On The Formulation of Bianchi Identity from Action Principle

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Short Report

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On The Formulation of Bianchi Identity from Action Principle

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Abstract

In this letter, we investigate the basic property of the Hilbert-Einstein action principle and its infinitesimal variation under suitable transformation of the metric tensor. We find that for the variation in action to be invariant, it must be a scalar so as to obey the principle of general covariance. From this invariant action principle, we eventually derive the Bianchi identity (where, both the 1st and 2nd forms are been dissolved) by using the Lie derivative and Palatini identity. Finally, from our derived Bianchi identity, splitting it into its components and performing cyclic summation over all the indices, we eventually can derive the covariant derivative of the Riemann curvature tensor.

This very formulation was first introduced by S Weinberg in case of a collision less plasma and gravitating system. We derive the Bianchi identity from the action principle via this approach; and hence the name *'Weinberg formulation of Bianchi identity'*.

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1. INTRODUCTION

The principle of general covariance (GC) along with the principle of equivalence (PE) plays a very crucial role in defining the parameters of a purely gravitational field. Following the principles of equivalence and general covariance, the affine connection (Γ) and the metric tensor ($g_{\mu\nu}$) are sufficient to describe all the intrinsic properties of a local inertial frame within any gravitational field, relatively respective.

In this letter, we are concerned to derive the Bianchi identity (dissolved for *1st* and *2nd* identities) from the Hilbert-Einstein (S_{HE}) or the gravitational action defined by the active lagrangian of the field. We perform an analogous gauge transformation for the metric tensor under consideration to construct an invariant action which is essentially be scalar. Our primary goal of this letter is to derive and formulate the Bianchi identity (in contracted and dissolved from) from the action principle due to purely gravitational lagrangian as proposed by Prof. Steven Weinberg first in *Gravitation and Cosmology, 1977*; and hence the name to be used as "*Weinberg formulation* of Bianchi identity".

All in this letter, we use the *hyperbolic metric signature*: $(+,-,-,-)$. The constituents of this letter are as: in first section we define S_{HE} and its variation properties, then we discuss the lie derivative for an infinitesimal vitiations for the metric tensor and its invariant transformation, next we derive the Bianchi identity. Finally, we discuss some mathematical properties of the *contracted Bianchi identities*.

2. MATHEMATICAL APPROACH

2.1 The Action Principle

We consider a purely gravitational field where the intrinsic properties of locally inertial frame by the virtue of transformation ($x^{\alpha} \to x^{\beta}$) within, is sufficiently defined by $g_{\alpha\beta}$ and $\Gamma^{\lambda}_{\alpha\beta}$. For such a field the action is given by

$$S_{HE} = \frac{1}{16\pi G} \int \sqrt{g(x)}\; R(x)\, d^4x \; . \tag{1}$$

Where, $R(x)$ is the *curvature scalar* defined as $R(x) = R \equiv R_{\alpha\beta} g^{\alpha\beta}$ or $R^{\alpha\beta} g_{\alpha\beta}$ .Where $R_{\alpha\beta}$ is the *Ricci tensor* defined as

$$R_{\alpha\beta} = R^{\beta}_{\alpha\beta\lambda} \equiv \frac{\partial \Gamma^{\lambda}_{\beta\alpha}}{\partial x^{\lambda}} - \frac{\partial \Gamma^{\lambda}_{\lambda\alpha}}{\partial x^{\beta}} + \Gamma^{\gamma}_{\beta\alpha}\Gamma^{\lambda}_{\lambda\gamma} - \Gamma^{\gamma}_{\lambda\alpha}\Gamma^{\lambda}_{\beta\gamma} \tag{2}$$

Next, an infinitesimal change $\delta(\sqrt{g(x)}\, R)$ is given as

$$\delta\left(\sqrt{g}\, R\right) = \sqrt{g}\, R^{\alpha\beta}\, \delta g_{\alpha\beta} + R\, \delta\sqrt{g} + \sqrt{g}\, g_{\alpha\beta}\, \delta R^{\alpha\beta} \; . \tag{3}$$

Now, using the *Palatini identity*, we have the infinitesimal change $\delta R^{\alpha\beta}$ or $\delta R_{\alpha\beta}$ as

$$\delta R_{\alpha\beta} = \left(\delta\, \Gamma^{\lambda}_{\alpha\lambda}\right)_{;\beta} + \left(\delta\, \Gamma^{\lambda}_{\alpha\beta}\right)_{;\lambda} \tag{4}$$

Using equation (4) for $\delta R_{\alpha\beta}$ in (3), we have the last term $\sqrt{g}\, g_{\alpha\beta}\, \delta R^{\alpha\beta}$ as

$$\sqrt{g}\, g_{\alpha\beta}\, \delta R^{\alpha\beta} = \sqrt{g}\, g_{\alpha\beta} \left[\left(\delta\, \Gamma^{\lambda}_{\alpha\beta}\right)_{;\beta} - \left(\delta\, \Gamma^{\lambda}_{\alpha\beta}\right)_{;\lambda}\right] = \sqrt{g}\left[\left(g_{\alpha\beta}\, \delta\, \Gamma^{\lambda}_{\alpha\beta}\right)_{;\beta} - \left(g_{\alpha\beta}\, \delta\, \Gamma^{\lambda}_{\alpha\beta}\right)_{;\lambda}\right] \tag{5}$$

Using the *Divergence theorem* for tensors in (5), we have

$$\sqrt{g}\, g_{\alpha\beta}\, \delta R^{\alpha\beta} = \frac{\partial}{\partial x^{\beta}}\left(\sqrt{g}\, g_{\alpha\beta}\, \delta\, \Gamma^{\lambda}_{\alpha\beta}\right) - \frac{\partial}{\partial x^{\lambda}}\left(\sqrt{g}\, g^{\alpha\beta}\, \delta\, \Gamma^{\lambda}_{\alpha\beta}\right) \tag{6}$$

And thus: $\delta\sqrt{g} = \frac{1}{2}\sqrt{g}\; g^{\alpha\beta}\, \delta g_{\alpha\beta}$ and $\delta g_{\alpha\beta} = -g^{\alpha\rho}\, g^{\beta\sigma}\, \delta g_{\rho\sigma}$

So, the infinitesimal change in Hilbert-Einstein action from (1) can be given as

$$\delta S_{HE} = \frac{1}{16\pi G} \int \sqrt{g(x)}\; \left[R^{\alpha\beta} - \frac{1}{2} g^{\alpha\beta} R\right] \delta g_{\alpha\beta}\, d^4x \tag{7}$$

2.2 Derivation of Bianchi Identity

Now, if we apply the principle of general covariance to equation (1) we may eventually conclude that for S_{HE} to be invariant, it must be a scalar quantity. Thus, from equation (5) it is evident that for any invariant transformation the change in action vanishes or $\delta S_{HE} \to 0$. However, this invariance is not altered by any variance in metric tensor as

$$g_{\alpha\beta}(x) \to g_{\alpha\beta}(x) + \delta g_{\alpha\beta}(x). \tag{8}$$

This analogous *gauge transformation* is accompanied by

$d^4x \to d^4x'$ or $\frac{\partial}{\partial x^{\alpha}} \to \frac{\partial}{\partial x'^{\alpha}}$.

Or $g_{\alpha\beta}(x) \to g'_{\alpha\beta}(x') \equiv g_{\rho\sigma} \frac{\partial x^{\rho}}{\partial x'^{\alpha}} \frac{\partial x^{\sigma}}{\partial x'^{\beta}} - \left[g'_{\alpha\beta}(x') - g'_{\alpha\beta}(x)\right]$. Now, if we consider a parametric transformation: $x^{\alpha} \to x^{\alpha} + \varepsilon^{\alpha}$, where ε^{α} is an arbitrary function.

Thus, the variation of $g_{\alpha\beta}$ in (8) is given by the *Lie derivative* for $\delta g_{\alpha\beta}(x)$ as

$$\delta g_{\alpha\beta}(x) = -g_{\alpha\lambda} \frac{\partial \varepsilon^{\lambda}(x)}{\partial x^{\beta}} - g_{\beta\lambda}(x) \frac{\partial \varepsilon^{\lambda}(x)}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}(x)}{\partial x^{\lambda}} \varepsilon^{\lambda}(x) \ . \tag{9}$$

So, in equation (7) as limit $(\delta S_{HE}) \to 0$,

$$0 = \frac{1}{16\pi G} \int \left[\sqrt{g(x)}\, R^{\alpha\beta} - \frac{1}{2}\sqrt{g(x)}\, g^{\alpha\beta}\, R\right] \delta g_{\alpha\beta}\, d^4x$$

$$= \int \left[\delta g_{\alpha\beta}\sqrt{g(x)}\, R^{\alpha\beta} - \frac{1}{2}\delta g_{\alpha\beta}\sqrt{g(x)}\, g^{\alpha\beta}\, R\right] d^4x \ .$$

Following integration by parts and having ε^{α} as arbitrary function in equation (9),

$$\delta g_{\alpha\beta}\sqrt{g(x)}\, R^{\alpha\beta} - \frac{1}{2}\delta g_{\alpha\beta}\sqrt{g(x)}\, g^{\alpha\beta}\, R = 0 \tag{10}$$

now from the *properties of tensor analysis* we have $\delta g_{\alpha\beta}\sqrt{g(x)} \to 1$. So (10) modifies to

$$R^{\alpha\beta} - \frac{1}{2}\, g^{\alpha\beta}\, R = 0 \tag{11}$$

So the left-hand side of the equation (11) is a constant. Hence, any covariant derivative w.r.t coordinate transformation (x^{α} or $x^{\beta} \to 0$) will tends to vanish. So

$$\left[R^{\alpha\beta} - \frac{1}{2}\, g^{\alpha\beta}\, R\right]_{;\alpha} = 0$$

Or expanding the indices further, we have

$$\left[R^{\alpha}_{\beta} - \frac{1}{2}\, g_{\alpha\beta} g^{\alpha\beta}\, R\right]_{;\alpha} = 0 \ . \tag{12}$$

Which is the contracted from of the Bianchi Identity. The validity of this equation lies in the fact that, we can eventually derive the 1st and 2nd Bianchi Identities from equation (12) with the contracted form of Ricci tensor (2) and the cyclic summation of the *Riemann-(Christoffel) curvature* tensor.

3. RESULT and DISCUSSION

Thus, from the derivation of the Bianchi identity from the Hilbert-Einstein action can be eventually derived as suggested by Weinberg, if we consider an invariant transformation in the action principle. However, if we consider the curvature tensor as $R_{\alpha\beta mn} = g_{\alpha s}\, R^{s}_{\beta mn}$, whose cyclic summation gives the complete Bianchi identities as

$$\textstyle\sum_{(\beta mn)} (R^{\alpha}_{\beta mn}) = 0 \quad \text{(1st identity)} \tag{13}$$

$$\textstyle\sum_{(mn\lambda)} R^{\alpha}_{\beta mn;\lambda} = 0 \quad \text{(2nd identity)} \tag{14}$$

Which eventually simplifies into

$$R^{\alpha}_{\beta mn} = -\, R^{\alpha}_{\beta nm} \tag{14a}$$

$$R_{\alpha\beta mn} = -\, R_{\beta\alpha nm} \tag{14b}$$

$$R_{\alpha\beta mn} = R_{mn\alpha\beta} \tag{14c}$$

Now, by symmetry of the Ricci tensor by (2) and (14c), we have

$$R_{\beta n} = g^{\alpha m}\, R_{\alpha\beta mn} \tag{15}$$

Then, we consider: $R^{\lambda}_{\beta\,;\lambda} = g^{\lambda n} R_{\beta n\,;\lambda} = g^{\lambda n}\, g^{\alpha m}\, R_{\alpha\beta mn\,;\lambda}$.

Then from equation (2) we simplify: $R^{\lambda}_{\beta\,;\lambda} = g^{\lambda n} g^{\alpha m}\, R_{mn\alpha\beta\,;\lambda}$. Now, using the 2nd Bianchi identity (14), we may infer that

$$R^{\lambda}_{\beta\,;\lambda} = g^{\lambda n} g^{\alpha m}\left(R_{mn\lambda\,;\alpha} + R_{mn\lambda;\beta}\right)$$

Now, according to equations (14a), (14b) and (15), the first term in the right hand side of the above equation becomes

$-g^{\alpha m}\, R_{m\beta\,;\alpha} = R^{n}_{\beta\,;n} \equiv R_{\,,\beta}$ .Hence,

$R^{m}_{\alpha\,;m} = \frac{1}{2}\, R_{\,,\alpha} \equiv \frac{1}{2}\, (\delta^{m}_{\alpha}\, R)_{\,;m}$. Where, δ^{m}_{α} is the Kronecker delta. (16)

Thus, the equation (12) can be re-written as (ignoring the index change/difference)

$\left[R^{\alpha}_{\lambda} - \frac{1}{2}\,\delta^{\alpha}_{\lambda}\, R\right]_{;\alpha} = 0$. From this equation, if we contract further followed by cyclic summation of the Bianchi identity(s), we eventually finds the covariant derivative of the Riemann curvature tensor as

$$R_{\lambda\alpha\beta\gamma\,;s} = \frac{1}{2}\frac{\partial}{\partial x^{s}}\left(\frac{\partial^{2} g_{\lambda\beta}}{\partial x^{\gamma}\,\partial x^{\alpha}} - \frac{\partial^{2} g_{\alpha\beta}}{\partial x^{\gamma}\,\partial x^{\lambda}} - \frac{\partial^{2} g_{\lambda\gamma}}{\partial x^{\alpha}\partial x^{\beta}} + \frac{\partial^{2} g_{\alpha\gamma}}{\partial x^{\beta}\,\partial x^{\lambda}}\right).$$

This above equation is most easily conceivable at a given point x adopting all locally inertial coordinate system in which the affine connection (but not necessary its derivatives) tends to vanish at x.

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BIBLIOGRAPHY:

Weinberg, S. (2015-reprint).*GRAVITATION AND COSMOLOGY: Principles and Applications of the General Theory of Relativity* (1st ed.). WILEY.