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7 How Can Group Theory be Generalized so Perhaps Providing Further Information About Our Universe?

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Abstract. Group theory is very familiar, perhaps too much so. We are thus prejudiced about it, leading to views that are far too narrow. Yet it is significantly richer than usually realized ([13]). Here we wish to understand the restrictions giving the familiar forms and how by changing these we can get added richness. Might these add to our knowledge of nature? A purpose of this note is to stimulate thinking about this.

7.1 Geometry, through its transformations groups, is very information, but so far not enough

It is clear that much (all?) of physics is determined by geometry, especially through its transformation groups ([13]; [4]; [5]; [7]; [8]; [9]; [10]; [11]; [12]; [3]; [6]). Yet it is necessary to go much further. Can additional progress be made using group theory? This is a very open question but worth exploring. One aspect to be explored is whether group theory itself can be generalized. That could be of interest for purely mathematical reasons. And it has many applications. Generalizing it can thus be useful in various ways. This we wish to explore here.

7.2 What is the best way to try to understand fundamental physics?

What is the best approach to try to understand physics? The big fad nowadays is to come up with the wildest, most unlikely ideas, ones furthest from reality, ones totally unrelated to anything known, ones having no rationale whatever. History and common sense show that this approach is destined to lead nowhere except to even more wild ideas (as it has).

Those who do that will find themselves badly cut by Occam's razor. Unfortunately a large part of the physics community is doing just that. Applying Occam's razor to the physics community will greatly help physics to advance.

Another approach that is very likely to lead to failure, certainly if there is no other rationale for it, is to base laws on how we measure, on ourselves. We do not determine the laws of nature (something many scientists, especially physicists, do not believe). How we measure is limited by physical laws, but does not limit them. Studying measurement can help us understand physics, but cannot determine it.

What then shall we do, what approach shall we use? The best approach is the most conservative using requirements that are certain, or at least likely, to be correct, or ones that deviate the least from these.

7.3 Reasonable requirements for developing theories

What requirements can we impose?

First is consistency. Fundamental physical theories must be consistent. (Phenomenological theories, classical physics is an example, can be inconsistent mishmashes.) This is more difficult than it might seem, so can be quite powerful.

Geometry imposes requirements, restrictions. Physics takes place in geometry so must be in accord with the rules it leads to. This also is powerful as we have seen (particularly) in the references.

What can we say about geometry? We always assume that it is a manifold (locally flat) and that its coordinates are real numbers (rather say than complex numbers or quaternions). It is very unlikely that physics would be possible otherwise, but this can be investigated. A fundamental property of geometry is its dimension. However it has long been known that physics would be impossible unless the dimension is 3+1 ([4]; [13]). This is required by consistency, illustrating its importance, for only with this dimension is a consistent physics possible.

7.4 What can we say about geometry?

Is space curved? The curvature of space is given by a function over it, the connection ([5]). Can every space that is a manifold be regarded as flat but with a function, the connection, so that all curved spaces can be reduced to flat ones with different such functions? This is an interesting question that we raise but do not try to answer here. Also (many) curved spaces can be mapped (in reasonable ways) into flat ones ([8]). Thus we consider only flat spaces, but these questions should be looked into.

What properties do flat spaces have? Beyond the reality of coordinates and the dimension is a most fundamental property: the transformation groups of the spaces (which are not symmetry groups ([13]), although it is quite interesting that they are that also). For our space, apparently the only one in which physics is possible, the largest symmetry group is the conformal group ([8]), which has subgroups the Poincaré group, its subgroup the Lorentz group and the subgroup of that, the rotation group (SO(3)). The last gives that angular momentum must be integral or half-odd-integral ([3]), illustrating how transformations limit physics. The massless representations of the Poincaré group determine electromagnetism and gravitation ([5]). Clearly these are quite informative, but clearly insufficient. It is possible that the conformal group can also be quite informative but how is less clear.

7.5 How might group theory be generalized?

Can we go further? What we wish to do here is study whether what is known about group theory can be generalized. We are all too familiar with semisimple groups, like the rotation and Lorentz groups. But group theory is far richer, even for these groups ([5]; [8]). Perhaps it is richer than we realize. That is what we consider here.

This may not have anything to do with fundamental laws. But it helps to understand group theory, decreasing prejudice and broadening our views, and produces interesting mathematical results. And they are likely to lead to useful, even important, applications.

7.6 Indexed groups

Start by considering the curve, $x = r\cos\theta$, $y = r\sin\theta$, which describes a circle. We move around a circle using the two-dimensional rotation group O(2). By putting a constant into the representation matrix we can generate an ellipse. But what about, say, the curve $x = r(\cos\theta)3$, $y = r(\sin\theta)3$. What set of transformations moves along this curve and why don't they form a group? Clearly there is an identity, we do not have to move, and for every transformation there is an inverse. Moreover the product of two transformations is a transformation; if we move from A to B and the from B to C, we can find a transformation from A to C. However the transformations are not associative. It is for this reason that they do not form a group. The transformation from B to C depends on where B is (it is in a sense history dependent, depending on the previous transformations). That is the operator going from B to C has a form that depends on B, unlike rotations. This causes associativity to fail.

Thus for a circle the product of the transformation matrix for θ_1 and for θ_2 is that for $\theta_1 + \theta_2$, which is not true for this curve.

While there are transformations along any (reasonable) n-dimensional surface it is only in special cases that they form a group (of the form usually considered). This emphasizes the relevance of associativity and the restrictions it places. Many of the properties of groups and their representations come from associativity. While restricting, it also allows us to obtain properties that are so useful in applications of groups.

How do we deal these more general transformations, say ones along arbitrary surfaces? We introduce the concept of indexed groups. To do this we assume that the transformations can be mapped (at least) one-to-one onto a *group* of transformations. For each general transformation we have a corresponding group element and this element is the index of the general transformation. Thus for three dimensions the index can be an element of the rotation group. For a group the matrix representing the transformation that is the product of two is the matrix product of the matrices of the two transformations. It is here that indexed representations differ. For these the matrix labeling the product is the matrix product of the two *labeling* matrices. But the product matrix of the indexed transformation is not the product of the matrices of the two transformations it is a product of. It is the index that is given by the product, not the transformation. With this definition of group product, differing from the normal definition, these transformations form a group. Associativity holds, since it follows from the associativity of the indexing group, but only because of the revised definition of a product.

To give a group we list its members and their products (thus the spaces on which they act). But now with each set of members we have an infinite number of product rules (determined by the mapping of the transformations into the indexing groups, of which there may be several) thus an infinite set of groups. Each n-dimensional surface has its own group.

7.7 Product rules determine groups

This shows how properties of a group are dependent on the definition of its product and how by revising this definition we can generalize the type of structures that form groups. This adds to the richness of group theory.

Groups which can be realized as matrices, thus whose products are matrix products, we call standard groups. Ones whose elements are indexed so whose products are given by the matrix products of their indices we call indexed groups.

Indexed transformation groups exist for any surface that can be mapped (properly, in a way that must be investigated) to the defining space of a Lie group. For the three-dimensional rotation group SO(3) that is a sphere, and there is a third parameter which can be considered as giving the direction of a vector at each point of the sphere. Then each point is mapped to a point on one of the generalized (overlying) space, and each direction to one on that (which perhaps might be considered as an internal symmetry). Likewise we can use SO(2,1) to get another set of such surfaces. So we have associated with each group an infinite set of groups, each given by different product rules (from different mappings), or another way of saying this, an infinite set of realizations or representations.

We assign to each group element a matrix, that of the regular (adjoint) representation. That this is possible follows from the group axioms. Then the group product is a matrix product, and this is the usual group product. We call this the standard product.

The rotation group, besides its defining representation of 3 X 3 matrices has an infinite set of others. Consider the 5 X 5 one, say. This is a subrepresentation of SO(5). We can map a surface to the defining surface of SO(5) and the group defined over it (one for each of the infinite number of such surfaces, ignoring aspects like inversions), form representations of SO(3), but with additional transformations which might be taken as internal ones. Since SO(3) has an infinite number of representations (and it is simple so other groups have infinite sets each of infinite numbers of representations) this admits a huge number of transformation sets to be defined over it.

But we can do more. The conformal group algebra is (isomorphic to) that of SO(4,2) and SU(3,1). The group algebras are the same but realized in terms of different variables. Instead of 4+2 real ones, or 3+1 complex ones, the conformal algebra is realized over 3+1 real ones ([8]). We might also realize it over more, rather than fewer, variables. These again can be taken as internal ones.

We now map surfaces (choosing from an infinite number) to the 3+1-dimensional real space on which the conformal group acts. The product of the transformations on these is given by the product of conformal transformations taking points and directions of the 3+1-space to others. These conformal transformations are the indices of the transformations on the preimage space. Note that we have three groups, SO(4,2), SU(3,1) and the conformal group, plus all their representations, which can act as indices. This shows the great richness introduced.

7.8 Groups of 3+1 space as illustrations

Thus there are two groups defined by a 3+1-dimensional real space, one, ISO(3,1) is the Poincaré group, the other is the conformal group. This itself shows richness known but not understood in group theory. Take a surface (one of an infinite number) that is mapped to real 3+1-dimensional space. The transformations on it can be indexed by the transformations of the base space. However there are two sets, the Poincaré group and the conformal group. So from ordinary space we can find two infinite sets of groups (realizations, representations). And the transformations of our space can be taken as subsets of larger groups, giving further labels and products, of infinite number. Some of this additional freedom might be relevant to internal transformations.

Why should we consider these, aside from their showing the assumptions? Groups are useful in many ways and these can extend their usefulness. For example special functions are group representation basis states and many properties can be derived from this. Generalizing the concepts of representation can lead to other special functions, perhaps with useful properties. However it is not clear that properties can be found for these as they can for standard representations, or indeed that they have simple properties. This must be investigated. Associativity is important in determining these properties, and allowing simple properties (that we can get rules for). This procedure allows such great generalization that it is likely that only a few cases, at most, can give simple rules. But there might be some and these could be useful.

Also physical objects are statefunctions ([13]) that are group representation basis states. By expanding the set of these we may be able to expand the set of objects that are such states. There are clear limitations as the known states are those of standard representations. It may be that the requirement that objects be observers, and conversely ([7]), provides strict limits. Yet this is not known and these new representations allow study of this. And some may have physical applications, perhaps to these fields.

7.9 Why standard products are matrix product and why are these usually relevant?

While these indexed groups (groups with indexed products) may seem unusual they raise the question why standard products are the relevant ones, for those cases in which they are? This has to be considered for each application. A general class is applications to geometry. The transformations, but only for certain geometries, have standard products. This is true for lines, circles, planes, spheres and generalizations. For these, why are the products of transformations the standard products?

Here symmetry enters. The action of a group transformation on the base space is independent of the point in that space — since all these points are identical. Thus a group transformation taking a point to another, acting on a second such transformation, gives a transformation with identical action (as can be seen with a circle). ¿ From this associativity follows. These transformations thus form the regular representation ([3]). But this representation can be given by matrices, and the group product is the same as a matrix product, the standard product.

Other spaces do not have symmetry so their transformations cannot be represented by matrices. Here we see how symmetry gives group operations, and limits these. For spaces without symmetry we must use other products.

7.10 Conclusion

Groups are determined very much by their products. Usually these give matrix products. Here we have considered a set of different product rules. These illustrate how such rules determine the properties of groups, and the role of symmetry in the standard product rules. Whether some of these generalized rules are useful has to be studied. Lie groups have an extensive structure including their algebras. Do these generalizations allow corresponding structures, including algebras? That is another field of study. There is much that can be done, some at least profitably.

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References

1. Borstnik, Norma Mankoc, Holger Bech Nielsen, Colin D. Froggatt, Dragan Lukman (2004), ”Proceedings to the 7th Workshop ’What comes beyond the Standard models’, July 19 – July 30. 2004, Bled, Slovenia”, Bled Workshops in Physics, Volume 5, #2, December.
2. Borstnik, Norma Mankoc, Holger Bech Nielsen, Colin D. Froggatt, Dragan Lukman (2005), ”Proceedings to the 8th Workshop ’What comes beyond the Standard models’, July 19 – July 29. 2005, Bled, Slovenia”, Bled Workshops in Physics, Volume 6, #2, December.
3. Mirman, R. (1995a), Group Theory: An Intuitive Approach (Singapore: World Scientific Publishing Co.).
4. Mirman, R. (1995b), Group Theoretical Foundations of Quantum Mechanics (Commack, NY: Nova Science Publishers, Inc.; republished by Backinprint.com).
5. Mirman, R. (1995c), Massless Representations of the Poincaré Group, electromagnetism, gravitation, quantum mechanics, geometry (Commack, NY: Nova Science Publishers, Inc.; republished by Backinprint.com).

6. Mirman, R. (1999), Point Groups, Space Groups, Crystals, Molecules (Singapore: World Scientific Publishing Co.).
7. Mirman, R. (2001a), Quantum Mechanics, Quantum Field Theory: geometry, language, logic (Huntington, NY: Nova Science Publishers, Inc.; republished by Backinprint.com).
8. Mirman, R. (2001b), Quantum Field Theory, Conformal Group Theory, Conformal Field Theory: Mathematical and conceptual foundations, physical and geometrical applications (Huntington, NY: Nova Science Publishers, Inc.; republished by Backinprint.com).
9. Mirman, R. (2004a), Geometry Decides Gravity, Demanding General Relativity — It Is Thus The Quantum Theory Of Gravity, in Borstnik, Nielsen, Froggatt and Lukman (2004), p. 84-93.
10. Mirman, R. (2004b), Physics Would Be Impossible in Any Dimension But 3+1 — There Could Be Only Empty Universes, in Borstnik, Nielsen, Froggatt and Lukman (2004), p. 94-101.
11. Mirman, R. (2004c), Conservation of Energy Prohibits Proton Decay, in Borstnik, Nielsen, Froggatt and Lukman (2004), p. 102-106.
12. Mirman, R. (2005), Are there Interesting Problems That Could Increase Understanding of Physics and Mathematics? in Borstnik, Nielsen, Froggatt and Lukman (2005), p. 72-78.
13. Mirman, R. (2006), Our Almost Impossible Universe: Why the laws of nature make the existence of humans extraordinarily unlikely (Lincoln, NE: iUniverse, Inc.)