universe

IMPACT FACTOR 2.5

CITESCORE 4.3

Article

Predictions for the Rare Kaon Decays $K_{S,L} \to \pi^0 \ell^+ \ell^-$ from QCD in the Limit of a Large Number of Colours

Giancarlo D'Ambrosio and Marc Knecht

Special Issue

CP Violation and Flavor Physics

Edited by

Dr. Andreas Trautner and Prof. Dr. Celso C. Nishi

MDPI

https://doi.org/10.3390/universe10120457 universe

MDPI

Article

Predictions for the Rare Kaon Decays $K_{S,L} \to \pi^0 \ell^+ \ell^-$ from QCD in the Limit of a Large Number of Colours

Giancarlo D'Ambrosio [1] and Marc Knecht [2,*]

[1] INFN-Sezione di Napoli, Complesso Universitario di Monte S. Angelo, Via Cintia Edificio 6, 80126 Napoli, Italy; gdambros@na.infn.it
[2] Centre de Physique Théorique, Aix-Marseille Univ./Univ. de Toulon/CNRS (UMR 7332), CNRS-Luminy Case 907, 13288 Marseille Cedex 9, France
* Correspondence: knecht@cpt.univ-mrs.fr

Abstract: The long-distance and non-local parts of the form factors describing the single-photon-mediated $K_{S,L} \to \pi^0 \gamma^* \to \pi^0 \ell^+ \ell^-$ ($\ell = e, \mu$) transitions in the standard model are addressed in QCD regarding the limit where the number N_c of colours becomes infinite. It is shown that this provides a suitable theoretical framework to study these decay modes and that it enables predicting the decay rates for $K_S \to \pi^0 \ell^+ \ell^-$. It also unambiguously predicts that the interference between the direct and indirect CP-violating contributions to the decay rate for $K_L \to \pi^0 \ell^+ \ell^-$ is constructive.

Keywords: high-energy physics; kaon decays

1. Introduction

Rare kaon decays remain a very active domain of research, with quite interesting perspectives for the future, as attested by several recent reports [1–8]. Since they are mediated by neutral currents, these processes are naturally suppressed in the standard model [9,10] and provide various ways to test the standard model's flavour structure. The fruitful completion of this research program requires a high level of precision regarding both the experimental measurements and theoretical predictions. This goal is about to be fulfilled on the theory side [5,11,12] in the case of the rare decay modes $K \to \pi \nu \bar{\nu}$, which are dominated by short-distance contributions, and the prospects to improve on the present experimental results [13,14] also appear to be quite promising [4,8]. Unfortunately, the situation is in a less satisfactory state, at least from the theoretical point of view, in the case of other rare kaon decay modes, whose amplitudes are instead dominated by a long-distance and non-local component that is governed by the non-perturbative dynamics of the strong interactions (QCD) at low energies.

In the present study, we wish to address this issue in the case of the decay modes of neutral kaons K_S and K_L into a neutral pion and a pair of charged leptons. In the case of the short-lived neutral kaon, we will consider the CP-conserving transition mediated by the exchange of a single virtual photon, $K_S \to \pi^0 \gamma^* \to \pi^0 \ell^+ \ell^-$, with $\ell = e, \mu$, and K_S being identified with the CP-even combination K_1^0 of K^0 and $\overline{K}^0$, i.e., using the convention $\mathrm{CP}|K^0\rangle = -|\overline{K}^0\rangle$,

$$|K_S\rangle \simeq |K_1^0\rangle = \frac{|K^0\rangle - |\overline{K}^0\rangle}{\sqrt{2}}. \tag{1}$$

In the case of the long-lived kaon, defined as

$$|K_L\rangle \simeq |K_2^0\rangle + \bar{\epsilon}|K_1^0\rangle, \quad |K_2^0\rangle = \frac{|K^0\rangle + |\overline{K}^0\rangle}{\sqrt{2}}, \tag{2}$$

the situation is reversed: while CP conservation requires the exchange of two virtual photons, $K_2^0 \to \pi^0 \gamma^* \gamma^* \to \pi^0 \ell^+ \ell^-$, the transition $K_2^0 \to \pi^0 \gamma^* \to \pi^0 \ell^+ \ell^-$ corresponds to a

check for updates

Citation: D'Ambrosio, G.; Knecht, M. Predictions for the Rare Kaon Decays $K_{S,L} \to \pi^0 \ell^+ \ell^-$ from QCD in the Limit of a Large Number of Colours. *Universe* **2024**, *10*, 457. https://doi.org/10.3390/universe10120457

Academic Editors: Andreas Trautner and Celso C. Nishi

Received: 18 November 2024
Revised: 10 December 2024
Accepted: 11 December 2024
Published: 13 December 2024

direct violation of CP [15]. It has been argued [16] and is usually admitted [17,18] that the corresponding contribution to the amplitude is dominated by short distances and is thus proportional, in the standard model, to $\mathrm{Im}\,\lambda_t > 0$, with $\lambda_t \equiv V_{td}V_{ts}^*$ being a product of CKM matrix elements [19]. Finally, the amplitude for this process receives a third component due to the CP violation in the mixing, which results from the non-vanishing parameter $\bar{\epsilon}$. For the latter, we will take [15,20,21]

$$\bar{\epsilon} \sim \frac{1+i}{\sqrt{2}}|\epsilon|, \;\; |\epsilon| = 2.228 \cdot 10^{-3}. \tag{3}$$

The branching ratio thus takes the form [16–18,22]

$$\mathrm{Br}(K_L \to \pi^0 \ell^+ \ell^-) = 10^{-12}\left[C_{\mathrm{mix}}^{(\ell)} + C_{\mathrm{int}}^{(\ell)}\frac{\mathrm{Im}\,\lambda_t}{10^{-4}} + C_{\mathrm{dir}}^{(\ell)}\left(\frac{\mathrm{Im}\,\lambda_t}{10^{-4}}\right)^2 + C_{\gamma^*\gamma^*}^{(\ell)}\right]. \tag{4}$$

The last term in this expression is the CP-conserving component. Phenomenological estimates have found that it is small in the case of $\ell = e$, $C_{\gamma^*\gamma^*}^{(e)} = \mathcal{O}(10^{-2})$ [15,16,23] and substantial in the case $\ell = \mu$, $C_{\gamma^*\gamma^*}^{(\mu)} = 5.2(1.6)$ [17,18]. The first term in Equation (4) provides the contribution from the indirect CP violation alone and can be expressed in terms of experimental quantities [16], the lifetimes $\tau(K_{S,L})$ of the neutral kaons, and the branching ratio for the CP-conserving transition $K_S \to \pi^0 \ell^+ \ell^-$,

$$C_{\mathrm{mix}}^{(\ell)} = 10^{12}|\bar{\epsilon}|^2 \frac{\tau(K_S)}{\tau(K_L)} \mathrm{Br}(K_S \to \pi^0 \ell^+ \ell^-). \tag{5}$$

The third term in Equation (4) is the contribution from the direct CP violation, while the second term provides the interference between the direct and indirect CP-violating contributions. Their dependence with respect to λ_t is shown explicitly. The coefficient $C_{\mathrm{int}}^{(\ell)}$ is provided as a phase-space integral whose integrand involves the amplitude of the decay $K_S \to \pi^0 \ell^+ \ell^-$. A crucial issue is whether this interference is constructive or destructive; from an experimental point of view, a constructive interference will be a key feature in order to overcome the important irreducible background induced by the $K_L \to \gamma\gamma\ell^+\ell^-$ decay [24] and thus provide access to an independent determination of $\mathrm{Im}\,\lambda_t$. This brief description of the decays $K_{S,L} \to \pi^0 \ell^+ \ell^-$ leaves us with a short list of questions to be answered:

- Can one predict $\mathrm{Br}(K_S \to \pi^0 \ell^+ \ell^-)$ (or even the decay distribution) in the standard model?
- Can the sign of C_{int} be predicted?
- Can one confirm that the long-distance component of the amplitude induced by the direct CP-violating contribution $K_2^0 \to \pi^0 \gamma^* \to \pi^0 \ell^+ \ell^-$ indeed remains negligible once the non-perturbative QCD effects are taken into account?

Answering these questions requires obtaining quantitative control of the non-perturbative aspects of QCD at low energies, a notoriously difficult task. The purpose of this paper is to show that this goal can be met in the limit where the number of colours N_c becomes infinite [25,26], a limit that has often provided relevant insights into the physical case $N_c = 3$. It turns out that in this large-N_c limit QCD leads to unambiguous positive answers for all three of the questions listed above. In order to demonstrate this, it is necessary that we first state more precisely how the large-N_c limit of QCD can be implemented in the case at hand.

2. Theoretical Framework

Long-distance-dominated rare kaon decays are traditionally addressed within the framework of the three-flavour low-energy expansion (ChPT) [27] extended to weak decays [28–32]. The lowest-order (one loop in this case) expressions of the amplitudes for $K \to \pi \ell^+ \ell^-$, $(K, \pi) = (K^\pm, \pi^\pm)$, (K_S, π^0) were obtained in Ref. [33] (see also [34]) and given in terms of form factors $\mathcal{W}_+(s)$ and $\mathcal{W}_S(s)$, where s denotes the square of the invariant mass of the di-lepton pair. A 'beyond-one-loop' representation of these form factors, accounting only for part of the pion loops at next-to-lowest order, was proposed in Ref. [22] and reads

$$\mathcal{W}_{+,S}(s) = G_F(M_K^2 a_{+,S} + b_{+,S}s) + \mathcal{V}_{+,S}^{\pi\pi}(s). \tag{6}$$

The neglected contributions from pion loops were shown to indeed be smallish in the whole range of energies corresponding to the relevant kinematic region [35]. The counter-terms at lowest and at next-to-lowest orders as well as the loops also involving kaons, i.e., from $K\overline{K}$ intermediate states (already at one loop) or from $K\pi$ intermediate states (starting at two loops), corresponding to higher thresholds sufficiently far away from the decay region, are described by a first-order polynomial in s. The expressions for the contributions $\mathcal{V}_+^{\pi\pi}(s)$ and $\mathcal{V}_S^{\pi\pi}(s)$ from the pion loops are provided in Ref. [22]. Focusing on $K_S \to \pi^0\ell^+\ell^-$, it turns out that $\mathcal{V}_S^{\pi\pi}(s)$ is suppressed since it proceeds through a $\Delta I = 3/2$ transition $K_S \to \pi^0\pi^+\pi^-$. Predicting the decay distribution and decay rate therefore amounts, in practice, to being able to predict the values of the two unknown parameters a_S and b_S. Quantitative information about a_S and b_S is not provided by ChPT itself and needs to be looked for in the non-perturbative regime of full QCD. This is where we can expect that the limit of a large number of colours may become useful. Indeed, these two constants, or more precisely the contributions from the counter-terms to them, are precisely what survives from the amplitude (6) at leading order in the limit $N_c \to \infty$ since

$$a_{+,S}, b_{+,S} \sim \mathcal{O}(N_c), \;\; \mathcal{V}_{+,S}^{\pi\pi}(s) \sim \mathcal{O}(N_c^0). \tag{7}$$

Obtaining the representation of the form factor $\mathcal{W}_S(s)$ in the large-N_c limit of QCD should therefore provide a good description of the amplitude in the decay region. In the remainder of this paper, we will outline the main steps of this endeavour, relying partly on Ref. [36], where a more detailed account will be provided, while here we merely discuss some phenomenological consequences. Before proceeding, let us mention that a similar procedure can be applied to the amplitude $\mathcal{W}_+(s)$ as well, and we briefly comment on it before concluding this study. A more detailed discussion of $\mathcal{W}_+(s)$ in the large-N_c limit will be provided in Ref. [36].

In the standard model, the structure of the amplitude $\mathcal{A}_S$ of the decay $K_S(k) \to \pi^0(p)\ell^+(p_+)\ell^-(p_-)$, with $k-p = p_+ + p_-$ and $s = (k-p)^2$, reads

$$\mathcal{A}_S = \mathcal{A}_S^{\text{SD;A}} - e^2\bar{\mathrm{u}}(p_{\ell^-})\gamma_\rho \mathrm{v}(p_{\ell^+})(k+p)^\rho \times \frac{\mathcal{W}_S(s)}{16\pi^2 M_K^2}. \tag{8}$$

Let us for the moment leave aside the short-distance part $\mathcal{A}_S^{\text{SD;A}}$ and concentrate on the form factor $\mathcal{W}_S(s)$. It comprises another local short-distance part but also a long-distance-dominated, non-local component,

$$\mathcal{W}_S(s) = \mathcal{W}_S^{\text{loc}}(s;\nu) + \mathcal{W}_S^{\text{non-loc}}(s;\nu). \tag{9}$$

The latter is provided by

$$\begin{aligned}\Big[s(k+p)_\rho - (M_K^2 - M_\pi^2)(k-p)_\rho\Big] \times \frac{W_S^{\text{non-loc}}(s;\nu)}{16\pi^2 M_K^2} \\ = i\int d^4x\, \langle \pi^0(p)|T\{j_\rho(0)\mathcal{L}_{\text{non-lept}}^{|\Delta S|=1}(x)\}|K_S(k)\rangle_{\overline{\text{MS}}},\end{aligned} \tag{10}$$

where j_ρ denotes the three-flavour electromagnetic current,

$$j_\rho(x) = \frac{2}{3}(\bar{u}\gamma_\rho u)(x) - \frac{1}{3}[(\bar{d}\gamma_\rho d)(x) + (\bar{s}\gamma_\rho s)(x)], \tag{11}$$

and $\mathcal{L}^{|\Delta S|=1}_{\text{non-lept}}$ is the order $\mathcal{O}(G_F)$ effective Lagrangian for $|\Delta S| = 1$ weak non-leptonic transitions below the charm-quark threshold,

$$\mathcal{L}^{|\Delta S|=1}_{\text{non-lept}} = -\frac{G_F}{\sqrt{2}} V_{us} V_{ud} \sum_{I=1}^{6} C_I(\nu) Q_I(\nu) + \text{H.c.} \tag{12}$$

The current–current four-quark operators Q_1 and Q_2 read (i and j are colour indices)

$$Q_1 = (\bar{s}^i u_j)_{V-A} (\bar{u}^j d_i)_{V-A}, \; Q_2 = (\bar{s}^i u_i)_{V-A} (\bar{u}^j d_j)_{V-A}. \tag{13}$$

The QCD-penguin operators $Q_{3,4,5,6}$ are provided in, e.g., Ref. [37]. In this same reference, the anomalous dimensions of these four-quark operators are also computed at next-to-leading order, which enables evolving the Wilson coefficients $C_I(\nu)$ from the electroweak scale $\nu = M_W$, where they are computed to order $\mathcal{O}(\alpha_s)$, down to the low scale $\nu \gtrsim 1$ GeV, thus including, in a renormalisation-group-improved perturbative way, with resummation of leading and next-to-leading logarithms, all contributions generated by the degrees of freedom between M_W and ν. For ν below 1 GeV, this perturbative treatment can no longer be trusted, and the contributions from degrees of freedom below 1 GeV are then provided by the non-perturbative matrix elements of the four-quark operators between hadronic states. While ChPT provides the contributions of the light pseudoscalar mesons to these matrix elements, it cannot account fully, that is otherwise than by largely unknown counter-terms [33,34], for those of the hadronic resonances in the 1 GeV region. As we will see, this is where the large-N_c limit steps in as an interesting alternative.

Notice that, although $\mathcal{L}^{|\Delta S|=1}_{\text{non-lept}}$ does not depend on the separation scale ν, the decomposition (9) does depend on it. This is a consequence of the fact that the definition of the non-local part $\mathcal{W}_S^{\text{non-loc}}$ of the form factor involves a time-ordered product that is singular at short distances [35,38], for instance (square brackets indicate colour-singlet quark bilinears)

$$\lim_{x\to 0} T\{j_\rho(x) Q_1(0)\} \sim -\frac{N_c}{18\pi^4} [\bar{s}\gamma_\mu(1-\gamma_5)d](0) \left(\delta^\mu_\rho \Box - \partial_\rho \partial^\mu\right) \frac{1}{(x^2)^2} + \cdots, \tag{14}$$

where the ellipsis denotes subdominant corrections. This requires that the time-ordered product in Equation (10) first be regularised (here, we have used dimensional regularisation) and eventually renormalised, here in the $\overline{\text{MS}}$ scheme, as indicated by the subscript on the right-hand side of Equation (10), leaving behind a dependence with respect to the associated renormalisation scale ν in $W_S^{\text{non-loc}}(s;\nu)$. In this renormalisation process, the divergent part in the time-ordered product in Equation (10) has to be absorbed by a local counter-term. The latter is provided by the Gilman–Wise operator Q_{7V} [39,40]; for a complete description of the form factor $\mathcal{W}_S(s)$ in the standard model, one also needs to consider contributions from

$$\mathcal{L}^{|\Delta S|=1}_{\text{lept}}(\nu) = -\frac{G_F}{\sqrt{2}} V_{us} V_{ud} [C_{7V}(\nu) Q_{7V} + C_{7A} Q_A] + \text{H.c.} \tag{15}$$

involving two additional local four-fermion operators with a mixed quark$\times$lepton content,

$$Q_{7V} = (\bar{\ell}\gamma_\mu \ell)[\bar{s}\gamma^\mu(1-\gamma_5)d], \quad Q_{7A} = (\bar{\ell}\gamma_\mu\gamma_5 \ell)[\bar{s}\gamma^\mu(1-\gamma_5)d]. \tag{16}$$

These operators are both finite, and the scale dependence of the Wilson coefficient $C_{7V}(\nu)$ can be interpreted as resulting from the absorption by a 'bare coupling' C_{7V}^{bare} of the local divergence of the time-ordered product in Equation (10). The scale dependence has to cancel between the two contributions once they are added up to form the physical form factor $\mathcal{W}_S(s)$ in Equation (9). A general discussion of how this happens, at least at next-to-leading order in perturbative QCD, can be found in Refs. [35,41], and it carries over to the limit $N_c \to \infty$ [36]. Finally, the operator Q_{7A} and its Wilson coefficients C_{7A} are defined at the electroweak scale and need not be renormalised in the standard model. This operator provides the contribution $\mathcal{A}_S^{\text{SD;A}}$ to the amplitude that was introduced in Equation (8),

$$\mathcal{A}_S^{\text{SD;A}} = \frac{G_F}{\sqrt{2}} V_{ud} V_{us} \text{Re}\, C_{7A} \times \bar{\text{u}}(p_{\ell^-})\gamma_\rho\gamma_5 \text{v}(p_{\ell^+}) \times [(k+p)^\rho f_+(s) + (k-p)^\rho f_-(s)]. \tag{17}$$

The form factors $f_\pm(s)$ are defined through (the minus sign is chosen such that the normalisation is $f_+(0) = 1$ in the limit of mass-degenerate u, d and s quarks)

$$\langle \pi^0(p)|[\bar{s}\gamma_\mu d](0)|K_S(k)\rangle = -[(k+p)_\mu f_+(s) + (k-p)_\mu f_-(s)]. \tag{18}$$

Having properly defined the form factor $\mathcal{W}_S(s)$ in terms of QCD matrix elements, we can now proceed with the evaluation of the latter, in the limit $N_c \to \infty$.

For the purpose of this study, we will concentrate on the contributions from the four-quark operators Q_1 and Q_2. Indeed, from the results provided in Ref. [37], one infers that the absolute values of the Wilson coefficients $C_{3,4,5,6}(\nu)$ at $\nu = 1$ GeV are smaller by at least one order of magnitude than the ones of the current–current operators at the same scale, so the contributions of the former can be neglected unless some of the corresponding matrix elements are enhanced. A more complete analysis [42], including all six operators, shows that this is not the case and that, in the large-N_c limit, the contribution from the QCD-penguin operators to the form factor $\mathcal{W}_S(s)$ is indeed marginal. Our task then reduces to computing the leading contributions, of order $\mathcal{O}(N_c)$, to the matrix elements $\langle \pi^0|T\{j_\mu(x)Q_{1,2}(0)\}|K_S\rangle$ when N_c becomes large. What makes this task possible is the fact that, in this limit, the four-quark operators factorise into the product of two-quark bilinears; gluon configurations that would break this factorisation are sub-leading in the $1/N_c$ expansion. To keep things as simple as possible, we only show the expressions obtained when the matrix γ_5 is handled in the 't Hooft–Veltman scheme [43,44]. Using naive dimensional regularisation [45] leads to additional terms in some of the matrix elements [36], to some extent compensated by the scheme dependence of the Wilson coefficients. For operator Q_1, one then obtains (from now on, all expressions, unless otherwise specified, will be understood to hold in the large-N_c limit, and the presence of sub-leading terms in the $1/N_c$ expansion will not be indicated explicitly)

$$\begin{aligned}
&\langle \pi^0(p)|T\{j_\rho(0)Q_1(x)\}|K^0(k)\rangle = \\
&= -\frac{2}{3}\langle \pi^0(p)|[\bar{s}\gamma_\nu d](0)|K^0(k)\rangle \times \langle 0|T\{[\bar{u}\gamma_\rho u](x)[\bar{u}\gamma_\nu u](0)\}|0\rangle \\
&\quad -\frac{2}{3}\langle 0|[\bar{s}\gamma_\mu\gamma_5 d](0)|K^0(k)\rangle \times \langle \pi^0(p)|T\{[\bar{u}\gamma_\rho u](x)[\bar{u}\gamma^\mu(1-\gamma_5)u](0)\}|0\rangle \\
&\quad +\frac{1}{3}\langle \pi^0(p)|[\bar{u}\gamma_\mu\gamma_5 u](0)|0\rangle \times \langle 0|T\{[\bar{d}\gamma_\rho d + \bar{s}\gamma_\rho s](x)[\bar{s}\gamma^\mu(1-\gamma_5)d](0)\}|K^0(k)\rangle.
\end{aligned} \tag{19}$$

The correlator appearing in the first term on the right-hand side of this expression, of the vacuum-polarization type, is divergent. This divergence reflects the short-distance singularity of time-ordered product (14) and has to be subtracted in the $\overline{\text{MS}}$ scheme, as explained previously. One also immediately notices that Q_2 cannot contribute to $\mathcal{W}_S(s)$ in the large-N_c limit. The reason for this is easy to understand: as can be seen from Equation (13), the operator Q_2 factorises into the product of two colour-singlet *charged* currents, $[\bar{s}\gamma_\mu(1-\gamma_5)u]$ and $[\bar{u}\gamma^\mu(1-\gamma_5)d]$, and it is not possible to construct non-vanishing matrix elements for these currents with only a neutral pion and a neutral kaon at disposal. After having used invariance under parity, charge conjugation, isospin symmetry, and applied Ward identities [36], the matrix element of the operator Q_1 in the large-N_c limit can be expressed in terms of the pion and kaon decay constants F_π and F_K, respectively, together with

(i) the properly renormalised vacuum-polarisation correlation function

$$i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{u}\gamma_\mu u](x)[\bar{u}\gamma_\nu u](0)\}|0\rangle_{\overline{\text{MS}}} = (q_\mu q_\nu - q^2\eta_{\mu\nu})\Pi_{\overline{\text{MS}}}(q^2;\nu); \tag{20}$$

(ii) the form factor $f_+(s)$ already defined in Equation (18);
(iii) the two vertex functions

$$\begin{aligned}\Gamma_\rho(q,k) &= i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{d}\gamma_\rho d](x)[\bar{s}i\gamma_5 d](0)\}|K^0(k)\rangle,\\ \tilde{\Gamma}_\rho(q,k) &= i\int d^4x\, e^{iq\cdot x}\langle 0|T\{[\bar{s}\gamma_\rho s](x)[\bar{s}i\gamma_5 d](0)\}|K^0(k)\rangle.\end{aligned} \tag{21}$$

These vertex functions each have a kaon pole at $(q-k)^2 = M_K^2$, whose residues involve kaon form factors $F_d^{K^0}(q^2)$ and $F_s^{K^0}(q^2)$, defined through the two matrix elements $\langle K^0|\bar{d}\gamma_\rho d|K^0\rangle$ and $\langle K^0|\bar{s}\gamma_\rho s|K^0\rangle$, respectively, with normalisations chosen such that $F_d^{K^0}(0) = -F_s^{K^0}(0) = 1$. Combined with the Ward identities these vertex functions satisfy, this leads to the convenient representations (m_s stands for the mass of the strange quark, while $\hat{m}$ denotes the common mass of the up and down quarks in the isospin limit)

$$\begin{aligned}(m_s+\hat{m})\Gamma_\rho(q,k) &= \sqrt{2}F_K M_K^2\frac{(2k-q)_\rho}{(q-k)^2-M_K^2}F_d^{K^0}(q^2)\\ &\quad+\sqrt{2}F_K M_K^2\frac{F_d^{K^0}(q^2)-1}{q^2}q_\rho\\ &\quad+\sqrt{2}[q^2k_\rho-(q\cdot k)q_\rho]\mathcal{P}(q^2,(q-k)^2),\\ (m_s+\hat{m})\tilde{\Gamma}_\rho(q,k) &= \sqrt{2}F_K M_K^2\frac{(2k-q)_\rho}{(q-k)^2-M_K^2}F_s^{K^0}(q^2)\\ &\quad+\sqrt{2}F_K M_K^2\frac{F_s^{K^0}(q^2)+1}{q^2}q_\rho\\ &\quad+\sqrt{2}[q^2k_\rho-(q\cdot k)q_\rho]\tilde{\mathcal{P}}(q^2,(q-k)^2).\end{aligned} \tag{22}$$

Putting everything together, the expression of the form factor $\mathcal{W}_S(s)$ at leading-order in the $1/N_c$ expansion reads

$$\begin{aligned}\frac{\mathcal{W}_S(s)}{16\pi^2M_K^2} &= -\frac{G_F}{\sqrt{2}}V_{us}V_{ud}\bigg\{f_+(s)\left[\frac{2}{3}C_1\Pi_{\overline{\rm MS}}(s;\nu)+\frac{{\rm Re}C_{7V}(\nu)}{4\pi\alpha}\right]\\ &\quad+\frac{2}{3}C_1\left[\frac{F_\pi F_K M_K^2}{M_K^2-M_\pi^2}\,\frac{F_d^{K^0}(s)+F_s^{K^0}(s)}{s}-\frac{F_\pi}{2}\mathcal{P}(s,M_\pi^2)-\frac{F_\pi}{2}\tilde{\mathcal{P}}(s,M_\pi^2)\right]\bigg\}.\end{aligned} \tag{23}$$

It can be shown that, in the large-N_c limit, this last expression does not depend on ν [36]. Moreover, the three form factors $f_+(q^2)$, $F_d^{K^0}(q^2)$, $F_s^{K^0}(q^2)$ and the vacuum-polarisation function $\Pi_{\overline{\rm MS}}(s;\nu)$ consist of an infinite number of poles due to zero-width mesonic resonances [25,26]. The three form factors behave in QCD like $\sim 1/q^2$ for large space-like values of q^2. Due to this smooth asymptotic behaviour, it is justified to retain only the lowest-lying resonance in each case [46], i.e., $K^*(892)$ for f_+, $\rho(770)/\omega(782)$ for $F_d^{K^0}$, and $\phi(1020)$ for $F_s^{K^0}$, i.e., (we take $M_\omega = M_\rho$),

$$f_+(q^2) = \frac{M_{K^*}^2}{M_{K^*}^2-q^2},\ F_s^{K^0}(q^2) = \frac{M_\phi^2}{q^2-M_\phi^2},\ F_d^{K^0}(q^2) = \frac{M_\rho^2}{M_\rho^2-q^2}. \tag{24}$$

On the other hand, the function $\Pi_{\overline{\rm MS}}(q^2;\nu)$ behaves as $\sim \ln(-q^2/\nu^2)$ when $q^2 \to -\infty$. Clearly, such logarithmic behaviour cannot be reproduced by a single resonance pole, and not even by a finite number of such poles, so a representation in terms of an infinite number of $J^{PC} = 1^{--}$ states cannot be avoided [26]. Fortunately, such representations have been discussed and constructed in the literature; see for instance Ref. [41] and the articles quoted therein. We will adopt the expression

$$\Pi_{\overline{\rm MS}}(q^2;\nu) = \frac{f_\rho^2 M_\rho^2}{M_\rho^2 - q^2} + \frac{9 f_\omega^2 M_\omega^2}{M_\omega^2 - q^2} + \frac{N_c}{12\pi^2}\left\{ -\ln(M^2/\nu) + \frac{5}{3} - \psi\left(3 - \frac{q^2}{M^2}\right)\right\}, \tag{25}$$

where ψ denotes the di-gamma function, and we have not shown $\mathcal{O}(\alpha_s N_c)$ corrections, which are known and included in the numerical analysis. The poles (in the large-N_c limit) due to the ρ and ω states have been shown explicitly. The couplings $f^2_{\rho,\omega}$ can be determined from the experimental decay widths $\Gamma(\rho,\omega \to e^+e^-)$. For $q^2 < 0$, $\psi(3 - q^2/M^2)$ is a smooth function, which has logarithmic asymptotic behaviour when $q^2 \to -\infty$,

$$\psi\left(3 - \frac{q^2}{M^2}\right) \sim \ln(-q^2/M^2) - \frac{5}{2}\frac{M^2}{q^2} + \mathcal{O}(M^4/q^4) \tag{26}$$

thus reproducing the leading perturbative expression of $\Pi_{\overline{\rm MS}}(q^2;\nu)$. For $q^2 > 0$, the di-gamma function sums a series of equidistant poles located at the values $q^2 = M_n^2 \equiv (n+2)M^2$,

$$\psi\left(3 - \frac{q^2}{M^2}\right) = -\gamma_E + \frac{3}{2} + \sum_{n\geq 1}\frac{1}{n+2}\frac{q^2}{q^2 - M_n^2}, \tag{27}$$

where γ_E is the Euler constant. We still need to fix the value of the mass scale M. This can be accomplished upon using the following constraint on the Adler function $\mathcal{A}(q^2) \equiv -q^2(\partial\Pi(q^2)/\partial q^2)$; for large Euclidian values of momentum q, the behaviour of $\mathcal{A}(q^2)$ in QCD cannot display a term $\sim 1/q^2$ in the chiral limit [46]. Neglecting $\mathcal{O}(\alpha_s N_c)$ corrections, this condition requires (we have taken $f_\omega M_\omega \sim f_\rho M_\rho/3$, as required in the combined large-N_c and isospin limits and as also reproduced by data)

$$M^2 = \frac{16\pi^2}{5}\frac{3}{N_c} f_\rho^2 M_\rho^2. \tag{28}$$

For $N_c = 3$ and $f_\rho M_\rho \sim 154$ MeV, this yields $M \sim 0.87$ GeV and $M_1 \sim 1.5$ GeV, which are quite reasonable values, the last one being comparable to the mass of the $\rho(1450)$, the first $J^{PC} = 1^{--}$ resonance after the $\rho(770)$ [21].

It remains to discuss the functions $\mathcal{P}(q^2,(q-k)^2)$ and $\tilde{\mathcal{P}}(q^2,(q-k)^2)$. These two functions account for the poles produced by zero-width radial excitations of the kaon, i.e., K', K'',.... The first of these states can, for instance, be identified with the $K(1460)$ resonance in the real world where $N_c = 3$. Two important observations concerning them can be made and exploited [42]. First, the behaviour of $\Gamma_\rho(q,k)$ and $\tilde{\Gamma}_\rho(q,k)$ at large space-like values of q^2, as determined by the operator-product expansion, shows that the leading short-distance term is saturated by the contribution due to their longitudinal parts, i.e., the kaon poles. Therefore, the functions $\mathcal{P}(q^2,(q-k)^2)$ and $\tilde{\mathcal{P}}(q^2,(q-k)^2)$ provide only subdominant contributions at short distances. Second, the poles due to the radial excitations of the kaon will come with the factors of the kaon poles replaced by $F_{K'}M^2_{K'}/[((q-k)^2 - M^2_{K'}]$, where $F_{K'}$ is the decay constant of the radial excitation K' of mass $M_{K'}$. Since these states do not become Goldstone bosons in the chiral limit, $F_{K'}$ must vanish linearly with vanishing quark masses. Indeed, estimates based on QCD sum rules [47–49] provide values much smaller than the kaon decay constant F_K for the first of these radial excitations, e.g., $F_{K'} = 21.4(2.8)$ MeV [48,49], and an even smaller value for the second radial excitation. In addition, the factor $M^2_{K'}$ in the residue of the pole is cancelled by the denominator when one eventually takes $q = k - p$, so $F_{K'}M^2_{K'}/[((q-k)^2 - M^2_{K'}]$ becomes $\sim -F_{K'}$. Barring any large enhancement due to the electromagnetic transition form factors $F^{K^0K'}_{u,s}(q^2)$ that replace $F^{K^0}_{u,s}(q^2)$, this indicates that the contributions of $\mathcal{P}(q^2,(q-k)^2)$ and

$\tilde{\mathcal{P}}(q^2, (q-k)^2)$ are highly suppressed as compared to the contributions from the kaon poles, which leads us to make the approximations $\mathcal{P}(q^2, (q-k)^2) \simeq 0$, $\tilde{\mathcal{P}}(q^2, (q-k)^2) \simeq 0$.

3. Phenomenological Consequences

We now have all the elements at our disposal in order to answer the three questions listed at the beginning of this paper. We use the values of the Wilson coefficients at the scale $\nu = 1$ GeV provided in Ref. [37], and the values of the remaining quantities are taken from Ref. [21]. The values shown below result from the average of those obtained with the 't Hooft–Veltman scheme and with the naive dimensional regularisation scheme.

- The predictions for the branching ratios read

$$\begin{aligned} \mathrm{Br}(K_S \to \pi^0 e^+ e^-)|_{m_{ee} > 165\ \mathrm{MeV}} &= 2.9(1.0) \cdot 10^{-9} \\ \mathrm{Br}(K_S \to \pi^0 e^+ e^-) &= 5.1(1.7) \cdot 10^{-9}, \\ \mathrm{Br}(K_S \to \pi^0 \mu^+ \mu^-) &= 1.3(0.4) \cdot 10^{-9}, \end{aligned} \tag{29}$$

where a conservative relative uncertainty of $\mathcal{O}(1/N_c) \sim 30\%$, accounting for sub-leading effects in the $1/N_c$ expansion, has been applied with $N_c = 3$. The first value agrees well with the measurement in the NA48/1 experiment [50] (the first error comes from statistics and the second from systematics)

$$\mathrm{Br}(K_S \to \pi^0 e^+ e^-)|_{m_{ee} > 165\ \mathrm{MeV}} = (3.0^{+1.5}_{-1.2} \pm 0.2) \cdot 10^{-9}. \tag{30}$$

When extrapolated to the full range of the di-lepton invariant mass m_{ee} with a form factor equal to unity (i.e., with $a_S = 1$, $b_S = 0$, and no pion loop), the total branching fraction is quoted as $\mathrm{Br}(K_S \to \pi^0 e^+ e^-) = (5.8^{+2.8}_{-2.3} \pm 0.8) \cdot 10^{-9}$ [50], which also agrees rather well with the value in Equation (29). The agreement is less optimal in the case of the decay into a muon pair, where the experimental value obtained by the NA48/1 collaboration is provided as [51] $\mathrm{Br}(K_S \to \pi^0 \mu^+ \mu^-) = (2.9^{+1.5}_{-1.2}(\mathrm{stat}) \pm 0.2(\mathrm{syst})) \cdot 10^{-9}$, but the uncertainties are still large. Finally, we also mention that, within the range $s \in [0, M_K^2]$, which covers the phase space of the $K_{S,L} \to \pi^0 \ell^+ \ell^-$ decays, the form factor (23) is well described by the quadratic polynomial

$$\mathcal{W}_S(s) \sim G_F[0.92 M_K^2 + 0.64 s + 0.39 s^2 / M_K^2]. \tag{31}$$

- The function $\mathcal{W}_S(s)$ being positive, cf. Equation (31), the coefficients $C_{\mathrm{int}}^{(\ell)}$ in Equation (4) are also positive. Numerically, we obtain

$$C_{\mathrm{int}}^{(e)} = +7.8(2.6)\, \frac{y_{7V}}{\alpha}, \; C_{\mathrm{int}}^{(\mu)} = +1.9(0.6)\, \frac{y_{7V}}{\alpha}, \tag{32}$$

where we have written (y_{7V} is positive [37])

$$V_{ud} V_{us} \mathrm{Im}\, C_{7V}(\nu = 1\ \mathrm{GeV}) = -(\mathrm{Im}\, \lambda_t) y_{7V}. \tag{33}$$

The interference between direct and indirect CP violation in the branching ratio for $K_L \to \pi^0 \ell^+ \ell^-$ is therefore unambiguously predicted to be constructive in the large-N_c limit of QCD.
- The amplitude of the CP-violating transition $K_2^0 \to \pi^0 \gamma^* \to \pi^0 \ell^+ \ell^-$ has the same structure as provided in Equation (8), provided one makes the replacements $C_1 \to 0$, $\mathrm{Re}\, C_{7X} \to i \mathrm{Im}\, C_{7X}$, $X = V, A$, in Equations (17) and (23). In addition, as already mentioned, the matrix elements of the QCD penguin operators show no particular enhancement as compared to the matrix element of Q_1 [42], while the imaginary parts of their Wilson coefficients at the scale $\nu = 1$ GeV are about one order of magnitude smaller (in absolute value) than $|\mathrm{Im}\, C_{7X}|/\alpha$ at the same scale [37]. The

approximation consisting of keeping only the contribution from the Gilman–Wise operators is therefore also supported by the large-N_c limit of QCD.

Before concluding, let us briefly discuss the case of $\mathcal{W}_+(s)$ in the context of the large-N_c limit. The main difference with $\mathcal{W}_S(s)$ lies in the fact that now operator Q_2 will contribute. Actually, the contribution of operator Q_1 to $\mathcal{W}_+(s)$ is now limited to the term proportional to $\Pi_{\overline{\rm MS}}(s;\nu)$ in Equation (23), whereas expressions similar to the remaining terms in this equation will instead be produced by Q_2. Since $C_1(\nu) \simeq -C_2(\nu)/2$ at $\nu \simeq 1$ GeV, this leads to an almost complete numerical cancellation between the two contributions [36], leaving only the small contributions from the QCD penguin operators as a remainder. The almost vanishing values of a_+ and b_+ predicted by the large-N_c limit thus do not at all account for the measured values [52], and sub-leading terms in the $1/N_c$ expansion must become important in this case. This is quite in line with the result of Ref. [35], where a crude unsubtracted dispersive evaluation of the contribution from two-pion states to $\mathcal{W}_+(s)$, suppressed in the large-N_c limit but this time enhanced by the $\Delta I = 1/2$ rule, produced values of a_+ and b_+ already reasonably close to the experimental ones.

4. Summary and Conclusions

To summarise, we have outlined the computation of the amplitudes for the kaon decay modes, $K_{S,L} \to \pi^0\gamma^* \to \pi^0\ell^+\ell^-$, in the large-$N_c$ limit of QCD. We have shown that this framework is predictive as it enables answering a few questions of phenomenological relevance for the possibility to experimentally probe the standard model's flavour structure at short distances. A more detailed account of the calculations and further implications will be provided in Ref. [36]. For completeness, let us also mention that the proposal [38,53,54] to investigate the $K \to \pi\ell^+\ell^-$ decay modes in the framework of lattice QCD is being actively pursued by the RBC and UKQCD collaborations. A first result for $\mathcal{W}_+(s)$ at $s/M_K^2 = 0.013(2)$ with physical values of the pion and kaon masses was published recently [55]. It corresponds to only a single lattice spacing and still shows quite large uncertainties due to the difficulty in extracting the signal from the statistical noise. Substantial improvements are, however, expected during the next decade for this and for other rare kaon decay modes [3,6]. In the meantime, the quest for a better theoretical and phenomenological understanding of rare kaon decay modes is certainly worth pursuing as well. The large-N_c limit may shed light on other processes than the ones studied here and bring to the fore interesting dynamical aspects and/or quantitative information. Of course, cancellations can also occur in other amplitudes than the one for $K^\pm \to \pi^\pm\ell^+\ell^-$, but a case-by-case study is probably required to eventually reveal which observables are actually affected.

Author Contributions: Conceptualization, M.K.; methodology, M.K.; validation, G.D. and M.K.; writing—original draft preparation, M.K.; writing—review and editing, G.D. and M.K. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in the study are included in the article. Further inquiries can be directed to the corresponding author.

Acknowledgments: One of us (M.K.) would like to thank the Department of Physics of the Università degli Studi Federico II di Napoli for the warm hospitality extended to him, and the Sezione di Napoli of INFN for financial support.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Alves Junior, A.A.; Bettler, M.O.; Rodríguez, A.B.; Vidal, A.C.; Chobanova, V.; Vidal, X.C.; Contu, A.; D'Ambrosio, G.; Dalseno, J.; Dettori, F.; et al. Prospects for Measurements with Strange Hadrons at LHCb. *J. High Energy Phys.* **2019**, *5*, 48. [CrossRef]
2. Goudzovski, E.; Passemar, E.; Aebischer, J.; Banerjee, S.; Bryman, D.; Buras, A.; Cirigliano, V.; Christ, N.; Dery, A.; Dettori, F.; et al. Weak Decays of Strange and Light Quarks. *arXiv* **2022**, arXiv:2209.07156.

3. Blum, T.; Boyle, P.; Bruno, M.; Christ, N.; Erben, F.; Feng, X.; Guelpers, V.; Hill, R.; Hodgson, R.; Hoying, D.; et al. Discovering new physics in rare kaon decays. *arXiv* **2022**, arXiv:2203.10998.
4. Cortina Gil, E. et al. [HIKE]. HIKE, High Intensity Kaon Experiments at the CERN SPS: Letter of Intent. *arXiv* **2022**, arXiv:2211.16586.
5. Aebischer, J.; Buras, A.J.; Kumar, J. On the Importance of Rare Kaon Decays: A Snowmass 2021 White Paper. *arXiv* **2022**, arXiv:2203.09524.
6. Anzivino, G.; Cuendis, S.A.; Bernard, V.; Bijnens, J.; Bloch-Devaux, B.; Bordone, M.; Brizioli, F.; Brod, J.; Camalich, J.M.; Ceccucci, A.; et al. Workshop summary: Kaons@CERN 2023. *Eur. Phys. J. C* **2024**, *84*, 377. [CrossRef]
7. D'Ambrosio, G.; Mahmoudi, F.; Neshatpour, S. Beyond the Standard Model prospects for kaon physics at future experiments. *J. High Energy Phys.* **2024**, *2*, 166. [CrossRef]
8. Nanjo, H. [KOTO]. KOTO II at J-PARC: Toward measurement of the branching ratio of $K_L \to \pi^0 \nu\bar{\nu}$. *J. Phys. Conf. Ser.* **2023**, *2446*, 012037. [CrossRef]
9. Glashow, S.L.; Iliopoulos, J.; Maiani, L. Weak Interactions with Lepton-Hadron Symmetry. *Phys. Rev. D* **1970**, *2*, 1285. [CrossRef]
10. Glashow, S.L.; Weinberg, S. Natural Conservation Laws for Neutral Currents. *Phys. Rev. D* **1977**, *15*, 1958. [CrossRef]
11. Buras, A.J.; Venturini, E. Searching for New Physics in Rare K and B Decays without $|V_{cb}|$ and $|V_{ub}|$ Uncertainties. *Acta Phys. Polon. B* **2021**, *53*, A1. [CrossRef]
12. Brod, J.; Gorbahn, M.; Stamou, E. Updated Standard Model Prediction for $K \to \pi\nu\bar{\nu}$ and ϵ_K. *arXiv* **2021**, arXiv:2105.02868.
13. Ahn, J.K. [KOTO]. Search for the $K_L \to \pi^0 \nu\bar{\nu}$ and $K_L \to \pi^0 X^0$ decays at the J-PARC KOTO experiment. *Phys. Rev. Lett.* **2019**, *122*, 021802. [CrossRef] [PubMed]
14. Cortina Gil, E. et al. [NA62]. Measurement of the very rare $K^+ \to \pi^+ \nu\bar{\nu}$ decay. *J. High Energy Phys.* **2021**, *6*, 93.
15. Donoghue, J.F.; Holstein, B.R.; Valencia, G. $K_L \to \pi^0 e^+ e^-$ as a Probe of CP Violation. *Phys. Rev. D* **1987**, *35*, 2769. [CrossRef]
16. Buchalla, G.; D'Ambrosio, G.; Isidori, G. Extracting short distance physics from $K_{L,S} \to \pi^0 e^+ e^-$ decays. *Nucl. Phys. B* **2003**, *672*, 387–408. [CrossRef]
17. Isidori, G.; Smith, C.; Unterdorfer, R. The Rare decay $K_L \to \pi^0 \mu^+ \mu^-$ within the SM. *Eur. Phys. J. C* **2004**, *36*, 57–66. [CrossRef]
18. Mescia, F.; Smith, C.; Trine, S. $K_L \to \pi^0 e^+ e^-$ and $K_L \to \pi^0 \mu^+ \mu^-$: A Binary star on the stage of flavor physics. *J. High Energy Phys.* **2006**, *8*, 88. [CrossRef]
19. Dib, C.; Dunietz, I.; Gilman, F.J. CP Violation in the $K_L \to \pi^0 \ell^+ \ell^-$ Decay Amplitude for Large M_t. *Phys. Lett. B* **1989**, *218*, 487–492. [CrossRef]
20. Buchalla, G.; Buras, A.J. K —> pi neutrino anti-neutrino and high precision determinations of the CKM matrix. *Phys. Rev. D* **1996**, *54*, 6782–6789. [CrossRef]
21. Workman, R.L. et al. [Particle Data Group]. Review of Particle Physics. *Prog. Theor. Exp. Phys.* **2022**, *2022*, 083C01.
22. D'Ambrosio, G.; Ecker, G.; Isidori, G.; Portolés, J. The Decays $K \to \pi l^+ l^-$ beyond leading order in the chiral expansion. *J. High Energy Phys.* **1998**, *8*, 4. [CrossRef]
23. Ecker, G.; Pich, A.; de Rafael, E. Radiative Kaon Decays and CP Violation in Chiral Perturbation Theory. *Nucl. Phys. B* **1988**, *303*, 665–702. [CrossRef]
24. Greenlee, H.B. Background to $K_L^0 \to \pi^0 ee$ From $K_L^0 \to \gamma\gamma ee$. *Phys. Rev. D* **1990**, *42*, 3724. [CrossRef] [PubMed]
25. 't Hooft, G. A Planar Diagram Theory for Strong Interactions. *Nucl. Phys. B* **1974**, *72*, 461–473. [CrossRef]
26. Witten, E. Baryons in the $1/n$ Expansion. *Nucl. Phys. B* **1979**, *160*, 57–115. [CrossRef]
27. Gasser, J.; Leutwyler, H. Chiral Perturbation Theory: Expansions in the Mass of the Strange Quark. *Nucl. Phys. B* **1985**, *250*, 465–516. [CrossRef]
28. Cronin, J.A. Phenomenological model of strong and weak interactions in chiral U(3) x U(3). *Phys. Rev.* **1967**, *161*, 1483. [CrossRef]
29. Kambor, J.; Missimer, J.H.; Wyler, D. The Chiral Loop Expansion of the Nonleptonic Weak Interactions of Mesons. *Nucl. Phys. B* **1990**, *346*, 17–64. [CrossRef]
30. Kambor, J.; Missimer, J.H.; Wyler, D. $K \to 2\pi$ and $K \to 3\pi$ decays in next-to-leading order chiral perturbation theory. *Phys. Lett. B* **1991**, *261*, 496–503. [CrossRef]
31. Esposito-Farèse, G. Right invariant metrics on SU(3) and one loop divergences in chiral perturbation theory. *Z. Phys. C* **1991**, *50*, 255–274. [CrossRef]
32. Ecker, G.; Kambor, J.; Wyler, D. Resonances in the weak chiral Lagrangian. *Nucl. Phys. B* **1993**, *394*, 101–138. [CrossRef]
33. Ecker, G.; Pich, A.; de Rafael, E. $K \to \pi$Lepton$^+$Lepton$^-$ Decays in the Effective Chiral Lagrangian of the Standard Model. *Nucl. Phys. B* **1987**, *291*, 692–719. [CrossRef]
34. Ananthanarayan, B.; Imsong, I.S. The 27-plet contributions to the CP-conserving $K \to \pi l^+ l^-$ decays. *J. Phys. G* **2012**, *39*, 095002. [CrossRef]
35. D'Ambrosio, G.; Greynat, D.; Knecht, M. On the amplitudes for the CP-conserving $K^\pm(K_S) \to \pi^\pm(\pi^0)\ell^+\ell^-$ rare decay modes. *J. High Energy Phys.* **2019**, *2*, 49. [CrossRef]
36. D'Ambrosio, G.; Knecht, M. Centre de Physique Théorique, Aix-Marseille Univ./Univ. de Toulon/CNRS (UMR 7332), CNRS-Luminy Case 907, 13288 Marseille Cedex 9, France. 2024, *article in preparation*.
37. Buras, A.J.; Lautenbacher, M.E.; Misiak, M.; Münz, M. Direct CP violation in $K_L \to \pi^0 e^+ e^-$ beyond leading logarithms. *Nucl. Phys. B* **1994**, *423*, 349–383. [CrossRef]
38. Isidori, G.; Martinelli, G.; Turchetti, P. Rare kaon decays on the lattice. *Phys. Lett. B* **2006**, *633*, 75–83. [CrossRef]

39. Gilman, F.J.; Wise, M.B. Effective Hamiltonian for Delta s = 1 Weak Nonleptonic Decays in the Six Quark Model. *Phys. Rev. D* **1979**, *20*, 2392. [CrossRef]
40. Gilman, F.J.; Wise, M.B. $K \to \pi e^+ e^-$ in the Six Quark Model. *Phys. Rev. D* **1980**, *21*, 3150. [CrossRef]
41. D'Ambrosio, G.; Greynat, D.; Knecht, M. Matching long and short distances at order $\mathcal{O}(\alpha_s)$ in the form factors for $K \to \pi \ell^+ \ell^-$. *Phys. Lett. B* **2019**, *797*, 134891. [CrossRef]
42. Knecht, M. Centre de Physique Théorique, Aix-Marseille Univ./Univ. de Toulon/CNRS (UMR 7332), CNRS-Luminy Case 907, 13288 Marseille Cedex 9, France. Unpublished Notes, 2024.
43. 't Hooft, G.; Veltman, M.J.G. Regularization and Renormalization of Gauge Fields. *Nucl. Phys. B* **1972**, *44*, 189–213. [CrossRef]
44. Breitenlohner, P.; Maison, D. Dimensional Renormalization and the Action Principle. *Commun. Math. Phys.* **1977**, *52*, 11–38. [CrossRef]
45. Chanowitz, M.S.; Furman, M.; Hinchliffe, I. The Axial Current in Dimensional Regularization. *Nucl. Phys. B* **1979**, *159*, 225–243. [CrossRef]
46. Peris, S.; Perrottet, M.; de Rafael, E. Matching long and short distances in large N(c) QCD. *J. High Energy Phys.* **1998**, *05*, 11. [CrossRef]
47. Narison, S.; Paver, N.; Treleani, D. Properties of the meson system and chiral symmetry breaking parameters from quantum chromodynamics. *Nuovo Cim. A* **1983**, *74*, 347–363. [CrossRef]
48. Maltman, K.; Kambor, J. On the longitudinal contributions to hadronic tau decay. *Phys. Rev. D* **2001**, *64*, 93014. [CrossRef]
49. Maltman, K.; Kambor, J. Decay constants, light quark masses and quark mass bounds from light quark pseudoscalar sum rules. *Phys. Rev. D* **2002**, *65*, 074013. [CrossRef]
50. Batley, J.R. et al. [NA48/1]. Observation of the rare decay $K_S \to \pi^0 e^+ e^-$. *Phys. Lett. B* **2003**, *576*, 43–54. [CrossRef]
51. Batley, J.R. et al. [NA48/1]. Observation of the rare decay $K_S \to \pi^0 \mu^+ \mu^-$. *Phys. Lett. B* **2004**, *599*, 197–211. [CrossRef]
52. Cortina Gil, E. et al. [NA62]. A measurement of the $K^+ \to \pi^+ \mu^+ \mu^-$ decay. *J. High Energy Phys.* **2022**, *11*, 11.
53. Christ, N.H. et al. [RBC and UKQCD]. Prospects for a lattice computation of rare kaon decay amplitudes: $K \to \pi \ell^+ \ell^-$ decays. *Phys. Rev. D* **2015**, *92*, 094512. [CrossRef]
54. Christ, N.H.; Feng, X.; Juttner, A.; Lawson, A.; Portelli, A.; Sachrajda, C.T. First exploratory calculation of the long-distance contributions to the rare kaon decays $K \to \pi \ell^+ \ell^-$. *Phys. Rev. D* **2016**, *94*, 114516. [CrossRef]
55. Boyle, P.A. et al. [RBC and UKQCD]. Simulating rare kaon decays $K^+ \to \pi^+ \ell^+ \ell^-$ using domain wall lattice QCD with physical light quark masses. *Phys. Rev. D* **2023**, *107*, L011503. [CrossRef]