Precision holography and supersymmetric theories on curved spaces

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The formulation of rigid supersymmetric field theories on curved space leads to a number of results on their strongly-interacting regime, crucial from both the mathematical and physical point of view, starting from Witten's topological twist of four-dimensional Yang–Mills theory. At the same time, strongly-coupled field theories may also be studied holographically via the AdS/CFT correspondence. The aim of this thesis is to study aspects of the holographic dictionary for supersymmetric theories on curved manifolds.

A key aspect of the correspondence is the renormalization of gravity observables, which is realized via holographic renormalization. If the dual boundary field theory is supersymmetric, it is natural to ask whether this scheme is compatible with the rigid supersymmetry at the curved boundary. The latter requires specific geometric structures, and general arguments imply that BPS observables, such as the partition function, are invariant under certain deformations of these structures. We may then formulate a precise check of the holographic dictionary by asking whether the dual holographic observables are similarly invariant, as the free energy of the gauge theory is identified with the holographically renormalized supergravity action.

In the first part of the thesis, we consider this question in $\mathcal{N}=4$ gauged supergravity in four and five dimensions for the holographic dual to the topological twists of $\mathcal{N}=4$ gauge theories on Riemannian three-manifolds and $\mathcal{N}=2$ gauge theories on Riemannian four-manifolds. We show that the renormalized on-shell action is independent of the metric on the boundary four-manifold, as required for a topological theory. We then go further, analyzing the geometry of supersymmetric bulk solutions. This allows us to show that the gravitational free energy of any smooth filling vanishes in both AdS_4/CFT_3 and AdS_5/CFT_4.

In the second part of the thesis, we study the same question in minimal $\mathcal{N}=2$ gauged supergravity in four and five dimensions. In four dimensions we show that holographic renormalization precisely reproduces the expected field theory results for the dependence of the partition function on the background. Surprisingly, in five dimensions we find that no choice of standard holographic counterterms is compatible with supersymmetry, which leads us to introduce novel finite boundary terms. For a class of solutions satisfying certain topological assumptions we provide some independent tests of these new boundary terms, in particular showing that they reproduce the expected VEVs of conserved charges. We also briefly comment on the relation between these terms and boundary supercurrent anomalies.

Statement of originality

This thesis contains material from works written in collaboration. Chapter 2 and 3 are mostly based, respectively, on

> P. Benetti Genolini, P. Richmond and J. Sparks, *Topological AdS/CFT*, JHEP **1712** (2017) 039, arXiv:1707.08575 [hep-th]. [40]

> P. Benetti Genolini, P. Richmond and J. Sparks, *Gravitational free energy in topological AdS/CFT*, arXiv:1804.08625 [hep-th]. [41]

Chapter 4 is mostly adapted from

> P. Benetti Genolini, D. Cassani, D. Martelli and J. Sparks, *The holographic supersymmetric Casimir energy*, Phys. Rev. D**95** no. 2, 021902, arXiv:1606.02724 [hep-th]. [39]

> P. Benetti Genolini, D. Cassani, D. Martelli and J. Sparks, *Holographic renormalization and supersymmetry*, JHEP **1702** (2017) 132, arXiv:1612.06761 [hep-th]. [38]

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οἶον τὸ γλυκύμαλον ἐρεύθεται ἄκρῳ ἐπ’ὔσδῳ
ἄκρον ἐπ’ἀκροτάτῳ

Contents

1 Introduction

1.1 The AdS/CFT correspondence

Gravity is intrinsically different from quantum field theory. Despite the numerous similarities in the classical formulation (think for instance to the relation between Yang–Mills theories and general relativity), quantum gravity and field theory differ at a very fundamental level, for gravity describes spacetime *itself* and thus self-interactions produce phenomena such as black holes. The appearance of black holes is at the root of a number of (conjectured) peculiar features of quantum gravity, such as the absence of global symmetries (for a review see [30]), and was the main stimulus for the development of the holographic principle [34, 210, 206]. The latter is one of the very few properties of quantum gravity we think we know, and succinctly states that the number of degrees of freedom of a gravitational system in a volume is bounded by the area of the boundary of the volume. Therefore, quantum gravity defies our naïve notion of locality derived from quantum field theory.[1]

String theory naturally unifies gravity and field theories in a framework that a number of highly non-trivial computations have shown to be self-consistent.[2] In the perturbative formulation of string theory on flat spacetime, the holographic principle is not immediately

[1] A more precise formulation involving the covariant entropy bound [56] is beyond the scope of this thesis, and we refer the reader to the relevant literature for more information; for a review see [57].

[2] For the purposes of this chapter, "string theory" indicates both the perturbative formulations of the ten-dimensional string theories, and the strongly-interacting completion and extension to eleven dimensions.

manifest. However, non-perturbative phenomena hint at a relation between string theory on a class of backgrounds and a conformal field theory [165, 116, 225]. Studying the low-energy limit of string theories in presence of black branes indicates that quantum gravity on AdS_{d+1} is equivalent to a conformal field theory on a d-dimensional space isomorphic to the boundary of AdS_{d+1}: this equivalence is the AdS/CFT correspondence.[3] More precisely, one connects independently decoupled sectors of the open and closed string descriptions of the same system. This correspondence looks holographic by construction, and a more careful analysis of concrete instances shows that the holographic principle is satisfied also quantitatively: the degrees of freedom of the bulk, being the same as the degrees of freedom of the boundary, saturate the holographic bound [207]. Moreover, this correspondence surprisingly defies our initial statement that gravity is intrinsically different from quantum field theory. In fact, on a certain class of spacetimes, a theory involving quantum gravity *is fully equivalent* to a particular quantum field theory.

The supergravity approximation to string theory is ten- or eleven-dimensional, so the full solution is of the form $\mathrm{AdS}_{d+1} \times X_p$, for a p-dimensional manifold X_p. The ur-examples of the original paper by Maldacena [165] are the maximally supersymmetric supergravity solutions that are products of an anti-de Sitter factor and spheres, the most studied one arguably being the duality between Type IIB on $\mathrm{AdS}_5 \times S^5$ with N units of flux through S^5 and four-dimensional $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills theory. However, we may generalize the setup by considering supergravity solutions with different internal spaces X_p, obtaining less supersymmetric field theories [225]. For instance, one may obtain dualities with four-dimensional $\mathcal{N} = 1$ field theories by considering Type IIB on backgrounds of the form $\mathrm{AdS}_5 \times X_5$ with X_5 being a Sasaki–Einstein manifolds: the physical interpretation would be a number of branes probing the singularity at the tip of the metric cone over X_5 [143, 148, 1, 181]. In the effort of geometrizing high-energy physics, this leads to a number of relations between the structure of the internal space and the dual field theory (e.g. the geometric dual to a maximization [135, 177]).

From the physics point of view, an appealing aspects of the AdS/CFT correspondence is that the dictionary found from string theory generally describes a strong-weak duality. The low-energy and weak-coupling regime of string theory, described by a semi-classical

[3]By and large, in this thesis we will work on Riemannian spaces. Therefore, to be precise, AdS_{d+1} does not refer to anti-de Sitter space in its cosmological setting, but rather to the hyperbolic space $\mathbb{H}^{d+1}$. However, "the H/CFT correspondence" doesn't have the same appeal as "AdS/CFT," so we will stick to the standard wording.

supergravity theory, is equivalent to the strong-coupling regime of the gauge theory. This is clearly seen in the example of the relation between Type IIB on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ super Yang–Mills, obtained by studying D3-branes at the origin of $\mathbb{R}^6 = C(S^5)$, the metric cone on the five-sphere.

As the quantization of string theory on $AdS_5 \times S^5$ is not yet fully understood, we should restrict to a regime where the quantum corrections can be ignored. This requires the open and closed string coupling to be small, that is $g_s \ll 1$. The D3-brane solution can be considered as a soliton in IIB supergravity, the low energy approximation of type IIB closed superstring theory. The approximation is valid provided the string scale $\sqrt{\alpha'}$ is negligible compared to the extended structure of the solution L, which is fixed by the equations of motion to be $L^4 = 4\pi g_s \alpha'^2 N$. Therefore, supergravity is a good approximation to the closed string picture if $g_s N \gg 1$. On the open string side, the D3-branes are four-dimensional objects where open strings can end. At low energies, we can neglect the massive string states, and the dynamics of the end-points of the open strings only involves the massless degrees of freedom, which describe $\mathcal{N} = 4$ $SU(N)$ super Yang–Mills theory. The gauge and string coupling can be identified by analysis of the DBI action of the branes, but also by identifying the complex moduli of type IIB supergravity and $\mathcal{N} = 4$ SYM, leading to $g_{\mathrm{YM}}^2 = 4\pi g_s$. We then have to rephrase the conditions for the weak-coupling of gravity in terms of g_{YM}: $g_{\mathrm{YM}} \ll 1$ and $g_{\mathrm{YM}}^2 N \gg 1$. This can only hold if $N \gg 1$. However, an old argument by 't Hooft explains that at large N the effective coupling of non-Abelian Yang–Mills theory is $\lambda = g_{\mathrm{YM}}^2 N$ [208], which for us implies that the gauge theory is strongly-coupled in the planar limit.

Strongly-interacting field theories are a largely unexplored territory, as by definition they lie beyond the validity of the perturbative regime. The AdS/CFT correspondence allows us to construct a dictionary between observables in the large N, but strongly-coupled, limit of a field theory and observables in the calculable limit of supergravity. Therefore, it opens the door to the investigation of a new domain in field theory. Said field theory appears as formulated on a space that is isomorphic to the boundary of the AdS part of the supergravity solution. Thus, possible backgrounds are round spheres, flat and hyperbolic spaces: the many faces of anti-de Sitter space, corresponding to different radial coordinates and slicings [93]. This list is fairly limited, and for various reasons one may want to consider more general backgrounds.[4] We may also generalize the correspondence in this direction.

Anti-de Sitter space (read Poincaré hyperbolic space) in $d+1$ dimensions can be described as the open unit ball B^{d+1} with metric $4g_{\mathbb{R}^{d+1}}/(1-|x|^2)^2$, which is normalized to have constant sectional curvature -1 (that is, the AdS radius is one). Note that if we multiply the metric by ρ^2, with $\rho = (1-|x|^2)/2$, we obtain the closed unit ball $\overline{B^{d+1}}$ with the flat Euclidean metric, which extends to the boundary $S^d = \partial B^{d+1}$ as the round metric g of constant sectional curvature $+1$. The crucial property of ρ is that it is positive on B^{d+1} and has a first order zero only on ∂B^{d+1}, where $\mathrm{d}\rho \neq 0$. However, any other function with these properties would lead to a smooth metric on $\overline{B^{d+1}}$, and would differ from ρ by a positive smooth function e^w, leading to a smooth metric $\mathrm{e}^{2w}g$. Since all extended metrics differ by a conformal transformation, only the boundary *conformal* manifold $(S^d, [g])$ is well-defined.

This construction can be generalized in a fairly straightforward way by using Penrose's idea of conformal infinity [193]. Let Y_{d+1} be the interior of a compact $(d+1)$-dimensional manifold with non-empty boundary $\partial Y_{d+1} \equiv M_d$. A complete Riemannian metric G on Y_{d+1} is *conformally compact* if there is a defining function ρ on $\overline{Y_{d+1}}$ such that the conformally equivalent metric

$$\widetilde{G} \;=\; \rho^2 G \qquad (1.1.1)$$

extends to a smooth metric on the compactification $\overline{Y_{d+1}}$. A defining function is a smooth non-negative function on $\overline{Y_{d+1}}$ with $\rho^{-1}(0) = \partial Y_{d+1}$ and $\mathrm{d}\rho \neq 0$ on ∂Y_{d+1}. As above, the induced metric $g = \widetilde{G}|_{\partial Y_{d+1}}$ is not uniquely defined, but the conformal class $[g]$ is, and the conformal manifold $(M_d, [g])$ is called the *conformal boundary* (or *infinity*) of (Y_{d+1}, G). A conformally compact manifold (Y_{d+1}, G) is *asymptotically locally hyperbolic* if it has asymptotic negative constant scalar curvature $R[G] \to -d(d+1)$. This corresponds to requiring $|\mathrm{d}\rho|_{\partial Y_{d+1}} = 1$. We may then find another smooth defining function z such that the metric in a neighbourhood of the boundary is expressed as

$$G \;=\; \frac{1}{z^2}(\mathrm{d}z^2 + \mathrm{g})\,, \qquad (1.1.2)$$

where g has an analytic expansion in z.[5] The structure of the expansion depends on the

[4]Different slicings of anti-de Sitter space correspond to different choices of radial coordinate, which in turn correspond to conformal transformations at the boundary. Indeed all the backgrounds collected above are locally conformally flat.

[5]It is easily checked that the scalar curvature for this metric does indeed asymptote to $-d(d+1)$ near the boundary.

dimensionality of the spacetime and on the field content of the bulk supergravity theory [93, 79, 212]. For instance, if $d+1$ is odd there are logarithmic terms that are known to be related to the conformal anomalies of the boundary field theory [225, 127]. The general framework is on firm mathematical grounds only in the case of pure gravity with a cosmological constant, when the metric G is Einstein and the existence of the expansion has been proved by Fefferman and Graham [96]. Theories with more fields, such as required by supersymmetry, have to be considered on a case by case basis, even though the structure may be similar, as we will see in the body of the thesis.

Therefore, we may generalize the AdS/CFT correspondence to include curved background manifolds by requiring the bulk to be asymptotically locally hyperbolic rather than just anti-de Sitter. Rephrased in these terms, the correspondence relates a field theory on a Riemannian manifold (M_d, g) to supergravity on an asymptotically locally hyperbolic space Y_{d+1} such that (M_d, g) arises as the conformal boundary. However, this leads to the next immediate question: how do we formulate the field theory on the curved manifold? Even more specifically, how do we formulate a supersymmetric field theory on the curved manifold preserving some supersymmetry?

1.2 Supersymmetry on curved spaces

The formulation of a field theory on a curved background is always ambiguous. We want to deform the theory by relevant operators that leave the short-distance behaviour of the theory unaltered, but we may rephrase the coupling of such operators in terms of curvature tensors, showing explicitly that there could be different curved-space completions of the same flat-space Lagrangian, as all the curvature terms vanish on flat $\mathbb{R}^d$. However, the earlier discussion leads us to look for a completion that leaves the curved-space theory supersymmetric.

The simplest way of formulating on a curved manifold (M_d, g) a generic field theory described by a Lagrangian $\mathcal{L}_{\mathbb{R}^d}$ is by minimal coupling: every instance of the flat space metric is replaced by g and every partial derivative is replaced by the covariant derivative ∇ obtained from the Levi-Civita connection of g. This is what we learn in a General Relativity course. However, it is not obvious that the resulting theory $\mathcal{L}_{M_d}$ will have the same symmetries as the flat-space theory. This is trivial: if the variation of the flat space

Lagrangian is a divergence on flat space, $\delta\mathcal{L}_{\mathbb{R}^d} = \partial_\mu(\cdots)^\mu$, it's not necessarily true that it will be a divergence on M_d, $\delta'\mathcal{L}_{M_d} \neq \nabla_\mu(\cdots)^\mu$.

The issue is directly related to the fact that the minimal coupling of a rigid supersymmetric theory on a spin manifold M_d requires the existence of a covariantly constant (or parallel) spinor on M_d representing the supersymmetry parameter of the supersymmetry transformation. The existence of a parallel spinor sets a very stringent condition on the geometry of the manifold: the holonomy has to be $SU(n) \subset SO(2n), Sp(n) \subset SO(4n), G_2 \subset SO(7), Spin(7) \subset SO(8)$ [216].[6] In four dimensions, for instance, a covariantly constant spinor on a compact manifold can only be found on flat tori T^4 and K3 surfaces with Calabi–Yau metrics.

Yet, there are ways around this. First of all, field theories that are supersymmetric and Weyl invariant on flat space can be placed on conformally flat spaces by virtue of their very own definition. Therefore, for instance, we can define a curved-space Lagrangian on any round sphere or $S^1 \times S^{d-1}$. Also, by exploiting the ambiguity in the curvature terms, we may construct *ad hoc* curved-space completions of the flat-space Lagrangian that preserve some supersymmetry on specific backgrounds. However, we are interested in more systematic approaches to non-minimal coupling.

1.2.1 Twisting

The guiding principle is to generalize the necessity of covariantly constant spinor. Supersymmetric field theories include a global symmetry, the R-symmetry, which is the outer automorphism group of the supersymmetry algebra. We may couple the R-symmetry to a background gauge field A and define a *twisted* supercharge with corresponding R-charged spinor parameter satisfying

$$(\nabla_\mu - \mathrm{i}A_\mu)\zeta \;=\; 0\,. \tag{1.2.1}$$

We can then choose a field configuration for the background A that cancels the spin connection part of the covariant derivative, thus leaving us with $\partial_\mu\zeta = 0$, which is solved on

[6] We expect a relation with holonomy, because a covariantly constant spinor is invariant under parallel transport and hence invariant under the holonomy group at any point. Moreover, the existence of a parallel spinor implies that the manifold is Ricci-flat (simply take the integrability condition). The four-classes of holonomy groups listed above are definitely Ricci-flat. However, is this the complete list? This question is still to be solved. In the non-compact case, it is known that there are Ricci-flat metrics with $SO(n)$ holonomy (and not a subgroup): the Euclidean Schwarzschild metric is a complete metric on $S^2 \times \mathbb{R}^2$ with holonomy $SO(4)$. In the compact case, it is an open question whether there are simply connected manifolds that have vanishing Ricci tensor but have generic holonomy.

any background by a constant spinor. The systematic method of formulating a field theory on an arbitrary curved manifold by coupling the R-symmetry to a background gauge field that cancels the spin connection goes under the name of *twisting*. Since it mixes part of the R-symmetry group with the structure group of the frame bundle of spacetime, this approach requires the starting field theory to have a sufficiently high amount of supersymmetry. Often, and quite surprisingly, it leads to sectors of the set of observables that depend only on a small part of the background geometry. Famously, for four-dimensional $\mathcal{N} = 2$ field theories, it leads to observables that reproduce the Donaldson invariants of the background four-manifold which, under some topological restrictions, only depend on the smooth structure [85, 219, 32]. For this reason, it is referred to as the *topological twist*. However, this is not always the case: one may twist an $\mathcal{N} = 1$ theory with $U(1)_R$ symmetry on Kähler surfaces (with $U(2)$-structure) obtaining observables that depend on the complex structure of the underlying manifold [137, 223].[7]

At times, it is easier to appeal to different equivalent descriptions of the topological twist. By phrasing it in terms of group theory, we consider the structure group $\mathcal{K}$ of the frame bundle of the background manifold and the R-symmetry group $\mathcal{H}$, and we aim to find a subgroup $\mathcal{K}'$ of $\mathcal{K} \times \mathcal{H}$ that is isomorphic to $\mathcal{K}$ but acts differently on the field theory. Specifically, we aim to find a twisted structure group $\mathcal{K}'$ such that a number of supercharges $\mathcal{Q}^I$ would transform as singlets under $\mathcal{K}'$ and in some representation of the leftover global symmetry group $\mathcal{H}'$. For instance, in the case of the Donaldson–Witten twist of four-dimensional $\mathcal{N} = 2$ SYM, we have $\mathcal{K} = Spin(4) \equiv SU(2)_\ell \times SU(2)_r$, $\mathcal{H} = SU(2)_R \times U(1)$, and $\mathcal{K}' = SU(2)_\ell \times (SU(2)_r \times SU(2)_R)_{\text{diag}}$.[8] The supercharges then become

[7]Here and throughout the thesis we consider the *full* topological twist, where we require the entire background to be arbitrary. However, one may also consider *partial* twistings, when the background is a product manifold and one twists only on, e.g., one of the factors X. In this case, the field theory observables will usually only depend in a simple way on the geometric structure of X. For instance, this is famously the case for class $\mathcal{S}$ theories obtained by a partial twist on a Riemann surface Σ of the worldvolume theory of a M5-brane wrapping $\mathbb{R}^{1,3} \times \Sigma$. The resulting theory only depends on the complex structure of Σ [104].

[8]We are being a bit cavalier here. Generically, the $SO(4)$ structure group of the frame bundle may not be lifted to $Spin(4) = SU(2) \times SU(2)$, with the obstruction being the second Stiefel–Whitney class $w_2(M_4) \in H^2(M_4, \mathbb{Z}_2)$. If this is not possible, and the manifold is not spin, we shall take the R-bundle to be just a $SO(3)_R$ bundle such that $w_2(P_R) = w_2(M_4)$, so that the spinors sections of $S^\pm \otimes V_R$ would exist. More details on the global structure will be discussed in the concrete examples in the body of the thesis.

	$\mathcal{K} \times \mathcal{H}$	$\mathcal{K}' \times \mathcal{H}'$	
Q^I_α	$(\mathbf{2},\mathbf{1},\mathbf{2})^{-1}$	$(\mathbf{2},\mathbf{2})^{-1}$	1-form G_μ
$\overline{Q}^I_{\dot\alpha}$	$(\mathbf{1},\mathbf{2},\mathbf{2})^{1}$	$(\mathbf{1},\mathbf{1})^{1} \oplus (\mathbf{1},\mathbf{3})^{1}$	scalar and self-dual 2-form $\mathcal{Q}, \chi^+_{\mu\nu}$

This is often the quickest method to determine the feasibility of a twist and the representations of the twisted field content, as we will see later in the thesis.

In terms of the geometry, we consider the $SO(d)$-structure of the generic Riemannian manifold and look for associated vector bundle V with connection ω_V. Supersymmetry then requires the introduction of a R-symmetry gauge principal bundle P_R, with associated vector bundle V_R and gauge connection ω_R. The twisting consists in the identification of the V_R and V bundles and of their connections. As for the four-dimensional $\mathcal{N} = 2$ twist, there we have a $Spin(4)$-structure with associated rank 3 vector bundles $\Lambda^\pm_2 M_4$ – the bundles of (anti-)self-dual 2-forms – which inherit the (anti-)self-dual part of the Levi–Civita connection $\omega_\pm$ (with respect to the Lie algebra-valued indices of the connection). Supersymmetry requires the existence of an $SU(2)_R$ principal bundle, to which we associate a rank 3 vector bundle $P_R \to M_4$ with connection ω_R. The twist is the identification of $\Lambda^+_2 M_4$ and P_R together with ω_+ and ω_R.[9]

Twisting allows us to define fermions on an arbitrary manifold: spinors would be sections of the spin bundle tensored with the R-symmetry bundle, even though the factors may not exist on their own. However, identifying spacetime and R-symmetry bundles allows one to define different sections of the tensor product bundle, which always exists.

The three reformulations of the topological twists above are all equivalent, and provide insights into different aspects of the twisted theories. We will see that we are going to need all of them as we go on.

1.2.2 General coupling to gravity and holography

One may also generalize this procedure. In a supersymmetric field theory, we always have a stress-energy tensor and a supersymmetric supercurrent, which (often) resides in the same supermultiplet. This supermultiplet contains a number of other bosonic and fermionic terms, which can be coupled to a number of bosonic and fermionic fields, $\mathcal{O}_B$ and $\mathcal{O}_F$.

[9]See footnote 8 for more precision.

For instance, $\mathcal{O}_B$ contain the variation of the background metric $\Delta g^{\mu\nu}$ that couples to the stress-energy tensor $T_{\mu\nu}$. In the deformed Lagrangian, the fermionic terms $\mathcal{O}_F$ are set to zero, and in order for the deformed theory to preserve supersymmetries $\mathcal{Q}^I$, their variation under $\mathcal{Q}^I$ should vanish. The equation $\delta_{\mathcal{Q}}\mathcal{O}_F = 0$ translates in a further generalization of (1.2.1), which schematically reads

$$(\nabla_\mu - \mathrm{i}A_\mu)\zeta = X_\mu \zeta \,, \tag{1.2.2}$$

where both A_μ and X_μ belong to the set $\mathcal{O}_B$. We can give a physical interpretation to (1.2.2) in terms of background supergravity [97]. In ordinary supergravity, the metric is dynamical, whereas now we want to view it as a classical background and allow it to take an arbitrary configuration (which is going to be the metric of the background space of our supersymmetric field theory). Of course, in supergravity the metric belongs to a supermultiplet together with other bosonic and fermionic fields. These will be our previous $\mathcal{O}_B$ and $\mathcal{O}_F$. Since these fields are not dynamical, they must reside in an off-shell supergravity multiplet, to which we couple the supermultiplet containing the stress-energy tensor. This construction constitutes a *rigid limit* of off-shell dynamical supergravity: we let the graviton and the other auxiliary fields of the multiplet fluctuate, then we freeze their degrees of freedom by sending the Planck mass to infinity, effectively decoupling gravity from the field theory. Looking for supersymmetric classical backgrounds requires the vanishing of the variation of the gravitino, which gives (1.2.2). Therefore, the rigid supersymmetry algebra arises as the subalgebra of the algebra of supergravity gauge transformations that leaves the background invariant. As in ordinary supergravity, the number of supercharges preserved by the background are determined by the number of solutions to the generalized Killing spinor equation (1.2.2). This construction provides a consistent way of formulating a supersymmetric field theory on a curved background, and we may then ask which backgrounds admit such completion. Again, as in ordinary supergravity, the allowed background can be found by studying the geometric conditions required in order to solve the generalized Killing spinor equation.

For concreteness, we will review the example of $\mathcal{N} = 1$ field theories in four dimensions with a $U(1)_R$ R-symmetry [97]. The stress-energy tensor resides in a $\mathcal{R}$-multiplet that in components contains [107, 151]

$$\left(j^{(R)}_\mu, T_{\mu\nu}, S_{\mu\alpha}, C_{\mu\nu}\right) ,$$

where $j^{(R)}_\mu$ is the $U(1)_R$-current, $T_{\mu\nu}$ the stress-energy tensor, $S_{\mu\alpha}$ the supercurrent, and $C_{\mu\nu}$ is a conserved 2-form current. A relevant off-shell formulation of four-dimensional $\mathcal{N} = 1$ supergravity is the new minimal formulation [5, 205]: the gravity multiplet of this theory contains

$$(A_\mu, g_{\mu\nu}, \Psi_{\mu\alpha}, B_{\mu\nu}) ,$$

where A_μ is a $U(1)$ gauge field, $g_{\mu\nu}$ is the graviton, $\Psi_{\mu\alpha}$ is the gravitino, and $B_{\mu\nu}$ is a 2-form gauge field. We may also dualize the field strength of the latter to a 1-form gauge field

$$V \;=\; \mathrm{i} * \mathrm{d}B\,,$$

which is obviously conserved.[10] Coupling to new minimal supergravity allows us to construct backgrounds for supersymmetric field theories by solving the generalized Killing spinor equation coming from the vanishing of the supersymmetry variation of the gravitino. In this context, we may consider supercharges of definite R-charge, corresponding to the following charged conformal Killing spinor equations

$$(\nabla_\mu \mp \mathrm{i}A_\mu)\zeta_\pm = \mp V_\mu(\sigma_\mp\sigma_\pm)\zeta_\pm\,, \tag{1.2.3}$$

where $\zeta_\pm$ are two-component spinors and the generators of the Clifford algebra are $(\sigma_\pm) = (\pm\sigma, -\mathrm{i}\mathbb{1}_2)$. If there are both supercharges, we may define a Killing vector bilinear $K = \zeta_+\sigma^i_+\zeta_-\partial_i$. The supersymmetry algebra is generated by the subalgebra of the infinite-dimensional supergravity gauge transformations that leaves invariant the classical background. In this case, acting on a field Φ of R-charge q, it reads

$$[\delta_{\zeta_+}, \delta_{\zeta_-}]\Phi \;=\; 2\mathrm{i}\left(\mathcal{L}_K - \mathrm{i}qK \lrcorner \left(A + \tfrac{3}{2}V\right)\right)\Phi\,, \qquad \delta^2_{\zeta_\pm} \;=\; 0\,.$$

Notice the appearance of the R-charge on the right-hand side of the transformation.

[10] Indeed

$$\nabla^\mu V_\mu \;=\; \mathrm{i} * \mathrm{d} * (*\mathrm{d}B) = \mathrm{i} * \mathrm{d}^2 B = 0\,.$$

We analyse the geometric conditions that (1.2.3) imposes on the background [90]. In the case of a single Killing spinor, say ζ_+, the background manifold turns out to be required to be complex with Hermitian metric adapted to the complex structure constructed from the spinor

$$I^i_{\ j} \ = \ -\frac{2\mathrm{i}}{|\zeta_+|^2}\zeta_+^\dagger(\sigma_+)^i_{\ j}\zeta_+ \, .$$

We may then turn the problem upside down and try to reconstruct the fields of new minimal supergravity from the geometry of the background (that is, g and I): curiously, (1.2.3) relates the divergence of I and V

$$\nabla_i I^i_{\ j} \ = \ 2V_j \, .$$

This makes beautifully clear the relation between the twist and the "rigid limit" for $\mathcal{N} = 1$ in four dimensions. Comparing (1.2.1) and (1.2.3), we notice that we recover the former when $V = 0$ in the latter, and indeed $V = 0$ if and only if the manifold is Kähler, which is the condition for the twist of $\mathcal{N} = 1$ SYM we saw in section 1.2.1! On the other hand, the gauge field V in new minimal supergravity allows us to extend the formulation to complex manifolds that are not Kähler, such as Hopf surfaces.

Notice that the analysis heavily depended on the choice of new minimal supergravity: different choices of off-shell formulations of supergravity lead to different completions of the stress-energy tensor multiplet and in general to different results. For instance, the analysis based on new minimal supergravity has an evident problem: it does not lead to an S^4 background. However, S^4 is conformally flat and we know that superconformal field theories always have a $U(1)_R$ symmetry. In fact, one may use different off-shell formulations of the same dynamical supergravity, such as the so-called conformal supergravity or old minimal, and indeed S^4 is among the backgrounds that one recovers by coupling to old minimal, but not new minimal [97].[11]

This approach has been generalized to a number of different dimensions and different supergravities (for a review see [89]), but for our purposes we are only going to need the three-dimensional case with the same amount of supersymmetry, studied in [66]. Three-dimensional field theories with two supercharges and a $U(1)_R$ symmetry may be formulated on manifolds admitting a transversely holomorphic foliation, with a metric

[11] We are being a bit sketchy for the sake of simplicity. It *may* be possible to recover S^4 from new minimal supergravity, but the conformally charged Killing spinor would have zeroes where some geometric objects constructed as spinor bilinears degenerate. This would be in analogy with recovering S^4 from conformal supergravity: the complement of the zero locus is $\mathbb{R}^4$, which is obviously complex [145].

that is transversely Hermitian. A transversely holomorphic foliation can be defined in terms of a foliated atlas. Let $\{U_\alpha\}$ be an open cover of a foliated manifold with local submersions $f_\alpha : U_\alpha \to \mathbb{C}^q$ such that the fibers of f_α are the leaves of the foliation (that is, we are immersing the space transverse to the foliation). We say that the foliation is *transversely holomorphic* if the local diffeomorphisms $\tau_{\beta\alpha} : f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$ such that $f_\beta = \tau_{\beta\alpha} f_\alpha$, are biholomorphisms for all α, β. Concretely, for a three-manifold this means that there exists a unit vector field X determining the foliation and a basic integrable complex structure J on the two-dimensional transverse spaces: $X \lrcorner J = \mathcal{L}_X J = 0$. In three dimensions, the only closed topological manifolds that admit a transversely holomorphic foliation are the total spaces of Seifert fibrations and torus bundles over a circle [58].

Finally, for superconformal field theories there is also a third completely different method to find supersymmetric backgrounds, based on a clever use of holography originated in [145]. By using the AdS/CFT correspondence, we may realize some superconformal field theories on curved manifolds at the boundary of asymptotically locally hyperbolic bulk spaces. Start with an appropriate $(d+1)$-dimensional supergravity with an AdS vacuum obtained by truncating the ten- or eleven-dimensional supergravity approximation to string theory, and impose that the bulk is a supersymmetric solution. This results in a set of generalized Killing spinor equations in the bulk. We may then consider such spinor equations in a neighbourhood of the conformal boundary using the (generalized) Fefferman–Graham expansion of the fields. The leading order contributions are going to be spinor equations on the conformal boundary M_d, and the existence on M_d of a superconformal field theory with a gravity dual requires the equations to have a solution. This in turn imposes some geometric requirements on M_d. More precisely, the AdS vacuum supergravity solution is dual to the superconformal field theory on flat space (or on a conformally flat manifold, cf. footnote 4). The asymptotically locally hyperbolic solution is dual to the relevant deformation of the SCFT required to formulate it on a curved background, e.g., the background gauge field coupling to the R-symmetry at the boundary extends to a non-trivial configuration of the bulk gauge field dual to the R-symmetry. *A priori*, the hope of consistency is the only reason why this latter holographic method should end with the same conclusions as the previous method, based on the rigid limit of supergravity: for instance, in five dimensions it is known that at the boundary of anti-de Sitter space one finds *conformal* supergravity and not new minimal [26].[12] However, as we will see in detail in the bulk of the thesis, at least for the instances we consider, this method leads to the same results as the previous one: the boundary Killing spinor equations may be expressed as the generalized Killing spinor equations of the appropriate supergravity of which we are taking the rigid limit.

1.3 Localization

The formulation of a supersymmetric field theory on a Riemannian compact curved manifold greatly improves the convergence properties of the path integral, making at least plausible the possibility of evaluating it in a rigorous way and computing some physical observables. This relies on a beautiful interplay of quantum field theory, geometry and supersymmetric ideas [221]. Let $\mathcal{E}$ be the integration space for the path integral of a quantum field theory with symmetry group F. If the action of F on $\mathcal{E}$ is free, it generates a fibration $\mathcal{E} \to \mathcal{E}/F$ and, by integrating first over the fiber, we may restrict the integral over $\mathcal{E}$ to an integral over $\mathcal{E}/F$ at the expense of an overall factor of $\mathrm{Vol}(F)$. For instance, for an F-invariant observable $\mathcal{O}$ this would give the one-point function

$$\int_{\mathcal{E}} \mathcal{D}X\, \mathrm{e}^{-S}\mathcal{O} \;=\; \mathrm{Vol}(F) \int_{\mathcal{E}/F} \mathcal{D}X'\, \mathrm{e}^{-S}\mathcal{O}\,.$$

However, if the symmetry is fermionic the volume of the group vanishes. Therefore, contributions to the one-point function of the observable may only come from the fixed locus of the supersymmetry $\mathcal{E}_0$, where the action is not free, and we say we have *localized* on this locus. In the generic case, we reduce to the evaluation of an integral over $\mathcal{E}_0$ consisting of a one-loop determinant. In some cases, aided by combinations of the symmetries of the background manifold and the supersymmetries, such integral may condense into a finite-dimensional one.

The idea of reducing the infinite-dimensional path integral to a much simpler (possibly finite-dimensional) integral already appears in [217] as a generalization of the Atiyah–Bott fixed point theorem. In that case, we are interested in the index of an operator over the infinite dimensional loop space $L(M_d)$ of a Riemannian manifold M_d (that is, the space of maps $S^1 \to M_d$), and it is shown that this reduces to a problem on the finite-dimensional

[12] In fact, for five-dimensional background spaces it is not clear that the two methods lead to the same result (see the discussion in [6]).

space of zeroes of the Killing vector of the canonical $U(1)$ action on $L(M_d)$.

However, the full power of the technique arguably becomes first manifest in the context of the topological twist in four dimensions [219]. After the topological twist, the path-integral of four-dimensional $\mathcal{N} = 2$ theory on an arbitrary background M_4 *localizes* onto Yang–Mills instantons on M_4, and correlation functions of a specific subset of operators localize to integrals of forms over the moduli space of said instantons. Even more surprisingly, these integrals may be evaluated (under some assumptions) and coincide with the Donaldson invariants of M_4 [223]! Moreover, the physical picture provides a much more powerful computational tool: since the theory is independent of the background metric g, we may consider a family of metrics $g_t = tg$ for $t \in \mathbb{R}$, thus allowing us to move from long to short distances. For $t \to 0$ we are studying the short-distance description, which is weakly coupled because of asymptotic freedom, and we have the classical picture of the Donaldson invariants in terms of $SU(2)$ instantons. However, we may follow the RG flow all the way to $t \to \infty$ and study the long-distance description of the *same* theory. The long-distance strongly-coupled description is in turn dual to a weakly-coupled Abelian theory [203]. This gives a different picture of the Donaldson invariants expressed in terms of solutions to PDEs involving *Abelian* gauge groups, the so-called Seiberg–Witten invariants. Since we have only followed an RG flow, we derive a conjectural equivalence between the two invariants [222, 180, 162].

A similar localization reasoning goes through for twisted four-dimensional $\mathcal{N} = 2$ theories on manifolds with a $U(1)$ isometry. In the language of section 1.2.1, the eight supercharges combine into a scalar, a 1-form and a self-dual 2-form of the twisted symmetry group $\mathcal{K}' = (SU(2)_\ell \times SU(2)_R)_{\text{diag}} \times SU(2)_r$. The scalar supercharge $\mathcal{Q}$ is the one familiar from the topological twist, but one may also contract the fermionic 1-form with the Killing vector of the $U(1)$ isometry, obtaining a new conserved supercharge taking into account the isometry. The paradigmatic example in this class is the so-called Ω-background, $\mathbb{R}^4$ with a $U(1) \times U(1)$ action, where one is able to compute the partition function, as it localizes on the moduli space of instantons equivariant with respect to the torus action [183, 184].

It is perhaps interesting to observe that the latter localization with respect to a $U(1)$ action on the manifold is a field theory counterpart of an older observation in the context of gravity [113]. For gravitational instantons with a $U(1)$ isometric action, the on-shell gravitational action can be computed in terms of geometric objects evaluated at the $U(1)$

fixed points, the nuts and bolts.[13]

It is often the case, and it surely is in all the examples mentioned above, that the localization computation is independent of the coupling parameter. Therefore, the localization of the path integral with respect to a symmetry provides crucial insight into the strong-coupling regime of the observables invariant under such symmetry. An important breakthrough came with Pestun's evaluation of the partition function and Wilson loop of $\mathcal{N} = 2^*$ theory on the four-sphere [194]. The entire idea is analogous to the original argument in [219]. Using a fermionic supersymmetry $\mathcal{Q}$, the action of the theory can be deformed by a $\mathcal{Q}$-exact term $t\mathcal{Q}V$, and the modified partition function $Z(t)$ is independent of t: the derivative $Z'(t)$ can be expressed as the path-integral of a $\mathcal{Q}$-exact object

$$Z'(t) = \int \mathcal{D}\Phi \, \mathcal{Q} \left(\exp^{-S-t\mathcal{Q}V} V \right) ,$$

but the integration over the field space reduces to a boundary integration by an analog of Stokes' theorem, and under some assumptions on the behaviour of the fields, this leads to zero. Then, if the bosonic part of $\mathcal{Q}V$ is positive, in the $t \to \infty$ limit the entire path localizes on the $\mathcal{Q}$-invariant subset, the BPS field configurations. This boils down to a (complicated) finite-dimensional integral for a matrix model, which can be at least put in a closed form. Pestun's result opened the doors to a number of computations of supersymmetry-protected observables in different dimensions, starting from three-dimensional $\mathcal{N} = 2$ Chern–Simons-matter models on the three-sphere [139] (for a review collecting part of the results in the topic see [195]).

Moreover, the extension of the spirit of the localization computations to more general backgrounds (based on the considerations made in the previous section) helps in investigating the dependence of the partition function of the field theory on the geometry of the background. Indeed, in certain cases one may rewrite the variations of the Lagrangians under specific variations of the background as $\mathcal{Q}$-exact terms, and in these cases the variation of the quantum observables would be zero by the same argument as above. For instance, in the topologically twisted theory, the entire stress-energy tensor is $\mathcal{Q}$-exact, which implies

[13] As already observed in the beginning, gravity and field theory are different. A field theory instanton (e.g. Yang–Mills instanton [209, 35]) is a solution to certain partial differential equations *on* a manifold, and sometimes gives us information about the topology of the background. A gravitational instanton is a spacetime *itself*: it is a connected manifold with a complete Einstein Riemannian metric (e.g. Euclidean Schwarzschild or the Gibbons–Hawking spaces) [124].

that the partition function (and a certain set of observables) is independent of the background metric. This is the case, again, under some assumptions on the topology of the manifold – in general there could be chambers and wall-crossing in the invariants [180]. In the four-dimensional $\mathcal{N} = 1$ case reviewed in section 1.2.2, the partition function only depends on the complex structure of the underlying manifold but not on the Hermitian metric [68]. Similarly, for the analogous case of three-dimensional $\mathcal{N} = 2$ theories summarised in the same section, the partition function only depends on the transversely holomorphic foliation of the background three-manifold, and not on the Hermitian transverse metric [68]. Note that, in addition to the aforementioned conditions required on the space of fields, $\mathcal{Q}$-exactness considerations require the path-integral measure to be consistent with supersymmetry: in the original Witten's case, this was justified *a posteriori* by the soundness of the mathematical formulation of Donaldson theory. However, in general one is not necessarily justified to make this assumption (see also footnote 2 of [68]).

Results from the strongly-coupled regime may also improve our understanding of quantum field theory itself. Most mathematically rigorous formulations of relativistic quantum field theory are based on the ideas of perturbation theory, and do not take into account the existence of *dualities*.[14] However, we have known since the 70s, and in a even more dramatic way since the Second Superstring Revolution, that the same physics can be described by wildly different mathematical formulations: for instance, fermions can be equivalent to bosons [71] and gauge redundancies of different types can be identified in certain regimes [202, 134]. Dualities such as the latter, of Seiberg-type, may be checked in greater detail by comparing the partition functions computed using localization, as they are valid for supersymmetric field theories [140]. Moreover, as the partition function is protected by supersymmetry, it can provide information about properties of the quantum field theory along the renormalization group flow, such as the R-symmetry [136].

Since localization computations provide a glimpse into the entire supersymmetric sector of the field theory observables, they are also crucial in establishing relations between theories even in different dimensions. This successful exploration of such correspondences is obviously guided by physical constructions, as in the notable case of compactifications of the M5-branes theory, but it heavily relies on the knowledge of the structure of the observables gained by localizing the path integral (the paradigmatic example being the

[14]In addition, observe that the very hard problem of constructing the path integral measure of the field theory collapses after localization to a well-defined measure.

consideration of the structure of the partition function of $\mathcal{N} = 2$ field theories on the four-sphere and its relation to Liouville theory [10, 227]).

1.4 Precision holography

In addition to all the above insight, an improved understanding of the strongly-coupled regime can also lead to a better picture and more refined checks of the AdS/CFT correspondence. The correspondence provides a dictionary between the operators of the two theories, and with the help of additional symmetries, such as supersymmetry, we may be able to sharpen the entries in this dictionary.

As already mentioned, the general prescription of the AdS/CFT correspondence is that quantum gravity on an asymptotically locally hyperbolic manifold Y_{d+1} is fully equivalent to a conformal field theory on a space isomorphic to the boundary of Y_{d+1}, so in particular the partition functions of the two theories (whatever this means) are the same.[15] Unfortunately, we have very little understanding of quantum gravity, let alone of its partition function. Therefore, we approximate it by supergravity, and the regime where the approximation holds corresponds to the strongly-coupled regime of the field theory (in an appropriate limit where the rank of the gauge group is large), as we saw in section 1.1.

Also, as we reviewed in section 1.2.2, rigid supersymmetry generically equips the background manifold (M_d, g), on which the gauge theory is defined, with certain additional geometric structure, such as an integrable complex structure for four-dimensional $\mathcal{N} = 1$ theories. In the gravitational dual description, one seeks asymptotically locally hyperbolic solutions to an appropriate supergravity theory in $d+1$ dimensions, where $(M_d, [g])$ arises as a conformal boundary. A saddle point approximation to quantum gravity in this bulk then identifies[16]

$$Z[M_d] \;=\; \sum \mathrm{e}^{-\mathbb{S}[Y_{d+1}]} \; . \tag{1.4.1}$$

Here, $Z[M_d]$ denotes the partition function of the gauge theory defined on M_d, while $\mathbb{S}[Y_{d+1}]$ is the holographically renormalized supergravity action, evaluated on an asymptotically

[15] In this framework, by “quantum gravity” we really mean string theory: it includes a quantized theory of pure gravity, but it also requires a number of other states and branes.

[16] We will focus on the on-shell action, corresponding to the partition function of the field theory, but an analogous construction would allow us to compute the holographic counterparts of correlators in field theory in terms of exchanges in the bulk (for instance, see the classic [100, 83] and the more recent results on four-point functions and quantum corrections [196, 7, 16]).

locally hyperbolic solution to the equations of motion of the $(d+1)$-dimensional theory. The manifold $M_d = \partial Y_{d+1}$ is the conformal boundary, with the boundary conditions for supergravity fields on Y_{d+1} fixed by the rigid background structure of M_d.

The general AdS/CFT relation is somewhat schematic, and both sides must be interpreted appropriately. The partition function of the field theory may suffer from ambiguities related to choices of renormalization schemes, and even in the case of topological field theories may be infinite due to the summing over topological sectors. Considering the right-hand side is the main aim of this dissertation.

The gravitational action on a space with boundaries has to be supplemented by a boundary term that makes the variational problem well-defined, reproducing the Einstein equations in the bulk. This is the Gibbons–Hawking–York term, proportional to the extrinsic curvature of the boundary [229, 112]. When evaluated on a solution, the sum of the on-shell action $I_{\text{o}-\text{s}}$ and the Gibbons–Hawking–York term I_{GHY} will generically diverge. The traditional method for removing this infinity is the background subtraction, that is, referring all the quantities to their analogues on a "reference" spacetime, such as flat space. Even though this prescription led to impressive agreements with quantum gravity computations [112], it suffers from an important drawback, as in general one may not embed a reference background in an arbitrary spacetime. However, for asymptotically hyperbolic spaces, there is a way out of the *impasse*: all the divergences can be expressed as local integrals of geometric quantities computed in terms of the induced metric on a surface of constant radial distance (where the radius can be identified e.g. with $1/z$ in the Fefferman–Graham expansion) [27, 93]. The *holographic renormalization* is the process of removing such divergences. Recall that near the boundary of any asymptotically locally hyperbolic space we may introduce a Fefferman–Graham-type expansion of the geometric quantities in terms of a coordinate z with $z = 0$ at the boundary. We cut off the bulk at a radius $z = \delta$ near the boundary, define Y_δ to be the internal region and evaluate $I_{\text{o}-\text{s}} + I_{\text{GHY}}$ on Y_δ and its boundary. Then, we introduce local counterterms I_{ct} constructed from the geometry induced on the hypersurface ∂Y_δ in order to remove the divergences that would arise by taking the limit where Y_δ covers the entire Y_{d+1}. This method provides a finite quantity

$$S = \lim_{\delta \to 0} \left(I_{\text{o}-\text{s}}|_{Y_\delta} + I_{\text{GHY}}|_{\partial Y_\delta} + I_{\text{ct}}|_{\partial Y_\delta} \right) ,$$

which is the holographically renormalized on-shell action appearing on the right-hand side of (1.4.1). Since the anti-de Sitter radius is identified with the energy scale of the field theory by holography [207], the holographic renormalization corresponds to regularization of the UV divergences in the dual field theory.

The method of holographic renormalization is a systematic method developed from the very beginnings of the subject with a variety of approaches. This mirrors the corresponding procedure in field theory and forms part of the foundations of the AdS/CFT correpondence [225, 116, 127, 27, 78, 79, 47, 48, 170, 204]. The *infinite* counterterms obtained via this approach are universal. However, the method leaves open the possibility of *finite* counterterms compatible with the requirements of the theory. The existence of these terms implies non-uniqueness of the renormalization scheme, and in such situations it is generally unclear how to match schemes on the two sides. Given that the classical gravitational description is typically valid only in a strong-coupling limit of the field theory, in general it is difficult to directly compute observables on both sides, and hence make precise quantitative comparisons.

However, the results obtained for supersymmetric quantum field theories defined on curved spaces, and specifically the exact computation of BPS observables such as the partition function by means of localization techniques, give us the possibility of sharpening (1.4.1): the field theory results may be compared with the holographic dual supergravity computations and provide a precision check of AdS/CFT.[17]

Comparison of the two sides of the AdS/CFT correspondence using localization brought a number of spectacular results, especially for AdS_4/CFT_3. To mention a few, the $N^{3/2}$ scaling of the free energy of the M2-brane theory [147] was recovered from a field theory computation in ABJM theory on S^3 [88], and the same free energy computation generalized to a number of cases of $\mathcal{N} = 3$ [128] and $\mathcal{N} = 2$ models [174]. Moreover, one may deform the spherical background, and compute via localization to matrix models the free energy

[17]Precision checks of the AdS/CFT correspondence have also been completed for AdS_3 and two-dimensional conformal field theories, starting from the fareytail of type IIB string theory on $AdS_3 \times S^3 \times K3$ [84]. However, because of the peculiar status of three-dimensional gravity and two-dimensional conformal field theory, that case is very different from what is considered in this thesis. In the case of specific highly symmetric field theories, such as $\mathcal{N} = 4$ SYM in four dimensions, one may take advantage of integrability in the planar limit, and compute the scaling dimensions of some local operators as a function of the effective coupling constant λ. This corresponds, via the AdS/CFT, to integrability of the non-linear sigma model of the worldsheet theory of type IIB string theory on $AdS_5 \times S^5$. Precision computations have been done in this setting achieving full agreement between the two sides of the correspondence. In this thesis we will consider field theories with less supersymmetry and the effective lower-dimensional supergravity description, for a review of the integrability phenomenon in the context of AdS/CFT see [33].

of Chern–Simons-matter theories on squashed spheres and more general three-manifolds [123, 11], matching precisely the holographic counterpart in the large N limit [172].

However, a more in-depth investigation of the precise match soon has to deal with at least some of the issues involved in (1.4.1), as any ambiguities in defining finite renormalized quantities in gravity are expected to be resolved in making such comparisons. Among them is the question of symmetries of the two sides: regularization schemes are usually expected to preserve the symmetries of the theory, and the choice of the radial cut-off in the holographic renormalization scheme is not necessarily compatible with supersymmetry. As well as trying to match precise quantities on both sides, there are more general predictions that may also be compared, such as the dependence of BPS observables on given sets of boundary data. These latter tests of the correspondence are inherently more robust than comparing observables in particular theories/backgrounds, and will therefore be a main focus of this dissertation.

There are numerous additional subtleties involved in (1.4.1) that we have not yet mentioned and that will not feature prominently in the main body of the work; the most glaring one being the domain of the sum on the right-hand side, which is not well understood. One should certainly include all saddle point solutions on smooth manifolds Y_{d+1}. However, the existence of such a filling immediately implies that M_d has trivial class in the oriented bordism group Ω_d^{SO}, in general constraining the choice of M_d. That said, various explicit examples (for example, [9, 8, 28]) suggest that requiring Y_{d+1} to be smooth is in any case too strong: one should allow for certain types of singular fillings of (M_d, g), and indeed these may even be the dominant contribution in (1.4.1) (especially for non-trivial topologies of M_d). There is no prescription on the inclusion of singular solutions, and the same is true of contributions from complex saddle points, that is, complex-valued metrics (such as, trivially, the Euclidean Kerr black hole). One would not expect a saddle point approximation to give necessarily a real solution, so why should that be the case for quantum gravity? There have been some speculations (e.g. [166]), but in this dissertation we will focus on real solutions. Even for smooth solutions, one may question the topology of the bulk: the supergravity action $\mathbb{S}$ typically scales with a positive power of N, and in the $N \to \infty$ limit only the solution of least action contributes to (1.4.1) at leading order, with contributions from other solutions being exponentially suppressed. Topology changes in the bulk conjecturally correspond to phase transitions in the dual field theory (e.g. the transition of $\mathcal{N} = 4$

SYM dual to the Hawking–Page transition for black holes in anti-de Sitter [126, 225]), but it is *a priori* not clear how to choose the right topology for the interior corresponding to field theories on the boundary, and this has non-trivial consequences for the uplift to string theory [125, 173, 214].

1.5 Outline

Following the previous boundary description of the context of the topic, the bulk of the thesis is divided in two parts, corresponding to the holographic study of the approaches to rigid supersymmetric field theories outlined earlier.

In the first part, we consider the holographic dual to the topological twist of supersymmetric field theories in three and four dimensions. In both cases, we begin by introducing the appropriate dual supergravity theory, the Fefferman–Graham expansion of the fields and holographically renormalizing the action. By expanding the bulk spinor equations, we show that on the conformal boundary we have the generalized Killing spinor equations of a conformal supergravity theory that admits the topological twist as its solution on any Riemannian three-/four-manifold. We then prove that the on-shell gravitational action, dual to the partition function of the boundary conformal field theory, is independent of the metric on the conformal boundary. More geometrically, we reformulate the conditions for a bulk supersymmetric solution in terms of a system of first-order differential equations. Using this system, we show that both in three and four dimensions the on-shell supergravity action vanishes for a *smooth* real solution dual to the boundary topological twist.

In the second part, we turn to approaches different from the topological twist. As already outlined, one may define a supersymmetric field theory on a curved d-manifold by coupling to d-dimensional supergravity and taking a rigid limit and also, for a conformal field theory, by studying the conformal boundary of a $(d+1)$-dimensional supergravity solution and appealing to the AdS/CFT correspondence. In three and four dimensions, the results of the two methods agree and we refer to them as “rigid supersymmetry.” In these two dimensions, the analysis of the dependence of field theory partition function on the background geometry leads to a set of *supersymmetric Ward identities*. The second part of the dissertation is concerned with the study of the holographic dual of such identities. After the introduction of the relevant supergravity theories in four and five dimensions and their holographic renormalization, we show that the gravitational on-shell action satisfies the supersymmetric Ward identities in the former case, but fails to do so in the latter. In four dimensions, we are then able to evaluate the on-shell action for a large class of self-dual solutions and match it to the field theory counterpart. In five dimensions, we have to introduce a set of novel (finite) boundary counterterms such that the improved on-shell action satisfies the supersymmetric Ward identities. Then, under some global assumptions we evaluate the renormalized on-shell action and compute the conserved charges, showing that they satisfy a BPS condition. Finally, we consider a number of examples, illustrating further the rôle of our new boundary terms and matching the gravity and field theory observables.

The AdS/CFT correspondence has been a source of great discoveries for the last twenty years. Yet there are many subtleties to be clarified in the dictionary between gravity and field theory. The work presented in this dissertation sheds some light on the gravity side of the duality by studying cases where supersymmetry is crucial to obtain a handle on the field theory. By doing so, it highlights many crucial points that are sometimes skirted – one might wryly condense the entire thesis as a long gloss over two footnotes (number 10 in [219] and number 2 in [68]). However, this is not the case, for as soon as one opens Pandora's box a number of questions arise, some of which are considered at the end of each part of the thesis.

Part I

Topological twist

2 Topological AdS_5/CFT_4

2.1 Introduction

In the first part of the thesis, we propose to formulate a "topological" version of AdS/CFT, where the boundary theory is a topological QFT (TQFT). A key motivation for studying AdS/CFT in this set up is that the field theory is potentially under complete control: observables are mathematically well-defined and exactly computable. TQFTs of this sort typically have a finite number of degrees of freedom, and in some instances can be solved completely.[1] These theories are often also of independent mathematical interest, since observables are topological/diffeomorphism invariants.

Under these desirable assumptions for the field theory, one can then focus on the dual gravitational description. In principle this is defined by a quantum gravity path integral, with boundary conditions determined by the observable one is computing. However, we have no precise definition of this, and in practice an appropriate strong-coupling (usually large rank N) limit of the QFT is described by supergravity. This classical limit is to be understood as a saddle point approximation to the quantum gravity path integral, where one instead finds classical solutions to supergravity with the appropriate boundary conditions.

[1] For example, the Donaldson–Witten twist of $\mathcal{N} = 4$ $SU(N)$ super-Yang–Mills is relevant for the set up in this chapter. For $N = 2$ the topological correlation functions have been computed explicitly for simply-connected spin four-manifolds of simple type in [154]; they may be written in terms of Abelian Seiberg–Witten invariants.

Yet, as already emphasised, even this is quite poorly understood. When the dual theory is a TQFT, in principle all observables are exactly computable in field theory for many classes of theories defined on different conformal boundary manifolds. Since the semi-classical gravity result must match the TQFT description, the AdS/CFT correspondence can potentially help to clarify the answers to some of the questions arising in the saddle point approximation and discussed in chapter 1.

Of course, one is tempted to push this line of argument further and speculate that this is a promising setting in which to try to formulate a topological form of quantum gravity on the AdS side of the correspondence. Such a theory should be completely equivalent to the dual TQFT description. At present this looks challenging, to say the least, but there is an analogous construction in topological string theory. Here $U(N)$ Chern–Simons gauge theory (a Schwarz-type TQFT) on a three-manifold M_3 is equivalent to open topological strings on T^*M_3 [224]. There is a large N duality relating this to a dual closed topological string description. For example, for $M_3 = S^3$ the closed strings propagate on the resolved conifold background, with N units of flux through the S^2 [188]. Here both sides are under computational control, and relate a TQFT to a topological sector of quantum gravity (string theory). This duality shares many features with AdS/CFT,[2] and might hint at how to attack the above problem.

For the time being, we begin much more modestly, setting up the basic problem in $\mathcal{N} = 4$ gauged supergravity in five dimensions. With appropriate boundary conditions this defines the Donaldson–Witten topological twist of the dual $\mathcal{N} = 2$ theory on the conformal boundary four-manifold, and we focus on the simplest observable, namely the partition function. Under AdS/CFT in the supergravity limit, minus the logarithm of the partition function is identified with the holographically renormalized supergravity action. We refer to this as the *gravitational free energy*.

The Donaldson–Witten twist may be interpreted as coupling the theory to a particular background $\mathcal{N} = 2$ conformal gravity multiplet, and in the next section we briefly review some aspects of the twisted theory relevant to the holographic construction. On the other hand, four-dimensional $\mathcal{N} = 2$ conformal gravity arises on the conformal boundary of asymptotically locally hyperbolic solutions to the Romans [197] $\mathcal{N} = 4^+$ gauged supergravity in five dimensions [186]. The real Euclidean signature version of this theory described

[2]This was emphasized by C. Vafa in his recent talk at the Princeton Workshop *20 Years Later: The Many Faces of AdS/CFT*.

in section 2.3 has, in addition to the bulk metric $G_{\mu\nu}$, an $SU(2)$ R-symmetry gauge field $\mathcal{A}^I_\mu$ ($I = 1, 2, 3$), a 1-form $\mathcal{C}$, and a scalar field X. (In general there is also a doublet of B-fields, but this is zero for the topological twist boundary condition, and moreover may be consistently set to zero in the Romans theory.)

The main property of a topological field theory is that appropriate correlation functions, including the partition function, are independent of any choice of metric. Assuming one is given an appropriate solution to the Romans theory with (M_4, g) as conformal boundary, we therefore expect the holographically renormalized action to be independent of g. Here one can mimic the field theory argument in [219], and attempt to show that arbitrary deformations $g_{ij} \to g_{ij} + \delta g_{ij}$ leave this action invariant. We have the general holographic Ward identity formula

$$\delta S \;=\; \int_{M_4} \mathrm{d}^4 x \sqrt{\det g} \left(\tfrac{1}{2} T_{ij}\, \delta g^{ij} + \mathscr{J}^i_I\, \delta A^I_i + \Xi\, \delta X_1 \right) \,. \tag{2.1.1}$$

Here S is the renormalized supergravity action of the Euclidean Romans theory, defined in section 2.3, while (g_{ij}, A^I_i, X_1) are the non-zero background fields in the $\mathcal{N} = 2$ conformal gravity multiplet for the topological twist. Equivalently, these arise as boundary values of the Romans fields: in particular A^I_i is simply the restriction of the bulk $SU(2)$ R-symmetry gauge field to the boundary at $z = 0$, while $X_1 = \lim_{z\to 0}(X-1)/z^2 \log z$. For the topological twist these quantities are all fixed by the choice of metric g_{ij}: A^I_i is fixed to be the right-handed spin connection, while $X_1 = -R/12$, where $R = R(g)$ is the Ricci scalar for g. Thus the variations of these fields appearing in (2.1.1) are all determined by the metric variation δg_{ij}. On the other hand, T_{ij}, $\mathscr{J}^i_I$ and Ξ are respectively the holographic vacuum expectation values (VEVs) of the operators for which these boundary fields are the sources. In particular T_{ij} is the holographic stress-energy tensor. As is well-known, the expansion of the equations of motion near $z = 0$ does not fix these VEVs in terms of boundary data on M_4, but rather they are only determined by regularity of the solution in the interior. Determining these quantities for fixed boundary data is thus an extremely non-linear problem. What allows progress in this case is supersymmetry: the partition function should be described by a supersymmetric solution to the Romans theory.[3] By similarly solving the Killing spinor

[3]If the dominant saddle point in the AdS/CFT relation (1.4.1) were non-supersymmetric, this would presumably be interpreted as spontaneous breaking of supersymmetry in the dual TQFT. This is certainly not expected in the case at hand, but would be interesting to investigate further.

equations in a Fefferman–Graham-like expansion, we are able to compute these VEVs for a general supersymmetric solution. This still leaves certain unknown data, ultimately determined by regularity in the interior, but remarkably these constraints are sufficient to prove that (2.1.1) is indeed zero, for arbitrary δg_{ij}! More precisely, in section 2.5, we show that the integrand on the right hand side is a total derivative, and its integral is then zero provided M_4 is closed, without boundary. The computation, although in principle straightforward, is not entirely trivial, and along the way we require some interesting identities that are specific to Riemannian four-manifolds (notably the quadratic curvature identity of Berger [45]).

We next analyse in more detail the geometry of supersymmetric solutions to the five-dimensional bulk supergravity theory in section 2.6. Because of the R-symmetry bundle, this geometry is characterized by what we call a *twisted* $Sp(1)$ structure satisfying a certain first-order differential system. Using these equations, remarkably we are able to show that the bulk on-shell action is always a total derivative. By carefully analysing the global structure of the twisted structure, and how this behaves where the bulk spinor becomes zero, this is shown to be globally a total derivative for any smooth solution. This is true on *any* five-manifold Y_5 that fills a four-manifold boundary $M_4 = \partial Y_5$. Moreover, on applying Stokes' theorem the bulk integral then always precisely cancels the boundary terms (including the holographic counterterms) in the action, with the net result being that the *gravitational free energy of any smooth solution is zero*!

These are the main results of the chapter, but they immediately raise a number of interesting questions. We postpone our discussion of these until the end of part I, after considering a lower-dimensional twist in the next chapter.

2.2 The Donaldson–Witten twist

In [219], Witten gave a physical construction of Donaldson invariants of four-manifolds [85, 86, 87] as certain correlation functions in a topological quantum field theory. This theory is constructed by taking pure $\mathcal{N} = 2$ Yang–Mills gauge theory and applying a topological twist: identifying a background $SU(2)$ R-symmetry gauge field with the right-handed spin connection results in a conserved scalar supercharge $\mathcal{Q}$, on any oriented Riemannian four-manifold (M_4, g). This has been reviewed in some detail in section 1.2.1.

The path integral of the twisted theory localizes onto Yang–Mills instantons, and correlation functions of $\mathcal{Q}$-invariant operators localize to integrals of certain forms over the instanton moduli space $\mathcal{M}$. These are precisely Donaldson's invariants of M_4. They are, under certain general conditions, independent of the choice of metric g on M_4, but in general depend on the diffeomorphism type of M_4. In particular, Donaldson invariants can sometimes distinguish manifolds which are homeomorphic but not diffeomorphic. That this is possible is because the instanton equations are PDEs, which depend on the differentiable structure. From the TQFT point of view, independence of the choice of metric follows by showing that metric deformations lead to $\mathcal{Q}$-exact changes in the integrand of the path integral. For example, the stress-energy tensor is $\mathcal{Q}$-exact, implying that the partition function is invariant under arbitrary metric deformations, and hence (at least formally) is a diffeomorphism invariant.

Donaldson–Witten theory is typically studied for pure $\mathcal{N} = 2$ Yang–Mills, with gauge group $\mathscr{G} = SU(2)$ or $\mathscr{G} = SO(3)$. However, the topological twist may be applied to any $\mathcal{N} = 2$ theory with matter, and also for any gauge group $\mathscr{G}$. For example, $\mathscr{G} = SU(N)$ Donaldson invariants were first studied in [168], with further mathematical work in [152]. In particular the latter reference contains some explicit large N results for the partition function on certain four-manifolds. As recounted in chapter 1, historically the development of Donaldson-like invariants took a rather different direction after the introduction of Seiberg–Witten invariants in [222]. The former may be expressed (conjecturally) in terms of the latter, but Seiberg–Witten theory is simpler and easier to compute with.

The Donaldson–Witten twist of $\mathcal{N} = 2$ gauge theories can also be understood as a special case of rigid supersymmetry. Soon after Witten's paper, Karlhede–Roček interpreted the construction as coupling the gauge theory to a background (*i.e.* non-dynamical) $\mathcal{N} = 2$ conformal gravity [142]. The background $SU(2)$ R-symmetry gauge field is part of this gravity multiplet, and is embedded into the spin connection in such a way that the Killing spinor equations of the theory admit a constant solution, leading to the conserved scalar supercharge $\mathcal{Q}$. There is also an auxiliary scalar field turned on in this background gravity multiplet, proportional to the Ricci scalar curvature of (M_4, g). Generalizing [142], $\mathcal{N} = 2$ theories may be coupled to a background $\mathcal{N} = 2$ conformal supergravity in the spirit of [97] reviewed in section 1.2.2 [146]. Generically this requires the existence of a conformal Killing vector on (M_4, g), but the topological twist arises as a degenerate special case, in which (M_4, g) is arbitrary.

2.2.1 Half-twisted $\mathcal{N} = 4$ super Yang–Mills

The procedure of topological twisting may also be applied to theories with different amounts of supersymmetry, and in various dimensions. For example, the larger $SU(4)$ R-symmetry of four-dimensional $\mathcal{N} = 4$ Yang–Mills leads to three inequivalent twists (two of them were constructed by Yamron in [228], and a third one is briefly mentioned there as a "private communication" from Witten). To classify them, it is easier to see the twists from the group-theoretic point of view: the spacetime symmetry group is still $\mathcal{K} = Spin(4) = SU(2)_\ell \times SU(2)_r$, but the R-symmetry group is $\mathcal{H} = Spin(6)_R = SU(4)_R$, and the supercharges transform under $\mathcal{K} \times \mathcal{H}$ as

$$Q_{I,\alpha} \quad (\mathbf{2}, \mathbf{1}, \overline{\mathbf{4}}) \,, \qquad \overline{Q}^I_{\dot{\alpha}} \quad (\mathbf{1}, \mathbf{2}, \mathbf{4}) \,. \tag{2.2.1}$$

One then looks for a homomorphism from $Spin(4)$ into $SU(4)_R$, and defines the twisted spacetime symmetry group to be the diagonal combination of $Spin(4)$ and the image of the homeomorphism in $SU(4)_R$, thus obtaining a new group isomorphic to $Spin(4)$, but a different physical theory (at least generically). Concretely, the twists can be characterised by the way the representation $\overline{\mathbf{4}}$ of $SU(4)_R$ transforms as a representation of $Spin(4)$, so we need to look for four-dimensional representations of $Spin(4)$, and then choose those giving a supercharge that is a scalar under $\mathcal{K}'$.[4] We obtain three inequivalent twists by the following four-dimensional representations of $\mathcal{K}$

$$\begin{aligned} &\text{(i)} \quad (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \,, \\ &\text{(ii)} \quad (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{2}, \mathbf{1}) \,, \\ &\text{(iii)} \quad (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \,, \end{aligned} \tag{2.2.2}$$

[4] The difference between $\mathbf{4}$ and $\overline{\mathbf{4}}$ is accounted for by the different charge under the remnant $U(1)$ symmetry of the twisted theory, which we are not mentioning in this section. For more details, see [141].

and obviously their mirrors (with the two $SU(2)$ factors exchanged). Under the twisted spacetime group $\mathcal{K}'$, the supercharges transform as

$$\begin{aligned} &\text{(i)} \quad 2(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{3},\mathbf{1}) \oplus 2(\mathbf{2},\mathbf{2})\,, \\ &\text{(ii)} \quad 2(\mathbf{1},\mathbf{1}) \oplus 2(\mathbf{3},\mathbf{1}) \oplus 2(\mathbf{2},\mathbf{2})\,, \\ &\text{(iii)} \quad (\mathbf{1},\mathbf{1}) \oplus 2(\mathbf{2},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{3}) \oplus (\mathbf{2},\mathbf{2})\,. \end{aligned} \tag{2.2.3}$$

It is easy to do some further group theory and show that these are the only four-dimensional representations that result in a scalar supercharge. For instance, obviously the trivial $\overline{\mathbf{4}} \to 4(\mathbf{1},\mathbf{1})$ does not produce scalar supercharges, and so does $(\mathbf{1},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})$, which yields

$$(\mathbf{2},\mathbf{1}) \oplus 2(\mathbf{1},\mathbf{2}) \oplus (\mathbf{2},\mathbf{3}) \oplus (\mathbf{1},\mathbf{4})\,. \tag{2.2.4}$$

The first twist in (2.2.2), which is the one privately communicated to Yamron by Witten, was originally studied by Marcus [167], and later found an application to the Geometric Langlands program thanks to the work of Kapustin and Witten [141]. The second twist is the Vafa–Witten twist, for which the only non-vanishing observable is the partition function corresponding to the Euler characteristic of a moduli space of instantons [215].[5] For both these twists there are two scalar supercharges. However, the third twist only contains one scalar supercharge, and is thus often referred to as the "half-twisted" $\mathcal{N} = 4$ theory. This is the theory relevant for this paper, and can be also obtained by viewing the $\mathcal{N} = 4$ theory as an $\mathcal{N} = 2$ theory coupled to an adjoint matter multiplet and applying the Donaldson–Witten twist [228]. An important restriction on the possible background manifolds is given by the fact that the half-twisted theory still contains spinor fields [228]. Therefore, it can only be defined on spin manifolds.

For general gauge group $\mathscr{G}$, the path integral localizes [153, 154] onto solutions to a non-Abelian [156] version of the Seiberg–Witten equations, in which the spinor field is in the adjoint representation of $\mathscr{G}$ (see also the review in [163]). In particular the (virtual) dimension of the relevant non-Abelian monopole moduli space $\mathcal{M}$ may be computed using index theory, leading to

$$\dim \mathcal{M} \;=\; -\tfrac{1}{4} \dim \mathscr{G} \cdot \left[2\chi(M_4) + 3\sigma(M_4)\right]\;. \tag{2.2.5}$$

[5] The virtual dimension of the relevant moduli space is exactly zero, so the only observable that can be non-vanishing is the partition function. Any other observable would vanish because of fermionic zero-modes.

Because of the associated fermion zero modes, the partition function of the theory vanishes unless the right hand side of (2.2.5) is also zero. On the other hand, when the right hand side of (2.2.5) is positive, one obtains non-zero invariants in the TQFT by inserting appropriate $\mathcal{Q}$-exact operators into the path integral.

An important observation is that (2.2.5) is independent of the topology of the gauge bundle over M_4, unlike the corresponding case for Donaldson theory (pure $\mathcal{N} = 2$ Yang–Mills with gauge group $\mathscr{G}$). Because of this, all choices of gauge bundle seem to contribute to the partition function at the same time. The left hand side of (1.4.1) then needs appropriately interpreting for such twists of four-dimensional $\mathcal{N} = 2$ SCFTs, as taken at face value it may be divergent. There is a standard way to deal with this, namely to refine the partition function via the $U(1)_R$ charge. For example, this is discussed at the end of section 2 of [117], and in [118].[6] This could play an important rôle in making sense also of the right hand side of (1.4.1) – we will briefly comment on this at the end of section 2.6.4.

As far as we are aware, computations of topological observables in the half-twisted $\mathcal{N} = 4$ theory, for general $\mathscr{G} = SU(N)$, have not been done explicitly. However, for $\mathscr{G} = SU(2)$ the partition function and topological correlation functions have been computed explicitly for simply-connected spin four-manifolds of simple type [154]. This is done by giving masses, explicitly breaking $\mathcal{N} = 4$ to $\mathcal{N} = 2$, leading to an $\mathcal{N} = 2$ gauge theory with a massive adjoint hypermultiplet, a twisted version of the $\mathcal{N} = 2^*$ theory. The twisted theory is still topological, and the relevant observables are written in terms of Seiberg–Witten invariants using the methods of [180]. Observables for the original theory are then identified with the massless limit of these formulae (when this makes sense), although the validity of this assertion is not completely clear. In any case, to compare to the holographic construction in this chapter one should compute the large N limit for gauge group $\mathscr{G} = SU(N)$. We note that an analogous large N limit of Donaldson invariants (for pure $\mathcal{N} = 2$ $SU(N)$ Yang–Mills) has been computed in [152]. Unlike the formula (2.2.5), here the dimension of the moduli space of instantons depends on the topology of the gauge bundle. One can then choose this bundle in such a way that $\dim \mathcal{M} = 0$. The partition function is a certain signed count of the points that make up $\mathcal{M}$, and the large N limit was computed for a certain class of

[6]The necessity of refinement is not clear in the literature, as the partition function for twisted $\mathcal{N} = 2$ SCFTs (for which the bundles would all be on the same footing) has been computed without refining and without divergences [179]. Moreover, for the case of the half-twist of $\mathcal{N} = 4$ SYM, the partition function is known on K3, and the instantons contribute weighted by their instanton number $q = e^{2\pi i\tau}$, where τ is the complexified gauge coupling. Therefore, the series already contains a counting parameter.

four-manifolds in [152].[7]

2.2.2 Uplifting

As we saw in chapter 1, in order to make quantitative comparisons between calculations on the two sides of the AdS/CFT correspondence, the holographic computation in the lower-dimensional effective supergravity needs to be embedded in string theory. For the case at hand this is straightforward, especially thanks to the amount of work done on five-dimensional supergravity dual to four-dimensional field theories. The relevant five-dimensional gauged supergravity we are interested in was constructed some time ago by Romans, it is usually referred to as $\mathcal{N} = 4^+$ model and admits a supersymmetric anti-de Sitter vacuum [197]. It is a consistent truncation of both Type IIB supergravity on S^5 [164], and also of eleven-dimensional supergravity on N_6 [110], where N_6 are the geometries classified by Lin–Lunin–Maldacena [160]. This means that any solution to the five-dimensional Romans theory uplifts (at least locally – see below) to a string/M-theory solution, and the details of the dual field theory are encoded in the geometry of the internal space involved in the uplifting.

In order to be concrete, let us focus on the case of $\mathcal{N} = 4$ Yang–Mills theory considered above. For $\mathscr{G} = SU(N)$, AdS/CFT should relate the large N limit of this theory to an appropriate class of solutions to the Romans $\mathcal{N} = 4^+$ theory in five dimensions, uplifted on S^5 to give full solutions of Type IIB string theory. This is where the restriction that M_4 is spin enters: if M_4 is not spin then the background $SU(2)$ R-symmetry gauge field we turn on to perform the twist is not globally a connection on an $SU(2)$ bundle over M_4. On the other hand, the Type IIB solution is an S^5 fibration over the filling Y_5, where $S^5 \subset \mathbb{C}^2 \oplus \mathbb{C}$, and $SU(2)$ acts on $\mathbb{C}^2$ in the fundamental representation. Thus if M_4 is not spin, this associated bundle is not well-defined. This is the gravity dual appearance of the requirement we saw directly in the TQFT: for the half-twist of $\mathcal{N} = 4$ Yang–Mills there are still spinors in the twisted theory, which only make sense if M_4 is spin. Finally, we note that for the large N limit of the $\mathscr{G} = SU(N)$ half-twisted $\mathcal{N} = 4$ Yang–Mills theory, a standard AdS/CFT

[7] In particular the final section of [152] computes the large N limit of the partition function Z for a four-manifold with boundary, constructed as $S^1 \times M_3$ where M_3 is a knot complement. One finds $Z \sim N \log \alpha$, where α is a certain knot invariant (the Mahler measure).

formula fixes the dual effective five-dimensional Newton constant as

$$\frac{1}{\kappa_5^2} = \frac{N^2}{4\pi^2} . \tag{2.2.6}$$

Similar remarks apply to twists of $\mathcal{N} = 2$ SCFTs with M-theory duals. Indeed, an important restriction on the class of $\mathcal{N} = 2$ gauge theories to which this holographic description applies is that they are conformal theories.[8] A large number of examples arise as class $\mathcal{S}$ theories [104], obtained by wrapping M5-branes over punctured Riemann surfaces, for which the gravity dual was found in [105] using the construction of [160]. Romans solutions uplift on the corresponding internal spaces N_6 to solutions of M-theory [110]. At the level of the five-dimensional theory, all that changes is the formula (2.2.6) for the effective Newton constant, which in general reads [127]

$$\frac{1}{\kappa_5^2} = \frac{\mathtt{a}}{\pi^2} , \tag{2.2.7}$$

where $\mathtt{a}$ is the $\mathtt{a}$ central charge. In the supergravity limit recall that $\mathtt{a} = \mathtt{c}$. For the above-mentioned M5-brane theories the central charge scales with N^3 as $N \to \infty$. Indeed, the partition function will *a priori* depend on both the choice of $\mathcal{N} = 2$ SCFT that is being twisted, and also on the four-manifold M_4 on which it is defined. The choice of theory corresponds to the choice of internal space in the uplifting to ten or eleven dimensions. The structure of the dual supergravity solution as a fibration of the internal space over the spacetime filling of M_4 then implies that the large N limits of the partition functions should also factorize. That is, the dependence on the choice of theory should only be visible via the central charge $\mathtt{a}$, which via (2.2.7) fixes the overall normalization of the supergravity action. On the other hand, the dependence on the choice of M_4 is then captured by the effective five-dimensional Romans theory we will describe.[9]

[8] In particular this is not true of pure $\mathcal{N} = 2$ Yang–Mills, from which the original Donaldson invariants are constructed.

[9] This structure can already be seen in the more general formula for $\dim \mathcal{M}$ given in [117]. For the general class of twisted field theories considered there, equation (2.42) of [117] implies that in the large N limit where $\mathtt{a} = \mathtt{c}$, one has $\dim \mathcal{M} = -\mathtt{a}[2\chi(M_4) + 3\sigma(M_4)]$, generalizing (2.2.5). The central charge appears as an overall factor, at large N.

2.3 Holographic supergravity theory

In section 2.3.1 we define a real Euclidean section of $\mathcal{N} = 4^+$ gauged supergravity in five dimensions. A Fefferman–Graham expansion of asymptotically locally hyperbolic solutions to this theory is constructed in section 2.3.2, for arbitrary conformal boundary four-manifold (M_4, g). Using this, in section 2.3.3 we holographically renormalize the action.

2.3.1 Euclidean Romans $\mathcal{N} = 4^+$ theory

The Lorentzian signature Romans $\mathcal{N} = 4^+$ theory [197] is a five-dimensional $SU(2) \times U(1)$ gauged supergravity which admits a supersymmetric AdS_5 vacuum. The bosonic sector comprises the metric $G_{\mu\nu}$, a dilaton ϕ, an $SU(2)_R$ Yang–Mills gauge field $\mathcal{A}^I_\mu$ $(I = 1, 2, 3)$, a $U(1)_R$ gauge field $\mathcal{A}_\mu$, and two real anti-symmetric tensors $B^\alpha_{\mu\nu}$, $\alpha = 4, 5$, which transform as a charged doublet under $U(1)_R \cong SO(2)_R$. It is convenient to introduce the scalar field $X \equiv \mathrm{e}^{-\frac{1}{\sqrt{6}}\phi}$ and the complex combinations $\mathcal{B}^\pm \equiv B^4 \pm \mathrm{i}B^5$. The associated field strengths are $\mathcal{F} = \mathrm{d}\mathcal{A}$, $\mathcal{F}^I = \mathrm{d}\mathcal{A}^I - \frac{1}{2}\epsilon^I{}_{JK}\mathcal{A}^J \wedge \mathcal{A}^K$, and $H^\pm = \mathrm{d}\mathcal{B}^\pm \mp \mathrm{i}\mathcal{A} \wedge \mathcal{B}^\pm$. We have set the gauged supergravity gauge coupling to 1.[10]

The bosonic action and equations of motion in Lorentzian signature appear in [164]. However, as we are interested in holographic duals to TQFTs defined on Riemannian four-manifolds, we require the Euclidean signature version of this theory. The Wick rotation in particular introduces a factor of i into the Chern–Simons couplings, leading to the Euclidean action

$$\begin{aligned} I \ = \ -\frac{1}{2\kappa_5^2}\int \Big[& R *1 - 3X^{-2}\mathrm{d}X \wedge *\mathrm{d}X + 4(X^2 + 2X^{-1}) *1 - \tfrac{1}{2}X^4\, \mathcal{F} \wedge *\mathcal{F} \\ & - \tfrac{1}{4}X^{-2}\left(\mathcal{F}^I \wedge *\mathcal{F}^I + \mathcal{B}^- \wedge *\mathcal{B}^+\right) + \tfrac{1}{8}\mathcal{B}^- \wedge H^+ - \tfrac{1}{8}\mathcal{B}^+ \wedge H^- - \tfrac{\mathrm{i}}{4}\mathcal{F}^I \wedge \mathcal{F}^I \wedge \mathcal{A}\Big] . \end{aligned} \tag{2.3.1}$$

Here $R = R(G)$ denotes the Ricci scalar of the metric $G_{\mu\nu}$, and $*$ is the Hodge duality operator acting on forms. The associated equations of motion are:[11]

$$\begin{aligned} \mathrm{d}(X^{-1} * \mathrm{d}X) \ &= \tfrac{1}{3}X^4\, \mathcal{F} \wedge *\mathcal{F} - \tfrac{1}{12}X^{-2}\left(\mathcal{F}^I \wedge *\mathcal{F}^I + \mathcal{B}^- \wedge *\mathcal{B}^+\right) \\ & \quad - \tfrac{4}{3}(X^2 - X^{-1}) * 1 \, , \end{aligned} \tag{2.3.2}$$

$$\mathrm{d}(X^{-2} * \mathcal{F}^I) \ = \ \epsilon^I{}_{JK}X^{-2} * \mathcal{F}^J \wedge \mathcal{A}^K - \mathrm{i}\mathcal{F}^I \wedge \mathcal{F} \, , \tag{2.3.3}$$

[10] In addition we have rescaled the $SU(2)_R$ gauge field and the anti-symmetric tensors by a factor of $1/\sqrt{2}$, compared to [164].

[11] Equation (2.3.3) incorporates a correction to the Lorentzian equation, in line with [110].

$$d(X^4 * \mathcal{F}) = -\tfrac{i}{4}\mathcal{F}^I \wedge \mathcal{F}^I - \tfrac{i}{4}\mathcal{B}^- \wedge \mathcal{B}^+ \,, \tag{2.3.4}$$

$$H^\pm = \pm X^{-2} * \mathcal{B}^\pm \,, \tag{2.3.5}$$

$$\begin{aligned} R_{\mu\nu} &= 3X^{-2}\partial_\mu X \partial_\nu X - \tfrac{4}{3}(X^2 + 2X^{-1})G_{\mu\nu} + \tfrac{1}{2}X^4\big(\mathcal{F}_\mu{}^\rho \mathcal{F}_{\nu\rho} - \tfrac{1}{6}G_{\mu\nu}\mathcal{F}^2\big) \\ &\quad + \tfrac{1}{4}X^{-2}\big(\mathcal{F}^{I\rho}_\mu \mathcal{F}^I_{\nu\rho} - \tfrac{1}{6}G_{\mu\nu}(\mathcal{F}^I)^2 + \mathcal{B}^-{}_{(\mu}{}^\rho \mathcal{B}^+_{\nu)\rho} - \tfrac{1}{6}G_{\mu\nu}\mathcal{B}^-{}_{\rho\sigma}\mathcal{B}^{+\rho\sigma}\big) \,. \end{aligned} \tag{2.3.6}$$

Here $\mathcal{F}^2 \equiv \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}$, $(\mathcal{F}^I)^2 \equiv \sum_{I=1}^3 \mathcal{F}^I_{\mu\nu}\mathcal{F}^{I\mu\nu}$. In general equations (2.3.2)–(2.3.6) are complex, and solutions will likewise be complex. However, note that setting $i\mathcal{A} \equiv \mathcal{C}$ effectively removes all factors of i. We may then consistently define a real section of this Euclidean theory in which all fields, and in particular $\mathcal{C}$ and $\mathcal{B}^\pm = B^4 \pm iB^5$, are real. We henceforth impose these reality conditions. Although globally $\mathcal{A}$ is a $U(1)_R$ gauge field in the original Lorentzian theory, after the above Wick rotation the real field $\mathcal{C} = i\mathcal{A}$ effectively becomes an $SO(1,1)_R$ gauge field. We may then think of $\mathcal{C}$ as a global 1-form, but for which the theory has a symmetry $\mathcal{C} \to \mathcal{C} - d\lambda$, for any global function λ. We denote the corresponding field strength as $\mathcal{G} \equiv d\mathcal{C} = i\mathcal{F}$.

In the Lorentzian theory the fermionic sector contains four gravitini and four dilatini, which together with the spinor parameters ϵ all transform in the fundamental **4** representation of the $Sp(2)_R$ global R-symmetry group. The $SU(2) \times U(1) \subset Sp(2)$ gauge symmetry arises as a gauged subgroup. Since $Sp(2) \cong Spin(5)$ it is natural to introduce the associated Clifford algebra Cliff(5,0), with generators Γ_A, $A = 1, \ldots, 5$, satisfying $\{\Gamma_A, \Gamma_B\} = 2\delta_{AB}$. We then decompose $I, J, K = 1, 2, 3$, transforming in the **3** of $SU(2)$, and $\alpha, \beta = 4, 5$ in the **2** of $U(1)$. In Euclidean signature the conditions for preserving supersymmetry are then the vanishing of the following supersymmetry variations of the gravitini and dilatini, respectively:

$$\begin{aligned} 0 &= D_\mu\epsilon + \tfrac{i}{3}\gamma_\mu\Big(X + \tfrac{1}{2}X^{-2}\Big)\Gamma_{45}\epsilon \\ &\quad + \tfrac{i}{24}(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho)\Big(X^{-1}\big(\mathcal{F}^I_{\nu\rho}\Gamma_I + B^\alpha_{\nu\rho}\Gamma_\alpha\big) + X^2\mathcal{F}_{\nu\rho}\Big)\epsilon \,, \end{aligned} \tag{2.3.7}$$

$$\begin{aligned} 0 &= \tfrac{\sqrt{3}}{2} i\gamma^\mu X^{-1}\partial_\mu X\epsilon + \tfrac{1}{\sqrt{3}}\Big(X - X^{-2}\Big)\Gamma_{45}\epsilon \\ &\quad + \tfrac{1}{8\sqrt{3}}\gamma^{\mu\nu}\Big(X^{-1}\big(\mathcal{F}^I_{\mu\nu}\Gamma_I + B^\alpha_{\mu\nu}\Gamma_\alpha\big) - 2X^2\mathcal{F}_{\mu\nu}\Big)\epsilon \,, \end{aligned} \tag{2.3.8}$$

where the covariant derivative is

$$D_\mu\epsilon \equiv \nabla_\mu\epsilon + \tfrac{1}{2}\mathcal{A}_\mu\Gamma_{45}\epsilon + \tfrac{1}{2}\mathcal{A}^I_\mu\Gamma_{I45}\epsilon \,. \tag{2.3.9}$$

Here γ_μ, $\mu = 1, \ldots, 5$, are generators of the Euclidean spacetime Clifford algebra, satisfying $\{\gamma_\mu, \gamma_\nu\} = 2G_{\mu\nu}$, where recall $G_{\mu\nu}$ is the metric. Given the gauging it is natural to introduce the following choice of generators:

$$\Gamma_I = \sigma_3 \otimes \sigma_I \,, \quad I = 1,2,3 \,, \qquad \Gamma_4 = \sigma_1 \otimes \mathbb{1}_2 \,, \qquad \Gamma_5 = \sigma_2 \otimes \mathbb{1}_2 \,, \tag{2.3.10}$$

where σ_I are the Pauli matrices, and $\mathbb{1}_2$ denotes the 2×2 identity matrix. In particular notice that $\Gamma_{45} = \mathrm{i}\sigma_3 \otimes \mathbb{1}_2$ squares to $-\mathbb{1}_4$, and we may write

$$\epsilon = \begin{pmatrix} \epsilon^+ \\ \epsilon^- \end{pmatrix} \,, \tag{2.3.11}$$

where the spinor doublets $\epsilon^\pm$ denote projections onto the $\pm\mathrm{i}$ eigenspaces of Γ_{45}, respectively. One then has

$$\Gamma_I \epsilon = \begin{pmatrix} \sigma_I \epsilon^+ \\ -\sigma_I \epsilon^- \end{pmatrix} \,, \qquad B^\alpha_{\mu\nu} \Gamma_\alpha \epsilon = \begin{pmatrix} \mathcal{B}^-_{\mu\nu} \epsilon^- \\ \mathcal{B}^+_{\mu\nu} \epsilon^+ \end{pmatrix} \,. \tag{2.3.12}$$

We next introduce the charge conjuguation matrix $\mathscr{C}$ for the Euclidean spacetime Clifford algebra. By definition $\gamma^*_\mu = \mathscr{C}^{-1} \gamma_\mu \mathscr{C}$, and one may choose Hermitian generators $\gamma^\dagger_\mu = \gamma_\mu$ together with the conditions $\mathscr{C} = \mathscr{C}^* = -\mathscr{C}^T$, $\mathscr{C}^2 = -1$. We may then define the following charge conjugate spinor in Euclidean signature

$$\epsilon^c \equiv (\sigma_3 \otimes \mathrm{i}\sigma_2) \mathscr{C} \epsilon^* \,. \tag{2.3.13}$$

It is straightforward to check that $(\epsilon^c)^c = \epsilon$. Moreover, provided $\mathcal{C} = \mathrm{i}\mathcal{A}$ and $\mathcal{B}^\pm$ (and all other bosonic fields) are real, then one can show that ϵ satisfies the gravitini and dilatini equations (2.3.7), (2.3.8) if and only if its charge conjugate ϵ^c satisfies the same equations. Given this property, we may consistently impose the symplectic Majorana condition $\epsilon^c = \epsilon$. We will be interested in solutions that satisfy these reality conditions.

2.3.2 Fefferman–Graham expansion

In this section we determine the Fefferman–Graham expansion [96] of asymptotically locally hyperbolic solutions to this Euclidean Romans theory. This is the general solution to the bosonic equations of motions (2.3.2)–(2.3.6), expressed as a perturbative expansion in a radial coordinate near the conformal boundary.

We take the form of the metric to be [96]

$$G_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \;=\; \frac{1}{z^2}\mathrm{d}z^2 + \frac{1}{z^2}\mathrm{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j \;=\; \frac{1}{z^2}\mathrm{d}z^2 + h_{ij}\mathrm{d}x^i\mathrm{d}x^j \;. \tag{2.3.14}$$

where the AdS radius $\ell = 1$, and in turn we have the expansion

$$\mathrm{g}_{ij} \;=\; \mathrm{g}^0_{ij} + z^2\mathrm{g}^2_{ij} + z^4\big(\mathrm{g}^4_{ij} + h^0_{ij}(\log z)^2 + h^1_{ij}\log z\big) + o(z^4)\;. \tag{2.3.15}$$

Here $\mathrm{g}^0_{ij} = g_{ij}$ is the boundary metric induced on the conformal boundary M_4 at $z = 0$.

It is convenient to introduce the inner product $\langle\alpha, \beta\rangle$ between two forms α, β via (A.2.1). The volume form for the five-dimensional bulk metric (2.3.14) is

$$\mathrm{vol}_5 \;=\; \frac{1}{z^5}\mathrm{d}z\wedge\mathrm{vol}_\mathrm{g} \;=\; \frac{1}{z^5}\mathrm{d}z\wedge\sqrt{\det\mathrm{g}}\,\mathrm{d}x^1\wedge\cdots\wedge\mathrm{d}x^4\;. \tag{2.3.16}$$

The determinant may then be expanded in a series in z, around that for g^0, as follows

$$\begin{aligned}\sqrt{\det\mathrm{g}} \;=\; \sqrt{\det\mathrm{g}^0}\,\Big[1 + \tfrac{z^2}{2}t^{(2)} + \tfrac{z^4}{2}\big(t^{(4)} - \tfrac{1}{2}t^{(2,2)} + \tfrac{1}{4}(t^{(2)})^2 \\ + u^{(0)}(\log z)^2 + u^{(1)}\log z\big)\Big] + o(z^4)\;.\end{aligned} \tag{2.3.17}$$

Here we have denoted $t^{(n)} \equiv \mathrm{Tr}\left[(\mathrm{g}^0)^{-1}\mathrm{g}^n\right]$, $u^{(n)} \equiv \mathrm{Tr}\left[(\mathrm{g}^0)^{-1}h^n\right]$ and $t^{(2,2)} \equiv \mathrm{Tr}\left[(\mathrm{g}^0)^{-1}\mathrm{g}^2\right]^2$.

The remaining bosonic fields are likewise expanded as follows:

$$X \;= 1 + z^2\left(X_1\log z + X_2\right) + z^4(X_3\log z + X_4) + o(z^4)\;, \tag{2.3.18}$$

$$\mathcal{A}^I \;= A^I + z^2(a^I_1\log z + a^I_2) + o(z^2)\;, \tag{2.3.19}$$

$$\mathcal{A} \;= \mathrm{a} + z^2(\mathrm{a}_1\log z + \mathrm{a}_2) + o(z^2)\;, \tag{2.3.20}$$

$$\mathcal{B}^\pm \;= \frac{1}{z}b^\pm + \mathrm{d}z\wedge b^\pm_1 + z(b^\pm_2\log z + b^\pm_3) + o(z)\;, \tag{2.3.21}$$

A priori there are additional terms that appear in these expansions. However, these may either be gauged away, or turn out to be set to zero by the equations of motion, and we have thus removed them in order to streamline the presentation.

We now substitute the above expansions into the equations of motion (2.3.2)–(2.3.6) and solve them order by order in the radial coordinate z in terms of the boundary data

$g^0 = g, X_1, A^I,$ a and $b^\pm$. This will leave a number of terms undetermined. For the Einstein equation (2.3.6) we will need the Ricci tensor of the metric (2.3.14):

$$R_{zz} = -\frac{4}{z^2} - \frac{1}{2}\Big(\mathrm{Tr}\left[g^{-1}\partial_z^2 g\right] - \tfrac{1}{z}\mathrm{Tr}\left[g^{-1}\partial_z g\right] - \tfrac{1}{2}\mathrm{Tr}\left[g^{-1}\partial_z g\right]^2\Big)\,, \tag{2.3.22}$$

$$\begin{aligned} R_{ij} = -\frac{4}{z^2}g_{ij} - \Big(&\tfrac{1}{2}\partial_z^2 g - \tfrac{3}{2z}\partial_z g - \tfrac{1}{2}(\partial_z g)g^{-1}(\partial_z g) + \tfrac{1}{4}(\partial_z g)\mathrm{Tr}\left[g^{-1}\partial_z g\right] \\ &- R(g) - \tfrac{1}{2z}g\mathrm{Tr}\left[g^{-1}\partial_z g\right]\Big)_{ij}\,, \end{aligned} \tag{2.3.23}$$

$$R_{zi} = -\frac{1}{2}(g^{-1})^{jk}\Big(\nabla_i g_{jk,z} - \nabla_k g_{ij,z}\Big)\,. \tag{2.3.24}$$

Here ∇ is the covariant derivative for g, and we have corrected the sign of $R(g)_{ij}$ and the right hand side of (2.3.24) compared to [212].

Examining first the equation (2.3.5) gives at leading order

$$*_{g^0}\, b^\pm \;=\; \mp b^\pm\,, \tag{2.3.25}$$

so that the boundary B-fields b^+, b^- are required to be anti-self-dual and self-dual, respectively. At subleading orders one finds

$$b_1^\pm \;=\; \mp *_{g^0}\left(\mathrm{d}b^\pm \mp \mathrm{i}a \wedge b^\pm\right)\,, \qquad *_{g^0}\, b_2^\pm \;=\; \pm(b_2^\pm - 2X_1\, b^\pm)\,. \tag{2.3.26}$$

In particular notice that the first equation fixes $b_1^\pm$ in terms of boundary data, while the second equation determines only the anti-self-dual/self-dual parts of $b_2^\pm$, respectively. An equation may also be derived for $b_3^\pm$, although we will not need this in what follows.

Next the gauge field equations (2.3.3), (2.3.4) determine

$$\begin{aligned} a_1 \;&=\; -\tfrac{1}{2} *_{g^0} \mathrm{d} *_{g^0} f + \tfrac{\mathrm{i}}{8} *_{g^0}\left(b^- \wedge b_1^+ + b^+ \wedge b_1^-\right)\,, \\ a_1^I \;&=\; -\tfrac{1}{2} *_{g^0} \mathcal{D} *_{g^0} F^I\,, \end{aligned} \tag{2.3.27}$$

in terms of boundary data, where the curvatures are $f \equiv \mathrm{d}a$, $F^I \equiv \mathrm{d}A^I - \frac{1}{2}\epsilon^I{}_{JK}A^J \wedge A^K$, and we have introduced a gauge covariant derivative with respect to the boundary $SU(2)$ field: $\mathcal{D}\alpha^I \equiv \mathrm{d}\alpha^I - \epsilon^I{}_{JK}A^J \wedge \alpha^K$. In addition we have the constraints

$$\mathrm{d} *_{g^0} a_2 \;=\; -\tfrac{\mathrm{i}}{8}F^I \wedge F^I\,, \qquad \mathcal{D} *_{g^0} a_2^I \;=\; 0\,, \tag{2.3.28}$$

which leave a_2 and a_2^I partially undetermined.

Turning next to the scalar equation of motion (2.3.2) we find

$$4X_3 = -\nabla^2 X_1 - 2\left(t^{(2)}X_1 - 2X_1^2\right) - \tfrac{1}{24}\left(\langle b^+, b_2^-\rangle_{g^0} + \langle b^-, b_2^+\rangle_{g^0}\right) , \tag{2.3.29}$$

$$\begin{aligned} 4X_4 = &-\nabla^2 X_2 - \left(t^{(2)}X_1 + 2t^{(2)}X_2 - X_1^2 - 4X_1X_2 + 4X_3\right) - \tfrac{1}{24}\langle F^I, F^I\rangle_{g^0} + \tfrac{1}{6}\langle f, f\rangle_{g^0} \\ &- \tfrac{1}{12}\langle b_1^+, b_1^-\rangle_{g^0} + \tfrac{1}{12}\langle b^-, g^2 \circ b^+\rangle_{g^0} - \tfrac{1}{24}\left(\langle b^+, b_3^-\rangle_{g^0} + \langle b^-, b_3^+\rangle_{g^0}\right) . \end{aligned} \tag{2.3.30}$$

We regard these as determining X_3, X_4 in terms of X_1 (a boundary field), and X_2 (which is undetermined by the equations of motion), together with the other fields in the expansion. In the second equation we have used the definition

$$(g^2 \circ \alpha)_{i_1 \cdots i_p} \equiv (g^2)_{[i_1}{}^{j}\alpha_{|j|i_2\cdots i_p]} , \tag{2.3.31}$$

where α is a p-form on M_4. Here indices are always raised with g^0, so $(g^2)_i{}^j \equiv (g^2)_{ik}(g^0)^{kj}$.

Finally, we introduce the matter-modified boundary Ricci tensor

$$\mathscr{R}_{ij} = \mathscr{R}_{ij}(g^0) \equiv R_{ij}(g^0) - \tfrac{1}{4}(b^+)_{(i}{}^{k}(b^-)_{j)k} . \tag{2.3.32}$$

Notice the scalar curvature is $\mathscr{R}(g^0) = R(g^0)$, due to the opposite duality properties (2.3.25) of $b^\pm$. From the ij component of the Einstein equation (2.3.6), using (2.3.23) gives

$$g_{ij}^2 = -\tfrac{1}{2}\left(\mathscr{R}_{ij} - \tfrac{1}{6}\, g_{ij}^0 \mathscr{R}\right) . \tag{2.3.33}$$

The right hand side is a matter-modified form of the Schouten tensor. From this expression we immediately deduce the traces

$$t^{(2)} = -\tfrac{1}{6}\mathscr{R} , \qquad t^{(2,2)} = \tfrac{1}{4}\left(\mathscr{R}_{ij}\mathscr{R}^{ij} - \tfrac{2}{9}\mathscr{R}^2\right) . \tag{2.3.34}$$

The zz component of the Einstein equation in (2.3.6), together with (2.3.22), determines the traces of higher order components in the expansion of the bulk metric:

$$u^{(0)} = -2X_1^2 , \tag{2.3.35}$$

$$u^{(1)} = -4X_1X_2 + \tfrac{1}{96}\left(\langle b^+, b_2^-\rangle_{g^0} + \langle b^-, b_2^+\rangle_{g^0}\right) , \tag{2.3.36}$$

$$\begin{aligned} 4t^{(4)} &= t^{(2,2)} - u^{(0)} - 3u^{(1)} - 3X_1^2 - 8X_2^2 - 12X_1X_2 + \tfrac{1}{12}\left(\langle \mathrm{f},\mathrm{f}\rangle_{\mathrm{g}^0} + \tfrac{1}{2}\langle F^I, F^I\rangle_{\mathrm{g}^0}\right) \\ &\quad - \tfrac{1}{6}\langle b_1^+, b_1^-\rangle_{\mathrm{g}^0} - \tfrac{1}{12}\langle b^-, (\mathrm{g}^2 \circ b^+)\rangle_{\mathrm{g}^0} + \tfrac{1}{24}\left(\langle b^+, b_3^-\rangle_{\mathrm{g}^0} + \langle b^-, b_3^+\rangle_{\mathrm{g}^0}\right) . \end{aligned} \tag{2.3.37}$$

Returning to the ij component we may determine the logarithmic terms in (2.3.15):

$$\begin{aligned} h_{ij}^0 &= \tfrac{1}{4}\mathrm{g}_{ij}^0(u^{(0)} + 2u^{(1)} + 8X_1X_2) \\ &\quad - \tfrac{1}{16}\left[(b^+)_{(i}{}^k(b_2^-)_{j)k} + (b^-)_{(i}{}^k(b_2^+)_{j)k} - \tfrac{1}{6}\mathrm{g}_{ij}^0\left(\langle b^+, b_2^-\rangle_{\mathrm{g}^0} + \langle b^-, b_2^+\rangle_{\mathrm{g}^0}\right)\right] , \end{aligned} \tag{2.3.38}$$

$$\begin{aligned} h_{ij}^1 &= -\tfrac{1}{2}h_{ij}^0 + \mathrm{g}_{ik}^2(\mathrm{g}^0)^{kl}\mathrm{g}_{lj}^2 + \tfrac{1}{4}\mathrm{g}_{ij}^0\left(4t^{(4)} - 2t^{(2,2)} + u^{(1)} + 8X_2^2\right) \\ &\quad + \tfrac{1}{4}\left(\nabla^k\nabla_i\mathrm{g}_{jk}^2 + \nabla^k\nabla_j\mathrm{g}_{ik}^2 - \nabla^2\mathrm{g}_{ij}^2 - \nabla_i\nabla_j t^{(2)}\right) - \tfrac{1}{8}\left((b_1^+)_{(i}(b_1^-)_{j)} - \tfrac{1}{3}\mathrm{g}_{ij}^0\langle b_1^+, b_1^-\rangle_{\mathrm{g}^0}\right) \\ &\quad + \tfrac{1}{8}\left[(b^-)_{(i|k|}(\mathrm{g}^2)^{kl}(b^+)_{j)l} - \tfrac{1}{3}\mathrm{g}_{ij}^0(b^-)_k{}^m(\mathrm{g}^2)^{kl}(b^+)_{lm}\right] \\ &\quad - \tfrac{1}{8}\left[(b^+)_{(i}{}^k(b_3^-)_{j)k} + (b^-)_{(i}{}^k(b_3^+)_{j)k} - \tfrac{1}{6}\mathrm{g}_{ij}^0\left(\langle b^+, b_3^-\rangle_{\mathrm{g}^0} + \langle b^-, b_3^+\rangle_{\mathrm{g}^0}\right)\right] \\ &\quad - \tfrac{1}{4}\left[\mathrm{f}_{ik}\mathrm{f}_j{}^k + \tfrac{1}{2}F_{ik}^I F_j^{Ik} - \tfrac{1}{6}\mathrm{g}_{ij}^0\left(\langle \mathrm{f},\mathrm{f}\rangle_{\mathrm{g}^0} + \tfrac{1}{2}\langle F^I, F^I\rangle_{\mathrm{g}^0}\right)\right] . \end{aligned} \tag{2.3.39}$$

The structure of the ij component of the Einstein equation in four dimensions is such that g^4 always appears with zero coefficient, and so is left undetermined. In the original literature [79] the iz component has been used to determine g^4 up to an arbitrary symmetric divergence-free tensor. However, in the supergravity we are considering the presence of a $(\log z)^2$ contribution to the bulk scalar field expansion means that X_2 appears without a derivative, which hence spoils this approach. In section 2.4.4 we will see that by imposing supersymmetry we obtain further constraints on the fields, and in particular this leads to an expression for g^4 in terms of other data.

2.3.3 Holographic renormalization

Having solved the bulk equations of motion to the relevant order, we are now in a position to holographically renormalize the Euclidean Romans theory. The bulk action (2.3.1) is divergent for an asymptotically locally hyperbolic solution, but can be rendered finite by the addition of appropriate local counterterms. The corresponding computations in Lorentzian signature have been carried out in [186].

We begin by taking the trace of the Einstein equation (2.3.6). Substituting the result together with (2.3.5) into the Euclidean action (2.3.1), we arrive at the bulk on-shell action

$$I_{\text{o-s}} = \frac{1}{2\kappa_5^2}\int_{Y_5}\Big[\tfrac{8}{3}(X^2+2X^{-1})*1+\tfrac{1}{3}X^4\mathcal{F}\wedge*\mathcal{F}+\tfrac{1}{6}X^{-2}\mathcal{F}^I\wedge*\mathcal{F}^I \\ -\tfrac{1}{12}X^{-2}\mathcal{B}^-\wedge*\mathcal{B}^+ +\tfrac{\mathrm{i}}{4}\mathcal{F}^I\wedge\mathcal{F}^I\wedge\mathcal{A}\Big]. \tag{2.3.40}$$

Here Y_5 is the bulk five-manifold, with boundary $\partial Y_5 = M_4$. In order to obtain the equations of motion (2.3.2)–(2.3.6) from the original bulk action (2.3.1) on a manifold with boundary, one has to add the Gibbons–Hawking–York term

$$I_{\text{GHY}} = -\frac{1}{\kappa_5^2}\int_{\partial Y_5}\mathrm{d}^4x\,\sqrt{\det h}\,K = \frac{1}{\kappa_5^2}\int_{\partial Y_5}\mathrm{d}^4x\,z\partial_z\sqrt{\det h}\,. \tag{2.3.41}$$

Here, more precisely, one cuts Y_5 off at some finite radial distance, or equivalently non-zero $z > 0$, and (M_4, h) is the resulting four-manifold boundary, with trace of the second fundamental form being K. Recall from (2.3.14) that $h_{ij} = \frac{1}{z^2}g_{ij}$.

The combined action $I_{\text{on-shell}} + I_{\text{GHY}}$ suffers from divergences as the conformal boundary is approached. To remove these divergences we use the standard method of holographic renormalization [93, 212, 79]. Namely, we introduce a small cut-off $z = \delta > 0$, and expand all fields via the Fefferman–Graham expansion of section 2.3.2 to identify the divergences. These may be cancelled by adding local boundary counterterms. We find

$$I_{\text{ct}} = \frac{1}{\kappa_5^2}\int_{\partial Y_5}\mathrm{d}^4x\,\sqrt{\det h}\,\Big\{3+\tfrac{1}{4}R(h)+3(X-1)^2-\tfrac{1}{32}\langle\mathcal{B}^-,\mathcal{B}^+\rangle_h \\ +\log\delta\Big[-\tfrac{1}{8}\Big(\mathscr{R}_{ij}(h)\mathscr{R}^{ij}(h)-\tfrac{1}{3}\mathscr{R}(h)^2\Big)+\tfrac{3}{2}(\log\delta)^{-2}(X-1)^2 \\ +\tfrac{1}{48}\langle H^-,H^+\rangle_h+\tfrac{1}{8}\langle\mathcal{F},\mathcal{F}\rangle_h+\tfrac{1}{16}\langle\mathcal{F}^I,\mathcal{F}^I\rangle_h\Big]\Big\}. \tag{2.3.42}$$

Notice the somewhat unusual form of the logarithmic term for the scalar field X, but cf. the expansion (2.3.18). As is standard, we have written the counterterm action (2.3.42) covariantly in terms of the induced metric h_{ij} on $M_4 = \partial Y_5$. The total renormalized action is then

$$\mathbb{S} = \lim_{\delta\to 0}\left(I_{\text{o-s}}+I_{\text{GHY}}+I_{\text{ct}}\right), \tag{2.3.43}$$

which by construction is finite.

The choice of local counterterms (2.3.42) defines a particular renormalization scheme, that is in some sense a "minimal scheme" in the case at hand. However, we are free to consider a non-minimal scheme where we add local counterterms to the action which remain finite as $\delta \to 0$. For the supergravity theory we are considering, the following are an independent set of finite counterterms that are both diffeomorphism and gauge invariant:[12]

$$\begin{aligned} I_{\text{ct, finite}} = & -\frac{1}{\kappa_5^2}\int_{\partial Y_5} \mathrm{d}^4x\,\sqrt{\det h}\,\Big[\zeta_1 R^2 + \zeta_2 C_{ijkl}C^{ijkl} + \zeta_3 \mathcal{F}_{ij}\mathcal{F}^{ij} + \zeta_4 \mathcal{F}^I_{ij}\mathcal{F}^{Iij} \\ & + \zeta_5 \mathcal{E} + \zeta_6 \mathcal{P} + \zeta_7 \epsilon^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl} + \zeta_8 \epsilon^{ijkl}\mathcal{F}^I_{ij}\mathcal{F}^I_{kl}\Big]\,. \end{aligned} \tag{2.3.44}$$

Here $\zeta_1, \ldots, \zeta_8$ are arbitrary constant coefficients, C_{ijkl} denotes the Weyl tensor of the metric h_{ij}, while the Euler scalar $\mathcal{E}$ and Pontryagin scalar $\mathcal{P}$ are defined by (A.1.2). In particular, notice that for compact $M_4 = \partial Y_5$ without boundary, the second line of (2.3.44) are all topological invariants: they are proportional to the Euler number $\chi(M_4)$, the signature $\sigma(M_4)$, and the Chern numbers $\int_{M_4} c_1(\mathcal{L})^2$, $\int_{M_4} c_2(\mathcal{V})$ respectively, where $\mathcal{L}$ and $\mathcal{V}$ denote the rank 1 and rank 2 complex vector bundles associated to the $U(1)_R$ and $SU(2)_R$ gauge bundles, respectively. In the real Euclidean theory in which we are working, recall that $\mathcal{F} = \mathrm{d}\mathcal{A}$ is globally exact (and purely imaginary), and in any case for the topological twist studied later in the chapter we will have $\mathcal{A}\mid_{M_4} = 0$. Being topological invariants, the variation of the action we shall compute in section 2.5 will be insensitive to the choice of constants $\zeta_5, \ldots, \zeta_8$.

As emphasized in the Introduction, in order to make quantitative comparisons in AdS/CFT it is important to match choices of renormalization schemes on the two sides. In particular, localization calculations in QFT make a (somewhat implicit) choice of scheme. In the case at hand, we note that in [73] a supersymmetric Rényi entropy, computed in field theory using localization, was successfully matched to a gravity calculation involving a supersymmetric black hole in the $\mathcal{N} = 4^+$ Romans theory. Here the supergravity action was computed using the minimal scheme. Our computation in section 2.5 will imply that this minimal scheme is indeed the correct one to compare to the topological twist of [219].

Given the renormalized action we may compute the following VEVs:

$$\begin{aligned} \langle T_{ij}\rangle &= \frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{ij}}\,, & \langle \Xi \rangle &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta X_1}\,, \\ \langle \mathscr{J}^i_I \rangle &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A^I_i}\,, & \langle \mathbb{J}^i \rangle &= \frac{1}{\sqrt{g}}\frac{\delta S}{\delta \mathrm{a}_i}\,. \end{aligned} \tag{2.3.45}$$

[12] We may also add finite local counterterms constructed from the B-field. For example, terms proportional to $\int_{\partial Y_5} \mathrm{d}^4x\sqrt{\det h}\,\langle H^-, H^+\rangle_h$, or $\int_{\partial Y_5} \mathrm{d}^4x\sqrt{\det h}\,R(h)\langle \mathcal{B}^-, \mathcal{B}^+\rangle_h$. However, for the topological twist we will later set the B-field to zero, and these terms will not be relevant to our discussion.

Here, as usual in AdS/CFT, the boundary fields $\mathrm{g}^0_{ij} = g_{ij}$, X_1, A^I_i and a_i act as sources for operators, and the expressions in (2.3.45) compute the vacuum expectation values of these operators. Similar expressions may also be written for the boundary fields $b^\pm$ for $\mathcal{B}^\pm$, but these will be zero for the topological twist of interest and play no rôle in the present chapter. Using the above holographic renormalization we may write (2.3.45) as the following limits:

$$\begin{aligned}\langle T_{ij}\rangle \;&=\; \frac{1}{\kappa_5^2}\lim_{\delta\to 0}\frac{1}{\delta^2}\Big[-K_{ij}+Kh_{ij}-\big(3+3(X-1)^2\big)h_{ij}+\tfrac{1}{2}\left(\mathscr{R}_{ij}(h)-\tfrac{1}{2}\mathscr{R}(h)\,h_{ij}\right)\\ &\quad+\log\delta\Big(\tfrac{1}{4}\mathscr{B}_{ij}(h)+\tfrac{1}{2}\mathcal{F}_{ik}\mathcal{F}_j{}^k-\tfrac{1}{8}h_{ij}\langle\mathcal{F},\mathcal{F}\rangle_h+\tfrac{1}{4}\mathcal{F}^I_{ik}\mathcal{F}^{Ik}_j-\tfrac{1}{16}h_{ij}\langle\mathcal{F}^I,\mathcal{F}^I\rangle_h\\ &\quad+\tfrac{1}{8}H^-_{ikl}H^+{}_j{}^{kl}-\tfrac{1}{48}h_{ij}\langle H^-,H^+\rangle_h-\tfrac{3}{2}(\log\delta)^{-2}(X-1)^2h_{ij}\Big)\Big]\,,\end{aligned}\qquad(2.3.46)$$

where K_{ij} is the second fundamental form of the cut-off hypersurface (M_4, h_{ij}) and the B-field modified Bach tensor is (cf. (2.3.32))

$$\begin{aligned}\mathscr{B}_{ij} \;&=\; -\tfrac{2}{3}\nabla_i\nabla_j\mathscr{R}-\nabla^2\Big(\mathscr{R}_{ij}-\tfrac{1}{6}h_{ij}\mathscr{R}\Big)+2\nabla_k\nabla_{(i}\mathscr{R}^k{}_{j)}-2\mathscr{R}_{ik}\mathscr{R}^k{}_j+\tfrac{2}{3}\mathscr{R}\mathscr{R}_{ij}\\ &\quad+\tfrac{1}{2}h_{ij}\Big(\mathscr{R}_{kl}\mathscr{R}^{kl}-\tfrac{1}{3}\mathscr{R}^2\Big)\,,\end{aligned}\qquad(2.3.47)$$

together with

$$\begin{aligned}\langle\Xi\rangle \;&=\; \frac{1}{\kappa_5^2}\lim_{\delta\to0}\frac{\log\delta}{\delta^2}\Big[-3X^{-2}\delta\partial_\delta X+6(X-1)+3(\log\delta)^{-1}(X-1)\Big]\,,\\ \langle\mathscr{J}^{Ii}\rangle \;&=\; \frac{1}{4\kappa_5^2}\lim_{\delta\to0}\frac{1}{\delta^4}\Big\{-*_h\Big[\mathrm{d}x^i\wedge(X^{-2}*_5\mathcal{F}^I+\mathrm{i}\mathcal{F}^I\wedge\mathcal{A})\Big]+\log\delta\,\mathcal{D}_j\mathcal{F}^{Iij}\Big\}\,,\\ \langle\mathbb{J}^i\rangle \;&=\; \frac{1}{2\kappa_5^2}\lim_{\delta\to0}\frac{1}{\delta^4}\Big[-*_h\Big(\mathrm{d}x^i\wedge X^4*_5\mathcal{F}\Big)+\log\delta\,\nabla_j\mathcal{F}^{ij}\Big]\,.\end{aligned}\qquad(2.3.48)$$

Here $*_h$ denotes the Hodge duality operator for the metric h_{ij}. A computation then gives the finite expressions

$$\begin{aligned}\langle T_{ij}\rangle \;&=\; \frac{1}{\kappa_5^2}\Big[2\mathrm{g}^4_{ij}+\tfrac{1}{2}h^1_{ij}-\tfrac{1}{2}(4t^{(4)}-2t^{(2,2)}-\tfrac{1}{2}u^{(1)})\mathrm{g}^0_{ij}-3\mathrm{g}^0_{ij}X_2^2-\mathrm{g}^2_{ij}t^{(2)}\\ &\quad+\tfrac{1}{4}\Big(\nabla^k\nabla_i\mathrm{g}^2_{jk}+\nabla^k\nabla_j\mathrm{g}^2_{ik}-\nabla^2\mathrm{g}^2_{ij}-\nabla_i\nabla_jt^{(2)}\Big)+\tfrac{1}{4}\mathrm{g}^0_{ij}\big(\mathrm{g}^2_{kl}R^{kl}\big)-\tfrac{1}{4}\mathrm{g}^2_{ij}R\\ &\quad-\tfrac{1}{8}\big[(b^+)_{(i}{}^k(b^-_3)_{j)k}+(b^-)_{(i}{}^k(b^+_3)_{j)k}-\tfrac{1}{2}\mathrm{g}^0_{ij}\big(\langle b^+,b^-_3\rangle_{\mathrm{g}^0}+\langle b^-,b^+_3\rangle_{\mathrm{g}^0}\big)\big]\\ &\quad+\tfrac{1}{8}\big[(b^+)_{(i|k|}(\mathrm{g}^2)^{kl}(b^-)_{j)l}-\tfrac{1}{2}\mathrm{g}^0_{ij}\langle b^-,(\mathrm{g}^2\circ b^+)\rangle_{\mathrm{g}^0}\big]\Big]\,,\end{aligned}\qquad(2.3.49)$$

$$\langle\Xi\rangle \;=\; \frac{3}{\kappa_5^2}X_2\,,\qquad(2.3.50)$$

$$\langle \mathscr{J}_i^I \rangle = -\frac{1}{4\kappa_5^2}\left[(a_1^I)_i + 2(a_2^I)_i - \mathrm{i}\big(*_4\,(\mathrm{a}\wedge F^I)\big)_i\right] , \tag{2.3.51}$$

$$\langle \mathbb{J}_i \rangle = -\frac{1}{2\kappa_5^2}\left[(\mathrm{a}_1)_i + 2(\mathrm{a}_2)_i\right] . \tag{2.3.52}$$

Notice that these expressions contain a number of terms that are not determined, in terms of boundary data, by the Fefferman–Graham expansion of the bosonic equations of motion. In particular the g^4_{ij} term in the stress-energy tensor T_{ij}, the scalar X_2 that determines Ξ, and a_2^I, a_2 appearing in the $SU(2)_R$ and $U(1)_R$ current, respectively. The general holographic Ward identity corresponding to the first three variations of the action is given by equation (2.1.1). We will need the expressions (2.3.49)–(2.3.51) in section 2.5.

2.4 Supersymmetric solutions

In this section we study supersymmetric solutions to the Euclidean $\mathcal{N} = 4^+$ theory. We begin in section 2.4.1 by deriving the Killing spinor equations on the conformal boundary, starting from the bulk equations (2.3.7), (2.3.8). We precisely recover the Euclidean $\mathcal{N} = 2$ conformal supergravity equations of [146]. In section 2.4.2 we then recall from [142] how the topological twist arises as a special solution to these Killing spinor equations, that exists on any Riemannian four-manifold (M_4, g). We rephrase this in terms of the quaternionic Kähler structure that exists on any such manifold, involving (locally) a triplet of self-dual 2-forms J^I. Finally, in section 2.4.4 we expand solutions to the bulk spinor equations in a Fefferman–Graham-like expansion.

2.4.1 Boundary spinor equations

We begin by expanding the bulk Killing spinor equations (2.3.7), (2.3.8) to leading order near the conformal boundary at $z = 0$. We will consequently need the Fefferman–Graham expansion of an orthonormal frame for the metric (2.3.14), (2.3.15), together with the associated spin connection. The following is a choice of frame $\mathrm{E}_\mu^{\overline{\mu}}$ for the metric (2.3.14):

$$\mathrm{E}_z^{\overline{z}} = \frac{1}{z}, \qquad \mathrm{E}_i^{\overline{z}} = \mathrm{E}_z^{\overline{i}} = 0, \qquad \mathrm{E}_i^{\overline{i}} = \frac{1}{z}\mathrm{e}_i^{\overline{i}} , \tag{2.4.1}$$

where $\mathsf{e}_i^{\bar{i}}$ is a frame for the z-dependent metric g. The latter then has the expansion (2.3.15), but for the present subsection we shall only need that

$$\mathsf{e}_i^{\bar{i}} \;=\; \mathrm{e}_i^{\bar{i}} + O(z^2)\;, \tag{2.4.2}$$

where $\mathrm{e}_i^{\bar{i}}$ is a frame for the boundary metric $\mathsf{g}^0 = g$. The non-zero components of the spin connection $\Omega_\mu{}^{\overline{\nu\rho}}$ at this order are correspondingly

$$\Omega_i{}^{\overline{zj}} \;=\; \frac{1}{z}\mathrm{e}_i^{\bar{j}} + O(z)\;, \qquad \Omega_i{}^{\overline{jk}} \;=\; (\omega^{(0)})_i{}^{\overline{jk}} + O(z^2)\;, \tag{2.4.3}$$

where $(\omega^{(0)})_i{}^{\overline{jk}}$ denotes the boundary spin connection.

The generators $\gamma_{\bar{\mu}}$ of the Clifford algebra Cliff$(5,0)$ in this frame are chosen to obey

$$\gamma_{\bar{z}} \;=\; \gamma_{\bar{1}\bar{2}\bar{3}\bar{4}}\;. \tag{2.4.4}$$

It follows that $\gamma_{\bar{z}}^2 = 1$, and we may identify $-\gamma_{\bar{z}}$ with the boundary chirality operator. The bulk Killing spinor is then expanded as

$$\epsilon \;=\; z^{-1/2}\varepsilon + z^{1/2}\eta + o(z^{1/2})\;. \tag{2.4.5}$$

As in (2.3.11), we may further decompose the spinors ε, η into their projections $\varepsilon^\pm$, $\eta^\pm$ onto the $\pm\mathrm{i}$ eigenspaces of Γ_{45}. At leading order in the z-component of the gravitino equation (2.3.7) one then finds

$$-\gamma_{\bar{z}}\varepsilon^\pm \;=\; \pm\varepsilon^\pm\;, \tag{2.4.6}$$

so that the Γ_{45} eigenvalue of the leading order spinor ε is correlated with its boundary chirality. Similarly, at the next order in the gravitino equation one finds the opposite correlation for the spinor η:

$$-\gamma_{\bar{z}}\eta^\pm \;=\; \mp\eta^\pm\,. \tag{2.4.7}$$

Recall that the boundary B-fields satisfy $*_4 b^\pm = \mp b^\pm$ (see (2.3.25)). This together with the chirality conditions (2.4.6) implies that

$$b^\pm \cdot \varepsilon^\pm \;=\; 0\,, \tag{2.4.8}$$

where $\cdot$ denotes the Clifford product defined in (A.2.3) (using the boundary frame). Using this, the leading order term in the i-component of the gravitino equation is then seen to be identically satisfied. The next order gives the pair of boundary Killing spinor equations:

$$\mathcal{D}_i^{(0)}\varepsilon^\pm - \tfrac{\mathrm{i}}{4}b^\mp_{ij}\gamma^j\varepsilon^\mp \mp \gamma_i\eta^\pm \;=\; 0\;, \tag{2.4.9}$$

where we have defined the covariant derivative

$$\mathcal{D}_i^{(0)} \;\equiv\; \nabla_i^{(0)} \pm \tfrac{\mathrm{i}}{2}\mathrm{a}_i + \tfrac{\mathrm{i}}{2}A_i^I\sigma_I\;. \tag{2.4.10}$$

Here $\nabla_i^{(0)}$ denotes the Levi-Civita spin connection of the boundary metric $\mathrm{g}^0_{ij} = g_{ij}$, and $\gamma_i = \gamma_{\bar{i}}\mathrm{e}^{\bar{i}}_i$, so that $\{\gamma_i,\gamma_j\} = 2g_{ij}$.

Turning to the bulk dilatino equation (2.3.8), the leading order term is in fact equivalent to the duality properties of $b^\pm$, given the chiralities of $\varepsilon^\pm$. At the next order we obtain the boundary dilatino equation

$$-\,\mathrm{f}\cdot\varepsilon^\pm \pm \tfrac{1}{2}F^I\sigma_I\cdot\varepsilon^\pm \mp 3\mathrm{i}X_1\,\varepsilon^\pm + \tfrac{1}{2}b^\mp\cdot\eta^\mp \mp \tfrac{1}{2}b_1^\mp\cdot\varepsilon^\mp \;=\; 0\;. \tag{2.4.11}$$

The supersymmetry equations for four-dimensional Euclidean off-shell $\mathcal{N}=2$ conformal supergravity have been studied[13] in [146], and our equations (2.4.9), (2.4.11) precisely reproduce the equations in this reference.[14] Notice in particular that one can solve for the (conformal) spinor η by taking the trace of (2.4.9) with γ^i, to obtain

$$\eta^\pm \;=\; \pm\tfrac{1}{4}\not{\mathcal{D}}^{(0)}\varepsilon^\pm\;, \tag{2.4.12}$$

where $\not{\mathcal{D}}^{(0)} \equiv \gamma^i\mathcal{D}_i^{(0)}$ is the Dirac operator. Taking the covariant derivative of (2.4.9) and using the integrability condition for $[\mathcal{D}_i^{(0)},\mathcal{D}_j^{(0)}]$ then leads to the following form of the dilatino equation

$$\not{\mathcal{D}}^{(0)}\not{\mathcal{D}}^{(0)}\varepsilon^\pm - \mathrm{i}\mathcal{D}_i(b^\mp)^i{}_j\gamma^j\varepsilon^\mp + \left(4X_1 + \tfrac{1}{3}R\right)\varepsilon^\pm \mp 2\mathrm{i}\,\mathrm{f}\cdot\varepsilon^\pm \;=\; 0\;, \tag{2.4.13}$$

where $R = R(g)$ is the Ricci scalar of the boundary metric. Requiring the boundary fields

[13] See [120] for related earlier work and [81] for a recent construction of Euclidean $\mathcal{N}=2$ conformal supergravity from a timelike reduction of a five-dimensional theory.

[14] The explicit notation change is $A_4^{\mathrm{KZ}} = -\mathrm{ia}$, $A^I_{\mathrm{KZ}} = A^I$, $T^\pm_{\mathrm{KZ}} = -b^\pm$, $\epsilon^{\mathrm{KZ}}_\pm = \varepsilon^\mp$, $\tilde{d}_{\mathrm{KZ}} = 2X_1$.

$g_{ij}, X_1, \mathrm{a}, A^I, b^\pm$ to solve the spinor equations (2.4.9), (2.4.11) for $\varepsilon^\pm$ in general imposes geometric constraints. Remarkably, in [146] it is shown that generically these conditions are equivalent to the boundary manifold (M_4, g) admitting a conformal Killing vector. However, the topological twist background of [142] arises as a very degenerate case, where in fact (M_4, g) may be an arbitrary Riemannian four-manifold. We turn to this case in the next subsection.

2.4.2 Topological twist

The topological twist background of [142] is obtained by setting in the first place

$$\varepsilon^- = 0\,, \qquad \mathrm{a} = 0\,, \qquad b^\pm = 0\,, \qquad \eta^\pm = 0\,. \tag{2.4.14}$$

The boundary Killing spinor equation (2.4.9) immediately implies that ε^+ is covariantly constant

$$\mathcal{D}_i^{(0)}\varepsilon^+ = 0\,. \tag{2.4.15}$$

The dilatino equation, in the form (2.4.13), then fixes

$$X_1 = -\tfrac{1}{12}R\,. \tag{2.4.16}$$

Recall that ε^+ is a doublet of positive chirality spinors: the Pauli matrices σ_I act on these doublet indices, while the Clifford matrices $\gamma_{\bar{i}}$ act on the spinor indices. We may write out the covariant derivative in (2.4.15) more explicitly by first introducing the following explicit Hermitian representation

$$\gamma_{\bar{a}} = \begin{pmatrix} 0 & \mathrm{i}\sigma_{\bar{a}} \\ -\mathrm{i}\sigma_{\bar{a}} & 0 \end{pmatrix}\,, \quad \gamma_{\bar{4}} = \begin{pmatrix} 0 & -\mathbb{1}_2 \\ -\mathbb{1}_2 & 0 \end{pmatrix}\,, \quad \gamma_{\bar{z}} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}\,. \tag{2.4.17}$$

Here $\bar{a} = 1, 2, 3$. Since $\gamma_{\bar{z}}\varepsilon^+ = -\varepsilon^+$, we may identify each of the two spinors in the doublet ε^+ with a two-component spinor, acted on by the second 2×2 block. With these choices (2.4.15) reads

$$\mathcal{D}_i^{(0)}\varepsilon^+ = \partial_i\varepsilon^+ + \tfrac{\mathrm{i}}{4}\eta^{\bar{a}}_{\bar{j}\bar{k}}\,(\omega^{(0)})_i^{\ \overline{jk}}\sigma_{\bar{a}}\varepsilon^+ + \tfrac{\mathrm{i}}{2}A_i^I\sigma_I\varepsilon^+ = 0\,, \tag{2.4.18}$$

where $\eta^{\bar{a}}_{\bar{i}\bar{j}}$ are the self-dual 't Hooft symbols defined in (A.2.2), and recall that $(\omega^{(0)})_i^{\ \overline{jk}}$ is the spin connection for the boundary metric g_{ij}. One may then solve (2.4.18) by taking

$$A_i^I = \frac{1}{2}\eta^I_{\overline{jk}}(\omega^{(0)})_i^{\overline{jk}}, \qquad (\varepsilon^+)^i{}_\alpha = (\mathrm{i}\sigma_2)^i{}_\alpha\, c\,. \qquad (2.4.19)$$

Here $i = 1,2$ labels the doublet indices, while $\alpha = 1,2$ labels the positive chirality spinor indices, and notice that the frame index $\bar{a} = 1,2,3$ is identified with the gauge indices $I = 1,2,3$. It is straightforward to check that (2.4.19) solves (2.4.18), for any constant c. The $SU(2)_R$ gauge field A^I given by (2.4.19) is precisely the right-handed part of the spin connection, where recall that $Spin(4) = SU(2)_\ell \times SU(2)_r$. Thus the $SU(2)_R$ gauge bundle is identified with $SU(2)_r$. This is a beautiful concrete realization of the geometric interpretation of the topological twist discussed at the end of section 1.2.1.

More invariantly, ε^+ is a section of $\mathcal{S}^+ \otimes \mathcal{V}$, where $\mathcal{S}^+$ denotes the positive chirality spinor bundle over M_4, while $\mathcal{V}$ is the rank 2 complex vector bundle for which A^I is an associated $SU(2)$ connection. *A priori* this makes sense globally only when M_4 is a spin manifold, when $\mathcal{S}^+$ and $\mathcal{V}$ both exist as genuine vector bundles. However, the topological twist (2.4.19) identifies $\mathcal{V}$ with $\mathcal{S}^+$, and their tensor product then always exists globally, even when M_4 is not spin.[15] This topological construction of a spin-type bundle on a manifold which is not necessarily spin was first suggested in [21], and is sometimes referred to as a $Spin_{\mathcal{G}}$ structure, where here the group $\mathcal{G} = SU(2)$. Perhaps more familiar is the Abelian case of $Spin^c$ structures, where instead $\mathcal{G} = U(1)$. (For example, this arises in Seiberg–Witten theory.) Note the consistency with the realization of the theory from branes: the half-twist is obtained by wrapping D3-branes on Cayley submanifolds in $Spin(7)$ manifolds, and their normal bundle indeed has the structure $\mathcal{S}^+ \otimes \mathcal{V}$ [46].

It will be convenient later to introduce the triplet of self-dual 2-forms

$$\mathrm{J}^I_{ij} \equiv \eta^I_{\overline{ij}}\,\mathrm{e}^{\bar{i}}_i\,\mathrm{e}^{\bar{j}}_j\,, \qquad (2.4.20)$$

where recall that $\mathrm{e}^{\bar{i}}_i$ is the boundary frame for g_{ij}. More explicitly, these read

$$\mathrm{J}^1 = \mathrm{e}^2\wedge\mathrm{e}^3 + \mathrm{e}^1\wedge\mathrm{e}^4\,, \quad \mathrm{J}^2 = \mathrm{e}^3\wedge\mathrm{e}^1 + \mathrm{e}^2\wedge\mathrm{e}^4\,, \quad \mathrm{J}^3 = \mathrm{e}^1\wedge\mathrm{e}^2 + \mathrm{e}^3\wedge\mathrm{e}^4\,. \qquad (2.4.21)$$

[15]There are various ways to see this. For example, the lack of a spin structure on M_4 is detected by a non-zero second Stiefel-Whitney class $w_2(M_4) \in H^2(M_4, \mathbb{Z}_2)$. Concretely this means the cocycle condition for the spin lift of the frame bundle fails up to some minus signs. However, if two copies are tensored together all such signs square to $+1$, and the tensor product is a well-defined bundle.

Of course, in general a frame $e_i^{\bar{i}}$ is only defined locally on M_4, in an appropriate open set, and likewise the J^I in (2.4.21) are then well-defined forms only locally. More globally, local frames are patched together with $SO(4)$. The spin cover is $Spin(4) \cong SU(2)_\ell \times SU(2)_r$, and the self-dual/anti-self-dual 2-forms are precisely the representations associated to $SO(3)_{\ell/r} = SU(2)_{\ell/r}/\mathbb{Z}_2$. In particular, the $\{J^I\}$ rotate as a 3-vector under $SO(3)_r \subset SO(4)$. In this sense the J^I in general don't exist individually as global 2-forms on M_4, but instead as a triplet of forms that rotate appropriately. We comment further on this below.

One can also write the J^I in terms of spinor bilinears. Recall from the end of section 2.3.1 that the bulk spinors satisfy a symplectic Majorana reality condition. In particular the boundary spinor ε^+ satisfies

$$(\varepsilon^+)^c \equiv i\sigma_2 \mathscr{C} (\varepsilon^+)^* = \varepsilon^+ , \qquad (2.4.22)$$

where recall that $\mathscr{C}$ is the charge conjugation matrix for the spacetime Clifford algebra. In the explicit basis (2.4.17) we may take

$$\mathscr{C} = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix} . \qquad (2.4.23)$$

Given the solution (2.4.19) one finds that the reality condition (2.4.22) is satisfied provided the constant $c \in \mathbb{R}$. Explicitly, the components of the doublet ε^+ are

$$(\varepsilon^+)^1 = (0,0,0,c)^T , \qquad (\varepsilon^+)^2 = (0,0,-c,0)^T . \qquad (2.4.24)$$

We then define the boundary spinor

$$\chi \equiv (\varepsilon^+)^1 . \qquad (2.4.25)$$

This has square norm $\bar{\chi}\chi = c^2$, where the bar denotes Hermitian conjugate, and χ of course has positive chirality, $-\gamma_{\bar{z}}\chi = \chi$. One easily checks that

$$J^2 + iJ^1 = \frac{1}{\bar{\chi}\chi} \bar{\chi}^c \gamma_{(2)} \chi , \qquad J^3 = \frac{i}{\bar{\chi}\chi} \bar{\chi} \gamma_{(2)} \chi , \qquad (2.4.26)$$

where $\chi^c \equiv \mathscr{C}\chi^*$.

From the original definition (2.4.20), the J^I inherit a number of algebraic identities from those for the 't Hooft symbols. For example,

$$J^I_{ij} J^I_{kl} \;=\; g_{ik}g_{jl} - g_{il}g_{jk} + \epsilon_{ijkl} \; . \tag{2.4.27}$$

Using the metric to raise an index, one obtains a triplet $(I^I)^i{}_j \equiv g^{ik}(J^I)_{kj}$ of endomorphisms of the tangent bundle of M_4. These satisfy the quaternionic algebra

$$I^I \circ I^J \;=\; -\delta^{IJ} - \epsilon^{IJ}{}_K I^K \; . \tag{2.4.28}$$

One also finds that

$$\nabla_i J^I_{jk} \;=\; \epsilon^I{}_{JK} A^J_i J^K_{jk} \; , \tag{2.4.29}$$

where the R-symmetry gauge field A^I here is precisely the right-handed spin connection given by the topological twist (2.4.19). Notice that we may correspondingly write the curvature as

$$F^I_{ij} \;=\; \tfrac{1}{2} J^I_{kl} R_{ij}{}^{kl} \; , \tag{2.4.30}$$

where R_{ijkl} is the boundary Riemann tensor.

In general a *quaternionic Kähler manifold* is a Riemannian manifold of dimension $4n$ with holonomy $Sp(n)\cdot Sp(1) \subset SO(4n)$.[16] Such manifolds admit, locally, a triplet of skew endomorphisms I^I of the tangent bundle satisfying (2.4.28), for which the corresponding triplet of 2-forms J^I satisfy (2.4.29). Here A^I is the Riemannian connection corresponding to the $Sp(1)$ part of this holonomy group. For $n = 1$ notice that $Sp(1) \cdot Sp(1) = SO(4)$, and such a structure exists on any Riemannian four-manifold (M_4, g) (as we have just seen). Crucially, the 2-forms (2.4.21) are not in general defined globally, but are (in our language) twisted by the R-symmetry gauge field, transforming as a vector under $SO(3)_R = SU(2)_R/\mathbb{Z}_2$. As such, they don't define a reduction of the structure group to $SU(2)_\ell$, as a global set of such forms would do. Indeed, the globally defined tensor on a quaternionic Kähler manifold is the 4-form $\Psi \equiv J^I \wedge J^I$ (summed over I), and in four dimensions ($n = 1$) this is proportional to the volume form. The stabiliser of Ψ is $Sp(n) \cdot Sp(1)$, which is $SO(4)$ when $n = 1$.

In dimensions $n \geq 2$ irreducible quaternionic Kähler manifolds are automatically Ein-

[16] See, for example, [201].

stein. Some authors choose to *define* a quaternionic Kähler four-manifold to be an Einstein manifold with self-dual Weyl tensor, but we shall not use this terminology.

2.4.3 $U(1)_R$ current

Before continuing to expand the spinor equations into the bulk, in this subsection we pause briefly to consider the VEV of the $U(1)_R$ current given by (2.3.52). In the topological twist background equation, (2.3.27) gives $\mathrm{a}_1 = 0$, so that $\langle \mathbb{J} \rangle = -\mathrm{a}_2/\kappa_5^2$. On the other hand, from (2.3.28) we obtain the $U(1)_R$ anomaly equation

$$\mathrm{d} *_4 \langle \mathbb{J} \rangle \;=\; \frac{\mathrm{i}}{8\kappa_5^2} F^I \wedge F^I \,, \tag{2.4.31}$$

where $*_4$ denotes the Hodge duality operator on (M_4, g). Using equations (2.4.30) and (2.4.27) this may be rewritten as

$$\mathrm{d} *_4 \langle \mathbb{J} \rangle \;=\; \frac{\mathrm{i}}{32\kappa_5^2} (\mathcal{E} + \mathcal{P})\, \mathrm{vol}_4 \,, \tag{2.4.32}$$

where $\mathcal{E}$ and $\mathcal{P}$ are the Euler and Pontryagin densities, (A.1.2). On a compact M_4 without boundary these integrate to $\int_{M_4} \mathcal{E}\, \mathrm{vol}_4 = 32\pi^2 \chi(M_4)$, $\int_{M_4} \mathcal{P}\, \mathrm{vol}_4 = 48\pi^2 \sigma(M_4)$, so that integrating (2.4.32) over M_4 gives[17]

$$\int_{M_4} \mathrm{d} *_4 \langle \mathbb{J} \rangle \;=\; \frac{\mathrm{i}\pi^2}{2\kappa_5^2} \left[2\chi(M_4) + 3\sigma(M_4) \right] \,. \tag{2.4.33}$$

It follows that if a_2, or equivalently $\langle \mathbb{J} \rangle$, is a *global* 1-form on M_4, then by Stokes' theorem the left hand side of (2.4.33) is zero, implying the topological constraint

$$2\chi(M_4) + 3\sigma(M_4) \;=\; 0 \,. \tag{2.4.34}$$

Indeed, in section 2.3.1 we noted that we are studying gravitational saddle points in the real Euclidean Romans theory, where the $U(1)_R$ gauge field $\mathcal{A}$ is a (purely imaginary) global 1-form. Related to this, the $U(1)_R$ symmetry effectively becomes an $SO(1,1)_R$ symmetry after Wick rotation, as also emphasized in [146] (see also [194]). A number of gravity

[17] A little less laboriously we can instead note that F^I is the curvature of the bundle of self-dual 2-forms $\Lambda_2^+ M_4$, and the integral of the right hand side of (2.4.31) is proportional to the first Pontryagin class $p_1(\Lambda_2^+ M_4) = 2\chi(M_4) + 3\sigma(M_4)$.

expressions that we shall obtain below only make sense if a_2 is interpreted as a global 1-form on M_4, at least in the set-up we have defined. Thus (2.4.34) already restricts the topology of M_4. Interestingly, the same formula appeared in section 2.2.1 in relation to the dual TQFT. Specifically, if (2.4.34) or, equivalently, (2.2.5) does not hold, the dimension of the relevant moduli space is non-vanishing, and the partition function is zero![18]

The two expressions are directly related, since the virtual dimension (2.2.5) of $\mathcal{M}$ computed in field theory is proportional to this integrated $U(1)_R$ anomaly. In the holographic set-up, we can see this explicitly by using the normalization of the effective gravity constant (2.2.6) In the large N limit, using (2.4.33) we may then write

$$\dim \mathcal{M} \ = \ 2\mathrm{i} \int_{M_4} \mathrm{d} *_4 \langle \mathbb{J} \rangle \ , \qquad (2.4.35)$$

in terms of the integrated (holographic) $U(1)_R$ anomaly.[19]

2.4.4 Supersymmetric expansion

In this section we continue to expand the bulk spinor equations to higher order in z. From this we extract further information about some of the fields which are not fixed, in terms of boundary data, by the bosonic equations of motion. We will continue to use the boundary conditions appropriate to the topological twist. In particular we note that the boundary B-fields $b^{\pm} = 0$ in this case, and that setting the bulk $\mathcal{B}^{\pm} = 0$ is a consistent truncation of the Euclidean $\mathcal{N} = 4^{+}$ theory. Moreover, in this case the bulk spinors $\epsilon^{\pm}$ satisfy decoupled equations, and since the leading order term $\varepsilon^{-} = 0$ it is then also consistent to set the bulk $\epsilon^{-} = 0$. We henceforth work in this truncated theory. This subsection is somewhat technical. All of the relevant formulas that we need in section 2.5 are in any case summarized in that section, and a reader uninterested in the details may safely skip the present subsection.

The frame, spin connection and spinor expansions beyond the leading order given in

[18] In passing we note that (2.4.34) corresponds (with an appropriate choice of orientation) to equality in the Hitchin–Thorpe inequality. In particular the only Einstein manifolds satisfying this condition are the flat torus, a K3 surface, or a quotient thereof [129]. A non-example is S^4, for which $2\chi(S^4) + 3\sigma(S^4) = 4$. On the other hand, for a complex surface (2.4.34) is equivalent to $\int_{M_4} c_1 \wedge c_1 = 0$, where $c_1 = c_1(M_4)$ is the first Chern class of the holomorphic tangent bundle (the anti-canonical class).

[19] Of course, the same formula holds for the $\mathcal{N} = 2$ SCFTs of class $\mathcal{S}$: one starts from the dimension of the moduli space derived in footnote 9 from (2.42) of [117] and uses (2.4.33) and (2.2.7).

section 2.4.1 will be needed, so we first give details of these. The frame expansion is

$$\mathrm{e}_i^{\bar{i}} \;=\; \mathrm{e}_i^{\bar{i}} + z^2(\mathrm{e}^{(2)})_i^{\bar{i}} + z^4\left[(\log z)^2(\mathring{\mathrm{e}}^{(4)})_i^{\bar{i}} + \log z\,(\tilde{\mathrm{e}}^{(4)})_i^{\bar{i}} + (\mathrm{e}^{(4)})_i^{\bar{i}}\right] + o(z^4)\,, \qquad (2.4.36)$$

where in particular $\mathrm{e}_i^{\bar{i}}$ is a frame for the boundary metric. The additional spin connection components we will need are

$$\Omega_i{}^{\overline{zi}} \;=\; \frac{1}{z}\mathrm{e}_i^{\bar{i}} - \tfrac{1}{2}\mathrm{g}^{jk}\mathrm{e}_j^{\bar{i}}\partial_z \mathrm{g}_{ik} \qquad \Omega_z{}^{\overline{ij}} \;=\; \mathrm{g}^{ij}\mathrm{e}_i^{[\bar{i}}\partial_z \mathrm{e}_j^{\bar{j}]}\,. \qquad (2.4.37)$$

The bulk spinor has $\epsilon^- = 0$ in our truncated theory, and we thus henceforth drop the superscript on $\epsilon^+ \to \epsilon$, $\varepsilon^+ \to \varepsilon$ (we hope this abuse of notation won't lead to any confusion). The bulk spinor then has the following expansion

$$\epsilon \;=\; z^{-1/2}\varepsilon + z^{3/2}\varepsilon^3 + z^{5/2}(\log z\,\tilde{\varepsilon}^5 + \varepsilon^5) + z^{7/2}\big((\log z)^2\,\mathring{\varepsilon}^7 + \log z\,\tilde{\varepsilon}^7 + \varepsilon^7\big) + o(z^{7/2})\,, \quad (2.4.38)$$

where ε is constant with positive chirality under $-\gamma_{\bar{z}}$. As in equation (2.4.22) the bulk spinor ϵ satisfies the reality condition

$$\epsilon^c \;\equiv\; \mathrm{i}\sigma_2\mathscr{C}\epsilon^* \;=\; \epsilon\,. \qquad (2.4.39)$$

We start by analysing the bulk dilatino equation. At lowest order we find

$$0 \;=\; X_1\,\varepsilon + \tfrac{\mathrm{i}}{6}F^I\cdot(\sigma^I\varepsilon) \;=\; \big(X_1 + \tfrac{1}{12}R\big)\,\varepsilon\,, \qquad (2.4.40)$$

which is satisfied identically, where we have used (2.4.16) and (2.4.30). At the next order we find

$$\mathrm{i}a_1^I\cdot(\sigma_I\varepsilon) = -\tfrac{1}{4}(\mathrm{d}R)\cdot\varepsilon\,. \qquad (2.4.41)$$

This is effectively a matrix equation, of which we shall see many more. Components of such equations may be extracted by first noting that

$$\varepsilon \;=\; \begin{pmatrix} \chi \\ -\mathscr{C}\chi^* \end{pmatrix}\,, \qquad (2.4.42)$$

in the notation of section 2.4.2. For example, one can then take the first component of (2.4.41), and apply $\bar{\chi}\gamma_j$ on the left. Taking the real part, and using the definitions (2.4.26) of

J^I in terms of spinor bilinears, one obtains

$$(a_1^I)^i \, \mathrm{J}_{ij}^I \;=\; \tfrac{1}{4}\nabla_j R \,. \tag{2.4.43}$$

We shall make use of similar manipulations throughout this subsection. Focusing on (2.4.43), recall that a_1^I is already fixed in terms of the $SU(2)$ covariant divergence of F^I, via equation (2.3.27). The latter reads $(a_1^I)_i = \frac{1}{2}\mathcal{D}^j F_{ij}^I$. Starting from this and (2.4.30), and using the identity $\alpha_{pq}\mathrm{J}_m^{I\,p}\mathrm{J}_n^{I\,q} = \alpha_{mn} - 2(*\alpha)_{mn}$, where α_{pq} is any 2-form, one can show that (2.4.43) is an identity. We may then differentiate (2.4.43) and, upon using the quaternionic Kähler equation (2.4.29), we obtain

$$(\mathcal{D}a_1^I)^{ij}\mathrm{J}_{ij}^I \;=\; -\tfrac{1}{4}\nabla^2 R \,. \tag{2.4.44}$$

This relation appears frequently hereafter.

At the next order in the dilatino equation we find an equation involving several undetermined fields:

$$\mathrm{i}a_2^I \cdot (\sigma_I \varepsilon) \;=\; \big(2\mathrm{i}\mathrm{a}_2 + 3\mathrm{d}X_2 + \tfrac{1}{8}\mathrm{d}R\big) \cdot \varepsilon \,, \tag{2.4.45}$$

from which we similarly extract

$$(a_2^I)^i \mathrm{J}_{ij}^I \;=\; -2\mathrm{i}(\mathrm{a}_2)_j - 3\nabla_j X_2 - \tfrac{1}{8}\nabla_j R \;. \tag{2.4.46}$$

From this expression, taking a covariant derivative and symmetrizing indices gives

$$3\nabla_i\nabla_j X_2 \;= \mathcal{D}_{(i}(a_2^I)^k \mathrm{J}_{j)k}^I - 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} - \tfrac{1}{8}\nabla_i\nabla_j R \;. \tag{2.4.47}$$

At higher order still we have

$$X_3\,\varepsilon \;=\; X_1(1+\gamma_{\bar{z}})\varepsilon^3 - \tfrac{\mathrm{i}}{12}\mathcal{D}a_1^I \cdot (\sigma_I \varepsilon) \;. \tag{2.4.48}$$

As ε has positive chirality we can act with $P_- = \frac{1}{2}(1+\gamma_{\bar{z}})$ to deduce that ε^3 also has positive chirality. It then follows that

$$X_3 \;=\; -\tfrac{1}{12}(\mathcal{D}a_1^I)^{ij}\mathrm{J}_{ij}^I \;=\; \tfrac{1}{48}\nabla^2 R \;. \tag{2.4.49}$$

where we have used (2.4.44). This expression for X_3 is equivalent to that in (2.3.29), for the topological twist. Finally, at order $\mathcal{O}(z^{7/2})$ we have

$$\begin{aligned} X_4\,\varepsilon \;&=\; -\tfrac{1}{2}X_3\,\varepsilon - \tfrac{1}{2}X_1\,\varepsilon^3 - \tfrac{\mathrm{i}}{12}\Big[(\mathcal{D}a_2^I)\cdot(\sigma_I\varepsilon) - 2\mathrm{f}_2\cdot\varepsilon + F^I\cdot(\sigma_I\varepsilon^3)\Big] \\ &\quad - \tfrac{\mathrm{i}}{12}\,\mathrm{e}_{\bar{i}}^{i}\,(\mathrm{e}^{(2)})_{\bar{j}}^{j}F_{ij}^{I}\gamma^{\bar{i}\bar{j}}(\sigma_I\varepsilon)\,. \end{aligned} \tag{2.4.50}$$

Here $\mathrm{e}_{\bar{i}}^{i}$ is the inverse frame to $\mathrm{e}_{i}^{\bar{i}}$, with $\mathrm{e}_{\bar{i}}^{i}$ and $(\mathrm{e}^{(2)})_{\bar{i}}^{i}$ being coefficients in its expansion, precisely as in (2.4.36). We have also defined $\mathrm{f}_2 = \mathrm{d}a_2$. Since ε^3 is so far undetermined, we cannot yet extract an expression for X_4. This concludes the expansion of the bulk dilatino equation.

Turning next to the bulk gravitino equation, at lowest order in the z direction we find, after using the fact that ε^3 has positive chirality, that

$$\varepsilon^3 \;=\; \tfrac{1}{48}R\,\varepsilon - \tfrac{1}{4}g^{ij}\,\mathrm{e}_{i}^{\bar{i}}\,(\mathrm{e}^{(2)})_{j}^{\bar{j}}\gamma_{\bar{i}\bar{j}}\,\varepsilon\,. \tag{2.4.51}$$

As a metric defines the frame only up to an arbitrary local $SO(4)$ rotation, it is convenient to gauge fix this arbitrariness. A consistent gauge choice is $(\mathrm{e}^{(2)})_{i}^{\bar{i}} = \frac{1}{2}(\mathrm{g}^2)^{\bar{i}}{}_{\bar{j}}\,\mathrm{e}_{i}^{\bar{j}}$ and $(\mathrm{e}^{(2)})_{\bar{i}}^{i} = -\frac{1}{2}\mathrm{e}_{\bar{j}}^{i}\,(\mathrm{g}^2)^{\bar{j}}{}_{\bar{i}}$, where recall that g^2 is fixed in terms of the boundary Schouten tensor via (2.3.33). This then implies that

$$g_{ij}\,\mathrm{e}_{\bar{i}}^{i}\,(\mathrm{e}^{(2)})_{\bar{j}}^{j} \;=\; -\tfrac{1}{2}\mathrm{g}^2_{\bar{i}\bar{j}}\,, \qquad g^{ij}\,\mathrm{e}_{i}^{\bar{i}}\,(\mathrm{e}^{(2)})_{j}^{\bar{j}} \;=\; \tfrac{1}{2}(\mathrm{g}^2)^{\bar{i}\bar{j}}\,, \tag{2.4.52}$$

and, being symmetric, their contraction with any anti-symmetric tensor automatically vanishes. Consequently, this gauge choice reduces the relation between the spinors ε and ε^3 to simply

$$\varepsilon^3 \;=\; \tfrac{1}{48}R\,\varepsilon\,. \tag{2.4.53}$$

Having found this relation we may substitute for ε^3 into the right hand side of (2.4.50), extract X_4 and then substitute for g^2, X_1, X_3 and F^I to obtain

$$X_4 \;=\tfrac{1}{288}R^2 - \tfrac{1}{48}R_{kl}R^{kl} - \tfrac{1}{96}\nabla^2 R - \tfrac{1}{12}\left(\mathcal{D}a_2^I\right)^{ij}\mathrm{J}_{ij}^{I}\,. \tag{2.4.54}$$

Here strictly speaking we have taken the real part of this equation, where the term involving f_2 is purely imaginary, and thus doesn't appear. Using the trace of (2.4.47), together with several other equations derived so far, one can check that the expression (2.4.54) for X_4

agrees with the expression (2.3.30), obtained from the equations of motion.

At the next orders we find

$$(5-\gamma_{\bar{z}})\,\varepsilon^5 \;=\; -2\,\tilde{\varepsilon}^5 + 2(\mathrm{i}a_2 + \mathrm{d}X_2)\cdot\varepsilon\,, \tag{2.4.55}$$

$$(5-\gamma_{\bar{z}})\,\tilde{\varepsilon}^5 \;=\; \tfrac{2\mathrm{i}}{3}a_1^I\cdot(\sigma_I\varepsilon) = -\tfrac{1}{6}\mathrm{d}R\cdot\varepsilon\,. \tag{2.4.56}$$

We could continue and analyse higher order terms in this z component of the gravitino equation, but the subsequent expressions are not required, nor particularly enlightening, and so we stop here.

The remaining equation to study is the i direction of the gravitino equation. Crucially this involves the spin connection components $\Omega_i{}^{\overline{zi}}$, which introduce the metric expansion fields from (2.3.15). Of course, the leading order equation is satisfied by construction. Remarkably, at the next order we find a non-trivial equation which is also identically satisfied given the chirality of ε^3 and the algebraic properties of the Riemann tensor. At the following order we find another condition on $\tilde{\varepsilon}^5$:

$$\gamma_{\bar{i}}\left[3\mathrm{i}(1+\gamma_{\bar{z}})\tilde{\varepsilon}^5 + a_1^I\cdot(\sigma_I\varepsilon)\right] \;=\; 0\,, \tag{2.4.57}$$

which, used in conjunction with (2.4.56), allows us to determine

$$\gamma_{\bar{z}}\tilde{\varepsilon}^5 \;=\; \tilde{\varepsilon}^5\,, \qquad \tilde{\varepsilon}^5 \;=\; -\tfrac{1}{24}\mathrm{d}R\cdot\varepsilon\,. \tag{2.4.58}$$

We now substitute $\tilde{\varepsilon}^5$ into equation (2.4.55):

$$(5-\gamma_{\bar{z}})\varepsilon^5 \;=\; \left(2\mathrm{i}a_2 + 2\mathrm{d}X_2 + \tfrac{1}{12}\mathrm{d}R\right)\cdot\varepsilon\,. \tag{2.4.59}$$

Acting on this last equation with $\gamma_{\bar{z}}$, and taking the difference, implies that ε^5 is a negative chirality spinor: $\gamma_{\bar{z}}\varepsilon^5 = \varepsilon^5$. We thus find

$$\varepsilon^5 \;=\; \left(\tfrac{\mathrm{i}}{2}a_2 + \tfrac{1}{2}\mathrm{d}X_2 + \tfrac{1}{48}\mathrm{d}R\right)\cdot\varepsilon\,. \tag{2.4.60}$$

At the next order we begin to see the metric fields appearing:

$$h^0_{\bar{i}\bar{j}}\gamma^{\bar{j}}\varepsilon \;=\; -\tfrac{1}{288}R^2\gamma_{\bar{i}}\varepsilon - \tfrac{1}{2}\gamma_{\bar{i}}(1+\gamma_{\bar{z}})\mathring{\varepsilon}^7\,. \tag{2.4.61}$$

Using the chiral projector P_- again we see that $\mathring{\varepsilon}^7$ has positive chirality, and we may extract h^0:

$$h^0_{ij} \;=\; -\tfrac{1}{288}R^2 g_{ij}\,. \tag{2.4.62}$$

This agrees with the expression $h^0_{ij} = -\frac{1}{2}g_{ij}X_1^2$, given by equation (2.3.38), derived from the expansion of the bosonic field equations. The next order gives

$$\begin{aligned} h^1_{\bar{i}\bar{j}}\gamma^{\bar{j}}\varepsilon \;=\; &-\tfrac{1}{2}\gamma_{\bar{i}}(1+\gamma_{\bar{z}})\,\tilde{\varepsilon}^7 - \tfrac{1}{2}h^0_{\bar{i}\bar{j}}\gamma^{\bar{j}}\varepsilon - X_1X_2\gamma_{\bar{i}}\,\varepsilon + \nabla_{\bar{i}}\tilde{\varepsilon}^5 + \tfrac{\mathrm{i}}{2}A^I_{\bar{i}}(\sigma_I\tilde{\varepsilon}^5) \\ &- \tfrac{\mathrm{i}}{24}X_1(\gamma_{\bar{i}}{}^{\bar{j}\bar{k}} - 4\delta^{\bar{j}}_{\bar{i}}\gamma^{\bar{k}})F^I_{\bar{j}\bar{k}}(\sigma_I\varepsilon) + \tfrac{\mathrm{i}}{24}(\gamma_{\bar{i}}{}^{\bar{j}\bar{k}} - 4\delta^{\bar{j}}_{\bar{i}}\gamma^{\bar{k}})(\mathcal{D}a^I_1)_{\bar{j}\bar{k}}(\sigma_I\varepsilon)\,. \end{aligned} \tag{2.4.63}$$

As before, we can show that $\tilde{\varepsilon}^7$ has positive chirality and hence drops out of (2.4.63). Now using the definition of $\tilde{\varepsilon}^5$ in (2.4.58) allows us to write everything acting on the spinor ε. After using the intermediate result

$$-\tfrac{1}{4}\mathrm{J}^I{}_{(i}{}^k(\mathcal{D}a^I_1)_{j)k} \;=\; -\tfrac{1}{8}\left(R_i{}^kR_{jk} + R_{iklj}R^{kl} - \nabla^2R_{ij} + \tfrac{1}{2}\epsilon_{(j|kmn|}R^{kl}R^{mn}{}_{i)l}\right)\,, \tag{2.4.64}$$

and substituting for the known expressions, we can then read off h^1_{ij}:

$$\begin{aligned} h^1_{ij} \;=\; &\tfrac{1}{192}g_{ij}R^2 + \tfrac{1}{12}g_{ij}RX_2 - \tfrac{1}{48}RR_{ij} - \tfrac{1}{24}\nabla_i\nabla_jR - \tfrac{1}{48}g_{ij}\nabla^2R \\ &- \tfrac{1}{8}\left(R_i{}^kR_{jk} + R_{iklj}R^{kl} - \nabla^2R_{ij} + \tfrac{1}{2}\epsilon_{(j|kmn|}R^{kl}R^{mn}{}_{i)l}\right)\,. \end{aligned} \tag{2.4.65}$$

Once again, we have found another expression for something we have already derived: h^1_{ij} is also given by equation (2.3.39). However, in this instance the equality of the two expressions (2.4.65) and (2.3.39) is non-trivial. It is equivalent to the equation

$$\begin{aligned} 0 \;=\; &(RR_{ij} - 2R_i{}^kR_{jk} + 2R_{iklj}R^{kl} + R_{mnik}R^{mn}{}_j{}^k) - \tfrac{1}{4}g_{ij}(R^2 - 4R_{kl}R^{kl} + R_{mnkl}R^{mnkl}) \\ &+ \tfrac{1}{2}\left[\epsilon_{mnpq}\left(-\tfrac{1}{4}g_{ij}R^{mn}{}_{kl}R^{pqkl} + g_{jk}R^{mn}{}_{il}R^{pqkl}\right) - 2\epsilon_{(j|kmn|}R^{kl}R^{mn}{}_{i)l}\right]\,. \end{aligned} \tag{2.4.66}$$

The first line quite remarkably is known to be zero for any Riemannian four-manifold, and is called Berger's identity [45]. One can also show that the second line is equal to zero, which amounts to an algebraic identity that holds for any tensor sharing the algebraic symmetries of the Riemann tensor.

Finally, at the last order we find[20]

$$
\begin{aligned}
(4\mathrm{g}^4_{\bar{i}\bar{j}} + h^1_{\bar{i}\bar{j}})\gamma^{\bar{j}}\varepsilon \;=\; & -2\gamma_{\bar{i}}(1+\gamma_{\bar{z}})\varepsilon^7 + 4\left(\nabla_{\bar{i}}\,\varepsilon^5 + \tfrac{\mathrm{i}}{2}A^I_{\bar{i}}(\sigma_I\varepsilon^5)\right) - 2X_2^2\gamma_{\bar{i}}\,\varepsilon - 2\mathrm{g}^2_{\bar{i}\bar{j}}\gamma^{\bar{j}}\varepsilon^3 \\
& + \tfrac{\mathrm{i}}{6}(\gamma_{\bar{i}}^{\;\overline{jk}} - 4\delta_{\bar{i}}^{[\bar{j}}\gamma^{\bar{k}]})\Big[(\mathcal{D}a^I_2)_{\overline{jk}}(\sigma_I\varepsilon) + (\mathrm{f}_2)_{\overline{jk}}\varepsilon + F^I_{\overline{jk}}(\sigma_I\varepsilon^3) - X_2F^I_{\overline{jk}}(\sigma_I\varepsilon) \\
& + 2\mathrm{e}^j_{\bar{j}}(\mathrm{e}^{(2)})^k_{\bar{k}}F^I_{jk}(\sigma_I\varepsilon)\Big] - 2\Big[\mathrm{e}^i_{\bar{i}}\,(\mathrm{e}^{(2)})^j_{\bar{j}} + (\mathrm{e}^{(2)})^i_{\bar{i}}\,\mathrm{e}^j_{\bar{j}}\Big]\mathrm{g}^2_{ij}\gamma^{\bar{j}}\varepsilon\,. \qquad (2.4.67)
\end{aligned}
$$

Again there is a positive chirality condition on ε^7 which removes it from the above equation. Using the many intermediate results we have derived, we then find

$$
\begin{aligned}
4\mathrm{g}^4_{ij} + h^1_{ij} \;=\; & 2\nabla_i\nabla_j\left(X_2 + \tfrac{1}{24}R\right) + 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} + \left(X_2 - \tfrac{1}{12}R\right)R_{ij} \\
& + g_{ij}\left(-\tfrac{1}{6}RX_2 - 2X_2^2 + \tfrac{1}{12}R_{kl}R^{kl}\right) + \tfrac{1}{4}R_{ik}R^k{}_j \qquad (2.4.68) \\
& - \tfrac{1}{8}\epsilon^{mnk}{}_jR_{mnli}R_k{}^l + \tfrac{1}{4}R_{iklj}R^{kl} + \tfrac{1}{3}[2\mathcal{D}a^I_2 - *(\mathcal{D}a^I_2)]_{(i|k|}\mathrm{J}^{Ik}{}_{|j)}\,.
\end{aligned}
$$

2.5 Metric independence

Our aim in this section is to show that, for any supersymmetric asymptotically locally hyperbolic solution to the Euclidean $\mathcal{N}=4^+$ supergravity theory, with the topologically twisted boundary conditions on an arbitrary Riemannian four-manifold (M_4, g), the variation (2.1.1) of the holographically renormalized action is identically zero. As explained in the introduction, this implies that the right hand side of (1.4.1) is independent of the choice of metric g, precisely as expected for the holographic dual of a topological QFT. We find that this is indeed the case, using the minimal holographic renormalization scheme described in section 2.3.3. We comment further on this at the end of section 2.5.2.

2.5.1 Variation of the action

As discussed in section 2.4.2, the Donaldson–Witten topological twist corresponds to the following boundary conditions on the supergravity fields on M_4:

$$
0 \;=\; b^{\pm} \;=\; \mathrm{a} \;=\; \varepsilon^- \,, \qquad X_1 \;=\; -\tfrac{1}{12}R\,, \qquad A^I \;=\; \tfrac{1}{2}\omega_i{}^{\overline{jk}}\mathrm{J}^I_{\overline{jk}}\,\mathrm{d}x^i\,. \qquad (2.5.1)
$$

Here the boundary Riemannian metric g_{ij} on M_4 is arbitrary, with $\omega_i{}^{\overline{jk}}$ being the spin connection, R being the Ricci scalar curvature, and the triplet of self-dual 2-forms J^I being

[20]Of course, knowing $h^1_{\bar{i}\bar{j}}$ we could write an expression for $\mathrm{g}^4_{\bar{i}\bar{j}}$ alone, but it is only the combination $4\mathrm{g}^4_{\bar{i}\bar{j}} + h^1_{\bar{i}\bar{j}}$ which we shall need in the next section.

given by (2.4.21). The holographic Ward identity for the variation of the renormalized action (2.3.43) with respect to general variations of the non-zero boundary fields is

$$\delta S \;=\; \delta_g S + \delta_{A^I} S + \delta_{X_1} S \;=\; \int_{\partial Y_5 = M_4} \mathrm{d}^4 x\, \sqrt{\det g}\, \Big[\tfrac{1}{2} T_{ij} \delta g^{ij} + \mathscr{J}^i_I \delta A^I_i + \Xi\, \delta X_1 \Big]\,. \qquad (2.5.2)$$

It is worth pausing to consider carefully why this equation holds. A variation of the boundary data on M_4 will induce a corresponding variation of the bulk solution that fills it. However, we are evaluating the action on a solution to the equations of motion, and by definition these are stationary points of the bulk action. Thus the resulting variation of the on-shell action is necessarily a boundary term, and this is the expression on the right hand side of (2.5.2). This argument requires that the equations of motion are solved everywhere in the interior of Y_5: if the latter has internal boundaries, or singularities, the above in general breaks down, and one will encounter additional terms around these boundaries/singularities on the right hand side of (2.5.2).

For the topological twist all boundary fields are determined by the metric g_{ij}. Since $X_1 = -\frac{1}{12} R$, to compute δX_1 we need the variation of the Ricci scalar:

$$\delta R \;=\; R_{ij} \delta g^{ij} + \nabla_i \left(g^{jk} \delta \Gamma^i_{jk} - g^{ij} \delta \Gamma^k_{jk} \right), \qquad (2.5.3)$$

with the variation of the Christoffel symbols being

$$\delta \Gamma^i_{jk} \;=\; \tfrac{1}{2} g^{il} \left(\nabla_k \delta g_{lj} + \nabla_j \delta g_{lk} - \nabla_l \delta g_{jk} \right)\,. \qquad (2.5.4)$$

After integrating by parts twice we obtain

$$\delta_{X_1} S \;=\; -\frac{1}{12} \int_{\partial Y_5} \left[\big(\Xi R_{ij} + g_{ij} \nabla^2 \Xi - \nabla_i \nabla_j \Xi \big) \delta g^{ij}\, \mathrm{vol}_4 + \frac{1}{\kappa_5^2} \mathscr{D}_{X_1} \mathrm{vol}_4 \right], \qquad (2.5.5)$$

where $\mathrm{vol}_4 \equiv \sqrt{\det g}\, \mathrm{d}^4 x$ is the Riemannian volume form on (M_4, g), and all geometric quantities appearing are computed using the boundary metric g_{ij}. Substituting the value of Ξ from (2.3.50) leads to

$$\delta_{X_1} S \;=\; -\frac{1}{4\kappa_5^2} \int_{\partial Y_5} \big[\big(X_2 R_{ij} + g_{ij} \nabla^2 X_2 - \nabla_i \nabla_j X_2 \big)\, \delta g^{ij}\, \mathrm{vol}_4 + \tfrac{1}{3} \mathscr{D}_{X_1} \mathrm{vol}_4 \big], \qquad (2.5.6)$$

where the total derivative term is

$$\mathscr{D}_{X_1} \;\equiv\; -3\nabla_i\Big[\nabla^k X_2 g^{ij}\delta g_{jk} - \nabla^i X_2 g^{jk}\delta g_{jk} - X_2 g^{jk} g^{il}(\nabla_k \delta g_{lj} - \nabla_l \delta g_{jk})\Big]\,. \qquad (2.5.7)$$

For δA_i^I we first need the variation of the spin connection. After a short calculation we have

$$\delta\omega_i{}^{\overline{jk}} \;=\; \tfrac{1}{2}\,\mathrm{e}^{l\bar{j}}\,\mathrm{e}^{m\bar{k}}\,(\nabla_m\delta g_{il} - \nabla_l\delta g_{im})\;. \qquad (2.5.8)$$

Thus

$$\delta A_i^I \;=\; \tfrac{1}{2}\delta\omega_i{}^{\overline{jk}}\mathrm{J}^I_{\overline{jk}} \;=\; \tfrac{1}{2}(\nabla_k\delta \mathrm{g}_{ij})\mathrm{J}^{Ijk}\,. \qquad (2.5.9)$$

After integrating by parts, the $SU(2)_R$ current contribution is hence

$$\delta_{A^I}\mathrm{S} \;=\; -\frac{1}{8\kappa_5^2}\int_{\partial Y_5}\Big\{\Big[\mathcal{D}^k(a_1^I + 2a_2^I)_i\,\mathrm{J}^I_{jk}\Big]\,\delta g^{ij}\,\mathrm{vol}_4 + \mathscr{D}_{A^I}\mathrm{vol}_4\Big\}, \qquad (2.5.10)$$

where we have substituted for the $SU(2)_R$ current using (2.3.51), and used the quaternionic Kähler identity (2.4.29). The object in square brackets is a tensor with indices ij: only the symmetric part contributes. The total derivative term is

$$\mathscr{D}_{A^I} \equiv\; \nabla_i\Big[(a_1^I + 2a_2^I)^k\mathrm{J}^{Iij}\delta g_{jk}\Big]\,. \qquad (2.5.11)$$

It remains to evaluate the stress-energy tensor contribution (2.3.49) and combine it with (2.5.6) and (2.5.10). Doing so leads to

$$\delta\mathrm{S} \;=\; \frac{1}{4\kappa_5^2}\int_{\partial Y_5}\Big(\mathcal{T}_{ij}\,\delta g^{ij}\,\mathrm{vol}_4 + \mathscr{D}_S\,\mathrm{vol}_4\Big)\;, \qquad (2.5.12)$$

where the total derivative term is

$$\mathscr{D}_S \;\equiv\; -\tfrac{1}{3}\mathscr{D}_{X_1} - \tfrac{1}{2}\mathscr{D}_{A^I}\;, \qquad (2.5.13)$$

and

$$\begin{aligned}\mathcal{T}_{ij} \;=\; & \big[4\mathrm{g}^4_{ij} + h^1_{ij} - 4g_{ij}\big(t^{(4)} - \tfrac{1}{2}t^{(2,2)} - \tfrac{1}{8}u^{(1)}\big) - 2\mathrm{g}^2_{ij}t^{(2)} - 6g_{ij}X_2^2 \\ & + \tfrac{1}{2}\big(\nabla^k\nabla_i \mathrm{g}^2_{jk} + \nabla^k\nabla_j \mathrm{g}^2_{ik} - \nabla^2\mathrm{g}^2_{ij} - \nabla_i\nabla_j t^{(2)}\big) - \tfrac{1}{2}\mathrm{g}^2_{ij}R + \tfrac{1}{2}g_{ij}\big(\mathrm{g}^2_{kl}R^{kl}\big)\big] \\ & - \big(X_2R_{ij} + g_{ij}\nabla^2 X_2 - \nabla_i\nabla_j X_2\big) - \tfrac{1}{2}\left[\mathcal{D}^k(a^I_1 + 2a^I_2)_{(i}\,\mathrm{J}^I{}_{j)k}\right] .\end{aligned} \tag{2.5.14}$$

Here the first two lines come from the stress-energy tensor (2.3.49), while the last line combines (2.5.6) and (2.5.10). Provided M_4 is a closed manifold, without boundary, the integral of the total derivative term is zero, and we have simply

$$\delta\mathbb{S} \;=\; \frac{1}{4\kappa_5^2}\int_{\partial Y_5 = M_4} \mathcal{T}_{ij}\,\delta g^{ij}\,\mathrm{vol}_4 \;. \tag{2.5.15}$$

The tensor $\mathcal{T}_{ij}$ is thus an *effective* stress-energy tensor, for variations of the renormalized on-shell action with respect to the boundary metric, all boundary data being determined by this choice of metric. Our claim that the on-shell action is invariant under an arbitrary metric deformation δg_{ij} is thus equivalent to the statement that $\mathcal{T}_{ij} \equiv 0$, for every Riemannian four-manifold. Remarkably, despite there being several undetermined quantities in (2.5.14), using the results of sections 2.3.3 and 2.4.4 we will show that indeed $\mathcal{T}_{ij} \equiv 0$ in the next subsection.

2.5.2 Proof that $\delta\mathbb{S}/\delta g_{ij} = 0$

We begin by substituting expressions from section 2.3.2 into (2.5.14), which recall follow from the Fefferman–Graham expansion of the bosonic equations of motion. In particular we substitute for $\nabla^2 X_2$ using equation (2.3.30), as well as various metric quantities, except for the combination $4\mathrm{g}^4_{ij} + h^1_{ij}$. With the topological twist boundary conditions (2.5.1) this leads to the expression

$$\begin{aligned}\mathcal{T}_{ij} \;=\; & \big(\tfrac{1}{12}R - X_2\big)\,R_{ij} - \tfrac{1}{2}R_{ik}R^k{}_j - \tfrac{1}{2}R_{iklj}R^{kl} - \tfrac{1}{4}\nabla_i\nabla_j R + \nabla_i\nabla_j\left(X_2 + \tfrac{1}{6}R\right) \\ & + \tfrac{1}{4}\nabla^2 R_{ij} + g_{ij}\left(2X_2^2 - \tfrac{1}{72}R^2 + \tfrac{1}{6}RX_2 - \tfrac{1}{24}\nabla^2 R + 4X_3 + 4X_4\right) \\ & + 4\mathrm{g}^4_{ij} + h^1_{ij} - \tfrac{1}{2}\left[\mathcal{D}^k(a^I_1 + 2a^I_2)_{(i}\,\mathrm{J}^I{}_{j)k}\right] .\end{aligned} \tag{2.5.16}$$

In particular we have used the identity

$$-\tfrac{1}{2}\nabla_k\nabla_{(i}R^k{}_{j)} \;=\; -\tfrac{1}{2}R_{ik}R^k{}_j - \tfrac{1}{2}R_{iklj}R^{kl} - \tfrac{1}{4}\nabla_i\nabla_j R \;, \tag{2.5.17}$$

in deriving (2.5.16).

The equations of motion, or equivalently supersymmetry conditions, determine

$$X_3 \;=\; \tfrac{1}{48}\nabla^2 R\,, \quad X_4 \;=\; \tfrac{1}{288}R^2 - \tfrac{1}{48}R_{kl}R^{kl} - \tfrac{1}{96}\nabla^2 R - \tfrac{1}{24}\left(\mathcal{D}a_2^I\right)^{ij} \mathrm{J}^I_{ij}\,. \tag{2.5.18}$$

On the other hand, in section 2.4.4 the expansion of the supersymmetry conditions led to the expression (2.4.68), which we repeat here:

$$\begin{aligned} 4\mathrm{g}^4_{ij} + h^1_{ij} \;&=\; 2\nabla_i\nabla_j\left(X_2 + \tfrac{1}{24}R\right) + 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} + \left(X_2 - \tfrac{1}{12}R\right)R_{ij} \\ &\quad + g_{ij}\left(-\tfrac{1}{6}RX_2 - 2X_2^2 + \tfrac{1}{12}R_{kl}R^{kl}\right) + \tfrac{1}{4}R_{ik}R^k{}_j \\ &\quad - \tfrac{1}{8}\epsilon^{mnk}{}_j R_{mnli}R_k{}^l + \tfrac{1}{4}R_{iklj}R^{kl} + \tfrac{1}{3}[2\mathcal{D}a_2^I - *(\mathcal{D}a_2^I)]_{(i|k|}\mathrm{J}^{Ik}{}_{|j)}\,. \end{aligned} \tag{2.5.19}$$

Substituting into (2.5.16), after several immediate cancellations we are left with

$$\begin{aligned} \mathcal{T}_{ij} \;&=\; \tfrac{1}{4}\nabla^2 R_{ij} - \tfrac{1}{8}\epsilon^{mnk}{}_j R_{mnpi}R_k{}^p - \tfrac{1}{4}R_{ik}R^k{}_j - \tfrac{1}{4}R_{iklj}R^{kl} + 3\nabla_i\nabla_j X_2 - \tfrac{1}{2}\mathcal{D}^k(a_1^I)_{(i}\mathrm{J}^I{}_{j)k} \\ &\quad + 2\mathrm{i}\nabla_{(i}(\mathrm{a}_2)_{j)} - \tfrac{1}{6}g_{ij}\left(\mathcal{D}a_2^I\right)^{kl} J^I_{kl} + \tfrac{1}{3}(2\mathcal{D}a_2^I - *\mathcal{D}a_2^I)_{(i|k|}\mathrm{J}^{Ik}{}_{j)} - \mathcal{D}^k(a_2^I)_{(i}\mathrm{J}^I{}_{j)k}\,. \end{aligned} \tag{2.5.20}$$

Using the expression

$$(a_1^I)_i \;=\; -\tfrac{1}{4}\mathrm{J}^I_{mn}\nabla_j R^{mnj}{}_i\,, \tag{2.5.21}$$

together with the contracted second Bianchi identity, we find that

$$\mathcal{D}^k(a_1^I)_i\mathrm{J}^I_{jk} \;=\; -\tfrac{1}{2}\epsilon_j{}^{kmn}\nabla_k\nabla_m R_{ni} - \tfrac{1}{2}\nabla^k\nabla^l R_{jkli}\,. \tag{2.5.22}$$

Substituting this expression, together with equation (2.4.47), into $\mathcal{T}_{ij}$ in (2.5.20), we arrive at

$$\begin{aligned} \mathcal{T}_{ij} \;&=\; \tfrac{1}{4}\nabla^2 R_{ij} - \tfrac{1}{8}\nabla_i\nabla_j R + \tfrac{1}{4}\nabla^k\nabla^l R_{jkli} - \tfrac{1}{4}R_{ik}R^k{}_j - \tfrac{1}{4}R_{iklj}R^{kl} \\ &\quad - \tfrac{1}{6}g_{ij}\left(\mathcal{D}a_2^I\right)^{kl}\mathrm{J}^I_{kl} + \tfrac{1}{3}[2\mathcal{D}a_2^I - *(\mathcal{D}a_2^I)]_{(i|k|}\mathrm{J}^{Ik}{}_{j)} - (\mathcal{D}a_2^I)_{(i|k|}\mathrm{J}^{Ik}{}_{j)} \\ &\quad + \tfrac{1}{8}\epsilon_j{}^{kmn}(2\nabla_k\nabla_m R_{ni} - R_{mni}{}^l R_{kl}) \\ &= 0\,. \end{aligned} \tag{2.5.23}$$

Here, remarkably, each of the three lines vanishes separately. The first line is zero using again (2.5.17) and the contracted second Bianchi identity, whilst the terms in the second line combine to give zero after using the self-duality property of the J^I tensors to remove the Hodge dual acting on the field strength $\mathcal{D}a_2^I$. The final line is zero after applying the Ricci identity for a rank 2 covariant tensor, followed by the first Bianchi identity and using the symmetry of the summed indices.

We emphasize again that this proof that $\delta S/\delta g_{ij} = 0$ uses the minimal holographic renormalization scheme defined in section 2.3.3. Up to finite counterterms in (2.3.44) that are topological invariants, which have identically zero variations, another choice of scheme would spoil the above result. Another important comment is that the original path integral arguments in [219] are essentially classical (see footnote 10 of [219]). In particular there might have been an anomaly, implying that the partition function (and other correlation functions) are not invariant under arbitrary metric deformations. In this case, the topological twist would not have led to a TQFT. This might seem like a strange comment, given that the topologically twisted $\mathcal{N} = 2$ Yang–Mills theory of [219] at least formally reproduces Donaldson theory, which of course certainly does rigorously define diffeomorphism invariants of M_4. However, it has recently been argued that precisely such an anomaly exists for four-dimensional rigid $\mathcal{N} = 1$ supersymmetry [191, 12]. The computations in these papers are in fact holographic, and rely on the fact that in AdS/CFT the semi-classical gravity computation is a fully quantum computation on the QFT side, including any potential anomalies. Specifically, it is argued that there is an anomalous transformation of the supercurrent under rigid supersymmetry on the conformal boundary, implying that the partition function is not invariant under certain metric deformations that are classically $\mathcal{Q}$-exact. These particular anomalous transformations were first discovered in [39, 38], via essentially the same computation we have followed in this chapter, although this was not interpreted as an anomaly in [39, 38] (this will be the content of chapter 4). Returning to our present problem, the QFT is in any case coupled to an $\mathcal{N} = 2$ conformal supergravity background, and for the $\mathcal{N} = 2$ topological twist we find no anomaly. In particular our topologically twisted supergravity theory, *formally* at least, defines a topological theory. We discuss this further in section 2.6.4.

2.6 Geometric reformulation

In this section we present a geometric reformulation of the bulk supersymmetry equations. In section 2.6.1 we describe how (twisted) differential forms built out of bilinears in the bulk spinor define a twisted $Sp(1)$ structure on Y_5, and in section 2.6.2 we then derive a set of first order differential constraints on this structure. On the conformal boundary this restricts to the quaternionic Kähler structure that exists on any oriented Riemannian four-manifold (M_4, g), described in section 2.4.2. In section 2.6.3, we use the information from the differential constraints to evaluate the gravitational free energy for smooth filling. Finally, we also discuss some general aspects of the filling problem in section 2.6.4.

2.6.1 Twisted $Sp(1)$ structure

Recall from section 2.3.1 that the bulk spinor ϵ of the Romans $\mathcal{N} = 4^+$ theory is originally a quadruplet of spinors. These split into two doublets $\epsilon^{\pm}$, with eigenvalues $\pm\mathrm{i}$ under Γ_{45} (see equation (2.3.11)). Beginning in section 2.4.2, we worked in a truncated theory in which $\mathcal{B}^{\pm} = 0$ and $\epsilon^- = 0$. We may then define

$$\epsilon^+ = \begin{pmatrix} \zeta \\ -\zeta^c \end{pmatrix} , \qquad (2.6.1)$$

where ζ is a spinor on Y_5, and recall that $\zeta^c \equiv \mathscr{C}\zeta^*$. Equation (2.6.1) is the solution to the symplectic Majorana condition $(\epsilon^+)^c = \epsilon^+$. More globally, and as on the conformal boundary M_4, the spinor ϵ^+ in (2.6.1) is a $Spin_{\mathscr{G}}$ spinor, where $\mathscr{G} = SU(2)_R$ – see section 2.4.2.

With this notation we may define the following (local) differential forms

$$\begin{aligned} S &\equiv \bar{\zeta}\zeta , & \mathcal{K} &\equiv \frac{1}{S}\bar{\zeta}\gamma_{(1)}\zeta , \\ \mathcal{J}^3 &\equiv \frac{\mathrm{i}}{S}\bar{\zeta}\gamma_{(2)}\zeta, & \mathcal{J}^2 + \mathrm{i}\mathcal{J}^1 &\equiv \frac{1}{S}\bar{\zeta^c}\gamma_{(2)}\zeta , \end{aligned} \qquad (2.6.2)$$

where in our Hermitian basis of Clifford matrices recall that a bar denotes Hermitian conjugate. There are a number of global comments to make. First, as in the discussion in section 2.4.2, the fact that ζ is globally a twisted spinor, rather than a spinor, means that (2.6.2) in general only locally defines an $SU(2) \cong Sp(1)$ structure.[21] More globally, the $\mathcal{J}^I$ are twisted via the $SU(2)_R$ symmetry, transforming as a triplet. We shall call this a *twisted* $Sp(1)$ structure. Another comment is that in any case the structure is well-defined only where $\zeta \neq 0$. In general there may be solutions to the spinor equations where $\zeta = 0$ on some locus. We should hence more precisely define $Y_5^{(0)} \equiv Y_5 \setminus \{\zeta = 0\}$, so that (2.6.2) is well-defined on $Y_5^{(0)}$. One will then need to impose certain boundary conditions on this

[21] A general discussion of global $Sp(1)$ structures on five-manifolds may be found in [72].

structure, near $\{\zeta = 0\}$, in order that the solution on Y_5 is appropriately regular. The bilinears (2.6.2) define a twisted $Sp(1)$ structure on $Y_5^{(0)}$.

Continuing the analysis of section 2.4.4 the expansion of the spinor (2.4.38) implies that near the conformal boundary

$$\begin{aligned}\zeta \;&= z^{-1/2}\chi + z^{3/2}\left(\tfrac{1}{48}R\right)\chi + z^{5/2}\left(-\tfrac{1}{24}\mathrm{d}R\,\log z + \tfrac{\mathrm{i}}{2}\mathrm{a}_2 + \tfrac{1}{2}\mathrm{d}X_2 + \tfrac{1}{48}\mathrm{d}R\right)_i\gamma^i\chi \\ &+ z^{7/2}\Big[-\tfrac{1}{1152}R^2\log^2 z + \tfrac{1}{48}\left(RX_2 + \tfrac{1}{16}R^2 - \tfrac{1}{4}\nabla^2 R\right)\log z \\ &\qquad - \tfrac{1}{8}\left(X_2^2 + \tfrac{1}{8}RX_2 + \tfrac{1}{128}R^2 - \tfrac{1}{96}\nabla^2 R - \tfrac{1}{24}R_{ij}R^{ij} - \tfrac{\mathrm{i}}{12}(\mathrm{d}\mathrm{a}_2)_{ij}\gamma^{ij}\right)\Big]\chi \\ &+ z^{7/2}\left[\tfrac{\mathrm{i}}{96}\left(\mathcal{D}a_2^I\right)_{ij}\gamma^{ij}(\sigma_I\epsilon^{-1})_1\right] + o(z^4)\,.\end{aligned} \tag{2.6.3}$$

where χ is the boundary spinor defined in section 2.4.2. In particular for the topological twist this is constant, with constant square norm $\bar{\chi}\chi = c^2$ (see equations (2.4.24), (2.4.25)). Without loss of generality we henceforth set $c = 1$, so that

$$S \;=\; \frac{1}{z} + \frac{z}{24}R + o(z^{5/2})\;. \tag{2.6.4}$$

In particular notice that $\zeta \neq 0$ near to the conformal boundary at $z = 0$. The last line of (2.6.3), seemingly, cannot be written in terms of the lowest order constant spinor χ, however it will not play a part in the following.

2.6.2 Differential system

Starting from the bulk Killing spinor equations (2.3.7), (2.3.8) one can derive a system of differential equations for the twisted $Sp(1)$ structure (2.6.2). In the notation (2.6.1) the spinor equations read

$$\begin{aligned}\nabla_\mu\zeta \;&=\; -\tfrac{\mathrm{i}}{2}\mathcal{A}_\mu\zeta + \tfrac{\mathrm{i}}{2}\left(\mathcal{A}^1_\mu - \mathrm{i}\mathcal{A}^2_\mu\right)\zeta^c - \tfrac{\mathrm{i}}{2}\mathcal{A}^3_\mu\zeta + \tfrac{1}{3}(X + \tfrac{1}{2}X^{-2})\gamma_\mu\zeta \\ &+ \tfrac{\mathrm{i}}{24}X^{-1}(\mathcal{F}^1_{\nu\rho} - \mathrm{i}\mathcal{F}^2_{\nu\rho})(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho)\zeta^c - \tfrac{\mathrm{i}}{24}\left(X^{-1}\mathcal{F}^3_{\nu\rho} + X^2\mathcal{F}_{\nu\rho}\right)(\gamma_\mu{}^{\nu\rho} - 4\delta^\nu_\mu\gamma^\rho)\zeta\,, \\ 0 \;&=\; \tfrac{3}{2}\mathrm{i}\,X^{-1}\partial_\mu X\gamma^\mu\zeta + \mathrm{i}(X - X^{-2})\zeta - \tfrac{1}{8}X^{-1}(\mathcal{F}^1_{\mu\nu} - \mathrm{i}\mathcal{F}^2_{\mu\nu})\gamma^{\mu\nu}\zeta^c \\ &+ \tfrac{1}{8}(X^{-1}\mathcal{F}^3_{\mu\nu} - 2X^2\mathcal{F}_{\mu\nu})\gamma^{\mu\nu}\zeta\,.\end{aligned} \tag{2.6.5}$$

As in section 2.3.1, it will be convenient to introduce the real 1-form

$$\mathcal{C} \;\equiv\; \mathrm{i}\mathcal{A}\,. \tag{2.6.6}$$

Using these equations, a standard calculation[22] leads to

$$X^{-2}\mathcal{K} \;= \mathrm{d}\log(XS) + \mathcal{C}\,, \tag{2.6.7}$$

together with the triplet of equations

$$\begin{aligned} \mathrm{d}(S\mathcal{J}^I) \;&=\; -\,\mathcal{C}\wedge S\mathcal{J}^I + (2X + X^{-2})\mathcal{K}\wedge S\mathcal{J}^I + \epsilon^I{}_{JK}\mathcal{A}^J\wedge S\mathcal{J}^K \\ &\quad + \tfrac{1}{2}X^{-1}S\,(*\mathcal{F}^I + \mathcal{K}\wedge\mathcal{F}^I)\,. \end{aligned} \tag{2.6.8}$$

Here the Hodge dual is constructed from the volume form $\mathrm{vol}_5 = -\mathcal{K}\wedge\mathrm{vol}_4$, where $\mathrm{vol}_4 \equiv \frac{1}{6}\mathcal{J}^I\wedge\mathcal{J}^I$. The sign here is chosen to match our earlier choice of orientation, via (2.3.16), as we shall see shortly.

We may read the first equation (2.6.7) as determining the 1-form $\mathcal{C}$ in terms of geometric data and the function X:

$$\mathcal{C} \;=\; X^{-2}\mathcal{K} - \mathrm{d}\log(XS)\,. \tag{2.6.9}$$

In particular, the associated flux is then

$$\mathcal{G} \;\equiv\; \mathrm{d}\mathcal{C} \;=\; \mathrm{i}\mathcal{F} \;=\; \mathrm{d}(X^{-2}\mathcal{K})\,. \tag{2.6.10}$$

Substituting (2.6.9) into (2.6.8), the latter simplifies to

$$\mathrm{d}\mathcal{J}^I \;=\; \epsilon^I{}_{JK}\mathcal{A}^J\wedge\mathcal{J}^K + (\mathrm{d}\log X + 2X\mathcal{K})\wedge\mathcal{J}^I + \tfrac{1}{2}X^{-1}(*\mathcal{F}^I + \mathcal{K}\wedge\mathcal{F}^I)\,. \tag{2.6.11}$$

Recall that in the original Lorentzian theory $\mathcal{A}$ is a $U(1)_R$ gauge field. In the real Euclidean section we have defined $\mathcal{C} = \mathrm{i}\mathcal{A}$, which is a real 1-form, but there is then a residual part of the (complexified) gauge symmetry $\mathcal{C}\to\mathcal{C}-\mathrm{d}\lambda$, where λ is a global real function. The fields transform as follows:

$$\zeta \;\to\; \mathrm{e}^{\lambda/2}\zeta\,, \qquad S \;\to\; \mathrm{e}^{\lambda}S\,, \qquad \mathcal{C} \;\to\; \mathcal{C}-\mathrm{d}\lambda\,, \tag{2.6.12}$$

[22] For example, see [8].

with everything else invariant. In particular it is immediate to see that (2.6.9), (2.6.11) are invariant under these gauge transformations. In our boundary value problem recall that we fixed $\mathcal{C}\mid_{M_4}=0$, and in order to preserve this gauge condition on the conformal boundary one should restrict to gauge transformations that vanish there, so that $\lambda\mid_{M_4}=0$. With this caveat, one might use this gauge freedom to effectively remove one of the functional degrees of freedom.

Let us look at the asymptotic form of the differential conditions near the conformal boundary at $z=0$. Recalling the Fefferman–Graham expansion of the fields (2.3.18)–(2.3.20), together with the topological twist boundary conditions (2.5.1), we have

$$\begin{aligned} X &= 1-\tfrac{1}{12}z^2\log z\, R+z^2X_2+\tfrac{1}{48}z^4\log z\,\nabla^2R \\ &\quad +z^4\Big(-\tfrac{1}{4}\nabla^2X_2-\tfrac{1}{48}\nabla^2R+\tfrac{1}{288}R^2-\tfrac{1}{48}R_{ij}R^{ij}-\tfrac{1}{192}(\mathcal{E}+\mathcal{P})\Big)+o(z^4)\,, \\ \mathcal{A}^I &= \tfrac{1}{2}\omega_i{}^{jk}\mathrm{J}^I_{jk}\,\mathrm{d}x^i-\tfrac{1}{4}z^2\log z\,\mathrm{J}^I_{mn}\nabla_jR^{mnj}{}_i\,\mathrm{d}x^i+z^2a^I_2+o(z^2)\,, \\ \mathcal{A} &= z^2\,\mathrm{a}_2+o(z^2)\,. \end{aligned} \tag{2.6.13}$$

Here R, R_{ij} and R_{mnij} are respectively the boundary Ricci scalar, Ricci and Riemann tensor and $\mathcal{E},\mathcal{P}$ are the Euler and Pontryagin densities constructed from these curvature tensors. The boundary spin connection is $\omega_i{}^{jk}$ and J^I are the triplet of boundary self-dual 2-forms. The 1-form ia_2 is real. Using also (2.6.4), equation (2.6.7) then implies that

$$\mathcal{K} \;=\; -\frac{\mathrm{d}z}{z}+z^2\big(-\tfrac{1}{12}\log z\,\mathrm{d}R+\mathrm{ia}_2+\mathrm{d}X_2+\tfrac{1}{24}\mathrm{d}R\big)+o(z^{5/2})\,. \tag{2.6.14}$$

Recall that in section 2.4.2 we defined the triplet of boundary almost complex structures $(\mathrm{I}^I)^i{}_j\equiv g^{ik}(\mathrm{J}^I)_{kj}$. If we define the boundary (almost) Ricci 2-forms

$$\rho^I_{ij} \;\equiv\; R_{k[i}(\mathrm{I}^I)^k{}_{j]}\,, \tag{2.6.15}$$

where R_{ij} is the boundary Ricci tensor, then similarly from the definition (2.4.26) we have

$$\begin{aligned} \mathcal{J}^I \;&=\; \frac{1}{z^2}\mathrm{J}^I+\tfrac{1}{12}R\,\mathrm{J}^I-\tfrac{1}{2}\rho^I \\ &\quad +z\mathrm{d}z\wedge\mathrm{I}^I\big(-\tfrac{1}{12}\log z\,\mathrm{d}R+\mathrm{ia}_2+\mathrm{d}X_2+\tfrac{1}{24}\mathrm{d}R\big)+o(z^{3/2})\,. \end{aligned} \tag{2.6.16}$$

Here $\mathrm{I}^I(\eta)_i = (\mathrm{I}^I)^j{}_i\eta_j$ for a 1-form η tangent to the boundary. It is interesting to note that the $O(1)$ terms in $\mathcal{J}^I$ above may also be written as $\frac{1}{12}R\,\mathrm{J}^I - \frac{1}{2}\rho^I = (\mathrm{g}^2 \circ J^I)$, where recall from equation (2.3.33) that g^2 is (minus) the Schouten tensor of the conformal boundary. From (2.6.11) we hence read off the leading order the boundary equation

$$\mathrm{dJ}^I \;=\; \epsilon^I{}_{JK}A^J \wedge \mathrm{J}^K \,. \tag{2.6.17}$$

Equation (2.6.17) follows from taking the skew symmetric part of (2.4.29). In fact since the exterior derivatives of the boundary $SU(2)$ structure J^I completely determine the intrinsic torsion (this is true for an $SU(n)$ structure in real dimension $2n$ [114]), it follows that (2.6.17) also implies (2.4.29).

We may always choose a frame $\mathscr{E}^{\overline{\mu}}_{\mu}$ for the bulk metric on Y_5 such that

$$\begin{aligned} \mathcal{K} \;&=\; -\mathscr{E}^5\,, & \mathcal{J}^1 \;&=\; \mathscr{E}^2\wedge\mathscr{E}^3 + \mathscr{E}^1\wedge\mathscr{E}^4\,,\\ \mathcal{J}^2 \;&=\; \mathscr{E}^3\wedge\mathscr{E}^1 + \mathscr{E}^2\wedge\mathscr{E}^4\,, & \mathcal{J}^3 \;&=\; \mathscr{E}^1\wedge\mathscr{E}^2 + \mathscr{E}^3\wedge\mathscr{E}^4\,. \end{aligned} \tag{2.6.18}$$

In particular (2.6.14) identifies $\mathscr{E}^5 \sim \mathrm{d}z/z$ to leading order, and the sign for $\mathcal{K}$ in (2.6.18) follows since $-\gamma_{\overline{z}}\chi = \chi$, where $\mathrm{E}^{\overline{z}} = \mathrm{d}z/z$. The volume form is $\mathrm{vol}_5 = \mathscr{E}^{12345}$. Notice that the expansions (2.6.14), (2.6.16) imply that in general we may not identify $\mathscr{E}^{\overline{\mu}}_{\mu}$ near the conformal boundary with the Fefferman–Graham frame $\mathrm{E}^{\overline{\mu}}_{\mu}$ in (2.4.1), except to leading order.

2.6.3 On-shell action

In the consistent truncation of the supergravity theory that we used to construct the dual to the topological twist, the on-shell action obtained using the Einstein equation reads (compare with (2.3.40))

$$\begin{aligned} I_{\text{o-s}} \;=\; \frac{1}{2\kappa_5^2}\int_{Y_5} \big[&\tfrac{8}{3}(X^2 + 2X^{-1}) *1 + \tfrac{1}{3}X^4\mathcal{F}\wedge *\mathcal{F} + \tfrac{1}{6}X^{-2}\mathcal{F}^I \wedge *\mathcal{F}^I \\ &+ \tfrac{\mathrm{i}}{4}\mathcal{F}^I\wedge\mathcal{F}^I\wedge\mathcal{A}\big]\,. \end{aligned} \tag{2.6.19}$$

However, by additionally using the scalar field equation (2.3.2) twice and (2.3.4) to rewrite the Chern–Simons term we arrive at the following simpler expression

$$I_{\text{o-s}} \;=\; \frac{1}{2\kappa_5^2}\int_{Y_5}\left[8X^{-1}*1-\mathrm{d}\big(2X^{-1}*\mathrm{d}X-X^4\mathcal{A}\wedge*\mathcal{F}\big)\right]. \qquad (2.6.20)$$

Now with some simple manipulation of the differential system (2.6.7)–(2.6.8) we can show that

$$\tfrac{1}{3}\mathrm{d}(X^{-2}\mathcal{J}^I\wedge\mathcal{J}^I) \;=\; -\,8X^{-1}*1\,, \qquad (2.6.21)$$

and immediately conclude that the on-shell action is (locally) exact;

$$I_{\text{o-s}} \;=\; -\frac{1}{2\kappa_5^2}\int_{Y_5}\mathrm{d}\big(\tfrac{1}{3}X^{-2}\mathcal{J}^I\wedge\mathcal{J}^I+2X^{-1}*\mathrm{d}X-X^4\mathcal{A}\wedge*\mathcal{F}\big)\,. \qquad (2.6.22)$$

In addition to $\mathcal{A}$ being a global 1-form, with $\mathcal{F}$ a global 2-form, we assume that $X>0$ is a smooth global function on Y_5. Further, note that $\mathcal{J}^I\wedge\mathcal{J}^I\propto*\mathcal{K}$ and $\mathcal{K}$ is fixed by (2.6.7) in terms of X, $\mathcal{A}$ and S. Hence $\mathcal{K}$ is globally defined as long as the spinor norm $S=\bar{\zeta}\zeta\neq0$. Therefore, we should more precisely work on $Y_5^{(0)}$, so that $(S,\mathcal{K},\mathcal{J}^I)$ are well-defined and the gravity solution is smooth. In summary, the on-shell action is globally exact apart from a set which we assume has zero measure. As in section 2.3.3, we cut off the bulk Y_5 at some small radius $z=\delta>0$, so that $\partial Y_5=M_\delta\equiv\{z=\delta\}\cong M_4$. Using Stokes' theorem, we may then write the on-shell action as integral over $\partial Y_5^{(0)}$:

$$I_{\text{o-s}} \;=\; -\frac{1}{2\kappa_5^2}\int_{\partial Y_5^{(0)}}\left[\tfrac{1}{3}X^{-2}\mathcal{J}^I\wedge\mathcal{J}^I+2X^{-1}*\mathrm{d}X-X^4\mathcal{A}\wedge*\mathcal{F}\right]. \qquad (2.6.23)$$

Here $\partial Y_5^{(0)}$ comprises the conformal boundary $M_4\cong M_\delta$, and the boundaries T_ϵ of the small tubular neighbourhoods of radius $\epsilon>0$ surrounding the subsets $Y_5\setminus Y_5^{(0)}$ where the spinor norm vanishes.

The above on-shell action must be supplemented by the standard Gibbons–Hawking–York term at the UV boundary, I_{GHY} as in section 2.3.3 and the divergences may be cancelled by adding the local boundary counterterms coming from the truncation of (2.3.42):

$$\begin{aligned} I_{\text{ct}} \;=\; & \frac{1}{\kappa_5^2}\int_{M_\delta} \mathrm{d}^4x\,\sqrt{\det h}\,\Big\{3+\tfrac{1}{4}R(h)+3(X-1)^2 \\ & + \log\delta\Big[-\tfrac{1}{8}\Big(R_{ij}(h)R^{ij}(h)-\tfrac{1}{3}R(h)^2\Big)+\tfrac{3}{2}(\log\delta)^{-2}(X-1)^2 \\ & + \tfrac{1}{8}\mathcal{F}_h^2+\tfrac{1}{16}(\mathcal{F}^I)_h^2\Big]\Big\}\,. \end{aligned} \tag{2.6.24}$$

As the on-shell actions given by (2.6.19) and (2.6.23) are equivalent, I_{ct} must also cancel divergences arising from the latter when supplemented by the common Gibbons–Hawking–York term. The total renormalized action is then

$$\mathbb{S} \;=\; \lim_{\delta\to 0}\,\left(I_{\text{o-s}}+I_{\text{GHY}}+I_{\text{ct}}\right)\,. \tag{2.6.25}$$

In order to calculate the UV contribution to $\mathbb{S}$ of the term $\frac{1}{3}X^{-2}\mathcal{J}^I\wedge\mathcal{J}^I$ in $I_{\text{o-s}}$ we use the expansion of the spinor (2.6.3) and the definition of the bilinears in (2.6.2), determining

$$\begin{aligned} \mathcal{J}^I\wedge\mathcal{J}^I\Big|_{z=\delta} \;=\; & \Big[\tfrac{6}{\delta^4}-\tfrac{1}{2\delta^2}R+\tfrac{1}{8}\left(\tfrac{1}{3}R^2-R_{ij}R^{ij}\right)-\tfrac{1}{24}R^2\,\log^2\delta+RX_2\,\log\delta \\ & +\tfrac{1}{128}\left(-384X_2^2+\mathcal{E}+\mathcal{P}\right)\Big]\mathrm{vol}_4+o(\delta^{1/2})\,. \end{aligned} \tag{2.6.26}$$

Here we have restricted the 2-forms to the boundary at constant $z=\delta$. On forming the exterior product there are several simplifications, in particular the anti-symmetric indices of $\mathrm{d}a_2$ and $\mathcal{D}a_2^I$ are traced over and do not contribute. This can also be shown by expanding the equation $\mathcal{K}\wedge\mathcal{J}^I\wedge\mathcal{J}^I=-6\mathrm{vol}_5$.

We are finally in the position to evaluate the UV contribution to the renormalized on-shell action (2.6.25). We find

$$\mathbb{S}^{\text{UV}} \;=\; \lim_{\delta\to 0}\frac{1}{\kappa_5^2}\int_{\partial Y_5}\Big[\log\delta\Big(\tfrac{1}{32}(\mathcal{E}+\mathcal{P})*_4 1+\tfrac{1}{24}\mathrm{d}*_4\mathrm{d}R\Big)-\tfrac{1}{48}\mathrm{d}*_4\mathrm{d}(R+24X_2)\Big]\,. \tag{2.6.27}$$

At first sight the $\log\delta$ term is problematic as it diverges. However, as we saw in section 2.4.3, the topological condition $\int_{\partial Y_5}(\mathcal{E}+\mathcal{P})*_4 1=0$ is required in order for $\mathcal{A}$ to be a global 1-form, or equivalently to have a non-zero partition function for the boundary TQFT. Moreover, the Ricci scalar is a globally defined function on M_δ, and consequently for boundaryless four-manifolds, i.e. $\partial M_\delta=0$, the second term vanishes on using Stokes' theorem. The same argument applies to the finite piece of $\mathbb{S}^{\text{UV}}$ as the bulk scalar X, and hence X_2, is a global smooth function. It follows that the UV contribution to the renormalized action is zero for smooth fillings.

That now leaves us with the contribution from the small tubular neighbourhood T_ϵ where the spinor norm vanishes:

$$S = \frac{1}{\kappa_5^2}\lim_{\epsilon\to 0}\int_{T_\epsilon}\left[-\tfrac{1}{6}X^{-2}\wedge\mathcal{J}^I\wedge\mathcal{J}^I - X^{-1}*\mathrm{d}X + \tfrac{1}{2}X^4\mathcal{A}\wedge*\mathcal{F}\right]. \tag{2.6.28}$$

However, this gives zero for a smooth solution. The contributions from the second and third forms vanish in the limit $\epsilon\to 0$: $\mathcal{A}$ is assumed to be a global smooth 1-form on Y_5, and the bosonic field is assumed to be smooth. In particular, $X = e^{\frac{1}{2}\phi}$, so $X > 0$ (indeed, bounded below by a positive constant since Y_5 is compact). Thus the integrals tend to zero as the volume enclosed by T_ϵ tends to zero. In addition, the first term may be written as

$$-\tfrac{1}{6}X^{-2}\mathcal{J}^I\wedge\mathcal{J}^I = X^{-2}*\mathcal{K}. \tag{2.6.29}$$

One may worry that $\mathcal{K}$ is not defined as the norm of the spinor vanishes. However, we may use (2.6.7) to rewrite the relation above as

$$X^{-2}*\mathcal{K} = (\mathrm{d}\log\rho + \mathcal{C})^\sharp \lrcorner \mathrm{vol}_5\,, \tag{2.6.30}$$

where $\rho = XS$ can be used as a radial coordinate near to the locus where the spinor vanishes, where $\rho = 0$, and one defines $T_\epsilon = \{\rho = \epsilon > 0\}$. It follows that (apart from a contribution from the smooth gauge field) $X^{-2}*K$ is directly proportional to the volume form $(\mathrm{d}\log\rho)^\sharp\lrcorner\mathrm{vol}_5$ induced on T_ϵ from the five-dimensional bulk metric. The integral hence vanishes in the limit $\epsilon\to 0$, where the volume of the tubular neighbourhood T_ϵ vanishes. We conclude that the renormalized action for any *smooth* supergravity solution is zero.

Topologically, a smooth filling Y_5 of M_4 exists if and only if the signature $\sigma(M_4) = 0$.[23] Together with the constraint (2.4.34), one necessarily has Euler number and signature of M_4 equal to zero: $\chi(M_4) = 0 = \sigma(M_4)$. Apart from this, no other topological assumption is made about M_4 or its filling in the above computation.

[23] In four dimensions, the oriented bordism group is $\Omega_4^{SO}\cong\mathbb{Z}$, with the map to the integers being given by the signature $\sigma(M_4) = b_2^+(M_4) - b_2^-(M_4) = \frac{1}{3}\int_{M_4}p_1(M_4)$, where p_1 denotes the first Pontryagin class. A generator of $\Omega_4^{SO}\cong\mathbb{Z}$ is the complex projective plane.

2.6.4 Filling problem

As explained in chapter 1, given a Riemannian four-manifold (M_4, g) as a fixed conformal boundary, at least to a zeroth order approximation in AdS/CFT one wants to find the least action supersymmetric solution to the five-dimensional $\mathcal{N} = 4^+$ supergravity theory, with this boundary data. Such a solution will be the dominant saddle point on the right hand side of (1.4.1).

As we have seen in the previous subsection, supersymmetric solutions on Y_5 are characterized geometrically in terms of a set of first order differential equations (2.6.9)-(2.6.11) for a certain twisted $Sp(1)$ structure. In particular there is a triplet of twisted 2-forms $\mathcal{J}^I$, $I = 1, 2, 3$, which locally at the conformal boundary restrict to an orthonormal set of self-dual 2-forms on (M_4, g). The differential equations become tautological on the boundary, and are equivalent to the fact that every oriented Riemannian four-manifold has a quaternionic Kähler structure, i.e. has holonomy group $Sp(1) \cdot Sp(1) \cong SO(4)$.[24] This differential system on Y_5, regarded as extending that on (M_4, g), clearly deserves closer study.

An important question is: what are the global constraints on Y_5? As already mentioned, a smooth filling Y_5 of M_4 exists if and only if the signature of the boundary four-manifold vanishes. Moreover, as explained in section 2.2.2, for solutions embedded in string theory one also needs these manifolds to be spin.[25] This restriction would seem to rule out many interesting four-manifolds.[26] However, as mentioned at the end of chapter 1 as well, requiring Y_5 to be smooth is almost certainly too strong. Already from AdS/CFT in other contexts, it is clear that the dominant saddle point contribution can be singular, and one might anticipate that this is somewhat generic, at least for general M_4. Perhaps the appropriate question is then: what are the relevant singularities of Y_5, for a given M_4? Mathematically one would need control over existence and uniqueness of the differential equations for the twisted $Sp(1)$ structure, for appropriate Y_5 (with singularities/appropriate internal boundary conditions) filling M_4. However, one might also anticipate that the supergravity action (2.3.43) could be evaluated without knowing the detailed form of the solution, but instead in terms of appropriate global data, and perhaps local data associated

[24]This result is parallel to the study of rigid supersymmetric backgrounds using holography for four-dimensional $\mathcal{N} = 1$ theories [145]. There, the boundary structure was found to be an integrable almost complex structure.

[25]The relevant spin bordism group is then $\Omega_4^{Spin} \cong \mathbb{Z}$, generated by a K3 surface, where the map to the integers is $\sigma(M_4)/16$.

[26]Although it leaves, for example, $M_4 = S^1 \times M_3$, for any oriented three-manifold M_3, and products of Riemann surfaces.

to singularities. Notice that one constraint on such singularities/internal boundaries is that they do not contribute to the variation of the action (2.5.2) – see the discussion after (2.5.2).[27]

Less ambitiously, one might also try to find explicit solutions; for example, via symmetry reduction so that the equations reduce to coupled ODEs. An obvious case is solutions with $Y_5 = S^1 \times B_4$, where B_4 is a four-ball so that $\partial Y_5 = M_4 = S^1 \times S^3$, and seek solutions invariant under $U(1) \times SU(2)$ (the latter acting on the left on $S^3 \cong SU(2)$).

In this case it seems that the refinement of the partition function discussed in section 2.2 could play an important rôle: the refined partition function is closely related to the Coulomb branch index, as explained in [82]. One might then try to reproduce this from a dual supergravity solution for which $Y_5 = S^1 \times B_4$. More generally, for a four-manifold $S^1 \times M_3$ with product metric both $\mathcal{E}$ and $\mathcal{P}$ vanish, and the holographic $U(1)_R$ current is conserved, as can be seen from (2.4.32). The associated conserved holographic R-charge might then provide a natural holographic correspondent to the refinement of the partition function for the twisted four-dimensional SCFT.

Finally, the present problem may be contrasted to the general hyperbolic filling problem described in [13]. Here one also begins with an arbitrary Riemannian (M_4, g), which is a conformal boundary, but one instead asks for the filling to be an Einstein metric of negative curvature. This problem is still quite poorly understood: there are in general obstructions and non-uniqueness, and one should at least impose that g has a conformal representative with positive scalar curvature [226] (physically, so that the CFT is stable). The geometric problem in the present chapter is likely to be much better behaved: the equations are first order, not second order, and the solutions should be dual to a TQFT.

[27] For example, the singularities in the gravity fillings in [9, 8] are isolated conical singularities. Provided the radial dependence of fields near to the singular point are no worse than for smooth fields in flat space, such singularities will not spoil the result (2.5.2).

3 Topological AdS_4/CFT_3

3.1 Introduction

In the previous chapter, we considered the gravitational dual to the Donaldson–Witten twist on four-dimensional backgrounds. Since four-manifolds are notoriously difficult, in this chapter we set up an analogous problem in one dimension lower. The relevant bulk supergravity theory is a Euclidean version of $\mathcal{N} = 4$ $Spin(4)$ gauged supergravity in four dimensions. As well as the metric, the bosonic content of the theory contains two scalar fields and two $SU(2)$ gauge fields. Here $Spin(4) = SU(2)_+ \times SU(2)_-$ is the spin double cover of $SO(4)$, and the fermions transform in the fundamental **4** representation of this R-symmetry group. The topological twist in particular identifies the boundary value of one of these two $SU(2)$ R-symmetry gauge fields with the spin connection of the conformal boundary three-manifold (M_3, g). There is then a consistent truncation in which the other $SU(2)$ gauge field is identically zero in the bulk. Such Witten-type twists of $\mathcal{N} = 4$ gauge theories in three dimensions have been studied in [51]. In the first part of the chapter we establish that the gravitational free energy of such solutions is indeed invariant under arbitrary deformations of the boundary three-metric on (M_3, g).

In analogy with the previous chapter, we next consider the geometry of supersymmetric solutions to the bulk supergravity theory. They are characterized by a *twisted identity structure*.

We show that a supersymmetric solution to the bulk supergravity equations equivalently satisfies a certain first order differential system for this twisted identity structure. As before, studying these equations allows us to show that the bulk on-shell action is always a total derivative, and careful consideration of the degeneracy locus of the frame shows that this holds globally for smooth solutions. Stokes' theorem then leaves us with a boundary integral that vanishes in perfect analogy with the results of the previous chapter. Therefore, in the case of a boundary three-dimensional TQFT as well, the gravitational free energy of any smooth solution is zero.

At first sight, this result and its counterpart obtained in the previous chapter are somewhat disappointing: the classical free energy is zero for smooth fillings, irrespective of their topology. Zero is a topological invariant, but not a very interesting one. However, if one believes that smooth real saddle points are the dominant saddle points in gravity, this is then a robust prediction for the large N limits of various classes of topologically twisted SCFTs, in both three and four dimensions. For example, since $\mathcal{N} = 4$ gauged supergravity in four dimensions [77] is a consistent truncation of eleven-dimensional supergravity on S^7 (or $S^7/\mathbb{Z}_k$) [75], as we discuss later in the chapter this leads to a prediction for the large N limit of the partition function of the topologically twisted ABJM theory, on any three-manifold M_3. On the other hand, with the exception of the $SU(N)$ Vafa–Witten partition function on $M_4 = \text{K3}$ discussed in section 3.7, to date none of these large N limits have been computed in field theory: such computations now become very pressing! It might be that these match our supergravity results for smooth solutions, but if not then one necessarily has to consider more general saddle points, allowing e.g. for appropriate singularities and/or complex saddle points. Notice that although our computation of the classical gravitational free energy will in general break down for such solutions, the result that this quantity is independent of boundary metric deformations is *a priori* a more general result.

The outline of the chapter is as follows. First, in section 3.2 we review the topological twists of three-dimensional supersymmetric field theories, as they are perhaps less well known than their four-dimensional relatives, and discuss the gravity dual to the ABJM theory. In section 3.3 we introduce the relevant four-dimensional $\mathcal{N} = 4$ Euclidean gauged supergravity. Surprisingly the supersymmetry transformations of this theory, as formulated in [75], do not appear in the literature, and we hence first fill this gap. After holographically renormalizing the action, in section 3.4 we identify the conformal boundary Killing spinor equations which admit a topological twist as a particular solution on any oriented Riemannian three-manifold (M_3, g). The bulk spinor equations are then expanded in a Fefferman–Graham-like expansion. In section 3.5 we prove that the gravitational free energy is independent of the metric g on M_3, following similar methods to the previous chapter. In section 3.6 we show that a supersymmetric solution to the bulk equations of motion equivalently satisfies a first order differential system of equations for a twisted identity structure. Using this we prove that the gravitational free energy of any *smooth* real solution is zero. We conclude in section 3.7 with a discussion of some of the issues of topological AdS/CFT that arose in this first part of the dissertation.

3.2 3d TQFTS and topological twists

We begin in section 3.2.1 by reviewing topological twists of three-dimensional supersymmetric QFTs. In section 3.2.2 we focus on the ABJM theory, its gravity dual, and the consistent truncation of eleven-dimensional supergravity on $S^7/\mathbb{Z}_k$ to four-dimensional $\mathcal{N}=4$ gauged supergravity.

3.2.1 Twisting $\mathcal{N}=4$ theories

One perspective on the topological twist is that it involves a modification of the global symmetry group of the theory, obtained by combining the spacetime symmetries with the R-symmetry of the theory. Concretely, one looks for group products such that a supercharge would transform as a singlet under an appropriate diagonal subgroup. In three dimensions every orientable manifold is spin.[1] Therefore, the frame bundle of any orientable three-manifold may be lifted to a $Spin(3) \cong SU(2)_E$, which constitutes the (Euclidean) spacetime symmetry.

On the other hand, the R-symmetry group of a three-dimensional field theory with $\mathcal{N}$ supersymmetries is $Spin(\mathcal{N})_R$. The minimal amount of supersymmetry required for a twist on a three-manifold of generic holonomy is $\mathcal{N}=4$:[2] in the $\mathcal{N}=3$ case the supercharges

[1]This follows from the fact that in three dimensions the second Stiefel–Whitney class is the square of the first Stiefel–Whitney class, $w_2 = w_1^2$. Since a manifold is orientable if and only if $w_1 = 0$, we see that an orientable three-manifold is automatically spin.

[2]If the manifold has $U(1)$ holonomy, one may twist with only $\mathcal{N}=2$ supersymmetry, in analogy with the corresponding four-dimensional case [137, 223]. Note that this is specific to the case of the full twist, and not the case of the partial twist, see footnote 7.

transform as $(\mathbf{2},\mathbf{3})$ under $SU(2)_E \times Spin(3)_R$, and in the tensor product there is no singlet $\mathbf{2}\otimes\mathbf{3} = \mathbf{2}\oplus\mathbf{4}$. The R-symmetry group of $\mathcal{N} = 4$ theories is $Spin(4)_R = SU(2)_+ \times SU(2)_-$, and the supercharges transform as doublets under each of the two factors. The $\mathcal{N} = 4$ multiplets are vector multiplets and hypermultiplets. The vector multiplet contains the gauge connection $\mathscr{A}$, a gaugino λ and three real scalars $\boldsymbol{\phi} = (\phi_1, \phi_2, \phi_3)$, respectively transforming under $SU(2)_E \times SU(2)_+ \times SU(2)_-$ as $(\mathbf{3},\mathbf{1},\mathbf{1})$, $(\mathbf{2},\mathbf{2},\mathbf{2})$ and $(\mathbf{1},\mathbf{3},\mathbf{1})$. The hypermultiplet contains two complex scalars q and two spinors ψ, each forming R-symmetry doublets, that is, transforming as $(\mathbf{1},\mathbf{1},\mathbf{2})$ and $(\mathbf{2},\mathbf{2},\mathbf{1})$. There is an outer automorphism of the superalgebra exchanging $SU(2)_+$ and $SU(2)_-$. Under this automorphism, a vector multiplet is taken to a twisted vector multiplet, whose scalars transform as $(\mathbf{1},\mathbf{1},\mathbf{3})$, and a hypermultiplet is taken to a twisted hypermultiplet, whose scalars and spinors form doublets, respectively, of $SU(2)_+$ and $SU(2)_-$. The field components of the twisted multiplets will be denoted by a tilde.

One may twist using either $SU(2)_+$ or $SU(2)_-$, obtaining generically inequivalent TQFTs. The inequivalence of the two twists is not immediate from the supercharges: they transform as $(\mathbf{2},\mathbf{2},\mathbf{2})$ under $SU(2)_E \times SU(2)_+ \times SU(2)_-$, so taking diagonal combinations of $SU(2)_E$ with either factors of the R-symmetry group leads to $(\mathbf{1},\mathbf{2}) \oplus (\mathbf{3},\mathbf{2})$. Nevertheless, the twisted fields transform differently in the two twists, as can be seen from the scalars. For instance, consider the scalars in a hypermultiplet q: after the two twists, they would transform as $(\mathbf{1},\mathbf{2})$ under $(SU(2)_E \times SU(2)_+)_{\text{diag}} \times SU(2)_-$, or $(\mathbf{2},\mathbf{1})$ under $(SU(2)_E \times SU(2)_-)_{\text{diag}} \times SU(2)_+$. On the other hand, because of the exchange of $SU(2)_+$ and $SU(2)_-$, the scalars in the twisted hypermultiplet transform in the opposite way. The same goes for vector multiplets and twisted vector multiplets: the scalars in a vector multiplet form a triplet under $SU(2)_+$ and a singlet under $SU(2)_-$, so they distinguish between the two twists, but the opposite is true of the scalars in the twisted vector multiplet.

In a three-dimensional $\mathcal{N} = 4$ super-Yang–Mills (SYM) theory, with a vector multiplet, the two twists are inequivalent. The first twist may also be recovered by dimensionally reducing the four-dimensional $\mathcal{N} = 2$ Donaldson–Witten twist. The resulting model is sometimes referred to as super-BF or super-IG model, and the partition function reproduces the Casson invariant of the background three-manifold [220, 49, 50]; and conjecturally, via renormalization group flow, the Rozansky–Witten invariants [52, 178].[3] The second twist,

[3]More precisely, the Casson invariant arises when the gauge group $\mathscr{G} \cong SU(2)$, for three-manifolds M_3

instead, is intrisically three-dimensional (it is not known to arise from the reduction of any four-dimensional theory) and supposedly provides a mirror-symmetric description of the Casson invariant [51]. There exists a third topologically twisted three-dimensional SYM theory with two twisted scalar supercharges, which may be obtained by a partial twist of three-dimensional $\mathcal{N} = 8$ SYM, or via dimensional reduction of the half-twist of four-dimensional $\mathcal{N} = 4$ SYM. It is closely related to the Casson model, but differs from it by the matter content [111].

In three dimensions it is also possible to couple Chern–Simons theory to free hypermultiplets to obtain $\mathcal{N} = 4$ supersymmetries [106], and twist the resulting theory [138, 149]. As in the previous case, if there are only untwisted or twisted hypermultiplets in the theory the two twists are inequivalent, and usually referred to as an A-twist and B-twist, respectively. However, in a theory with both hypers and twisted hypers, the difference between the two twists amounts to the exchange between the untwisted and twisted matter. Therefore, one may consider a twist by a single factor in $Spin(4)_R$ and exchange the "quality" of the hypermultiplets, obtaining theories, called AB-models, which have both types of hypermultiplets. For concreteness, after the twist, an AB-model contains matter transforming under $(SU(2)_E \times SU(2)_+)_{\text{diag}} \times SU(2)_-$ as

$$\begin{aligned} q &: (\mathbf{1},\mathbf{1},\mathbf{2}) \longmapsto (\mathbf{1},\mathbf{2}), \\ \psi &: (\mathbf{2},\mathbf{2},\mathbf{1}) \longmapsto (\mathbf{1},\mathbf{1}) \oplus (\mathbf{3},\mathbf{1}), \\ \tilde{q} &: (\mathbf{1},\mathbf{2},\mathbf{1}) \longmapsto (\mathbf{2},\mathbf{1}), \\ \tilde{\psi} &: (\mathbf{2},\mathbf{1},\mathbf{2}) \longmapsto (\mathbf{2},\mathbf{2}). \end{aligned} \tag{3.2.1}$$

Therefore, the bosonic fields are two scalars and a spinor, whilst the fermionic fields are a scalar, a 1-form and two spinors. Chern–Simons-matter theories with $\mathcal{N} > 4$ contain an equal number of untwisted and twisted hypermultiplets, so the symmetry between the A and B twist is automatically implemented.

In the present chapter, we will be particularly interested in topological twists of the ABJM theory [4] (see [159] for twists of the BLG [23, 25, 24, 121] models).[4] Classically

with the same homology groups as S^3. It was originally defined in terms of the combinatorics of $SU(2)$-representations of $\pi_1(M_3)$. However, the Casson invariant naturally generalizes to the Lescop invariant, which is defined on any oriented three-manifold. Moreover, the TQFT Casson model suggests an extension of this invariant to any gauge group $\mathscr{G}$.

[4]The BLG models are Chern–Simons-matter theories with manifest $\mathcal{N} = 8$ supersymmetry and concretely describe two M2-branes. On the other hand, ABJM theories, in the UV, are $\mathcal{N} = 6$ $U(N)_k \times U(N)_{-k}$ Chern–

this theory has $\mathcal{N} = 6$ supersymmetry, so let us consider topological twists of $\mathcal{N} = 6$ Chern–Simons-matter theories. Here the R-symmetry group is $Spin(6)_R \cong SU(4)$, and there are two decompositions

$$\begin{aligned} &\text{(i)} \quad SU(4) \longrightarrow SU(2) \times SU(2)\,, \\ &\text{(ii)} \quad SU(4) \longrightarrow SU(2) \times SU(2) \times U(1)\,. \end{aligned} \tag{3.2.2}$$

In the first case we are viewing $SU(4) \cong Spin(6)$ as a double cover of $SO(6) \longmapsto SO(3) \times SO(3)$, the latter being the two diagonal 3×3 blocks. In the second case instead the two copies of $SU(2)$ are the two diagonal 2×2 blocks in $SU(4)$. Alternatively, projecting to $SO(6)$ the second decomposition is simply $SO(6) \longmapsto SO(4) \times SO(2)$, with the obvious $4 + 2$ block decomposition, where $SU(2) \times SU(2) \cong Spin(4)$ is the double cover of $SO(4)$, and $U(1) \cong SO(2)$. The supercharges transform in the **6** of $SU(4)$, which decompose under the above as

$$\begin{aligned} &\text{(i)} \quad \mathbf{6} \longrightarrow (\mathbf{1},\mathbf{3}) \oplus (\mathbf{3},\mathbf{1})\,, \\ &\text{(ii)} \quad \mathbf{6} \longrightarrow (\mathbf{2},\mathbf{2})_0 \oplus (\mathbf{1},\mathbf{1})_{+2} \oplus (\mathbf{1},\mathbf{1})_{-2}\,. \end{aligned} \tag{3.2.3}$$

In the first case it is clear that a twist with $SU(2)_E$ does not lead to any scalar supercharge, while for the second twist one reduces to the AB-model [149].

It is not completely clear what the observables of the topologically twisted Chern–Simons-matter theories compute. In [149] it was argued that the A-model is related via the novel Higgs mechanism [182] to the super-BF theory obtained by twisting $\mathcal{N} = 4$ SYM, and thus computes the Casson invariant of the background three-manifold. Similarly, the mathematical content of the observables of the topological models of [138] is also currently unclear.

The group-theoretic point of view on the topological twist considered above is not the only possible viewpoint. One may also describe the topological twist in the context of background rigid supersymmetry. Three-dimensional field theories with $\mathcal{N} = 2$ have been extensively studied in the context of rigid supersymmetry, both from holography [145, 131] and by coupling to supergravity [66]. However, the same cannot be said for $\mathcal{N} = 4$ theories. We will find very concretely that the topological twist corresponds to

Simons-matter theories describing N M2-branes for any N. For $k = 1, 2$, the supersymmetry is enhanced to $\mathcal{N} = 8$. For certain values of N, k there exist equivalences between the BLG, ABJM and ABJ models [3, 157, 31, 2].

identifying the boundary value of one $SU(2)$ factor of the gauged R-symmetry with the spin connection. This allows us to construct a solution to the Killing spinor equation obtained from three-dimensional $\mathcal{N} = 4$ conformal supergravity, in analogy with the standard approach.

3.2.2 The ABJM theory and its supergravity dual

The AdS/CFT correspondence has been especially influential in the context of three-dimensional field theories. In particular the $\mathrm{AdS}_4 \times S^7$ near-horizon geometry describing a stack of N M2-branes provided strong evidence for the existence of a strongly-coupled maximally supersymmetric conformal field theory with $N^{3/2}$ degrees of freedom. After initial work by Bagger–Lambert–Gustavsson [23, 25, 24, 121], the worldvolume theory of N M2-branes probing $\mathbb{C}^4/\mathbb{Z}_k$ was eventually found ten years ago by Aharony–Bergman–Jafferis–Maldacena [4].

The ABJM theory in flat spacetime $\mathbb{R}^{1,2}$ is conjectured to be holographically dual to M-theory on $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. In order to study the gravity dual of the field theory defined on different manifolds M_3 in the large N limit, one may consider a consistent truncation of eleven-dimensional supergravity on S^7, or $S^7/\mathbb{Z}_k$, to an effective four-dimensional bulk supergravity theory. Such a consistent truncation has been found in [75], where it is shown that any solution to the four-dimensional $\mathcal{N} = 4$ supergravity theory of Das–Fischler–Roček [77] uplifts to an eleven-dimensional solution. In particular this supergravity theory has a $Spin(4) \cong SU(2) \times SU(2)$ gauged R-symmetry, where the massless gauge fields arise, as usual in Kaluza–Klein reduction, from a corresponding isometry of the internal space. Specifically, the uplifting/reduction ansatz in [75] identifies the $SU(2) \times SU(2)$ isometry as acting in the $\mathbf{2}$ of each factor in $\mathbb{C}^4 \equiv \mathbb{C}^2 \times \mathbb{C}^2$, where the internal space S^7 is the unit sphere in $\mathbb{C}^4$. This description makes it clear that one may also replace the internal space by $S^7/\mathbb{Z}_k$, where the $\mathbb{Z}_k$ acts on the coordinates of $\mathbb{C}^4$ via the diagonal action $z^i \mapsto \mathrm{e}^{2\pi \mathrm{i}/k} z^i$. This manifestly commutes with the $SU(2) \times SU(2) \subset SU(4) \curvearrowright \mathbb{C}^4$ action above. There is another notable geometric symmetry, namely the $\mathbb{Z}_2$ that acts by exchanging the two copies of $\mathbb{C}^2$ in $\mathbb{C}^4$, and thus exchanges the $SU(2)$ isometries. This symmetry is then inherited by the four-dimensional $\mathcal{N} = 4$ gauged supergravity theory.

According to the holographic dictionary, symmetries of the eleven-dimensional solution correspond to symmetries of the field theory. In particular the $SU(2) \times SU(2)$ isometry of the internal space, which becomes a $Spin(4)_R$ gauged R-symmetry of the consistently truncated four-dimensional theory, corresponds to the $Spin(4)_R$ R-symmetry of the field theory dual. The $\mathbb{Z}_2$ that acts as an outer automorphism, exchanging the group factors in $Spin(4)_R \subset Spin(6)_R$, is indeed a symmetry of the $\mathcal{N} = 6$ ABJM theory, since the latter has an equal number of untwisted and twisted hypermultiplets, in $\mathcal{N} = 4$ language, and therefore its matter content is symmetric under the exchange of $SU(2)_+$ and $SU(2)_-$ [130].

In the rest of the chapter we will work entirely within the Das–Fischler–Roček four-dimensional $\mathcal{N} = 4$ gauged supergravity theory. Any solution to this theory, for a bulk asymptotically locally hyperbolic four-manifold Y_4, automatically uplifts on $S^7/\mathbb{Z}_k$ to give a gravity dual to the ABJM theory defined on the conformal boundary $M_3 = \partial Y_4$. In particular we note that the effective four-dimensional Newton constant is

$$\frac{1}{2\kappa_4^2} = \frac{k^{1/2}}{12\sqrt{2}\pi} N^{3/2} . \tag{3.2.4}$$

3.3 Holographic supergravity theory

We begin in section 3.3.1 by defining a real Euclidean section of $\mathcal{N} = 4$ gauged supergravity in four dimensions and determine the fermionic supersymmetry transformations. A Fefferman–Graham expansion of asymptotically locally hyperbolic solutions to this theory is constructed in section 3.3.2, for arbitrary conformal boundary three-manifold (M_3, g). Using this, in section 3.3.3 we holographically renormalize the action.

3.3.1 Euclidean $\mathcal{N} = 4$ gauged supergravity

As outlined so far, holographic duals to three-dimensional SCFTs with a $Spin(4) = SU(2)_+ \times SU(2)_-$ R-symmetry should be solutions of a four-dimensional $\mathcal{N} = 4$ $SU(2) \times SU(2)$ gauged supergravity. As discussed in the previous subsection, the Das–Fischler–Roček [77] theory has a supersymmetric AdS_4 vacuum and was shown in [75] to be a consistent truncation of eleven-dimensional supergravity on $S^7/\mathbb{Z}_k$.

In Lorentzian signature the bosonic sector of this $\mathcal{N} = 4$ supergravity theory comprises the metric $G_{\mu\nu}$, two real scalars ϕ, φ which together parametrize an $SL(2, \mathbb{R})$ coset, and two triplets of $SU(2)$ gauge fields $\mathcal{A}^I_\mu$, $\hat{\mathcal{A}}^I_\mu$ ($I = 1,2,3$). The associated field strengths are

$$\mathcal{F}^I \equiv \mathrm{d}\mathcal{A}^I + \tfrac{1}{2}\mathfrak{g}\,\epsilon^{IJK}\mathcal{A}^J \wedge \mathcal{A}^K\,, \qquad \hat{\mathcal{F}}^I \equiv \mathrm{d}\hat{\mathcal{A}}^I + \tfrac{1}{2}\mathfrak{g}\,\epsilon^{IJK}\hat{\mathcal{A}}^J \wedge \hat{\mathcal{A}}^K\,, \tag{3.3.1}$$

and we have taken equal gauge couplings $\mathfrak{g}$ for each of the $SU(2)$ factors in the non-simple gauge group. It is convenient to introduce the scalar field $X \equiv \mathrm{e}^{\frac{1}{2}\phi}$ and define $\tilde{X} \equiv X^{-1}q$ where $q^2 \equiv 1 + \varphi^2 X^4$. The bosonic action and equations of motion in Lorentzian signature appear in [75]. However, as we are interested in holographic duals to TQFTs defined on Riemannian three-manifolds, we require a Euclidean signature version of this theory. After a Wick rotation the action becomes

$$\begin{aligned} I \;=\; -\frac{1}{2\kappa_4^2}\int \big[& R * 1 - 2X^{-2}\mathrm{d}X \wedge *\mathrm{d}X - \tfrac{1}{2}X^4\mathrm{d}\varphi \wedge *\mathrm{d}\varphi + \mathfrak{g}^2(8 + 2X^2 + 2\tilde{X}^2) * 1 \\ & - \tfrac{1}{2}X^{-2}\big(\mathcal{F}^I \wedge *\mathcal{F}^I + \mathrm{i}\varphi X^2 \mathcal{F}^I \wedge \mathcal{F}^I\big) - \tfrac{1}{2}\tilde{X}^{-2}\big(\hat{\mathcal{F}}^I \wedge *\hat{\mathcal{F}}^I - \mathrm{i}\varphi X^2 \hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^I\big)\big]\,. \end{aligned} \tag{3.3.2}$$

Here $R = R(G)$ denotes the Ricci scalar of the metric $G_{\mu\nu}$, and $*$ is the Hodge duality operator acting on forms. The equations of motion which follow from this action are:[5]

$$\begin{aligned} E_X:\ 0 &= \mathrm{d}(X^{-1} * \mathrm{d}X) - \tfrac{1}{2}X^4\mathrm{d}\varphi \wedge *\mathrm{d}\varphi + \mathfrak{g}^2\big(X^2 - X^{-2}(1-\varphi^2X^4)\big) * 1 \\ &\quad + \tfrac{1}{4}X^{-2}\mathcal{F}^I \wedge *\mathcal{F}^I - \tfrac{1}{4}X^2(1-\varphi^2X^4)q^{-4}\hat{\mathcal{F}}^I \wedge *\hat{\mathcal{F}}^I + \tfrac{\mathrm{i}}{2}\varphi\tilde{X}^{-4}\hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^I\,, \end{aligned} \tag{3.3.3}$$

$$\begin{aligned} E_\varphi:\ 0 &= \mathrm{d}(X^4 * \mathrm{d}\varphi) + 4\mathfrak{g}^2X^2\varphi * 1 - \tfrac{\mathrm{i}}{2}\mathcal{F}^I \wedge \mathcal{F}^I \\ &\quad + \varphi X^2\tilde{X}^{-4}\hat{\mathcal{F}}^I \wedge *\hat{\mathcal{F}}^I + \tfrac{\mathrm{i}}{2}(1-\varphi^2X^4)\tilde{X}^{-4}\hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^I\,, \end{aligned} \tag{3.3.4}$$

$$E_{\mathcal{A}^I}:\ 0 = D(X^{-2} * \mathcal{F}^I) + \mathrm{i}\mathrm{d}\varphi \wedge \mathcal{F}^I\,, \tag{3.3.5}$$

$$E_{\hat{\mathcal{A}}^I}:\ 0 = \hat{D}(\tilde{X}^{-2} * \hat{\mathcal{F}}^I) - \mathrm{i}\mathrm{d}(\varphi X^2\tilde{X}^{-2}) \wedge \hat{\mathcal{F}}^I\,, \tag{3.3.6}$$

$$\begin{aligned} E_G:\ 0 &= R_{\mu\nu} + \mathfrak{g}^2G_{\mu\nu}(4 + X^2 + \tilde{X}^2) - 2X^{-2}\partial_\mu X\partial_\nu X - \tfrac{1}{2}X^4\partial_\mu\varphi\partial_\nu\varphi \\ &\quad - \tfrac{1}{2}X^{-2}\big(\mathcal{F}^I_{\mu\rho}\mathcal{F}^{I\rho}_\nu - \tfrac{1}{4}G_{\mu\nu}(\mathcal{F}^I)^2\big) - \tfrac{1}{2}\tilde{X}^{-2}\big(\hat{\mathcal{F}}^I_{\mu\rho}\hat{\mathcal{F}}^{I\rho}_\nu - \tfrac{1}{4}G_{\mu\nu}(\hat{\mathcal{F}}^I)^2\big)\,. \end{aligned} \tag{3.3.7}$$

Here $(\mathcal{F}^I)^2 \equiv \sum_{I=1}^3 \mathcal{F}^I_{\mu\nu}\mathcal{F}^{I\mu\nu}$, $(\hat{\mathcal{F}}^I)^2 \equiv \sum_{I=1}^3 \hat{\mathcal{F}}^I_{\mu\nu}\hat{\mathcal{F}}^{I\mu\nu}$ and the Bianchi identities define the $SU(2)$ covariant derivatives

$$B_{\mathcal{A}^I}:\ D\mathcal{F}^I \equiv \mathrm{d}\mathcal{F}^I + \mathfrak{g}\,\epsilon^{IJK}\mathcal{A}^J \wedge \mathcal{F}^K = 0\,, \tag{3.3.8}$$

[5]The Einstein equation (3.3.7) incorporates the potential-like term which is missing from the Lorentzian version in [75].

$$B_{\hat{\mathcal{A}}^I}: \ \hat{D}\hat{\mathcal{F}}^I \equiv \mathrm{d}\hat{\mathcal{F}}^I + \mathfrak{g}\,\epsilon^{IJK}\hat{\mathcal{A}}^J \wedge \hat{\mathcal{F}}^K = 0\,. \tag{3.3.9}$$

In general, equations (3.3.3)–(3.3.7) are complex, and solutions will likewise be complex. However, note that taking the axion φ to be purely imaginary effectively removes all factors of i. Note also that the action and equations of motion are invariant under the $\mathbb{Z}_2$ symmetry: $\mathfrak{g} \to -\mathfrak{g}$, $\mathcal{A}^I \to -\mathcal{A}^I$, $\hat{\mathcal{A}}^I \to -\hat{\mathcal{A}}^I$. There is a second $\mathbb{Z}_2$ symmetry, discussed in section 3.2.2, which corresponds to the field theory outer automorphism exchanging the group factors in $Spin(4)_R \cong SU(2)_+ \times SU(2)_-$. This second $\mathbb{Z}_2$ symmetry acts on the supergravity fields as $X \to \tilde{X}$, $\varphi X^2 \to -\varphi X^2$, $\mathcal{A}^I \to \hat{\mathcal{A}}^I$ and $\hat{\mathcal{A}}^I \to \mathcal{A}^I$. Whilst not manifest in the action and equations of motion, it can be made so upon rewriting the scalar kinetic terms in (3.3.2) as $2X\tilde{X}\mathrm{d}X \wedge *\mathrm{d}\tilde{X} - \frac{1}{2}\mathrm{d}(\varphi X^2) \wedge *\mathrm{d}(\varphi X^2)$.

In the Lorentzian theory the fermionic sector contains four gravitini, ψ^a_μ, and four dilatini, χ^a, which together with the spinor parameters ϵ^a all transform in the fundamental **4** representation of the $Spin(4)$ global R-symmetry group, which we label by $a = 1, \ldots, 4$. The supersymmetry transformations are not given in [75] and the form of the action is different to that appearing in the original literature [77]; in particular the parametrization of the scalars and their coupling to the gauge fields is different. We cannot, therefore, simply take the supersymmetry transformations given in [77]. Of course, the two actions represent the same theory but presumably in different symplectic duality frames, and possibly with different gauge fixed $SL(2,\mathbb{R})$ scalar coset representatives. Instead of translating between the different presentations in Lorentzian signature and then Wick rotating to the Euclidean, we have instead derived the conditions for preserving supersymmetry by a different method.

We started with a general ansatz for the gravitino and dilatino variations and then acted on the dilatino with the Dirac operator, adding additional field dependent multiples of the dilatino variation in order to recover a subset of the bosonic equations of motion (3.3.3)–(3.3.7). This essentially shows that the dilatino field equation (in a bosonic background) maps to some of the bosonic field equations. Computing the integrability condition on the spinor parameter, which can be rephrased in terms of the free Rarita–Schwinger equation for the gravitino, and adding further dilatino variations recovers the remaining bosonic equations of motion. Hence the fermionic field equations map to bosonic ones, i.e. the theory is supersymmetric. At the end of this analysis we find:

$$\begin{aligned}\delta\psi^a_\mu \;=\; 0 \;&= \mathcal{D}_\mu\epsilon^a - \tfrac{1}{8\sqrt{2}}\eta^I_{ab}X^{-1}\mathcal{F}^I_{\nu\lambda}\Gamma^{\nu\lambda}\Gamma_\mu\epsilon^b + \tfrac{1}{8\sqrt{2}}\bar{\eta}^I_{ab}X^{-1}\tilde{X}^{-2}\hat{\mathcal{F}}^I_{\nu\lambda}\Gamma^{\nu\lambda}\Gamma_\mu\big[1+\mathrm{i}\varphi X^2\Gamma_5\big]\epsilon^b \\ &\quad + \tfrac{\mathrm{i}}{4}X^2\partial_\mu\varphi\Gamma_5\epsilon^a - \tfrac{1}{2\sqrt{2}}\mathfrak{g}\big[(X+X^{-1}) - \mathrm{i}\varphi X\Gamma_5\big]\Gamma_\mu\epsilon^a\,, \end{aligned} \tag{3.3.10}$$

$$\begin{aligned}\delta\chi^a \;=\; 0 \;&= \tfrac{1}{8}\eta^I_{ab}X^{-1}\mathcal{F}^I_{\nu\lambda}\Gamma^{\nu\lambda}\epsilon^b + \tfrac{1}{8}\bar{\eta}^I_{ab}X^{-1}\tilde{X}^{-2}\hat{\mathcal{F}}^I_{\nu\lambda}\big[1-\mathrm{i}\varphi X^2\Gamma_5\big]\Gamma^{\nu\lambda}\epsilon^b \\ &\quad + \tfrac{1}{\sqrt{2}}\big[X^{-1}\partial_\nu X + \tfrac{\mathrm{i}}{2}X^2\partial_\nu\varphi\Gamma_5\big]\Gamma^\nu\epsilon^a + \tfrac{1}{2}\mathfrak{g}\big[(X-X^{-1}) + \mathrm{i}\varphi X\Gamma_5\big]\epsilon^a\,. \end{aligned} \tag{3.3.11}$$

Here the gauge covariant derivative acting on the supersymmetry parameter is

$$\mathcal{D}_\mu\epsilon^a \;=\; \nabla_\mu\epsilon^a - \tfrac{1}{2}\mathfrak{g}\,\eta^I_{ab}\mathcal{A}^I_\mu\epsilon^b + \tfrac{1}{2}\mathfrak{g}\,\bar{\eta}^I_{ab}\hat{\mathcal{A}}^I_\mu\epsilon^b\,, \tag{3.3.12}$$

and η^I_{ab}, $\bar{\eta}^I_{ab}$ are respectively the self-dual/anti-self-dual 't Hooft symbols of (A.2.2). In addition, Γ_μ, $\mu = 1,\ldots,4$, are generators of the Euclidean spacetime Clifford algebra, satisfying $\{\Gamma_\mu, \Gamma_\nu\} = 2G_{\mu\nu}$, and we define $\Gamma_5 \equiv -\Gamma_{1234}$. Note that the $\mathbb{Z}_2$ symmetry that reverses the signs of $\mathfrak{g}$ and the two $SU(2)$ gauge fields is also a symmetry of these supersymmetry equations, provided one combines it with $\Gamma^\mu \to -\Gamma^\mu$.

For the purpose of completeness, we note that the transformations satisfy

$$\begin{aligned}&\Gamma^\mu\mathcal{D}_\mu\delta\chi^a + \tfrac{3\mathrm{i}}{4}X^2\partial_\mu\varphi\Gamma^\mu\Gamma_5\delta\chi^a \\ &\quad= \tfrac{1}{\sqrt{2}}E_X\epsilon^a - \tfrac{\mathrm{i}}{2\sqrt{2}}X^{-2}E_\varphi\Gamma_5\epsilon^a \\ &\qquad + \tfrac{1}{8}\eta^I_{ab}X^{-1}(B_{\mathcal{A}^I})_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\epsilon^b + \tfrac{1}{8}\bar{\eta}^I_{ab}X^{-1}\tilde{X}^{-2}(B_{\hat{\mathcal{A}}^I})_{\mu\nu\lambda}\Gamma^{\mu\nu\lambda}\big[1-\mathrm{i}\varphi X^2\Gamma_5\big]\epsilon^b \\ &\qquad + \tfrac{1}{4}\eta^I_{ab}X(E_{\mathcal{A}^I})_\mu\Gamma^\mu\epsilon^b + \tfrac{1}{4}\bar{\eta}^I_{ab}X^{-1}(E_{\hat{\mathcal{A}}^I})_\mu\Gamma^\mu\big[1-\mathrm{i}\varphi X^2\Gamma_5\big]\epsilon^b\,, \end{aligned} \tag{3.3.13}$$

and

$$\begin{aligned}&\Gamma^\nu[\mathcal{D}_\mu,\mathcal{D}_\nu]\epsilon^a - \sqrt{2}X^{-1}\partial_\mu X\delta\chi^a + \tfrac{\mathrm{i}}{\sqrt{2}}X^2\partial_\mu\varphi\Gamma_5\delta\chi^a - \tfrac{1}{2}\mathfrak{g}\big[(X-X^{-1}) + \mathrm{i}\varphi X\Gamma_5\big]\Gamma_\mu\delta\chi^a \\ &\qquad + \tfrac{1}{8}\eta^I_{ab}X^{-1}\mathcal{F}^{I\nu\rho}\Gamma_{\nu\rho}\Gamma_\mu\delta\chi^b + \tfrac{1}{8}\bar{\eta}^I_{ab}X^{-1}\tilde{X}^{-2}\hat{\mathcal{F}}^{I\nu\rho}\big[1-\mathrm{i}\varphi X^2\Gamma_5\big]\Gamma_{\nu\rho}\Gamma_\mu\delta\chi^b \\ &\quad= \tfrac{1}{2}(E_G)_{\mu\nu}\Gamma^\nu\epsilon^a - \tfrac{1}{8\sqrt{2}}\eta^I_{ab}X^{-1}(B_{\mathcal{A}^I})^{\nu\rho\sigma}\Gamma_{\nu\rho\sigma}\Gamma_\mu\epsilon^b \\ &\qquad + \tfrac{1}{8\sqrt{2}}\bar{\eta}^I_{ab}X^{-1}\tilde{X}^{-2}(B_{\hat{\mathcal{A}}^I})^{\nu\rho\sigma}\Gamma_{\nu\rho\sigma}\Gamma_\mu\big[1+\mathrm{i}\varphi X^2\Gamma_5\big]\epsilon^b \\ &\qquad - \tfrac{1}{4\sqrt{2}}\eta^I_{ab}X(E_{\mathcal{A}^I})^\nu\Gamma_\nu\Gamma_\mu\epsilon^b + \tfrac{1}{4\sqrt{2}}\bar{\eta}^I_{ab}X^{-1}(E_{\hat{\mathcal{A}}^I})^\nu\Gamma_\nu\Gamma_\mu[1+\mathrm{i}\varphi X^2\Gamma_5]\epsilon^b\,. \end{aligned} \tag{3.3.14}$$

In deriving these conditions we have not needed to specify the type of spinor we are using. Later, in section 3.4, we will deal with a truncation of this theory in which one triplet of gauge fields is set to zero and the spinors are taken to be symplectic-Majorana.

3.3.2 Fefferman–Graham expansion

In this section we determine the Fefferman–Graham expansion of asymptotically locally hyperbolic solutions to this Euclidean supergravity theory. This is the general solution to the bosonic equations of motion (3.3.3)–(3.3.7), expressed as a perturbative expansion in a radial coordinate near the conformal boundary.

We take the form of the metric to be

$$G_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu \;=\; \frac{1}{z^2}\mathrm{d}z^2+\frac{1}{z^2}\mathrm{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j \;=\; \frac{1}{z^2}\mathrm{d}z^2+h_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \qquad (3.3.15)$$

The AdS radius $\ell=1$, and in turn we have the expansion

$$\mathrm{g}_{ij} \;=\; \mathrm{g}^0_{ij}+z^2\mathrm{g}^2_{ij}+z^3\mathrm{g}^3_{ij}+o(z^3)\,. \qquad (3.3.16)$$

Here $\mathrm{g}^0_{ij}=g_{ij}$ is the boundary metric induced on the conformal boundary M_3 at $z=0$. The volume form for the four-dimensional bulk metric (3.3.15) is

$$\mathrm{vol}_4 \;=\frac{1}{z^4}\mathrm{d}z\wedge\mathrm{vol}_\mathrm{g} \;=\; \frac{1}{z^4}\mathrm{d}z\wedge\sqrt{\det\mathrm{g}}\,\mathrm{d}x^1\wedge\mathrm{d}x^2\wedge\mathrm{d}x^3\,. \qquad (3.3.17)$$

The determinant may then be expanded in a series in z, around that for g^0, as follows

$$\sqrt{\det\mathrm{g}} \;=\; \sqrt{\det\mathrm{g}^0}\left[1+\tfrac{z^2}{2}t^{(2)}+\tfrac{z^3}{2}t^{(3)}\right]+o(z^3)\,. \qquad (3.3.18)$$

Here we have denoted $t^{(n)}\equiv\mathrm{Tr}\left[(\mathrm{g}^0)^{-1}\mathrm{g}^n\right]$ and indices are always raised with g^0.

The remaining bosonic fields are likewise expanded as follows:

$$X \;=1+zX_1+z^2X_2+z^3X_3+o(z^3)\,, \qquad (3.3.19)$$

$$\varphi \;=z\varphi_1+z^2\varphi_2+z^3\varphi_3+o(z^3)\,, \qquad (3.3.20)$$

$$\mathcal{A}^I \;=A^I+z\mathrm{a}^I_1+z^2\mathrm{a}^I_2+o(z^2)\,, \qquad (3.3.21)$$

$$\hat{\mathcal{A}}^I \;=\hat{A}^I+z\hat{\mathrm{a}}^I_1+z^2\hat{\mathrm{a}}^I_2+o(z^2)\,. \qquad (3.3.22)$$

We have chosen a gauge in which all dz terms in the gauge field expansions are set to zero.

We now substitute the above expansions into the equations of motion (3.3.3)–(3.3.7) and solve them order by order in the radial coordinate z in terms of the boundary data $\mathrm{g}^0 = g, X_1, \varphi_1, A^I$ and $\hat{A}^I$. For the Einstein equation (3.3.7) we will need the Ricci tensor of the metric (3.3.15):

$$R_{zz} = -\frac{3}{z^2} - \frac{1}{2}\Big(\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z^2\mathrm{g}\right] - \tfrac{1}{z}\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right] - \tfrac{1}{2}\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right]^2\Big), \tag{3.3.23}$$

$$\begin{aligned} R_{ij} = &-\frac{3}{z^2}\mathrm{g}_{ij} - \Big(\tfrac{1}{2}\partial_z^2\mathrm{g} - \tfrac{1}{z}\partial_z\mathrm{g} - \tfrac{1}{2}(\partial_z\mathrm{g})\mathrm{g}^{-1}(\partial_z\mathrm{g}) + \tfrac{1}{4}(\partial_z\mathrm{g})\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right] \\ &- R(\mathrm{g}) - \tfrac{1}{2z}\mathrm{g}\mathrm{Tr}\left[\mathrm{g}^{-1}\partial_z\mathrm{g}\right]\Big)_{ij}, \end{aligned} \tag{3.3.24}$$

$$R_{zi} = -\frac{1}{2}(\mathrm{g}^{-1})^{jk}\Big(\nabla_i \mathrm{g}_{jk,z} - \nabla_k \mathrm{g}_{ij,z}\Big), \tag{3.3.25}$$

where ∇ is the covariant derivative for g.

Examining first the axion equation (3.3.4) gives at the first two orders

$$0 = (1-2\mathfrak{g}^2)\varphi_1, \qquad 0 = (1-2\mathfrak{g}^2)(2X_1\varphi_1 + \varphi_2), \tag{3.3.26}$$

which can be solved by setting $\mathfrak{g} = \pm\frac{1}{\sqrt{2}}$. These equations fix the gauging coupling in terms of the AdS_4 length scale, which we have set to unity. At even higher order we find

$$\nabla^2\varphi_1 = 2\mathfrak{g}^2\Big(\varphi_1(t^{(2)} + 2X_1^2 + 4X_2) + 4X_1\varphi_2 + 2\varphi_3\Big). \tag{3.3.27}$$

Moving on to the dilaton equation (3.3.3) we find

$$0 = (1-2\mathfrak{g}^2)X_1, \qquad 0 = (1-2\mathfrak{g}^2)(X_2 - \tfrac{1}{2}X_1^2 + \tfrac{1}{4}\varphi_1^2), \tag{3.3.28}$$

which are again solved by $\mathfrak{g} = \pm\frac{1}{\sqrt{2}}$ together with

$$\nabla^2 X_1 = 2\mathfrak{g}^2\Big(2X_3 + X_1(t^{(2)} + 2X_1^2 - 2X_2 + \varphi_1^2) + \varphi_1\varphi_2\Big) - 2\varphi_1(X_1\varphi_1 + \varphi_2). \tag{3.3.29}$$

Next the $\mathcal{A}^I$ gauge field equation (3.3.5) yields

$$0 = D *_{\mathrm{g}^0} \mathrm{a}_1^I, \qquad \mathrm{a}_2^I = X_1\mathrm{a}_1^I + \tfrac{1}{2} *_{\mathrm{g}^0} D *_{\mathrm{g}^0} F^I - \tfrac{\mathrm{i}}{2}\varphi_1 *_{\mathrm{g}^0} F^I, \tag{3.3.30}$$

where the curvature is $F^I \equiv \mathrm{d}A^I + \frac{1}{2}\mathfrak{g}\,\epsilon^{IJK}A^J \wedge A^K$. Notice that a_1^I, and hence a_2^I, is partially undetermined. Similarly, the other gauge field equation (3.3.6) gives

$$0 \;=\; \hat{D} *_{g^0} \hat{a}^I_1 \,, \qquad \hat{a}^I_2 \;=\; -X_1 \hat{a}^I_1 + \tfrac{1}{2} *_{g^0} \hat{D} *_{g^0} \hat{F}^I + \tfrac{i}{2}\varphi_1 *_{g^0} \hat{F}^I \,, \tag{3.3.31}$$

with $\hat{F}^I \equiv \mathrm{d}\hat{A}^I + \frac{1}{2}\mathfrak{g}\,\epsilon^{IJK}\hat{A}^J \wedge \hat{A}^K$.

The non-trivial information from the ij component of the Einstein equation (3.3.7), using (3.3.24), is

$$g^2_{ij} \;=\; -\left[R_{ij}(g^0) - \tfrac{1}{4} g^0_{ij} R(g^0)\right] - g^0_{ij}\left(\tfrac{1}{2}X_1^2 + \tfrac{1}{8}\varphi_1^2\right) , \tag{3.3.32}$$

which is again a matter-modified version of the boundary Schouten tensor. From this expression we immediately deduce that the trace of g^2_{ij} is

$$t^{(2)} \;=\; -\tfrac{1}{4}R(g^0) - \tfrac{3}{2}X_1^2 - \tfrac{3}{8}\varphi_1^2 \,. \tag{3.3.33}$$

The zz component of the Einstein equation in (3.3.7), together with (3.3.23), determines the trace of the highest order component in the expansion of the bulk metric:

$$t^{(3)} \;=\; \tfrac{4}{3}X_1^3 - \tfrac{2}{3}X_1(4X_2 + \varphi_1^2) - \tfrac{2}{3}\varphi_1\varphi_2 \,. \tag{3.3.34}$$

3.3.3 Holographic renormalization

Having solved the bulk equations of motion to the relevant order, we are now in a position to holographically renormalize the Euclidean $\mathcal{N} = 4$ gauged supergravity theory. The bulk action (3.3.2) is divergent for an asymptotically locally hyperbolic solution, but can be rendered finite by the addition of appropriate local counterterms. We begin by taking the trace of the Einstein equation (3.3.7). Substituting the result into the Euclidean action (3.3.2) with $\mathfrak{g} = \pm\frac{1}{\sqrt{2}}$, we arrive at the bulk on-shell action

$$\begin{aligned} I_{\text{o-s}} \;=\; \frac{1}{2\kappa_4^2}\int_{Y_4} \Big[& -(4 + X^2 + \tilde{X}^2) * 1 - \tfrac{1}{2}X^{-2}\big(\mathcal{F}^I \wedge *\mathcal{F}^I + i\varphi X^2 \mathcal{F}^I \wedge \mathcal{F}^I\big) \\ & - \tfrac{1}{2}\tilde{X}^{-2}\big(\hat{\mathcal{F}}^I \wedge *\hat{\mathcal{F}}^I - i\varphi X^2 \hat{\mathcal{F}}^I \wedge \hat{\mathcal{F}}^I\big)\Big] \,. \end{aligned} \tag{3.3.35}$$

Here Y_4 is the bulk four-manifold, with boundary $\partial Y_4 = M_3$. In order to obtain the equations of motion (3.3.3)–(3.3.7) from the original bulk action (3.3.2) on a manifold with boundary, one has to add the Gibbons–Hawking–York term

$$I_{\text{GHY}} = -\frac{1}{\kappa_4^2}\int_{\partial Y_4} \mathrm{d}^3x\,\sqrt{\det h}\,K = \frac{1}{\kappa_4^2}\int_{\partial Y_4} \mathrm{d}^3x\, z\partial_z\sqrt{\det h}\,. \tag{3.3.36}$$

More precisely one cuts Y_4 off at some finite radial distance, or equivalently non-zero $z > 0$, and (M_3, h) is the resulting three-manifold boundary, with trace of the second fundamental form being K. Recall from (3.3.15) that $h_{ij} = \frac{1}{z^2}g_{ij}$.

The combined action $I_{\text{o-s}} + I_{\text{GHY}}$ suffers from divergences as the conformal boundary is approached, which are removed by the standard method of holographic renormalization. As before, we introduce a small cut-off $z = \delta > 0$, and expand all fields via the Fefferman–Graham expansion of section 3.3.2 to identify the divergences. These may be cancelled by adding local boundary counterterms:

$$I_{\text{ct}} = \frac{1}{\kappa_4^2}\int_{\partial Y_4} \mathrm{d}^3x\,\sqrt{\det h}\left[2 + \tfrac{1}{2}R(h) + (X-1)^2 + \tfrac{1}{4}\varphi^2\right]. \tag{3.3.37}$$

As is standard, we have written the counterterm action (3.3.37) covariantly in terms of the induced metric h_{ij} on $M_3 = \partial Y_4$. The total renormalized finite action is then

$$S = \lim_{\delta\to 0}\left(I_{\text{o-s}} + I_{\text{GHY}} + I_{\text{ct}}\right). \tag{3.3.38}$$

The choice of counterterms (3.3.37) defines a particular renormalization scheme. For this theory there are other local, gauge invariant counterterms that one can construct from the boundary fields, that have non-zero (and finite) limits as $\delta \to 0$. It is straightforward to check that there are no such finite counterterms constructed without using the scalar fields; but including the latter we may write down finite counterterms proportional to the boundary integrals of φ^3, $(X-1)^3$, $\varphi R(h)$, *etc.* There are also local but non-gauge invariant terms that one might consider. For example, boundary Chern–Simons terms for the $SU(2)$ gauge fields, and the boundary gravitational Chern–Simons term. However, such terms would change the gauge invariance of the theory, and we shall hence not consider them further.[6] In principle we should use a *supersymmetric* holographic renormalization scheme, but in

[6]The topological twist will later identify one boundary $SU(2)$ gauge field with the boundary spin connection of (M_3, g), so that these Chern–Simons terms are the same. Moreover, since any oriented three-manifold is parallelizable there is always a globally defined frame. Choosing such a frame then allows one to interpret the gravitational Chern–Simons term as a global 3-form on M_3. However, its integral depends on the choice of framing.

the absence of a prescription for this we shall use the minimal scheme with counterterms (3.3.37) in the remainder of the chapter, cf. the discussion in [39, 38, 191, 12]. In any case, for the topological twist boundary condition the boundary values φ_1, X_1 of φ and X will be zero, and the above-mentioned finite gauge invariant counterterms are all zero.

Given the renormalized action we may compute the following vacuum expectation values (VEVs):

$$\begin{aligned} \langle T_{ij}\rangle &= \frac{2}{\sqrt{g}}\frac{\delta \mathbb{S}}{\delta g^{ij}}\,, \qquad \langle \Xi\rangle = \frac{1}{\sqrt{g}}\frac{\delta \mathbb{S}}{\delta X_1}\,, \qquad \langle \Sigma\rangle = \frac{1}{\sqrt{g}}\frac{\delta \mathbb{S}}{\delta \varphi_1}\,, \\ \langle \mathscr{J}^i_I\rangle &= \frac{1}{\sqrt{g}}\frac{\delta \mathbb{S}}{\delta A^I_i}\,, \qquad \langle \hat{\mathscr{J}}^i_I\rangle = \frac{1}{\sqrt{g}}\frac{\delta \mathbb{S}}{\delta \hat{A}^I_i}\,. \end{aligned} \tag{3.3.39}$$

Here, as usual in AdS/CFT, the boundary fields g_{ij}, X_1, φ_1, A^I_i and $\hat{A}^I_i$ act as sources for operators, and the expressions in (3.3.39) compute the VEVs of these operators. Using the above holographic renormalization we may write (3.3.39) as the following limits:

$$\begin{aligned} \langle T_{ij}\rangle &= \frac{1}{\kappa_4^2}\lim_{\delta\to 0}\frac{1}{\delta}\Big[-K_{ij}+Kh_{ij}+R_{ij}(h)-\tfrac{1}{2}h_{ij}R(h)+h_{ij}(-2-(X-1)^2-\tfrac{1}{4}\varphi^2)\Big]\,, \\ \langle \Xi\rangle &= \frac{1}{\kappa_4^2}\lim_{\delta\to 0}\frac{1}{\delta^2}\Big[-2\delta X^{-2}\partial_\delta X+2(X-1)\Big]\,, \\ \langle \Sigma\rangle &= \frac{1}{\kappa_4^2}\lim_{\delta\to 0}\frac{1}{\delta^2}\Big[-\tfrac{1}{2}\delta X^4\partial_\delta\varphi+\tfrac{1}{2}\varphi\Big]\,, \\ \langle \mathscr{J}^{Ii}\rangle &= \frac{1}{2\kappa_4^2}\lim_{\delta\to 0}\frac{1}{\delta^3}\Big[-*_h\Big(\mathrm{d}x^i\wedge\big(X^{-2}*_4\mathcal{F}^I+\mathrm{i}\varphi\mathcal{F}^I\big)\Big)\Big]\,, \\ \langle \hat{\mathscr{J}}^{Ii}\rangle &= \frac{1}{2\kappa_4^2}\lim_{\delta\to 0}\frac{1}{\delta^3}\Big[-*_h\Big(\mathrm{d}x^i\wedge\big(\tilde{X}^{-2}*_4\hat{\mathcal{F}}^I-\mathrm{i}\varphi X^2\tilde{X}^{-2}\hat{\mathcal{F}}^I\big)\Big]\,. \end{aligned} \tag{3.3.40}$$

Here K_{ij} is the second fundamental form of the cut-off hypersurface (M_3, h_{ij}), and $*_h$ denotes the Hodge duality operator for the metric h_{ij}. A computation then gives the finite expressions

$$\langle T_{ij}\rangle = \frac{1}{\kappa_4^2}\Big[\tfrac{3}{2}g^3_{ij}-\tfrac{1}{2}g^0_{ij}\big(3t^{(3)}+4X_1X_2+\varphi_1\varphi_2\big)\Big]\,, \tag{3.3.41}$$

$$\langle \Xi\rangle = \frac{1}{\kappa_4^2}\big(4X_1^2-2X_2\big)\,, \tag{3.3.42}$$

$$\langle \Sigma\rangle = -\frac{1}{\kappa_4^2}\big(2X_1\varphi_1+\tfrac{1}{2}\varphi_2\big)\,, \tag{3.3.43}$$

$$\langle \mathscr{J}_i^I \rangle \ = \ -\frac{1}{2\kappa_4^2}(\mathrm{a}_1^I)_i \, , \tag{3.3.44}$$

$$\langle \hat{\mathscr{J}}_i^I \rangle \ = \ -\frac{1}{2\kappa_4^2}(\hat{\mathrm{a}}_1^I)_i \, . \tag{3.3.45}$$

Notice that each of these expressions contains terms that are not determined, in terms of boundary data, by the Fefferman–Graham expansion of the bosonic equations of motion. In particular the g^3_{ij} term in the stress-energy tensor T_{ij}, the scalars X_2, φ_2 that determine respectively Ξ, Σ, and a_1^I, $\hat{\mathrm{a}}_1^I$ appearing in the $SU(2)_R$ currents.

As a quick check/application of these formulae, consider a boundary Weyl transformation $\delta\sigma$ under which $\delta g^{ij} = 2g^{ij}\delta\sigma$, the scalars X_1, φ_1 have Weyl weight 1: $\delta X_1 = X_1\delta\sigma$, $\delta\varphi_1 = \varphi_1\delta\sigma$ and the gauge fields Weyl weight 0. Then it is a simple exercise to show that

$$\delta_\sigma S \ = \ \int_{\partial Y_4} \mathrm{vol}_g \left[\tfrac{1}{2} T_{ij}\delta g^{ij} + \Xi\delta X_1 + \Sigma\delta\varphi_1 + \mathscr{J}_i^I \delta A^{Ii} + \hat{\mathscr{J}}_i^I \delta \hat{A}^{Ii} \right] \ = \ 0 \, , \tag{3.3.46}$$

which is consistent with the fact that there is no conformal anomaly in three-dimensional SCFTs.

3.4 Supersymmetric solutions

In this section we study supersymmetric solutions to the Euclidean $\mathcal{N} = 4$ supergravity theory. We begin in section 3.4.1 by deriving the Killing spinor equations on the conformal boundary from the bulk supersymmetry equations, and then compare them to the component form equations of off-shell three-dimensional $\mathcal{N} = 4$ conformal supergravity. In section 3.4.2 we describe how the topological twist arises as a special solution to these Killing spinor equations, that exists on any Riemannian three-manifold (M_3, g). Finally, in section 3.4.3 we expand solutions to the bulk spinor equations in a Fefferman–Graham-like expansion.

3.4.1 Boundary spinor equations

We begin by introducing the charge conjugation matrix $\mathscr{C}$ for the Euclidean spacetime Clifford algebra. By definition $\Gamma_\mu^* = \mathscr{C}^{-1}\Gamma_\mu\mathscr{C}$, and one may choose Hermitian generators $\Gamma_\mu^\dagger = \Gamma_\mu$ together with the conditions $\mathscr{C} = \mathscr{C}^* = -\mathscr{C}^T$, $\mathscr{C}^2 = -1$. We may then define spinors in Euclidean signature to satisfy the symplectic-Majorana condition

$$\epsilon^a \;\equiv\; \Omega^a{}_b \mathscr{C}(\epsilon^b)^* \,, \tag{3.4.1}$$

with $\Omega = \sigma_3 \otimes \mathrm{i}\sigma_2$. It is straightforward to check that when $\hat{\mathcal{A}}^I = 0$, and provided the axion φ is purely imaginary with all other bosonic fields being real, the supersymmetry variations (3.3.10), (3.3.11) are compatible with this symplectic-Majorana condition. We will be interested in solutions that satisfy these reality conditions, and henceforth work in the truncation of the bulk supergravity theory for which the triplet of $SU(2)$ gauge fields $\hat{\mathcal{A}}^I_\mu$ is set to zero. For completeness we record here the truncated bulk supersymmetry conditions:

$$\begin{aligned} 0 \;=&\; \nabla_\mu \epsilon^a - \tfrac{1}{2}\mathfrak{g}\eta^I_{ab}\mathcal{A}^I_\mu \epsilon^b - \tfrac{1}{8\sqrt{2}}\eta^I_{ab} X^{-1}\mathcal{F}^I_{\nu\lambda}\Gamma^{\nu\lambda}\Gamma_\mu \epsilon^b + \tfrac{\mathrm{i}}{4}X^2\partial_\mu\varphi\Gamma_5\epsilon^a \\ &- \tfrac{1}{2\sqrt{2}}\mathfrak{g}\big[(X+X^{-1}) - \mathrm{i}\varphi X\Gamma_5\big]\Gamma_\mu\epsilon^a \,, \end{aligned} \tag{3.4.2}$$

$$\begin{aligned} 0 \;=&\; \tfrac{1}{8}\eta^I_{ab}X^{-1}\mathcal{F}^I_{\nu\lambda}\Gamma^{\nu\lambda}\epsilon^b + \tfrac{1}{\sqrt{2}}\big[X^{-1}\partial_\nu X + \tfrac{\mathrm{i}}{2}X^2\partial_\nu\varphi\Gamma_5\big]\Gamma^\nu\epsilon^a \\ &+ \tfrac{1}{2}\mathfrak{g}\big[(X-X^{-1}) + \mathrm{i}\varphi X\Gamma_5\big]\epsilon^a \,. \end{aligned} \tag{3.4.3}$$

We next expand the bulk Killing spinor equations (3.4.2), (3.4.3) to leading order near the conformal boundary at $z = 0$. We will consequently need the Fefferman–Graham expansion of an orthonormal frame for the metric (3.3.15), (3.3.16), together with the associated spin connection. The following is a choice of frame $\mathrm{E}^{\overline{\mu}}_\mu$ for the metric (3.3.15):

$$\mathrm{E}^{\overline{z}}_z \;=\; \frac{1}{z}\,, \qquad \mathrm{E}^{\overline{z}}_i \;=\; \mathrm{E}^{\overline{i}}_z \;=\; 0\,, \qquad \mathrm{E}^{\overline{i}}_i \;=\; \frac{1}{z}\mathsf{e}^{\overline{i}}_i \,, \tag{3.4.4}$$

where $\mathsf{e}^{\overline{i}}_i$ is a frame for the z-dependent metric g. The latter then has the expansion (3.3.16), but for the present subsection we shall only need that

$$\mathsf{e}^{\overline{i}}_i \;=\; \mathrm{e}^{\overline{i}}_i + O(z^2)\,, \tag{3.4.5}$$

where $\mathrm{e}^{\overline{i}}_i$ is a frame for the boundary metric $\mathsf{g}^0 = g$. The non-zero components of the spin connection $\Omega_\mu{}^{\overline{\nu\rho}}$ at this order are correspondingly

$$\Omega_i{}^{\overline{zj}} \;=\; \frac{1}{z}\mathrm{e}^{\overline{j}}_i + O(z)\,, \qquad \Omega_i{}^{\overline{jk}} \;=\; \omega_i{}^{\overline{jk}} + O(z^2)\,, \tag{3.4.6}$$

where $\omega_i^{\overline{jk}}$ denotes the boundary spin connection.

We take as the generators of the Clifford algebra the following

$$\Gamma_{\bar{1}} \equiv \Gamma_{\bar{z}} = \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & -\mathbb{1}_2 \end{pmatrix}, \quad \Gamma_{\overline{1+i}} = \begin{pmatrix} 0 & \sigma_{\bar{i}} \\ \sigma_{\bar{i}} & 0 \end{pmatrix}, \tag{3.4.7}$$

so that

$$\Gamma_5 = \begin{pmatrix} 0 & -\mathrm{i}\mathbb{1}_2 \\ \mathrm{i}\mathbb{1}_2 & 0 \end{pmatrix}, \tag{3.4.8}$$

and

$$\mathscr{C} = \begin{pmatrix} \mathrm{i}\sigma_2 & 0 \\ 0 & -\mathrm{i}\sigma_2 \end{pmatrix}, \tag{3.4.9}$$

where $\sigma_{\bar{i}}$ the usual Pauli matrices. The bulk Killing spinor is then expanded as

$$\epsilon^a = z^{-1/2}\varepsilon^a + z^{1/2}\zeta^a + o(z^{1/2}). \tag{3.4.10}$$

From the z-component of the gravitino equation (3.4.2) one then finds

$$0 = -z^{-1/2}\tfrac{1}{2}(\mathbb{1} \pm \Gamma_{\bar{z}})\varepsilon^a + z^{1/2}\left[\tfrac{1}{2}(\mathbb{1} \mp \Gamma_{\bar{z}})\zeta^a + \tfrac{\mathrm{i}}{4}\varphi_1\Gamma_5(\mathbb{1} \pm \Gamma_{\bar{z}})\varepsilon^a\right] + o(z^{1/2}), \tag{3.4.11}$$

with the upper/lower signs corresponding to taking $\mathfrak{g} = \pm\frac{1}{\sqrt{2}}$. We can then satisfy this equation by taking ε^a to have a definite chirality under $\Gamma_{\bar{z}}$ and ζ^a to have the opposite chirality. Recall that there is a $\mathbb{Z}_2$ symmetry of the action, equations of motion, and supersymmetry equations, that sends $\mathfrak{g} \to -\mathfrak{g}$, $\mathcal{A}^I \to -\mathcal{A}^I$, $\Gamma^\mu \to -\Gamma^\mu$. Using this, without loss of generality we set $\mathfrak{g} = -\frac{1}{\sqrt{2}}$ from now on, so that ε^a has positive $\Gamma_{\bar{z}}$ chirality and ζ^a negative chirality, and we write them as

$$\varepsilon^a = \begin{pmatrix} \varepsilon^a_L \\ 0 \end{pmatrix}, \qquad \zeta^a = \begin{pmatrix} 0 \\ \zeta^a_R \end{pmatrix}. \tag{3.4.12}$$

The leading order term in the i-component of the gravitino equation is then seen to be identically satisfied. The next order gives the boundary Killing spinor equation (KSE):

$$0 \;=\; \nabla_i^A \varepsilon_L^a + \sigma_i \zeta_R^a - \tfrac{1}{4}\varphi_1 \sigma_i \varepsilon_L^a\,. \tag{3.4.13}$$

Here $\nabla_i^A \varepsilon_L^a = \nabla_i \varepsilon_L^a + \frac{1}{2\sqrt{2}}\eta^I_{ab} A^I_i \varepsilon^b_L$, where the covariant derivative is with respect to the Levi-Civita spin connection of the boundary metric $\mathrm{g}^0_{ij} = g_{ij}$, and $\sigma_i = \sigma_{\bar{i}}\, \mathrm{e}^{\bar{i}}_i$, so that $\{\sigma_i, \sigma_j\} = 2g_{ij}$. Note that after redefining the conformal spinor parameter such that $\tilde{\zeta}^a_R = \zeta^a_R - \frac{1}{4}\varphi_1 \varepsilon^a_L$, the boundary KSE becomes

$$0 \;=\; \nabla_i^A \varepsilon_L^a + \sigma_i \tilde{\zeta}_R^a\,. \tag{3.4.14}$$

This is the equation which results from setting to zero the gravitino supersymmetry variation of off-shell 3d $\mathcal{N} = 4$ conformal supergravity [29].

Turning to the bulk dilatino equation (3.4.3), the leading order term is equivalent to the chirality property of ε^a. At the next order we obtain two conditions, corresponding to the left and right-handed components

$$0 \;=\; -\tfrac{1}{\sqrt{2}}\varphi_1 \zeta^a_R - \tfrac{1}{2\sqrt{2}}(X_1^2 - 2X_2)\varepsilon^a_L + \tfrac{1}{2\sqrt{2}}\partial_i \varphi_1 \sigma^i \varepsilon^a_L + \tfrac{1}{8}\eta^I_{ab} F^I_{ij}\sigma^{ij}\varepsilon^b_L\,, \tag{3.4.15}$$

$$0 \;=\; \sqrt{2}X_1 \zeta^a_R + \tfrac{1}{2\sqrt{2}}(X_1\varphi_1 + \varphi_2)\varepsilon^a_L - \tfrac{1}{\sqrt{2}}\partial_i X_1 \sigma^i \varepsilon^a_L + \tfrac{1}{4}\eta^I_{ab}(\mathrm{a}^I_1)_i \sigma^i \varepsilon^b_L\,. \tag{3.4.16}$$

After the redefinition of the conformal spinor parameter and Hodge dualising one term these read

$$0 \;=\; -\tfrac{1}{\sqrt{2}}\varphi_1 \tilde{\zeta}^a_R - \tfrac{1}{2\sqrt{2}}\big(\tfrac{1}{2}\varphi_1^2 + X_1^2 - 2X_2\big)\varepsilon^a_L + \tfrac{1}{2\sqrt{2}}\partial_i \varphi_1 \sigma^i \varepsilon^a_L + \tfrac{1}{8}\eta^I_{ab} F^I_{ij}\sigma^{ij}\varepsilon^b_L\,, \tag{3.4.17}$$

$$0 \;=\; \sqrt{2}X_1 \tilde{\zeta}^a_R + \tfrac{1}{\sqrt{2}}\big(X_1\varphi_1 + \tfrac{1}{2}\varphi_2\big)\varepsilon^a_L - \tfrac{1}{\sqrt{2}}\partial_i X_1 \sigma^i \varepsilon^a_L - \tfrac{\mathrm{i}}{8}\eta^I_{ab}(*\mathrm{a}^I_1)_{ij}\sigma^{ij}\varepsilon^b_L\,. \tag{3.4.18}$$

These equations are not equivalent, and matching them to the single algebraic condition arising from setting a three-dimensional dilatino variation to zero is not therefore entirely straightforward. The Weyl multiplet of off-shell $\mathcal{N} = 4$ conformal supergravity contains two auxiliary scalar fields S_1, S_2 of Weyl weight 1 and 2 respectively, and generically six gauge fields. The vanishing of the dilatino supersymmetry transformation [29] when one triplet of gauge fields is turned off is, schematically,

$$0 = S_1 \tilde{\xi}^a + S_2 \varepsilon^a + \partial_i S_1 \sigma^i \varepsilon^a + \eta^I_{ab} F^I_{ij} \sigma^{ij} \varepsilon^b \,. \tag{3.4.19}$$

Clearly (3.4.17) is of this form once we identify $S_1 \sim \varphi_1$, $S_2 \sim \frac{1}{2}\varphi_1^2 + X_1^2 - 2X_2$. However, (3.4.18) does not match so neatly as $*\mathrm{a}_1^I$ is not a field strength. Moreover, our spinor expansion should recover a single equation, and so it is perhaps some linear combination of (3.4.17) and (3.4.18) that reproduces (3.4.19). In any case, it is not clear that the leading order dilatino equation should match this particular off-shell formulation of $\mathcal{N} = 4$ conformal supergravity.

3.4.2 Topological twist

Recall that the boundary Killing spinor equation (3.4.13) written in full is

$$0 = \partial_i \varepsilon^a_L + \tfrac{1}{4}\omega_i{}^{\overline{jk}} \sigma_{\overline{jk}} \varepsilon^a_L + \tfrac{1}{2\sqrt{2}} \eta^I_{ab} A^I_i \varepsilon^b_L + \sigma_i \xi^a_R - \tfrac{1}{4}\varphi_1 \sigma_i \varepsilon^a_L \,. \tag{3.4.20}$$

To solve this equation with a topological twist, we begin by setting the boundary scalar φ_1 and conformal spinor parameter ξ^a_R to zero. We then identify the boundary $SU(2)$ gauge field with the spin connection as follows

$$A^I_i = \tfrac{1}{\sqrt{2}} \epsilon^I{}_{\overline{jk}} \omega_i{}^{\overline{jk}} \,. \tag{3.4.21}$$

The constant spinor which solves the Killing spinor equation is then

$$\varepsilon^a_L = \mathrm{i}\sigma^a \begin{pmatrix} w \\ \mathrm{i}\bar{w} \end{pmatrix} , \tag{3.4.22}$$

where w is any complex number and

$$(\sigma^a) = (\sigma^1, \sigma^2, \sigma^3, -\mathrm{i}\mathbb{1}_2) \,. \tag{3.4.23}$$

It is useful to note that the 't Hooft symbol action on ε_L^a may be exchanged for the Pauli matrix action:

$$\eta^I_{ab}\varepsilon_L^b \;=\; -\,\mathrm{i}\sigma^I\varepsilon_L^a\,. \tag{3.4.24}$$

We have solved the leading order KSE. Turning to the algebraic spinor equations we note that, in general, the conformal spinor parameter ξ_R^a can be solved for by taking the σ^i trace of the KSE (3.4.13). Substituting this generic expression for ξ_R^a into (3.4.15) and rescaling by $\sqrt{2}$ leads to

$$0 \;=\; -\,\varphi_1\nabla\!\!\!\!/^A\varepsilon_L^a+\tfrac{1}{2}[\nabla\!\!\!\!/^A,\nabla\!\!\!\!/^A]\varepsilon_L^a+\tfrac{1}{2}\partial_i\varphi_1\sigma^i\varepsilon_L^a+\tfrac{1}{4}(3\varphi_1^2-2X_1^2+4X_2+R)\varepsilon_L^a\,, \tag{3.4.25}$$

with $R=R(g)$ the boundary Ricci scalar. Specialising to the field configuration which solves the boundary KSE above, this simplifies to

$$0 \;=\tfrac{1}{4}(-2X_1^2+4X_2+R)\varepsilon_L^a\,, \tag{3.4.26}$$

and therefore fixes

$$X_2 \;=\; \tfrac{1}{4}(2X_1^2-R)\,. \tag{3.4.27}$$

The other algebraic relation (3.4.16) now reads

$$0 \;=\tfrac{1}{2\sqrt{2}}\varphi_2\varepsilon_L^a-\tfrac{1}{\sqrt{2}}\partial_iX_1\sigma^i\varepsilon_L^a+\tfrac{1}{4}\eta^I_{ab}(\mathrm{a}_1^I)_i\sigma^i\varepsilon_L^b\,. \tag{3.4.28}$$

Here recall that a_1^I is (proportional to) the VEV of the remaining $SU(2)_R$ current. One can use (3.4.24) to swap the 't Hooft symbol for a Pauli matrix, plus the usual relation

$$\sigma_{\bar{i}}\sigma_{\bar{j}} \;=\; \delta_{\overline{ij}}+\mathrm{i}\epsilon_{\overline{ijk}}\sigma_{\bar{k}}\,. \tag{3.4.29}$$

The resulting equation takes the algebraic form

$$c_b\sigma^b\varepsilon_L^a \;=0\,, \tag{3.4.30}$$

where (σ^b) are the extended Pauli matrices (3.4.23), and the coefficients c_b are *real*. In particular here we use that φ_2 is purely imaginary. Using the solution (3.4.22), one can easily check that as long as $w \neq 0$ equation (3.4.30) implies that $c_a = 0$ for all $a = 1, 2, 3, 4$. We thus conclude the equations

$$\varphi_2 = \frac{\mathrm{i}}{\sqrt{2}}(\mathrm{a}_1^I)_{\bar{i}}\delta_I^{\bar{i}}, \qquad \partial_{\bar{i}}X_1 = \frac{1}{2\sqrt{2}}\epsilon_{\bar{i}\bar{j}I}(\mathrm{a}_1^I)^{\bar{j}}. \tag{3.4.31}$$

Note here the trace over frame indices and $SU(2)_R$ indices in the expression for φ_2: this makes sense globally, since the topological twist identifies the gauge bundle with the spin bundle. Having identified indices we may view $(\mathrm{a}_1^I)^{\bar{i}}$ as a two-tensor.

3.4.3 Supersymmetric expansion

In this section we continue to expand the bulk spinor equations to higher order in z. From this we extract further information about some of the fields which are not fixed, in terms of boundary data, by the bosonic equations of motion. We will continue to use the boundary conditions appropriate to the topological twist. The frame, spin connection and spinor expansions beyond the leading order given in section 3.4.1 will be needed, so we first give details of these. The frame expansion is

$$\mathrm{e}_i^{\bar{i}} = \mathrm{e}_i^{\bar{i}} + \tfrac{1}{2}z^2(\mathrm{g}^2)^{\bar{i}}{}_{\bar{j}}\mathrm{e}_i^{\bar{j}} + z^3(\mathrm{e}^{(3)})_i^{\bar{i}} + o(z^3), \tag{3.4.32}$$

where in particular $\mathrm{e}_i^{\bar{i}}$ is a frame for the boundary metric and we have used a local SO(3) rotation to gauge fix the order z^2 term. The additional spin connection components we will need are

$$\Omega_i{}^{\overline{zi}} = \frac{1}{z}\mathrm{e}_i^{\bar{i}} - \tfrac{1}{2}\mathrm{g}^{jk}\mathrm{e}_j^{\bar{i}}\partial_z\mathrm{g}_{ik}, \qquad \Omega_z{}^{\overline{ij}} = \mathrm{g}^{ij}\mathrm{e}_i^{[\bar{i}}\partial_z\mathrm{e}_j^{\bar{j}]}. \tag{3.4.33}$$

The bulk spinor then has the following expansion

$$\epsilon^a = z^{-1/2}\varepsilon^a + z^{3/2}\varepsilon_3^a + z^{5/2}\varepsilon_5^a + o(z^{5/2}), \tag{3.4.34}$$

where ε^a are constant with positive chirality under $\Gamma_{\bar{z}}$.

The remaining orders of the bulk dilatino equation give us

$$0 = \tfrac{1}{2\sqrt{2}}(X_1^3 - 4X_1X_2 + 4X_3)\varepsilon_L^a + \tfrac{1}{2\sqrt{2}}\partial_{\bar{i}}\varphi_2\sigma^{\bar{i}}\varepsilon_L^a + \tfrac{1}{8}\eta_{ab}^I\big((F_1^I)_{\overline{ij}} - X_1F_{\overline{ij}}^I\big)\sigma^{\overline{ij}}\varepsilon_L^b, \tag{3.4.35}$$

$$0 = -\sqrt{2}X_1\varepsilon^a_{3,R} - \frac{1}{2\sqrt{2}}(3X_1\varphi_2 + 2\varphi_3)\varepsilon^a_L + \frac{1}{\sqrt{2}}\left(\partial_{\bar{i}}X_2 - X_1\partial_{\bar{i}}X_1\right)\sigma^{\bar{i}}\varepsilon^a_L - \frac{1}{4}\eta^I_{ab}\left(2(\mathrm{a}^I_2)_{\bar{i}} - X_1(\mathrm{a}^I_1)_{\bar{i}}\right)\sigma^{\bar{i}}\varepsilon^b_L\,, \tag{3.4.36}$$

where $F^I_1 = D\mathrm{a}^I_1 \equiv \mathrm{d}\mathrm{a}^I_1 - \frac{1}{\sqrt{2}}\epsilon^{IJK}A^J \wedge \mathrm{a}^K_1$. The remaining gravitino expansions give

$$0 = \varepsilon^a_{3,L} + \tfrac{1}{8}X_1^2\varepsilon^a_L - \frac{1}{16\sqrt{2}}\eta^I_{ab}F^I_{\bar{i}\bar{j}}\sigma^{\bar{i}\bar{j}}\varepsilon^b_L\,, \tag{3.4.37}$$

$$0 = \varepsilon^a_{3,R} - \tfrac{1}{4}\varphi_2\varepsilon^a_L + \frac{1}{4\sqrt{2}}\eta^I_{ab}(\mathrm{a}^I_1)_{\bar{i}}\sigma^{\bar{i}}\varepsilon^b_L\,, \tag{3.4.38}$$

$$0 = \tfrac{1}{2}\mathrm{g}^2_{\bar{i}\bar{j}}\sigma^{\bar{j}}\varepsilon^a_L + \tfrac{1}{4}X_1^2\sigma_{\bar{i}}\varepsilon^a_L - \frac{1}{8\sqrt{2}}\eta^I_{ab}F^I_{\bar{j}\bar{k}}\sigma^{\bar{j}\bar{k}}\sigma_{\bar{i}}\varepsilon^b_L\,, \tag{3.4.39}$$

$$0 = \varepsilon^a_{5,L} - \tfrac{1}{12}(X_1^3 - 2X_1X_2)\varepsilon^a_L - \frac{1}{24\sqrt{2}}\eta^I_{ab}\left((F^I_1)_{\bar{i}\bar{j}} - X_1F^I_{\bar{i}\bar{j}}\right)\sigma^{\bar{i}\bar{j}}\varepsilon^b_L\,, \tag{3.4.40}$$

$$0 = \varepsilon^a_{5,R} - \tfrac{1}{8}(3X_1\varphi_2 + 2\varphi_3)\varepsilon^a_L + \frac{1}{8\sqrt{2}}\eta^I_{ab}\left(2(\mathrm{a}^I_2)_{\bar{i}} - X_1(\mathrm{a}^I_1)_{\bar{i}}\right)\sigma^{\bar{i}}\varepsilon^b_L\,, \tag{3.4.41}$$

$$0 = \sigma_{\bar{i}}\varepsilon^a_{5,R} + \nabla^A_{\bar{i}}\varepsilon^a_{3,L} + \tfrac{1}{4}\omega^{(2)\bar{j}\bar{k}}_{\bar{i}}\sigma_{\bar{j}\bar{k}}\varepsilon^a_L - \tfrac{1}{4}(X_1\varphi_2 + \varphi_3)\sigma_{\bar{i}}\varepsilon^a_L - \frac{1}{4\sqrt{2}}\eta^I_{ab}\left((\mathrm{g}^2)_{\bar{i}}^{\ \bar{j}}A^I_{\bar{j}} - X_1(\mathrm{a}^I_1)_{\bar{i}}\right)\varepsilon^b_L + \frac{1}{4\sqrt{2}}\eta^I_{ab}\left(2(\mathrm{a}^I_2)_{\bar{j}} - X_1(\mathrm{a}^I_1)_{\bar{j}}\right)\sigma^{\bar{i}\bar{j}}\varepsilon^b_L\,, \tag{3.4.42}$$

$$0 = \tfrac{3}{4}\mathrm{g}^3_{\bar{i}\bar{j}}\sigma^{\bar{j}}\varepsilon^a_L + \nabla^A_{\bar{i}}\varepsilon^a_{3,R} - \tfrac{1}{4}(X_1^3 - 2X_1X_2)\sigma_{\bar{i}}\varepsilon^a_L - \tfrac{1}{4}\partial_{\bar{i}}\varphi_2\varepsilon^a_L - \frac{1}{8\sqrt{2}}\eta^I_{ab}\left((F^I_1)_{\bar{j}\bar{k}} - X_1F^I_{\bar{j}\bar{k}}\right)\sigma^{\bar{j}\bar{k}}\sigma_{\bar{i}}\varepsilon^b_L\,. \tag{3.4.43}$$

From the topological twist condition (3.4.21) the boundary gauge field strength is

$$F^I_{\bar{i}\bar{j}} = \frac{1}{\sqrt{2}}\epsilon^I{}_{\overline{kl}}R_{\bar{i}\bar{j}}{}^{\overline{kl}}\,. \tag{3.4.44}$$

Substituting this and the expressions for X_2, a^I_1 and φ_2 into (3.4.37), (3.4.38) allows us to identify

$$\varepsilon^a_{3,L} = -\tfrac{1}{16}(2X_1^2 - R)\,\varepsilon^a_L\,, \qquad \varepsilon^a_{3,R} = \tfrac{1}{2}\varphi_2\,\varepsilon^a_L - \tfrac{1}{2}\partial_{\bar{i}}X_1\sigma^{\bar{i}}\varepsilon^a_L\,. \tag{3.4.45}$$

We also find that equation (3.4.39) is identically satisfied given the expression (3.3.32) for g^2 found in solving the Einstein equation. Equations (3.4.40) and (3.4.41) are solved by removing the unknown quantities F^I_1, a^I_2 using (3.4.35) and (3.4.36):

$$\varepsilon^a_{5,L} = -\tfrac{1}{24}(X_1R - 2X_1^3 + 8X_3)\varepsilon^a_L - \tfrac{1}{12}\partial_{\bar{i}}\varphi_2\sigma^{\bar{i}}\,\varepsilon^a_L\,, \tag{3.4.46}$$

$$\varepsilon^a_{5,R} = \tfrac{1}{2}(2X_1\varphi_2 + \varphi_3)\varepsilon^a_L - \tfrac{1}{16}\partial_{\bar{i}}(2X_1^2 - R)\,\sigma^{\bar{i}}\varepsilon^a_L\,. \tag{3.4.47}$$

We will not solve (3.4.42) as knowledge of a^I_2 or $\omega^{(2)}$ is not relevant for our purposes. Turning now to (3.4.43), using previous results we can re-express this particular equation as

$$\begin{aligned} 0 \;&=\; \big[\tfrac{3}{4}\mathrm{g}^3_{\bar{i}\bar{j}} - \tfrac{1}{2}\nabla_{\bar{i}}\partial_{\bar{j}}X_1 - \tfrac{1}{8}X_1R\delta_{\bar{i}\bar{j}}\big]\sigma^{\bar{j}}\varepsilon^a_L \\ &\quad + \tfrac{1}{4}\partial_{\bar{i}}\varphi_2\varepsilon^a_L - \tfrac{1}{8\sqrt{2}}\eta^I_{ab}\big((F^I_1)_{\bar{j}\bar{k}} - X_1F^I_{\bar{j}\bar{k}}\big)\sigma^{\bar{j}\bar{k}}\sigma_{\bar{i}}\varepsilon^b_L\,. \end{aligned} \tag{3.4.48}$$

By taking the real part we can extract the remaining term in the Fefferman–Graham expansion of the bulk metric

$$\begin{aligned} \mathrm{g}^3_{\bar{i}\bar{j}} = \tfrac{2}{3}\nabla_{\bar{i}}\partial_{\bar{j}}X_1 + \tfrac{1}{6}X_1R\delta_{\bar{i}\bar{j}} + \tfrac{1}{6\sqrt{2}}(F_{1(\bar{i}})^{\overline{kl}}\epsilon_{\bar{j})\overline{kl}} - \tfrac{1}{3\sqrt{2}}(F_1^{\bar{k}})^{\bar{l}}{}_{(\bar{i}}\epsilon_{\bar{j})\overline{kl}} \\ - X_1\big[\tfrac{1}{6\sqrt{2}}(F_{(\bar{i}})^{\overline{kl}}\epsilon_{\bar{j})\overline{kl}} - \tfrac{1}{3\sqrt{2}}(F^{\bar{k}})^{\bar{l}}{}_{(\bar{i}}\epsilon_{\bar{j})\overline{kl}}\big]\,. \end{aligned} \tag{3.4.49}$$

3.5 Metric independence

Our aim in this short section is to show that, for any supersymmetric asymptotically locally hyperbolic solution to the Euclidean $\mathcal{N}=4$ supergravity theory, with the topologically twisted boundary conditions on an arbitrary Riemannian three-manifold (M_3, g), the variation with respect to the arbitrary boundary metric of the holographically renormalized action is identically zero.

An arbitrary deformation of the renormalized action can be written as

$$\delta S \;=\; \int_{\partial Y_4 = M_3} \mathrm{d}^3x\,\sqrt{\det g}\Big[\tfrac{1}{2}T_{ij}\delta g^{ij} + \Xi\delta X_1 + \Sigma\delta\varphi_1 + \mathscr{J}_i{}^I\delta A^{Ii} + \hat{\mathscr{J}}_i{}^I\delta\hat{A}^{Ii}\Big]\,. \tag{3.5.1}$$

For the topological twist we set $\varphi_1 = 0$ and $A^I_i = \frac{1}{\sqrt{2}}\epsilon^I{}_{\overline{jk}}\omega_i{}^{\overline{jk}}$, together with truncating the bulk $SU(2)$ triplet $\hat{\mathcal{A}}^I = 0$. At this point we have not chosen a value for the freely specifiable boundary field X_1 which, recall, has Weyl weight 1. In order for δX_1 to be relatable to δg^{ij}, X_1 must be a scalar function built from the boundary curvature tensors, R_{ijkl}, R_{ij} and R. However, from these tensors we cannot construct a Weyl weight 1 object. Consequently we choose to set $X_1 = 0$ as part of the topological twist boundary conditions.

To evaluate δA^I_i we require the variation of the boundary spin connection in terms of the boundary metric:

$$\delta\omega_i{}^{\overline{jk}} \;=\; \tfrac{1}{2}\mathrm{e}^{j\bar{j}}\mathrm{e}^{k\bar{k}}(\nabla_k\delta g_{ij} - \nabla_j\delta g_{ik})\,. \tag{3.5.2}$$

Thus

$$\delta A^I_i \;=\; \tfrac{1}{\sqrt{2}}\epsilon^I{}_{\overline{jk}}\delta\omega_i{}^{\overline{jk}} \;=\; \tfrac{1}{\sqrt{2}}\epsilon^I{}_{\overline{jk}}\mathrm{e}^{j\bar{j}}\mathrm{e}^{k\bar{k}}\nabla_k\delta g_{ij}\,, \tag{3.5.3}$$

and the variation of the action for the topological twist boundary conditions reduces to

$$\delta\mathbb{S} = \int_{M_3}\Big[\Big(\tfrac{1}{2}T_{ij} - \tfrac{1}{\sqrt{2}}\nabla^k(\mathscr{J}_{Ii}\epsilon^I{}_{\overline{jk}}\mathrm{e}^{\bar{j}}_j\mathrm{e}^{\bar{k}}_k)\Big)\,\delta g^{ij} + \nabla_k\Big(\tfrac{1}{\sqrt{2}}\epsilon^I{}_{\overline{jk}}\,\mathscr{J}^i_I\mathrm{e}^{j\bar{j}}\mathrm{e}^{k\bar{k}}\delta g_{ij}\Big)\Big]\,\mathrm{vol}_3\,, \tag{3.5.4}$$

where we have introduced $\mathrm{vol}_3 \equiv \sqrt{\det g}\,\mathrm{d}^3x$. Dropping the total derivative, which is zero for the closed three-manifolds we are considering, and inserting the expressions for the stress-energy tensor and $SU(2)$ current from (3.3.41) and (3.3.44) gives

$$\delta\mathbb{S} \;=\; \frac{1}{4\kappa_4^2}\int_{M_3}\mathcal{T}_{ij}\delta g^{ij}\mathrm{vol}_3\,, \tag{3.5.5}$$

where the effective stress-energy tensor is

$$\mathcal{T}_{ij} \;= 3\mathrm{g}^3_{ij} + \tfrac{1}{\sqrt{2}}\nabla^k\big(\epsilon_{Ik(i}\,(\mathrm{a}^I_1)_{j)}\big)\,. \tag{3.5.6}$$

Note that because we have identified spacetime and R-symmetry indices, the covariant derivative in $\mathcal{T}_{ij}$ acts on both the I and i indices of $(\mathrm{a}^I_1)_i$. Inserting the expression for g^3_{ij} from (3.4.49) when $X_1 = 0$ gives

$$\mathcal{T}_{ij} \;= \mathrm{e}^{\bar{i}}_i\mathrm{e}^{\bar{j}}_j\big[\tfrac{1}{2\sqrt{2}}(F_{1(\bar{i}})^{\overline{kl}}\epsilon_{\bar{j})\overline{kl}} - \tfrac{1}{\sqrt{2}}(F^{\bar{k}}_1)^{\bar{l}}{}_{(\bar{i}}\epsilon_{\bar{j})\overline{kl}}\big] + \tfrac{1}{\sqrt{2}}\nabla^k\big(\epsilon_{Ik(i}(\mathrm{a}^I_1)_{j)}\big)\,. \tag{3.5.7}$$

Expanding the field strengths we have

$$\begin{aligned}2\sqrt{2}\mathcal{T}_{ij} \;&= \mathrm{e}^{\bar{i}}_i\mathrm{e}^{\bar{j}}_j\big[\nabla^{\bar{k}}(\mathrm{a}_{1(\bar{i}})^{\bar{l}}\epsilon_{\bar{j})\overline{kl}} + (\omega^{\bar{k}})_{(\bar{i}}{}^I(\mathrm{a}_{1|I|})^{\bar{l}}\epsilon_{\bar{j})\overline{kl}} + 2\nabla_{[\bar{l}}(\mathrm{a}^{\bar{k}}_1)_{(\bar{i}]}\epsilon_{\bar{j})}{}^{\bar{l}}{}_{\bar{k}} + 2(\omega_{[\bar{l}})^{\bar{k}I}(\mathrm{a}_{1|I|})_{(\bar{i}]}\epsilon_{\bar{j})}{}^{\bar{l}}{}_{\bar{k}}\big]\\ &\quad + 2\nabla^k\big(\epsilon_{Ik(i}(\mathrm{a}^I_1)_{j)}\big)\,.\end{aligned} \tag{3.5.8}$$

Here covariant derivatives of $(\mathrm{a}^I_1)_i$ in the first line are understood to act with respect to the index outside the bracket only, in contrast to the action on the second line. By carefully expanding, using the definition of the spin connection as the connection of the frame bundle, and recalling from section 3.4.2 that when $X_1 = 0$, $(\mathrm{a}_1^I)_i$ is symmetric in I and i indices, we find delicate cancellations and ultimately that $\mathcal{T}_{ij} = 0$. Notice this is true for an arbitrary background closed three-manifold (M_3, g), and that while the Fefferman–Graham expansion does not determine $(\mathrm{a}_1^I)_i$, nevertheless the expression for $\mathcal{T}_{ij}$ is identically zero.

In analogy with our comments after (2.5.2), we close this section by commenting on more precisely when the derivation in this section holds, and in particular when the formula (3.5.1) holds. The latter computes the variation δS of the on-shell action. A variation of the boundary fields induces a corresponding variation of the bulk fields. Since the background solution that we are varying about solves the bulk equations of motion, crucially the bulk contribution to the resulting variation of the on-shell action is zero (by definition, this bulk integrand multiplies the bulk equations of motion). Thus δS is necessarily a boundary term, and for smooth saddle point solutions dual to the vacuum, one expects the only boundary to be the conformal boundary $\partial Y_4 = M_3$. Equation (3.5.1) is the resulting boundary expression. However, as in five dimensions, this computation would also hold if the bulk solution is singular, or has internal boundaries, provided these do not contribute a corresponding surface term in the interior, in addition to (3.5.1). The internal boundary conditions for fields are clearly then relevant, but if one is going to allow internal singularities/boundaries of this type in a putative saddle point, the absence of these additional surface terms is a fairly clear constraint.

3.6 Geometric reformulation

In this section we first reformulate the bulk supersymmetry conditions (3.4.2), (3.4.3) in terms of a local identity structure. We then use this structure in section 3.6.2 to determine the renormalized on-shell action for *any* smooth filling with topological twist boundary conditions.

3.6.1 Twisted identity structure

Recall that the bulk spinor is originally a quadruplet of Dirac spinors, and we halved the number of degrees of freedom by requiring that it solve the symplectic-Majorana condition

(3.4.1). Therefore, the quadruplet of spinors has the form

$$\epsilon^a = \left(\epsilon^1, -(\epsilon^1)^c, \epsilon^2, (\epsilon^2)^c\right)^{\mathrm{T}}, \qquad (3.6.1)$$

where $\epsilon^{1,2}$ are Dirac spinors on the four-manifold Y_4 and the charge conjugate is $\epsilon^c = \mathscr{C}\epsilon^*$. Notice that the Weyl condition imposed with Γ_5 acting on the spinor indices is not compatible with the topological twist. One sees this from the expressions (3.4.8) and (3.4.12): the leading order term in the expansion of the bulk spinor is chiral if and only if it is zero. However, we may instead act with Γ_5 on the R-symmetry indices of the spinor and require

$$(\Gamma_5)^a{}_b\epsilon^b = \pm\epsilon^a. \qquad (3.6.2)$$

This condition is compatible with the gravitino and dilatino equations (3.4.2) and (3.4.3), since Γ_5 commutes with the self-dual 't Hooft symbols. Projecting onto the subspaces with positive or negative "internal chirality" in (3.6.2) further reduces the bulk spinor to

$$\epsilon^a = (\zeta, -\zeta^c, \pm\mathrm{i}\zeta, \mp\mathrm{i}\zeta^c)^{\mathrm{T}}. \qquad (3.6.3)$$

Using the single Dirac spinor ζ, we may define the following (local) differential forms

$$\begin{aligned} S &\equiv \bar{\zeta}\zeta, & P &\equiv \bar{\zeta}\Gamma_5\zeta, \\ K \equiv \frac{1}{S}\bar{\zeta}\Gamma_{(1)}\zeta, \quad V^1 \mp \mathrm{i}V^3 &\equiv \frac{\mathrm{i}}{S}\overline{\zeta^c}\Gamma_{(1)}\Gamma_5\zeta, & V^2 &\equiv \frac{\mathrm{i}}{S}\bar{\zeta}\Gamma_{(1)}\Gamma_5\zeta, \end{aligned} \qquad (3.6.4)$$

where a bar denotes Hermitian conjugation. Globally, the full bulk spinor is a section of $Spin(Y_4) \otimes E$, where E is a real rank 4 vector bundle associated to the principal $SU(2)_R$ bundle. By considering the change between local trivializations of the spinor under the $SU(2)_R \subset Spin(4)$, one can check that S and P are global smooth functions. Moreover, K is a global 1-form on $Y_4 \setminus \{S = 0\}$, whilst (V^1, V^2, V^3) are sections of $\Omega^1(Y_4 \setminus \{S = 0\}) \otimes V$, where V is the rank 3 vector bundle associated to the $SO(3)_R = SU(2)_R/\mathbb{Z}_2$.

In order to have a globally well-defined bulk spinor ϵ^a, we have to lift the $SO(3)_R$ bundle acting on V to an $SU(2)_R$ bundle acting on E. Moreover, we should define the spinor in the first place, thus lifting the orthonormal frame bundle of the tangent bundle to a $Spin(4)$ frame bundle. In both cases, the obstruction to the lifting is the second Stiefel–Whitney class of the real vector bundles, that is, $w_2(V), w_2(Y_4) \in H^2(Y_4, \mathbb{Z}_2)$. However, because the full bulk spinor is a section of $Spin(Y_4) \otimes E$, we only need

$$w_2(V) \;=\; w_2(Y_4)\,, \tag{3.6.5}$$

in order for the tensor product of the "virtual" bundles to be defined. As in the previous case, we say that the bulk spinor is a $Spin_{SU(2)}$ spinor.

Geometrically, a single Dirac spinor in four dimensions defines a local identity structure on the four-manifold, or equivalently a local orthonormal frame. In order to construct it, we split the bulk spinor into its components with positive and negative chirality under Γ_5, $\zeta = \zeta_+ + \zeta_-$, and define

$$\eta_\pm \;\equiv\; \frac{\zeta_\pm}{\sqrt{S_\pm}}\,, \tag{3.6.6}$$

where $S_\pm \equiv \overline{\zeta_\pm}\zeta_\pm$. Then an orthonormal frame can be defined by

$$\mathrm{i}\mathrm{E}^2 - \mathrm{E}^4 \;\equiv\; \overline{\eta_-}\Gamma_{(1)}\eta_+\,, \qquad \mathrm{i}\mathrm{E}^1 - \mathrm{E}^3 \;\equiv\; \overline{\eta_-^c}\Gamma_{(1)}\eta_+\,, \tag{3.6.7}$$

and we choose the orientation induced by the volume form E^{4123}. We also define the function θ by

$$\cos^2\frac{\theta}{2} \;\equiv\; \frac{S_+}{S}\,, \qquad \sin^2\frac{\theta}{2} \;\equiv\; \frac{S_-}{S}\,. \tag{3.6.8}$$

We may then re-express the local differential forms above in terms of the frame as

$$P \;=\; S\cos\theta\,, \qquad K \;=\; -\sin\theta\,\mathrm{E}^4\,, \qquad V^I \;=\; -\sin\theta\,\mathrm{E}^I\,, \quad I = 1,2,3\,. \tag{3.6.9}$$

This canonical frame degenerates at $\theta = 0, \pi$, where the spinor has positive/negative chirality, and also when $S = 0$, where the spinor vanishes. The subset of Y_4 with these points excluded will be denoted $Y_4^{(0)}$. From the global considerations above it then follows that E^4 is a global 1-form on $Y_4^{(0)}$, and E^I are sections of $\Omega^1(Y_4^{(0)}) \otimes V$. Therefore, the E^I rotate into each other in the fundamental representation of $SO(3)_R$ between local trivializations, and the orthonormal frame is not global in general.

Starting with the bulk Killing spinor equations (3.4.2) and (3.4.3), we may find a set of

Killing spinor equations for ζ. Choosing negative internal chirality in (3.6.2), they read

$$\begin{aligned}\nabla_\mu\zeta \;=\;& -\tfrac{\mathrm{i}}{2\sqrt{2}}\mathcal{A}^2_\mu\zeta - \tfrac{\mathrm{i}}{2\sqrt{2}}(\mathcal{A}^1_\mu + \mathrm{i}\mathcal{A}^3_\mu)\zeta^c + \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}\mathcal{F}^2_{\nu\lambda}\Gamma^{\nu\lambda}\Gamma_\mu\zeta - \tfrac{\mathrm{i}}{4}X^2\partial_\mu\varphi\,\Gamma_5\zeta \\ &+ \tfrac{\mathrm{i}}{8\sqrt{2}}X^{-1}(\mathcal{F}^1_{\nu\lambda} + \mathrm{i}\mathcal{F}^3_{\nu\lambda})\Gamma^{\nu\lambda}\Gamma_\mu\zeta^c - \tfrac{1}{4}(X + X^{-1})\Gamma_\mu\zeta - \tfrac{\mathrm{i}}{4}\varphi X\Gamma_\mu\Gamma_5\zeta\,,\end{aligned} \tag{3.6.10}$$

$$\begin{aligned}0 \;=\;& \tfrac{1}{\sqrt{2}}X^{-1}\partial_\nu X\Gamma^\nu\zeta + \tfrac{\mathrm{i}}{8}X^{-1}\mathcal{F}^2_{\nu\lambda}\Gamma^{\nu\lambda}\zeta + \tfrac{\mathrm{i}}{8}X^{-1}(\mathcal{F}^1_{\nu\lambda} + \mathrm{i}\mathcal{F}^3_{\nu\lambda})\Gamma^{\nu\lambda}\zeta^c \\ &- \tfrac{\mathrm{i}}{2\sqrt{2}}X^2\partial_\nu\varphi\Gamma^\nu\Gamma_5\zeta - \tfrac{1}{2\sqrt{2}}(X - X^{-1})\zeta - \tfrac{\mathrm{i}}{2\sqrt{2}}\varphi X\Gamma_5\zeta\,.\end{aligned} \tag{3.6.11}$$

From these equations, one can use standard spinor bilinear manipulations to obtain differential conditions for the frame and the fields:

$$\mathrm{d}(XS) \;=\; S\sin\theta\,\mathrm{E}^4\,, \tag{3.6.12}$$

$$\mathrm{d}(XS\cos\theta) \;=\; \tfrac{1}{\sqrt{2}}S\sin\theta\,\mathrm{E}_I\lrcorner\mathcal{F}^I\,, \tag{3.6.13}$$

$$\begin{aligned}-D(S\sin\theta\,\mathrm{E}^I) \;=\;& \tfrac{1}{\sqrt{2}}X^{-1}S(*\mathcal{F}^I - \cos\theta\mathcal{F}^I) \\ &+ (X + X^{-1})S\left(\mathrm{E}^{I4} - \tfrac{1}{2}\cos\theta\,\epsilon^{IJK}\mathrm{E}^{JK}\right) \\ &+ \mathrm{i}\varphi XS\left(\cos\theta\,\mathrm{E}^{I4} - \tfrac{1}{2}\epsilon^{IJK}\mathrm{E}^{JK}\right)\,,\end{aligned} \tag{3.6.14}$$

$$\begin{aligned}\mathrm{d}\varphi \;=\;& \tfrac{\mathrm{i}}{\sqrt{2}}X^{-3}\csc\theta\,\mathrm{E}_J\lrcorner\left(\mathcal{F}^J + \cos\theta * \mathcal{F}^J\right) \\ &+ X^{-3}\csc\theta\left(\mathrm{i}X(X - X^{-1})\cos\theta - \varphi X^2\right)\mathrm{E}^4\,,\end{aligned} \tag{3.6.15}$$

$$\begin{aligned}\mathrm{d}X \;=\;& -\tfrac{1}{2\sqrt{2}}\csc\theta\,\mathrm{E}_J\lrcorner\left(\cos\theta\,\mathcal{F}^J + *\mathcal{F}^J\right) \\ &- \tfrac{1}{2}\csc\theta\left(X(X - X^{-1}) - \mathrm{i}\varphi X^2\cos\theta\right)\mathrm{E}^4\,.\end{aligned} \tag{3.6.16}$$

Here the covariant derivative acting on E^I is $D\mathrm{E}^I \equiv \mathrm{d}\mathrm{E}^I - \frac{1}{\sqrt{2}}\epsilon^{IJK}\mathcal{A}^J \wedge \mathrm{E}^K$. We may in particular combine these equations to obtain an expression for φ:

$$\varphi \;=\; \mathrm{i}X^{-2}\cos\theta + \alpha(XS)^{-1}\,, \tag{3.6.17}$$

where $\alpha \in \mathrm{i}\mathbb{R}$, and we have used that everything in this last equation is globally defined to integrate, assuming that Y_4 is path-connected.

The system of equations (3.6.12)–(3.6.16) is in fact necessary and sufficient to have a supersymmetric solution to the bulk equations of motion. There are several steps involved in showing this. Firstly, we note that for a Dirac spinor ζ the set $\{\zeta, \zeta^c, \Gamma_\mu \zeta\, \Gamma_\mu \zeta^c\}$ spans the spinor space. Thus contracting the dilatino equation (3.6.11) with the Hermitian conjugate of each element of this set gives a collection of equations which are equivalent to the dilatino equation. In turn, these equations can be shown to be equivalent to (3.6.15) and (3.6.16). On the other hand, since we have a (local) identity structure, the intrinsic torsion is determined by the exterior derivatives in (3.6.12)–(3.6.14). It follows that (3.6.12)–(3.6.16) are equivalent to the Killing spinor equations. One next considers the truncated integrability conditions derived from (3.3.13) and (3.3.14). From these it is straightforward to show that the Killing spinor equations imply the equations of motion, while the Bianchi identity for $\mathcal{F}^I$ has to be imposed additionally. In particular the proof of this uses the fact that the bulk spinor ζ is Dirac. The upshot is that the complete system of equations to solve is given by the first order differential system (3.6.12)–(3.6.16).

It is interesting, especially in light of the computation of the on-shell action in the next section, to consider the expansion of the bilinear equation near the boundary. Using the Fefferman–Graham coordinate z, the bulk spinor ζ has the expansion

$$\zeta \;=\; z^{-1/2}\begin{pmatrix}\chi\\ 0\end{pmatrix} + z^{3/2}\begin{pmatrix}\frac{1}{16}R\,\chi\\ \frac{1}{2}\varphi_2\,\chi\end{pmatrix} + z^{5/2}\begin{pmatrix}-\frac{1}{12}\partial_{\bar{i}}\varphi_2\,\sigma^{\bar{i}}\chi\\ \frac{1}{16}\partial_{\bar{i}}R\,\sigma^{\bar{i}}\chi\end{pmatrix} + o(z^{5/2})\,, \tag{3.6.18}$$

where χ is a constant 2-component spinor given by

$$\chi \;=\; \begin{pmatrix} c\\ -\mathrm{i}c\end{pmatrix}\,, \tag{3.6.19}$$

with $c \in \mathbb{R}$ (compare with (3.4.22) with $c = -\overline{w}$). Without loss of generality, we may set $c = 1$ in the following, and the norm of the spinor takes the form

$$S \;=\; \frac{2}{z} + \frac{z}{4}R + o(z^2)\,. \tag{3.6.20}$$

We also find

$$\begin{aligned} X &= 1 - \frac{z^2}{4}R + o(z^3)\,, \\ \varphi &= \frac{\mathrm{i}}{\sqrt{2}} z^2 (a_1^I)_I + o(z^3)\,, \\ t^{(2)} &= -\frac{1}{4}R\,, \qquad t^{(3)} \;=\; 0\,. \end{aligned} \tag{3.6.21}$$

The vanishing of φ_1 allows us to fix the constant α in (3.6.17): expanding the latter equation leads to $\varphi_1 = \alpha/2$, so under the assumption of the topological twist, $\alpha = 0$. In a neighbourhood of the conformal boundary, the bulk frame has the form

$$\begin{aligned} \mathrm{E}^I &= \frac{1}{z}\mathrm{e}^I + \frac{z}{2}\left(\mathrm{g}^2 \circ \mathrm{e}^I\right) + o(z)\,, \\ \mathrm{E}^4 &= -\frac{\mathrm{d}z}{z} - \frac{z^2}{8}\mathrm{d}R + o(z^2)\,. \end{aligned} \tag{3.6.22}$$

Near the boundary, the leading order of the equations (3.6.12)–(3.6.16) is trivial apart from (3.6.14), which corresponds to the condition that e^I satisfy the first Cartan's structural equation

$$\mathrm{d}\mathrm{e}^I + \omega^I{}_J \wedge \mathrm{e}^J \;=\; 0\,. \tag{3.6.23}$$

Here the spin connection $\omega^I{}_J$ arises from the topological twist boundary condition for the gauge field (3.4.21). In some sense (3.6.23) is a redundant equation, simply stating that the frame defined by supersymmetry is compatible with the boundary metric. As in the AdS_5/CFT_4 example, the bulk differental equations are tautological on the boundary, where they simply define a (twisted) frame for the three-manifold M_3.

3.6.2 On-shell action

Thanks to these results, we can now greatly simplify the expression for the on-shell action. We start with the expression (3.3.35) and set $\hat{\mathcal{F}}^I = 0$, obtaining

$$I_{\text{o-s}} \;=\; -\frac{1}{2\kappa_4^2}\int_{Y_4}\big[-(4 + X^2 + X^{-2} + \varphi^2 X^2) * 1 - \tfrac{1}{2}X^{-2}\big(\mathcal{F}^I \wedge *\mathcal{F}^I + \mathrm{i}\varphi X^2 \mathcal{F}^I \wedge \mathcal{F}^I\big)\big]\,. \tag{3.6.24}$$

Then, using (3.3.3) and (3.3.4), we may exchange the gauge field contribution for an exact term

$$I_{\text{o-s}} \;=\; -\frac{1}{2\kappa_4^2}\int_{Y_4}\big[-(4+2X^{-2}+2\varphi^2X^2)*1+\mathrm{d}\big(2X^{-1}*\mathrm{d}X-\varphi X^4*\mathrm{d}\varphi\big)\big]\,. \qquad (3.6.25)$$

Notice that, using the equations for the orthonormal frame and (3.6.17), we can write

$$\mathrm{d}\left(X^{-1}*K\right) \;=\; -\left(2+X^{-2}\sin^2\theta\right)*1\,, \qquad (3.6.26)$$

and this, using the expression (3.6.17) for φ, is exactly (modulo a numerical factor) the potential term in the on-shell action (3.6.25). Therefore, the on-shell action is exact

$$I_{\text{o-s}} \;=\; -\frac{1}{\kappa_4^2}\int_{Y_4}\mathrm{d}\left(X^{-1}*K+X^{-1}*\mathrm{d}X-\tfrac{1}{2}\varphi X^4*\mathrm{d}\varphi\right)\,. \qquad (3.6.27)$$

The global arguments discussed above imply that the 4-form in the action

$$\mathrm{Y} \;\equiv\; X^{-1}*K+X^{-1}*\mathrm{d}X-\tfrac{1}{2}\varphi X^4*\mathrm{d}\varphi\,, \qquad (3.6.28)$$

is globally well-defined on $Y_4^{(0)}$. In what follows we assume that the subset of Y_4 where the spinor becomes chiral or zero is measure zero. As in section 3.3.3, we cut off the bulk Y_4 at some small radius $z=\delta>0$, so that $\partial Y_4 = M_\delta \equiv \{z=\delta\}\cong M_3$. We may then appeal to Stokes' theorem and write the on-shell action as integrals over the conformal boundary $M_3\cong M_\delta$, and over the boundaries T_ϵ of the small tubular neighbourhoods of radius $\epsilon>0$ surrounding the subsets $Y_4\setminus Y_4^{(0)}$ where the frame degenerates. Let us consider first the contribution from the conformal boundary: using the expansion of the spinor (3.6.18) and of the fields (3.6.21), it is easy to show that near the conformal boundary

$$\mathrm{Y} \;=\; \left(\frac{1}{\delta^3}-\frac{3}{8\delta}R+o(1)\right)*_{g^0}1\,. \qquad (3.6.29)$$

To this we should add the contributions from the Gibbons–Hawking–York term (3.3.36) and the counterterms (3.3.37), which in a neighbourhood of the boundary are

$$I_{\text{GHY}} \;=\; \frac{1}{\kappa_4^2}\int_{M_3}\left(-\frac{3}{\delta^3}+\frac{1}{8\delta}R+o(1)\right)*_{g^0}1\,, \qquad (3.6.30)$$

$$I_{\text{ct}} \;=\; \frac{1}{\kappa_4^2}\int_{M_3}\left(\frac{2}{\delta^3}+\frac{1}{4\delta}R+o(1)\right)*_{g^0}1\,. \tag{3.6.31}$$

Once we take into account the change in sign of the on-shell terms, due to the orientation of the bulk compared to the orientation of the boundary, the contribution to the renormalized action from the conformal boundary is zero in the limit $\delta \to 0$.

Therefore, the renormalized gravitational action only receives contributions from the subsets where the frame degenerates:

$$S \;=\; \frac{1}{\kappa_4^2}\lim_{\epsilon\to 0}\int_{\mathrm{T}_\epsilon}\Upsilon\,, \tag{3.6.32}$$

where the limit collapses the small neighbourhood around the degeneration locus. As in section 2.6.3, these contributions are zero. That is, a supergravity solution with a smooth metric and smooth bosonic fields. Clearly the last two forms in Υ, which only involve X, φ, are well-defined if the bosonic fields are smooth. The last two terms in Υ therefore provide zero contribution when integrated over a subset of vanishing measure. The only non-trivial contribution could arise from $X^{-1} * K$.

Consider first the subset where the spinor is chiral but non-vanishing. While changing from local $SU(2)_R$ gauge patches of definition for ϵ^a, ζ is a linear combination of ζ and ζ^c, but note that in four dimensions $\Gamma_5\zeta = \pm\zeta$ if and only if $\Gamma_5\zeta^c = \pm\zeta^c$. Therefore, spacetime chirality is a well-defined global concept for the $Spin_{SU(2)}$ spinor. If the spinor is chiral but non-vanishing, $S \neq 0$ and the bilinears K and V^I vanish, so $X^{-1} * K$ is zero there, and the integral is zero.

Secondly, consider the subset where the spinor is vanishing. Note that we may write

$$X^{-1} * K = -X^{-1}\sin\theta\,\mathrm{E}^4 \lrcorner\,\mathrm{vol}_4\,. \tag{3.6.33}$$

Using (3.6.12) we then in turn have

$$X^{-1}\sin\theta\,\mathrm{E}^4 \;=\; \mathrm{d}\log\rho\,, \qquad \text{where} \quad \rho \;\equiv\; XS\,. \tag{3.6.34}$$

We may thus use $\rho > 0$ as a radial coordinate near to the where the spinor vanishes at $\rho = 0$, and more precisely define $\mathrm{T}_\epsilon = \{\rho = \epsilon > 0\}$. It follows that $X^{-1} * K$ is the product of a bounded function $X^{-1}\sin\theta$ (as long as $X > 0$ is smooth), and the volume form $\mathrm{E}^4 \lrcorner\,\mathrm{vol}_4$

induced on T_ϵ from the four-dimensional bulk metric. The integral hence vanishes in the limit $\epsilon \to 0$, exactly as in section 2.6.3.

We conclude that the renormalized action for any *smooth* supergravity solution is zero. In particular, we have made no assumptions at all here on the topology of M_3, or of its path-connected filling Y_4 with $\partial Y_4 = M_3$.

3.7 Discussion

In the first part of the thesis, we have defined and studied a holographic dual to the topological twist of $\mathcal{N} = 2$ gauge theories on Riemannian four-manifolds and $\mathcal{N} = 4$ gauge theories on Riemannian three-manifolds, and verified that the renormalized gravitational free energy is independent of the boundary metric in both cases. We have also reformulated the bulk supersymmetry equations in terms of G-structures twisted by R-symmetry bundles, and used these structures to prove that the gravitational free energy of all smooth bulk fillings, irrespective of their topology, is zero. Let us emphasize one more time that the latter result does not make the former result of sections 2.5 and 3.5 redundant: the computation of the variation of the gravitational free energy holds for smooth solutions, but *a priori* it is more general. Metric-independence will still hold for singular solutions, provided the additional surface terms around the singularities are zero. In fact if one allows singular saddle point solutions at all, this should be a clear constraint.

The results presented here raise a number of interesting questions and directions for future research. In general the classical supergravity limit of the AdS/CFT correspondence identifies

$$- \log Z_{\mathrm{QFT}} \; = \; \mathcal{S} \; . \tag{3.7.1}$$

Here on the right hand side we have the least action solution to the given filling problem in the bulk supergravity, while the left hand side is understood to be the leading term in the corresponding strong coupling (typically large rank N) limit of the QFT partition function. For example, uplifting the four-dimensional $\mathcal{N} = 4$ gauged supergravity solutions to M-theory on $S^7/\mathbb{Z}_k$ leads to the effective four-dimensional Newton constant in (3.2.4), which scales as $N^{3/2}$. The latter multiplies the holographically renormalized on-shell action $\mathcal{S}$ on the right hand side of (3.7.1). On the other hand, in this chapter we have shown that this gravitational free energy is always zero, for any smooth supergravity filling of any conformal boundary three-manifold M_3. We have already noted that every oriented three-manifold is spin, but another important topological fact is that every such three-manifold bounds a smooth four-manifold (which may be taken to be spin). There is thus no topological obstruction to finding such a bulk filling of M_3. Of course, an important assumption here is that there exist smooth fillings that solve the supergravity equations, with prescribed conformal boundary (M_3, g). We have recast the supergravity equations as the first order differential system (3.6.12)–(3.6.16), and thus existence and uniqueness theorems for solutions to these equations will play an important rôle. Given that such solutions are supersymmetric and are dual to a topologically twisted theory, one naturally expects better behaviour than the non-supersymmetric Einstein filling problem, typically studied by mathematicians. In any case, assuming that such smooth fillings are the dominant saddle points in (3.7.1), the results of this chapter imply that the large N limit of the topologically twisted ABJM partition function is $o(N^{3/2})$, for any three-manifold M_3. This should be contrasted with the non-twisted partition function on (for example) S^3, where both sides of (3.7.1) agree and equal $\frac{\pi\sqrt{2k}}{3}N^{3/2}$ in the large N limit [88]. It thus remains an interesting open problem to compute the large N limit of the topologically twisted ABJM theory, on a three-manifold M_3, and compare with our holographic result. Moreover, if the leading classical saddle point indeed contributes zero, the next obvious step is to try to compute the subleading term, as a correction to the supergravity limit. Since by construction everything is a topological invariant, this may well be possible.

Similar remarks apply to the Donaldson–Witten twist studied holographically in the previous chapter. Here the bulk five-dimensional $\mathcal{N} = 4^+$ gauged supergravity solutions uplift on S^5 to solutions of type IIB supergravity, where now the five-dimensional Newton constant is given by (2.2.6).[7] The resulting solutions are holographically dual to the Donaldson–Witten twist of $\mathcal{N} = 4$ SYM on the conformal boundary four-manifold M_4. Similar remarks apply to those made in the paragraph above, although there is an important difference: the partition function is only non-zero when $2\chi(M_4) + 3\sigma(M_4) = 0$, and moreover M_4 bounds a smooth five-manifold if and only if $\sigma(M_4) = 0$. The fact that the gravitational free energy is zero for smooth fillings, as shown in section 2.6.3, is therefore only directly applicable when $\chi(M_4) = 0 = \sigma(M_4)$. In this case, the topologically twisted partition function of $\mathcal{N} = 4$ SYM should be $o(N^2)$, assuming the dominant saddle point

[7] As already noted, one may also uplift to solutions of M-theory, which are dual to $\mathcal{N} = 2$ theories of class $\mathcal{S}$ with N^3 scaling, but we won't discuss this further here.

solution is indeed smooth.

On the other hand, the Donaldson–Witten twisted partition function has been computed, for general rank gauge group $\mathscr{G} = SU(N)$, on $M_4 = \mathrm{K3}$ in [215, 211]. This follows from the fact that on the hyperKähler K3 manifold the Donaldson–Witten and Vafa–Witten twists are equivalent (and in fact equivalent to the untwisted theory). However, $|\sigma(\mathrm{K3})| = 16$ and a smooth filling by Y_5 does not exist in this case, so there is no obvious classical gravity solution to compare to. Nevertheless, the partition function is (for N prime) [215, 211]

$$Z(\mathrm{K3}) \ = \ \frac{1}{N^2} G(q^N) + \frac{1}{N} \sum_{I=1}^{N} G\left(\omega^I q^{1/N}\right) \ , \tag{3.7.2}$$

where $q = \exp(2\pi \mathrm{i} \tau)$, with $\tau = \frac{\theta}{2\pi} + \frac{4\pi \mathrm{i}}{g_{\mathrm{YM}}^2}$ the usual complexified gauge coupling, $\omega = \exp(2\pi \mathrm{i}/N)$, and $G(q) = 1/\eta^{24}(\tau)$, with η the Dedekind eta-function. Taking the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ fixed and large, the $N \to \infty$ limit is dominated by the first term in (3.7.2), resulting in the leading order behaviour

$$\log Z(\mathrm{K3}) \ \sim \ \frac{8\pi^2 N^2}{\lambda} \ . \tag{3.7.3}$$

As mentioned above, in general the classical gravitational free energy is order N^2, which for smooth fillings of M_4 we have shown is multiplied by zero for the holographic Donaldson–Witten twist. However, there is no such smooth filling of $M_4 = \mathrm{K3}$, so it is not clear what the dual classical solution should be. Perhaps one should allow for certain singular Y_5, and/or fill the boundary $S^5 \times \mathrm{K3}$ with a topology that is not simply an S^5 bundle over Y_5. These would lie outside the class of smooth solutions to the consistently truncated five-dimensional $\mathcal{N} = 4^+$ gauged supergravity we have studied. That said, a perhaps naive interpretation of (3.7.3) is that the leading classical $O(N^2)$ term is indeed zero, with the N^2/λ term being a subleading string correction to this. This particular example clearly deserves much further study.

Perhaps the most immediate generalization of the computations of topological AdS/CFT in five bulk dimensions would be to the so-called Ω-background of [184] mentioned in section 1.3. Here (M_4, g, ξ) is an arbitrary Riemannian four-manifold, equipped with a Killing vector field ξ. As for the pure topological twist, this geometry also arises by coupling an $\mathcal{N} = 2$ gauge theory to a certain background of $\mathcal{N} = 2$ conformal supergravity, and is briefly mentioned at the end of section 3 of [146]. The non-zero Killing vector ξ requires turning on a boundary B-field: specifically one needs to take b^- (or b^+) proportional to the self-dual (or anti-self-dual) part of the 2-form $d\xi^\flat$, where $\xi^\flat$ is the Killing 1-form dual to ξ. Correspondingly, both boundary spinor doublets ε^+ and ε^- are now non-zero, and one needs to work with the full Romans theory, rather than the truncated version with $\mathcal{B}^\pm = 0$ we used from section 2.4.2 onwards. One expects the supergravity action now to depend on the choice of Killing vector ξ on M_4, but otherwise not on the metric. One should thus look at metric deformations $g_{ij} \to g_{ij} + \delta g_{ij}$, where $\mathcal{L}_\xi \, \delta g_{ij} = 0$.

One may also want to consider the other (generically inequivalent) topological twists of $\mathcal{N} = 4$ Yang–Mills. The two twists not considered here are the Vafa–Witten twist [215], and the twist studied by Kapustin–Witten in [141]. In particular in the former theory the only non-trivial observable is the partition function, and this has been studied for gauge group $\mathscr{G} = SU(N)$ in [155, 211]. These twists require the larger $SU(4)_R$ R-symmetry of the $\mathcal{N} = 4$ theory, meaning for the holographic dual one needs to start with a Euclidean form of $\mathcal{N} = 8$ gauged supergravity theory. Optimistically, one might hope to embed within the $SU(4) \sim SO(6)$ truncation of the latter theory studied in [76], which is a consistent truncation of Type IIB supergravity on S^5, and contains the five-dimensional Romans $\mathcal{N} = 4^+$ theory (with zero B-field) as a further truncation.

More generally, there are a wide variety of possible topologically twisted theories in diverse spacetime dimensions. One could ask if zero action/gravitational free energy for smooth supergravity solutions dual to TQFTs is a general property. Perhaps this is specific to cases in which the preserved supercharge $\mathcal{Q}$ in the TQFT satisfies $\mathcal{Q}^2 = 0$, which is generally not the case. The apparent simplicity of our results suggests there should be a more elegant way to set up the holographic problem. Recall that in field theory, invariance of the TQFT partition function with respect to metric deformations crucially relies on the stress-energy tensor being $\mathcal{Q}$-exact. We have shown the corresponding result holographically, but in a less direct manner. It is natural to conjecture that a topological sector of gauged supergravities, in this holographic setting, may be similarly described using a boundary BRST symmetry [218, 133, 200, 22, 80].

Finally, in these chapters we have focused exclusively on the partition function. However, in general TQFTs have non-trivial topological correlation functions, involving the insertion of $\mathcal{Q}$-invariant operators into the path integral. For example, this is true of Donaldson theory, where such insertions are required to obtain non-zero invariants in field theory whenever $\dim \mathcal{M} = d > 0$, due to fermion zero modes. Geometrically these invariants arise as the integral of a d-form over $\mathcal{M}$, where this top form is itself constructed as a wedge product of certain closed forms. The operators are constructed via a descent procedure [219]. It would be very interesting to understand the holographic dual computation of these correlation functions. Of course, correlation functions are well studied in AdS/CFT. In the present setting one would again hope to be able to work in a truncated supergravity theory, containing the fields whose boundary values act as sources for the operators (so, concretely, the boundary conditions for the supergravity fields would be different from those considered in the last two chapters). Being topological, the correlation functions should be independent of the positions at which the local operators are inserted, and also independent of the metric. These statements might be proven along similar lines to the present dissertation.

Part II

Rigid supersymmetry

4 Holographic Renormalization and Supersymmetry

4.1 Introduction

Holographic observables in the AdS/CFT correspondence typically need regularizing, and the very structure of anti-de Sitter space provides a renormalization method. As we saw in the previous part of the thesis, the infinite local boundary counterterms found via the holographic renormalization are universal, but there exist finite counterterms as well. Such ambiguities in the renormalization scheme can be clarified by comparing specific observables on the two sides of the correspondence, for instance those protected by supersymmetry, and particularly by requiring them to depend in the same way on the background. In the previous part of the dissertation, we saw that the on-shell supergravity action of $\mathcal{N} = 4$ gauged supergravity in four and five dimensions, renormalized using the minimal scheme, is independent of the boundary metric provided we impose the boundary conditions corresponding to the topological twist of the boundary field theory. In this chapter, we will study minimal $\mathcal{N} = 2$ gauged supergravity in four and five dimensions, whose bosonic sectors are simply Einstein–Maxwell theory with a negative cosmological constant (and Chern–Simons coupling in dimension five). Solutions to these theories uplift either to M-theory or to type II string theory, and there are large classes corresponding to known field theory duals.

Asymptotically locally AdS supersymmetric solutions induce a rigid supersymmetric structure on the conformal boundary, which has been studied in both Lorentzian and Euclidean signature [145, 61].[1] The boundaries M_3 of asymptotically locally hyperbolic supersymmetric solutions to four-dimensional supergravity have metric of the form

$$ds^2_3 \quad = \quad (d\psi + a)^2 + 4e^w dz d\bar{z} \; . \tag{4.1.1}$$

Here ∂_ψ is a nowhere zero Killing vector on M_3, and we have used the freedom to make conformal transformations to take this to be a unit norm vector. This generates a transversely holomorphic foliation of M_3, allowing one to introduce a canonical local transverse complex coordinate z. The function $w = w(z, \bar{z})$ is in general a local transverse function, while $a = a_z(z, \bar{z})dz + \overline{a_z(z, \bar{z})}d\bar{z}$ is a local 1-form. We may also write $da = iu\, e^w dz \wedge d\bar{z}$, where $u = u(z, \bar{z})$. In addition to the background metric (4.1.1) there is also a non-dynamical Abelian R-symmetry gauge field, which arises as the restriction of the bulk Maxwell field to the conformal boundary and whose form is specified by supersymmetry.

It is a general result of [68, 67] that the partition function of any $\mathcal{N} = 2$ field theory in three dimensions, with a choice of Abelian R-symmetry coupling to the background R-symmetry gauge field, depends on the above background geometry only through the choice of transversely holomorphic foliation. Concretely, this means that the field theory partition function is invariant under deformations $w \to w + \delta w$, $u \to u + \delta u$, where $\delta w(z, \bar{z})$, $\delta u(z, \bar{z})$ are *arbitrary* smooth global functions on M_3, invariant under ∂_ψ. This is proven by showing that these deformations of the background geometry lead to $\mathcal{Q}$-exact deformations of the Lagrangian, where $\mathcal{Q}$ is a supercharge, and a standard argument then shows that the partition function is invariant. This general result has also been borne out by explicit computations of localized partition functions (such as [11], where M_3 has the topology of S^3).

The field theory results in the previous paragraph then lead to a very concrete prediction: the holographically renormalized on-shell action of a supersymmetric asymptotically locally hyperbolic solution to four-dimensional supergravity, with conformal boundary M_3 and metric (4.1.1), should be invariant under the arbitrary deformations $w \to w + \delta w$, $u \to u + \delta u$ defined above. As we shall review, in four dimensions holographic renormalization

[1] Asymptotically locally AdS manifolds are the Lorentzian version of the asymptotically locally hyperbolic manifolds defined in section 1.1.

leads to a unique set of standard counterterms for minimal $\mathcal{N} = 2$ gauged supergravity – there are no finite ambiguities[2] – and we prove that the renormalized on-shell action has indeed the expected invariance properties. Since we do this for an *arbitrary* solution, and *arbitrary* deformation, this constitutes a robust check of the AdS/CFT correspondence, in particular that holographic renormalization corresponds to the (unique) supersymmetric renormalization scheme employed implicitly in the localization computations. We also go further, and show that the on-shell action itself correctly evaluates to the large N field theory partition function obtained from localization, in the cases where this is known.

The corresponding situation for five-dimensional supergravity turns out to be more involved. We will consider Euclidean conformal boundaries M_4 given by the direct product of a circle S^1 with M_3 equipped with the metric (4.1.1), although we shall later generalize this slightly to a simple class of twisted backgrounds in which S^1 is fibred over M_3; the boundary value of the Abelian gauge field in the supergravity multiplet is again determined by supersymmetry. The general dependence of the four-dimensional field theory partition function on the background is similar to the one in three dimensions: for $\mathcal{N} = 1$ theories with an R-symmetry (and thus for any $\mathcal{N} = 1$ superconformal field theory), the supersymmetric partition function is invariant under deformations $w \to w + \delta w$, $u \to u + \delta u$ [68, 67, 19]. Although contrastingly with the three-dimensional case these "supersymmetric Ward identities" *a priori* only hold up to anomalies and local finite counterterms, it was shown in [18] that the supersymmetric renormalization scheme used in field theory is unique, i.e. free of ambiguities. Moreover the background M_4 we consider is such that there are no Weyl and R-symmetry anomalies [63].[3] Therefore the statement on invariance of the partition function should hold exactly in our set-up.

In five-dimensional supergravity, holographic renormalization contains a set of diffeomorphism-invariant and gauge-invariant local boundary terms corresponding *a priori* to the same ambiguities and anomalies as in field theory [225, 127, 27]. One might thus have expected that there is a unique linear combination of the finite holographic counterterms that matches the supersymmetric field theory scheme, i.e. such that the renormalized action is invariant under deformations $w \to w + \delta w$, $u \to u + \delta u$ of M_4. Surprisingly, we find that *no choice* of these counterterms has this property. If the AdS/CFT correspondence is to hold, we

[2]More precisely there are no finite diffeomorphism-invariant and gauge-invariant local counterterms constructed using the bosonic supergravity fields.

[3]See the discussion at the end of the chapter for some brief remarks on the possibility of supercurrent anomalies.

must conclude that holographic renormalization *breaks supersymmetry* in this case (or, perhaps more precisely, is not compatible with the four-dimensional supersymmetry determining the Ward identities above). However, remarkably we are able to write down a set of non-standard, finite boundary terms that do not correspond to the usual diffeomorphism and gauge invariant terms and that give the on-shell action the expected invariance properties.

The approach we follow in our supergravity analysis starts in Lorentzian signature. In particular we will rely on the existing classification of Lorentzian supersymmetric solutions to minimal gauged supergravity [108] to construct a general asymptotically locally AdS solution in a perturbative expansion near the boundary. Then we perform a Wick rotation; this generally leads to complex bulk solutions, however we focus on a class with real Euclidean conformal boundary $M_4 \cong S^1 \times M_3$.

The fact that supersymmetric holographic renormalization is more subtle in five dimensions was already anticipated, and in fact the issue can be illustrated by considering the simple case of AdS_5. In global coordinates, and after compactifying the Euclidean time, the conformal boundary of AdS_5 can be taken to be $M_4 \cong S^1 \times S^3$, with a round metric on S^3. This space is expected to be dual to the vacuum of a superconformal field theory (SCFT) on M_4. In this background, such theories develop a non-ambiguous non-zero vacuum expectation value (VEV) for both the energy and the R-charge operators [19, 18]. On the other hand, standard holographic renormalization unambiguously yields a vanishing electric charge for AdS_5, which leads to an immediate contradiction with the field theory result. In fact this mismatch holds much more generally than just for AdS_5 space. For instance, in [64] a family of five-dimensional supergravity solutions was constructed, where the conformal boundary comprises a squashed S^3, and it was found that no choice of standard holographic counterterms correctly reproduced the supersymmetric partition function and the corresponding VEV of the energy (the supersymmetric Casimir energy). Our general results summarized above explain all these discrepancies, and moreover the new counterterms we have introduced solve *all* of these issues. In fact we go further, and show that for a general class of solutions satisfying certain topological assumptions (which may be argued to be required for the solution to correspond to the vacuum state of the dual SCFT), our holographically renormalized VEVs of conserved charges quantitatively reproduce the expected field theory results.

The rest of the chapter is organized as follows. In section 4.2 we review the relevant field theory backgrounds and the properties of supersymmetric partition functions. In section 4.3 we present our four-dimensional supergravity analysis, showing in particular that standard holographic renormalization does satisfy the supersymmetric Ward identities, and evaluating the on-shell action for a large class of self-dual solutions. In section 4.4 we turn to five-dimensional supergravity. We prove that standard holographic renormalization fails to satisfy the supersymmetric Ward identities and we introduce the new boundary terms curing this issue. Then under some global assumptions we evaluate the renormalized on-shell action and compute the conserved charges, showing that they satisfy a BPS condition. Section 4.5 discusses a number of examples in five dimensions, illustrating further the rôle of our new boundary terms and making contact with the existing literature. In section 4.6, we conclude and consider some of the questions raised by this work. Finally, appendix B.1 illustrates our construction of the five-dimensional perturbative solution, and appendix B.2 discusses the Killing spinors at the boundary.

4.2 Field theory

In this chapter we are interested in the holographic duals to both three-dimensional and four-dimensional supersymmetric field theories, defined on general classes of rigid supersymmetric backgrounds. More precisely, these are three-dimensional $\mathcal{N} = 2$ theories and four-dimensional $\mathcal{N} = 1$ theories, in both cases with a choice of Abelian R-symmetry. For superconformal field theories, relevant for AdS/CFT, this R-symmetry will be the superconformal R-symmetry. Whilst in the previous chapters we considered topologically twisted theory that can be formulated on any Riemannian manifold, putting such theories on curved backgrounds in a way that preserves supersymmetry requires particular geometric structures. As reviewed in sections 1.2.2, there are two general approaches: one can either couple the field theory to supergravity, and take a rigid limit in which the supergravity multiplet becomes a set of non-dynamical background fields; or take a holographic approach, realizing the background geometry as the conformal boundary of a holographic dual supergravity theory [97, 145, 90, 66]. In the case at hand, both lead to the same results, although the holographic approach will be particularly relevant for this chapter.

We will focus on backgrounds admitting two supercharges of opposite R-charge. The resulting geometric structures in three and four dimensions are very closely related, and this will allow us to treat some aspects in parallel. In particular certain objects will appear in both dimensions, and we will use a common notation – the dimension should always be clear from the context.

4.2.1 Three-dimensional backgrounds

The three-dimensional geometries of interest belong to a general class of real supersymmetric backgrounds, admitting two supercharges related to one another by charge conjugation [66]. If ζ denotes the Killing spinor then there is an associated Killing vector

$$\xi = \zeta^\dagger \sigma^i \zeta\, \partial_i = \partial_\psi \, . \tag{4.2.1}$$

In an orthonormal frame here the Clifford algebra generators σ^a may be taken to be the Pauli matrices, where $a = 1, 2, 3$ is an orthonormal frame index. The Killing vector (4.2.1) is nowhere zero, and thus defines a foliation of the three-manifold M_3. This foliation is transversely holomorphic, with transverse local complex coordinate z. In terms of these coordinates the background metric is

$$ds_3^2 = \Omega^2 \left[(d\psi + a)^2 + 4e^w dz d\bar{z} \right] \, . \tag{4.2.2}$$

Here $\Omega = \Omega(z, \bar{z})$ is a conformal factor, which is a global nowhere zero function on M_3, $w = w(z, \bar{z})$ is in general a local transverse function, while $a = a_z(z, \bar{z})dz + \overline{a_z(z, \bar{z})}d\bar{z}$ is a local 1-form. The metric and Riemannian volume form on the two-dimensional leaf space are

$$ds_2^2 = 4e^w dz d\bar{z} \, , \qquad \text{vol}_2 = 2i\, e^w dz \wedge d\bar{z} \, . \tag{4.2.3}$$

Notice that a is not gauge invariant under local diffeomorphisms of ψ. On the other hand the 1-form

$$\eta \equiv d\psi + a \tag{4.2.4}$$

is a global almost contact form on M_3, where the Killing vector $\xi = \partial_\psi$ is the associated *Reeb vector field*. It will be convenient to write

$$\mathrm{d}\eta = \mathrm{d}a = \mathrm{i}\, u\, \mathrm{e}^{w} \mathrm{d}z \wedge \mathrm{d}\bar{z} , \tag{4.2.5}$$

where $u = u(z, \bar{z})$ is a global function that parametrizes the gauge-invariant data in a.

Since we are mainly interested in conformal theories with gravity duals, we will (without loss of generality) henceforth set the conformal factor $\Omega \equiv 1$. With this choice, the non-dynamical R-symmetry gauge field that couples to the R-symmetry current is

$$A = \frac{u}{4}(\mathrm{d}\psi + a) + \frac{\mathrm{i}}{4}(\partial_{\bar{z}} w \mathrm{d}\bar{z} - \partial_z w \mathrm{d}z) + \gamma\, \mathrm{d}\psi + \mathrm{d}\lambda . \tag{4.2.6}$$

Notice this is determined entirely by the metric data in (4.2.2), apart from the last two terms which are locally pure gauge. Here $\lambda = \lambda(z, \bar{z})$, and the constant γ will play a particularly important rôle in this chapter.[4]

4.2.2 Four-dimensional backgrounds

There is a related class of rigid four-dimensional supersymmetric backgrounds, first discussed in [145, 90]. These again have two supercharges of opposite R-charge, with corresponding Killing spinors $\zeta_\pm$. We use the spinor conventions of [90, 19], in which the positive/negative chirality $\zeta_\pm$ are two-component spinors with corresponding Clifford algebra generated by $(\sigma_\pm)^a = (\pm\boldsymbol{\sigma}, -\mathrm{i}\mathbb{1}_2)$, where $a = 1, \ldots, 4$ is an orthonormal frame index and $\boldsymbol{\sigma} = (\sigma^1, \sigma^2, \sigma^3)$ are the Pauli matrices. In particular the generators of $SU(2)_{\ell/r} \subset \mathrm{Spin}(4) \cong SU(2)_\ell \times SU(2)_r$ are $(\sigma_\pm)^{ab} = \frac{1}{4}\left(\sigma_\pm^a \sigma_\mp^b - \sigma_\pm^b \sigma_\mp^a\right)$. As in (4.2.1) we may define the vector field

$$K = \zeta_+ \sigma_+^i \zeta_- \, \partial_i . \tag{4.2.7}$$

This is a *complex* Killing vector, satisfying $K^i K_i = 0$. Following [19, 176], and to parallel the three-dimensional discussion in section 4.2.1, we consider a restricted class of these

[4] Compared to the conventions of [94, 95], we have reversed the overall sign of A. However, as noted in the first of these references, for real A sending $A \to -A$ is a symmetry of the Killing spinor equation, provided one also charge conjugates the spinor $\zeta \to \zeta^c$. This $\mathbb{Z}_2$ symmetry also reverses the sign of the Killing vector (4.2.1).

backgrounds in which the metric on M_4 takes the product form

$$\mathrm{d}s_4^2 \quad = \quad \mathrm{d}\tau^2 + (\mathrm{d}\psi + a)^2 + 4\mathrm{e}^w \mathrm{d}z\mathrm{d}\bar{z} \ . \qquad (4.2.8)$$

Thus $M_4 \cong S^1 \times M_3$, where $\tau \in [0, \beta)$ parametrizes the circle $S^1 = S^1_\beta$. More generally one can also introduce an overall conformal factor $\Omega = \Omega(z, \bar{z})$, as in (4.2.2), and the τ direction may be fibred over M_3, as we will discuss later in section 4.4.5. The complex Killing vector (4.2.7) takes the form

$$K \quad = \quad \frac{1}{2}(\xi - \mathrm{i}\partial_\tau) \ , \qquad (4.2.9)$$

where again $\xi = \partial_\psi$. The induced geometry on M_3, on a constant Euclidean time slice $\tau =$ constant, is identical to that for rigid supersymmetry in three dimensions. Moreover, the non-dynamical R-symmetry gauge field is

$$A \ = \ \frac{u}{4}(\mathrm{d}\psi + a) + \frac{\mathrm{i}}{4}(\partial_{\bar{z}} w \mathrm{d}\bar{z} - \partial_z w \mathrm{d}z) + \gamma \, \mathrm{d}\psi + \mathrm{d}\lambda + \frac{\mathrm{i}}{8} u \, \mathrm{d}\tau - \mathrm{i}\gamma' \mathrm{d}\tau \ . \qquad (4.2.10)$$

We stress that this is the gauge field of background *conformal* supergravity, rather than the gauge field of new minimal supergravity [5, 205] used in [90]. The former arises as the restriction of the bulk graviphoton to the conformal boundary in the holographic approach to rigid supersymmetry [145, 61]. Notice that setting $\tau =$ constant, (4.2.10) reduces to the three-dimensional gauge field (4.2.6). The last term in (4.2.10), proportional to the (real) constant γ', is again locally pure gauge, although via a complex gauge transformation. In contrast to three dimensions here A is generically complex, although after a Wick rotation $\tau = \mathrm{i}t$ to Lorentzian signature it becomes real.

The geometry we have described above is *ambi-Hermitian*: the two Killing spinors $\zeta_\pm$ equip M_4 with two commuting integrable complex structures

$$(I_\pm)^i{}_j \quad = \quad -\frac{2\mathrm{i}}{|\zeta_\pm|^2} \, \zeta_\pm^\dagger (\sigma_\pm)^i{}_j \, \zeta_\pm \ . \qquad (4.2.11)$$

The metric (4.2.8) is Hermitian with respect to both of these, but where the induced orientations are opposite. The complex Killing vector (4.2.7) has Hodge type $(0, 1)$ with respect to both complex structures. On the other hand, the local 1-form $\mathrm{d}z$ has Hodge type

$(1,0)$ with respect to I_+, but Hodge type $(0,1)$ with respect to I_-.

4.2.3 Examples

In both cases the geometry involves a three-manifold M_3, equipped with a transversely holomorphic foliation generated by the real Killing vector $\xi = \partial_\psi$. Any such three-manifold, with any compatible metric of the form (4.2.2), defines a rigid supersymmetric background in both three and four dimensions. If all its orbits close ξ generates a $U(1)$ isometry, and the quotient space $\Sigma_2 = M_3/U(1)$ is an orbifold Riemann surface, with induced metric (4.2.3). Such three-manifolds are classified, and are known as *Seifert fibred three-manifolds*. If ξ has a non-closed orbit then M_3 admits at least a $U(1)^2$ isometry, meaning that the transverse metric $\mathrm{d}s_2^2$ also admits a Killing vector.

The simplest example has $M_3 \cong S^3$, with ξ generating the Hopf fibration of the round metric on S^3.[5] In this case $\Sigma_2 \cong S^2$, equipped with its round metric. More generally one can think of $S^3 \subset \mathbb{C} \oplus \mathbb{C}$, and take

$$\xi \quad = \quad b_1 \partial_{\varphi_1} + b_2 \partial_{\varphi_2} \, , \tag{4.2.12}$$

where φ_1, φ_2 are standard 2π periodic azimuthal angles on each copy of $\mathbb{C}$. For $b_1 = \pm b_2$ this is again the Hopf action on S^3, but for b_1/b_2 irrational the flow of ξ is irregular, with generically non-closed orbits. In this case ψ and $\arg z$ are not good global coordinates on the three-sphere. It is straightforward to write down the general form of a compatible smooth metric in this case, of the form (4.2.2) – see [19]. From the perspective of complex geometry, these manifolds with $S^1 \times S^3$ topology (and largely arbitrary Hermitian metric) are primary *Hopf surfaces*.

A large and interesting class of examples are given by links of weighted homogeneous hypersurface singularities. Here one begins with $\mathbb{C}^3$ with a weighted $\mathbb{C}^*$ action $(Z_1, Z_2, Z_3) \to (q^{w_1} Z_1, q^{w_2} Z_2, q^{w_3} Z_3)$, where $w_i \in \mathbb{N}$ are the weights, $i = 1, 2, 3$, and $q \in \mathbb{C}^*$. The hypersurface is the zero set

$$X \;=\; \{f \;=\; 0\} \;\subset\; \mathbb{C}^3 \, , \tag{4.2.13}$$

[5] Throughout the chapter, the symbol $\cong$ means “diffeomorphic to”. In general, $M_d \cong S^d$ does not imply that the metric is the round metric on S^d; we will always specify when this is the case.

where $f = f(Z_1, Z_2, Z_3)$ is a polynomial satisfying

$$f(q^{w_1} Z_1, q^{w_2} Z_2, q^{w_3} Z_3) \quad = \quad q^d f(Z_1, Z_2, Z_3) \;, \qquad (4.2.14)$$

where $d \in \mathbb{N}$ is the degree. For appropriate choices of f the *link*

$$M_3 \quad = \quad X \cap \{|Z_1|^2 + |Z_2|^2 + |Z_3|^2 \;=\; 1\} \qquad (4.2.15)$$

is a smooth three-manifold. Moreover, the weighted $\mathbb{C}^*$ action induces a $U(1)$ isometry of the metric (induced from the flat metric on $\mathbb{C}^3$), and the associated Killing vector ξ naturally defines a transversely holomorphic foliation of M_3. Here $\Sigma_2 = M_3/U(1)$ is the orbifold Riemann surface given by $\{f = 0\}$ in the corresponding weighted projective space $\mathbb{WCP}^2_{[w_1,w_2,w_3]}$. This construction covers all spherical three-manifolds S^3/Γ_{ADE}, but also many three-manifolds with infinite fundamental group. One can further generalise this construction by considering links of complete intersections, i.e. realizing X as the zero set of m weighted homogeneous polynomials in $\mathbb{C}^{2+m}$.

4.2.4 A global restriction

If we take the product $X_0 \equiv \mathbb{R}_{>0} \times M_3$, then we may pair the Reeb vector ξ with a radial vector $r\partial_r$, where r is the standard coordinate on $\mathbb{R}_{>0}$. Notice this is particularly natural in four dimensions, where we may identify $\tau = \log r$, with $X_0 = \mathbb{R}_{>0} \times M_3$ being a covering space for $M_4 = S^1 \times M_3$. Then X_0 is naturally a complex manifold, with the complex vector field $\xi - \mathrm{i} r\partial_r$ being of Hodge type $(0,1)$. In fact X_0 may be equipped with either the I_+ or the I_- complex structure, with the former more natural in the sense that z is a local holomorphic coordinate with respect to I_+. In the following we hence take the I_+ complex structure.

The examples in section 4.2.3 all share a common feature: in these cases the complex surface X_0 admits a global holomorphic $(2,0)$-form. That is, its canonical bundle $\mathcal{K}$ is (holomorphically) trivial. This is obvious for S^3, where $X_0 \cong \mathbb{C}^2 \setminus \{0\}$, while for links of homogeneous hypersurface singularities X we may identify $X_0 = X \setminus \{o\}$, where the isolated singular point o is at the origin $\{Z_1 = Z_2 = Z_3 = 0\}$ of $\mathbb{C}^3$. In this case the holomorphic $(2,0)$-form is $\Psi = \mathrm{d}Z_1 \wedge \mathrm{d}Z_2/(\partial f/\partial Z_3)$ in a patch where $\partial f/\partial Z_3$ is nowhere zero. One can easily check that Ψ patches together to give a smooth holomorphic volume form on X_0. Such singularities X are called *Gorenstein*.

As shown in [176], the 1-form A in (4.2.6) is (in our sign conventions) a connection on $\mathcal{K}^{1/2}$. It follows that when the canonical bundle of X_0 is trivial A may be taken to be a *global* 1-form (this is true on M_3 or on $M_4 \cong S^1 \times M_3$). This global restriction on A will play an important rôle in certain computations later. For example, the computation of the supersymmetric Casimir energy in [176] requires this additional restriction on $M_4 \cong S^1 \times M_3$, and the same condition will also be needed in our evaluations of the renormalized gravitational actions in four and five dimensions. That said, other computations will not require this restriction, and we shall always make clear when we need the global restriction of this section, and when not.

As explained in [176], when the canonical bundle of X_0 is trivial the constant γ in (4.2.6), (4.2.10) may be identified with $\frac{1}{2}$ the charge of the holomorphic $(2,0)$-form Ψ under the Reeb vector ξ. Thus for example we have

$$\gamma = \begin{cases} \frac{1}{2}(b_1+b_2)\ , & S^3 \text{ with Reeb vector } \xi = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} \\ \frac{1}{2}b(-d+\sum_{i=1}^3 w_i)\ , & M_3 = \text{link of weighted homogeneous} \\ & \qquad \text{hypersurface singularity, } \xi = b\chi\ . \end{cases} \tag{4.2.16}$$

Here in the second example the normalized generator of the $U(1) \subset \mathbb{C}^*$ action for the link has been denoted by χ, and b is an arbitrary scale factor. The local function $\lambda(z,\bar{z})$ in (4.2.6), (4.2.10) is chosen so that A is a global 1-form on M_3. The form of this depends on the choice of transverse coordinate z, and then λ is fixed uniquely up to a shift by a global function on M_3 that is invariant under ξ: this is just a small gauge transformation of A. Finally, on $M_4 \cong S^1 \times M_3$ the constant γ' is fixed by requiring the Killing spinors $\zeta_\pm$ to be invariant under ∂_τ. This is necessary in order that the Killing spinors survive the compactification of $\mathbb{R} \times M_3$ to $S^1 \times M_3$. In fact as we show in appendix B.2 this sets $\gamma' = 0$, but it will be convenient to keep this constant since the more general background with S^1 fibred over M_3 we will discuss in section 4.4.5 will require $\gamma' \neq 0$.

In order to compute the four- and five-dimensional on-shell supergravity actions later in the chapter, we will also need some further expressions for the constant γ. Since we may always approximate an irregular Reeb vector field (with generically non-closed orbits) by a quasi-regular Reeb vector field (where all orbits close), there is no essential loss of generality in assuming that ξ generates a $U(1)$ isometry of M_3. Equivalently, M_3 is the total space of a $U(1)$ principal orbibundle over an orbifold Riemann surface Σ_2 with metric (4.2.3) (which is smooth where $U(1)$ acts freely on M_3). Since the orbits of $\xi = \partial_\psi$ close, for a generic orbit we may write $\psi \sim \psi + 2\pi/b$, with $b \in \mathbb{R}_{>0}$ a constant. This allows us to write the following relation between the almost contact volume and characteristic class

$$\frac{b^2}{(2\pi)^2} \int_{M_3} \eta \wedge \mathrm{d}\eta \;\; = \;\; \int_{\Sigma_2} c_1(\mathcal{L}) \ , \tag{4.2.17}$$

where $c_1(\mathcal{L}) \in H^2(\Sigma_2, \mathbb{Q})$ is the first Chern class of $\mathcal{L}$, the orbifold line bundle associated to $S^1 \hookrightarrow M_3 \to \Sigma_2$. If the $U(1)$ action generated by ξ is free, then Σ_2 is a smooth Riemann surface and the right hand side of (4.2.17) is an integer; more generally it is a rational number. Analogously, by definition the first Chern class of Σ_2 is the first Chern class of its anti-canonical bundle, which integrates to

$$\int_{\Sigma_2} c_1(\Sigma_2) \;\equiv\; \int_{\Sigma_2} c_1\left(\mathcal{K}_{\Sigma_2}^{-1}\right) \;\; = \;\; \frac{1}{4\pi} \int_{\Sigma_2} R_{2d} \, \mathrm{vol}_2 \ . \tag{4.2.18}$$

Here $R_{2d} = -\Box w$ is the scalar curvature of the metric (4.2.3) on Σ_2, expressed in terms of the two-dimensional Laplace operator $\Box \equiv \mathrm{e}^{-w} \partial^2_{z\bar{z}}$ (we are using the notation $\partial^2_{z\bar{z}} \equiv \partial_z \partial_{\bar{z}}$). Equivalenty we may write this as an integral over M_3:

$$\int_{\Sigma_2} c_1(\Sigma_2) \; = \; \frac{b}{8\pi^2} \int_{M_3} R_{2d} \, \eta \wedge \mathrm{vol}_2 \ . \tag{4.2.19}$$

Given these preliminary formulas, we next claim that the expression (4.2.6) for A describes a globally defined 1-form on M_3 if and only if γ is given by

$$\gamma \;\; = \;\; -\frac{b}{2} \frac{\int_{\Sigma_2} c_1(\Sigma_2)}{\int_{\Sigma_2} c_1(\mathcal{L})} \; = \; -\frac{1}{4} \frac{\int_{M_3} R_{2d} \, \eta \wedge \mathrm{vol}_2}{\int_{M_3} \eta \wedge \mathrm{d}\eta} \ . \tag{4.2.20}$$

To see this, recall from our discussion above that $2A$ is a connection on the canonical bundle $\mathcal{K}$ of X_0. The latter is (by assumption) holomorphically trivial, with global holomorphic section a $(2,0)$-form Ψ. It follows that 2γ may be identified with the charge of Ψ under the Reeb vector $\xi = \partial_\psi$ [176]. On the other hand, Ψ in turn may be constructed as a section of the canonical bundle $\mathcal{K}_{\Sigma_2}$ of Σ_2, tensored with a section of some power of $\mathcal{L}^*$, say $(\mathcal{L}^*)^p$, where $\mathcal{L}^*$ is the bundle dual to $\mathcal{L}$. The former must be dual line bundles in order that Ψ is globally defined as a form, meaning that

$$p\, c_1(\mathcal{L}^*) \;=\; -c_1(\mathcal{K}_{\Sigma_2}) \;=\; c_1(\Sigma_2)\,. \tag{4.2.21}$$

Since $\exp(b\,\mathrm{i}\psi)$ is a section of $\mathcal{L}$, which has charge b under $\xi = \partial_\psi$, and $c_1(\mathcal{L}^*) = -c_1(\mathcal{L})$, this means that the charge of Ψ is fixed to be

$$2\gamma \;=\; b\,p \;=\; -b\,\frac{\int_{\Sigma_2} c_1(\Sigma_2)}{\int_{\Sigma_2} c_1(\mathcal{L})}\,. \tag{4.2.22}$$

Rearranging gives (4.2.20). We stress again that although we have derived (4.2.20) for quasi-regular Reeb vector fields, by continuity the expression for γ given by the first equality holds also in the irregular case.

These Seifert invariants are readily computed for particular examples. For example, in section 4.2.3 we considered $M_3 \cong S^3$ with Reeb vector $\xi = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$, where φ_1, φ_2 are standard 2π periodic coordinates. The foliation is quasi-regular when $b_1/b_2 = p/q \in \mathbb{Q}$ is rational. Taking $p, q \in \mathbb{N}$ with no common factor, we have the so-called "spindle" $\Sigma_2 = S^3/U(1)_{p,q} \cong \mathbb{WCP}^1_{[p,q]}$. This weighted projective space is topologically a two-sphere, but with orbifold singularities with cone angles $2\pi/p$ and $2\pi/q$ at the north and south poles, respectively. Recalling that $\mathcal{L}$ is the line bundle associated to $S^1 \hookrightarrow S^3 \to \Sigma_2$, it is straightforward to compute that

$$\int_{\Sigma_2} c_1(\mathcal{L}) \;=\; -\frac{1}{pq}\,, \qquad \int_{\Sigma_2} c_1(\Sigma_2) \;=\; \frac{p+q}{pq}\,. \tag{4.2.23}$$

Similarly, for M_3 a link of a weighted homogeneous hypersurface singularity, described in section 4.2.3, one finds

$$\int_{\Sigma_2} c_1(\mathcal{L}) \;=\; -\frac{d}{w_1 w_2 w_3}\,, \qquad \int_{\Sigma_2} c_1(\Sigma_2) \;=\; \frac{d(-d+\sum_{i=1}^3 w_i)}{w_1 w_2 w_3}\,. \tag{4.2.24}$$

These invariants are also often referred to as the *virtual degree* and *virtual Euler characteristic* of the weighted homogeneous hypersurface singularity, respectively. Notice that (4.2.23) may be derived from (4.2.24) as a special case: we may take weights $(w_1, w_2, w_3) = (p, q, 1)$, together with the polynomial $f(Z_1, Z_2, Z_3) = Z_3$, which has degree $d = 1$. The zero set of f is then $\mathbb{C}^2$, with coordinates Z_1, Z_2, with weighted Reeb vector $\xi = p\partial_{\varphi_1} + q\partial_{\varphi_2}$.

Finally, it is worth pointing out there are interesting examples that are not covered by the restriction we make in this section. In particular setting the connection 1-form $a = 0$ gives a direct product $M_3 \cong S^1 \times \Sigma_2$, but unless $\Sigma_2 \cong T^2$ the canonical bundle of X_0 is non-trivial (being the pull back of the canonical bundle of Σ_2). This rules out $M_3 \cong S^1 \times S^2$, where the Reeb vector rotates the S^1. In this case A is a unit charge Dirac monopole on S^2. Localized gauge theory partition functions on such backgrounds have been computed in [43, 44, 69].

4.2.5 The partition function and supersymmetric Casimir energy

The general results of [68, 67] imply that the supersymmetric partition function of an $\mathcal{N} = 2$ theory on M_3, or an $\mathcal{N} = 1$ theory on $M_4 \cong S^1 \times M_3$, depends on the choice of background only via the transversely holomorphic foliation of M_3. Concretely, this means that the partition function is invariant under deformations $w \to w + \delta w$, $u \to u + \delta u$, where $\delta w(z, \bar{z})$, $\delta u(z, \bar{z})$ are *arbitrary* smooth global functions on M_3, invariant under $\xi = \partial_\psi$. Rigid supersymmetric backgrounds M_4 with a single supercharge ζ are in general Hermitian, and more generally the partition function is insensitive to Hermitian metric deformations and depends on the background only via the complex structure (up to local counterterms and anomalies) [68]. It is important to note that these statements are valid when the new minimal formulation of four-dimensional supergravity [205] (or its three-dimensional analogue) is used to couple the field theory to the curved background. We will refer to these results as *supersymmetric Ward identities*.

The Lagrangians for general vector and chiral multiplets on these backgrounds may be found in the original references cited above. In [68, 67] the strategy is to show that deformations of the background geometry that leave the transversely holomorphic foliation (or more generally in four dimensions the complex structure) fixed are $\mathcal{Q}$-exact. A standard argument then shows that the partition function is invariant under such deformations (up to invariance of the measure).

These general statements are supported by explicit computations of localized partition functions. In three dimensions the simplest case is $M_3 \cong S^3$, with general Reeb vector (4.2.12). This was studied in [11]. The partition function of a general $\mathcal{N} = 2$ gauge theory coupled to arbitrary matter localizes to a matrix model for the scalar in the vector multipet, where this matrix model depends on the background geometry only via b_1, b_2. The large N limit was computed for a broad class of Chern–Simons-matter theories in [172] using saddle point methods. The final result for the free energy $\mathscr{F} = -\log Z$ in the large N limit is

$$\mathscr{F} = \frac{(b_1 + b_2)^2}{4b_1 b_2} \cdot \frac{4\pi^2}{\kappa_4^2} \ . \tag{4.2.25}$$

Here

$$\mathscr{F}_{S^3_{\rm round}} = \frac{4\pi^2}{\kappa_4^2} \tag{4.2.26}$$

is the free energy on the round S^3, which scales as $N^{3/2}$ [88], where κ_4^2 is the four-dimensional effective coupling constant of the gravity dual. The partition function has also been computed on (round) Lens spaces $S^3/\mathbb{Z}_p$ in [42, 9]. Here the partition function localizes onto flat gauge connections, and thus splits into a sum over topological sectors. However, in the large N limit of the ABJM theory studied in [9] it was shown that only certain flat connections contribute, all giving the same contribution as the trivial flat connection. The upshot is that the large N free energy is simply $\frac{1}{p}$ times the free energy on S^3. As far as the author is aware, there are no explicit results for the partition function, or its large N limit, on more general links of homogeneous hypersurface singularities. However, it is tempting to conjecture that for appropriate classes of theories with large N gravity duals, the large N free energy may be computed from the sector with trivial gauge connection. The one-loop determinants here should be relatively straightforward to compute, in contrast to the full partition function which localizes onto solutions of the Bogomol'nyi equation, i.e. flat connections (on a closed three-manifold).

The partition function for general $\mathcal{N} = 1$ theories with an R-symmetry, defined on Hopf surfaces $M_4 \cong S^1 \times S^3$, was computed using localization in [19] (the chiral multiplet was also studied in [70]). With two supercharges of opposite R-charge one localizes onto flat gauge connections, which on $S^1 \times S^3$ amount to a constant component of the dynamical gauge field along S^1. The resulting matrix model is similar to that in three dimensions, albeit with additional modes along S^1, and indeed in [19] the results of [11] were used. Besides checking explicitly that the supersymmetric partition function depends on the transversely holomorphic foliation defined by the Reeb vector (4.2.12) on $M_3 \cong S^3$ and not on the choice of Hermitian metric on the Hopf surface, the main result of [19] was that the partition function factorizes as

$$Z_{S^1_\beta \times S^3} \quad = \quad \mathrm{e}^{-\beta E_{\mathrm{susy}}} \cdot \mathcal{I} \,, \tag{4.2.27}$$

where $\mathcal{I}$ is the supersymmetric index originally defined in [199, 144] and

$$E_{\mathrm{susy}} \quad = \quad \frac{2}{27}\frac{(b_1+b_2)^3}{b_1 b_2}(3\mathtt{c}-2\mathtt{a}) + \frac{2}{3}(b_1+b_2)(\mathtt{a}-\mathtt{c}) \tag{4.2.28}$$

was dubbed the *supersymmetric Casimir energy*. Here, $\mathtt{a}$ and $\mathtt{c}$ are the usual trace anomaly coefficients for a four-dimensional SCFT; more generally, for a supersymmetric theory with a choice of R-symmetry one should replace $\mathtt{a}$ and $\mathtt{c}$ in (4.2.28) by the corresponding 't Hooft anomaly formulae, involving traces over the R-charges of fermions. This result has been argued to be scheme-independent, provided one uses a supersymmetric regularization scheme, hence E_{susy} is an intrinsic observable [20, 18]. One can see that E_{susy} corresponds to a Casimir energy by showing that it is the vacuum expectation value of the Hamiltonian generating translations along the Euclidean time, in the limit $\beta \to \infty$ [161, 18].

For field theories admitting a large N gravity dual in type IIB supergravity, to leading order in the large N limit one has $\mathtt{a} = \mathtt{c} = \pi^2/\kappa_5^2$, where κ_5^2 is the five-dimensional gravitational coupling constant and we have set the AdS radius to 1. Moreover, one can see that the index $\mathcal{I}$ does not contribute at leading order [144]. Then at large N the field theory partition function reduces to

$$-\frac{1}{\beta}\log Z_{S^1_\beta \times S^3} \;=\; E_{\mathrm{susy}} \;=\; \frac{2(b_1+b_2)^3}{27 b_1 b_2}\frac{\pi^2}{\kappa_5^2}\,. \tag{4.2.29}$$

The right hand side is expressed in terms of the five-dimensional gravitational coupling constant, and one of our aims will be to reproduce this formula from a dual supergravity computation. For the locally conformally flat $S^1_\beta \times S^3_{r_3}$, where $M_3 \cong S^3_{r_3}$ is equipped with the standard round metric of radius r_3, we have $b_1 = b_2 = 1/r_3$, leading to

$$-\frac{1}{\beta}\log Z_{S^1_\beta \times S^3_{r_3}} \;=\; E_{\mathrm{susy}, S^1_\beta \times S^3_{r_3}} \;=\; \frac{16}{27 r_3}\frac{\pi^2}{\kappa_5^2}\,. \tag{4.2.30}$$

Following [18, 161], in [176] the supersymmetric Casimir energy was studied on the more general class of $M_4 \cong S^1_\beta \times M_3$ backgrounds, by reducing to a supersymmetric quantum mechanics.[6] The short multiplets that contribute to E_{susy} were shown to be in 1-1 correspondence with holomorphic functions on $X_0 \cong \mathbb{R}_{>0} \times M_3$, with their contribution being determined by the charge under the Reeb vector ξ. This makes it manifest that E_{susy} depends on the background only via the choice of transversely holomorphic foliation on M_3. From this it follows that E_{susy} may be computed from an index-character that counts holomorphic functions on X_0 according to their Reeb charge. Again, more precisely this is true in the sector with trivial flat gauge connection, while more generally one should look at holomorphic sections of the corresponding flat holomorphic vector bundles. In any case, in the sector with trivial flat connection on M_3 one can use this result to show that for links of homogeneous hypersurface singularities

$$E_{\text{susy}} = \frac{2b}{27}\frac{d\,c_1^3}{w_1 w_2 w_3}(3\mathrm{c} - 2\mathrm{a}) + \frac{b}{3}\frac{d\,c_1}{w_1 w_2 w_3}(c_1^2 - c_2)(\mathrm{a} - \mathrm{c}) \ . \qquad (4.2.31)$$

Here we have defined

$$c_1 = -d + \sum_{i=1}^{3} w_i \ , \qquad c_2 = -d^2 + \sum_{i=1}^{3} w_i^2 \ . \qquad (4.2.32)$$

In particular, c_1 is precisely the charge of the holomorphic $(2,0)$-form under the generator χ of the $U(1)$ action. Equivalently, this is the orbifold first Chern number of the orbifold anti-canonical bundle of the orbifold Riemann surface $\Sigma_2 = M_3/U(1)$, which is an integer version of the second invariant in (4.2.24). Again, for theories with a large N gravity dual, in the large N limit this becomes

$$E_{\text{susy}} = \frac{2b}{27}\frac{d\,c_1^3}{w_1 w_2 w_3}\frac{\pi^2}{\kappa_5^2} \ . \qquad (4.2.33)$$

Assuming that the dominant contribution comes from this sector with trivial flat connection, (4.2.33) is hence the prediction for the gravity dual.

An aim of this chapter will be to reproduce these field theory results holographically from supergravity.

[6] Other methods to extract the supersymmetric Casimir energy on Hopf surfaces use equivariant integration of anomaly polynomials [53] or exploit properties of the supersymmetric index [17, 59]. See also [185] for localization on backgrounds with more general topologies.

4.3 Four-dimensional supergravity

In this section we are interested in the gravity duals to three-dimensional $\mathcal{N} = 2$ field theories on the backgrounds M_3 described in section 4.2.1. The gravity solutions are constructed in $\mathcal{N} = 2$ gauged supergravity in four dimensions. The general form of (real) Euclidean supersymmetric solutions to this theory was studied in [91]. In particular they admit a Killing vector, which for asymptotically locally Euclidean AdS solutions restricts on the conformal boundary M_3 to the Killing vector ξ defined in (4.2.1). Indeed, we will see that the conformal boundary of a general supersymmetric supergravity solution is equipped with the same geometric structure described in section 4.2.1. We show that the renormalized on-shell supergravity action, regularized according to standard holographic renormalization, depends on the boundary geometric data only via the transversely holomorphic foliation, thus agreeing with the general field theory result summarized in section 4.2.5. Moreover, for *self-dual* supergravity solutions we show that the holographic free energy correctly reproduces the localized field theory results (in the cases where these are available) described in section 4.2.5. We thus find very general agreement between large N localized field theory calculations, on general supersymmetric backgrounds M_3, and dual supergravity computations.

4.3.1 Supersymmetry equations

The Euclidean action for the bosonic sector of four-dimensional $\mathcal{N} = 2$ gauged supergravity [99] is

$$I = -\frac{1}{2\kappa_4^2}\int d^4x\sqrt{G}\left(R + 6 - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\right) . \qquad (4.3.1)$$

Here $R = R(G)$ is the Ricci scalar of the four-dimensional metric $G_{\mu\nu}$, $\mathcal{F} = d\mathcal{A}$ is the field strength of the Abelian graviphoton $\mathcal{A}$, and the cosmological constant has been normalized to $\Lambda = -3$. The equations of motion are

$$\begin{aligned} R_{\mu\nu} + 3G_{\mu\nu} &= 2\left(\mathcal{F}_\mu{}^\rho\mathcal{F}_{\nu\rho} - \frac{1}{4}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}G_{\mu\nu}\right) , \\ d *_4 \mathcal{F} &= 0 . \end{aligned} \qquad (4.3.2)$$

A supergravity solution is supersymmetric if it admits a non-trivial Dirac spinor ϵ satisfying the Killing spinor equation

$$\left(\nabla_\mu + \frac{\mathrm{i}}{4}\mathcal{F}_{\nu\rho}\Gamma^{\nu\rho}\Gamma_\mu + \frac{1}{2}\Gamma_\mu + \mathrm{i}\mathcal{A}_\mu\right)\epsilon = 0\,, \tag{4.3.3}$$

where Γ_μ generate Cliff(4) in an orthonormal frame, so $\{\Gamma_\mu, \Gamma_\nu\} = 2G_{\mu\nu}$. Locally, any such solution can be uplifted to a supersymmetric solution of eleven-dimensional supergravity in a number of ways, as explained in [109]. Strictly speaking the latter reference discusses the Lorentzian signature case, while the corresponding Euclidean signature result was studied in [95]. We also note that there may be global issues in uplifting some solutions, as discussed in detail in [173]. However, these considerations will not affect any of the statements and results in the present chapter.

The general form of real Euclidean supersymmetric solutions to this theory was studied in [91]. There is a canonically defined local coordinate system in which the metric takes the form

$$\mathrm{d}s^2_4 \;=\; \frac{1}{y^2UV}(\mathrm{d}\psi + \phi)^2 + \frac{UV}{y^2}(\mathrm{d}y^2 + 4\mathrm{e}^W\mathrm{d}z\mathrm{d}\bar{z})\,. \tag{4.3.4}$$

Here $\xi = \partial_\psi$ is a Killing vector, arising canonically as a bilinear from supersymmetry, and $W = W(y, z, \bar{z})$, $U = U(y, z, \bar{z})$, $V = V(y, z, \bar{z})$, while ϕ is a local 1-form satisfying $\xi \lrcorner \phi = 0$ and $\mathcal{L}_\xi\phi = 0$. In addition, the following equations should be imposed:

$$U \;=\; 1 - \frac{y}{4}\partial_y W + \frac{f}{2}\,, \tag{4.3.5}$$

$$\partial^2_{z\bar{z}}W + \mathrm{e}^W\left[\partial^2_{yy}W + \frac{1}{4}(\partial_y W)^2 + 3y^{-2}f^2\right] \;=\; 0\,, \tag{4.3.6}$$

$$\begin{aligned}\partial^2_{z\bar{z}}f + \frac{\mathrm{e}^W}{y^2}\Bigg[f\,(f^2+2) - y\left(2\partial_y f + \frac{3}{2}f\partial_y W\right) + \\ + y^2\left(\partial^2_{yy}f + \frac{3}{2}\partial_y W\partial_y f + \frac{3}{2}f\partial^2_{yy}W + \frac{3}{4}f(\partial_y W)^2\right)\Bigg] \;=\; 0\,,\end{aligned} \tag{4.3.7}$$

$$\begin{aligned}\mathrm{d}\phi \;=\; \mathrm{i}\,UV\Bigg[&\partial_z \log\frac{V}{U}\,\mathrm{d}y\wedge\mathrm{d}z - \partial_{\bar{z}}\log\frac{V}{U}\,\mathrm{d}y\wedge\mathrm{d}\bar{z} \\ &+ 2\,\mathrm{e}^W\left(\partial_y\log\frac{V}{U} + \frac{2}{y}(U - V)\right)\mathrm{d}z\wedge\mathrm{d}\bar{z}\Bigg]\,,\end{aligned} \tag{4.3.8}$$

where we have introduced $f \equiv U - V$. The first equation (4.3.5) defines U in terms of W and f, and we could therefore use it to substitute in (4.3.8) and conclude that the entire geometry is fixed by a choice of W and f (apart from a possible gauge transformation/diffeomorphism on ϕ). In deriving this form of the solutions, (4.3.5), (4.3.6) and (4.3.8) follow from imposing the Killing spinor equation (4.3.3), while (4.3.7) is required for the equation of motion for $\mathcal{F}$ (the Maxwell equation) to be satisfied.

The graviphoton is determined by the above geometry, and is given by

$$\mathcal{A} = \frac{1}{2y}\frac{f}{U(U-f)}(d\psi+\phi)+\frac{i}{4}(\partial_{\bar{z}}W d\bar{z}-\partial_z W dz) \ . \qquad (4.3.9)$$

In general this expression is only valid locally, and we will see later that we need to perform a local gauge transformation in order that $\mathcal{A}$ is regular.

A rich subclass of solutions are the *self-dual* solutions, studied in [92, 94]. Here one imposes $\mathcal{F}$ to be anti-self-dual, which together with supersymmetry implies that the metric has anti-self-dual Weyl tensor [92]. We adopt the same abuse of terminology as [94], and refer to these as "self-dual" solutions. This amounts to setting

$$f = \frac{y}{2}\partial_y W \qquad \text{(self-dual case)}. \qquad (4.3.10)$$

This in turn fixes $U \equiv 1$, and therefore self-dual solutions to $\mathcal{N}=2$ gauged supergravity in four dimensions are completely specified by a single function $W = W(y, z, \bar{z})$, which solves (4.3.6). This turns out to be the $SU(\infty)$ Toda equation.[7]

4.3.2 Conformal boundary

In order to apply the gauge/gravity correspondence we require the solutions described in the previous subsection to be asymptotically locally hyperbolic. This is naturally imposed, with the coordinate $1/y$ playing the rôle of the radial coordinate. Indeed, there is then a conformal boundary at $y = 0$, and the metric has the leading asymptotic form $\frac{dy^2}{y^2}+\frac{1}{y^2}ds^2_{M_3}$. More precisely, this all follows if we assume that $W(y, z, \bar{z})$, $f(y, z, \bar{z})$ are analytic functions in y around $y = 0$:[8]

$$\begin{aligned} W(y,z,\bar{z}) &= w_{(0)}(z,\bar{z})+y w_{(1)}(z,\bar{z})+\frac{y^2}{2}w_{(2)}(z,\bar{z})+\mathcal{O}(y^3) \ , \\ f(y,z,\bar{z}) &= f_{(0)}(z,\bar{z})+y f_{(1)}(z,\bar{z})+\frac{y^2}{2}f_{(2)}(z,\bar{z})+\frac{y^3}{6}f_{(3)}(z,\bar{z})+\mathcal{O}(y^4) \ , \end{aligned} \qquad (4.3.11)$$

[7] Of course for self-dual solutions the Maxwell equation is automatic, and indeed one can check that, with (4.3.10) imposed, equation (4.3.7) is implied by the other equations.

[8] Note that this is not true in general. For more details see section 3 of [94].

and the 1-form ϕ can be expanded as

$$\phi(y,z,\bar{z}) \;=\; a_{(0)}(z,\bar{z}) + y a_{(1)}(z,\bar{z}) + \frac{y^2}{2} a_{(2)}(z,\bar{z}) + \mathcal{O}(y^3)\,. \tag{4.3.12}$$

This implies that to leading order

$$ds_4^2 \;=\; [1+\mathcal{O}(y)]\frac{dy^2}{y^2} + y^{-2}[(d\psi + a_{(0)})^2 + 4e^{w_{(0)}} dz d\bar{z} + \mathcal{O}(y)]\,, \tag{4.3.13}$$

confirming that the metric is indeed asymptotically locally hyperbolic around the boundary $\{y = 0\}$. A natural choice of metric (rather than conformal class of metrics) on the boundary is therefore

$$ds^2_{M_3} \;=\; (d\psi + a_{(0)})^2 + 4e^{w_{(0)}} dz d\bar{z}\,. \tag{4.3.14}$$

The boundary 1-form $\eta \equiv d\psi + a_{(0)}$ has exterior derivative

$$d\eta \;=\; 2i\, e^{w_{(0)}} f_{(1)}\; dz \wedge d\bar{z}\,, \tag{4.3.15}$$

as can be seen by expanding (4.3.8) to leading order and using $f_{(0)} = 0$, the latter coming from the leading order term in (4.3.6). More specifically, η is a global almost-contact 1-form and ξ is its Reeb vector field, as

$$\xi \lrcorner \eta \;=\; 1\,, \qquad \xi \lrcorner d\eta \;=\; 0\,. \tag{4.3.16}$$

On the conformal boundary ξ is nowhere vanishing, which implies that it foliates M_3. This Reeb foliation is transversely holomorphic, with locally defined complex coordinate z. The leading term of the expansion of the bulk Abelian graviphoton is

$$A_0 \;\equiv\; \mathcal{A}\,|_{\{y=0\}} = \frac{f_{(1)}}{2}\left(d\psi + a_{(0)}\right) + \frac{i}{4}\left(\partial_{\bar{z}} w_{(0)} d\bar{z} - \partial_z w_{(0)} dz\right), \tag{4.3.17}$$

where as usual this expression is only valid locally, and we are free to perform (local) gauge transformations.

Of course, we see immediately that we recover the rigid supersymmetric geometry of M_3 described in section 4.2.1. More precisely, comparing (4.3.14) and (4.2.2) we identify

$a_{(0)} = a$, $w_{(0)} = w$, with the choice of conformal factor $\Omega = 1$ so that the Killing vector ξ has length 1 (as usual in AdS/CFT, the conformal factor Ω on the boundary appears as a Weyl rescaling of the radial coordinate $y \to \Omega^{-1} y$). Moreover, comparing (4.3.15) and (4.2.5) we see that

$$f_{(1)} \quad = \quad \frac{1}{2} u \, . \tag{4.3.18}$$

Finally, the background R-symmetry gauge field arises as the restriction to the conformal boundary of the bulk Abelian graviphoton, as shown by comparing (4.3.17) and (4.2.6). Thus we identify $A_0 = A$ (up to local gauge transformations).

By expanding (4.3.6), (4.3.7) and (4.3.8) to higher order we obtain the relations

$$w_{(2)} \quad = \quad -\mathrm{e}^{-w_{(0)}} \partial^2_{z\bar{z}} w_{(0)} - 3 f^2_{(1)} - \frac{1}{4} w^2_{(1)} \, , \tag{4.3.19}$$

$$f_{(3)} \quad = \quad -3\mathrm{e}^{-w_{(0)}} \partial^2_{z\bar{z}} f_{(1)} - \frac{9}{4} f_{(1)} \left(w^2_{(1)} + 2 w_{(2)} \right) - 3 f^3_{(1)} - \frac{9}{4} f_{(2)} w_{(1)} \, , \tag{4.3.20}$$

$$\phi_{(2)} \quad = \quad \mathrm{i} \left(\partial_{\bar{z}} f_{(1)} \mathrm{d}\bar{z} - \partial_z f_{(1)} \mathrm{d}z \right) \, . \tag{4.3.21}$$

This (and expansions to higher orders) allows us to see an interesting difference between the self-dual and non-self-dual case. In general a representative of the boundary conformal class is fixed by the choice of two basic functions $w_{(0)} = w$ and $f_{(1)} = u/2$. However, in the general case there are in addition two free functions in the expansion into the bulk, namely $w_{(1)}$ and $f_{(2)}$, that appear in the Taylor expansions of W and f in the inverse radial coordinate y. In general these functions are not determined by the conformal boundary data, but only by regularity of the solution in the deep interior of the bulk solution. However, given $w_{(0)}$, $w_{(1)}$, $f_{(1)}$ and $f_{(2)}$, the series solutions of W and f are then uniquely fixed by the supersymmetry equations/equations of motion. On the other hand, in the self-dual case, instead f and W are related by (4.3.10), so that the coefficients of the power series expansion $f_{(n)}$ and $w_{(n)}$ are related by

$$f_{(n)} \quad = \quad \frac{n}{2} w_{(n)} \qquad \text{(self-dual case)} \, . \tag{4.3.22}$$

Thus the gravitational filling of a given conformal boundary has a unique power series solution with self-dual metric, while there is no such uniqueness in the general case.

4.3.3 Holographic renormalization

The Euclidean supergravity action (4.3.1), with the Gibbons–Hawking–York term added to obtain the equations of motion (4.3.2) on a manifold with boundary, diverges for asymptotically locally hyperbolic solutions. However, we can use (the by now standard) holographic renormalization to remove these divergences.

In order to obtain a finite value for the on-shell action we need to consider a cut off space Y_δ, where the y coordinate extends to $y = \delta$, and add to the regularized action the appropriate local counterterms on the hypersurface $M_\delta = \{y = \delta\}$. One then sends $\delta \to 0$. Explicitly, we write the bulk action as

$$I = I_{\text{grav}} + I_{\text{gauge}} \, , \tag{4.3.23}$$

where

$$I_{\text{grav}} = -\frac{1}{2\kappa_4^2}\int_{Y_\delta} \mathrm{d}^4x\,\sqrt{G}\,(R+6)\;, \qquad I_{\text{gauge}} = \frac{1}{2\kappa_4^2}\int_{Y_\delta} \mathrm{d}^4x\,\sqrt{G}\,\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\,. \tag{4.3.24}$$

As we are considering a manifold with boundary we must add the Gibbons–Hawking–York term to make the variational problem well-defined

$$I_{\text{GHY}} = -\frac{1}{\kappa_4^2}\int_{M_\delta} \mathrm{d}^3x\,\sqrt{h}\,K\,. \tag{4.3.25}$$

Here h is the induced metric on M_δ, and K is the trace of the second fundamental form of M_δ with the induced metric. Finally, we add the counterterms

$$I_{\text{ct}} = \frac{1}{\kappa_4^2}\int_{M_\delta} \mathrm{d}^3x\,\sqrt{h}\left(2 + \frac{1}{2}R\right), \tag{4.3.26}$$

where here R is the scalar curvature of h. These counterterms cancel the power-law divergences in the action. Note the absence of logarithmic terms, which are known to be related to the holographic Weyl anomaly, as the boundary is three-dimensional and therefore there is no conformal anomaly. The on-shell action is the limit of the sum of the four terms above

$$S = \lim_{\delta\to 0}\left(I_{\text{o-s}} + I_{\text{GHY}} + I_{\text{ct}}\right)\,. \tag{4.3.27}$$

The holographic energy-momentum tensor is the quasi-local energy-momentum tensor of the gravity solution; that is, the variation of the on-shell gravitational action with respect to the boundary metric g_{ij}, $i, j = 1, 2, 3$, on M_3:

$$T_{ij} \; = \; \frac{2}{\sqrt{g}} \frac{\delta \mathbb{S}}{\delta g^{ij}} \; . \tag{4.3.28}$$

The holographic energy-momentum tensor can be expressed as a limit of a tensor defined on any surface of constant $y = \delta$. In our case this is

$$T_{ij} \; = \; \frac{1}{\kappa_4^2} \lim_{\delta \to 0} \frac{1}{\delta} \left(-K_{ij} + K\, h_{ij} - 2h_{ij} + R_{ij} - \frac{1}{2} R\, h_{ij} \right) \; , \tag{4.3.29}$$

where the tensors in the bracket are computed on M_δ using h_{ij}, the induced metric. One can define a holographic $U(1)_R$ current in a similar way as

$$j^i \; = \; \frac{1}{\sqrt{g}} \frac{\delta \mathbb{S}}{\delta A_i} \; , \tag{4.3.30}$$

where $A = A_0$ is the boundary R-symmetry gauge field. In three boundary dimensions, this current can be extracted from the expansion of the bulk Abelian graviphoton as

$$\mathcal{A} \; = \; A_0 - \frac{1}{2} \kappa_4^2\, j\, y + \mathcal{O}\left(y^2\right) . \tag{4.3.31}$$

The holographic energy-momentum tensor and R-current are identified with the expectation values of the respective field theory operators in the state dual to the supergravity solution under study.

From the definitions, a variation of the renormalized on-shell action can be expressed as

$$\delta \mathbb{S} \quad = \quad \int_{M_3} \mathrm{d}^3 x \, \sqrt{g} \left(\frac{1}{2} T_{ij} \delta g^{ij} + j^i \delta A_{0\, i} \right) \; . \tag{4.3.32}$$

This formula can be used to check several holographic Ward identities. Invariance of the action under a boundary gauge transformation gives the conservation equation of the holographic R-current

$$\nabla_i j^i \; = \; 0 \; . \tag{4.3.33}$$

Invariance under boundary diffeomorphisms generated by arbitrary vectors on M_3 leads to the conservation equation for the holographic energy-momentum tensor,[9]

$$\nabla^i T_{ij} \;=\; F_{0\,ij} j^i \,, \tag{4.3.34}$$

where $F_0 = \mathrm{d}A_0$. Performing a Weyl transformation at the boundary $\delta g_{ij} = 2g_{ij}\delta\sigma$, $\delta A_0 = 0$, for infinitesimal parameter function σ, we obtain for the trace of the holographic energy-momentum tensor,

$$T_i{}^i \;=\; 0 \,, \tag{4.3.35}$$

consistently with the fact that there is no conformal anomaly in three-dimensional SCFTs.

As reviewed in section 4.2, the field theory supersymmetric Ward identities of [68, 67] imply that the supersymmetric partition function of $\mathcal{N} = 2$ theories on M_3 depends on the background only via the transversely holomorphic foliation of M_3. AdS/CFT thus implies that the holographically renormalized on-shell supergravity action evaluated on a solution with boundary M_3 should also depend on the geometric data of M_3 only through its transversely holomorphic foliation. Concretely, this means that the on-shell action should be invariant under arbitrary deformations $w_{(0)} \to w_{(0)} + \delta w_{(0)}, a_{(0)} \to a_{(0)} + \delta a_{(0)}$, where $\delta w_{(0)}(z,\bar{z})$ is an arbitrary smooth basic global function on M_3, and $\delta a_{(0)}(z,\bar{z})$ is an arbitrary smooth basic global 1-form on M_3. Recall that the Reeb foliation induces a basic cohomology on M_3: a p-form α on M_3 is called *basic* if $\xi \lrcorner \alpha = 0$, $\mathcal{L}_\xi \alpha = 0$, and the set of basic forms $\Omega^\bullet_B$ together with the exterior derivative $\mathrm{d}_B = \mathrm{d}|_{\Omega^\bullet_B}$ constitute the basic de Rham complex.

We may now check this directly by evaluating (4.3.32) for the general class of supersymmetric solutions described in sections 4.3.1, 4.3.2. The holographic R-current is obtained from the subleading term in the expansion (4.3.31), and a computation reveals that this is given by

$$j \;=\; -\frac{1}{2\kappa_4^2}\left[\left(f_{(2)} + f_{(1)} w_{(1)}\right)\eta + \mathrm{d}^c_B w_{(1)}\right] \,. \tag{4.3.36}$$

[9]This is easily seen by recalling that if v^i is the boundary vector generating the diffeomorphism, then $\delta g^{ij} = -2\nabla^{(i}v^{j)}$ and $\delta A_i = v^j \nabla_j A_i + \nabla_i v^j A_j$.

We find that the holographic energy-momentum tensor (4.3.29) evaluates to

$$\begin{aligned} 4\kappa_4^2\, T \;=\; & -\left[2f_{(1)}\left(f_{(2)}+f_{(1)}w_{(1)}\right)+\Box w_{(1)}\right]\eta^2 \\ & +2\left(w_{(1)}\mathrm{d}^c_B f_{(1)}+\mathrm{d}^c_B f_{(2)}\right)\odot\eta+\partial_B w_{(0)}\odot\partial_B w_{(1)}+\bar\partial_B w_{(0)}\odot\bar\partial_B w_{(1)} \\ & +2\mathrm{e}^{w_{(0)}}\left[2f_{(1)}\left(f_{(2)}+f_{(1)}w_{(1)}\right)+\Box w_{(1)}\right]\mathrm{d}z\mathrm{d}\bar z\,, \end{aligned} \tag{4.3.37}$$

where $\odot$ denotes the symmetrized tensor product with weight 1/2. In writing these expressions we have used the almost contact form on M_3, η, the differential operators of the basic cohomology, $\mathrm{d}_B = \partial_B + \bar\partial_B, \mathrm{d}^c_B = \mathrm{i}\left(\bar\partial_B - \partial_B\right)$, and the transverse Laplacian $\Box = \mathrm{e}^{-w_{(0)}}\partial^2_{z\bar z}$.

We next plug these expressions for the holographic energy-momentum tensor and R-current in (4.3.32). We assume that the boundary M_3 is closed, which allows us to use Stokes' theorem to simplify expressions. Moreover the resulting integrand can be simplified by recalling that all functions are basic, as is the deformation $\delta a_{(0)}$. We find that the general variation of the on-shell action is

$$\delta S \;=\; \frac{\mathrm{i}}{2\kappa_4^2}\int_{M_3}\eta\wedge\mathrm{d}_B\left[\left(f_{(2)}+w_{(1)}f_{(1)}\right)\delta a_{(0)}+\frac{1}{2}*_2\left(\delta w_{(0)}\,\mathrm{d}_B w_{(1)}\right)\right]\,. \tag{4.3.38}$$

Notice this *a priori* depends on the non-boundary functions $w_{(1)}$, $f_{(2)}$, which (with the exception of self-dual solutions) are not determined by the boundary data, but only via regularity of the supergravity solution in the deep interior.

However, this expression vanishes because of an analogue of Stokes' theorem, valid for almost contact structures (for instance, it can be found as Lemma 9.1 of [103]). Let X be a $(2m+1)$-dimensional manifold with almost contact 1-form η: if α is a basic $(2m-1)$-form, then

$$\int_X \eta\wedge\mathrm{d}_B\alpha \;=\; 0\,. \tag{4.3.39}$$

The vanishing of the variation of the action $\delta S = 0$ under arbitrary deformations of the background that leave the transversely holomorphic foliation fixed is a very general check of the AdS/CFT relation (1.4.1): it shows that both sides depend on the same data, which *a priori* is far from obvious. Anticipating the (contrasting) results in $\mathrm{AdS}_5/\mathrm{CFT}_4$ we shall obtain later in the chapter, we might also stress that this means that standard holographic renormalization agrees with the supersymmetric renormalization scheme used in the boundary three-dimensional field theory to obtain the results of [68].

In the next section we go further, and show that for a suitable class of solutions the holographically renormalized action reproduces the known field theory results, the latter obtained by supersymmetric localization methods.

4.3.4 Evaluation of the on-shell action

In this section we evaluate the regularized on-shell action (4.3.27) for a class of *self-dual* supersymmetric asymptotically locally hyperbolic solutions. The supergravity equations are simpler in the self-dual case, and moreover the geometry is better understood; there are also more known examples [94]. However, explicit families of non-self-dual supersymmetric solutions are known [173], and it would be interesting to generalise the computations in this section to cover the general case.

As already mentioned the self-dual condition fixes $U \equiv 1$, so that the metric locally takes the form

$$\mathrm{d}s^2 \;=\; \frac{1}{y^2 V}\left(\mathrm{d}\psi+\phi\right)^2+\frac{V}{y^2}\left(\mathrm{d}y^2+4\mathrm{e}^W\mathrm{d}z\mathrm{d}\bar{z}\right) . \qquad (4.3.40)$$

The graviphoton is

$$\mathcal{A} \;=\; \frac{1}{2y}\frac{1-V}{V}\left(\mathrm{d}\psi+\phi\right)+\frac{\mathrm{i}}{4}\left(\partial_{\bar{z}}W\mathrm{d}\bar{z}-\partial_z W\mathrm{d}z\right)+\gamma\,\mathrm{d}\psi+\mathrm{d}\lambda\ , \qquad (4.3.41)$$

where $\lambda = \lambda(y, z, \bar{z})$ is a local basic function. Moreover, the following equations should be imposed

$$\begin{aligned} V &= 1-\frac{1}{2}y\partial_y W\ ,\\ \mathrm{d}\phi &= \mathrm{i}\,\partial_z V\mathrm{d}y\wedge\mathrm{d}z-\mathrm{i}\,\partial_{\bar{z}}V\mathrm{d}y\wedge\mathrm{d}\bar{z}+2\mathrm{i}\,\partial_y\left(V\mathrm{e}^W\right)\mathrm{d}z\wedge\mathrm{d}\bar{z}\ ,\\ 0 &= \partial^2_{z\bar{z}}W+\partial^2_y\mathrm{e}^W\ . \end{aligned} \qquad (4.3.42)$$

Here the first equation may be used to eliminate V in terms of $W = W(y, z, \bar{z})$, the second equation simply fixes $\mathrm{d}\phi$, while the final equation is the $SU(\infty)$ Toda equation. We begin by following part of the global analysis in [94] – the latter reference focused on solutions with $U(1)^2$ isometry and M_4 diffeomorphic to a ball, with conformal boundary $M_3 \cong S^3$, but in fact a number of key arguments go through more generally.

First we recall that the coordinate y may be more invariantly defined as

$$y^2 \;=\; \frac{2}{\|\Xi\|^2}\,, \qquad \text{where} \qquad \Xi \;\equiv\; \frac{1}{2}\left(\mathrm{d}\xi^\flat + *_4\mathrm{d}\xi^\flat\right)_+ \,. \tag{4.3.43}$$

Here the self-dual 2-form Ξ is called a *twistor*, and is constructed from the Killing 1-form $\xi^\flat = (1/y^2V)(\mathrm{d}\psi + \phi)$ dual to the Killing vector $\xi = \partial_\psi$. The conformal boundary is at $y = 0$. Assuming the metric is regular in the interior, the twistor form is then also regular, and thus y is non-zero in the interior. There can potentially be points at which $\|\Xi\| = 0$, where y then diverges, and indeed there are smooth solutions for which this happens. However, this can *only* happen at fixed points of the Killing vector ξ – see the discussion in section 3.4 of [94]. It follows that y is a globally well-defined non-zero function on the interior of $M_4 \setminus \{\xi = 0\}$. These self-dual solutions are also (locally) *conformally Kähler*, with Kähler 2-form

$$\omega \;=\; -y^3\Xi \;=\; \mathrm{d}y \wedge (\mathrm{d}\psi + \phi) + V\mathrm{e}^W\, 2\mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}\,. \tag{4.3.44}$$

It follows from the first equality that ω is also well-defined on the interior of $M_4 \setminus \{\xi = 0\}$. Since $\mathrm{d}y = -\xi \lrcorner \omega$, we see that y is also a Hamiltonian function for ξ, and in particular is a Morse–Bott function. In particular this implies that y has no critical points on $M_4 \setminus \{\xi = 0\}$. We may hence extend the y coordinate from the conformal boundary $y = 0$ up to some $y = y_0 > 0$ in the interior, where on the locus $y = y_0$ the Killing vector ξ has a fixed point (this may include $y_0 = \infty$). Moreover, the preimage of $(0, y_0)$ in M_4 is topologically simply a product, $(0, y_0) \times M_3$, where the Killing vector is tangent to M_3 and has no fixed points.

With these global properties in hand, we can now proceed to compute the regularized on-shell action. We deal with each term in turn. Consider first the gravitational part of the action. Using the equation of motion we may write $R(G) = -12$, so that on-shell

$$I_{\text{grav}} \;=\; \frac{3}{\kappa_4^2}\int_{M_\delta} \mathrm{vol}_4\,, \tag{4.3.45}$$

where the Riemannian volume form is

$$\mathrm{vol}_4 \;=\; \frac{1}{y^4}\mathrm{d}y \wedge (\mathrm{d}\psi + \phi) \wedge V\mathrm{e}^W 2\mathrm{i}\, \mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{4.3.46}$$

We can write this as an exact form

$$-3\mathrm{vol}_4 \;=\; \mathrm{dY}, \tag{4.3.47}$$

with

$$\mathrm{Y} \;=\; \frac{1}{2y^2}(\mathrm{d}\psi+\phi)\wedge\mathrm{d}\phi+\frac{1}{y^3}(\mathrm{d}\psi+\phi)\wedge V\mathrm{e}^W 2\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}\,. \tag{4.3.48}$$

The global arguments above imply that Y is well-defined everywhere on $M_4\setminus\{\xi=0\}$: in the first term y is a global regular function and ξ does not vanish, guaranteeing that $\mathrm{d}\psi+\phi$ is a global 1-form. The second term is simply $1/y^3(\mathrm{d}\psi+\phi)\wedge\omega$, which is also globally well-defined and regular on $M_4\setminus\{\xi=0\}$. Having written the volume form as a globally exact form on $M_4\setminus\{\xi=0\}$, we can then use Stokes' theorem to write (4.3.45) in terms of integrals over the conformal boundary $M_3\cong\{y=\delta\}$, and over the boundary T_ϵ of a small tubular neighbourhood of radius ϵ around the fixed point set of ξ. Using the expansion of the Toda equation (4.3.42) and (4.3.39) near the conformal boundary, we can simplify the resulting expression to

$$I_{\mathrm{grav}} \;=\; \frac{1}{\kappa_4^2}\frac{1}{\delta^3}\int_{M_3}\eta\wedge\mathrm{vol}_2+\frac{3}{4\kappa_4^2}\frac{1}{\delta^2}\int_{M_3}w_{(1)}\,\eta\wedge\mathrm{vol}_2-\frac{1}{\kappa_4^2}\int_{\mathrm{T}_\epsilon}\mathrm{Y}\,. \tag{4.3.49}$$

Here vol_2 is the two-dimensional volume form (4.2.3) (with $w_{(0)}=w$). In general the fixed point set of ξ may have a number of connected components, consisting either of fixed points (NUTs) or fixed two-dimensional surfaces (bolts). More precisely the last term in (4.3.49) is then a sum over connected components, and the integral should be understood as a limit $\lim_{\epsilon\to 0}\int_{\mathrm{T}_\epsilon}$.

The first two divergent terms in (4.3.49) are cancelled by the Gibbons–Hawking–York term and the local counterterms (4.3.26), which in a neighbourhood of infinity become

$$\begin{aligned} I_{\mathrm{GHY}}+I_{\mathrm{ct}} = &-\frac{1}{32\kappa_4^2}\int_{M_3}\left(w_{(1)}^3+4w_{(1)}\Box w_{(0)}\right)\eta\wedge\mathrm{vol}_2-\frac{1}{\kappa_4^2}\frac{1}{\delta^3}\int_{M_3}\eta\wedge\mathrm{vol}_2\\ &-\frac{3}{4\kappa_4^2}\frac{1}{\delta^2}\int_{M_3}w_{(1)}\,\eta\wedge\mathrm{vol}_2\,, \end{aligned} \tag{4.3.50}$$

where again $\Box = \mathrm{e}^{-w_{(0)}}\partial^2_{z\bar{z}}$. Overall, the contribution from gravity is hence

$$I_{\mathrm{grav}} + I_{\mathrm{GHY}} + I_{\mathrm{ct}} \;=\; -\frac{1}{32\kappa_4^2}\int_{M_3}\left(w_{(1)}^3 + 4w_{(1)}\Box w_{(0)}\right)\eta\wedge \mathrm{vol}_2 - \frac{1}{\kappa_4^2}\int_{\mathrm{T}_\epsilon}\Upsilon\,. \tag{4.3.51}$$

Next we turn to the contribution of the gauge field to the on-shell action. Here for the first time in this section we impose the additional global assumption in section 4.2.4: that is, we take $A = A_0 = \mathcal{A}\mid_{y=0}$ to be a global 1-form on the conformal boundary M_3. Equivalently, $M_4\mid_{(0,y_0)}\cong (0,y_0)\times M_3$ is conformally Kähler, and we are imposing that the associated canonical bundle is trivial. If this is true throughout $M_4\setminus\{\xi=0\}$ then $\mathcal{F}=\mathrm{d}\mathcal{A}$ is globally exact on the latter,[10] and we may again use Stokes' theorem to deduce

$$I_{\mathrm{gauge}} \;=\; -\frac{1}{\kappa_4^2}\int_{M_4}\mathcal{F}\wedge\mathcal{F} \;=\; \frac{1}{\kappa_4^2}\int_{M_3}A_0\wedge F_0 - \frac{1}{\kappa_4^2}\int_{\mathrm{T}_\epsilon}\mathcal{A}\wedge\mathcal{F}\,. \tag{4.3.52}$$

In order to further evaluate the first term on the right hand side of (4.3.52), recall that in the self-dual case the boundary gauge field is

$$A_0 \;=\; \frac{1}{4}w_{(1)}\eta + \frac{\mathrm{i}}{4}(\partial_{\bar{z}}w_{(0)}\mathrm{d}\bar{z} - \partial_z w_{(0)}\mathrm{d}z) + \gamma\,\mathrm{d}\psi + \mathrm{d}\lambda\,. \tag{4.3.53}$$

Carefully integrating by parts then leads to

$$\begin{aligned}\frac{1}{\kappa_4^2}\int_{M_3}A_0\wedge F_0 = &-\frac{\gamma}{4\kappa_4^2}\int_{M_3}R_{2d}\,\eta\wedge\mathrm{vol}_2\\ &+\frac{1}{32\kappa_4^2}\int_{M_3}\left(w_{(1)}^3 + 4w_{(1)}\Box w_{(0)}\right)\eta\wedge\mathrm{vol}_2\,.\end{aligned} \tag{4.3.54}$$

Here the first term arises by noting that $R_{2d} = -\Box w_{(0)}$ is the scalar curvature for Σ_2. Notice that the second term perfectly cancels the same term in (4.3.51). In general the total action, obtained by summing (4.3.51) and (4.3.52), is thus

$$\mathrm{S} \;=\; -\frac{\gamma}{4\kappa_4^2}\int_{M_3}R_{2d}\,\eta\wedge\mathrm{vol}_2 - \frac{1}{\kappa_4^2}\int_{\mathrm{T}_\epsilon}(\Upsilon + \mathcal{A}\wedge\mathcal{F})\,. \tag{4.3.55}$$

This hence splits into a term evaluated at the conformal boundary M_3, and an integral around the fixed points of ξ.

We may next further evaluate the first term on the right hand side of (4.3.55) using some

[10] If the canonical bundle is non-trivial in the interior of $M_4\setminus\{\xi=0\}$ there would also be contributions from Dirac strings, but we shall not consider that further here.

of the results of section 4.2.4. As argued there, since we may approximate an irregular Reeb vector field by quasi-regular Reeb vectors, there is no essential loss of generality (for the formulas that follow) in assuming that M_3 is quasi-regular. This means that M_3 is the total space of a circle orbibundle over an orbifold Riemann surface Σ_2, with associated line orbibundle $\mathcal{L}$. Combining equations (4.2.19) and (4.2.20) then allows us to write the action (4.3.55) as

$$S = \frac{\pi^2}{\kappa_4^2}\frac{\left(\int_{\Sigma_2} c_1(\Sigma_2)\right)^2}{\int_{\Sigma_2} c_1(\mathcal{L})} - \frac{1}{\kappa_4^2}\int_{T_\epsilon}(Y + \mathcal{A}\wedge\mathcal{F}) \ . \tag{4.3.56}$$

The contribution of the conformal boundary is now written purely in terms of topological invariants of the Seifert fibration structure of M_3. We will not attempt to evaluate the contributions around the fixed points in (4.3.56) in general – this would take us too far from our main focus. Instead we will follow the computation in [94], where M_4 has the topology of a ball, with a single fixed point at the origin (a NUT). In this case $\mathcal{A}$ is a global 1-form on M_4, and correspondingly $\int_{T_\epsilon}\mathcal{A}\wedge\mathcal{F} = 0$. Similarly, since the Kähler form ω is smooth near the NUT, one can argue that the second term in Y in (4.3.48) does not contribute to the (limit of the) integral in (4.3.56). However, the first term in Y *does* contribute. Using Stokes' theorem we may write this as

$$-\frac{1}{\kappa_4^2}\int_{T_\epsilon} Y = -\frac{1}{\kappa_4^2}\cdot\frac{1}{2y_{\text{NUT}}^2}\int_{M_3}\eta\wedge d\eta \ , \tag{4.3.57}$$

where y_{NUT} is the function y evaluated at the NUT. Since the Reeb vector ξ has norm $\|\xi\| \sim r$ near the NUT, where r denotes geodesic distance from the NUT, one concludes from the form of the metric (4.3.40) that $V \sim r^{-2}$. Since $\xi \lrcorner \mathcal{A}$ is necessarily zero at the NUT in order that $\mathcal{A}$ is smooth there, from (4.3.41) we hence deduce that

$$0 = -\frac{1}{2y_{\text{NUT}}} + \gamma \ , \tag{4.3.58}$$

which allows us to relate y_{NUT} to γ.[11] Thus we may also express the contribution to the

[11] The same formula was derived in [94] using a different, much longer, route. In the latter reference it was concluded that all cases where $b_1/b_2 > 0$, and $b_1/b_2 = -1$, are regular. The case $b_1/b_2 = -1$ is qualitatively different from the former: the NUT is a point at infinity in the conformal Kähler metric, and the Kähler metric is asymptotically locally Euclidean. The instanton is regular at the NUT because it vanishes there, and $V \sim r^2$, so (4.3.58) does not hold. Nevertheless, a careful analysis shows that the action evaluates to (4.3.60).

action from the NUT (4.3.57) purely in terms of topological invariants of M_3:

$$-\frac{1}{\kappa_4^2}\int_{T_\epsilon} \Upsilon \;=\; -\frac{1}{\kappa_4^2}\cdot 2\gamma^2\cdot\frac{(2\pi)^2}{b^2}\int_{\Sigma_2} c_1(\mathcal{L}) \;=\; -\frac{2\pi^2}{\kappa_4^2}\frac{\left(\int_{\Sigma_2} c_1(\Sigma_2)\right)^2}{\int_{\Sigma_2} c_1(\mathcal{L})}\,. \tag{4.3.59}$$

Thus in this case the total action (4.3.56) becomes simply

$$\mathcal{S} \;=\; -\frac{\pi^2}{\kappa_4^2}\frac{\left(\int_{\Sigma_2} c_1(\Sigma_2)\right)^2}{\int_{\Sigma_2} c_1(\mathcal{L})}\,. \tag{4.3.60}$$

Using (4.2.23) we reproduce the result of [94], where recall that $b_1/b_2 = p/q$. However, we can now generalise this further: in the above computation all that we needed was the existence of a supergravity solution with topology $X = C(M_3)$, a real cone over M_3, where the tip of the cone is the only fixed point of ξ, hence a NUT. If M_3 is not diffeomorphic to S^3 this will not be smooth at the NUT, but we can formally consider such singular solutions. The assumptions we made about the behaviour of the metric near to this point are then satisfied if the metric is conical near to the NUT. In this situation all of the above steps are still valid, and we obtain the same formula (4.3.60) for the action.

In general

$$\int_{\Sigma_2} c_1(\Sigma_2) \;=\; 2-2g-n+\sum_{I=1}^{n}\frac{1}{k_I}\,, \tag{4.3.61}$$

where the smooth Riemann surface associated to Σ_2 has genus g, and there are n orbifold points with cone angles $2\pi/k_I$, $k_I \in \mathbb{N}$, $I = 1,\ldots,n$. When the first Chern class above is positive, Σ_2 hence necessarily has genus $g = 0$ and so is topologically S^2. It then follows that $M_3 \cong S^3/\Lambda$, where Λ is a finite group. This shows that the class of weighted homogeneous hypersurface singularities with $-d + \sum_{i=1}^3 w_i > 0$ have links M_3 which are all quotients of S^3 by finite groups. Corresponding supergravity solutions can hence be constructed very simply as quotients by Λ of smooth solutions M_4 with ball topology. The supergravity action should then be $1/|\Lambda|$ times the action for the ball solution. It is simple to check this is indeed the case from the formula (4.3.60). For weighted hypersurface singularities this reads

$$\mathcal{S} \;=\; \frac{4\pi^2}{\kappa_4^2}\frac{d\left(-d+\sum_{i=1}^3 w_i\right)^2}{4w_1w_2w_3}\,. \tag{4.3.62}$$

As summarized in [176], we may construct supersymmetric quotients $M_3 \cong S^3/\Lambda$ where $\Lambda = \Lambda_{\mathrm{ADE}} \subset SU(2)$. These may equivalently be realized as links of ADE hypersurface singularities, and one can check that indeed

$$\frac{4w_1w_2w_3}{d\left(-d+\sum_{i=1}^3 w_i\right)^2} \quad = \quad |\Lambda_{\mathrm{ADE}}| \ . \tag{4.3.63}$$

For example, the E_8 singularity has weights $(w_1, w_2, w_3) = (6, 10, 15)$ and degree $d = 30$, for which the left hand side of (4.3.63) gives $|\Lambda_{E_8}| = 120$, which is the order of the binary icosahedral group.

Our formula for the action (4.3.60) reproduces all known large N field theory results, summarized in section 4.2.5. In particular, we may realize squashed three-spheres, with rational Reeb vector $\xi = b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2}$, where $b_1/b_2 = p/q \in \mathbb{Q}$, as links of hypersurface singularities with weights $(w_1, w_2, w_3) = (p, q, 1)$ and degree $d = 1$, for which (4.3.62) reproduces the field theory result (4.2.25). Similarly, we may realize Lens spaces $L(p, 1) = S^3/\mathbb{Z}_p = S^3/\Lambda_{A_{p-1}}$ as links of A_{p-1} singularities, with weights $(w_1, w_2, w_3) = (2, p, p)$ and degree $d = 2p$. Here $|\Lambda_{A_{p-1}}| = p$, and we reproduce the field theory result of [9] that the large N free energy is simply $\frac{1}{p}$ times the free energy on S^3. The formula (4.3.60) was derived by assuming supergravity solutions with appropriate general properties exist. For more general M_3, and in particular for M_3 with *negative* $c_1(\Sigma_2)$, more work needs to be done to investigate such solutions. We leave this interesting question for future work.

4.4 Five-dimensional supergravity

In the remaining part of the chapter we turn to five-dimensional supergravity. We start by constructing a very general asymptotically locally AdS supersymmetric solution of minimal gauged supergravity, in a perturbative expansion near the conformal boundary. Then we perform holographic renormalization, extract the holographic energy-momentum tensor and R-current and compare with the field theory results reviewed in section 4.2. We will show that standard holographic renormalization violates the field theory supersymmetric Ward identities. However, we will prove that the latter can be restored by introducing new, unconventional boundary terms.

4.4.1 The perturbative solution

Differently from what we did in four-dimensional supergravity, we will initially work in Lorentzian signature $(-,+,+,+,+)$ and discuss an analytic continuation later. In this way we take advantage of the known technology for constructing the solution and postpone the complexification of the supergravity fields.

The bosonic action of minimal gauged supergravity in five dimensions reads [119][12]

$$I = \frac{1}{2\kappa_5^2}\int\left[\mathrm{d}^5x\sqrt{G}\left(R - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu} + 12\right) - \frac{8}{3\sqrt{3}}\mathcal{A}\wedge\mathcal{F}\wedge\mathcal{F}\right]. \tag{4.4.1}$$

Here $R = R(G)$ denotes the Ricci scalar of the five-dimensional metric $G_{\mu\nu}$, $G = |\det G_{\mu\nu}|$, $\mathcal{A}$ is the Abelian graviphoton and $\mathcal{F} = \mathrm{d}\mathcal{A}$. Moreover, κ_5^2 is the five-dimensional gravitational coupling constant, and the cosmological constant has been normalized to $\Lambda = -6$. The Einstein and Maxwell equations read

$$R_{\mu\nu} + 2\mathcal{F}_{\mu\rho}\mathcal{F}^{\rho}{}_{\nu} + G_{\mu\nu}\left(4 + \frac{1}{3}\mathcal{F}_{\rho\sigma}\mathcal{F}^{\rho\sigma}\right) = 0\,, \tag{4.4.2}$$

$$\mathrm{d}*\mathcal{F} + \frac{2}{\sqrt{3}}\mathcal{F}\wedge\mathcal{F} = 0\,. \tag{4.4.3}$$

All solutions of these equations uplift to solutions of type IIB supergravity [60, 109].[13]

A bosonic field configuration is supersymmetric if there exists a non-trivial Dirac spinor ϵ satisfying the generalised Killing spinor equation

$$\left[\nabla_\mu + \frac{\mathrm{i}}{4\sqrt{3}}\left(\Gamma_\mu{}^{\nu\lambda} - 4\delta_\mu^\nu\Gamma^\lambda\right)\mathcal{F}_{\nu\lambda} - \frac{1}{2}\left(\Gamma_\mu - 2\sqrt{3}\,\mathrm{i}\mathcal{A}_\mu\right)\right]\epsilon = 0\,, \tag{4.4.4}$$

where the Γ_μ generate Cliff(1,4), with $\{\Gamma_\mu, \Gamma_\nu\} = 2G_{\mu\nu}$. The conditions for a bosonic supersymmetric solution were worked out in [108] and discussed further in [62]. The solutions relevant to us are those in the timelike class of [108] and are largely determined by a certain four-dimensional Kähler structure. In appendix B.1 we review such conditions and solve them in a perturbative expansion. A suitable ansatz for the Kähler structure eventually yields a metric and a gauge field on the conformal boundary of the five-dimensional solution which, after a Wick rotation, match the field theory Euclidean background fields (4.2.8),

[12] This section is independent of section 4.3. We will thus adopt the same notation for the five-dimensional supergravity fields as for the four-dimensional ones with no risk of confusion.

[13] As for the four-dimensional supergravity solutions discussed in section 4.3, this statement holds locally, see e.g. [62] for some global issues.

(4.2.10). Here we present the final result after having cast it in Fefferman–Graham form, which is most convenient for extracting the holographic data.

The Fefferman–Graham form of the five-dimensional metric is[14]

$$\mathrm{d}s_5^2 \; = \; \frac{\mathrm{d}\rho^2}{\rho^2} + h_{ij}(x,\rho)\mathrm{d}x^i\mathrm{d}x^j \, , \tag{4.4.5}$$

with the induced metric on the hypersurfaces at constant ρ admitting the expansion

$$h(x,\rho) \; = \; \frac{1}{\rho^2}\left[\mathrm{g}^0 + \mathrm{g}^2\rho^2 + \left(\mathrm{g}^4 + \tilde{h}^0\log\rho^2\right)\rho^4 + \mathcal{O}(\rho^5)\right] . \tag{4.4.6}$$

The gauge field is of the form

$$\mathcal{A}(x,\rho) \; = \; A_0 + \left(A_2 + \tilde{A}_2\log\rho^2\right)\rho^2 + \mathcal{O}(\rho^3) \, , \tag{4.4.7}$$

with $\mathcal{A}_\rho = 0$.

The hypersurfaces at constant ρ will be described by coordinates $x^i = \{t, z, \bar{z}, \psi\}$. As discussed in detail in appendix B.1, we find that the solution depends on six arbitrary functions $u(z,\bar{z})$, $w(z,\bar{z})$, $k_1(z,\bar{z})$, $k_2(z,\bar{z})$, $k_3(z,\bar{z})$, $k_4(z,\bar{z})$. The functions u and w control the boundary geometry and will be referred to as the *boundary data*; these are the same functions appearing in the field theory background (4.2.8), (4.2.10). The functions k_1, k_2, k_3, k_4 first show up in the h^0 and A_2 subleading terms of the Fefferman–Graham expansion and will be denoted as the *non-boundary data* of the solution.

The first two terms in the expansion of the induced metric read

$$\begin{aligned} \mathrm{g}^0 \; &= \; -\mathrm{d}t^2 + (\mathrm{d}\psi + a)^2 + 4\mathrm{e}^w\mathrm{d}z\mathrm{d}\bar{z} \, , \\ \mathrm{g}^2 \; &= \; \frac{8\Box w + u^2}{96}\mathrm{d}t^2 - \frac{8\Box w + 7u^2}{96}(\mathrm{d}\psi + a)^2 + \frac{16\Box w + 5u^2}{24}\mathrm{e}^w\mathrm{d}z\mathrm{d}\bar{z} \\ &\quad - \frac{1}{4}(*_2\mathrm{d}u)(\mathrm{d}\psi + a) \, , \end{aligned} \tag{4.4.8}$$

where a satisfies (4.2.5) as in the field theory background. Moreover, $\Box = \mathrm{e}^{-w}\partial^2_{z\bar{z}}$ is the Laplacian of the two-dimensional part of the boundary metric g^0, which coincides with

[14] We use ρ instead of z to denote the Fefferman–Graham coordinate in this section in order to avoid confusion with the z coordinate on Σ_2

(4.2.3), and we are using the notation

$$*_2\mathrm{d} \;=\; \mathrm{i}(\mathrm{d}\bar{z}\,\partial_{\bar{z}} - \mathrm{d}z\,\partial_z)\;. \tag{4.4.9}$$

One can check that g^2 is determined by the Schouten tensor of g^0 [79, 212]

$$\mathrm{g}^2_{ij} \;=\; -\frac{1}{2}\left(R_{ij} - \frac{1}{6}R\,\mathrm{g}^0_{ij}\right)^{(0)}\;. \tag{4.4.10}$$

Here and in the formulae below, a superscript $^{(0)}$ outside the parenthesis reminds the reader that all quantities within the parenthesis are computed using the boundary metric g^0 (and, as far as the formulae below are concerned, the boundary gauge field A_0).

In order to determine the on-shell action and the holographic charges we will also need the $\tilde{h}^0$ and g^4 terms in the Fefferman–Graham expansion (4.4.6). We have explicitly verified that $\tilde{h}^0$ is determined by the boundary data as

$$\tilde{h}^0_{ij} \;=\; -\frac{1}{8}\left(B_{ij} + 8F_{ik}F_j{}^k - 2\mathrm{g}^0_{ij}F_{kl}F^{kl}\right)^{(0)}\,, \tag{4.4.11}$$

where B_{ij} is the Bach tensor, see appendix A.1 for its definition. Recalling that the variation of the integrated Euler density vanishes identically in four dimensions, we can write

$$\tilde{h}^0_{ij} \;=\; \frac{1}{16\sqrt{\mathrm{g}^0}}\frac{\delta}{\delta \mathrm{g}^{0\,ij}}\int \mathrm{d}^4x\sqrt{\mathrm{g}^0}\left(-\mathcal{E} + C_{klmn}C^{klmn} - 8F_{kl}F^{kl}\right)^{(0)}\;. \tag{4.4.12}$$

Hence $\tilde{h}^0_{ij}$ is proportional to the metric variation of the integrated holographic Weyl anomaly,[15] a fact that for vanishing gauge field was first observed in [79].

As for g^4_{ij}, this contains the four non-boundary functions k_1, k_2, k_3, k_4, as well as the boundary functions u, w (hit by up to six derivatives); we will not give its explicit expression here as it is extremely cumbersome. As a sample we provide two simple relations between some of the components:

$$\begin{aligned}\mathrm{g}^4_{tt} - \mathrm{g}^4_{\psi\psi} \;=\; & -k_3 + \frac{1}{6}k_2^2 + \frac{1}{24}\Box k_2 + \frac{1}{24}(2\Box w + u^2)k_2 + \frac{17}{6144}u^4 - \frac{3}{256}\Box u^2 \\ & + \frac{1}{96}\mathrm{e}^{-w}\partial_z u\partial_{\bar{z}}u + \frac{1}{192}\left(u^2\Box w - \frac{5}{2}\Box^2 w - (\Box w)^2\right)\,,\end{aligned} \tag{4.4.13}$$

[15] The functional being varied is also the action of four-dimensional conformal supergravity.

$$\mathrm{g}^4_{tt} + \mathrm{g}^4_{\psi\psi} - 2\mathrm{g}^4_{t\psi} = -\frac{1}{2}uk_1 - \frac{1}{6}u^2k_2 + \frac{1}{128}u^4 + \frac{1}{48}u^2\Box w \,. \tag{4.4.14}$$

We also checked that the trace is determined by boundary data as

$$\mathrm{g}^{0\,ij}\mathrm{g}^4_{ij} = \frac{1}{48}\left(4R_{ij}R^{ij} - R^2\right)^{(0)} \,. \tag{4.4.15}$$

As a consequence of supersymmetry, the gauge field is entirely determined by the metric and does not contain new functions (apart for the gauge choice to be discussed momentarily). In particular, A_0 and $\tilde{A}_2$ just depend on the boundary metric functions, while A_2 also depends on three of the four non-boundary functions, that is k_1, k_2, k_3. The explicit expressions are

$$A_0 = -\frac{1}{\sqrt{3}}\left[-\frac{1}{8}u\,\mathrm{d}t + \frac{1}{4}u(\mathrm{d}\psi + a) + \frac{1}{4}*_2\mathrm{d}w + \mathrm{d}\lambda + \gamma\,\mathrm{d}\psi + \gamma'\mathrm{d}t\right] , \tag{4.4.16}$$

$$\tilde{A}_2 = \frac{1}{32\sqrt{3}}\left[-\Box u\,\mathrm{d}t + \left(2\Box u - u\Box w - \frac{1}{2}u^3\right)(\mathrm{d}\psi + a) + *_2\mathrm{d}\left(2\Box w + u^2\right)\right] , \tag{4.4.17}$$

$$\begin{aligned} A_2 = \frac{1}{64\sqrt{3}}\Bigg[&\left(96k_1 + 32uk_2 - 4u\Box w - \frac{3}{2}u^3\right)\mathrm{d}t - *_2\mathrm{d}\left(32k_2 + u^2\right) \\ &+ \frac{1}{u}\Bigg(128k_3 - 32uk_1 - \frac{64}{3}k_2^2 + 16\Box k_2 - \frac{32}{3}k_2\Box w - 16u^2k_2 + 3\Box(\Box w + u^2) \\ &- 2(\Box w)^2 - \frac{5}{3}u^2\Box w - 3\mathrm{e}^{-w}\partial_z u\partial_{\bar z}u - \frac{5}{12}u^4\Bigg)(\mathrm{d}t + \mathrm{d}\psi + a)\Bigg] \,. \end{aligned} \tag{4.4.18}$$

Clearly, upon performing the Wick rotation $t = -\mathrm{i}\tau$ we can identify $\mathrm{g}^0 = g$, $A_0 = -\frac{1}{\sqrt{3}}A$, where g and A were given in (4.2.8), (4.2.10) and define the four-dimensional SCFT background. We recall that the last three terms in (4.4.16) are gauge choices: γ, γ' are two constants while λ is a function of $z, \bar{z}$; these will play an important rôle in the following.

One can check that

$$(\tilde{A}_2)_i = -\frac{1}{4}(\nabla^j F_{ji})^{(0)} \,. \tag{4.4.19}$$

In analogy with $\tilde{h}^0$, we see that $\tilde{A}_2$ is obtained by varying the integrated holographic Weyl anomaly, this time with respect to the boundary gauge field A_0.

Generically, the boundary is not conformally flat and the solution is asymptotically *locally* AdS$_5$. In the particular case where the boundary is conformally flat and the boundary gauge field strength vanishes — i.e. when the solution is asymptotically AdS rather than asymptotically locally AdS — both $\tilde{h}^0$ and $\tilde{A}_2$ vanish. This is in agreement with the general fact that the logarithmic terms in the Fefferman–Graham expansion vanish for a conformally flat boundary.

The solutions described above preserve at least (and generically no more than) two real supercharges. We have also verified that the five-dimensional metric and gauge field discussed above satisfy the Einstein and Maxwell equations at order $\mathcal{O}(\rho^3)$, which is the highest we have access to given the order at which we worked out the solution.

4.4.2 Standard holographic renormalization

Following the standard procedure of holographic renormalization,[16] a finite on-shell action S is obtained by considering a regularized five-dimensional space Y_δ where the radial coordinate ρ does not extend until the conformal boundary at $\rho = 0$ but is cut off at $\rho = \delta$, so that $M_4 = \partial Y_5 = \lim_{\delta\to 0} \partial Y_\delta \equiv \lim_{\delta \to 0} M_\delta$. Then one evaluates the limit

$$S \ = \ \lim_{\delta\to 0} \left(I_{\text{o}-\text{s}} + I_{\text{GHY}} + I_{\text{ct}} + I_{\text{ct,finite}} \right) \ . \tag{4.4.20}$$

Here, $I_{\text{o}-\text{s}}$ is the bulk action (4.4.1) evaluated over Y_δ. I_{GHY} is the Gibbons–Hawking–York boundary term, which makes the Dirichlet variational problem for the metric well-defined and reads

$$I_{\text{GHY}} \ = \ \frac{1}{\kappa_5^2} \int_{M_\delta} \mathrm{d}^4x \sqrt{h}\, K \,, \tag{4.4.21}$$

where $K = h^{ij} K_{ij}$ is the trace of the extrinsic curvature $K_{ij} \ = \ -\frac{1}{2}\frac{\partial h_{ij}}{\partial \rho}$ of M_δ. The counterterm action I_{ct} is a boundary term cancelling all divergences that appear in $I_{\text{o}-\text{s}} + I_{\text{GHY}}$ as $\delta \to 0$; it reads

$$I_{\text{ct}} \ = \ -\frac{1}{\kappa_5^2} \int_{M_\delta} \mathrm{d}^4x \sqrt{h} \left[3 + \frac{1}{4} R + \frac{1}{16} \left(\mathcal{E} - C_{ijkl} C^{ijkl} + 8 \mathcal{F}_{ij} \mathcal{F}^{ij} \right) \log \delta \right] \ . \tag{4.4.22}$$

The first two terms cancel power-law divergences while the logarithmically divergent term removes the holographic Weyl anomaly. Here, $\mathcal{E}$ is the Euler density and C_{ijkl} is the Weyl tensor of the induced metric, see appendix A.1 for their definition. Note that since $\sqrt{h}(\mathcal{E} - C_{ijkl}C^{ijkl} + 8\mathcal{F}_{ij}\mathcal{F}^{ij})$ remains finite as $\delta \to 0$, it can equivalently be computed using the boundary metric g^0 and gauge field A_0.

Finally, $I_{\text{ct,finite}}$ comprises local counterterms that remain finite while sending $\delta \to 0$. In general, these may describe ambiguities in the renormalization scheme or be necessary in

[16] See [212, 48] for the modifications due to the inclusion of a Maxwell field.

order to restore some desired symmetry that is broken by the rest of the action. In our case, requiring diffeomorphism and gauge invariance the linearly independent such terms may be parameterized as

$$I_{\text{ct,finite}} = \frac{1}{\kappa_5^2}\int_{M_\delta} \mathrm{d}^4x\sqrt{h}\left(\varsigma\, R^2 - \varsigma'\mathcal{F}_{ij}\mathcal{F}^{ij} + \varsigma''\, C_{ijkl}C^{ijkl}\right), \tag{4.4.23}$$

where ς, ς', ς'' are *a priori* arbitrary numerical constants.[17]

The holographic energy-momentum tensor is defined as the variation of the on-shell action with respect to the boundary metric[18]

$$T_{ij} = -\frac{2}{\sqrt{g}}\frac{\delta S}{\delta g^{ij}}, \tag{4.4.24}$$

and can be computed by means of the general formula

$$\begin{aligned} T_{ij} = \frac{1}{\kappa_5^2}\lim_{\delta\to 0}\frac{1}{\delta^2}\Bigg[& -K_{ij} + Kh_{ij} - 3h_{ij} + \frac{1}{2}\left(R_{ij} - \frac{1}{2}R\,h_{ij}\right) \\ & + \frac{1}{4}\left(B_{ij} + 8\mathcal{F}_{ik}\mathcal{F}_j{}^k - 2h_{ij}\mathcal{F}_{kl}\mathcal{F}^{kl}\right)\log\delta \\ & + \left(2\varsigma H_{ij} + 4\varsigma'' B_{ij} + \varsigma'\left(4\mathcal{F}_{ik}\mathcal{F}_j{}^k - h_{ij}\mathcal{F}_{kl}\mathcal{F}^{kl}\right)\right)\Bigg], \end{aligned} \tag{4.4.25}$$

where all quantities in the square bracket are evaluated on M_δ, and we refer to appendix A.1 for the definition of the tensor H_{ij}.

The holographic $U(1)_R$ current is defined as

$$j^i = \frac{1}{\sqrt{g}}\frac{\delta S}{\delta A_i}, \tag{4.4.26}$$

Note that we defined the variation in terms of the rescaled boundary gauge field $A = -\sqrt{3}A_0$. In this way the holographic R-current is normalised in the same way as the field

[17] We could also include in the linear combination the terms $\int \mathrm{d}^4x\sqrt{h}\mathcal{E}$, $\int \mathrm{d}^4x\sqrt{h}\mathcal{P}$ and $\int \mathrm{d}^4x\sqrt{h}\epsilon^{ijkl}\mathcal{F}_{ij}\mathcal{F}_{kl}$, where $\mathcal{P}$ is the Pontryagin density on M_δ, however these are topological quantities that have a trivial variation; moreover, as we will see below they vanish identically in the geometries of interest for this chapter.

[18] The minus sign that appears here, as opposed to the corresponding equation (4.3.28) in four-dimensional supergravity, is due to the different signatures. In particular compare the four-dimensional Euclidean supergravity action (4.3.1) with the five-dimensional Lorentzian action (4.4.1). On Wick rotation of the latter the conventions are compatible.

theory R-current. This yields the expression:

$$j^i = -\frac{2}{\sqrt{3}\kappa_5^2} \lim_{\delta\to 0} \frac{1}{\delta^4} \left\{ *_4 \left[\mathrm{d}x^i \wedge \left(*_5 \mathcal{F} + \frac{4}{3\sqrt{3}} \mathcal{A} \wedge \mathcal{F} \right) \right] + \nabla_j \mathcal{F}^{ji} \log\delta + 2\varsigma' \nabla_j \mathcal{F}^{ji} \right\} , \quad (4.4.27)$$

where the first term comes from varying the bulk action $I_{\text{o-s}}$, the second from I_{ct} and the third from $I_{\text{ct,finite}}$.

Given the definitions (4.4.24) and (4.4.26), the variation of the renormalized on-shell action under a generic deformation of the boundary data can be expressed via the chain rule as

$$\delta S \;=\; \int_{\partial Y_5} \mathrm{d}^4 x \sqrt{g} \left(-\frac{1}{2} T_{ij} \delta g^{ij} + j^i \delta A_i \right) \,. \quad (4.4.28)$$

Starting from this formula, one can check several Ward identities holding in the holographic renormalization scheme defined above. Invariance of the action under a boundary diffeomorphism generated by an arbitrary vector on ∂Y_5 yields the expected conservation equation for the holographic energy-momentum tensor,

$$\nabla^i T_{ij} \;=\; F_{ji} j^i - A_j \nabla_i j^i \,. \quad (4.4.29)$$

Studying the variation of the on-shell action under a boundary Weyl transformation such that $\delta g_{ij} = 2 g_{ij} \delta\sigma$, $\delta A_i = 0$, one finds for the trace of the holographic energy-momentum tensor [127]:

$$T_i{}^i \;=\; \frac{1}{16\kappa_5^2} \left(-\mathcal{E} + C_{ijkl} C^{ijkl} - \frac{8}{3} F_{ij} F^{ij} \right) - \frac{12\varsigma}{\kappa_5^2} \Box R \,, \quad (4.4.30)$$

which reproduces the known expression for the Weyl anomaly of a superconformal field theory [15, 63], with the standard identifications $\mathtt{a} = \mathtt{c} = \pi^2/\kappa_5^2$. Studying the variation under a gauge transformation at the boundary one obtains for the divergence of the holographic R-current [225, 63]:

$$\nabla_i j^i \;=\; \frac{1}{27\kappa_5^2} \epsilon^{ijkl} F_{ij} F_{kl} \,, \quad (4.4.31)$$

which again is consistent with the chiral anomaly of the superconformal R-symmetry.

4.4.3 The new boundary terms

We now specialize to the family of asymptotic supersymmetric solutions constructed in section 4.4.1 and test whether the supersymmetric Ward identities reviewed in section 4.2 are satisfied holographically. We will consider variations of the boundary functions that preserve the complex structure(s), and compute the corresponding variation of the on-shell action via (4.4.28). As discussed in section 4.2, the input from field theory is that this variation should vanish if supersymmetry is preserved. *A priori* one might expect that there is at least a choice of the ς-coefficients in the standard finite counterterms (4.4.23) such that the supersymmetric Ward identity is satisfied. However, we will show that this is *not* the case and that *new*, non-standard finite counterterms are required.

Before going into this, it will be useful to notice that the boundary metric and gauge field in (4.4.8), (4.4.16) satisfy

$$\mathcal{E} \;=\; \mathcal{P} \;=\; \epsilon^{ijkl} F_{ij} F_{kl} \;=\; 0\,, \tag{4.4.32}$$

where $\mathcal{P}$ is the Pontryagin density on ∂M. Moreover, supersymmetry implies [63]

$$C_{ijkl} C^{ijkl} - \frac{8}{3} F_{ij} F^{ij} \;=\; 0\,. \tag{4.4.33}$$

It follows that (4.4.29)–(4.4.31) simplify to

$$\nabla_i j^i \;=\; 0\,, \qquad \nabla^i T_{ij} \;=\; F_{ji} j^i\,, \qquad T_i{}^i \;=\; -\frac{12\varsigma}{\kappa_5^2} \Box R\,. \tag{4.4.34}$$

Relation (4.4.33) also implies that by redefining the coefficients ς', ς'' we can set $\varsigma'' = 0$ in the finite counterterm action (4.4.23) as well as in all its variations that preserve supersymmetry at the boundary. Below we will assume this has been done.

As explained in section 4.2.5, a variation of the boundary data that preserves the complex structures $I_\pm$ on the boundary corresponds to deformations $u \to u + \delta u$, $w \to w + \delta w$ such that $\delta u = \delta u(z, \bar{z})$ and $\delta w = \delta w(z, \bar{z})$ are *globally well-defined* functions. In the following we study the consequences of such variations. We will also assume that ∂Y_5 is closed and that the non-boundary functions k_1, k_2, k_3, k_4 are globally well-defined functions of their arguments $z, \bar{z}$. This will allow us to apply Stokes' theorem on the boundary and discard several total derivative terms.

We first vary w keeping the 1-form a fixed. From (4.2.5), we see that this is possible provided the variation preserves $\mathrm{e}^{w}u$, hence we also need to take $\delta u = -u\,\delta w$. After dropping several total derivative terms involving the boundary functions and $k_2(z,\bar{z})$, we find that the corresponding variation of the on-shell action is:

$$\begin{aligned}\delta_w \mathbb{S} = \frac{1}{2^6 3\kappa_5^2}\int_{\partial Y_5} \mathrm{d}^4x\sqrt{g}\,\delta w\Big[&\left(-1+96\varsigma-16\varsigma'\right)u^2 R_{2d}-\frac{1}{2}\left(1-96\varsigma+28\varsigma'\right)\Box u^2\\ &+\frac{1}{32}\left(19-288\varsigma+192\varsigma'\right)u^4-\frac{8}{9}(\gamma+2\gamma')\left(2uR_{2d}+2\Box u-u^3\right)\\ &-12\varsigma' u\Box u+8(-24\varsigma+\varsigma')(R_{2d}^2+2\Box R_{2d})\Big]\,,\end{aligned} \tag{4.4.35}$$

where we recall that $R_{2d}=-\Box w$ is the Ricci scalar of the two-dimensional metric (4.2.3). If instead we vary u while keeping w fixed we obtain

$$\begin{aligned}\delta_u \mathbb{S} \;=\; \frac{1}{2^9 3^2\kappa_5^2}\int_{\partial Y_5} \mathrm{d}^4x\sqrt{g}\,\delta u\Big[&24\left(1-96\varsigma+16\varsigma'\right)uR_{2d}+288\varsigma'\Box u\\ &-\left(19-288\varsigma+192\varsigma'\right)u^3-\frac{32}{3}(\gamma+2\gamma')(3u^2-4R_{2d})\Big]\,,\end{aligned} \tag{4.4.36}$$

where again we dropped many total derivative terms, some of which containing the non-boundary data k_2, k_3. In order to do this, we used that δa is globally defined; this follows from the assumption that the complex structures are not modified.

Inspection of (4.4.35), (4.4.36) shows that there exists no choice of the coefficients ς,ς' such that $\delta_w\mathbb{S}=\delta_u\mathbb{S}=0$. Therefore we conclude:

Standard holographic renormalization does not satisfy the field theory supersymmetric Ward identities.

Remarkably, we find that this can be cured by introducing *new* finite terms. Both variations $\delta_w\mathbb{S}$ and $\delta_u\mathbb{S}$ vanish if we take $\varsigma=\varsigma'=0$ (that is, if we set $I_{\text{ct,finite}}=0$) *and* add to the on-shell action the new terms

$$\Delta I_{\text{new}} \;=\; \frac{1}{2^{11}3^2\kappa_5^2}\int_{\partial Y_5}\mathrm{d}^4x\sqrt{g}\left[19u^4-48u^2R_{2d}+\frac{128}{3}(2\gamma'+\gamma)(u^3-4uR_{2d})\right]. \tag{4.4.37}$$

In other words, *the new renormalized action*

$$\mathbb{S}_{\text{susy}} \;=\; \lim_{\delta\to 0}\left(I_{\text{o}-\text{s}}+I_{\text{GHY}}+I_{\text{ct}}\right)+\Delta I_{\text{new}} \tag{4.4.38}$$

does satisfy the supersymmetric Ward identities. We claim that this is the correct supersymmetric on-shell action that should be compared with the supersymmetric field theory partition function.

It should be clear that the terms ΔI_{new} cannot be written as local actions that are: *i*) invariant under four-dimensional diffeomorphisms, *ii*) invariant under gauge transformations of A, and *iii*) constructed using the boundary metric, the boundary gauge field and their derivatives only. If this was the case, ΔI_{new} would fall in the family of standard finite counterterms (4.4.23), which we have just proven not to be possible. We will comment on this issue in the conclusions. Here we make a first step towards clarifying it by observing that the gauge-dependent part of ΔI_{new} — that is, the term containing the gauge parameters γ, γ' — has to come from a term linear in the boundary gauge potential $A = -\sqrt{3}A_0$. So we may write

$$\Delta I_{\text{new}} \;=\; \frac{1}{\kappa_5^2}\int_{\partial Y_5}\left(A\wedge\Phi+\Psi\right)\,, \tag{4.4.39}$$

where Ψ is gauge-invariant. Matching this with (4.4.37), we obtain

$$\begin{aligned}\Phi \;&=\; \frac{1}{2^3 3^3}\left(u^3-4uR_{2d}\right)\mathrm{i}\,\mathrm{e}^{w}\mathrm{d}z\wedge\mathrm{d}\bar{z}\wedge\left(2\mathrm{d}\psi-\mathrm{d}t\right)\,,\\ \Psi \;&=\; \frac{1}{2^{11}3^2}\left(19u^4-48u^2R_{2d}\right)\mathrm{d}^4x\sqrt{g}\,.\end{aligned} \tag{4.4.40}$$

Notice that $\mathrm{d}\Phi = 0$, so ΔI_{new} is invariant under small gauge transformations. However, it depends on the choice of flat connection for A when ∂Y_5 has one-cycles. Also notice that (4.4.39) implies that ΔI_{new} yields a new contribution to the holographic R-current (4.4.26). Below we will show that this modifies the R-charge precisely as demanded by the superalgebra.

4.4.4 Evaluation of the on-shell action

In this section we evaluate the renormalized supergravity action (4.4.38) on the class of five-dimensional solutions constructed above. Since this involves performing a bulk integral, *a priori* one would need to know the full solution in the interior, while we just have it in a perturbative expansion near the boundary. However, we show that under certain global assumptions the on-shell action reduces to a boundary term that can be evaluated exactly as a function of boundary data only.

The assumptions consist in requiring that the solution caps off regularly and with no boundary in the interior, and that the graviphoton $\mathcal{A}$ is a global 1-form.[19] As shown in [64], this allows to express the bulk action of supersymmetric solutions in the timelike class as the boundary term

$$I_{\text{o-s}} \;=\; \frac{1}{3\kappa_5^2}\int_{M_\delta}\left(\mathrm{d}y\wedge P\wedge J-2\mathcal{A}\wedge *_5\mathcal{F}\right)\,, \tag{4.4.41}$$

where the coordinate y, the Ricci 1-form potential P and the Kähler form J are those of the "canonical structure" dictated by supersymmetry [108] and are defined in appendix B.1.1. We remark that while demanding that $\mathcal{A}$ is a global 1-form we are also taking P as a global 1-form, see eq. (B.1.6). Notice this implies that the canonical bundle of the $4d$ Kähler metric is trivial, cf. an analogous global assumption in section 4.3. The integral on the hypersurface M_δ at constant ρ can be explicitly evaluated for our solution after passing to Fefferman–Graham coordinates as discussed in appendix B.1.2.

Even if the on-shell action is now reduced to a boundary term, generically it still depends on the arbitrary non-boundary functions appearing in the solution. We now generalize an argument given in [64] and show that the assumption of global regularity also entails a relation between these non-boundary functions and the boundary ones that is precisely sufficient for determining the on-shell action.

Let $\mathcal{C}$ be a Cauchy surface (namely, a hypersurface at constant t), with boundary $M_3=\mathcal{C}\cap\partial Y_5$, and consider the Page charge[20]

$$\Theta \;=\; \int_{M_3}\Big(*_5\mathcal{F}+\frac{2}{\sqrt{3}}\mathcal{A}\wedge\mathcal{F}\Big)\,. \tag{4.4.42}$$

Since $\mathcal{A}$ is globally defined and ∂Y_5 is by assumption the only boundary of the space, we can apply Stokes' theorem and then use the Maxwell equation to infer that Θ must vanish:

$$\Theta \;=\; \int_{M_3}\Big(*_5\mathcal{F}+\frac{2}{\sqrt{3}}\mathcal{A}\wedge\mathcal{F}\Big) \;=\; \int_{\mathcal{C}}\Big(\mathrm{d}*_5\mathcal{F}+\frac{2}{\sqrt{3}}\mathcal{F}\wedge\mathcal{F}\Big) \;=\; 0\,. \tag{4.4.43}$$

We now replace the Fefferman–Graham expansion of the graviphoton field strength

$$\mathcal{F} \;=\; \mathrm{d}A_0+\rho^2\left(\mathrm{d}A_2+\mathrm{d}\tilde{A}_2\log\rho^2\right)+2\rho\mathrm{d}\rho\wedge\left(A_2+\tilde{A}_2+\tilde{A}_2\log\rho^2\right)+o(\rho^2) \tag{4.4.44}$$

[19] For example this excludes supersymmetric black hole solutions [122, 65].

[20] This the name reserved for a charge that is localized and conserved, but not gauge invariant (see also [189, 169]).

and its Hodge dual restricted to the hypersurfaces at constant ρ,

$$(*_5\mathcal{F})\big|_{\mathrm{d}\rho=0} = 2 *_{g^0} \left(A_2 + \tilde{A}_2 + \tilde{A}_2 \log \rho^2\right) + \mathcal{O}(\rho) \ , \tag{4.4.45}$$

where $*_{g^0}$ is the Hodge star of the boundary metric g^0.[21] It is easy to see that expression (4.4.42) then becomes

$$\Theta = \int_{M_3} \left(2\,\mathrm{vol}_3 \big(A_2 + \tilde{A}_2\big)_t + \frac{2}{\sqrt{3}} A_0 \wedge \mathrm{d}A_0\right) , \tag{4.4.46}$$

where we are using the notation $\mathrm{vol}_3 \equiv \mathrm{d}^3x\sqrt{g_3}$ for the Riemannian volume form on M_3. The condition $\Theta = 0$ is thus equivalent to the statement that the integrated time component of A_2, which *a priori* is controlled by non-boundary data and is thus not fixed by the equations of motion, is actually determined by boundary data. Evaluating this on our perturbative solution, we find the following integral relation between the non-boundary functions k_1, k_2, k_3 and the boundary functions u, w:

$$\begin{aligned} 0 = \Theta = \frac{1}{96\sqrt{3}} \int_{M_3} \mathrm{vol}_3 \Big[&\frac{1}{u}\Big(384\, k_3 - 64k_2^2 + 48\Box k_2 + 32k_2 R_{2d} + 9\mathrm{e}^{-w}\partial_z u \partial_{\bar{z}} u \\ &- 9\Box R_{2d} - 6R_{2d}^2\Big) + 48uk_2 - \frac{15}{4}u^3 + 192k_1 \\ &+ 6\,\mathrm{e}^{\frac{1}{3}w}\big[\nabla_z\big(\mathrm{e}^{-\frac{4}{3}w}\partial_{\bar{z}} u\big) + c.c.\big] + (13u - 16\gamma)R_{2d}\Big] \\ - \frac{1}{6\sqrt{3}} \int_{M_3} \mathrm{d}\psi \wedge \mathrm{d}\big[u(\mathrm{d}\lambda - \gamma\, a)\big] \ . \end{aligned} \tag{4.4.47}$$

We can now give our result for the renormalized on-shell action. Adding up all contributions to (4.4.38), including the new counterterms (4.4.37), and without making further assumptions, we obtain

$$\begin{aligned} S_{\text{susy}} = \frac{\int \mathrm{d}t}{27\kappa_5^2} \Big\{ &\int_{M_3} \mathrm{vol}_3 \left[(\gamma' - \gamma)\gamma R_{2d} + \frac{9}{8}\Box\,(4k_2 - \gamma u)\right] \\ &+ \frac{1}{64}\int_{M_3} \mathrm{d}\left[\mathrm{d}\psi \wedge \big(96k_2 + 12R_{2d} - 3u^2 + 16(\gamma' - \gamma)u\big)\,(4\mathrm{d}\lambda - 4\gamma a + *_2 \mathrm{d}w)\right] \\ &+ 6\sqrt{3}(\gamma' - \gamma)\,\Theta \Big\} \ . \end{aligned} \tag{4.4.48}$$

The Laplacian term in the first line and the whole integrand in the second line are total

[21] Note that the logarithmic divergence drops out of the quantities we are interested in. Indeed, recalling (4.4.19), we see that $*_{g^0}\tilde{A}_2 \propto (\mathrm{d} * F)^{(0)}$ is a total derivative, hence it drops from any boundary integral.

derivatives of globally defined quantities and therefore vanish upon integration. The term Θ in the third line, given by (4.4.47), also vanishes as just seen. So we obtain a very simple expression for the on-shell action, depending on boundary data only:

$$S_{\text{susy}} = \frac{(\gamma' - \gamma)\gamma}{27\kappa_5^2} \int \mathrm{d}t \int_{M_3} \mathrm{vol}_3\, R_{2d} \,. \tag{4.4.49}$$

We next implement the analytic continuation $t = -\mathrm{i}\tau$, which renders the boundary metric Euclidean, and assume that τ parameterizes a circle of length β. The expression for the on-shell action thus becomes[22]

$$S_{\text{susy}} = \frac{\beta(\gamma - \gamma')\gamma}{27\kappa_5^2} \int_{M_3} \mathrm{vol}_3\, R_{2d} \,. \tag{4.4.50}$$

It is interesting to note that, as we show in appendix B.2, the flat connection parameters γ and γ' also correspond to the charge of the boundary Killing spinor ζ_+ under ∂_ψ and $\mathrm{i}\partial_\tau$, respectively. Hence $\gamma - \gamma'$ is twice the charge of ζ_+ under the complex Killing vector K introduced in section 4.2.2.

Recall from section 4.2.4 that the requirement that the boundary gauge field is globally defined fixes γ as

$$\gamma = -\frac{1}{4} \frac{\int_{M_3} \mathrm{vol}_3\, R_{2d}}{\int_{M_3} \eta \wedge \mathrm{d}\eta} \,. \tag{4.4.51}$$

Recalling (4.2.4), (4.2.5), the contact volume of M_3 appearing in the denominator can also be expressed as $\int_{M_3} \eta \wedge \mathrm{d}\eta = \frac{1}{2} \int_{M_3} \mathrm{vol}_3\, u$.

As far as the bosonic solution is concerned, expression (4.4.50) makes sense for any value of γ'. However, for S_{susy} to be the on-shell action of a proper supersymmetric solution we also need to impose that the Killing spinors are independent of τ, so that they remain globally well-defined when this coordinate is made compact. Since γ' is the charge of the Killing spinors under $\mathrm{i}\partial_\tau$, we must take $\gamma' = 0$.

We conclude that for a regular, supersymmetric five-dimensional asymptotically locally AdS solution satisfying the global conditions above, and such that the conformal boundary has a direct product form $S^1 \times M_3$, the supersymmetric on-shell action is given by

$$S_{\text{susy}} = \frac{\beta\gamma^2}{27\kappa_5^2} \int_{M_3} \mathrm{vol}_3\, R_{2d} \,, \tag{4.4.52}$$

[22]The overall sign change comes from the identification $\mathrm{i}S_{\text{Lorentzian},\, t=-\mathrm{i}\tau} = -S_{\text{Euclidean}}$.

where γ is fixed as in (4.4.51). Note that because of the dependence on γ^2, $\mathbb{S}_{\text{susy}}$ cannot itself be written as a local term in four dimensions.

In section 4.5 we will show that this result precisely matches the large N limit of the SCFT partition function in all known examples (and beyond).

4.4.5 Twisting the boundary

We can easily discuss a slightly more general class of solutions, having different boundary geometry. This is obtained by making the local change of coordinates

$$\tau \rightarrow \cos\alpha\,\tau\,, \qquad \psi \rightarrow \psi + \sin\alpha\,\tau\,, \tag{4.4.53}$$

where $0 < \alpha < \pi/2$ is a real parameter.[23] Then the old boundary metric and gauge field (4.2.8), (4.2.10) become

$$\mathrm{d}s_4^2 = (\mathrm{d}\tau + \sin\alpha\,(\mathrm{d}\psi + a))^2 + \cos^2\alpha\,(\mathrm{d}\psi + a)^2 + 4\mathrm{e}^{w}\mathrm{d}z\mathrm{d}\bar{z}\,, \tag{4.4.54}$$

$$\begin{aligned} A &= (\mathrm{i}\cos\alpha + 2\sin\alpha)\,\frac{u}{8}\mathrm{d}\tau + \frac{u}{4}(\mathrm{d}\psi + a) + \frac{1}{4} *_2\mathrm{d}w \\ &\quad + (\gamma\sin\alpha - \mathrm{i}\gamma'\cos\alpha)\mathrm{d}\tau + \gamma\,\mathrm{d}\psi + \mathrm{d}\lambda\,. \end{aligned} \tag{4.4.55}$$

Although this configuration is locally equivalent to the original one, if we take for the new coordinates the same identifications as for the old ones (in particular $\tau \sim \tau + \beta$, $\psi \sim \psi$ as one goes around the S^1 parameterised by τ one full time), then the new boundary geometry with $\alpha \neq 0$ is globally distinct from the original one. From (4.4.54) we see that the S^1 parameterised by τ is fibered over M_3, although in a topologically trivial way since $\mathrm{d}\psi + a$ is globally defined; moreover, the term $(\mathrm{d}\psi + a)^2$ in the M_3 part of the metric is rescaled by a factor $\cos^2\alpha$. We will denote as "twisted" the new four-dimensional background (4.4.54), (4.4.55), as well as the corresponding five-dimensional solution obtained by implementing the transformation (4.4.53) in the bulk.[24] In fact we can show that the complex structure of the twisted boundary is inequivalent to the complex structure with $\alpha = 0$. Recall from section 4.2.2 that four-dimensional field theory backgrounds with two supercharges of

[23] In Lorentzian signature, the change of coordinates reads $t \to \cosh\alpha_{\mathrm{L}}\,t$, $\psi \to \psi + \sinh\alpha_{\mathrm{L}}\,t$, with α_{L} constant. This is related to (4.4.53) by $t = -\mathrm{i}\tau$ and $\alpha_{\mathrm{L}} = \mathrm{i}\alpha$.

[24] An equivalent description would be to maintain the metric and gauge field (4.2.8), (4.2.10) and modify the identifications for the periodic coordinates, so that going around the circle parameterised by τ also advances the coordinate ψ in M_3. This is what is commonly known as twisting, see e.g. [68].

opposite R-charge admit a globally defined, complex Killing vector K holomorphic with respect to two complex structures $I_\pm$. For our untwisted background, this was given in (4.2.9). For the twisted background, and in terms of a coordinate $\tilde{\tau} = \tau/\beta$ with canonical unit periodicity, it reads

$$K \;=\; \frac{1}{2\beta\cos\alpha}\left(\beta e^{i\alpha}\,\partial_\psi - i\,\partial_{\tilde{\tau}}\right)\,. \tag{4.4.56}$$

We infer that $\beta e^{i\alpha}$ is a complex structure parameter of the background (while the overall factor in K does not affect the complex structure). Depending on the specifics of M_3, the background may admit additional complex structure moduli, however the one discussed here is a universal modulus of manifolds with $S^1 \times M_3$ topology and metric (4.4.54).

The results of [68] then imply that the supersymmetric partition function on the twisted background should be related to the one on the untwisted background by replacing $\beta \to \beta e^{i\alpha}$. It would be interesting to check this expectation by an explicit localization computation. To date, only partial localization computations have been carried out for four-dimensional supersymmetric field theories on similarly twisted backgrounds [70].[25]

We can compare with the on-shell action of the twisted bulk solutions. This is evaluated in the same way as for $\alpha = 0$, with just two differences: $i)$ the volume form on M_3 is rescaled by a factor $\cos\alpha$, and $ii)$ the boundary Killing spinors are independent of the new time coordinate for a different value of γ': as discussed in appendix B.2, now we must take

$$\gamma' \;=\; -i\,\gamma\tan\alpha\,. \tag{4.4.57}$$

Starting from (4.4.50) it is thus easy to see that the net result of the twist by α is to multiply the on-shell action of the untwisted solution by a phase:

$$\mathbb{S}_{\text{susy},\,\alpha} \;=\; e^{i\alpha}\,\mathbb{S}_{\text{susy},\,\alpha=0}\,, \tag{4.4.58}$$

where $\mathbb{S}_{\text{susy},\,\alpha=0}$ is given by (4.4.52). Here the imaginary part is a consequence of the choice of γ', that is of the way the terms depending on large gauge transformations $A \to A + \text{const}\,d\tau$ are fixed in the on-shell action. Effectively, the phase $e^{i\alpha}$ can be seen as a complexification of β. So we find that the twisting has the same consequence for the on-shell action as expected

[25] In [19] the two complex structure parameters p, q of primary Hopf surfaces were assumed real, however in appendix D therein it was discussed how to generalize the background so that p, q become generally complex. It would be interesting to evaluate the partition function of general supersymmetric gauge theories on such backgrounds.

for the field theory partition function: the parameter β is replaced by $\beta e^{i\alpha}$.

Besides being interesting *per se*, this complexification of the on-shell action will serve as a tool for computing the charges below.

4.4.6 Conserved charges

We now compute the holographic conserved charges taking into account the contribution of the new counterterms ΔI_{new} and verify that they satisfy the expected BPS condition.

Let us first consider the currents defined by standard holographic renormalization. Recall from (4.4.34) that the R-current j^i is conserved and thus provides a conserved R-charge. In addition, given any boundary vector v preserving the boundary fields, i.e. such that $\mathcal{L}_v g = \mathcal{L}_v A = 0$, we can introduce the current

$$Y^i \;=\; v^j (T_j{}^i + A_j j^i) \; . \tag{4.4.59}$$

Using the modified conservation equation of the energy-momentum tensor in (4.4.34), it is easy to see that Y^i is conserved and thus defines a good charge for the symmetry associated with v.
Although we do not know how exactly the new counterterms affect the energy-momentum tensor (because we do not know the variation of ΔI_{new} with respect to the metric), we will show how the relevant charges can be computed anyway by varying the on-shell action with respect to appropriate parameters. We will just need to assume that ΔI_{new} *can* be expressed as a quantity invariant under diffeomorphisms and small gauge transformations, constructed from the boundary metric and the boundary gauge field (and necessarily other boundary fields), so that the chain rule (4.4.28) and the conservation equations make sense also after $\mathbb{S}$ is replaced by $\mathbb{S}_{\text{susy}}$, and T_{ij}, j^i are replaced by their supersymmetric counterparts defined by varying $\mathbb{S}_{\text{susy}}$.

We will discuss the charges for the untwisted background with $\alpha = 0$, although it would be straightforward to extend this to general α. The background with $\alpha \neq 0$ will however play a rôle in the computation of the angular momentum.

R-charge The supersymmetric holographic R-charge is defined as

$$Q_{\text{susy}} \;=\; -\int_{M_3} \text{vol}_3\, j^t_{\text{susy}} \;=\; -\mathrm{i} \int_{M_3} \text{vol}_3\, j^\tau_{\text{susy}} \;, \tag{4.4.60}$$

where

$$j^i_{\text{susy}} = j^i + \Delta j^i \tag{4.4.61}$$

is the sum of the current (4.4.27), obtained in a minimal holographic renormalization scheme, and

$$\Delta j^i = \frac{1}{\sqrt{g}} \frac{\delta}{\delta A_i} \Delta I_{\text{new}} \,. \tag{4.4.62}$$

Using (4.4.27), the former contribution is found to be

$$\begin{aligned} \int_{M_3} \text{vol}_3\, j^t &= \frac{2}{\sqrt{3}\kappa_5^2}\,\Theta + \frac{1}{108\kappa_5^2}\int_{M_3} \mathrm{d}\psi \wedge \mathrm{d}\left[u(4\mathrm{d}\lambda - 4\gamma a + *_2 \mathrm{d}w)\right] \\ &+ \frac{1}{216\kappa_5^2}\int_{M_3} \text{vol}_3 \left(8\gamma R_{2d} + 4u R_{2d} - u^3\right) \,, \end{aligned} \tag{4.4.63}$$

where Θ is again given by expression (4.4.47). Both Θ and the other integral in the first line vanish due to the global assumptions we made in section 4.4.4, so the R-charge in a minimal holographic renormalization scheme is given by the second line only. The shift in the current due to the new counterterms can be read from (4.4.39), (4.4.40) and leads to

$$\int_{M_3} \text{vol}_3\, \Delta j^t = \frac{1}{216\kappa_5^2}\int_{M_3} \text{vol}_3 \left(-4u R_{2d} + u^3\right) \,. \tag{4.4.64}$$

Adding the two contributions up, the expression for the supersymmetric holographic R-charge simplifies to

$$Q_{\text{susy}} = -\frac{\gamma}{27\kappa_5^2}\int_{M_3} \text{vol}_3\, R_{2d} = -\frac{1}{\beta\gamma}\mathcal{S}_{\text{susy}} \,. \tag{4.4.65}$$

We notice that a faster way to arrive at the same result is to take the derivative $\frac{1}{\beta}\frac{\partial}{\partial\gamma'}$ of the action (4.4.50). Indeed, a variation of the parameter γ' amounts to shift by a constant the time component of the gauge field, which computes the electric charge.

Energy We define the energy H of the supergravity solution as the charge associated with the Killing vector ∂_t (or ∂_τ in Euclidean signature). This is given by

$$H = \int_{M_3} \text{vol}_3 \left(T_{tt} + A_t j_t\right) = \int_{M_3} \text{vol}_3 \left(-T_{\tau\tau} + A_\tau j_\tau\right) \,. \tag{4.4.66}$$

Since we wish to compute the supersymmetric energy, we need to use the supersymmetric versions of the energy-momentum tensor and R-current, which receive contributions from the new boundary terms ΔI_{new}. Although we do not know the contribution to the holographic energy-momentum tensor, we notice that the chain rule (4.4.28) implies that H is obtained by simply varying the on-shell action with respect to β. This is easily seen by rescaling τ so that it has fixed unit periodicity while β appears in the expressions for the metric and gauge field. Hence we obtain

$$H_{\text{susy}} = \frac{\partial}{\partial\beta}\mathbb{S}_{\text{susy}} = \frac{1}{\beta}\mathbb{S}_{\text{susy}} . \tag{4.4.67}$$

Angular momentum We denote as angular momentum the charge associated with $-\partial_\psi$. This is given by

$$J = -\int_{M_3} \text{vol}_3 \left(T_{t\psi} + A_\psi j_t\right) = -\text{i}\int_{M_3} \text{vol}_3 \left(T_{\tau\psi} - A_\psi j_\tau\right) . \tag{4.4.68}$$

Again we can circumvent the problem that we do not know how ΔI_{new} affects the energy-momentum tensor by varying the supersymmetric on-shell action with respect to a parameter. In this case the relevant parameter is α introduced via the twisting transformation of section 4.4.5. Using the chain rule (4.4.28) and recalling (4.4.54), (4.4.55), we find that the variation of the on-shell action with respect to α (keeping γ' fixed) gives:

$$\frac{\partial}{\partial\alpha}\mathbb{S}_{\text{susy}}\Big|_{\alpha=0} = \int \text{d}^4x\sqrt{g}\left(-T_{\tau\psi} + A_\psi j_\tau\right)_{\alpha=0} = -\text{i}\beta J_{\text{susy}} , \tag{4.4.69}$$

where as indicated all quantities are evaluated at $\alpha = 0$, namely in the original, untwisted, background. On the other hand, we can vary the explicit expression for $\mathbb{S}_{\text{susy}}$. Since γ' is kept fixed, we just need to vary the overall factor $\cos\alpha$. This gives $\frac{\partial}{\partial\alpha}\mathbb{S}_{\text{susy}}|_{\alpha=0} = 0$ and thus we conclude that

$$J_{\text{susy}} = 0 , \tag{4.4.70}$$

that is all untwisted solutions have vanishing angular momentum.

BPS relation In summary, we obtained the following expressions for the holographic charges associated with our supersymmetric, untwisted solutions:

$$H_{\rm susy} \;=\; -\gamma\, Q_{\rm susy} \;=\; \frac{1}{\beta}\mathbb{S}_{\rm susy}\,, \qquad\qquad J_{\rm susy} \;=\; 0\,. \tag{4.4.71}$$

Via the AdS/CFT correspondence, these should be identified with the vacuum expectation values of the dual SCFT operators. The SCFT superalgebra implies that the latter satisfy the BPS relation

$$\langle H\rangle + \langle J\rangle + \gamma\langle Q\rangle \;=\; 0\,, \tag{4.4.72}$$

see appendix B.2 for its derivation. Of course, here it is assumed that the vacuum expectation values are computed in a supersymmetric scheme. We see that the holographic charges (4.4.71) do indeed satisfy the condition. This can be regarded as a further check that the proposed boundary terms $\Delta I_{\rm new}$ restore supersymmetry.

4.5 Examples in five dimensions

We now discuss some examples of increasing complexity. This will offer the opportunity to illustrate further the rôle of the new boundary terms and make contact with the existing literature.

4.5.1 AdS_5

It is instructive to start by discussing the simplest case, that is AdS_5 space.

Euclidean AdS_5 is just five-dimensional hyperbolic space. In global coordinates, the unit metric can be written as

$$\mathrm{d}s^2_5 \;=\; \frac{\mathrm{d}\rho^2}{\rho^2} + \left(\frac{1}{\rho}+\frac{\rho}{4r_3^2}\right)^2\mathrm{d}\tau^2 + \left(\frac{1}{\rho}-\frac{\rho}{4r_3^2}\right)^2\mathrm{d}s^2_{S^3}\,, \tag{4.5.1}$$

where

$$\mathrm{d}s^2_{S^3} \;=\; \frac{r_3^2}{4}\left[\left(\mathrm{d}\tilde{\psi}+\cos\theta\mathrm{d}\varphi\right)^2+\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right] \tag{4.5.2}$$

is the round metric on a three-sphere of radius r_3, with canonical angular coordinates $\theta \in [0,\pi]$, $\varphi \in [0,2\pi]$, $\tilde{\psi} \in [0,4\pi]$. Here ρ is a Fefferman–Graham radial coordinate, extending from the conformal boundary at $\rho = 0$ until $\rho = 2r_3$, where the three-sphere shrinks to zero size. The conformal boundary is $\mathbb{R} \times S^3$, equipped with the conformally-flat metric

$$ds_4^2 \;=\; d\tau^2 + ds_{S^3}^2 \;. \tag{4.5.3}$$

We compactify the Euclidean time so that $\tau \sim \tau + \beta$ and the boundary becomes $S^1_\beta \times S^3_{r_3}$. For the relevant Killing spinors to be independent of time, we need to switch on a flat gauge field on S^1,

$$-\sqrt{3}\mathcal{A} \;=\; A \;=\; -\frac{\mathrm{i}}{2r_3}\,d\tau \;. \tag{4.5.4}$$

It is natural to assume that AdS_5 is dual to the vacuum state of a SCFT living on the conformal boundary $S^1_\beta \times S^3_{r_3}$.[26] In the following we illustrate how *the on-shell action and the holographic charges of AdS_5 match the SCFT supersymmetric vacuum expectation values only after holographic renormalization is supplemented with our new boundary terms.*

In the standard scheme of section 4.4.2, the renormalized on-shell action and holographic energy are found to be

$$\mathbb{S} \;=\; \beta H \;=\; \frac{3(1-96\varsigma)\beta}{4r_3}\frac{\pi^2}{\kappa_5^2} \;, \tag{4.5.5}$$

while both the angular momentum J and the holographic R-charge Q vanish. The latter value follows from formula (4.4.27) using $\mathcal{F} = 0$. Thus, by dialing ς the holographic energy H may be set either to agree with $Q = 0$, so that the BPS condition stating the proportionality between energy and charge is satisfied, or with the field theory result in (4.2.30), but not with both. Hence even in the simple example of AdS we see that standard holographic renormalization disagrees with the supersymmetric field theory results.

Let us describe how this discrepancy is solved by the new terms introduced in section 4.4.3. Starting from the general boundary geometry (4.2.8), (4.2.10) we take $u = \text{const} = -\frac{4}{r_3}$, $e^{\frac{w}{2}} = \frac{r_3}{2}\frac{1}{1+|z|^2}$, and make the change of coordinate $z = \cot\frac{\theta}{2}\,e^{-\mathrm{i}\varphi}$, $\psi = \frac{r_3}{2}\tilde{\psi}$. Then the two-dimensional metric, its curvature and the volume form are

$$ds_2^2 = \frac{r_3^2}{4}(d\theta^2 + \sin^2\theta d\varphi^2) \;, \qquad R_{2d} = \frac{8}{r_3^2} \;, \qquad \mathrm{vol}_2 = \frac{r_3^2}{4}\sin\theta\, d\theta \wedge d\varphi \;, \tag{4.5.6}$$

and eq. (4.2.5) for the connection 1-form a is solved by $a = \frac{r_3}{2}\cos\theta d\varphi$. Moreover to recover

[26] The possibility that a different asymptotically AdS supergravity solution may be dual to the SCFT vacuum on $S^1_\beta \times S^3_{r_3}$ was considered in [62]. The analysis of that chapter, though not exhaustive, indicates that this is not the case, and strongly suggests that AdS is the natural candidate.

the correct gauge field we need to take

$$\gamma = \frac{1}{r_3}\,, \qquad \gamma' = 0\,, \qquad \lambda = -\frac{\varphi}{2}\,, \tag{4.5.7}$$

the value of γ being in agreement with (4.4.51). In this way our general boundary metric and gauge field reduce to (4.5.3), (4.5.4).

The new boundary terms (4.4.39) then evaluate to (after Wick rotation):

$$\Delta I_{\text{new}} = -\frac{17\beta}{108 r_3}\frac{\pi^2}{\kappa_5^2}\,, \tag{4.5.8}$$

so that we obtain for the supersymmetric on-shell action of AdS_5:

$$\mathbb{S}_{\text{susy}} = \mathbb{S}_{\varsigma=0} + \Delta I_{\text{new}} = \frac{16\,\beta}{27 r_3}\frac{\pi^2}{\kappa_5^2}\,. \tag{4.5.9}$$

This result also follows directly from (4.4.52) since AdS_5 satisfies all global assumptions that were made in section 4.4.4 to derive it.[27] Then the energy is just $H = \frac{1}{\beta}\mathbb{S}_{\text{susy}}$ and the angular momentum vanishes, $J = 0$.

Using eq. (4.4.64), we see that the new terms also shift the value of the holographic R-charge from zero to

$$Q_{\text{susy}} = -\frac{16}{27}\frac{\pi^2}{\kappa_5^2}\,. \tag{4.5.10}$$

Therefore we have found for the supersymmetric energy, charge and angular momentum:

$$H_{\text{susy}} = -\frac{1}{r_3}Q_{\text{susy}} = \frac{16}{27 r_3}\frac{\pi^2}{\kappa_5^2}\,, \qquad J_{\text{susy}} = 0\,. \tag{4.5.11}$$

Besides respecting the BPS condition, these values precisely match the supersymmetric field theory vacuum expectation values of [19, 18], cf. eq. (4.2.30) for the energy.

It is worth pointing out that the choice (4.5.4) for the flat gauge field does not affect the conserved charges of AdS_5 computed via standard holographic renormalization, while it plays a crucial rôle in our new boundary terms. Indeed in the formulae of section 4.4.2

[27] For generic asymptotically AdS solutions, conformal flatness of the boundary metric (4.2.8) on $S^1_\beta \times M_3$ amounts to $u = \text{const}$ and $R_{2d} = \frac{u^2}{2}$; it also implies $\mathrm{d}A = 0$. Then from (4.4.51) we find $\gamma = -\frac{u}{4}$. If the solution satisfies the global assumptions made in section 4.4.4, our formula (4.4.52) applies and the supersymmetric on-shell action reads

$$\mathbb{S}_{\text{susy}} = \frac{\beta u^4}{2^5 3^3 \kappa_5^2}\int_{M_3}\text{vol}_3\,.$$

For a round sphere $M_3 \cong S^3_{r_3}$, we set $u = -\frac{4}{r_3}$, $\int_{S^3}\text{vol}_3 = 2\pi^2 r_3^3$ and the result (4.5.9) follows.

the only term potentially affected by a flat gauge connection is the bulk Chern–Simons term $\int \mathcal{A} \wedge \mathcal{F} \wedge \mathcal{F}$, which however vanishes on AdS$_5$ as $\mathcal{F} = 0$. On the other hand, ΔI_{new} in (4.4.39) depends on a flat connection on S^1 since the 3-form Φ does not vanish on the S^3 at the boundary of AdS$_5$, and this affects the holographic charges. In particular, it gives the full answer for the holographic R-charge associated with AdS$_5$.

4.5.2 Twisted AdS$_5$

We can take advantage of the very explicit example of AdS$_5$ to further illustrate the twisting of section 4.4.5.

Starting from the AdS$_5$ metric (4.5.1), (4.5.2) we make the change of coordinates

$$\tau \;\to\; \cos\alpha\,\tau\,, \qquad \tilde{\psi} \;\to\; \tilde{\psi} + \frac{2}{r_3}\sin\alpha\,\tau\,, \tag{4.5.12}$$

with $0 < \alpha < \pi/2$. Then the new bulk metric reads

$$\begin{aligned} \mathrm{d}s_5^2 \;=\; & \frac{\mathrm{d}\rho^2}{\rho^2} + \left(\frac{1}{\rho} + \frac{\rho}{4r_3^2}\right)^2 \cos^2\alpha\,\mathrm{d}\tau^2 \\ & + \left(\frac{1}{\rho} - \frac{\rho}{4r_3^2}\right)^2 \frac{r_3^2}{4}\left[\left(\mathrm{d}\tilde{\psi} + \frac{2}{r_3}\sin\alpha\,\mathrm{d}\tau + \cos\theta\mathrm{d}\varphi\right)^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right]. \end{aligned} \tag{4.5.13}$$

The new boundary metric may be written as

$$\mathrm{d}s_4^2 = \left[\mathrm{d}\tau + \frac{r_3}{2}\sin\alpha\big(\mathrm{d}\tilde{\psi} + \cos\theta\mathrm{d}\varphi\big)\right]^2 + \frac{r_3^2}{4}\left[\cos^2\alpha\big(\mathrm{d}\tilde{\psi} + \cos\theta\mathrm{d}\varphi\big)^2 + \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2\right]. \tag{4.5.14}$$

Since we do not transform the range of the coordinates, i.e. we take $\tau \in [0,\beta]$, $\tilde{\psi} \in [0,4\pi]$ also after the transformation, the new geometry is globally distinct from the original one. However, both the boundary and the bulk metric remain regular.[28] The choice of boundary gauge field A ensuring that the Killing spinors are independent of the new time coordinate on S^1 was explained in section 4.4.5, cf. eqs. (4.4.55), (4.4.57). For AdS$_5$ this also corresponds to the bulk gauge field:

$$-\sqrt{3}\mathcal{A} \;=\; A \;=\; \frac{\mathrm{i}}{2r_3}\left(-\cos\alpha + 2\mathrm{i}\sin\alpha\right)\mathrm{d}\tau\,. \tag{4.5.15}$$

[28]Regularity of the boundary metric follows from the fact that $\mathrm{d}\tilde{\psi} + \cos\theta\mathrm{d}\varphi$ is globally defined. Regularity of the bulk metric as $\rho \to 2r_3$ can be seen by noting that the $G_{\tau\tau}$ component remains finite, that the components $G_{\rho\rho}, G_{\theta\theta}, G_{\varphi\varphi}, G_{\tilde{\psi}\tilde{\psi}}$ and $G_{\tilde{\psi}\varphi}$ asymptote to the metric on the cone on a round S^3 (i.e. the flat metric on $\mathbb{R}^4$), and finally that the $G_{\tau\varphi}, G_{\tau\theta}$ components go to zero. It follows that as $\rho \to 2r_3$ the space looks like $S^1 \times \mathbb{R}^4$.

Note that this has both a real and an imaginary part.

The on-shell action in the standard holographic scheme is found to be

$$\mathcal{S} \;=\; \cos\alpha\,\frac{3(1-96\varsigma)\beta}{4r_3}\frac{\pi^2}{\kappa_5^2}\;, \tag{4.5.16}$$

as the only consequence of the twist in the computation is to rescale the volumes by $\cos\alpha$. The new boundary terms (4.4.39) are evaluated as for untwisted AdS_5, except that one must implement the transformation (4.5.12) and use the gauge field (4.5.15). This gives

$$\Delta I_{\rm new} \;=\; \left(-\frac{17}{108}\cos\alpha+\frac{16}{27}\,\mathrm{i}\,\sin\alpha\right)\frac{\beta}{r_3}\frac{\pi^2}{\kappa_5^2}\;. \tag{4.5.17}$$

Then the supersymmetric on-shell action evaluates to

$$\mathcal{S}_{\rm susy} \;=\; \mathcal{S}_{\varsigma=0}+\Delta I_{\rm new} \;=\; \frac{16\beta\,\mathrm{e}^{\mathrm{i}\alpha}}{27r_3}\frac{\pi^2}{\kappa_5^2}\;. \tag{4.5.18}$$

This illustrates in a concrete example the general result of section 4.4.5 that the on-shell action in the twisted background is related to the one in the untwisted background by the replacement $\beta\to\mathrm{e}^{\mathrm{i}\alpha}\beta$.

4.5.3 A simple squashing of AdS_5

A different one-parameter supersymmetric deformation of AdS_5 was presented in [64]. In this solution, the boundary geometry is non conformally flat as $S^3\subset\partial AdS_5$ is squashed. The squashing is such that the Hopf fibre of $S^1\hookrightarrow S^3\to S^2$ is rescaled with respect to the S^2 base by a parameter v, and thus defines a Berger sphere S^3_v with $SU(2)$-invariant metric. The boundary metric then reads

$$\mathrm{d}s_4^2 \;=\; \mathrm{d}\tau^2+\frac{r_3^2}{4}\left[v^2\big(\mathrm{d}\tilde\psi+\cos\theta\mathrm{d}\varphi\big)^2+\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2\right]\;, \tag{4.5.19}$$

which for $v=1$ reduces to (4.5.2), (4.5.3). The boundary geometry is controlled by the three parameters β, r_3, v, however the complex structure on the boundary is determined just by the ratio $\frac{\beta}{vr_3}$ specifying the relative size of S^1_β to the Hopf fibre, hence the supersymmetric field theory partition function depends on these parameters only through this combination [68, 19].

As for the solutions in section 4.4.1, the supergravity solution of [64] was constructed in Lorentzian signature and then analytically continued so that the boundary is Riemannian, while the bulk metric becomes complex. It is known analytically at first order in the squashing and numerically for finite v. While we refer to [64] for more details, here it will be sufficient to mention that the solution is regular and such that the global assumptions made in section 4.4.4 to derive the on-shell action formula (4.4.52) are satisfied. In fact, as already mentioned, the strategy followed in section 4.4.4 is a generalization of the one in [64]. Since its near-boundary behaviour falls in the larger family of perturbative solutions constructed in the present chapter, the solution of [64] also provides a concrete example that the latter can admit a smooth completion in the interior also when the boundary is not conformally flat.

While the field theory results predict that the on-shell action only depends on the ratio $\frac{\beta}{vr_3}$, it was found in [64] that after performing standard holographic renormalization this depends both on $\frac{\beta}{vr_3}$ and v. Indeed, in a minimal scheme where the finite counterterms (4.4.23) are set to zero one obtains[29]

$$S_{\rm min} \;=\; \frac{8v\beta}{r_3}\left(\frac{2}{27v^2}+\frac{2}{27}-\frac{13}{108}v^2+\frac{19}{288}v^4\right)\frac{\pi^2}{\kappa_5^2}\,, \tag{4.5.20}$$

so only the first term in parenthesis has the correct dependence on $\frac{\beta}{vr_3}$. In addition, it was shown in [64, sect. 5.3] that there is no combination of the ordinary finite counterterms (4.4.23) that cancels all but the first term in (4.5.20). It was then proposed that a new counterterm should be added, and it was found that a certain term involving the Ricci form, combined with the standard finite counterterms, does the job (cf. eq. (5.51) therein). However, in the light of our more general analysis that specific prescription turns out incorrect, as the proposed term does not evaluate to $\Delta I_{\rm new}$ in (4.4.37) for the more general boundary metric and gauge field considered in the present chapter. This also follows from the fact that the term proposed in [64] is gauge invariant, while in order to adjust the holographic R-charge so that the BPS condition is satisfied a dependence on large gauge transformations is needed. Therefore while the idea of correcting the holographic renormalization scheme by new boundary terms survives and is much strengthened by the general analysis performed in the present chapter, a covariant form for these terms remains to be found.

[29] See also. eq. (4.15) of [64]. The present variables are obtained setting $\Delta_t^{\rm there} = \frac{v}{r_3}\beta$ and $8\pi G/\ell^2 = \kappa_5^2$.

Let us show how ΔI_{new} removes the terms in (4.5.20) not depending solely on $\frac{\beta}{v r_3}$. The metric (4.5.19) on $S^1 \times S^3_v$ is obtained from our general boundary metric (4.2.8) by modifying slightly the transformations made for the example of AdS$_5$. Again we take $\mathrm{e}^{\frac{w}{2}} = \frac{r_3}{2}\frac{1}{1+|z|^2}$ and $z = \cot\frac{\theta}{2}\,\mathrm{e}^{-\mathrm{i}\varphi}$, so that the two-dimensional formulae (4.5.6) hold the same. Choosing $u = -\frac{4v}{r_3}$, the connection 1-form a can be taken $a = \frac{v r_3}{2}\cos\theta \mathrm{d}\varphi$, while the coordinate on the Hopf fibre with canonical period 4π is $\tilde{\psi} = \frac{2}{v r_3}\psi$. In this way (4.2.8) reduces to (4.5.19). Also choosing

$$\gamma \;=\; \frac{1}{v r_3}\,, \qquad\qquad \gamma' \;=\; 0\,, \qquad\qquad \lambda = -\frac{\varphi}{2}\,, \tag{4.5.21}$$

where again the value of γ is in agreement with (4.4.51), the boundary gauge field (4.2.10) reduces to the $SU(2)$-invariant expression[30]

$$-\sqrt{3}A_0 \;=\; A \;=\; -\frac{\mathrm{i}\,v}{2r_3}\mathrm{d}\tau + \frac{1}{2}(1-v^2)(\mathrm{d}\tilde{\psi} + \cos\theta \mathrm{d}\varphi)\,. \tag{4.5.22}$$

Then our formula (4.4.52) for the supersymmetric on-shell action evaluates to

$$S_{\text{susy}} \;=\; \frac{16\beta}{27 v r_3}\frac{\pi^2}{\kappa_5^2}\,, \tag{4.5.23}$$

that only depends on $\frac{\beta}{v r_3}$ as predicted by the field theory arguments. In fact our new counterterms evaluate to

$$\Delta I_{\text{new}} \;=\; -\frac{8v\beta}{r_3}\left(\frac{2}{27} - \frac{13}{108}v^2 + \frac{19}{288}v^4\right)\frac{\pi^2}{\kappa_5^2}\,, \tag{4.5.24}$$

which precisely accounts for the difference between (4.5.20) and (4.5.23). One could also consider twisting this five-dimensional solution by the parameter α as discussed in section 4.4.5 and further illustrated in the example of AdS$_5$, thus introducing an overall phase $\mathrm{e}^{\mathrm{i}\alpha}$ in the on-shell action.

Eq. (4.4.71) gives for the holographic charges:

$$H_{\text{susy}} \;=\; -\frac{1}{v r_3}Q_{\text{susy}} \;=\; \frac{16}{27 v r_3}\frac{\pi^2}{\kappa_5^2}\,, \qquad\qquad J_{\text{susy}} \;=\; 0\,. \tag{4.5.25}$$

[30] These boundary fields agree with those of [64] upon identifying $\psi^{\text{there}} = \tilde{\psi}$, $t^{\text{there}} = \frac{iv}{r_3}\tau$ and $a_0^{\text{there}} = \frac{r_3}{2}$.

The electric charge given in [64, sect. 4] reads in the present normalization

$$Q^{\text{there}} \;=\; -\frac{16\pi^2}{27\kappa_5^2}(v^2-1)^2\,, \tag{4.5.26}$$

while the shift (4.4.64) due to our new boundary terms evaluates to

$$\Delta Q \;=\; -\int \text{vol}_3\,\Delta j^t \;=\; \frac{16\pi^2}{27\kappa_5^2}(v^4-2v^2)\,, \tag{4.5.27}$$

therefore $Q^{\text{there}}+\Delta Q$ matches the supersymmetric charge in (4.5.25). When comparing (4.5.25) with the energy and angular momentum computed in [64] one needs to take into account both the contribution of the new boundary terms and the fact that in [64] these quantities were defined in terms of the energy-momentum tensor alone (which for the present solution still yields conserved quantities), while here we presented the charges (4.4.66), (4.4.68) computed from the current (4.4.59) that is always conserved in the presence of a general background gauge field.

4.5.4 Hopf surfaces at the boundary

We can also evaluate our on-shell action formula (4.4.52) for the more general boundary geometry with $S^1\times S^3$ topology considered in [19]. Contrarily to the previous examples in this section, in this case we do not have a general proof of existence of regular bulk fillings satisfying all the global properties we required in section 4.4.4 to evaluate the on-shell action. However, we are going to show that if we assume that such supergravity solutions exist, then eq. (4.4.52) gives the correct holographic dual of the supersymmetric Casimir energy of [19, 18].

In [19] the three-sphere is described as a torus foliation: the torus coordinates are $\varphi_1\in[0,2\pi]$, $\varphi_2\in[0,2\pi]$, while the remaining coordinate is $\hat\rho\in[0,1]$.[31] The four-dimensional complex manifolds with topology $S^1\times S^3$ are Hopf surfaces, and in [19] the complex structure moduli are two real, positive parameters $\beta b_1,\beta b_2$ (as above, β denotes the circumpherence of the S^1 parameterized by τ). These characterize the choice of complex

[31] The coordinate $\hat\rho$ is defined on the four-dimensional boundary and should not be confused with the radial coordinate ρ used elsewhere in this chapter.

Killing vector (4.2.7) as

$$K = \frac{1}{2}\left(\partial_\psi - \mathrm{i}\,\partial_\tau\right) = \frac{1}{2}\left(b_1\partial_{\varphi_1} + b_2\partial_{\varphi_2} - \mathrm{i}\,\partial_\tau\right) . \tag{4.5.28}$$

The four-dimensional metric is taken as

$$\begin{aligned} \mathrm{d}s_4^2 &= \Omega^2\left[\mathrm{d}\tau^2 + (\mathrm{d}\psi + a_\chi \mathrm{d}\chi)^2 + \Omega^{-2}f^2\mathrm{d}\hat{\rho}^2 + c^2\mathrm{d}\chi^2\right] \\ &= \Omega^2\mathrm{d}\tau^2 + f^2\mathrm{d}\hat{\rho}^2 + m_{IJ}\mathrm{d}\varphi_I\mathrm{d}\varphi_J , \end{aligned} \tag{4.5.29}$$

where $I, J = 1, 2$. The first line is the canonical form dictated by supersymmetry (with $\mathrm{d}s_2^2 = \Omega^{-2}f^2\mathrm{d}\hat{\rho}^2 + c^2\mathrm{d}\chi^2$), while the expression in the second line is convenient for discussing global properties, since it uses periodic coordinates. When passing from the first to the second expressions one identifies the coordinates as

$$\psi = \frac{1}{2}\left(\frac{\varphi_1}{b_1} + \frac{\varphi_2}{b_2}\right) , \qquad \chi = \frac{1}{2}\left(\frac{\varphi_1}{b_1} - \frac{\varphi_2}{b_2}\right) \tag{4.5.30}$$

and the functions as

$$a_\chi = \frac{1}{\Omega^2}\left(b_1^2 m_{11} - b_2^2 m_{22}\right) , \qquad c = \frac{2b_1 b_2}{\Omega^2}\sqrt{\det m_{IJ}} . \tag{4.5.31}$$

Moreover supersymmetry imposes the relation

$$\Omega^2 = b^I m_{IJ} b^J , \tag{4.5.32}$$

which ensures Hermiticity of the metric. Here, f and m_{IJ} are functions of $\hat{\rho}$ satisfying suitable boundary conditions at $\hat{\rho} = 0$ and $\hat{\rho} = 1$ so that the metric is regular and describes a smooth S^3 topology. As $\hat{\rho} \to 0$, one requires that

$$f \to f_2 , \quad m_{11} \to m_{11}(0) , \quad m_{22} = (f_2\hat{\rho})^2 + \mathcal{O}(\hat{\rho}^3) , \quad m_{12} = \mathcal{O}(\hat{\rho}^2) , \tag{4.5.33}$$

where $f_2 > 0$ and $m_{11}(0) > 0$ are constants, and similarly for $\hat{\rho} \to 1$ (see [19]).

In principle our on-shell action formula (4.4.52) is derived for a boundary metric of the type (4.2.8), thus with trivial conformal factor $\Omega = 1$, however we now show that the same formula gives the correct result even for general Ω if it is evaluated using the metric in the square bracket of (4.5.29).[32]

Using the expressions above, we can compute

$$\int_{M_3} \mathrm{vol}_3 R_{2d} = -\int \partial_{\hat\rho}\left(\frac{c\,\Omega}{f}\,\partial_{\hat\rho}\log c^2\right)\mathrm{d}\hat\rho\wedge\mathrm{d}\chi\wedge\mathrm{d}\psi = -\frac{4\pi^2}{b_1 b_2}\left[\frac{\Omega}{f}\,\partial_{\hat\rho} c\right]_{\hat\rho=0}^{\hat\rho=1} = 8\pi^2\frac{b_1+b_2}{b_1 b_2}\,, \tag{4.5.34}$$

where in the last equality we used the behaviour of the functions at the extrema of the $\hat\rho$ interval. Similarly,

$$\int_{M_3} \eta\wedge\mathrm{d}\eta = \int \partial_{\hat\rho} a_\chi\,\mathrm{d}\hat\rho\wedge\mathrm{d}\chi\wedge\mathrm{d}\psi = \frac{2\pi^2}{b_1 b_2} a_\chi\Big|_{\hat\rho=0}^{\hat\rho=1} = -\frac{4\pi^2}{b_1 b_2}\,. \tag{4.5.35}$$

Then formula (4.4.51) for γ gives

$$\gamma = \frac{1}{2}(b_1+b_2) \tag{4.5.36}$$

and the on-shell action (4.4.52) evaluates to

$$\mathbb{S}_{\mathrm{susy}} = \frac{2\beta}{27}\,\frac{(b_1+b_2)^3}{b_1 b_2}\,\frac{\pi^2}{\kappa_5^2}, \tag{4.5.37}$$

which perfectly matches the field theory prediction (4.2.29).[33] This result was the main point emphasized in our short communication [39].

4.5.5 General M_3

In section 4.4.4 we derived the general formula (4.4.52) for the supersymmetric on-shell action (evaluated with our new counterterms). Here the conformal boundary has topology $S^1\times M_3$, and the derivation of the formula requires certain global assumptions about the topology of the five-dimensional bulk supergravity solution that fills this boundary. In particular, we required the graviphoton field $\mathcal{A}$ to be a global 1-form. Particular explicit examples have been studied in the subsections above. In this subsection we present a more general but abstract analysis, and show that our supergravity result (4.4.52) always

[32] Otherwise one can choose m_{IJ} so that (4.5.32) is satisfied with $\Omega=1$, which is not a serious loss of generality since it still allows for general b_1, b_2.

[33] This agrees with eq. (5.18) of [19], upon identifying $|b_I|^{\mathrm{there}} = \frac{\beta}{2\pi} b_I^{\mathrm{here}}$ and $8\pi G^{\mathrm{there}} = \kappa_5^2$.

reproduces the supersymmetric Casimir energy, as computed in field theory in [176].[34]

We begin by rewriting the supersymmetric on-shell supergravity action (4.4.52) in terms of Seifert invariants of M_3. In particular, using equations (4.2.19) and (4.2.20) we may write

$$\mathbb{S}_{\text{susy}} \;=\; \frac{2\pi^2 b\beta}{27\kappa_5^2}\frac{\left(\int_{\Sigma_2} c_1(\Sigma_2)\right)^3}{\left(\int_{\Sigma_2} c_1(\mathcal{L})\right)^2}\,. \tag{4.5.38}$$

Recall here that ψ has period $2\pi/b$, so that the Reeb vector $\xi = \partial_\psi = b\chi$, where χ is the normalized vector field which exponentiates to the corresponding $U(1)$ action on M_3.

Under the same global assumptions on $M_4 \cong S^1_\beta \times M_3$, the supersymmetric Casimir energy E_{susy} was computed in field theory in [176]. More precisely, in the path integral sector with trivial flat gauge connection on M_3, E_{susy} may be computed from an index-character that counts holomorphic functions on $X_0 \cong \mathbb{R}_{>0} \times M_3$. The formula for weighted homogeneous hypersurface singularities was given in equation (4.2.31), with large N limit (4.2.33). Substituting for $\int_{\Sigma_2} c_1(\Sigma_2)$ and $\int_{\Sigma_2} c_1(\mathcal{L})$ for hypersurface singularities using formulas (4.2.24), the supergravity result (4.5.38) precisely agrees with the large N field theory computation of βE_{susy}, with E_{susy} given by (4.2.33)!

This agreement between exact field theory and supergravity calculations is already remarkable. However, we can go further and present a very general derivation of this agreement, based on a formula for the index-character appearing in [177]. Recall first that the $U(1)$ Seifert action on M_3 extends to a holomorphic $\mathbb{C}^*$ action on $X_0 = \mathbb{R}_{>0} \times M_3$, and hence on $X = C(M_3)$. Following [176, 177], we denote the index-character that counts holomorphic functions on X (or equivalently X_0) according to their weights under $q \in \mathbb{C}^*$ by $C(\bar{\partial}, q, X)$. If the $U(1) \subset \mathbb{C}^*$ action is *free*, meaning that $\Sigma_2 = M_3/U(1)$ is a smooth Riemann surface, then we may write

$$\begin{aligned} C(\bar{\partial}, q, X) &= \sum_{k\geq 0} q^k \int_{\Sigma_2} \mathrm{e}^{-kc_1(\mathcal{L})} \cdot \mathrm{Todd}(\Sigma_2) && (4.5.39)\\ &= \sum_{k\geq 0} q^k \int_{\Sigma_2} \left[-k\,c_1(\mathcal{L}) + \frac{1}{2}c_1(\Sigma_2)\right]\,. && (4.5.40) \end{aligned}$$

The first equality is the Riemann–Roch theorem, and the second equality uses Todd $= 1 + \frac{1}{2}c_1 + \cdots$, where the higher order terms do not contribute in this dimension. We may

[34] There are caveats to this statement, that we will clarify below.

then sum the series for $|q| < 1$ to obtain the formula

$$C(\bar{\partial}, q, X) \quad = \quad \frac{\int_{\Sigma_2} c_1(\Sigma_2) - q\left(\int_{\Sigma_2} 2c_1(\mathcal{L}) + c_1(\Sigma_2)\right)}{2(1-q)^2} \,. \tag{4.5.41}$$

We emphasize that this formula is valid for regular Reeb vector fields, so that Σ_2 is a smooth Riemann surface, and is *not* valid in the quasi-regular case, where Σ_2 has orbifold singularities. However, as we shall explain below, one may effectively still use this formula to compute the large N supersymmetric Casimir energy even in the general quasi-regular case.

The full character that computes the supersymmetric Casimir energy is given by [176]

$$C(q, \mu, X) \quad = \quad q^{-\int_{\Sigma_2} c_1(\Sigma_2)/2 \int_{\Sigma_2} c_1(\mathcal{L})} \cdot \mu \cdot C(\bar{\partial}, q, X) \,. \tag{4.5.42}$$

Here the power of q in the first factor is precisely γ/b, which arises as $\frac{1}{2}$ the charge of the holomorphic $(2,0)$-form under the canonically normalized vector field χ. The supersymmetric Casimir energy is then obtained by setting $q = \mathrm{e}^{tb}$, $\mu = \mathrm{e}^{-tu}$, where $u = (r-1)\gamma$ for a matter multiplet of R-charge r, and extracting the coefficient of $-t$ in a Laurent series about $t = 0$. For field theories with a large N gravity dual in type IIB supergravity one has $\mathtt{a} = \mathtt{c} = \pi^2/\kappa_5^2$, where the trace anomaly coefficients may in turn be expressed in terms of certain cubic functions of the R-charges $(r-1)$ of fermions. Using this prescription applied to (4.5.42), (4.5.41), we find that the large N field theory result gives

$$E_{\mathrm{susy}} \quad = \quad \frac{2\pi^2 b}{27\kappa_5^2} \frac{\left(\int_{\Sigma_2} c_1(\Sigma_2)\right)^3}{\left(\int_{\Sigma_2} c_1(\mathcal{L})\right)^2} \,, \tag{4.5.43}$$

so that the supergravity action $\mathcal{S}_{\mathrm{susy}}$ in (4.5.38) agrees with βE_{susy} computed in field theory.

Although (4.5.41) only holds in the regular case, in fact this formula is sufficient to compute the correct large N supersymmetric Casimir energy in (4.5.43) in the general quasi-regular case. The point is that when Σ_2 has orbifold singularities there are additional contributions to Riemann–Roch formula (4.5.41). However, also as in [177], the general form of these contributions is such that they do not contribute to the relevant limit that gives (4.5.43). Thus the latter formula holds in general (we have already shown independently that it holds for homogeneous hypersurface singularities, which are generically not regular).

Finally, although the agreement of the two computations is remarkable, without more work it is also somewhat formal. In particular, in the field theory computation we have assumed that the sector with trivial flat gauge connection dominates at large N, while the general supergravity computation assumes the existence of an appropriate solution with the required global properties. Known examples suggest that these are not unreasonable assumptions, but there is clearly a need for further work to clarify how general a result this is. We leave these interesting questions for future work.

4.6 Discussion

Since the early days of the AdS/CFT correspondence, it has been clear that in order to define observables holographically, infinities have to be subtracted [225, 127, 27]. These initial findings developed into the systematic framework of holographic renormalization, which has taken various incarnations [79, 47, 48, 78, 170, 204, 192, 187]. Despite the fact that this has proved to be very robust as a method for subtracting infinities in the context of asymptotically locally hyperbolic solutions, the problem of matching *finite* boundary terms in holographic computations to choices of renormalization schemes in quantum field theory has remained a subtle question requiring further study. Recent exact results in supersymmetric quantum field theories, in part obtained through the technique of localization, have sharpened this question within a large class of holographic constructions. In this chapter, we have presented a systematic study of the interplay of holographic renormalization and supersymmetry, in the context of minimal $\mathcal{N} = 2$ gauged supergravity theories in four and five dimensions. These theories are consistent truncations of eleven-dimensional and type IIB supergravity on very general classes of internal manifolds with known field theory duals. They thus give access to a vast set of examples of supersymmetric gauge/gravity dual pairs, where both sides are well understood [198, 172, 175, 173, 171, 132, 64, 94, 62, 39].

In this chapter we have made certain simplifying assumptions; in particular our studies apply to asymptotically locally hyperbolic solutions of the given supergravities, where the boundary geometry admits at least a pair of Killing spinors. Under these assumptions, our main results may be summarized as follows. In four-dimensional minimal $\mathcal{N} = 2$ gauged supergravity, the on-shell action, renormalized using standard counterterms, is supersymmetric. In particular, as expected, we did not find any ambiguities related to finite counterterms.[35] In five-dimensional minimal gauged supergravity, we showed that there is *no choice* of standard finite counterterms (i.e. four-dimensional diffeomorphism and gauge invariants) that renders the holographically renormalized on-shell action compatible with the boundary supersymmetry obtained by coupling to off-shell new minimal supergravity. Thus, surprisingly, standard holographic renormalization breaks supersymmetry in five dimensions. We then found a specific set of new boundary terms that restores supersymmetry of the on-shell action, as well as the validity of certain supersymmetric Ward identities inferred from field theory [68, 67]. We provided some independent tests of these new terms, illustrating their application in smooth asymptotically locally AdS solutions with topology $\mathbb{R} \times \mathbb{R}^4$.

Although our analysis provides a very strong evidence that in order to formulate holographic renormalization in a supersymmetric fashion a new set of boundary terms is needed, a more fundamental understanding of the origin of these terms is clearly desirable. As already mentioned at the end of section 2.5, after the publication of the work in this chapter in [39, 38], the same expressions for the variation of the on-shell action (4.4.35), (4.4.36) were recovered independently in [191], but interpreted in a different way. Since under AdS/CFT semi-classical gravity computations correspond to quantum field theory computations, the non-vanishing variation of the on-shell action was interpreted as an anomalous variation of the fermionic part of the supercurrent on the boundary. Led by this result, Papadimitriou concluded that rigid supersymmetry is anomalous on generic non-Ricci-flat backgrounds, and so that the results in [68] on the dependence of BPS observables on the background are flawed (as noticed in footnote 2 of [68], the authors require the absence of such anomalies) – a similar computation for a non-minimal gauged supergravity was then carried out in [12]. As written in these latter articles, the interpretation of the non-vanishing variation in terms of a supercurrent anomaly cast doubts on localization computations for four-dimensional field theories. However, there are subtle nuances in these results and some room for maneuvering: for instance, bulk minimal gauged supergravity reduces at the boundary to conformal supergravity, whereas the localization computations are done using the coupling to (non-dynamical) new minimal supergravity.[36] Moreover, it is paramount to

[35]This situation is radically different in supergravity models coupled to matter. The interplay of holographic renormalization and supersymmetry in the presence of scalar fields has been discussed for conformally flat boundaries in [47, 102, 54, 55, 101, 150].

[36]Notice that this difference is crucial for the case of the topological twist: there, we found that the inclusion of any additional finite counterterm would have spoiled the result of the independence of the gravitational free

emphasize that all the works cited assumed the validity of the gauge/gravity dictionary, and used this to either obtain constraints on the gravity side from exact results originally derived on the field theory side (as in this chapter), or find supersymmetric anomalies in field theory from gravity computations (as in [191, 12]). It is still an open problem to perform a first principles analysis of supersymmetry of supergravities in asymptotically locally hyperbolic space-times, or to directly derive this anomalous transformation from the QFT in a new minimal supergravity background.

Let us mention some possible avenues that could be pursued to achieve the former goal. A direct approach to retrieve the correct boundary terms is to work on a space with a boundary at a finite distance and to impose that the combination of bulk plus boundary supergravity action is invariant under supersymmetry (of course the bulk action is invariant under supersymmetry *up to boundary terms*). Notice that, in different situations, this approach has been recently advocated in [14, 101]. One could also attempt to derive the boundary terms by enforcing the holographic Ward identities stemming from supersymmetry, using the Hamilton–Jacobi approach [170, 190]. It may also be fruitful to extend to higher dimensions the approach of [37, 115], where the standard holographic counterterms in three-dimensional[37] $\mathcal{N} = 1$ supergravity were argued to preserve supersymmetry, by working in an off-shell formulation. It would be very interesting to see whether any of these methods, or possibly others, may be used to shed light on the origin of the boundary terms proposed in the present chapter.

We conclude by alluding to a few possible generalizations of our results. Perhaps the most straightforward extension will be to lift the simplifying assumption that the metric on the four-dimensional conformal boundary is of a direct product type $S^1 \times M_3$. We expect that the new boundary terms arising from this analysis will be more general than those found presently, and this could help achieving a better understanding of them. One could also study the consequences on such terms following from a Weyl transformation of the boundary metric. In minimal gauged supergravity, to complete the program we initiated it will be necessary to address the supersymmetric solutions in the null class [108], which are known to comprise asymptotically locally AdS solutions. Another obvious generalization

energy from the background metric. In the language of this chapter, the supersymmetric Ward identity was satisfied in the minimal scheme. However, the topological twist can be obtained as a special case of the rigid limit coupling to *conformal* supergravity. Is it possible that this difference is crucial?

[37] An off-shell formulation of four dimensional supergravity in the presence of a boundary has been considered in [36], however as far as we are aware the application to the study of holographic renormalization is lacking in the literature.

would be to investigate similar gauged supergravities in three, six, and seven space-time dimensions. In particular, it is expected that defining two- and six-dimensional SCFTs in curved backgrounds leads to suitable versions of the supersymmetric Casimir energy [53], and reproducing these in dual holographic computations remains an open problem. The fact that in odd bulk dimension one has anomalies and ambiguities in holographic renormalization suggests that at least in these dimensions a supersymmetric formulation of holographic renormalization will lead to a set of new boundary terms, analogous to those we uncovered in five-dimensional supergravity.

Finally, we emphasize that in the derivation of the boundary terms, we made no assumptions on the properties of the supersymmetric solutions in the bulk. In particular, our boundary terms should be included in holographic studies of supersymmetric solutions with topologies different from $\mathbb{R} \times \mathbb{R}^4$. For example, it will be nice to investigate how the analysis of the properties of supersymmetric asymptotically locally AdS black holes [122, 65] (or topological solitons [74, 62]) will be affected by our findings.

A

Some conventions

A.1 Curvature tensors

Our sign convention on the Riemann tensor is fixed by

$$R^i{}_{jkl} = \partial_k \Gamma^i_{jl} + \Gamma^i_{km}\Gamma^m_{jl} - k \leftrightarrow l \ ,$$

and the Ricci tensor is $R_{ij} = R^k{}_{ikj}$. We next give some formulae by specializing to four dimensions. The Weyl tensor of a metric g_{ij} and its square are given by

$$\begin{aligned} C_{ijkl} &= R_{ijkl} - g_{i[k}R_{l]j} + g_{j[k}R_{l]i} + \frac{1}{3}R\, g_{i[k}g_{l]j} \ , \\ C_{ijkl}C^{ijkl} &= R_{ijkl}R^{ijkl} - 2R_{ij}R^{ij} + \frac{1}{3}R^2 \ . \end{aligned} \tag{A.1.1}$$

The Euler and Pontryagin densities can be written as

$$\mathcal{E} = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2 \ , \qquad \mathcal{P} = \frac{1}{2}\epsilon^{ijkl}R_{ijmn}R_{kl}{}^{mn} \ . \tag{A.1.2}$$

From the metric and the Levi-Civita symbol we can construct four linearly independent functionals: $\int d^4x\sqrt{g}\,\mathcal{E}$ (proportional to the Euler characteristic), $\int d^4x\sqrt{g}\,\mathcal{P}$ (proportional to the signature invariant), $\int d^4x\sqrt{g}\,C_{ijkl}C^{ijkl}$ (the conformal gravity action) and $\int d^4x\sqrt{g}\,R^2$

(which is neither topological nor conformal). While the metric variation of the first and the second vanishes identically in four dimensions, varying the third defines the Bach tensor

$$\begin{aligned} B_{ij} &= -\frac{1}{2\sqrt{g}}\frac{\delta}{\delta g^{ij}}\int \mathrm{d}^4x\sqrt{g}\,C_{klmn}C^{klmn} \\ &= \frac{1}{3}\nabla_i\nabla_j R - \nabla^2 R_{ij} + \frac{1}{6}g_{ij}\,\nabla^2 R - 2R_{ikjl}R^{kl} + \frac{2}{3}RR_{ij} + \frac{1}{2}g_{ij}\left(R_{kl}R^{kl} - \frac{1}{3}R^2\right). \end{aligned} \tag{A.1.3}$$

This is covariantly conserved and traceless. Varying the fourth functional yields the tensor

$$H_{ij} = -\frac{1}{\sqrt{g}}\frac{\delta}{\delta g^{ij}}\int \mathrm{d}^4x\sqrt{g}\,R^2 \;=\; 2\nabla_i\nabla_j R - 2g_{ij}\nabla^2 R + \frac{1}{2}g_{ij}R^2 - 2R\,R_{ij}\,,$$

which is covariantly conserved and satisfies $H_i{}^i = -6\nabla^2 R$.

A.2 Hodge dual conventions

The Hodge dual is defined for two k-forms α and β by

$$\alpha\wedge *\beta \;=\; \frac{1}{k!}\langle\alpha,\beta\rangle\,\mathrm{vol}_g \;=\; \frac{1}{k!}\alpha_{a_1\cdots a_k}\beta^{a_1\cdots a_k}\,\mathrm{vol}_g\,, \tag{A.2.1}$$

where $\langle\cdot,\cdot\rangle$ is the inner product induced by g on the fibers of $\Lambda^k(M)$. Then

$$**\alpha \;=\; (-1)^{k(n-k)}\alpha \quad \forall\alpha\in\Omega^k(M)\,, \qquad X\lrcorner\,\mathrm{vol}_g \;=\; *X^\flat \quad \forall X\in\mathfrak{X}(M)\,.$$

The components of the codifferential of a k-form α satisfy

$$(*\mathrm{d}*\alpha)_{a_1\cdots a_{k-1}} \;=\; (-1)^{k(n-k)+n+1}\nabla^b\alpha_{a_1\cdots a_{k-1}b}\,.$$

The self-dual and anti-self-dual 't Hooft symbols are defined by

$$\eta^a_{ij} \;=\; \epsilon_{aij4} + \delta_{ai}\delta_{j4} - \delta_{aj}\delta_{i4}\,, \qquad \overline{\eta}^a_{ij} \;=\; \epsilon_{aij4} - \delta_{ai}\delta_{j4} + \delta_{aj}\delta_{i4}\,, \tag{A.2.2}$$

where $a = 1,2,3$, and $i,j = 1,2,3,4$. The Clifford product is defined, for $\alpha\in\Omega^k(M)$ and ψ a spinor on M, by

$$\alpha\cdot\psi \equiv \frac{1}{k!}\alpha_{a_1\cdots a_k}\gamma^{a_1\cdots a_k}\psi\,. \tag{A.2.3}$$

B Details on Holographic Renormalization and Supersymmetry

B.1 Construction of the five-dimensional solution

B.1.1 The general equations

In this appendix we provide details on how our five-dimensional supersymmetric solution is constructed. We start by summarizing the conditions for bosonic solutions of minimal gauged supergravity in five dimensions to be supersymmetric, first obtained in [108] and recently revisited in [62]. The analysis of [108] shows that the supersymmetry equation (4.4.4) implies the existence of a Killing vector field V that is either timelike or null. In this thesis we just consider the timelike case. Choosing coordinates such that $V = \partial/\partial y$, the five-dimensional metric takes the form

$$\mathrm{d}s_5^2 = -f^2 \left(\mathrm{d}y + \omega\right)^2 + f^{-1}\,\mathrm{d}s_B^2\ , \qquad \text{(B.1.1)}$$

where $\mathrm{d}s_B^2$ is a Kähler metric on a four-dimensional base B transverse to V, while f and ω are a positive function and a 1-form on B, respectively. We will work with a Kähler form J that is anti-self-dual on B, namely, $*_B J = -J$, so that the orientation on B is fixed as

$\mathrm{vol}_B = -\frac{1}{2}J \wedge J$. We will also need the Ricci form $\mathcal{R}$ and its potential P, satisfying $\mathcal{R} = \mathrm{d}P$. The Ricci form is defined as $\mathcal{R}_{mn} = \frac{1}{2}R_{mnpq}J^{pq}$, where R_{mnpq} is the Riemann tensor of the Kähler metric and $m, n = 1, \ldots, 4$ are curved indices on B. The Ricci potential also appears in the relation $\nabla_m \Omega_{np} + \mathrm{i}P_m \Omega_{np} = 0$, where ∇_m is the Levi-Civita connection of the Kähler metric and Ω is a complex $(2,0)$-form normalized as $\Omega \wedge \overline{\Omega} = 2J \wedge J$.

The geometry of the Kähler base determines the whole solution. The function f in (B.1.1) is given by

$$f \ = \ -\frac{24}{R} \,, \tag{B.1.2}$$

where R is the Ricci scalar of the Kähler metric, and is required to be non-zero everywhere. The equations for the 1-form ω are

$$\mathrm{d}\omega + *_B \mathrm{d}\omega \ = \ \frac{R}{24}\Big(\mathcal{R} - \frac{1}{4}RJ\Big) \,, \tag{B.1.3}$$

and

$$(\mathrm{d}\omega)_{mn}J^{mn} \ = \ -\frac{1}{12}\left(\frac{1}{2}\nabla^2 R + \frac{2}{3}R_{mn}R^{mn} - \frac{1}{3}R^2\right) \,. \tag{B.1.4}$$

It was shown in [62] that for these conditions to admit a solution the Kähler metric on B must necessarily satisfy the highly non-trivial sixth-order equation[1]

$$\nabla^2\left(\frac{1}{2}\nabla^2 R + \frac{2}{3}R_{mn}R^{mn} - \frac{1}{3}R^2\right) + \nabla^m(R_{mn}\partial^n R) \ = \ 0 \,. \tag{B.1.5}$$

Finally, the expression for the Maxwell field strength is

$$\mathcal{F} \ = \ -\sqrt{3}\,\mathrm{d}\Big[f(\mathrm{d}y + \omega) + \frac{1}{3}P\Big] \,. \tag{B.1.6}$$

The solutions obtained from (B.1.1)–(B.1.6) preserve at least (and generically no more than) two real supercharges.

B.1.2 The perturbative solution

We will make the assumption that the four-dimensional base B admits an isometry. This is motivated by the fact that (after Wick rotation) we want the boundary metric to reproduce the field theory background metric (4.2.8), and has the obvious advantage of simplifying

[1] The specialization of this equation for a particular Kähler metric appeared earlier in [98].

the supersymmetry equations. With no further loss of generality, for the metric on B we can choose

$$\mathrm{d}s_B^2 \;=\; U(r,z,\bar{z})^2\left[\frac{\mathrm{d}r^2}{r^2}+4r^2W(r,z,\bar{z})^2\mathrm{d}z\mathrm{d}\bar{z}\right]+\frac{r^4}{U(r,z,\bar{z})^2}(\mathrm{d}\hat{\psi}+\phi)^2\;, \qquad \text{(B.1.7)}$$

where z is a complex coordinate, $\hat{\psi}$ is the Killing coordinate (to be redefined later) and r will play the rôle of the radial coordinate. Moreover, $U(r,z,\bar{z})$, $W(r,z,\bar{z})$ are functions while ϕ is a $\hat{\psi}$-independent 1-form transverse to $\partial/\partial\hat{\psi}$. This type of metric ansatz has been studied by [158, 213] where it is shown to be the *generic* form satisfying our assumptions. The explicit powers of r in (B.1.7) have been introduced for convenience: they are chosen so that the asymptotic expansions of U and W start at order one – see below. We fix the orientation choosing the volume form on B as

$$\mathrm{vol}_B \;=\; 2\mathrm{i}r^3U^2W^2\mathrm{d}z\wedge\mathrm{d}\bar{z}\wedge\mathrm{d}\hat{\psi}\wedge\mathrm{d}r\;. \qquad \text{(B.1.8)}$$

The ansatz for the Kähler form is

$$J \;=\; 2\mathrm{i}r^2U^2W^2\,\mathrm{d}z\wedge\mathrm{d}\bar{z}+r\,\mathrm{d}r\wedge(\mathrm{d}\hat{\psi}+\phi)\;, \qquad \text{(B.1.9)}$$

which defines an almost complex structure, *i.e.* $J_m{}^pJ_p{}^n = -\delta_m{}^n$. The metric is Kähler if $\mathrm{d}J=0$ and the almost complex structure $J_m{}^n$ is integrable. Together, these two conditions are equivalent to imposing

$$\mathrm{d}\phi \;=\; \frac{1}{r}\partial_r\left(r^2U^2W^2\right)2\mathrm{i}\,\mathrm{d}z\wedge\mathrm{d}\bar{z}+\mathrm{i}(\mathrm{d}\bar{z}\,\partial_{\bar{z}}-\mathrm{d}z\,\partial_z)U^2\wedge\frac{\mathrm{d}r}{r^3}\;, \qquad \text{(B.1.10)}$$

which determines the connection 1-form ϕ in terms of other metric data. Acting on this equation with the exterior derivative, we find the integrability condition

$$\partial_z\partial_{\bar{z}}U^2+r^3\partial_r\left[r^{-1}\partial_r(r^2U^2W^2)\right] \;=\; 0\;, \qquad \text{(B.1.11)}$$

which constrains the functions U, W. Using (B.1.10), the Ricci scalar of the Kähler metric can be written as

$$R \;=\; -\frac{2}{r^2U^2W^2}\left[\partial_z\partial_{\bar{z}}\log W+\partial_r\left(rW\partial_r(r^3W)\right)+W\partial_r(r^3W)\right]\;, \qquad \text{(B.1.12)}$$

and the Ricci connection as

$$P = -\frac{1}{U^2 W}\partial_r(r^3 W)(\mathrm{d}\hat{\psi}+\phi) - \mathrm{i}(\mathrm{d}\bar{z}\,\partial_{\bar{z}} - \mathrm{d}z\,\partial_z)\log W\,, \tag{B.1.13}$$

with the Ricci form following from $\mathcal{R} = \mathrm{d}P$.

We will solve the supersymmetry equations in an asymptotic expansion around $r = \infty$. To do so, we express all functions entering in the ansatz in a suitable expansion involving powers of $1/r$ and $\log r$. The requirement that the solution be AlAdS$_5$ fixes the leading order terms in the expansions, as explained in detail in [61].

For the function $U(r, z, \bar{z})$ we take:

$$\begin{aligned} U &= \sum_{m\geq 0}\sum_{0\leq n\leq m} U_{2m,n}\frac{(\log r)^n}{r^{2m}} \\ &= U_{0,0} + \frac{1}{r^2}(U_{2,0} + U_{2,1}\log r) + \frac{1}{r^4}(U_{4,0} + U_{4,1}\log r + U_{4,2}(\log r)^2) + \ldots\,, \end{aligned} \tag{B.1.14}$$

with $U_{2m,n} = U_{2m,n}(z, \bar{z})$. Similarly, for W we take

$$W = W_{0,0} + \frac{1}{r^2}(W_{2,0} + W_{2,1}\log r) + \frac{1}{r^4}(W_{4,0} + W_{4,1}\log r + W_{4,2}(\log r)^2) + \ldots\,, \tag{B.1.15}$$

with all coefficients also being functions of $z, \bar{z}$. As for the 1-form ϕ, note that by redefining the coordinate $\hat{\psi}$ in (B.1.7) we can always take the radial component $\phi_r = 0$, namely we can take $\phi = \phi_z(r, z, \bar{z})\mathrm{d}z + \overline{\phi_z(r, z, \bar{z})}\mathrm{d}\bar{z}$. The expansion of ϕ_z is analogous to those of U and W (albeit with complex coefficients), in particular it starts at order $\mathcal{O}(1)$.

We also need to expand the 1-form ω appearing in the five-dimensional metric (B.1.1). By a redefinition of the coordinate y we can always choose $\omega_r = 0$. Then ω can be parameterized as

$$\omega = c(r, z, \bar{z})(\mathrm{d}\hat{\psi} + \phi) + C_z(r, z, \bar{z})\mathrm{d}z + \overline{C_z(r, z, \bar{z})}\mathrm{d}\bar{z}\,. \tag{B.1.16}$$

The expansion of the real function c starts at order $\mathcal{O}(r^2)$,

$$c = c_{-2,0}\,r^2 + (c_{0,0} + c_{0,1}\log r) + \frac{1}{r^2}\left(c_{2,0} + c_{2,1}\log r + c_{2,2}(\log r)^2\right) + \ldots\,, \tag{B.1.17}$$

and a similar expansion is taken for C_z.

We next solve order by order the conditions on the four-dimensional metric on B.

The explicit expressions are too cumbersome to be presented here and can only be dealt with using a computer algebra system like *Mathematica*; we will nevertheless describe in detail the procedure we followed. The constraints on the four-dimensional base metric amount to the equation (B.1.10) for ϕ, its integrability condition (B.1.11), and the sixth-order equation (B.1.5). We start from (B.1.11), that we solve for $U_{2,1}$, $U_{4,0}$, $U_{4,1}$, $U_{4,2}$, $U_{6,0}$, $U_{6,1}$, $U_{6,2}$, $U_{6,3}$ in terms of $U_{0,0}$, $U_{2,0}$ and the coefficients of W. Then we solve the sixth-order equation (B.1.5) at the first two non-trivial orders, which are $\mathcal{O}(1/r)$ and $\mathcal{O}(1/r^3)$ (together with the associated logarithmic terms). This fixes $W_{4,2}$, $W_{6,1}$, $W_{6,2}$, $W_{6,3}$ in terms of $U_{0,0}$, $U_{2,0}$, $W_{0,0}$, $W_{2,0}$, $W_{2,1}$, $W_{4,0}$, $W_{4,1}$, $W_{6,0}$, which thus remain undetermined at this stage. Finally we solve (B.1.10) for ϕ; the latter is explicitly determined, up to the leading $\mathcal{O}(1)$ term $\phi_{0,0}$, which has to obey the equation

$$d\phi_{0,0} \;=\; 4\mathrm{i}\,(U_{0,0}W_{0,0})^2 dz \wedge d\bar{z}\;. \tag{B.1.18}$$

Having fulfilled the constraints on the four-dimensional base B with metric (B.1.7), we can solve the equations (B.1.3), (B.1.4) for the connection ω. Using the ansatz (B.1.16), these become equations for c and C_z, that again we can solve order by order. We find that both c and C_z are fully determined (in particular, the divergent $\mathcal{O}(r^2)$ term in the expansion of $C_z dz + \overline{C_z} d\bar{z}$ vanishes), except for the $\mathcal{O}(1)$ term $C_{0,0}$ in the expansion of $C_z dz + \overline{C_z} d\bar{z}$, which is left free. In addition, from the $\mathcal{O}(\log r/r^2)$ term in the expansion of (B.1.3) we obtain a differential equation involving $U_{0,0}, W_{0,0}, W_{2,0}, W_{2,1}, W_{4,1}$ and $C_{0,0}$, that can most easily be solved for $W_{4,1}$ as the latter appears linearly and with no derivatives.[2]

We can next obtain the function f from (B.1.2). This concludes the construction of the metric (B.1.1) and the gauge field (B.1.6) near to $r \to \infty$. At leading order, we find that the five-dimensional metric is

$$ds_5^2 \;=\; \frac{dr^2}{r^2} + r^2 ds_4^2\;, \tag{B.1.19}$$

[2]This is a new constraint on the Kähler base metric, that may be unexpected since we have already solved all the conditions reviewed above for obtaining a supersymmetric solution from such metric. There is no contradiction here: *a priori* we could avoid to further constrain the Kähler metric by interpreting the equation under examination as a differential equation for the boundary function $C_{0,0}$. However, shortly we will impose a boundary condition setting $C_{0,0} = 0$; consistency with the present equation then fixes $W_{4,1}$.

where the metric ds_4^2 on the conformal boundary is

$$
\begin{aligned}
ds_4^2 \;=\; & \frac{1}{4U_{0,0}^4 W_{0,0}^2}\left[2W_{0,0}W_{2,1} - 2\mathrm{i}U_{0,0}^2(\mathrm{d}C_{0,0})_{z\bar{z}} - \partial_z\partial_{\bar{z}}\log W_{0,0}\right]\left(\mathrm{d}\hat{\psi} + \phi_{0,0}\right)^2 \\
& - 2\left(\mathrm{d}y + C_{0,0}\right)\left(\mathrm{d}\hat{\psi} + \phi_{0,0}\right) + 4W_{0,0}^2\mathrm{d}z\mathrm{d}\bar{z}\,.
\end{aligned} \tag{B.1.20}
$$

This is in agreement with the general form of a supersymmetric Lorentzian boundary metric, as can be seen by comparing with [61, eq. (4.12)]. In fact, it is even too general for our purposes, as it does not admit a simple Wick rotation to Euclidean signature. In order to be able to perform a simple Wick rotation and match (4.2.8), we will fix part of the free functions in (B.1.20) as

$$
C_{0,0} \;=\; 0\,, \qquad W_{2,1} \;=\; 2U_{0,0}^4 W_{0,0} + \frac{1}{2W_{0,0}}\partial_z\partial_{\bar{z}}\log W_{0,0}\,. \tag{B.1.21}
$$

In this way, the perturbative solution takes a simpler form, and only depends on the free functions $U_{0,0}$, $U_{2,0}$, $W_{0,0}$, $W_{2,0}$, $W_{4,0}$, $W_{6,0}$, where $U_{0,0}$ and $W_{0,0}$ are boundary data, while the remaining four functions only appear at subleading order in the five-dimensional metric. For convenience we will rename the boundary data as

$$
U_{0,0} \;=\; \frac{1}{2}u^{1/2}\,, \qquad W_{0,0} \;=\; \mathrm{e}^{w/2}\,, \qquad \phi_{0,0} \;=\; a \;=\; a_z\mathrm{d}z + \overline{a_z}\mathrm{d}\bar{z}\,, \tag{B.1.22}
$$

and the subleading functions as

$$
U_{2,0} \;=\; \mathrm{e}^{w/2}k_1\,, \quad W_{2,0} \;=\; \mathrm{e}^{w/2}k_2\,, \quad W_{4,0} = \mathrm{e}^{w/2}k_3\,, \quad W_{6,0} = \mathrm{e}^{w/2}k_4\,, \tag{B.1.23}
$$

where we recall that all functions depend on $z, \bar{z}$. Also redefining the Killing coordinates $\{y, \hat{\psi}\}$ into new coordinates $\{t, \psi\}$ as

$$
y \;=\; t\,, \qquad \hat{\psi} \;=\; \psi + t\,, \tag{B.1.24}
$$

the boundary metric becomes

$$
ds_4^2 = -\mathrm{d}t^2 + (\mathrm{d}\psi + a)^2 + 4\mathrm{e}^w\mathrm{d}z\mathrm{d}\bar{z}\,, \tag{B.1.25}
$$

with eq. (B.1.18) now being

$$\mathrm{d}a \;=\; \mathrm{i}\, u\, \mathrm{e}^{w} \mathrm{d}z \wedge \mathrm{d}\bar{z}\,. \tag{B.1.26}$$

At leading order, the gauge field strength reads

$$\mathrm{d}A_0 \;=\; -\frac{1}{\sqrt{3}}\,\mathrm{d}\left[-\frac{u}{8}\mathrm{d}t + \frac{u}{4}(\mathrm{d}\psi + a) + \frac{1}{4} *_2 \mathrm{d}w\right]\,, \tag{B.1.27}$$

where we denote $*_2\mathrm{d} = \mathrm{i}(\mathrm{d}\bar{z}\,\partial_{\bar{z}} - \mathrm{d}z\,\partial_z)$. The corresponding gauge potential is determined up to a gauge choice that will play an important rôle. We see that after taking $t = -\mathrm{i}\tau$, these agree with the field theory background fields (4.2.8), (4.2.10).

At subleading order the canonical form (B.1.1) of our five-dimensional metric is not of the Fefferman–Graham type (4.4.5), (4.4.6). Besides being more standard, the latter is desirable as it makes it simpler to extract the holographic data from the solution. We find that Fefferman–Graham coordinates are reached after implementing a suitable asymptotic transformation, sending $\{t, z^{\text{old}}, \psi^{\text{old}}, r\}$ into $\{t, z^{\text{new}}, \psi^{\text{new}}, \rho\}$ and having the form:

$$\begin{aligned}
r &= \frac{1}{\rho}\big[1 + \rho^2(m_{r,2,0} + m_{r,2,1}\log\rho) + \rho^4(m_{r,4,0} + m_{r,4,1}\log\rho + m_{r,4,2}(\log\rho)^2) + \mathcal{O}(\rho^5)\big],\\
z^{\text{old}} &= z^{\text{new}} + \rho^4\left(m_{z,4,0} + m_{z,4,1}\log\rho\right) + \mathcal{O}(\rho^5)\,,\\
\psi^{\text{old}} &= \psi^{\text{new}} + \rho^4\left(m_{\psi,4,0} + m_{\psi,4,1}\log\rho\right) + \mathcal{O}(\rho^5)\,,
\end{aligned} \tag{B.1.28}$$

where all the m coefficients are specific functions of $z, \bar{z}$. It should be noted that the conformal boundary, originally located at $r = \infty$, is now found at $\rho = 0$. In section 4.4.1 we give further details on the subleading terms in the metric and in the gauge field in Fefferman–Graham coordinates. There we drop the label "new", being understood that we always work in the new, Fefferman–Graham coordinates. Notice that since the metric can be cast in Fefferman–Graham form it is asymptotically locally anti-de Sitter.

B.2 Supersymmetry at the boundary

B.2.1 Killing spinors

At the boundary of a five-dimensional asymptotically locally anti-de Sitter solution, the supersymmetry condition (4.4.4) gives rise to the charged conformal Killing spinor equation

$$\nabla_i^A \zeta_\pm = -\frac{1}{4}\sigma_{\pm i}\sigma_\mp^j \nabla_j^A \zeta_\pm \,, \tag{B.2.1}$$

where we are using the two-component spinor notation introduced in section 4.2.2 and $\nabla_i^A \zeta_\pm = (\nabla_i \mp \mathrm{i}A_i)\,\zeta_\pm$ is the spinor covariant derivative, with ∇_i the Levi-Civita connection constructed with the boundary vierbein and $A = -\sqrt{3}A^{(0)}$ the canonically normalized gauge connection. This holds both in Euclidean and Lorentzian signature, for details see [145] and [61], respectively. Here we are identifying the $\Gamma^1, \Gamma^2, \Gamma^3, \Gamma^4$ matrices of Cliff(5) with those of Cliff(4), and the Γ^5 of Cliff(5) with the chirality matrix of Cliff(4); then we pass to two-component notation. The same equation ensures that some supersymmetry is preserved when a four-dimensional SCFT is coupled to background conformal supergravity, and (for spinors with no zeros) can be mapped into the equation arising when one couples the theory to new minimal supergravity [145, 90, 61].

One can see that the four-dimensional metric (4.2.8) and gauge field (4.2.10) allow for solutions to (B.2.1) and thus define a supersymmetric field theory background as well as supersymmetric boundary conditions for the bulk supergravity fields. Our scope here is to illustrate the gauge choice that makes the spinors independent of the coordinate τ, so that they are globally well-defined when this is made compact.

We choose the vierbein

$$e^1 + \mathrm{i}\,e^2 = 2\,\mathrm{e}^{\frac{w}{2}}\mathrm{d}z\,, \qquad e^3 = \mathrm{d}\psi + a\,, \qquad e^4 = \mathrm{d}\tau\,. \tag{B.2.2}$$

By studying (B.2.1) we find that in the generic case where u is non-constant, the solution reads

$$\zeta_+ = \frac{1}{\sqrt{2}}\,\mathrm{e}^{\gamma'\tau+\mathrm{i}\gamma\psi+\mathrm{i}\lambda}\begin{pmatrix}0\\1\end{pmatrix}, \qquad \zeta_- = \frac{1}{\sqrt{2}}\,\mathrm{e}^{-\gamma'\tau-\mathrm{i}\gamma\psi-\mathrm{i}\lambda}\begin{pmatrix}1\\0\end{pmatrix}, \tag{B.2.3}$$

where we have fixed an arbitrary overall constant. In the special case $u = \text{const}$ there exist additional solutions, however this enhancement of supersymmetry is not relevant for the present work and we will not discuss it further.

Kosmann's spinorial Lie derivative along a vector v is defined as

$$\mathcal{L}_v \zeta_\pm = v^i \nabla_i \zeta_\pm + \frac{1}{2} \nabla_i v_j \sigma_\pm^{ij} \zeta_\pm \,. \tag{B.2.4}$$

For the Killing vectors in our background, we find:

$$\begin{aligned} \mathcal{L}_{\partial_\psi} \zeta_\pm &= \partial_\psi \zeta_\pm = \pm \mathrm{i} \gamma \, \zeta_\pm \,, \\ \mathcal{L}_{\partial_\tau} \zeta_\pm &= \partial_\tau \zeta_\pm = \pm \gamma' \, \zeta_\pm \,, \end{aligned} \tag{B.2.5}$$

hence $\pm\gamma$ and $\pm\gamma'$ are the charge of the spinors $\zeta_\pm$ under ∂_ψ and $\mathrm{i}\partial_\tau$, respectively. It follows that the condition for $\zeta_\pm$ to be independent of τ is

$$\gamma' = 0 \,. \tag{B.2.6}$$

B.2.2 Superalgebra

The algebra of field theory supersymmetry transformations generated by a pair of spinors ζ_+, ζ_- solving (B.2.1) reads [145, 90, 61] (see also [64, sect. 5.1] for some more details):

$$[\delta_{\zeta_+}, \delta_{\zeta_-}]\Phi = 2\mathrm{i} \left(\mathcal{L}_K - \mathrm{i} \, q \, K \lrcorner A^{\mathrm{nm}} \right) \Phi \,, \qquad \delta^2_{\zeta_\pm} = 0 \,, \tag{B.2.7}$$

where $\mathcal{L}_K$ denotes the Lie derivative along the complex Killing vector K defined in (4.2.7) and q is the R-charge of a generic field Φ in the field theory. The gauge field A^{nm} is defined as $A^{\mathrm{nm}} = A + \frac{3}{2} V^{\mathrm{nm}}$, where V^{nm} is a well-defined 1-form satisfying

$$\nabla^i V_i^{\mathrm{nm}} = 0 \,, \qquad 2\mathrm{i} \, \sigma^i_\mp V_i^{\mathrm{nm}} \zeta_\pm = \pm \sigma^i_\mp \nabla_i^A \zeta_\pm \,. \tag{B.2.8}$$

This actually only fixes $K^i V_i^{\mathrm{nm}}$. In this way, A^{nm} and V^{nm} can be interpreted as the auxiliary fields of background new minimal supergravity (hence the label "nm").

Let us now evaluate these quantities in our background (4.2.8), (4.2.10). With the choice (B.2.3), the vector K takes precisely the form (4.2.9), $K = \frac{1}{2}(\partial_\psi - \mathrm{i}\partial_\tau)$, while its dual 1-form is

$$K^\flat = \frac{1}{2} \left(\mathrm{d}\psi + a - \mathrm{i} \, \mathrm{d}\tau \right) \,. \tag{B.2.9}$$

As long as $u \neq 0$ this has non-vanishing twist,

$$K^\flat \wedge \mathrm{d}K^\flat \;=\; \frac{\mathrm{i}}{4} u\, \mathrm{e}^{w}\, (\mathrm{d}\psi - \mathrm{i}\,\mathrm{d}\tau) \wedge \mathrm{d}z \wedge \mathrm{d}\bar{z}\;. \tag{B.2.10}$$

As discussed in [61], after Wick rotating to Lorentzian signature by $\tau = \mathrm{i}t$ this implies that the five-dimensional bulk solution falls in the timelike class of [108]. Eqs. (B.2.8) for V^{nm} are solved by

$$V^{\mathrm{nm}} \;=\; -\frac{u}{4}(\mathrm{d}\psi + a) + \kappa\, K^\flat\;, \tag{B.2.11}$$

where κ is an undetermined complex function satisfying $K^i\partial_i\kappa = 0$. Then A^{nm} reads:

$$A^{\mathrm{nm}} \;=\; A + \frac{3}{2}V^{\mathrm{nm}} \;=\; \frac{1}{2}\,(3\kappa - u)\, K^\flat + \frac{\mathrm{i}}{4}(\mathrm{d}\bar{z}\,\partial_{\bar{z}}w - \mathrm{d}z\,\partial_z w) - \mathrm{i}\gamma'\mathrm{d}\tau + \gamma\,\mathrm{d}\psi + \mathrm{d}\lambda\;. \tag{B.2.12}$$

Contracting with K gives $K \lrcorner A^{\mathrm{nm}} \;=\; \frac{1}{2}\,(\gamma - \gamma')$. Note from (B.2.5) that this is also the charge of the Killing spinor under K, $\mathcal{L}_K\zeta_+ = \frac{\mathrm{i}}{2}(\gamma - \gamma')\zeta_+$.

We conclude that in the background of interest, and with the choice (B.2.6), the superalgebra reads

$$[\delta_{\zeta_+}, \delta_{\zeta_-}]\Phi \;=\; \mathrm{i}\Big(- \mathrm{i}\mathcal{L}_{\partial_\tau} + \mathcal{L}_{\partial_\psi} - \mathrm{i}\gamma\, q\Big)\Phi\;. \tag{B.2.13}$$

Passing to the corresponding generators gives

$$\{\mathcal{Q}_+, \mathcal{Q}_-\} \;=\; H + J + \gamma\, Q\;, \tag{B.2.14}$$

where H and J are the charges associated with ∂_τ and $-\partial_\psi$, respectively, while Q is the R-charge. Taking the expectation value in a supersymmetric vacuum leads to the BPS condition

$$\langle H\rangle + \langle J\rangle + \gamma\langle Q\rangle \;=\; 0\;. \tag{B.2.15}$$

B.2.3 Twisted background

For the twisted background (4.4.54), (4.4.55), requiring that the Killing spinors $\zeta_\pm$ are independent of the new time coordinate and recalling relations (B.2.5), valid in the old coordinates, immediately leads to $\gamma' = -\mathrm{i}\,\gamma\tan\alpha$. It is also straightforward to implement the change of coordinates and obtain the new K (given in (4.4.56)) and the new form of the superalgebra.

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