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Discrete quantum gravity

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Abstract. A review is given of a number of approaches to discrete quantum gravity, with a restriction to those likely to be relevant in four dimensions.

This paper is dedicated to Rafael Sorkin on the occasion of his sixtieth birthday.

1. Introduction

Quantum gravity involves the quantisation of geometry, described in general relativity by the metric, so a complete theory should not depend on any fixed background spacetime. The general covariance of classical relativity is also an aspect which should be preserved in the quantum theory. Finally, since gravity is perturbatively non-renormalisable, perturbation theory is not a useful avenue to pursue. Thus our aim is to find a background-independent, diffeomorphism-invariant and non-perturbative theory of quantum gravity.

There are many approaches to formulating a theory of quantum gravity, and a number of them involve discretisation. There are quite radically different ways of regarding the discretisation; if one believes that spacetime really is discrete at the smallest scales [1], then the discretisation is necessary for a fundamental theory, and the continuum will appear as an approximation at larger length scales. On the other hand, and perhaps more usually, the discretisation is used as a practical tool; this may involve providing an approximation scheme used in classical numerical relativity or a regularisation scheme in quantum gravity. Whatever the motivation, the discretisation builds on related branches of mathematics (piecewise linear spaces and topology, and the geometric notion of intrinsic curvature on polyhedra) and on lattice techniques used in physics, for example lattice QCD.

We shall give a brief description of a number of discrete approaches to quantum gravity. The majority have developed from Regge calculus, which will be described first. Most use the sum over histories approach to calculate the partition function or transition amplitude, although of course discretisation can also be used in the canonical approach (see for example the section on consistent discretisations).

For reasons of space, the list of references is very far from being complete but it is hoped that enough are given to point the interested reader in the right direction.

2. Review of discrete quantum gravity

2.1. Regge calculus

In 1961, Regge [2] formulated a discrete version of “general relativity without coordinates” as a tool for numerical relativity. The basic idea of Regge calculus, as it came to be known, is

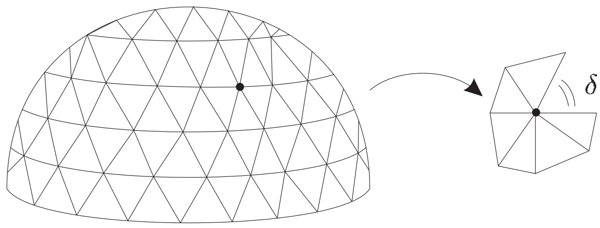

Figure 1. A geodesic dome, with the triangles meeting at one vertex projected onto a flat plane.

to consider discrete spacetimes with curvature restricted to subspaces of codimension 2, the "hinges". It can be visualised as the gluing together of flat simplices to approximate curved spaces or spacetimes. The curvature depends on the deficit or gap angles at the hinges. For example, in two dimensions, a geodesic dome (Figure 1) consists of a network of flat triangles. By projecting the triangles meeting at a vertex onto a flat plane, one can see that the (intrinsic) curvature is not at the edges, but at the vertices, and a measure of it is given by the deficit angle

$$\delta = 2\pi - \sum (vertex\ angles). \tag{1}$$

It is straightforward to extend this to general dimension d. The building blocks are flat d-simplices (simplices are used because their edge lengths specify their geometry completely), and the curvature is at the $(d-2)$-dimensional simplices where they meet. The analogue of the Einstein action

$$S = \frac{1}{2}\int R\sqrt{g}d^d x \tag{2}$$

is given by

$$S_R = \sum_{hinges\ h} V_h \delta_h, \tag{3}$$

where V_h is the volume of the hinge and δ_h is the deficit angle there, equal to 2π minus the sum of the dihedral angles between the faces of the simplices meeting at that hinge. The variables are the edge lengths (the analogues of the metric in the continuum theory) and the discrete version of Einstein's equations is obtained from the principle of stationary action by varying with respect to the edge lengths:

$$\sum_h \frac{\partial V_h}{\partial l_i}\epsilon_h = 0, \tag{4}$$

where we have used the result [2] that the variation of the dihedral angles gives zero, when summed over each simplex. This formalism can be used to construct piecewise linear Einstein spaces.

The use of Regge calculus in quantum gravity has been mainly in the sum over histories approach. It provides a straightforward way of calculating the action for a discrete space, and the sum over histories is implemented by integrating over edge lengths. There is no general agreement on the form of the measure. A comparison with the DeWitt measure in the continuum suggests some power of the volume associated with each edge, and this, together with a function implementing the triangle inequalities, has been used widely in numerical simulations of quantum gravity. It has also been claimed that a discrete version of the Fadeev-Popov determinant should be included [3] but this has not been implemented numerically.

The complexity of the Regge action (in particular through the dependence of deficit angles on edge lengths) makes analytic calculations difficult in all but the most symmetric configurations. Most analytic calculations have involved perturbation theory about a classical background. For example, in the earliest work in four dimensions [4], it was shown that the discrete propagator agrees with the continuum one in the weak field limit. (The *first* quantum application of Regge calculus was in three dimensions, the Ponzano-Regge model which is described in Section 2.3.)

Quantum cosmology provides an example of the computational use of Regge calculus. In the minisuperspace models developed mainly by Hartle [5], the wave function of the universe is calculated from the path integral with the Regge action, for a chosen simplicial complex. The infinite number of degrees of freedom in a continuum geometry are replaced by the finite number of variables describing that complex.

Numerical simulations of discrete quantum gravity have been performed by several groups, for example in Vienna (Beirl et al. [6]) and in Irvine (Hamber et al. [7]) I will describe the method of the latter group. A four-dimensional Euclidean lattice of hypercubes is divided into 4-simplices (24 to each hypercube) by drawing in diagonals. The partition function is given by

$$Z_L = \int \prod_s [V(S)]^\sigma \prod_{ij} dl_{ij}^2 \ \Theta(l_{ij}^2) \ exp\left(- \sum_h (\lambda V_h - k\delta_h A_h + \frac{a\delta_h^2 A_h^2}{V_h}) \right), \tag{5}$$

a lattice version of the continuum expression

$$Z_C = \int \prod_x (\sqrt{g(x)})^\sigma \prod_{\mu\geq\nu} dg_{\mu\nu}(x) \ exp\left(- \int d^4x\sqrt{g}(\lambda - \frac{k}{2}R + \frac{a}{4}R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}) \right), \tag{6}$$

where $k = 1/8\pi G$, λ is the cosmological constant and a the coefficient of the curvature-squared term, included originally to ensure positivity of the action [8]. $V(s)$ is the volume of the 4-simplex s and $\Theta(l_{ij}^2)$ is a function imposing the triangle inequalities.

Monte Carlo simulations are performed; from a flat space configuration, changes are made to the edge lengths, which are rejected if they increase the action, and accepted accepted with a certain probability if they decrease it. The system evolves to an equilibrium configuration about which it makes quantum fluctuations, and expectation values of operators, like curvature and volume, can be calculated. Finite size scaling and the renormalisation group can be used to obtain phases and relations involving critical exponents. For example, on a 16^4 lattice, with $\lambda = 1$ and $a = 0$, Hamber [9] obtained for $G < G_c$ a weak coupling phase, with degenerate geometry behaving like a spiky branched polymer. For $G > G_c$, there was a strong coupling phase, with smooth geometry at large scales and small negative average curvature. The phase transition occurred at $k_c \approx 0.0636$, and the correlation length exponent ν, defined by

$$\xi \sim (k_c - k)^{-\nu}, \tag{7}$$

where ξ is the correlation length, was approximately 1/3.

The canonical approach to Regge calculus has been explored in detail by Khatsymovsky [10].

2.2. Dynamical triangulations

In this approach, the space or spacetime is divided into flat simplices and the Regge action, with cosmological constant, is used, but in contrast with more usual Regge calculus spaces, all the simplices are taken to be equilateral. In this case, the action simplifies greatly. For example, in four-dimensional Euclidean space,

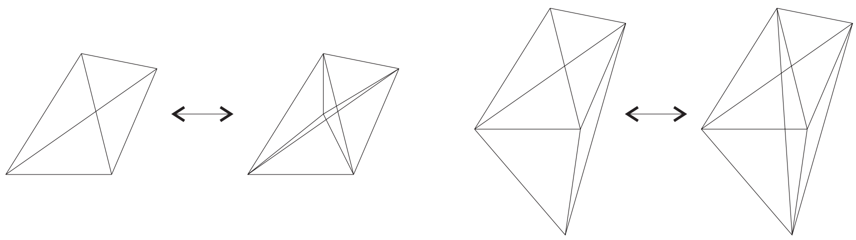

Figure 2. The (1-4) and (2-3) Pachner moves.

$$S_{DT} = K_4 N_4 - K_2 N_2, \tag{8}$$

where N_2 and N_4 are the numbers of traingles and 4-simplices, respectively, K_2 is inversely proportional to G and K_4 is proportional to λ.

The sum over histories no longer involves summing/integrating over edge lengths, but *summing over triangulations* with the topology kept fixed. This is deemed to be another way of summing over geometries. The triangulations are changed using Pachner moves, which are ergodic (meaning that any topologically equivalent triangulation may be reached by a sequence of such moves). For example, in three dimensions, there are two types of Pachner move, and their inverses (see Figure 2). In the (1-4) move, a new vertex is placed inside a tetrahedron and joined to each of the original vertices, dividing the tetrahedron into four new tetrahedra. In the (2-3) move, for two tetrahedra meeting on a common triangle, their opposite vertices (those not on the common triangle) are joined and the space is now divided into three tetrahedra meeting on that newly constructed edge.

Work on *Euclidean* dynamical triangulations, mainly by Ambjørn and collaborators [11] has produced interesting results in two, three and four dimensions, but the main stumbling-block was that no second order phase transition (needed for there to be a continuum limit) was found in four dimensions.

Simulations by Egawa et al. [12] showed a first order phase transition for pure gravity; this became second order when matter fields were included.

More recently, Ambjørn et al. have considered *Lorentzian* dynamical triangulations [13] and these seem at last to be producing a recognisable four-dimensional universe. They use two types of 4-simplices, one labelled (4,1), which has 6 spacelike edges and 4 timelike ones, and the other labelled (3,2) which has 4 spacelike and 6 timelike edges. The spacelike edges have $l^2 = a^2$, and the timelike ones $l^2 = -\alpha a^2$. Spacetime with topology $[0, 1] \times S^3$ or $S^1 \times S^3$ is foliated with slices of proper time τ, and filled in with (4,1) and (3,2) simplices (Figure 3). An analytic continuation is made to the Euclidean domain to do the calculations, with action

$$S_E = -(K_0 + 6\Delta)N_0 + K_4(N_4^{(4,1)} + N_4^{(3,2)}) + \Delta(2N_4^{(4,1)} + N_4^{(3,2)}), \tag{9}$$

where K_0, K_4 and Δ depend on G, λ and α. Δ gives a measure of the asymmetry between the lengths of the spacelike and timelike edges in the simplicial geometry. Monte Carlo simulations are used to extract information about the geometry of the ground state (which has constant spacetime volume). The number of 4-simplices is up to 362×10^3. The (1-5) move is not allowed since the vertices are restricted to be on surfaces of integer proper time, so it is not completely clear that the moves are still ergodic.

The resulting spacetime structure shows three distinct regions (see Figure 4). Region A (Figure 6), corresponding to large K_0, shows distinct spatial slices, giving structure reminiscent of branched polymers. Region B (figure 5), for which K_0 and Δ are both small, has a

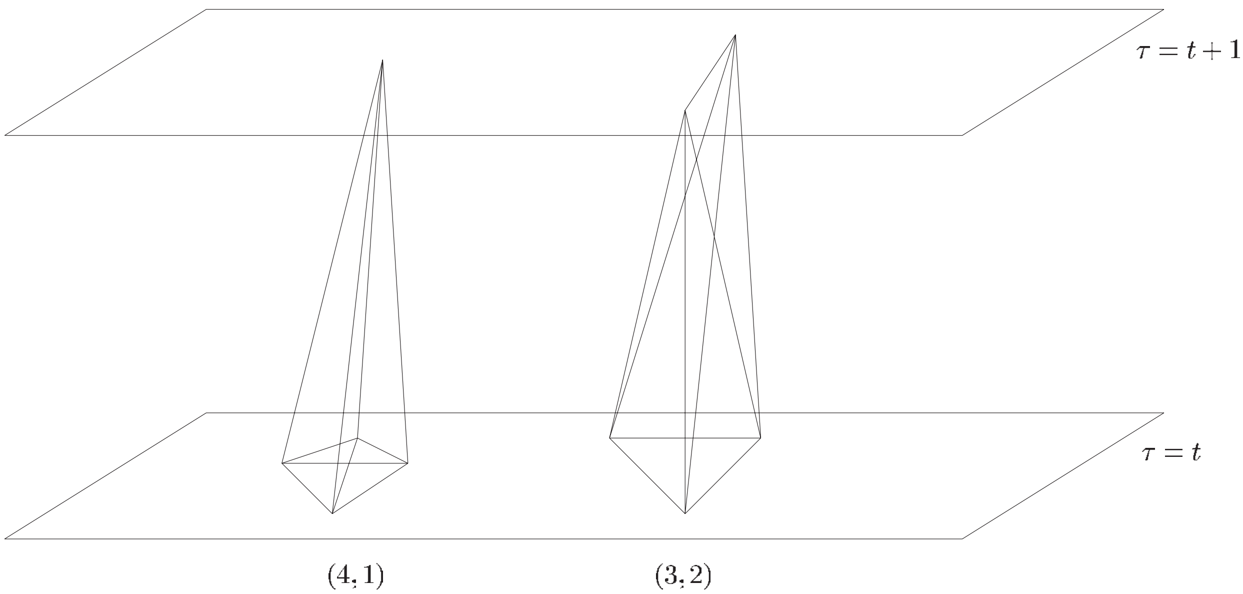

Figure 3. Typical 4-simplices of types (4,1) and (3,2) in the spacetime between two hypersurfaces of constant proper time.

single spatial slice and corresponds to the crumpled phase, exhibiting "spontaneous dimensional reduction". Neither of these regions has any macroscopic extended spacetime geometry. On the other hand, region C (Figure 7), which has $\Delta > 0$ and $\alpha > -1$ has a stable extended 4-geometry, suggestive of the real universe.

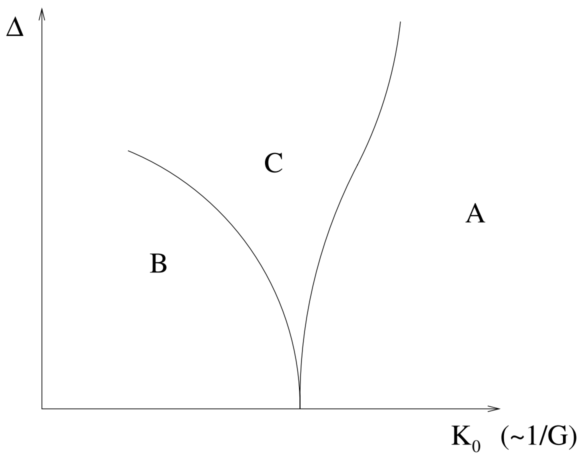

Figure 4. Distinct regions in the phase diagram.

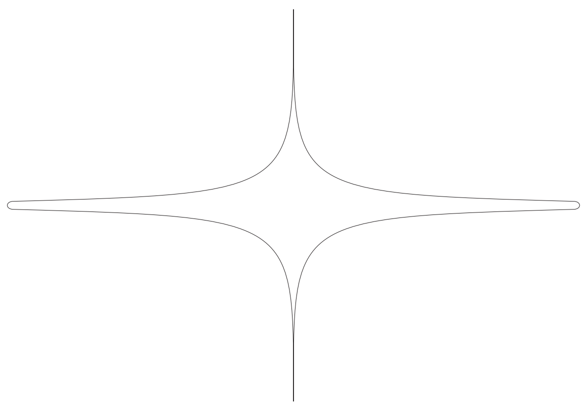

Figure 5. Cross-section of a typical spacetime geometry in Region B.

Detailed study of the geometry of region C shows a classical macroscopic structure but highly non-classical microscopic behaviour. The scaling dimension, extracted from the way the volume-volume correlator scales with the number of 4-simplices, is

$$D_H \approx 3.8 \pm 0.25, \tag{10}$$

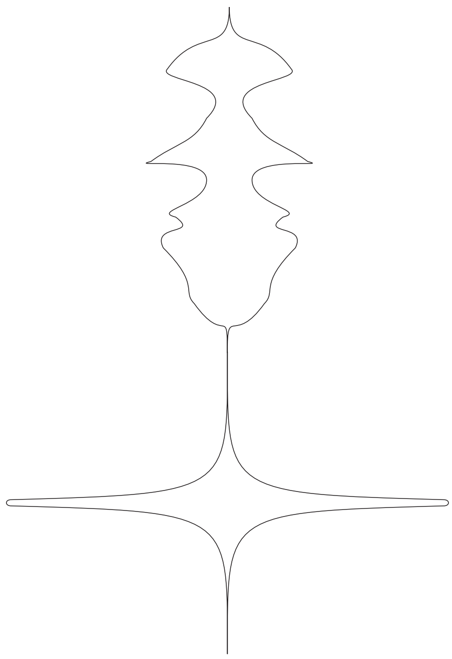

Figure 6. Cross-section of a typical spacetime geometry in Region A.

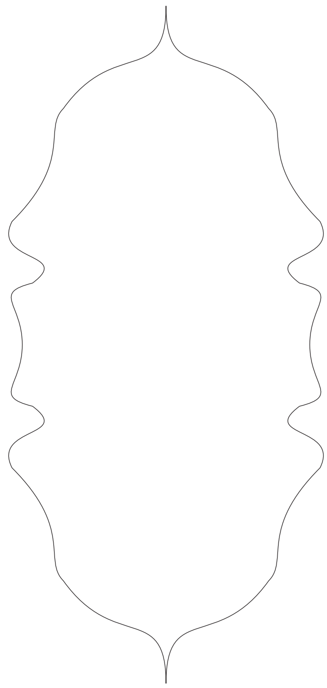

Figure 7. Cross-section of a typical spacetime geometry in Region C.

compatible with $d = 4$ at large scales. On the other hand, the spectral dimension, found from the return probability in a diffusion process with time σ, is

$$D_s(\sigma \to \infty) = 4.02 \pm 0.1, \qquad D_s(\sigma = 0) = 1.80 \pm 0.25. \tag{11}$$

This means that d is approximately 4 at large distances but 2 at short distances.

The effective action, written in terms of the scale factor, is

$$S^{eff}_{V_4} = \frac{1}{G} \int_0^t d\tau \ \left(a(\tau)(da/d\tau)^2 + a(\tau) - \lambda a^3(\tau) \right). \tag{12}$$

which is precisely minus the corresponding minisuperspace action in quantum cosmology. This is reminiscent of the non-perturbative cure of the conformal divergence in the Euclidean path integral found by Dasgupta and Loll [14].

The geometry of the spatial slices also exhibits interesting behaviour. For $\tau = constant$ slices, the Hausdorff dimension is $d_h \approx 3$ and the spectral dimension $d_s = 1.56 \pm 0.1$. For thick slices with $\Delta\tau = 1$, $d_H = 4.01 \pm 0.05$ and $d_S = 2.0 \pm 0.2$, which is highly non-classical, and baby universes are found to be suppressed. The emergence of scale-dependent dimensions is very significant and should be compared with work by Lauscher and Reuter [15] on dimensional reduction using the exact renormalisation group.

Clearly there are many open questions still to be investigated (for example, whether the fixed-time foliation is overly restrictive, whether Newton's law emerges in some limit), but there has been very substantial progress in this approach.

2.3. Spin foam models

This is another exciting and active field, which also grew out of Regge calculus. In 1968, in a paper on the asymptotic behaviuor of 6j-symbols [16], Ponzano and Regge formulated the

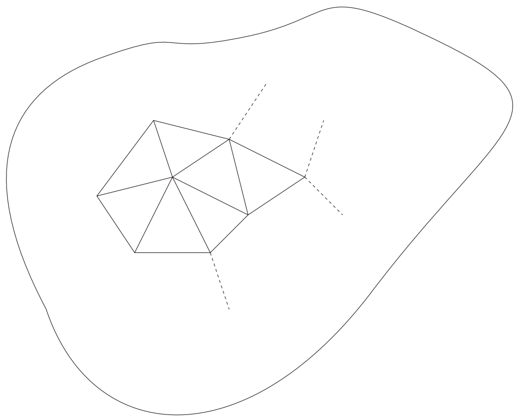

Figure 8. A triangulated 3-manifold.

following model. Triangulate a 3-manifold (Figure 8), and label each edge with a representation of $SU(2)$, $j_i = \{0, 1/2, 1, ...\}$. Assign a 6j-symbol, $\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix}$ (a generalised Clebsch-Gordan coefficient, which relates bases of states when three angular momenta are added) to each tetrahedron. Form the state sum

$$Z = \sum_{j_i} \prod_i (2j_i + 1)(-1)^\chi \prod_{tetrahedra} \{6j\}, \tag{13}$$

where the χ in the phase factor is a function of the j_i. This sum is infinite in many cases, but it has some very interesting properties. In particular, the semi-classical limit exhibits a connection with quantum gravity. The edge lengths can be thought of as $\hbar(j_i + 1/2)$, and the limit is obtained by keeping these quantities finite while $\hbar$ tends to zero and j_i tends to infinity. Ponzano and Regge showed that for large j_i, the asymptotic behaviour of the 6j-symbol is

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12\pi V}} cos \left(\sum_i j_i \theta_i + \pi/4 \right) \tag{14}$$

where V is the volume of the corresponding tetrahedron and θ_i the exterior dihedral angle at edge i. In the sum over edge lengths, the large values dominate, so the sum over the j_i in the state sum can be replced by an integral over the edge lengths, and the asymptotic formula used. The state sum then contains a term proportional to

$$\int \prod_i dj_i (2j_i + 1) \prod_{tetrahedra\ k} \frac{1}{\sqrt{V}} exp(iS_R), \tag{15}$$

which looks precisely like a Feynman sum over histories with the Regge action in three dimensions, and with the other terms contributing to the measure.

In 1990, Turaev and Viro [17] constructed a model which turned out to be a regularised version of the Ponzano-Regge model. This was achieved by replacing $SU(2)$ by the quantum group $U_q(sl2)$, which has only a finite number of representations. The Turaev-Viro state sum is a manifold invariant, independent of triangulation. This can be proved by using relations between the q-deformed 6j-symbols; the Biedenharn-Elliott relation corresponds exactly to the (2-3) Pachner move, while the (1-4) move corresponds to using both the Biedenharn-Elliott relation and orthogonality. The Turaev-Viro invariant has been shown to be equivalent to a Feynman path integral with the Chern-Simons action for $SU_k(2) \bigotimes SU_{-k}(2)$. This in turn corresponds to three-dimensional gravity with a cosmological constant term given by $\lambda = (4\pi/k)^2$.

There has been much work on these three-dimensional models, so-called spin networks. Freidel and collaborators [18] have shown how to regularise the Ponzano-Regge model by gauge fixing, and how to obtain Feynman graphs for the matter incorporated. There has been work on quantum cosmology using the Ponzano-Regge model [19].

The obvious challenge has been to extend these models to four dimensions. Early work by Ooguri [20] and Crane and Yetter [21] produced triangulation independent models, but they were thought not to capture the essential features of four-dimensional gravity. Barrett and Crane [22] introduced a model with "relativistic spin networks". The triangles are labelled by two representations of $SU(2)$ (prompted by the fact that $SO(4)$ has spin covering $SU(2) \bigotimes SU(2)$), and they showed that simple representations were required, where the two spin labels are equal. This has an analogy in the BF formulations of gravity in four dimensions. It has been shown [23] that the 10j-symbol appearing in the Barrett-Crane vertex behaves asymptotically like $exp(\pm i S_R)$ but also contains a contribution from degenerate simplices. The Barrett-Crane model has been extended to the Lorentzian case [24] and a way of incorporating causality formulated [25].

Many variations on the Barrett-Crane model have been constructed, and they are known collectively as spin foam models. They are background independent, and the spin foam is a 2-complex (which may or may not arise from the dual of a triangulation), with vertices v, edges e and faces f. Representations of a chosen group are assigned to the faces, and a sum over those representations performed to obtain the partition function, which has the general form

$$Z = \sum_{reps J} \prod_f A_f(J_f) \prod_e A_e(J_{f|e}) \prod_v A_v(J_{f|v}), \tag{16}$$

with specified amplitudes for faces, edges and vertices. The states are spin networks on the boundary, labelled by representations. In four dimensions, these models appear to be finite [26], but not independent of triangulation, which may not be a problem if one regards the existence of gravitons a reason for four dimensional gravity not to be described by a topological quantum field theory.

It is hard to do analytic calculations with spin foam models but some progress is being made. Outstanding problems include the incorporation of all types of matter (for work on this using the group field theory approach, see papers by Oriti and Ryan in these proceedings), and the need for a continuum limit.

2.4. *Gauge theory formulation*

Ideas from lattice gauge theories have contributed to formulations of discrete gravity, and Caselle, D'Adda and Magnea [27] have constructed a description of gravity as a gauge theory of the Poincaré group. Spacetime is again triangulated with flat n-simplices, each containing an orthonormal reference frame. The frames are related by a Poincaré transformation on each (n-1)-face. In their second order formulation, where the variables are the edge lengths of the simplices, the action is given by

$$S = -\frac{1}{2} \sum_h Tr(U^{(h)}_{\alpha\alpha} \mathcal{V}^{(h)}(\alpha)), \tag{17}$$

where $U^{(h)}_{\alpha\alpha}$ is an orthogonal matrix, the product of the rotation matrices Λ around the dual plaquette for the hinge h, and $\mathcal{V}^{(h)}(\alpha)$ is the oriented volume of that hinge. This action is independent of the starting simplex α in each dual plaquette. With a particular choice of coordinates, this reduces to

$$S = \sum_h sin\delta(h)V(h), \tag{18}$$

where $\delta(h)$ is the deficit angle of hinge h and V its volume. For small deficit angles, this reduces to the Regge action.

In their first order formulation, bivectors $b^a_{\alpha\beta}(\alpha)$ are defined, orthogonal to the face between simplices α and β. The rotation matrices Λ and the bivectors b are then treated as independent variables, subject to the constraints

$$b^a_{\alpha\beta}(\alpha) = \Lambda^a_b(\alpha,\beta)b^b_{\beta\alpha}(\beta), \tag{19}$$

$$\sum b^a_{\alpha\beta}(\alpha) = 0. \tag{20}$$

A volume tensor $\mathcal{W}$ is defined in a complicated way, to be independent of the starting simplex in the dual plaquette. The action is then

$$S = -\frac{1}{2}\sum_h Tr(U^{(h)}_{\alpha\alpha}\mathcal{W}^{(h)}(\alpha)). \tag{21}$$

Deriving the field equations from this is quite challenging, but D'Adda and Gionti [28] have shown that the Regge connection is the solution for the first order field equations for small deficit angles. The coupling to fermionic matter is straightforward, and numerical calculations with the formalism should be possible.

2.5. Consistent discretisations

The consistent discretisation of Gambini and Pullin [29] is a very different form of discrete gravity; it uses the canonical approach rather than the sum over histories, and it is not geometric in the Regge calculus sense. The aim is to deal with two major problems. Firstly, a classical problem: in numerical relativity, the evolution does not in general preserve the constraints. Secondly, in approaches in the past to discrete canonical quantum gravity, the constraints do not usually close. The basic idea of the approach is to discretise the action appropriately, then the equations of motion derived from it should be consistent. Use is made of discrete canonical transformations.

There have already been a number of successful applications, at the classical level in the numerical evolution of nonlinear systems, Gowdy spacetimes for example, and at the quantum level where the problem of the constraints is automatically solved, for example in the Friedman model.

2.6. Causal sets

Causal sets have no *a priori* relationship with any of the other approaches to discrete quantum gravity described so far, although some use of spin networks has been made in attempts to do calculations. The approach [30] is a very radical one, starting not with more usual constructs like dimension, manifold or triangulations, but with just the two concepts of discreteness and order. The discretisation involves a random set of points, which are sprinkled according to a Poisson process (giving a Lorentz invariant distribution). The order is implemented using a binary relation, ("precedes"), which satisfies three axioms:

$$(1)\ transitivity : x \prec y \prec z \Rightarrow x \prec z \tag{22}$$

$$(2)\ irreflexivity : x \not\prec x \tag{23}$$

(3) $local\ finiteness: the\ number\ of\ y\ such\ that\ x \prec y \prec z,\ for\ any\ given\ x\ and\ z,\ is\ finite.$ (24)

The order relation leads to the construction of light rays, and hence to the equivalent of light cones. In the continuum, the light cones determine the structure of a spacetime up to the conformal factor, and similarly for causal sets, the volume cannot be determined that way. One can argue (on dimensional grounds, or from the entropy of a black hole) that the length scale should be of the order of $\sqrt{k\hbar}$, which is about $10^{-23}cm$. There are various possible definitions of dimension, but the number 4 has not emerged naturally as yet.

The dynamics of causal sets takes place through the process of "sequential growth". New elements are "born", subject to discrete general covariance (the labelling of points being pure gauge). Bell causality, which means that a birth in one region cannot be influenced by other births in regions spacelike to the first, leads to a formula for the transition probability between various states.

The main result so far in causal sets is that the cosmological constant, Λ, fluctuates about its target value of zero, and the current order of magnitude for the fluctuations is obtained. This is very impressive, but there are still many oustnding problems in the appraoch. For example, it might be necessary to use coarse graining to form a manifold from a causal set.

3. Conclusions

We have reviewed a number of approaches to discrete quantum gravity. There are others which have not been included , for example, 't Hooft's polygonal approach, which is specific to (2+1) dimensions, and leads to non-commutative spaces [31].

Discrete quantum gravity is certainly a very active area of research, incorporating ideas from a number of different areas of mathematics and physics. The most outstanding recent result is probably in Lorentzian dynamical triangulations. The result on the cosmological constant in causal sets is also very exciting. As with all approaches, there are still a great many problems to solve (for example, obtaining Newton's law in the appropriate limit, and incorporating all types of matter).

If we understand the interrelationships between these approaches (see Figure 9), we shall understand the individual approaches better too. For example, understanding why there seems to be a second order phase transition in simulations of Euclidean Regge calculus, but not in Euclidean dynamical triangulations, should throw light on the relation between their universality classes.

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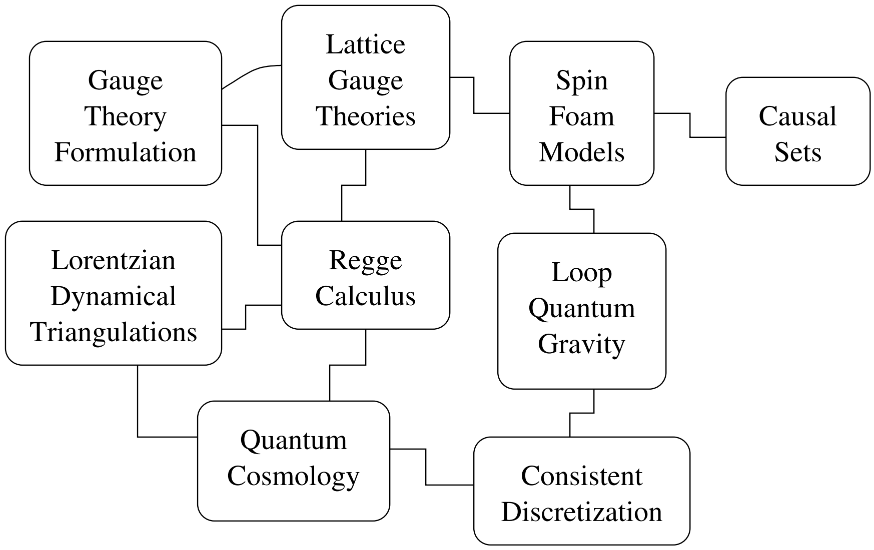

Figure 9. Some of the interrelationships between discrete approaches to quantum gravity.

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