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Inhomogeneous generalization of Einstein's static universe with Sasakian space

Hideki Ishihara* and Satsuki Matsuno

Department of Mathematics and Physics, Graduate School of Science, Nambu Yoichiro Institute of Theoretical and Experimental Physics (NITEP), Osaka City University, Osaka 558-8585, Japan

*E-mail: ishihara@osaka-cu.ac.jp

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We construct exact static inhomogeneous solutions to Einstein's equations with counter flow of particle fluid and a positive cosmological constant by using the Sasaki metrics on three-dimensional spaces. The solutions, which admit an arbitrary function that denotes the inhomogeneous number density of particles, are a generalization of Einstein's static universe. For some examples of explicit solutions, we discuss non-linear density contrast and deviation of the metric functions.

Subject Index E01

1. Introduction

In the general theory of relativity, the investigation of solutions to Einstein's equations is an important task in understanding the structure of the universe. Since it is hard to solve Einstein's equations, which are non-linear field equations with constraints, almost all solutions are found under simplification using isometries.

Among the most important exact solutions with matter sources are cosmological solutions of the Friedmann–Lemaître–Robertson–Walker metric, which describe a homogeneous and isotropic universe. Less-symmetric solutions are provided by the Lemaître–Tolman–Bondi solutions [1–3], where spherically symmetric dust fluid is a source of gravity. It is striking that the solutions admit arbitrary functions. The best-known generalization of the Lemaître–Tolman–Bondi solutions are Szekeres's solutions [4,5], which admit no geometrical symmetry.

In the solutions noted above, the matter sources are characterized by vanishing vorticity. In contrust, we propose exact solutions to Einstein's equations with a fluid of particles moving along geodesics with non-vanishing vorticity.

The total spacetimes of the solutions are direct products of time and static three-dimensional space. We take the space homothetic to a three-dimensional Sasakian space [6,7], and construct exact solutions with inhomogeneous fluid with vorticity. The solutions admit an arbitrary function that describes the density of the fluid.

2. Metric with Sasakian space

We consider a static metric

$$ds^2 = -dt^2 + ds_M^2, \tag{1}$$

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where the metric of the three-dimensional space, M, is given by

$$ds_M^2 = a^2\left(d\theta^2 + h(\theta,\phi)^2\, d\phi^2\right) + b^2(d\psi + f(\theta,\phi)\, d\phi)^2. \tag{2}$$

In Eq. (2), a and b are constants, and $f(\theta, \phi)$ and $h(\theta, \phi)$ are functions to be determined later. The metric in Eq. (1) admits two unit Killing vectors,

$$\xi_{(t)} = \partial_t, \qquad \xi_{(\psi)} = \frac{1}{b}\partial_\psi. \tag{3}$$

The space M is a fiber bundle: a one-dimensional fiber with the coordinate ψ on a two-dimensional base space, N, with coordinates (θ, ϕ). We take one-form basis as

$$\sigma^0 := dt, \qquad \sigma^1 := ad\theta, \qquad \sigma^2 := ah(\theta,\phi)d\phi, \qquad \sigma^3 := b(d\psi + f(\theta,\phi)\, d\phi), \tag{4}$$

so that the metric in Eq. (1) with Eq. (2) is rewritten as

$$g_{ab} = -\sigma_a^0 \otimes \sigma_b^0 + g_{ab}^M, \qquad g_{ab}^M = \sigma_a^1 \otimes \sigma_b^1 + \sigma_a^2 \otimes \sigma_b^2 + \sigma_a^3 \otimes \sigma_b^3. \tag{5}$$

Assuming that the relation between the functions f and h is

$$h(\theta,\phi) = \partial_\theta f(\theta,\phi), \tag{6}$$

we have

$$d\sigma^3 = \frac{b}{a^2}\sigma^1 \wedge \sigma^2 \tag{7}$$

and $\sigma^3 \wedge d\sigma^3 \neq 0$. The manifold M that admits such a one-form is called a contact manifold, and it is known that the three-dimensional space (M, g^M) in the form of Eq. (2) with the condition in Eq. (6), which admits the unit Killing vector, is homothetic to a three-dimensional Sasakian space. Equation (7) means the existence of vorticity of the vector field $\xi_{(\psi)}$, which is metric dual to σ^3.

The scalar curvature of the two-dimensional base space N is

$$R_N = -\frac{2}{a^2}\frac{\partial_\theta^2 h(\theta,\phi)}{h(\theta,\phi)}, \tag{8}$$

and the Ricci curvature tensor of the total spacetime with respect to the basis in Eq. (4) is given by

$$R_{ab} = \left(-\frac{b^2}{2a^4} + \frac{1}{2}R_N\right)\left(\sigma_a^1 \otimes \sigma_b^1 + \sigma_a^2 \otimes \sigma_b^2\right) + \frac{b^2}{2a^4}\sigma_a^3 \otimes \sigma_b^3. \tag{9}$$

3. Counter flow fluid

We consider a counter flow fluid that consists of collisionless particles: one component, labeled with $+$, flows in the direction of $\xi_{(\psi)}$, and the other, labeled with $-$, flows in the opposite direction. Namely, the four-velocities, parametrized by the proper time, are given by

$$\begin{aligned} u_+^a &= \frac{1}{\sqrt{1-v^2}}\xi_{(t)}^a + \frac{v}{\sqrt{1-v^2}}\xi_{(\psi)}^a, \\ u_-^a &= \frac{1}{\sqrt{1-v^2}}\xi_{(t)}^a - \frac{v}{\sqrt{1-v^2}}\xi_{(\psi)}^a, \end{aligned} \tag{10}$$

where v is a function that depends only on θ and ϕ. Each particle with $u_\pm$ obeys the geodesic equation,

$$u_\pm^a \nabla_a u_\pm^b = 0. \tag{11}$$

As for the congruence of the geodesics with the tangent vectors in Eq. (10), we see that the expansion vanishes, and the shear does not vanish if v is not a constant. The vorticity that comes from $\xi_{(\psi)}$ is non-vanishing if $v \neq 0$.

The number densities of counter flow particles are assumed to be $n_+ = n_- = n/2$, where n is a function on N. Then, the energy–momentum tensor of the particle fluid is

$$T^{ab} = \frac{1}{2}mn(u_+^a \otimes u_+^b + u_-^a \otimes u_-^b) = mn\left(\frac{1}{1-v^2}\xi_{(t)}^a \otimes \xi_{(t)}^b + \frac{v^2}{1-v^2}\xi_{(\psi)}^a \otimes \xi_{(\psi)}^b\right), \tag{12}$$

and

$$\mathrm{tr}\, T = -mn. \tag{13}$$

The total angular momentum vanishes by the counter flow. Taking the limit $m \to 0$ and $v^2 \to 1$ with $m/(1-v^2) = $ ‖nite, we can consider the energy–momentum tensor of null particles moving along the fiber.

4. Einstein's equation

From Eqs. (9) and (12), Einstein's equation with a cosmological constant,

$$R_{ab} = T_{ab} - \frac{1}{2}(\mathrm{tr}\, T)g_{ab} + \Lambda g_{ab}, \tag{14}$$

yields

$$0 = \frac{1}{2}mn(\theta,\phi)\left(\frac{1+v(\theta,\phi)^2}{1-v(\theta,\phi)^2}\right) - \Lambda, \tag{15}$$

$$-\frac{b^2}{2a^4} + \frac{1}{2}R_N = \frac{1}{2}mn(\theta,\phi) + \Lambda, \tag{16}$$

$$\frac{b^2}{2a^4} = \frac{1}{2}mn(\theta,\phi)\left(\frac{1+v(\theta,\phi)^2}{1-v(\theta,\phi)^2}\right) + \Lambda. \tag{17}$$

Here and hereafter, we set $8\pi G = 1$. Taking a combination of Eqs. (15) and (17), we have

$$\Lambda = \frac{b^2}{4a^4} > 0, \tag{18}$$

and from Eq. (15) we see that the function $v(\theta, \phi)$ is expressed by the function $n(\theta, \phi)$ as

$$v^2(\theta,\phi) = \frac{2\Lambda - mn(\theta,\phi)}{2\Lambda + mn(\theta,\phi)}. \tag{19}$$

Then, $mn(\theta, \phi)$ should be in the range

$$0 \le mn(\theta,\phi) \le 2\Lambda. \tag{20}$$

Under the relations in Eqs. (18) and (19), the four-dimensional Einstein equations reduce to a simple equation,

$$R_N(\theta,\phi) = mn(\theta,\phi) + 6\Lambda. \tag{21}$$

We call this the "reduced Einstein equation" on the two-dimensional base space that means the scalar curvature of N is equal to the mass density of particles plus the cosmological constant.

Since $R_N(\theta, \phi)$ is positive everywhere on N, we hereafter consider N, as far as it is simply connected, to be homeomorphic to the two-dimensional sphere. We integrate Eq. (21) on N as

$$\int_N R_N(\theta,\phi)dS = \int_N (mn(\theta,\phi) + 6\Lambda)\,dS. \tag{22}$$

By using the Gauss–Bonnet theorem, the left-hand side of Eq. (22) is 8π. Introducing the average of the number density by

$$\langle n\rangle := A_N^{-1}\int_N n(\theta,\phi)dS, \tag{23}$$

where A_N denotes the surface area of the base space N, we have

$$m\langle n\rangle + 6\Lambda = 8\pi A_N^{-1}, \tag{24}$$

no matter whether $n(\theta, \phi)$ is inhomogeneous.

The reduced Einstein equation in Eq. (21) with Eq. (8) is written in the form

$$\partial_\theta^2 h(\theta, \phi) + a^2 w(\theta, \phi) h(\theta, \phi) = 0, \tag{25}$$

$$w(\theta, \phi) := \frac{1}{2} mn(\theta, \phi) + 3\Lambda. \tag{26}$$

We should note that Eq. (25) is a linear ordinary differential equation with respect to θ for every fixed value of the coordinate ϕ.

We take (θ, ϕ) to be the geodesic polar coordinate system; then, the function $h(\theta, \phi)$ should satisfy

$$h(0, \phi) = 0, \qquad h(\theta, \phi + 2\pi) = h(\theta, \phi), \tag{27}$$

and

$$\partial_\theta h(0, \phi) = 1 \tag{28}$$

in order to avoid the conical singularities at the pole $\theta = 0$.

5. Examples

Here, we present simple examples of global solutions. We consider (θ, ϕ) as spherical coordinates on the base space N, homeomrphic to S^2, in the range $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, where $\theta = 0, \pi$ correspond to the north and south poles. The function h should satisfy

$$h(\theta, \phi + 2\pi) = h(\theta, \phi) \tag{29}$$

and

$$h(0, \phi) = h(\pi, \phi) = 0, \qquad \partial_\theta h(0, \phi) = -\partial_\theta h(\pi, \phi) = 1, \tag{30}$$

so that the coordinate singularities at each pole can be removed.

5.1 *Homogeneous cases*

In the case that the number density of the particles, n, is constant, Eq. (21) means $R_N = \text{const.}$, i.e. the two-dimensional base space N is a homogeneous S^2 with radius a, and $A_N = 4\pi a^2$. Since $n = \langle n\rangle$, Eq. (24) leads to

$$w = \frac{1}{2} mn + 3\Lambda = \frac{1}{a^2}, \tag{31}$$

and then we have

$$h = \sin\theta \tag{32}$$

as the solution to Eq. (25) with the boundary conditions in Eqs. (29) and (30), and the function f is given by

$$f = -\cos\theta. \tag{33}$$

With the help of Eq. (18), the metric becomes

$$ds^2 = -dt^2 + a^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 + 4a^2\Lambda (d\psi - \cos\theta \, d\phi)^2 \right). \tag{34}$$

We assume the fiber is S^1, so that the three-dimensional space M is a Hopf fiber bundle[1] that describes a squashed S^3. The "aspect ratio" of the radius of the S^1 fiber to the radius of the S^2 base space is given by Eq. (31) as

$$\frac{b}{a} = 2a\sqrt{\Lambda} = \sqrt{\frac{8\Lambda}{mn + 6\Lambda}} = \sqrt{1 + \frac{v^2}{2 + v^2}} \geq 1. \tag{35}$$

Namely, M is a "prolate" three-dimensional sphere for non-vanishing v, where the metric admits five Killing vectors: $\xi_{(t)}$, $\xi_{(\psi)}$, and three on the base space S^2.

In the null-particle limit, i.e. $m \to 0$ and $v^2 \to 1$, the aspect ratio takes the maximum value, $2/\sqrt{3}$. On the other hand, in the case that the particles are at rest, i.e. $v = 0$ and $mn = 2\Lambda$, the aspect ratio becomes 1 and we have

$$ds^2 = -dt^2 + \frac{1}{4\Lambda}\left(d\theta^2 + \sin^2\theta\, d\phi^2 + (d\psi - \cos\theta\, d\phi)^2\right). \tag{36}$$

This is the metric of Einstein's static universe, where the three-dimensional space is a round S^3. This spacetime admits seven Killing vectors: $\xi_{(t)}$ and six on S^3 including $\xi_{(\psi)}$.

5.2 *Axisymmetric cases*

We consider the case that the system is inhomogeneous but symmetric under a rotation of ϕ. Then, the functions n and h depend only on θ, and Eq. (25) reduces to

$$\frac{d^2h(\theta)}{d\theta^2} + a^2 w(\theta) h(\theta) = 0, \qquad w(\theta) = \frac{1}{2}mn(\theta) + 3\Lambda. \tag{37}$$

The boundary conditions for $h(\theta)$ are

$$h(0) = h(\pi) = 0, \tag{38}$$

and $h(\theta)$ should be non-vanishing in the region $0 < \theta < \pi$. The ordinary differential equation in Eq. (37) with the boundary conditions in Eq. (38) is a Sturm–Liouville problem, where a^2 is the eigenvalue and $w(\theta)$ is the weight function.

At the north and south poles, regularity of the geometry requires

$$\partial_\theta h(0) = 1, \qquad \partial_\theta h(\pi) = -1, \tag{39}$$

and the smoothness of the number density requires

$$\partial_\theta n(0) = \partial_\theta n(\pi) = 0. \tag{40}$$

As a special example, we consider

$$n(\theta) = n_0 - n_1 \cos(2\theta), \tag{41}$$

where Eq. (20) requires that n_0 and n_1 are constants satisfying

$$0 \leq n_0 - |n_1|, \qquad n_0 + |n_1| \leq 2\Lambda/m. \tag{42}$$

In this case, Eq. (37) reduces to the Mathieu equation in the form

$$\frac{d^2h}{d\theta^2} + \big(p - 2q\cos(2\theta)\big)h = 0, \tag{43}$$

where p and q are constant parameters given by

$$p := \left(3\Lambda + \frac{1}{2}mn_0\right)a^2, \qquad q := \frac{1}{4}mn_1 a^2. \tag{44}$$

[1] Indeed, if M is simply connected and complete, it has been proved that the fiber is S^1 and M is a Hopf bundle [8].

The solutions without a node that satisfy Eqs. (38) and (39) are

$$h(\theta) = C\, se_1(q,\theta), \tag{45}$$

where $se_1(q, \theta)$ is the odd Mathieu function of order 1, and C is the normalization constant given by

$$\frac{1}{C} = \frac{d}{d\theta} se_1(q,\theta)\Big|_{\theta=0}. \tag{46}$$

For given n_0 and n_1, the parameter a is determined so that p should be the characteristic value of $se_1(q, \theta)$; then, $h(\theta)$ satisfies Eqs. (38) and (39).

The function $f(\theta)$ is a primitive function of $Cse_1(q, \theta)$. Since we have assumed that M is a simply connected S^1 bundle on an S^2 base space, the first Chern number is 1. We rescale ψ and f as

$$\sigma_3 = b(d\psi + f(\theta)d\phi) = \bar{b}(d\bar{\psi} + \bar{f}(\theta)d\phi), \tag{47}$$

so that

$$\frac{1}{4\pi} \int_N d\left(\bar{f}(\theta)d\phi\right) = 1. \tag{48}$$

The metrics composed of the functions $h(\theta)$ and $\bar{f}(\theta)$ have three Killing vectors: $\xi_{(t)}$, $\xi_{(\psi)}$, and ∂_ϕ.

We consider the case that the mass density varies maximally in Eq. (20), namely $mn_{\rm min} = 0$ and $mn_{\rm max} = 2\Lambda$. Setting $n_0 = |n_1| = \Lambda/m$, we have two cases: (i) $q = \frac{1}{4}\Lambda a^2$ and (ii) $q = -\frac{1}{4}\Lambda a^2$:

(i) $n(\theta) = \Lambda(1 - \cos 2\theta)$ (sparse at the poles and dense at the equator),
(ii) $n(\theta) = \Lambda(1 + \cos 2\theta)$ (dense at the poles and sparse at the equator).

In these cases, $p = \frac{7}{2}\Lambda a^2$ should be the the characteristic value of the Mathieu functions $se_1\left(\pm\frac{1}{4}\Lambda a^2, \theta\right)$; then, a and related quantities are determined numerically as

(i) $a = 0.5162\Lambda^{-1/2}$, $A_N = 1.09003\Lambda^{-1}\pi$, $m\langle n\rangle = 1.33922\Lambda$,
(ii) $a = 0.5545\Lambda^{-1/2}$, $A_N = 1.19876\Lambda^{-1}\pi$, $m\langle n\rangle = 0.673552\Lambda$.

As a reference, $a = (1/2)\Lambda^{-1/2}$, $A_N = \Lambda^{-1}\pi$, and $m\langle n\rangle = mn = 2\Lambda$ for Einstein's static universe. While the geometrical quantities take similar values in these cases, i.e. $a \sim 0.5\Lambda^{-1/2}$, $A_N \sim \Lambda^{-1}$, the averaged mass density of case (i) is almost double that of case (ii).

5.3 *Non-axisymmetric cases*

On the assumption of the metric in Eq. (2) with the boundary conditions in Eqs. (38) and (39), the ϕ = const. curves, which connect the north and south poles, are geodesics on the base space N, and all these curves have the same length, πa. Then, the inhomogeneous global solutions obtained in this paper are such a class of special solutions.

Although it is possible, in principle, to solve Eq. (25) with Eq. (26) for a given smooth function $n(\theta, \phi)$, it is hard to represent the solutions by using well-known special functions. Starting from $f(\theta, \phi)$, however, we can easily present a set of functions $h(\theta, \phi)$ and $n(\theta, \phi)$ expressed by combinations of the trigonometric functions as exact solutions.

As an exact solution, we present the metric functions

$$f(\theta, \phi) = -\cos\theta + \beta \sin^5\theta \cos\phi, \tag{49}$$

$$h(\theta, \phi) = \sin\theta + 5\beta \sin^4\theta \cos\theta \cos\phi, \tag{50}$$

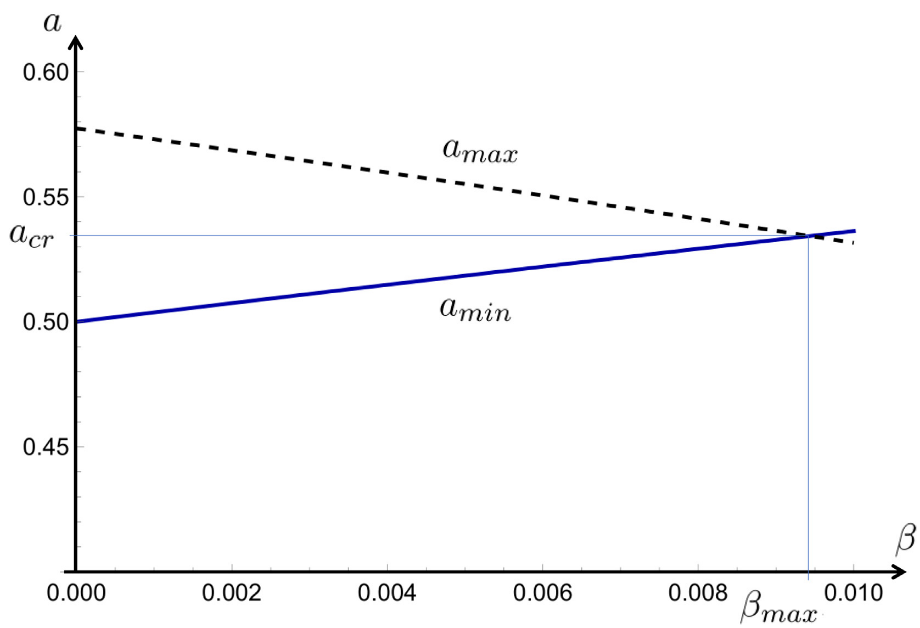

Fig. 1. The upper bound, $a_{\max}$, and lower bound, $a_{\min}$, of a depicted as functions of β. At $\beta = \beta_{\max}$, $a_{\max}$ and $a_{\min}$ coincide with $a_{\rm cr}$.

and the mass density function

$$mn(\theta, \phi) = -6\Lambda + \frac{1}{a^2}\left(2 - \frac{30\beta \sin 4\theta \cos\phi}{1 + 5\beta \sin^3\theta \cos\theta \cos\phi}\right), \tag{51}$$

where β is a positive parameter that denotes the amplitude of inhomogeneity. The metrics composed of the functions in Eqs. (49) and (50) have only two Killing vectors, $\xi_{(t)}$ and $\xi_{(\psi)}$, if $\beta \neq 0$, while in the special case $\beta = 0$ the solutions reduce to the homogeneous cases discussed above.

For the functions in Eqs. (49), (50), and (51), which have inhomogeneity, the surface area A_N and averaged mass density $m\langle n\rangle$ are obtained as

$$A_N = 4\pi a^2, \qquad m\langle n\rangle = \frac{2}{a^2} - 6\Lambda. \tag{52}$$

These quantities, explicitly independent of the parameter β, are the same forms as in the homogeneous case.

The parameter β and a are limited as $0 \leq \beta \leq \beta_{\max}$ and $a_{\min} \leq a \leq a_{\max}$, so that $mn(\theta, \phi)$ satisfies Eq. (20). In the case that $mn(\theta, \phi)$ varies maximally, namely it takes 0 and 2Λ elsewhere, β becomes the upper bound $\beta_{\max}$, and $a_{\min}$ and $a_{\max}$ coincide with a value $a_{\rm cr}$. In Fig. 1, $a_{\min}$ and $a_{\max}$ are shown as functions of β, where $\beta_{\max} \sim 0.009436$ and $a_{\rm cr} \sim 0.5343$, numerically.[2] It is interesting that even for the non-linear density contrast, $(n_{\max} - n_{\min})/(n_{\max} + n_{\min}) = 1$, Eqs. (50) and (49) with $\beta = \beta_{\rm cr}$ mean that the deviation of the metric functions is small, of the order of 1/100.

6. Summary

We have constructed exact static inhomogeneous solutions to Einstein's equations with counter flow of particle fluid and a positive cosmological constant. The three-dimensional space of the solution is homothetic to a Sasakian space that consists of S^1 fibers on a S^2 base space.

[2] A rough estimation of $\beta_{\max}$ and $a_{\rm cr}$ is given in the appendix.

The solutions admit two unit Killing vector fields: a timelike Killing vector field of the static spacetime, and a spacelike Killing vector field that is tangent to the fiber. The unit Killing vector tangent to the fiber, which is metric dual to the contact form of the three-dimensional space, is a geodesic tangent and has non-vanishing rotation. Particles of the fluid move along geodesics whose tangent vectors are linear combinations of the two Killing vectors metioned above. Then, the geodesic congruences of the particles have non-vanishing vorticity.

On these assumptions, we have obtained reduced Einstein equations on the two-dimensional base space that relate the scalar curvature with the mass density of the particles and the cosmological constant. The equation has the form of a linear differential equation for the metric function. We have found exact solutions to the differential equation, where the number density of particles has non-linear inhomogeneity denoted by an arbitrary function on the base space. The solutions are inhomogeneous generalizations of Einstein's static universe. We have presented examples of exact solutions explicitly, and we observed that the deviation of the metric is small, of the order of 1/100, for non-linear density contrast of the particles.

As is well known, Einstein's static universe is dynamically unstable. Similarly, the solutions obtained in this paper would be unstable. It is an interesting problem to extend the solutions to expanding ones with inhomogeneity.

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Appendix A. Rough estimation of $\boldsymbol{\beta}_{\max}$ and $\boldsymbol{a}_{\mathrm{cr}}$ in the model in Sect. 5.3

The upper and lower limits of a are determined by $mn = 0$ and $mn = 2\Lambda$, respectively. We then have

$$a_{\max}^2 = \frac{1}{3\Lambda} w(\theta_-, \phi_-; \beta), \qquad a_{\min}^2 = \frac{1}{4\Lambda} w(\theta_+, \phi_+; \beta), \tag{A1}$$

where

$$w(\theta, \phi; \beta) := 1 - \frac{15\beta \sin 4\theta \cos\phi}{1 + 5\beta \sin^3\theta \cos\theta \cos\phi}, \tag{A2}$$

and (θ_+, ϕ_+) and (θ_-, ϕ_-) give the maximum and mminimum of $w(\theta, \phi; \beta)$, respectively. We see that

$$\partial_\phi w(\theta, \phi; \beta) = 0 \qquad \text{for } \phi = 0, \pi, \tag{A3}$$

and $w(\theta, \phi; \beta)$ is invariant under

$$\phi \to \phi + \pi, \qquad \theta \to \pi - \theta, \tag{A4}$$

so we fix $\phi = 0$. The parameter β should be small for positive $w(\theta, \phi; \beta)$; then, the minimum of $w(\theta, \phi; \beta)$ is attained for

$$\sin 4\theta \sim 1, \qquad \sin^3\theta \cos\theta < 0, \tag{A5}$$

and the maximum for

$$\sin 4\theta \sim -1, \qquad \sin^3\theta \cos\theta < 0. \tag{A6}$$

Then, we have, approximately,

$$(\theta_+, \phi_+) \sim \left(\frac{7}{8}\pi, 0\right), \qquad (\theta_-, \phi_-) \sim \left(\frac{5}{8}\pi, 0\right). \tag{A7}$$

We expand Eq. (A1) by the small parameter β up to the second order as

$$a_{\max}^2 \sim \frac{1}{3\Lambda}(1 - 15\beta\sin(4\theta_-) + 75\beta^2\sin(4\theta_-)\sin^3\theta_-\cos\theta_-),$$

$$a_{\min}^2 \sim \frac{1}{4\Lambda}(1 - 15\beta\sin(4\theta_+) + 75\beta^2\sin(4\theta_+)\sin^3\theta_+\cos\theta_+). \tag{A8}$$

For small β, $a_{\max}$ and $a_{\min}$ are almost linear functions of β, as seen in Fig. 1. By equating $a_{\max}^2$ and $a_{\min}^2$, and using Eq. (A7), we can estimate $\beta_{\max} \sim 0.00944$, and $a_{\rm cr} = a_{\min} = a_{\max} \sim 0.534$.

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