Nonuniqueness of a System of Harmonic Coordinates in General Relativity and of a Similar System in Bimetric Theory and in Theory with Two Connections

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Abstract

Preceding investigations of external Schwarzschild, Nordström–Reissner and Kerr problems have led to the conclusion that a system of harmonic coordinates in general relativity is not unique. Consideration of the external Schwarzschild problem, solved by Chernikov in his new theory with two connections but one metric (1993) (this solution is valid for Rosen–1980 Bimetric general relativity too) supports a similar conclusion.

In 1939 Fock supposed [1] that the system of harmonic coordinates is privileged and unique up to Lorentz transformations. The harmonic condition was used by Einstein [2] in approximate consideration and then exactly formulated by De Donder [3] as follows:

$$\Box x^\mu(\overline{x}^\sigma) \equiv \frac{1}{\sqrt{-\overline{g}}} \frac{\partial}{\partial \overline{x}^\beta} \left(\overline{g}^{\alpha\beta} \sqrt{-\overline{g}} \frac{\partial x^\mu}{\partial \overline{x}^\alpha} \right) = 0; \quad \alpha, \beta = 0, 1, 2, 3. \quad (1)$$

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where $\bar{x}^\sigma$ are original arbitrary coordinates, and x^μ are harmonic by definition (1). The Fock conception is based on the hypothesis of uniqueness of the solution of a wave equation of the type (1) if some additional conditions are fulfilled (they are a sort of radiation conditions). In 1979 Mishnajevsky and Ramm made an attempt [4] to prove this hypothesis when the gravitational field is constant in time, i.e. $\frac{\partial}{\partial t} g^{\mu\nu} = 0$. But their positive conclusion seems to be wrong.

Really, the external Schwarzschild solution in "rectilinear" harmonic coordinates ct, x^i has the form [5], [6]

$$ds^2 = \left(1 - \frac{2\alpha}{\rho(r)}\right)(cdt)^2 - \frac{\rho^2(r)}{r^2}\left[\delta_{ik} + \left(\frac{\rho'^2 r^2}{\rho^2(r) - 2\alpha\rho(r)} - 1\right)\frac{x_i x_k}{r^2}\right]dx^i dx^k,$$ (2)

$$r \equiv \sqrt{x^i x^i}, \quad \alpha \equiv \frac{Gm}{c^2}, \quad \rho' \equiv \frac{d\rho(r)}{dr}, \quad i, k = 1, 2, 3,$$

where the function $\rho(r)$ (or its inverse $r(\rho)$) is defined by the correspondence condition with Newton theory when $r \to \infty$

$$lim_{r\to\infty} \frac{\rho(r)}{r} = 1$$ (3)

and by the De Donder condition (1) which becomes

$$\frac{d}{d\rho}\left[(\rho^2 - 2\alpha\rho)\frac{dr(\rho)}{d\rho}\right] - 2r(\rho) = 0.$$ (4)

So we have the one–parameter family of solutions of this equation

$$r(\rho) = \rho - \alpha + C\left(\frac{\rho - \alpha}{2\alpha}ln\mid\frac{\rho}{R - 2\alpha}\mid -1\right), \quad C = const.$$ (5)

Lanozos [5] and then Rosen [7] and Fock [1] used only one choice $c = 0$.

But some investigations of the full (internal and external) Schwarzschild problem [8] give $c \neq 0$ and its value depends on characteristics of the central body.

Note, a transformation of the coordinates inside the family is not Lorentzian and even not linear, it is like a similarity transformation, $\bar{r} = \bar{r}(r)$.

The same situation takes place for the external Nordsröm–Reissner solution in the harmonic coordinates [9]

$$ds^2 = \left(1 - \frac{2\alpha}{\rho(r)} + \frac{\epsilon^2}{\rho^2(r)}\right)(cdt)^2$$

$$- \frac{\rho^2(r)}{r^2}\left[\delta_{ik} + \left(\frac{\rho'^2 r^2}{\rho^2(r) - 2\alpha\rho(r)} - 1\right)\frac{x_i x_k}{r^2}\right]dx^i dx^k.$$ (6)

$\epsilon^2 \equiv \frac{Ge^2}{c^4}$, e — electric charge. The De Donder condition (1) gives

$$\frac{d}{d\rho}\left[(\rho^2 - 2\alpha\rho + \epsilon^2)\frac{dr(\rho)}{d\rho}\right] - 2r(\rho) = 0,$$

from which we have the one–parameter family of solutions (say, when $\alpha > \epsilon$)

$$r(\rho) = \rho - \alpha + C\left(\frac{\rho - \alpha}{2\sqrt{\alpha^2 - \epsilon^2}}ln\mid\frac{\rho - \alpha + \sqrt{\alpha^2 - \epsilon^2}}{\rho - \alpha - \sqrt{\alpha^2 - \epsilon^2}}\mid -1\right).$$ (7)

There are known parameterless harmonic coordinates by Ding Hao-Gang [10] for the Kerr solution

$$x^1 = \sqrt{(r - \alpha)^2 + a^2}sin\theta cos[\varphi - \Phi(r)],$$
$$x^2 = \sqrt{(r - \alpha)^2 + a^2}sin\theta sin[\varphi - \Phi(r)],$$
$$x^3 = (r - \alpha)cos\theta,$$

$$\Phi \equiv -\int_r^\infty \frac{a\alpha^2 dr}{\triangle(\triangle + \alpha^2)}, \triangle \equiv r^2 + a^2 - 2\alpha r, c = 1.$$

where r, θ, φ are the Boyer–Lindquist space coordinates, $a = \frac{J}{mc}$, J is the angular momentum. At the same time, there are known Ruiz [11] many-parameter harmonic coordinates for the Kerr solution

$$x^0 = t, \quad x^1 = h(r, \theta)cos\varphi, \quad x^2 = h(r, \theta)sin\varphi, \quad x^3 = f(r, \theta).$$

For example, if $\mid a \mid < \alpha$ then

$$f(r, \theta) = a_0 + (r - \alpha)cos\theta + \sum_{l=0}^\infty b_l Q_l\left(\frac{\alpha - r}{\sqrt{\alpha^2 - a^2}}\right)P_l(cos\theta),$$

here P_l and Q_l are the Legendre polynomials.

Now we try to investigate the same problem for the new generalization of Einstein's theory, the theory with two connections (but one metric) by Chernikov [12]. One of the main purposes of this theory was the construction of the tensor generalization of the Einstein gravitational energy–momentum pseudotensor. In one variant of this theory, the second ("background") connection $\hat{\Gamma}^{\mu}_{\nu\lambda}$ is christoffelian and defined by the metric in the spherical space coordinates

$$d\hat{s}^2 = \hat{g}_{\mu\nu}dx^{\mu}dx^{\nu} = (cdt)^2 - dr^2 - \left(ksh\frac{r}{k}\right)^2 d\Omega^2, \quad d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2, \tag{8}$$

whence its spatial part describes the Lobachevski space (k is the Lobachevsky constant). Note that in this case the equations of the theory would coincide with the equations of Rosens [13] bimetric general relativity with the background metric (8).

For the static case, we take the field interval in the form

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu} = V^2(r)(cdt)^2 - F^2(r)dr^2 - H^2(r)dr^2. \tag{9}$$

Now, in this variant of the theory there exists (as a sequence of the basic theory equations) a condition

$$\frac{d}{dr}\left(F^{-1}H^2V\right) - FVksinh\frac{2r}{k} = 0, \tag{10}$$

which in the limit $k \to \infty$ turns into the De Donder harmonic condition (1).

The external Schwarzschild problem for this variant was solved by Chernikov [12] with the help of the additional condition $FV = 1$, and he found the solution

$$ds^2 = P^{-2}sinh\left(\frac{r-\hat{\alpha}}{k}\right)sinh^{-1}\left(\frac{r+\hat{\alpha}}{k}\right)(cdt)^2 -$$

$$-P^2sinh^{-1}\left(\frac{r-\hat{\alpha}}{k}\right)sinh\left(\frac{r+\hat{\alpha}}{k}\right)dr^2 -$$

$$-P^2\left(ksinh\frac{r+\hat{\alpha}}{k}\right)^2 d\Omega^2, \quad P \equiv exp\left(-\frac{\hat{\alpha}}{k}\right), \quad \frac{k}{2}sinh\frac{2\hat{\alpha}}{k} = \frac{Gm}{kc^2}. \tag{11}$$

When $k \to \infty$, it turns into the known Schwarzschild solution written in the spherical space coordinates, corresponding harmonic x^{μ} [7]

$$x^0 = ct, x^1 = rsin\theta cos\varphi, x^2 = rsin\theta sin\varphi, x^3 = rcos\theta.$$

Now let us turn to the question on the uniqueness of these "extraharmonic" coordinates. As the theory is generally covariant, we shall search for a transformation to a new space coordinate $R = R(r)$ or its inverse $r = r(R)$. The interval (9) turns into

$$ds^2 = \overline{V}^2(R)(cdt)^2 - \overline{F}^2(R)\left(\frac{dr}{dR}\right)^2 dR^2 - \overline{H}^2(R)d\Omega^2,$$

$$\overline{V}(R) = V[r(R)], \quad \overline{F}(R) = F[r(R)], \quad \overline{H}(R) = H[r(R)],$$

and the background interval (8) turns into

$$d\hat{s}^2 = (cdt)^2 - \left(\frac{dr}{dk}\right)^2 dk^2 - k^2 sh\frac{r(R)}{k}d\Omega^2.$$

The extraharmonic condition now is

$$\frac{d}{dR}\left(\overline{F}^{-1}\overline{H}^2\overline{V}\right) - \overline{FV}\frac{dr}{dR}ksinh\frac{2r(R)}{k} = 0,$$

from where, since for (11) $\overline{FV} = FV = 1$, we have

$$\frac{d}{dR}\left(H^2V^2\right) - k\frac{dr}{dR}sinh\frac{2r(R)}{k} = 0. \tag{12}$$

As for (11)

$$H^2V^2 = k^2sinh\frac{r-\bar{r}}{k}sinh\frac{r+\bar{r}}{k} = \frac{k^2}{2}\left(cosh\frac{2r}{k} - cosh\frac{2\bar{r}}{k}\right),$$

Then (12) becomes the identity. This means that an arbitrary an arbitrary (monotonous and smooth) transformation $R = R(r)$ leads to a new system of extraharmonic coordinates.

Whence, the situation for this region is analogous to the Schwarzschild case in general relativity.

As by the transformation $R = R(r)$, the background connection (and background metric, if exists) should be correspondingly transformed, one can say that in these theories the question of uniqueness of the coordinate system is transferred to another plane [14] and connected with a possibility of choosing a background connection and its properties.

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