi.org/10.1016/j.physleta.2004.07.010

79. D. Jiulin, The nonextensive parameter and Tsallis distribution for self-gravitating systems. EPL **67**, 893 (2004). https://doi.org/10.1209/epl/i2004-10145-2

80. S. Martínez, F. Nicolás, F. Pennini, A. Plastino, Tsallis' entropy maximization procedure revisited. Phys. A Stat. Mech. Appl. **286**(3–4), 489–502 (2000). https://doi.org/10.1016/s0378-4371(00)00359-9

81. E.K. Lenzi, R.S. Mendes, L.R. da Silva, Statistical mechanics based on Renyi entropy. Phys. A **280**, 337 (2000). https://doi.org/10.1016/S0378-4371(00)00007-8

82. G.L. Ferri, S. Martínez, A. Plastino, The role of constraints in Tsallis' nonextensive treatment revisited. Phys. A **347**, 205 (2005)

83. S. Abe, S. Martínez, F. Pennini, A. Plastino, Classical gas in nonextensive optimal lagrange multipliers formalism. Phys. Lett. A **278**(5), 249–254 (2001). https://doi.org/10.1016/s0375-9601(00)00780-5

84. G.L. Ferri, S. Martínez, A. Plastino, Equivalence of the four versions of tsallis's statistics. J. Stat. Mech. Theory Exp. **2005**(04), P04009 (2005). https://doi.org/10.1088/1742-5468/2005/04/p04009

85. Q.A. Wang, M. Pezeril, L. Nivanen, A. Le Méhauté, Nonextensive distribution and factorization of the joint probability. Chaos Solitons Fractals **13**(1), 131 (2002). https://doi.org/10.1016/S0960-0779(00)00244-7

86. Q.A. Wang, L. Nivanen, A.L. Méhauté, M. Pezeril, On the generalized entropy pseudoadditivity for complex systems. J. Phys. A Math. Gen. **35**(33), 7003–7007 (2002). https://doi.org/10.1088/0305-4470/35/33/304

87. S.R. Naik, H.J. Haubold, On the q-laplace transform and related special functions. Axioms **5**(3) (2016). https://doi.org/10.3390/axioms5030024

88. D. Lynden-Bell, R. Wood, A. Royal, The gravo-thermal catastrophe in isothermal spheres and the onset of red-giant structure for stellar systems. Mon. Not. R. Astron. Soc. **138**(4), 495 (1968). https://doi.org/10.1093/mnras/138.4.495

89. P. Chunaksorn, R. Nakarachinda, P. Wongjun, q-equilibrium of gas in spacetime of multi-horizon black holes (2024). arXiv:2402.13742

90. C. Tsallis, L.J.L. Cirto, Black hole thermodynamical entropy. Eur. Phys. J. C **73**(7) (2013). https://doi.org/10.1140/epjc/s10052-013-2487-6

91. C. Tsallis, Black hole entropy: a closer look. Entropy **22**, 17 (2019). https://doi.org/10.3390/e22010017

92. P. Pessoa, B. Arderucio Costa, Comment on "Black Hole Entropy: A Closer Look". Entropy **22**(10), 1110 (2020). https://doi.org/10.3390/e22101110

93. C. Tsallis, Reply to Pessoa, P.; Arderucio Costa, B. Comment on "Tsallis, C. Black Hole Entropy: A Closer Look. Entropy 2020, 22, 17". Entropy **23**(5), 630 (2021). https://doi.org/10.3390/e23050630

94. P. Pessoa, B. Arderucio Costa, S. Pressé, Revisiting Claims in *Black Hole Entropy: A Closer Look* (2022)

95. L. Smarr, Mass formula for kerr black holes. Phys. Rev. Lett. **30**, 71 (1973). https://doi.org/10.1103/PhysRevLett.30.71

96. S.W. Hawking, D.N. Page, Thermodynamics of black holes in anti-De Sitter space. Commun. Math. Phys. **87**, 577 (1983). https://doi.org/10.1007/BF01208266

97. R. Banerjee, B.R. Majhi, Quantum tunneling and back reaction. Phys. Lett. B **662**, 62 (2008). https://doi.org/10.1016/j.physletb.2008.02.044

98. R. Banerjee, B.R. Majhi, Quantum tunneling beyond semiclassical approximation. JHEP **06**, 095 (2008). https://doi.org/10.1088/1126-6708/2008/06/095

99. R.K. Kaul, P. Majumdar, Logarithmic correction to the Bekenstein–Hawking entropy. Phys. Rev. Lett. **84**, 5255 (2000). https://doi.org/10.1103/PhysRevLett.84.5255

100. M. Domagala, J. Lewandowski, Black hole entropy from quantum geometry. Class. Quantum Gravity **21**, 5233 (2004). https://doi.org/10.1088/0264-9381/21/22/014

101. X. He, B. Wang, R.G. Cai, C.Y. Lin, Signature of the black hole phase transition in quasinormal modes. Phys. Lett. B **688**, 230 (2010). https://doi.org/10.1016/j.physletb.2010.04.006

102. Y. Liu, D.C. Zou, B. Wang, Signature of the Van der Waals like small-large charged AdS black hole phase transition in quasinormal modes. JHEP **09**, 179 (2014). https://doi.org/10.1007/JHEP09(2014)179

103. M. Chabab, H. El Moumni, S. Iraoui, K. Masmar, Behavior of quasinormal modes and high dimension RN-AdS black hole phase transition. Eur. Phys. J. C **76**(12), 676 (2016). https://doi.org/10.1140/epjc/s10052-016-4518-6

104. D.C. Zou, Y. Liu, R.H. Yue, Behavior of quasinormal modes and Van der Waals-like phase transition of charged AdS black holes in massive gravity. Eur. Phys. J. C **77**(6), 365 (2017). https://doi.org/10.1140/epjc/s10052-017-4937-z

105. S.W. Wei, Y.X. Liu, Photon orbits and thermodynamic phase transition of d-dimensional charged AdS black holes. Phys. Rev. D **97**(10), 104027 (2018). https://doi.org/10.1103/PhysRevD.97.104027

106. M. Zhang, S.Z. Han, J. Jiang, W.B. Liu, Circular orbit of a test particle and phase transition of a black hole. Phys. Rev. D **99**(6), 065016 (2019). https://doi.org/10.1103/PhysRevD.99.065016

107. M. Zhang, M. Guo, Can shadows reflect phase structures of black holes? Eur. Phys. J. C **80**(8), 790 (2020). https://doi.org/10.1140/epjc/s10052-020-8389-5

108. A. Belhaj, L. Chakhchi, H. El Moumni, J. Khalloufi, K. Masmar, Thermal image and phase transitions of charged AdS black holes using shadow analysis. Int. J. Mod. Phys. A **35**(27), 2050170 (2020). https://doi.org/10.1142/S0217751X20501705

109. Y.Z. Du, H.F. Li, F. Liu, L.C. Zhang, Photon orbits and phase transition for non-linear charged anti-de Sitter black holes. JHEP **01**, 137 (2023). https://doi.org/10.1007/JHEP01(2023)137

110. X. Guo, Y. Lu, B. Mu, P. Wang, Probing phase structure of black holes with Lyapunov exponents. JHEP **08**, 153 (2022). https://doi.org/10.1007/JHEP08(2022)153

111. S. Yang, J. Tao, B. Mu, A. He, Lyapunov exponents and phase transitions of Born-Infeld AdS black holes. JCAP **07**, 045 (2023). https://doi.org/10.1088/1475-7516/2023/07/045

112. X. Lyu, J. Tao, P. Wang, Probing the thermodynamics of charged Gauss Bonnet AdS black holes with the Lyapunov exponent. Eur. Phys. J. C **84**(9), 974 (2024). https://doi.org/10.1140/epjc/s10052-024-13354-9

113. A.N. Kumara, S. Punacha, M.S. Ali, Lyapunov exponents and phase structure of Lifshitz and hyperscaling violating black holes. JCAP **07**, 061 (2024). https://doi.org/10.1088/1475-7516/2024/07/061

114. T. Hale, D. Kubiznak, J. Menšíková, Optical properties of black holes in regularized Maxwell theory. Phys. Rev. D **109**(8), 084061 (2024). https://doi.org/10.1103/PhysRevD.109.084061

115. C. Promsiri, W. Horinouchi, E. Hirunsirisawat, Observing black hole phase transitions in extended phase space and holographic thermodynamics approaches from optical features (2024). arXiv:2409.01582