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Special Issue

Advances in Nonlinear Systems and Symmetry/Asymmetry

Edited by

Dr. Nannan Zhao and Prof. Dr. Zhongkui Sun

MDPI

https://doi.org/10.3390/sym17010081

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Derivation of Tensor Algebra as a Fundamental Operation—The Fermi Derivative in a General Metric Affine Space

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Abstract: The aim of this work is to demonstrate that all linear derivatives of the tensor algebra over a smooth manifold M can be viewed as specific cases of a broader concept—the operation of derivation. This approach reveals the universal role of differentiation, which simplifies and generalizes the study of tensor derivatives, making it a powerful tool in Differential Geometry and related fields. To perform this, the generic derivative is introduced, which is defined in terms of the quantities $Q_k^{(i)}(X)$. Subsequently, the transformation law of these quantities is determined by the requirement that the generic derivative of a tensor is a tensor. The quantities $Q_k^{(i)}(X)$ and their transformation law define a specific geometric object on M, and consequently, a geometric structure on M. Using the generic derivative, one defines the tensor fields of torsion and curvature and computes them for all linear derivatives in terms of the quantities $Q_k^{(i)}(X)$. The general model is applied to the cases of Lie derivative, covariant derivative, and Fermi derivative. It is shown that the Lie derivative has non-zero torsion and zero curvature due to the Jacobi identity. For the covariant derivative, the standard results follow without any further calculations. Concerning the Fermi derivative, this is defined in a new way, i.e., as a higher-order derivative defined in terms of two derivatives: a given derivative and the Lie derivative. Being linear derivative, it has torsion and curvature tensor. These fields are computed in a general affine space from the corresponding general expressions of the generic derivative. Applications of the above considerations are discussed in a number of cases. Concerning the Lie derivative, it is been shown that the Poisson bracket is in fact a Lie derivative. Concerning the Fermi derivative, two applications are considered: (a) the explicit computation of the Fermi derivative in a general affine space and (b) the consideration of Freedman–Robertson–Walker spacetime endowed with a scalar torsion field, which satisfies the Cosmological Principle and the computation of Fermi derivative of the spatial directions defining a spatial frame along the cosmological fluid of comoving observers. It is found that torsion, even in this highly symmetric case, induces a kinematic rotation of the space axes, questioning the interpretation of torsion as a spin. Finally it is shown that the Lie derivative of the dynamical equations of an autonomous conservative dynamical system is equivalent to the standard Lie symmetry method.

Keywords: derivation; Lie derivative; covariant derivative; Fermi derivative; torsion; metricity; Poisson derivative; Poisson bracket; curvature; Friedman-Robertson-Walker

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Academic Editors: Nannan Zhao, Zhongkui Sun and Abraham A. Ungar

Received: 21 November 2024
Revised: 25 December 2024
Accepted: 31 December 2024
Published: 7 January 2025

Citation: Tsamparlis, M. Derivation of Tensor Algebra as a Fundamental Operation—The Fermi Derivative in a General Metric Affine Space. *Symmetry* **2025**, *17*, 81. https://doi.org/10.3390/sym17010081

1. Introduction

In Differential Geometry and in Physics, the derivatives mainly used on smooth manifolds are the Lie derivative and the covariant derivative. This is due to their crucial role in studying differential equations and stating the dynamic equations. Specifically, the Lie derivative is the main tool for the determination of symmetries of geometric objects like metric, connection, etc., for the invariance of differential equations and their classification according to their invariance under a specific Lie algebra. On the other hand, the covariant derivative is used in the formulation of dynamic equations and the conserved currents. Both derivatives are entangled in applications in Differential Geometry and in Physics, for example, in the determination of the first integrals of differential equations and the conserved currents of dynamic equations.

Apart from these derivatives, in practice, other derivatives have been introduced, each one serving special purposes. A few of them are as follows:

a. The Fermi derivative. To study the kinematics of a spacetime fluid, one has to adopt along the 4-velocity u^a of the observers a non-rotating frame, which is the closest to an inertial frame [1]. The condition for that is that the Fermi derivative with respect to u^a of the spatial vectors defining the 3D-frame vanishes.

b. The exterior derivative defined on the exterior algebra of differential forms over a smooth manifold. This is the unique linear map which satisfies a graded version (of grade 1) of the Leibnitz law and squares to zero, i.e., $d^2 = 0$. For a function f, it is the standard differential df. For a 1-form $\theta = a_i dx^i$, the exterior derivative is the 2-form $d\theta = a_{[i,j]} dx^i dx^j$, where square brackets denote antisymmetrization. The exterior derivative is used in the Cartan formulation of Differential Geometry to define the connection 1-form, and from that, the torsion 2-form and the curvature 3-form. Further applications are the study of the geometry of surfaces and the determination of independence of vector fields. The applications of the exterior derivative in Physics cover the whole field of Physics. In Classical Mechanics, the exterior derivative defines the symplectic structure of phase space. In Electromagnetism, the Faraday 2-form F is introduced, in terms of which Maxwell's equations are stated very compactly as exterior derivatives of F. In Fluid Dynamics, the vorticity 2-form ω, which is used to study the rotational motion of fluid particles, is defined as the exterior derivative of the velocity 1-form v. Other applications concern the geometric theory of defects [2] and certain formulations of topological quantum field theory, where the exterior derivative is used to describe topological invariants of the spacetime manifold [3].

c. The interior product, which is a degree -1 derivation on the exterior algebra, is defined by the contraction of a form with a vector field. Together with the exterior derivative and the Lie derivative, it forms a Lie superalgebra.

d. The Fréchet derivative, defined on Banach spaces, generalizes the derivative of a real-valued function of a single real variable to the case of a vector-valued function of multiple real variables, and is widely used to define the functional derivative in the calculus of variations [4].

Apart from these derivatives, others have been introduced for special needs. It is important to formulate these derivatives under a common scheme so that the deeper significance of each and the real purpose for its introduction will be revealed. Furthermore, the interrelation of different derivatives will lead to new derivatives and applications. Finally, it is possible that derivatives which are believed to be new are in fact combinations of existing derivatives.

A striking example in that direction is the Fermi derivative. It is widely believed that this derivative is relevant to the covariant derivative of the Riemannian spacetime of General Relativity. As will be shown, this is not true. The Fermi derivative is the combination of a general linear derivative and the Lie derivative; therefore applies to all linear derivatives and all affine spaces, which are not necessarily Riemannian. Because the Fermi derivative concerns the generalization of inertial frames in General Relativity, its generalization to affine spaces means that it can be used to define the inertial frames in the alternative theories of gravitation, where the geometry of spacetime is not assumed to be Riemannian, e.g., in the Einstein–Cartan theory.

The aim of the present work is to demonstrate that all linear derivatives of the tensor algebra can be viewed as specific cases of a broader concept—the operation of derivation. The focus is thus on derivation as a fundamental operation rather than on the specific derivatives. In particular, it demonstrates that all linear derivatives have torsion and curvature; therefore, these tensor fields are not exclusively a property of the covariant derivative, as it is commonly believed.

It is apparent that it is not possible to address all the linear derivatives in a single work; therefore, in order to proceed, one has to select a subset of relevant linear derivatives. In the present work, we select the fundamental and the most widely used derivatives, namely the Lie derivative, the covariant derivative, and the Fermi derivative. This choice is justified by their complementary significance in the study of applications. Indeed, the Lie derivative is used for the invariance (i.e., symmetry) of geometric objects, which define the differential structure of the "configuration" space, i.e., the manifold where the dynamical system evolves. The covariant derivative is used in the formulation of dynamics, that is, the evolution of a dynamical system in the given "configuration" space; and the Fermi derivative is used in the kinematics of a dynamical system along the world line of the observers in "configuration" space. Furthermore, in the study of Lie symmetries of differential equations and the determination of their first integrals, a combination of the Lie derivative and the covariant derivative is used.

As a rule, these three linear derivatives are defined and studied separately. For example, the Lie derivative and its applications have been studied in depth in the classical book of Yano [5]. On the other hand, both derivatives are studied in different levels of detail in all textbooks on Differential Geometry and General Relativity [6–9]. Finally, the Fermi derivative, being an element of relativistic kinematics, is studied only in General Relativity books (e.g., [1]).

The scenario for the development of these three derivatives within the concept of linear derivation proceeds with the following steps:

a. The linear derivation along a vector field X is defined by a set of linear maps over a smooth manifold M. This defines the generic derivative abstractly.

b. To associate the generic derivative with a geometric object and make calculations possible, one considers a coordinate system x^i in M and assigns the components of the generic derivative in the chart x^i to be the quantities $Q^{(i)}_k(X)$.

c. Requiring that the generic derivative of a tensor results in a tensor, the transformation law of the components $Q^{(i)}_k(X)$ is determined. This transformation associates the generic linear derivative with a geometric object whose components are the quantities $Q^{(i)}_k(X)$. It is found that the geometric object associated with the generic derivative is not necessarily a tensor. This is why the upper index in $Q^{(i)}_k$ is enclosed in parentheses.

d. With each linear derivative over M, two tensor fields are associated, corresponding to the commutativity properties of partial differentiation, that is, $\partial_x\partial_y = \partial_y\partial_x$ and $\partial^2_{xy} = \partial^2_{yx}$. One field is the torsion tensor, which measures the failure of the derivative to commute, and the second field is the curvature tensor, which measures the deviation from "flatness".

e. A particular linear derivative is defined by a specific set of quantities $Q^{(i)}_k(X)$, which transform as the components of the generic derivative.

Steps a and b are discussed in Section 2, step c in Section 3, and step d in Section 4. Following this, we consider special linear derivatives over M by defining special sets of quantities $Q_k^{(i)}(X)$.

In Section 5.1, the Lie derivative is defined. It is shown that it has non-vanishing torsion and zero curvature, with the latter being equivalent to the Jacobi identity for the Lie bracket.

In Section 5.2, the covariant derivative is introduced and discussed briefly because it is very well known.

The main new result of the present work is in Section 5.3, where the Fermi derivative is introduced from a completely new perspective. It is shown that this derivative is a higher-order derivative in the sense that it is defined in terms of two derivatives: one general derivative (not necessarily the covariant derivative) and the Lie derivative. In Sections 5.3.1 and 5.3.2, we compute the torsion and the curvature tensors of the Fermi derivative. It is found that, in general, the torsion of the Fermi derivative does not vanish even when the torsion tensor of the general derivative defining the Fermi derivative vanishes. We continue with special Fermi derivatives by specifying the general linear derivative defining the Fermi derivative. In Section 6, we consider the Fermi derivative defined by the Lie derivative and the covariant derivative. Using the Fermi derivative as the second derivative, one may consider the Fermi derivative of the Fermi derivative, and so on. This is performed in Section 7. It follows that the Fermi derivative is not a single derivative but a derivative generating Fermi derivatives.

The remaining sections refer to applications of the previous general results.

In Section 8.1, we introduce the general Poisson bracket of two functions h, f by setting

$$(h, f) = \omega^{ij}(x) h_{,i} f_{,j}$$

where ω^{ij} is an arbitrary constant tensor field, which is not necessarily antisymmetric, and the comma indicates partial derivatives. It is shown that this derivative is a Lie derivative and satisfies a Jacobi-like identity due to the vanishing of the curvature of the Lie derivative. If one specializes further to $\omega^{ij} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$, where I_n is the $n \times n$ unit matrix, then one finds the standard Poisson bracket, and ω^{ij} defines a symplectic structure on the manifold.

In Section 8.2, we demonstrate the general approach to the Fermi derivative in two cases. In the first case, we compute the Fermi derivative in a general 2D affine space. In the second case, we consider a problem from cosmology. Specifically, we assume the Friedman–Robertson–Walker spacetime endowed with a scalar torsion, which is in agreement with the Cosmological Principle. We calculate the Fermi derivative for the comoving observers and show that the torsion, even in this highly symmetric spacetime, produces a rotation which, however, is of a kinematic nature. This shows that torsion is not necessarily a dynamic field associated with spin as proposed in the Einstein–Cartan theory [10].

In Section 9, the role of the Lie derivative in the determination of the symmetries of dynamical equations is shown to be equivalent to the standard Lie symmetry methods used in the determination of the Lie symmetries of differential equations.

Finally, in Section 10, we draw our conclusions.

2. Definition of the General Derivative

We start with the well-known and standard definition of the linear derivative of tensor algebra.

Definition 1. *On the tensor algebra $T(M)$ of a manifold M, we define the derivative D_X with respect to the vector field X to be the automorphism*

$$D_X : T(M) \to T(M)$$

satisfying the following properties:

1. *$D_X : T^r_s(M) \to T^r_s(M)$, that is, D_X preserves the type of tensor fields.*
2. *The function D_X is R-linear, that is, $\forall a \in R$ and $\forall S, N \in T^r_s(M)$, and the following relation holds:*
$$D_X(aS + N) = aD_X S + D_X N$$
3. *It satisfies the Leibnitz rule with respect to the tensor product, that is, $\forall\ S \in T^r_s(M)$ and $N \in T^m_n(M)$*
$$D_X(S \otimes N) = D_X S \otimes N + S \otimes D_X N.$$
4. *D_X (constant tensor) $= 0$ for all constant tensor fields.*

The quantity $D_X T \quad \forall T \in T^r_s(M)$ is called the D-derivative of the tensor field T with respect to X.

From Definition 1 (properties 2, 3), it follows that in order to define a derivative D_X on the tensor algebra $T(M)$, it is sufficient to define its action on the elements of the ring of functions $F(M)$ and the elements (vector fields) of the module $T^1_0(M)$.

Definition 2. *Consider a chart (U, ϕ) in M with coordinates x^i and the vector field $\mathbf{X} = X^i \partial_i$. We define the action of D_X on the functions in $F(M)$ and on the vector fields $Y = Y^i \partial_i$ in (U, ϕ) by the relations*

$$D_X(f) = \mathbf{X}(f) \quad \forall f \in F(M) \tag{1}$$

$$D_X(\mathbf{Y}) = Q^{(i)}_k(X) Y^k \partial_i, \quad \forall \mathbf{X}, \mathbf{Y} \in T^1_0(M). \tag{2}$$

The index (i) is enclosed in parentheses because, as will be shown, in general, it is not a tensorial index. This means that the n^2 quantities $Q^{(i)}_k$ are not generally components of a tensor field of type (1, 1) (this is possible but it is not relevant to the present discussion). The quantities $Q^{(i)}_j$ define a geometric object on M which is identified by their transformation rules (to be determined).

It is easy to show that a linear derivation satisfies the following properties:

$$Q^i_{.sj}(X + Y) = Q^i_{.sj}(X) + Q^i_{.sj}(Y)$$
$$Q^i_{.sj}(\alpha X) = \alpha Q^i_{.sj}(X)$$

or, in an obvious notation,

$$\begin{aligned} D^1_{(X+Y)} Z &= D^1_X Z + D^1_Y Z \\ D^1_{\alpha X} Z &= \alpha D^1_X Z \end{aligned}$$

all $Z \in F^1_0(M)$.

From (1) and (2), we have the following results:

$$D_{\partial_j} x^i = \delta^i_j \tag{3}$$

$$D_X(\partial_k) = Q^{(i)}_k(X) \partial_i. \tag{4}$$

$$D_X(dx^i) = -Q_j^{(i)} dx^j \tag{5}$$

Using the above and applying the Leibnitz rule, one computes the derivative $D_X T$ of an arbitrary tensor field $T = T^{i_1 \ldots i_r}_{j_1 \ldots j_s} \partial_{i_1} \otimes \ldots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}$ of type (r,s) defined in the chart (U, ϕ):

$$\begin{aligned}(D_X T)^{i_1 \ldots i_r}_{j_1 \ldots j_s} \quad = \quad & [T^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} X^k + T^{k \ldots i_r}_{j_1 \ldots j_s} Q_k^{(i_1)}(X) + \ldots + T^{i_1 \ldots k}_{j_1 \ldots j_s} Q_k^{(i_r)}(X) \\ & - T^{i_1 \ldots i_r}_{k \ldots j_s} Q_{j_1}^{(k)}(X) - \ldots - T^{i_1 \ldots i_r}_{j_1 \ldots k} Q_{j_s}^{(k)}(X)] \partial_{i_1} \otimes \ldots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}. \end{aligned} \tag{6}$$

3. The Transformation of the Quantities $Q_k^{(i)}$

The transformation of the basis vectors is $\partial_{i'} = J^i_{i'} \partial_i$, where $J^i_{i'}$ is the Jacobian. We compute

$$D_{\mathbf{X}}(\partial_{i'}) = D_{\mathbf{X}}(J^i_{i'} \partial_i) = D_{\mathbf{X}}(J^i_{i'})\partial_i + J^i_{i'} D_{\mathbf{X}}(\partial_i) = J^i_{i',j} X^j \partial_i + J^i_{i'} Q_i^{(j)}(X) \partial_j.$$

We define the transformed quantities by the following requirement:

$$D_X(\partial_{i'}) = Q_{i'}^{(j')}(X) \partial_{j'} = Q_{i'}^{(j')}(X) J^i_{j'} \partial_i.$$

Equating the two expressions of $D_{\mathbf{X}}(\partial_{i'})$ in the last two relations and using the fact that X is arbitrary follows the transformation rule:

$$Q_{i'}^{(j')}(X) = J_i^{j'} \left(J^i_{i',k} X^k + J^j_{i'} Q_j^{(i)}(X) \right) \tag{7}$$

Relation (7) defines the type of geometric object $Q_i^{(j)}(X)$. We infer that the quantities $Q^{(i)}_{(j)k}$ define a geometric object, which *in general*, is not a tensor field.

4. The Torsion Tensor and the Curvature Tensor of the Generic Derivative

We continue with the commutation of the first and the second derivatives of a general derivative $Q_k^{(i)}$.

4.1. The Torsion Tensor

Consider the derivative with components $Q_j^{(i)}$ and two vector fields $\mathbf{V}$ and $\mathbf{W}$, which define the derivations $D_{\mathbf{V}} : (V^i, Q_j^{(i)}(\mathbf{V}))$ and $D_{\mathbf{W}} : (W^i, Q_j^{(i)}(\mathbf{W}))$ (with the same $Q_j^{(i)}$!). We consider the commutator (this corresponds to the property $\partial_x \partial_y = \partial_y \partial_x$ of partial derivative)

$$D_{\mathbf{V}}\mathbf{W} - D_{\mathbf{W}}\mathbf{V}.$$

We compute:

$$\begin{aligned} D_{\mathbf{V}}\mathbf{W} - D_{\mathbf{W}}\mathbf{V} &= [W^i_{,j} V^j + Q_j^{(i)}(\mathbf{V}) W^j - V^i_{,j} W^j - Q_j^{(i)}(\mathbf{W}) V^j] \partial_i \\ &= [\mathbf{W}, \mathbf{V}] + [Q_j^{(i)}(\mathbf{V}) W^j - Q_j^{(i)}(\mathbf{W}) V^j] \partial_i \end{aligned} \tag{8}$$

where $[\mathbf{W}, \mathbf{V}] = [W^i_{,j} V^j - V^i_{,j} W^j] \partial_i$ is the Lie bracket of the vector fields $\mathbf{V}, \mathbf{W}$.

We define the tensor field of type $(1,2)$:

$$T_D(\mathbf{V}, \mathbf{W}) = D_{\mathbf{V}}\mathbf{W} - D_{\mathbf{W}}\mathbf{V} - [\mathbf{V}, \mathbf{W}] \tag{9}$$

which is called the **torsion of the derivative** D.

In terms of the quantities $Q_j^{(i)}(\mathbf{V}), Q_j^{(i)}(\mathbf{W})$ from (8) and (9), we have

$$T_D(\mathbf{V},\mathbf{W}) = [Q_j^{(i)}(\mathbf{V})W^j - Q_j^{(i)}(\mathbf{W})V^j]\partial_i. \tag{10}$$

4.2. The Curvature Tensor

We consider two vector fields $\mathbf{V} = V^i\partial_i$, $\mathbf{W} = W^i\partial_i$ and the commutator (this is the analog of the property of the partial derivative $\frac{\partial^2 f}{\partial_x \partial_y} = \frac{\partial^2 f}{\partial_y \partial_x}$):

$$\mathbf{D_V D_W} - \mathbf{D_W D_V}$$

Let $\mathbf{U} = U^i\partial_i$ be a third vector field. A rather long computation gives (see Appendix A):

$$(\mathbf{D_V D_W} - \mathbf{D_W D_V})\mathbf{U} = D_\mathbf{V} D_\mathbf{W}\mathbf{U} - D_\mathbf{W} D_\mathbf{V}\mathbf{U} = U^i_{,j}[\mathbf{V},\mathbf{W}]^j + H^i_j U^j \partial_i \tag{11}$$

where

$$H^i_j = Q^i_j(\mathbf{W})_{,k}V^k - Q^i_j(\mathbf{V})_{,k}W^k + Q^k_j(\mathbf{W})Q^i_k(\mathbf{V}) - Q^k_j(\mathbf{V})Q^i_k(\mathbf{W}).$$

The derivative $D_{[\mathbf{V},\mathbf{W}]}$ defined by the vector field $[\mathbf{V},\mathbf{W}] = (V^j W^i_{,j} - W^j V^i_{,j})\partial_i = [\mathbf{V},\mathbf{W}]^i\partial_i$ is

$$D_{[\mathbf{V},\mathbf{W}]}U = U^i_{,j}[\mathbf{V},\mathbf{W}]^j\partial_i + U^j Q^i_j([\mathbf{V},\mathbf{W}])\partial_i.$$

We define the geometric object

$$R_D(\mathbf{V},\mathbf{W}) = D_\mathbf{V} D_\mathbf{W} - D_\mathbf{W} D_\mathbf{V} - D_{[\mathbf{V},\mathbf{W}]} \tag{12}$$

and obtain the general result:

$$R_D(\mathbf{V},\mathbf{W})\mathbf{U} = [Q_j^{(i)}(\mathbf{W})_{,k}V^k - Q_j^{(i)}(\mathbf{V})_{,k}W^k + Q_j^{(k)}(\mathbf{W})Q_k^{(i)}(\mathbf{V}) - Q_j^{(k)}(\mathbf{V})Q_k^{(i)}(\mathbf{W}) - Q_j^{(i)}([\mathbf{V},\mathbf{W}])]U^j\partial_i. \tag{13}$$

The geometric object $R_D(\mathbf{V},\mathbf{W})$ is a tensor of type (1,3), which is called the curvature tensor of the derivation D.

5. Specific Derivatives

Every set of quantities $Q_j^{(i)}$ that transform according to (7) defines a linear derivative over the chart (U,ϕ) of M. In the following subsections, we consider special linear derivatives that play a major role in Geometry and Physics.

5.1. The Lie Derivative

The Lie derivative is the only derivative that is defined on a manifold without introducing an extra structure (equivalently geometric object). This is the reason that it is used in the study of fundamental structures over M.

Consider a coordinate chart with coordinates x^i and let the vector field $\mathbf{X} = X^i\partial_{x^i}$. Define the quantities

$$Q_j^{L(i)}(\mathbf{X}) = -X^i_{,j} \tag{14}$$

so that

$$D^L_X(\partial_j) = -X^i_{,j}\partial_i. \tag{15}$$

In order the quantities $Q_j^{L(i)}(X)$ to define a derivation on M, they must transform in accordance to (7).

To prove this, we note that because X^i are the components of a vector field, they transform as follows:

$$X^{i'} = J^{i'}_i X^i.$$

Using this, we find

$$X^{i'}_{,j'} = (J^{i'}_i X^i)_{,j'} = J^{i'}_{i,j'} X^i + J^{i'}_i X^i_{,j'} = J^{i'}_i J^j_{j'} X^i_{,j} - J^{i'}_k J^k_{j',i} X^i \Rightarrow$$

$$(-X^{i'}_{,j'}) = J^{i'}_i J^j_{j'} (-X^i_{,j}) + J^{i'}_i J^i_{j',k} X^k$$

which is compatible with (7). Therefore, the quantities $-X^i_{,j}$ define a derivative called the **Lie derivative** with respect to the vector field $\mathbf{X} = X^i \partial_{x^i}$. We denote the Lie derivative as $D_{\mathbf{X}} = L_{\mathbf{X}}$.

Using the general result (6), we write

$$\begin{aligned} L_{\mathbf{X}} T = [&T^{i_1 \ldots i_r}_{j_1 \ldots j_s, k} X^k - T^{k \ldots i_r}_{j_1 \ldots j_s} X^{i_1}_{,k} - \ldots - T^{i_1 \ldots k}_{j_1 \ldots j_s} X^{i_r}_{,k} + T^{i_1 \ldots i_r}_{k \ldots j_s} X^k_{,j_1} \\ &+ \ldots + T^{i_1 \ldots i_r}_{j_1 \ldots k} X^k_{,j_s}] \partial_{i_1} \otimes \ldots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}. \end{aligned} \quad (16)$$

In order to compute the torsion tensor and the curvature tensor of the Lie derivative, we use the general results of (10) and (13).

Concerning the torsion, we have

$$T^i_L(\mathbf{V}, \mathbf{W}) = Q^{(i)}_j(V) W^j - Q^{(i)}_j(W) V^j = -V^i_{,j} W^j + W^i_{,j} V^j = [W, V]^i. \quad (17)$$

that is, the torsion of the Lie derivative is the commutator of the involved vector fields.

Concerning the curvature of the Lie derivative, using (12) and (14) we find:

$$\begin{aligned} R_L(\mathbf{X}, \mathbf{Y})\mathbf{Z} &= L_{\mathbf{X}} L_{\mathbf{Y}} \mathbf{Z} - L_{\mathbf{Y}} L_{\mathbf{X}} \mathbf{Z} - L_{[\mathbf{X}, \mathbf{Y}]} \mathbf{Z} \\ &= [\mathbf{X}, [\mathbf{Y}, \mathbf{Z}]] + [\mathbf{Y}, [Z, \mathbf{X}]] + [\mathbf{Z}, [\mathbf{X}, \mathbf{Y}]]. \end{aligned}$$

The rhs is zero according to the Jacobi identity. Therefore, the curvature of the Lie derivative vanishes. This shows the deeper geometric meaning of the Jacobi identity.

The Lie derivative is used in the study of Lie symmetries of geometric objects (including differential equations) [11–13].

5.2. The Covariant Derivative

The covariant derivative is a linear derivative defined in terms of the connection coefficients Γ^i_{jk} as follows:

$$Q^{\nabla(i)}_j(\mathbf{X}) = \Gamma^i_{kj} X^k. \quad (18)$$

In order the quantities $Q^{\nabla(i)}_j(\mathbf{X})$ to define a linear derivation, they must transform according to (7). This requires that the functions Γ^i_{jk} must transform as follows:

$$\Gamma^{j'}_{i'k'} = J^{j'}_i \left(J^k_{k'} J^i_{i',k} + J^k_{k'} J^j_{i'} \Gamma^i_{jk} \right). \quad (19)$$

We denote the covariant derivative with respect to the vector field $\mathbf{X}$ by $\nabla_{\mathbf{X}}$. For the covariant derivative, we have the well-known results obtained directly from (6), (10) and (13):

$$T^i_{\nabla jk} = \Gamma^i_{kj} - \Gamma^i_{jk} \quad (20)$$

$$R^i_{\nabla jmk} = \Gamma^i_{jk,m} - \Gamma^i_{jm,k} + \Gamma^i_{ms} \Gamma^s_{kj} - \Gamma^i_{ks} \Gamma^s_{mj}. \quad (21)$$

$$\nabla_X T = (T^{i_1 \ldots i_r}_{j_1 \ldots j_s, h} + T^{k \ldots i_r}_{j_1 \ldots j_s}\Gamma^{i_1}_{hk} + \ldots - T^{i_1 \ldots i_r}_{j_1 \ldots k}\Gamma^{k}_{hj_s})X^h \partial_{i_1} \otimes \ldots \otimes \partial_{i_r} \otimes dx^{j_1} \otimes \ldots \otimes dx^{j_s}. \quad (22)$$

The covariant derivative is used in the formulation of the dynamic equations, whereas the Lie derivative is used in the study of symmetries of differential equations and numerous other applications.

5.3. The Fermi Derivative

Consider a coordinate chart (U, ϕ) with coordinates x^i and let the vector field $\mathbf{X} = X^i \partial_{x^i}$. Suppose in (U, ϕ) , we define a derivation with components $Q_j^{(i)}(X)$, and let $Q_j^{L(i)}(X)$ be the components of the Lie derivative in (U, ϕ) defined by $\mathbf{X}$. Using the two derivatives, one defines a new derivative, the Fermi derivative:

$$D^F_X(\partial_i) = Q^{F(j)}{}_i(X)\partial_j, \quad \forall \mathbf{X} \in T^1_0(M) \quad (23)$$

where the quantities, i.e., components, $Q_j^{F(i)}(X)$ are given by the following formula:

$$Q^{F(j)}{}_i(X) = Q^{(j)}{}_i(X) + X^s g_{sr}(Q_t^{(r)}(X) - Q^{L(r)}{}_t(X))(g^{tj}X_i - \delta_i^t X^j). \quad (24)$$

The Fermi derivative is a "higher-order linear derivative" because it is defined in terms of pairs of other types of linear derivatives.

In order the Fermi derivative to be well defined, the quantities $Q^{F(j)}{}_i(X)$ must transform as in (7), that is, it must hold

$$Q^{F(j)}{}_i(X) = J^j_{j'} J^{i'}_i Q^{F(j)'}{}_{i'}(X) + J^j_{j'} J^{j'}_{i,k'} X^{k'}. \quad (25)$$

In Appendix B, it is shown that this is satisfied.

We have the obvious relations:

$$D^F_X f = f_{,i} X^i = \mathbf{X}(f), \quad \forall f \in F(M)$$

$$D^F(x^i) = x^i_{,j} X^j = \delta^i_j X^j = X^i$$

$$D^F_X(\partial_k) = \left[Q^{(i)}{}_k(X) + X^s g_{sr}(Q_j^{(r)} - Q_j^{L(r)})(g^{ji}X_k - \delta^j_k X^i)\right]\partial_i. \quad (26)$$

$$D^F(dx^j) = -Q^{F(j)}{}_i dx^i. \quad (27)$$

As an application and for comparison with the existing results, we compute the Fermi derivative $D^F_{\mathbf{X}}(\mathbf{Y})$ of the vector field $\mathbf{Y} = Y^i \partial_i$.

We set $Q^L_{st}(X) = g_{sr} Q_t^{Lr}(X)$, and after a formal calculation (see Appendix C), we find

$$D^F_{\mathbf{X}}(\mathbf{Y}) = D_{\mathbf{X}}(\mathbf{Y}) + g(\mathbf{X}, \mathbf{Y}) D_{\mathbf{X}}(\mathbf{X}) - g(\mathbf{Y}, D_{\mathbf{X}}(\mathbf{X}))\mathbf{X}. \quad (28)$$

This formula coincides with the definition given in p. 80 of [1] in the special case that the derivation $Q^{(j)}{}_i$ is the Riemann covariant derivative. However, the present result holds for a general linear derivative D and it is not restricted to the covariant derivative only (e.g., [14–22]).

5.3.1. The Torsion of the Fermi Derivative

We have from relation (9)

$$T_D(\mathbf{V}, \mathbf{W}) = \mathbf{D}^F_{\mathbf{V}}\mathbf{W} - \mathbf{D}^F_{\mathbf{W}}\mathbf{V} - [\mathbf{V}, \mathbf{W}] = [Q^i_j(\mathbf{V})W^j - Q^i_j(\mathbf{W})V^j]\partial_i.$$

where $\mathbf{V}, \mathbf{W}$ are vector fields. In index-free notation, we find

$$\begin{aligned} T_{D^F}(\mathbf{V},\mathbf{W}) &= \mathbf{D}^F_{\mathbf{V}}\mathbf{W} - \mathbf{D}^F_{\mathbf{W}}\mathbf{V} - [\mathbf{V},\mathbf{W}] \\ &= D_{\mathbf{V}}(\mathbf{W}) + g(\mathbf{V},\mathbf{W})D_{\mathbf{V}}(\mathbf{V}) - g(\mathbf{W}, D_{\mathbf{V}}(V))\mathbf{V} \\ &- D_{\mathbf{W}}(\mathbf{V}) - g(\mathbf{W},\mathbf{V})D_{\mathbf{W}}(\mathbf{W}) + g(\mathbf{V}, D_{\mathbf{W}}(\mathbf{W}))\mathbf{W} - [\mathbf{V},\mathbf{W}] \\ &= T_D(\mathbf{V},\mathbf{W}) + g(\mathbf{V},\mathbf{W})[D_{\mathbf{V}}(\mathbf{V}) - D_{\mathbf{W}}(\mathbf{W})] + g(\mathbf{V}, D_{\mathbf{W}}(\mathbf{W}))\mathbf{W} - g(\mathbf{W}, D_{\mathbf{V}}(\mathbf{V}))\mathbf{V}. \end{aligned}$$

We note that even if $T_D = 0$, $T_{D^F} \neq 0$, unless it holds that $D_{\mathbf{V}}(\mathbf{V}) = D_{\mathbf{W}}(\mathbf{W}) = 0$; that is, the vector fields $\mathbf{V}, \mathbf{W}$ are autoparallel in the derivation D.

5.3.2. The Curvature of the Fermi Derivative

In the case of the Fermi derivative, the general relation (12) gives:

$$\begin{aligned} R_{D^F}(\mathbf{V},\mathbf{W})\mathbf{U} = [&{Q^F}_j{}^i(\mathbf{W})_{,k}V^k - {Q^F}_j{}^i(\mathbf{V})_{,k}W^k + {Q^F}_j{}^k(\mathbf{W})Q^{Fi}_k(\mathbf{V}) - Q^{Fk}_j(\mathbf{V})Q^{Fi}_k(\mathbf{W}) \\ &- Q^{Fi}_j([\mathbf{V},\mathbf{W}])]U^j\partial_i \end{aligned}$$

The detailed calculation in terms of $R_D(\mathbf{V},\mathbf{W})\mathbf{U}$ is cumbersome and it is better to be performed in each specific case.

6. The Fermi Derivative of the Lie and the Covariant Derivative

We consider the Fermi derivative defined by the Lie and the covariant derivative. For the Lie derivative, the Fermi derivative reduces to the Lie derivative, as expected. For the covariant derivative, we have ${Q^{\nabla(i)}}_k(X) = \Gamma^j_{ki}X^k$ and ${Q^{L(i)}}_k(X) = -X^j_{,i}$. Then, from (24), we obtain the following:

$$\begin{aligned} {Q^{F(\nabla)(j)}}_i(X) &= \Gamma^j_{ki}X^k + X^s g_{sr}(\Gamma^r_{kt}X^k + X^r_{,t})(g^{tj}X_i - \delta^t_i X^j) \\ &= X^k\Big(\Gamma^j_{ki} + X^s X_i g_{sr} g^{tj}\Gamma^r_{kt} - X^s X^j g_{sr}\Gamma^r_{ki} + X_i g_{kr} g^{tj} X^r_{,t} - X^j g_{kr} X^r_{,i}\Big). \end{aligned}$$

The terms:

$$\begin{aligned} & X^k X^s X_i g_{sr} g^{tj}\Gamma^r_{kt} + X^k X_i g_{kr} g^{tj} X^r_{,t} \\ = \;& X_i g^{tj}(X^k X^s g_{sr}\Gamma^r_{kt} + X^k g_{kr} X^r_{,t}) \\ = \;& X_i g^{tj} X^s g_{sr}(X^r_{,t} + X^k\Gamma^r_{kt}) = X_i g^{tj} X^s g_{sr} X^r_{|t} \end{aligned}$$

where $X^r_{|t}$ denotes the covariant derivative. The terms:

$$\begin{aligned} & -X^s X^k X^j g_{sr}\Gamma^r_{ki} - X^k X^j g_{kr} X^r_{,i} \\ = \;& -X^s X^k X^j g_{sr}\Gamma^r_{ki} - X^s X^j g_{sr} X^r_{,i} = -X^s X^j g_{sr}(X^r_{,i} + \Gamma^r_{ki}) \\ = \;& -X^s X^j g_{sr} X^r_{|i}. \end{aligned}$$

Therefore, the components of the Fermi derivative of the covariant derivative are:

$$ {Q^{F(\nabla)(j)}}_i(X) = X^k\Gamma^j_{ki} + X_i g^{tj} X^s g_{sr} X^r_{|t} - X^s X^j g_{sr} X^r_{|i} = X^k\Big[\Gamma^j_{ki} + g_{kr}(X_i g^{tj} X^r_{|t} - X^j X^r_{|i})\Big] \quad (29)$$

The general Formula (29) produces the result (28) when applied to the vector field $\mathbf{Y}$.

The Fermi derivative of the covariant derivative has been considered extensively in Gravitational Physics.

7. The Fermi Derivative of the Fermi Derivative

The Fermi derivative is itself a linear derivative; therefore, it is possible to define the Fermi derivative by using it as the second linear derivative. The determination of the components ${Q_X^{F^2}}_i{}^j$ of the Fermi derivative along the Fermi derivative is given by the following expression:

$$Q^{F^2}{}_i{}^j = Q^F{}_i{}^j + X^s(Q^F{}_{st} - Q^L{}_{st})(g^{tj}X_i - \delta_i^t X^j)$$

where $Q^F{}_i{}^j(X) = Q_i{}^j(X) + X^s(Q_{st} - Q^L{}_{st})(g^{tj}X_i - \delta_i^t X^j)$. We compute (see Appendix D)

$$Q^{F^2}{}_i{}^j = (2 + X^r X_r)Q^F{}_i{}^j - (1 + X^r X_r)Q_i{}^j. \tag{30}$$

We note that the Lie derivative cancels out.

It is evident that one may follow the same routine and introduce the $Q^{F^m}{}_i{}^j$, where $m = 2, 3, 4, \ldots$

In the following in order to appreciate the above results, we consider a number of applications.

8. Applications

8.1. The Poisson Derivative as a Lie Derivative

Consider a chart (U, ϕ) on a smooth $2n-$dimensional manifold M with coordinates $\{x^i\}$. For every C^∞ function f, we define a vector field P_f^i on M by the rule

$$P_f^i \equiv \omega^{ij}\frac{\partial f}{\partial x^i}\partial_j \tag{31}$$

where ω^{ij} is a constant tensor field of type $(0, 2)$ (not necessarily antisymmetric!). The Lie derivative in (U, ϕ) defined by the vector field P_f^μ is given by the quantities $Q_\nu^\mu(f) \equiv -\frac{\partial P_f^\mu}{\partial x^\nu} = \omega^{\mu\rho}\frac{\partial^2 f}{\partial x^\rho \partial x^\nu}$. This new Lie derivative is called the general Poisson derivative generated by f, or simply, the general Poisson derivative.

The general Poisson derivative of a smooth function $h \in F(M)$ is computed as follows:

$$D_{P_f}h = h_{,i}P_f^i = \omega^{ij}(x)h_{,i}f_{,j} = [h, f]. \tag{32}$$

Being a Lie derivative, the torsion of this derivative gives nothing new. However, the vanishing of the curvature of the Lie derivative gives the Jacobi identity:

$$[[h, f], g] + [[f, g], f] + [[g, f], h] = 0. \tag{33}$$

Two main possibilities arise. One is to assume that ω^{ij} is symmetric, and the other, that ω^{ij} is antisymmetric. To our knowledge, the first case has not been considered in the literature. The latter defines the well-known Poisson geometry and ω^{ij} defines a symplectic structure on the manifold M.

If we specialize the manifold M to be the configuration space of a dynamic system with the coordinates $x^i = p^i$, $x^{n+i} = q^i$ $i = 1, 2, \ldots, n$ $(= \dim M)$ to be the canonical conjugate pairs and define

$$\omega^{ij} = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \tag{34}$$

we recover the well-known result

$$D_{P_f}h = (P^{\mu}_{f}\partial_{\mu})h = \omega^{\mu\nu}(\xi)\frac{\partial h}{\partial x^{\mu}}\frac{\partial f}{\partial x^{\nu}} = \sum_{i=1}^{n}\left[\frac{\partial h}{\partial q_i}\frac{\partial f}{\partial p_i} - \frac{\partial h}{\partial p_i}\frac{\partial f}{\partial q_i}\right] = \{h, f\} \tag{35}$$

and the subsequent Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h.f\}\} + \{h, \{f, g\}\} = 0. \tag{36}$$

In order to compute the generalized Poisson derivative of a tensor field, one applies the general Formula (16) of Lie derivative. For example, for a vector field $\mathbf{V} = V^{\mu}\frac{\partial}{\partial x^{\mu}}$, one has

$$D_{P_f}V^{\mu} = \begin{cases} \{V^{\mu}, f\} - f_{,(n+i)\nu}V^{\nu}, & \text{for } \mu = i \\ \{V^{\mu}, f\} + f_{,i\nu}V^{\nu}, & \text{for } \mu = n+i. \end{cases} . \tag{37}$$

8.2. The Fermi Derivative in a Metric Affine Space with Torsion and Metricity

Consider a $2D$ space M with coordinates $\{x^1, x^2\}$ endowed with the Euclidean metric δ_{ij} and the covariant derivative with connection coefficients Γ^i_{jk} $i, j = 1, 2$, which are not all zero. Because $\Gamma^i_{jk} \neq 0$, this space is not Riemannian and, in general, has non-zero torsion and metricity. Therefore, this case is not covered by the standard Riemannian approach. To compute the components of the Fermi derivative, we use the general Formula (29). We find

$$Q^{F(\nabla)(j)}{}_{i}(X) = X^k\Gamma^j_{ki} + X_i g^{tj}X^s g_{sr}X^r_{|t} - X^sX^j g_{sr}X^r_{|i} = X^k\left[\Gamma^j_{ki} + g_{kr}(X_i g^{tj}X^r_{|t} - X^jX^r_{|i})\right] \tag{38}$$

Let us compute the quantities $Q^{F(\nabla)(j)}{}_{i}(X)$, where $X = \partial_1, \partial_2$, assuming that $g(\partial_1, \partial_2) = 0$; that is, ∂_1 is normal in the Euclidean metric to the vector ∂_2. We note that the components of ∂_1 are δ^i_1; therefore,

$$\begin{aligned} Q^{F(\nabla)(j)}{}_{i}(\partial_1) &= X^k\Gamma^j_{ki} + X_i g^{tj}X^s g_{sr}X^r_{|t} - X^sX^j g_{sr}X^r_{|i} \\ &= \delta^k_1\Gamma^j_{ki} + X_i g^{tj}X^s g_{sr}(\Gamma^r_{mt}X^m) - X^sX^j g_{sr}(\Gamma^r_{mi}X^m) \\ &= \Gamma^j_{1i} + \delta_{1i}\delta^{tj}\delta^s_1\delta_{sr}(\Gamma^r_{mt}\delta^m_1) - \delta^s_1\delta^j_1\delta_{sr}\Gamma^r_{mi}\delta^m_1 \\ &= \Gamma^j_{1i} + \delta_{1i}\delta^{tj}\Gamma^1_{1t} - \delta^j_1\Gamma^1_{1i}. \end{aligned}$$

From this, it follows that

$$\begin{aligned} Q^{F(\nabla)(1)}{}_{1}(\partial_1) &= \Gamma^1_{11} + \Gamma^1_{11} - \Gamma^1_{11} = \Gamma^1_{11} \\ Q^{F(\nabla)(1)}{}_{2}(\partial_1) &= \Gamma^1_{12} + \delta_{12}\delta^{t1}\Gamma^1_{t1} - \Gamma^1_{12} = 0 \\ Q^{F(\nabla)(2)}{}_{1}(\partial_1) &= \Gamma^2_{11} + \delta_{11}\delta^{t2}\Gamma^1_{1t} - \delta^2_1\Gamma^1_{11} = \Gamma^2_{11} + \Gamma^1_{12} \\ Q^{F(\nabla)(2)}{}_{2}(\partial_1) &= \Gamma^2_{12} + \delta_{12}\delta^{t2}\Gamma^1_{t1} - \delta^2_1\Gamma^1_{21} = \Gamma^2_{12}. \end{aligned}$$

Concerning the vector $D^{F(\nabla)}_{\partial_1}(\partial_2)$, we have

$$\begin{aligned} D^{F(\nabla)}_{\partial_1}(\partial_2) &= Q^{F(\nabla)(j)}_{2}(\partial_1)\partial_j \\ &= Q^{F(\nabla)(1)}{}_{2}(\partial_1)\partial_1 + Q^{F(\nabla)(2)}{}_{2}(\partial_1)\partial_2 \\ &= \Gamma^2_{12}\partial_2. \end{aligned}$$

Working in a similar manner one computes the Fermi derivative of a general tensor field over a non-Riemannian space, for example, in the spacetime of the Einstein–Cartan theory or in non-Riemannian spacetimes of other alternative theories of gravity.

8.3. The Freedman–Robertson–Walker Spacetime with Torsion

As a further application, we compute the Fermi derivative in the Freedman–Robertson–Walker (FRW) spacetime with torsion. This spacetime has been considered in the literature [10], but not the Fermi derivative in this spacetime. We consider the standard FRW spacetime metric

$$ds^2 = -dt^2 + R^2(t)(dr^2 + r^2 d\Omega^2) \quad (39)$$

where $\{t, r, \theta, \phi\}$ are coordinates and $d\Omega^2 - d\theta^2 + \sin^2\theta d\phi^2$ is the 2-sphere. $R(t)$ is the scale factor. This spacetime has a high degree of symmetries, as follows:

a. It is conformally flat and therefore admits $\frac{1}{2}(4+1)x(4+2) = 15$ conformal Killing vectors.

b. It satisfies the Cosmological Principle, which states that spacetime is 1 + 3 decomposable by the gradient timelike Killing vector ∂_t, whereas the 3-spaces normal to the vector ∂_t are 3D spacelike maximally symmetric spaces, which admit $\frac{1}{2}3x(3+1) = 6$ spacelike Killing vectors: three vectors for the homogeneity (translations) and three for the isotropy (rotations).

The comoving observers in this model have 4-velocity $u^a = \delta^a_t$ and the projection tensor $h_{ab} = g_{ab} + u_a u_b$ projecting normal to these observers is given by the expression

$$h_{ab} = diag(\frac{R^2(t)}{1-kr^2}, R^2(t)r^2, R^2(t)r^2\sin^2\theta). \quad (40)$$

In this spacetime, we assume that besides the metric, there is a torsion tensor field S^a_{bc} which we require that satisfies the Cosmological Principle. This type of torsion is the minimal requirement for the generalization of the standard FRW cosmological model. It has been shown [23] that in this case, the torsion tensor S^a_{bc} must be of the form

$$S^a_{bc} = 2\phi(x)h^a_{[b}u_{c]} \quad (41)$$

where $\phi(x)$ is a smooth function of the coordinates. It follows that in the coordinates $\{t, r, \theta, \phi\}$, the non-zero components of S^a_{bc} are the $S^\mu_{t\mu}$=ϕ, where Greek indices take the values $1, 2, 3$ and correspond to the coordinates r, θ, ϕ, respectively.

In order to compute the Fermi derivative, we have to compute the connection coefficients. For that, it is helpful to recall the following well-known result.

Theorem 1. *The connection coefficients Γ^i_{jk} in a (symmetric) metric space with metric $g_{ij}(x)$ satisfy the identity*

$$\Gamma^i_{jk} = \widehat{\Gamma}^a_{bc} + K^i{}_{jk} - \Delta^i_{jk} \quad (42)$$

where the quantities $\widehat{\Gamma}^a_{bc}$, $K^i{}_{jk}$, and Δ^i_{jk} are defined as follows:

$$\widehat{\Gamma}^a_{bc} = \frac{1}{2}g^{ir}(g_{jr,k} + g_{kr,j} - g_{jk,r}) \quad \text{(Riemannian connection - Christofell symbols)} \quad (43)$$

$$K^i{}_{jk} = S^i{}_{jk} + S_{kj}{}^i + S_{jk}{}^i \quad \text{(Cartan's contorsion tensor)} \quad (44)$$

$$\Delta^i_{jk} = \frac{1}{2}g^{ir}(g_{jr|k} + g_{kr|j} - g_{jk|r}) \quad \text{(Associated metricity tensor).} \quad (45)$$

$g_{jr|k}$ denotes covariant differentiation with respect to the index k. The quantities S^i_{jk} are given by the relation

$$S^i_{jk} = \frac{1}{2}T^i{}_{jk} = \frac{1}{2}(\Gamma^i_{jk} - \Gamma^i_{kj}) \quad (46)$$

where the quantities $T^i{}_{jk}$ are the components of the torsion tensor. Furthermore, the tensors $K^i{}_{jk}, \Delta^i_{jk}$ satisfy the index symmetries

$$K_{ijk} = -K_{jik} \tag{47}$$

and

$$\Delta^i_{jk} = \Delta^i_{kj}. \tag{48}$$

In a FRW spacetime with torsion given by (41), the Cartan's contorsion tensor is computed to be

$$K^a_{bc} = 4\phi(\delta^a_c u_b - h_{bc}u^a). \tag{49}$$

Then, from Theorem 1, it follows that the connection coefficients are

$$\Gamma^a_{bc} = \widehat{\Gamma}^a_{bc} + 4\phi(\delta^a_c u_b - h_{bc}u^a) \tag{50}$$

where $\widehat{\Gamma}^a_{bc}$ are the Riemannian connection coefficients (Christofell symbols) given by (43).

We compute the Fermi derivative along the comoving observers with 4-velocity $u^a = \delta^a_t$. The vanishing of this derivative defines the Fermi-transported (i.e., non-rotating) frames for these observers in this spacetime.

The Fermi derivative defined by the covariant derivative is given by (29).

$$Q^{F(\nabla)(j)}{}_i(X) = X^k\left[\Gamma^j_{ki} + g_{kr}(X_i g^{tj} X^r_{|t} - X^j X^r_{|i})\right]$$

Setting $X^a = u^a$, we have

$$Q^{F(\nabla)(j)}{}_i(u) = u^k\left[\Gamma^j_{ki} + g_{kr}(u_i g^{mj} u^r_{|m} - u^j u^r_{|i})\right]$$

where $u^r_{|i} = u^r_{,i} + \Gamma^r_{si}u^s = \Gamma^r_{si}u^s$. Therefore,

$$Q^{F(\nabla)(j)}{}_i(u) = u^k\left[\Gamma^j_{ki} + u^s g_{kl}(u_i\Gamma^l_{sm}g^{mj} - u^j\Gamma^l_{si})\right]. \tag{51}$$

Replacing Γ^j_{ki} from (50) and using $u^r u_r = -1$, we find

$$Q^{F(\nabla)(j)}{}_i(u) = u^k\left[\widehat{\Gamma}^j_{ki} + u^s g_{kl}(u_i\widehat{\Gamma}^l_{sm}g^{mj} - u^j\widehat{\Gamma}^l_{si})\right] - 4\phi\delta^j_i. \tag{52}$$

Using $u^a = \delta^a_t$, this reduces to

$$\begin{aligned} Q^{F(\nabla)(j)}{}_i(u) &= \widehat{\Gamma}^j_{ti} + g_{tl}(u_i\widehat{\Gamma}^l_{tm}g^{mj} - u^j\widehat{\Gamma}^l_{ti}) - 4\phi\delta^j_i \\ &= \widehat{\Gamma}^j_{ti} - (u_i\widehat{\Gamma}^t_{tm}g^{mj} - u^j\widehat{\Gamma}^t_{ti}) - 4\phi\delta^j_i \end{aligned} \tag{53}$$

In order to find the rotation of the radial direction along u^a, we compute the Fermi derivative of ∂_r along u^a. We have

$$\begin{aligned} Q^{F(\nabla)(j)}{}_i(u)(\partial_r) &= Q^{F(\nabla)(j)}{}_r(u)\partial_j \\ &= Q^{F(\nabla)(t)}{}_r(u)\partial_t + Q^{F(\nabla)(r)}{}_r(u)\partial_r + Q^{F(\nabla)(\theta)}{}_r(u)\partial_\theta + Q^{F(\nabla)(\phi)}{}_r(u)\partial_\phi. \end{aligned} \tag{54}$$

We compute from (53)

$$\begin{aligned} Q^{F(\nabla)(t)}{}_r(u) &= 2\widehat{\Gamma}^t_{tr} = 0 \\ Q^{F(\nabla)(r)}{}_r(u) &= \widehat{\Gamma}^r_{tr} - 4\phi = -4\phi \\ Q^{F(\nabla)(\theta)}{}_r(u) &= \widehat{\Gamma}^\theta_{tr} = 0 \\ Q^{F(\nabla)(\phi)}{}_r(u) &= \widehat{\Gamma}^\phi_{tr} = 0 \end{aligned}$$

Replacing in (54) we find:

$$Q^{F(\nabla)(j)}{}_{i}(u)(\partial_r) = -4\phi\partial_r.$$

which implies that torsion induces the rotation of the radial direction. This associates torsion with a kinematic phenomenon not with a dynamic field (spin), at least in the case of the highly symmetric FRW spacetime.

9. The Distinct Role of Lie and the Covariant Derivative

As aforementioned, the role of the Lie derivative is to formulate the symmetries, i.e., invariance of geometric objects, whereas the main role of the covariant derivative is to formulate the equations of a dynamical system and their first integrals. In this section, we demonstrate how the two roles interact and lead to conditions whose solution makes the determination of first integrals of the dynamical equations possible.

The dynamical equations of an autonomous conservative dynamical system are

$$\ddot{q}^a = -\Gamma^a_{bc}\dot{q}^b\dot{q}^c - V^{,a} \tag{55}$$

where the potential $V = V(q)$ and Γ^a_{bc} are the Riemann connection coefficients with respect to the kinetic metric γ_{ab} defined by the kinetic energy of the system. We set the velocity $u^a \equiv \dot{q}^a$ and Equation (55) is written equivalently as

$$u^a_{;b}u^b = -V^{,a}. \tag{56}$$

We consider the vector $X = \xi(t)\partial_t + \eta^i(q)\partial_{q^i}$, which generates the point transformation $\Delta t = \varepsilon\xi(t)$ and $\Delta q^i = \varepsilon\eta^i$. A dot over a symbol indicates total derivative, for example,

$$\dot{\eta}^i = \frac{\partial\eta^i}{\partial t} + \frac{\partial\eta^i}{\partial q^j}\dot{\eta}^j.$$

Because the system is autonomous, the condition for X to be a Lie point symmetry of (56) is

$$L_\eta\left(u^a_{;b}u^b + V^{,a}\right) = 0 \tag{57}$$

along solutions of (55).

We compute

$$\varepsilon L_\eta u^a = \varepsilon u^a_{,b}\eta^b - \varepsilon\eta^a_{,b}u^b = u^a_{,b}\Delta q^b - (\Delta q^a)^{\cdot} = -u^a(\Delta t)^{\cdot} = -\varepsilon\dot{\xi}u^a$$

Therefore,

$$L_\eta u^a = -\dot{\xi}u^a \tag{58}$$

where we use the identity of the variational calculus

$$\Delta\dot{q}^a - (\Delta q^a)^{\cdot} = -\dot{q}^a(\Delta t)^{\cdot}.$$

From the Ricci identity,

$$\begin{aligned} L_{\vec{\eta}}\nabla_k T^{i_1\ldots i_r}_{j_1\ldots j_s} - \nabla_k L_{\vec{\eta}} T^{i_1\ldots i_r}_{j_1\ldots j_s} &= L_{\vec{\eta}}\left(\Gamma^{i_1}_{\ell k}\right)T^{\ell i_2\ldots i_r}_{j_1 j_2\ldots j_s} + \ldots + L_{\vec{\eta}}\left(\Gamma^{i_r}_{\ell k}\right)T^{i_1\ldots i_{r-1}\ell}_{j_1\ldots j_{s-1}j_s} \\ &\quad - L_{\vec{\eta}}\left(\Gamma^{\ell}_{j_1 k}\right)T^{i_1 i_2\ldots i_r}_{\ell j_2\ldots j_s} - \ldots - L_{\vec{\eta}}\left(\Gamma^{\ell}_{j_s k}\right)T^{i_1\ldots i_{r-1}i_r}_{j_1\ldots j_{s-1}\ell} \end{aligned} \tag{59}$$

we have

$$L_\eta(u^a_{;b}) = (L_\eta u^a)_{;b} + (L_\eta\Gamma^a_{bc})u^c.$$

Then, (57), along solutions of (56), gives

$$\begin{aligned} 0 &= L_\eta\left(u^a_{;b}u^b - V^{,a}\right) = L_\eta\left(u^a_{;b}\right)u^b + u^a_{;b}L_\eta\left(u^b\right) - L_\eta(V^{,a}) \\ &= (L_\eta u^a)_{;b}u^b + (L_\eta\Gamma^a_{bc})u^b u^c - \dot{\xi}u^a_{;b}u^b - L_\eta V^{,a} \\ &= (L_\eta\Gamma^a_{bc})u^b u^c - 2\dot{\xi}_{;b}u^b + \dot{\xi}u^a_{;b}u^b - L_\eta V^{,a} \\ &= (L_\eta\Gamma^a_{bc})u^b u^c - 2\phi_{,b}u^a u^b - 2\phi u^a_{;b}u^b - 2\phi u^a_{;b}u^b - L_\eta V^{,a} \\ &= [L_\eta\Gamma^a_{bc} - 2\dot{\xi}_{,b}\delta^a_c]u^b u^c - L_\eta V^{,a} - 2\dot{\xi}V^{,a}. \end{aligned}$$

This expression must be identically zero for all u^a; therefore, each term for different powers of u^a must vanish. This gives the following necessary and sufficient conditions for (56) to admit the Lie symmetry X:

$$L_\eta V^{,a} + 2\dot{\xi}V^{,a} = 0$$
$$L_\eta\Gamma^i_{jk} = \dot{\xi}_{,k}\delta^i_j + \delta^i_k\dot{\xi}_{,j}$$

These conditions coincide with the conditions which are found if one applies the standard Lie symmetry approach [11]. From the above, it follows that:

The Lie symmetry condition for autonomous conservative dynamical systems is equivalent to the Lie derivative of the dynamical equations.

From the second condition, it follows that the vector $\eta^i\partial_{q^i}$ is a projective collineation of the kinetic metric with the projection function $\dot{\xi}$. The first condition constrains the potential V with the function $\dot{\xi}$ and the projective vector η^i. A solution of this system of equations can be found in many publications. Concerning the first integrals, they are the Noether symmetries, which have been shown to coincide with the homothetic algebra (a subalgebra of the projective algebra) of the kinetic metric [24].

10. Conclusions

The concept of a universal approach to linear derivatives of tensor algebra through the introduction of the generic linear derivative offers a deeper understanding and profound generalization of the notion of the concept of a derivative. The key points of this paper are as follows:

a. The association of the generic linear derivative with a geometric object with components $Q_j^{\ (i)}$.

b. The introduction of torsion and curvature tensors for all linear derivatives of tensor algebra, not just for the covariant derivative as traditionally believed.

c. The definition of a specific linear derivative by the introduction of a specific set of quantities $Q_j^{\ (i)}$, which transform according to (7).

The generic derivative approach has been applied to the main derivatives used in practice, that is, the Lie derivative, the covariant derivative, and the Fermi derivative.

For each derivative, the following results were obtained:

Lie Derivative:

1. The Lie derivative has non-vanishing torsion.
2. The curvature of the Lie derivative vanishes due to the Jacobi identity.
3. The Poisson bracket is a manifestation of the Lie derivative. The Jacobi identity for the Poisson bracket follows naturally from the vanishing of the curvature of the Lie derivative.

Covariant Derivative:

The covariant derivative involves the connection coefficients $Q_j^{(i)}(X) = \Gamma^i_{kj} X^k$, which reduce to Christofell symbols in the case of the Riemannian derivative. The various formulae of the latter follow without any extra calculation.

Fermi derivative:

The main contribution of the present work, apart from the introduction of the general approach to derivation, is the Fermi derivative. This derivative is used in General Relativity to define the propagation of a "non-rotating" orthonormal spatial frame along the world line of an observer. In the standard literature, the Fermi derivative is defined in terms of the (Riemannian) covariant derivative.

The Fermi derivative is a second-order linear derivative in the sense that it combines a given derivative, which is not necessarily the covariant derivative, and the Lie derivative. The Fermi derivative has both torsion and curvature. Furthermore, the Fermi derivative, being a linear derivative, can be iterated to produce a series of higher-order Fermi derivatives for every given Fermi derivative.

Overall, the universal approach to derivation using the non-tensorial quantities $Q_j^{(i)}$ provides a versatile and powerful framework for constructing new geometric structures on a manifold. The dynamic equations of the theories of Physics are mainly based on the covariant derivative, whereas the Lie symmetries of these equations are mainly based on the Lie derivative. Using the universal approach, it is possible that one could use the present results and construct new theories of Physics. The same applies to Differential Geometry, where already other types of derivative have been introduced.

One final point is the derivatives of non-tensorial geometric objects. These derivatives are not—and probably cannot–be defined the way we used to define the linear derivative of tensors. This is the case even for the tensor densities, which are geometric objects very near to tensors [25]. For these derivatives, it makes no sense to define the torsion tensor and the curvature tensor, which are fundamental in the development of physical theories. However, as it is well known, these derivatives do play a major role in the studies of Geometry and Physics.

Funding: This research received no external funding.

Data Availability Statement: No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Conflicts of Interest: The author declares no conflict of interest.

Appendix A

Proof of Relation (11). Consider two vector fields $\mathbf{V} = V^i \partial_i$ and $\mathbf{W} = W^i \partial_i$, and the commutator:

$$\mathbf{D_V D_W} - \mathbf{D_W D_V}$$

Let $\mathbf{U} = U^i \partial_i$ be a third vector field. We compute

$$
\begin{aligned}
(\mathbf{D_V D_W} - \mathbf{D_W D_V})\mathbf{U} = & D_\mathbf{V} D_\mathbf{W} \mathbf{U} - D_\mathbf{W} D_\mathbf{V} \mathbf{U} \\
&= D_\mathbf{V}(U^i_{,j} W^j \partial_i + U^j Q^i_j(\mathbf{W})\partial_i) - D_\mathbf{W}(U^i_{,j} V^j \partial_i + U^j Q^i_j(\mathbf{V})\partial_i) \\
&= U^i_{,jk} V^k W^j \partial_i + U^i_{,j} W^j_{,k} V^k \partial_i + U^i_{,j} W^j Q^k_i(\mathbf{V})\partial_k \\
+ U^j_{,k} V^k Q^i_j(\mathbf{W})\partial_i + & U^j Q^i_j(\mathbf{W})_{,k} V^k \partial_i + U^j Q^i_j(\mathbf{W}) Q^k_i(\mathbf{V})\partial_k \\
&- U^i_{,jk} W^k V^j \partial_i - U^i_{,j} V^j_{,k} W^k \partial_i - U^i_{,j} V^j Q^k_i(\mathbf{W})\partial_k \\
- U^j_{,k} W^k Q^i_j(\mathbf{V})\partial_i & - U^j Q^i_j(\mathbf{V})_{,k} W^k \partial_i - U^j Q^i_j(\mathbf{V}) Q^k_i(\mathbf{W})\partial_k \\
&= U^i_{,j} W^j_{,k} V^k \partial_i - U^i_{,j} V^j_{,k} W^k \partial_i + U^j Q^i_j(\mathbf{W})_{,k} V^k \partial_i - U^j Q^i_j(\mathbf{V})_{,k} W^k \partial_i \\
&\quad + U^j Q^k_j(\mathbf{W}) Q^i_k(\mathbf{V})\partial_i - U^j Q^k_j(\mathbf{V}) Q^i_k(\mathbf{W})\partial_i \\
&= U^i_{,j}(V^k W^j_{,k} - W^k V^j_{,k})\partial_i + [Q^i_j(\mathbf{W})_{,k} V^k - Q^i_j(\mathbf{V})_{,k} W^k \\
&\quad + Q^k_j(\mathbf{W}) Q^i_k(\mathbf{V}) - Q^k_j(\mathbf{V}) Q^i_k(\mathbf{W})] U^j \partial_i \\
&= U^i_{,j}(V^k W^j_{,k} - W^k V^j_{,k})\partial_i + H^i_j U^j \partial_i \\
&= U^i_{,j}[\mathbf{V}, \mathbf{W}]^j + H^i_j U^j \partial_i.
\end{aligned}
$$

where

$$
H^i_j = Q^i_j(\mathbf{W})_{,k} V^k - Q^i_j(\mathbf{V})_{,k} W^k + Q^k_j(\mathbf{W}) Q^i_k(\mathbf{V}) - Q^k_j(\mathbf{V}) Q^i_k(\mathbf{W}).
$$

□

Appendix B

Proof of (25). Note: in order to ease the notation, we omit the parentheses in the upper index of ${Q^{(j)}}_i$.

$$
\begin{aligned}
{Q^{Fj}}_i &= J^j_{j'} J^{i'}_i {Q^{j'}}_{i'} + J^j_{j'} J^{j'}_{i,k'} X^{k'} + X^s g_{tr}(J^r_{r'} J^{s'}_s Q^{r'}_{s'} + J^r_{r'} J^{r'}_{s,k'} X^{k'} \\
&\quad - J^r_{l'} J^{m'}_s {Q^{Ll'}}_{m'} - J^r_{l'} J^{l'}_{s,k'} X^{k'})(g^{tj} X_i - \delta^t_i X^j) \\
&= J^j_{j'} J^{i'}_i Q^{j'}_{i'} + J^j_{j'} J^{j'}_{i,k'} X^{k'} + J^s_{k'} X^{k'} J^{t'}_t J^{a'}_r g_{t'a'} J^r_{r'} J^{s'}_s (Q^{r'}_{s'} - {Q^{Lr'}}_{s'})(g^{tj} X_i - \delta^t_i X^j) \\
&= J^j_{j'} J^{i'}_i Q^{j'}_{i'} + J^j_{j'} J^{j'}_{i,k'} X^{k'} \\
&\quad + X^{s'} J^{t'}_t g_{t'r'}(Q^{r'}_{s'} - {Q^{Lr'}}_{s'})(J^t_{k'} J^j_{j'} g^{k'j'} J^{i'}_i X_{i'} - J^t_{k'} J^{i'}_i \delta^{k'}_{i'} J^j_{j'} X^{j'}) \\
&= J^j_{j'} J^{i'}_i Q^{j'}_{i'} + J^j_{j'} J^{j'}_{i,k'} X^{k'} + X^{s'} J^{t'}_t g_{t'r'} J^t_{k'} J^j_{j'} J^{i'}_i (Q^{r'}_{s'} - {Q^{Lr'}}_{s'})(g^{k'j'} X_{i'} - \delta^{k'}_{i'} X^{j'}) \\
&= J^j_{j'} J^{i'}_i Q^{j'}_{i'} + J^j_{j'} J^{j'}_{i,k'} X^{k'} + J^j_{j'} J^{i'}_i X^{s'} g_{k'r'}(Q^{r'}_{s'} - {Q^{Lr'}}_{s'})(g^{k'j'} X_{i'} - \delta^{k'}_{i'} X^{j'}) \\
&= J^j_{j'} J^{i'}_i [{Q_{i'}}^{j'} + X^{s'} g_{s'r'}(Q^{r'}_{s't'} - {Q^{Lr'}}_{s't'})(g^{t'j'} X_{i'} - \delta^{t'}_{i'} X^{j'})] + J^j_{j'} J^{j'}_{i,k'} X^{k'} \\
&= J^j_{j'} J^{i'}_i {Q^{Fj'}}_{i'} + J^j_{j'} J^{j'}_{i,k'} X^{k'}.
\end{aligned}
$$

□

Appendix C

Proof of (28). We set $Q^L_{st}(X) = g_{sr}Q^{Lr}_t(X)$ and have

$$
\begin{aligned}
D^F_{\mathbf{X}}(\mathbf{Y}) &= D^F_{\mathbf{X}}(Y^i\partial_i)\\
&= Y^i_{,j}X^j\partial_i + Y^jQ^F{}_j{}^i\partial_i\\
&= Y^i_{,j}X^j\partial_i + Y^jQ_j{}^i\partial_i + Y^jX^s(Q_{st} - Q^L{}_{st})(g^{ti}X_j - \delta^t_jX^i)\partial_i\\
&= D_X(\mathbf{Y}) + Y^jX^sg_{tr}(Q_s{}^r - Q^L{}_s{}^r)(g^{ti}X_j - \delta^t_jX^i)\partial_i\\
&= D_X(\mathbf{Y}) + Y^jX^sg_{tr}[Q_s{}^r - (-X^r_{,s})](g^{ti}X_j - \delta^t_jX^i)\partial_i\\
&= D_X(\mathbf{Y}) + Y^jX^sg_{tr}Q_s{}^rg^{ti}X_j\partial_i + Y^jX^sg_{tr}X^r_{,s}g^{ti}X_j\partial_i\\
&\quad - Y^jX^sQ_s{}^rg_{tr}\delta^t_jX^i\partial_i - Y^jX^sg_{tr}X^r_{,s}\delta^t_jX^i\partial_i\\
&= D_X(\mathbf{Y}) + Y^jX^sQ_s{}^iX_j\partial_i + Y^jX^sX^i_{,s}X_j\partial_i - Y^jX^sg_{jr}Q_s{}^r\mathbf{X} - Y^jX^sg_{jr}X^r_{,s}\mathbf{X}\\
&= D_X(\mathbf{Y}) + Y^jX^sQ_s{}^ig_{jr}X^r\partial_i + Y^jX^sX^i_{,s}g_{jr}X^r\partial_i - g_{jr}Y^j(X^sX^r_{,s} + X^sQ_s{}^r)\mathbf{X}\\
&= D_X(\mathbf{Y}) + g_{jr}Y^jX^r(X^sQ_s{}^i\partial_i + X^sX^i_{,s}\partial_i) - g_{jr}Y^j(X^sX^r_{,s} + X^sQ_s{}^r)\mathbf{X}\\
&= .D(\mathbf{Y}) + g(\mathbf{X},\mathbf{Y})\Big[X^s(Q_s{}^i\partial_i + X^i_{,s}\partial_i)\Big] - g_{jr}Y^j(X^sX^r_{,s} + X^sQ_s{}^r)\mathbf{X}\\
&= D_{\mathbf{X}}(\mathbf{Y}) + g(\mathbf{X},\mathbf{Y})D_{\mathbf{X}}(\mathbf{X}) - g_{jr}Y^j(D_{\mathbf{X}}X)^r\mathbf{X}.
\end{aligned}
$$

□

Appendix D

Proof of (30).

$$
\begin{aligned}
Q^{F^2}{}_i{}^j(X) &= Q^F{}_i{}^j + X^s(Q^F{}_{st} - Q^L{}_{st})(g^{tj}X_i - \delta^t_iX^j)\\
&= Q_i^j + X^s(Q_{st} - Q^L{}_{st})(g^{tj}X_i - \delta^t_iX^j)\\
&\quad + X^s[Q_{st} + X^r(Q_{rk} - Q^L{}_{rk})(\delta^k_tX_s - \delta^k_sX_t) - Q^L{}_{st}](g^{tj}X_i - \delta^t_iX^j)\\
&= Q_i^j + X^s(g^{tj}X_i - \delta^t_iX^j)[2(Q_{st} - Q^L{}_{st})\\
&\quad + X^rQ_{rt}X_s - X^rQ_{rs}X_t - X^rQ^L{}_{rt}X_s + X^rQ^L_{rs}X_t]\\
&= Q_i^j + 2X^s(Q_{st} - Q^L{}_{st})(g^{tj}X_i - \delta^t_iX^j) + X^rX_rg^{tj}X_iX^s(Q_{st} - Q^L{}_{st})\\
&\quad - X^rX_r\delta^t_iX^jX^s(Q_{st} - Q^L{}_{st})\\
&= Q_i^j + 2X^s(Q_{st} - Q^L{}_{st})(g^{tj}X_i - \delta^t_iX^j) + X^rX_rX^s(Q_{st} - Q^L{}_{st})(g^{tj}X_i - \delta^t_iX^j)\\
&= Q_i^j + (2 + X^rX_r)X^s(Q_{st} - Q^L{}_{st})(g^{tj}X_i - \delta^t_iX^j)\\
&= Q_i^j + (2 + X^rX_r)(Q^F{}_i{}^j - Q_i^j)\\
&= (2 + X^rX_r)Q^F{}_i{}^j - (2 + X^rX_r)Q_i^j + Q_i^j\\
&= (2 + X^rX_r)Q^F{}_i{}^j - (1 + X^rX_r)Q_i^j.
\end{aligned}
$$

□

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