Warped Throat Solutions in String Theory and Their Cosmological Applications

Anatoly Dymarsky

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Abstract

This thesis is devoted to a study of certain examples of gauge/string duality related to warped throat backgrounds in string theory. Namely, we consider a family of IIB SUGRA solutions dual to a moduli space of certain cascading $\mathcal{N} = 1$ gauge theory. This theory exhibits rich low-energy behavior, including chiral symmetry breaking and confinement. The first part of this thesis is focused on the gravity dual description of these phenomena. In particular, we discuss string theory description of the continuous gauge theory moduli space, evaluate the tension of BPS domain wall, and calculate baryonic condensates. The second part of the thesis is devoted to the embedding of the warped throat backgrounds into flux compactifications. To this end we calculate the nonperturbative superpotential of the D3-D7 system on warped conic geometries. This superpotential plays an important role in fixing Kähler moduli and is an important ingredient in constructing consistent compactification scenarios. In the last part of the thesis we apply this superpotential to a particular cosmological inflation scenario based on the dynamics of a D3-brane moving along the throat. We conclude that the realization of stringy inflation within this scenario is possible only around an inflection point of the potential and requires a fine tuning of the parameters.

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Contents

Chapter 1

Introduction

String theory was originally proposed in the late 60's as a theory of strong interactions. However, some problems with these applications and the discovery of QCD, led a change of its purpose in 1974. In these new applications the string tension was scaled up by 38 orders of magnitude, and string theory became a leading candidate for quantum gravity and the unified theory of all other interaction. Its role was solidified in the mid-80's when new models and constraints on string theory were understood. Since string theory is formulated in ten dimensions, the main goal was to compactify extra dimensions into a suitable manifold $\mathcal{M}$, leaving four non-compact dimensions describing the observable Universe. Although significant progress was made in understanding various aspects of string theory compactifications, the goal of finding the unified theory still seems too ambitious to be achieved in the near future.

In an unexpected twist, about a decade ago the *AdS/CFT correspondence* [1, 2, 3] returned string theory to its role in studying the strong coupling dynamics of gauge theories. The idea behind the AdS/CFT correspondence is to consider the stack of N D3-branes placed in the flat ten-dimensional space [4, 5]. For very large N, D3-branes produce back reaction on the metric

$$\mathrm{d}s_{10}^2 = h^{-1/2}dx^2 + h^{1/2}\left(dr^2 + ds_{\Omega_5}^2\right) \ ,$$
$$h(r) = 1 + \frac{R^4}{r^4} \ , \quad R^4 \equiv 4\pi g_s \alpha'^2 N \ , \qquad (1.1)$$

and induce the R-R flux $C_4 = dx^0 \wedge .. \wedge dx^3 h^{-1}$. The dynamics of the effective field theory on the branes can be described by String Theory in the curved space (1.1). The effective field theory on the branes is still coupled to gravity at this point. The crucial step is to consider the low-energy limit which decouples the field theory from gravity. On the string theory side this is equivalent to taking a formal limit $r \to 0$ [1]

$$h(r) \to \frac{R^4}{r^4} \ . \tag{1.2}$$

The resulting geometry is a product of AdS_5 and S^5. The field theory on the branes is $\mathcal{N} = 4$ SYM, which can be identified by counting supersymmetries of the background. The conformal symmetry of the theory $SO(4,2)$ is realized through the geometrical symmetries of AdS_5. Because of $\mathcal{N} = 4$ supersymmetry this background is believed to be the solution to string theory at the quantum level. This setup leads to the original AdS/CFT conjecture that the $\mathcal{N} = 4$ SYM in the planar limit $N \to \infty$ and fixed t' Hooft coupling $\lambda = g_{YM}^2 N$ is dual to the IIB String Theory on $AdS_5 \times S^5$ of radius $R^4 = 4\pi\alpha'^2\lambda$.

The duality in question is weak-strong, i.e. the perturbative phase of one theory matches the nonperturbative phase of the other. One the one hand when λ is small, the Feynman diagram expansion in field theory is convergent and hence the dynamics is controllable. At the same time the radius of geometry (1.1) is small and all higher corrections in α' are important making dual string theory non-perturbative. On the other hand when $\lambda \to \infty$, we do not have any suitable technique to deal with field theory, but the curvature of the dual background is small everywhere and string theory can be successfully approximated by supergravity in the semi-classical regime. The latter correspondence can be extended to the non-conformal theories and constitutes the gauge/gravity duality.

The AdS/CFT correspondence was studied extensively over the last ten years.

Perhaps one of the most important achievements in this area is the recently developed technique of calculating anomalous dimensions of gauge-invariant operators with the help of auxiliary spin chains (see [6] for an overview of the method). The newly discovered spin-chains [7] are believed to provide a smooth extrapolation of the theory between the weak and strong coupling regimes [8]. This is one of the few known examples of a theoretical approach which can successfully interpolate to the strong coupling regime.

The dynamics of $\mathcal{N} = 4$ SYM is drastically different from the one of confining gauge theory, like QCD, because of conformal and super symmetries. In order to generalize gauge/string duality to a more "realistic" field theory, one needs to construct an example with few or no supersymmetries.

To this end one can consider D3-branes at conical singularity called the conifold

$$\sum_{i=1}^{4} z_i^2 = 0 \ , \tag{1.3}$$

rather than putting them into the flat space. The resulting theory is a certain superconformal $\mathcal{N} = 1$ gauge theory with $SU(N) \times SU(N)$ gauge group [9]. It is discussed in detail in section (2.1.1). The resulting geometry

$$\begin{aligned} \mathrm{d}s_{10}^2 &= h^{-1/2} dx^2 + h^{1/2} \left(dr^2 + ds_{T^{1,1}}^2 \right) \ , \\ h(r) &= 1 + \frac{R^4}{r^4} \ , \quad R^4 \equiv 4\pi g_s \frac{27}{16} \alpha'^2 N \ , \\ C_4 &= dx^0 \wedge .. \wedge dx^3 h^{-1} \ , \end{aligned} \tag{1.4}$$

is similar to (1.1) with five-sphere S^5 replaced by the Sasaki-Einstein manifold $T^{1,1}$. The resulting six-cone over $T^{1,1}$ is a toric Calabi-Yau with Ricci-flat metric $ds_6^2 = dr^2 + ds_{T^{1,1}}^2$. This example was generalized to the case of D3-branes probing arbitrary toric Calabi-Yau singularity. The dual quiver field theory, i.e. the symmetries, field content, corresponding charges and superpotential of field theory can be unambiguously determined from the geometrical data [10].

The field theories resulting from toric Calabi-Yau singularities are superconformal. It is desirable to get rid of conformal symmetries to extend gauge/string duality to the theories with confinement. It was done, for example, in the case of field theory on conifold singularity introduced above. The gauge group $SU(N) \times SU(N)$ can be extended by adding M extra colors $SU(N+M) \times SU(N)$. This breaks conformal symmetry and the theory becomes confining in the IR. It exhibits a cascade of Seiberg dualities [11] which can be described by introducing an effective scale-dependent number of colors N. On the gravity side, the running of N_{eff} is accommodated by the radius dependence of warp-factor $h(r)$. In the case of $N = kM$ for integer k the theory is confining in the IR. Because of dimensional transmutation the dual geometry (1.3) is modified near the tip $r \to 0$ by a dimensionful parameter ε

$$\sum_{i=1}^{4} z_i^2 = \varepsilon^2 \ . \tag{1.5}$$

This changes the topology at the tip $r = 0$ and leads to the smooth supergravity solution [12]. The metric of the singular conifold $(dr^2 + r^2 \mathrm{d}s^2_{T^{1,1}})$ from (1.4) is transformed into a Ricci-flat metric $\mathrm{d}s^2_{\mathcal{M}}$ on the deformed conifold (1.5). In the planar limit $M \to \infty$ and for large $g_s M$, the curvature is small everywhere and higher α' corrections are negligible. Therefore the string theory can be truncated to supergravity and the $SU((k+1)M) \times SU(kM)$ theory can be successfully described via gauge/gravity duality.

The cascading $SU((k+1)M) \times SU(kM)$ has rich IR dynamics resembling many features of non-SUSY gauge theory. Nevertheless it is different in many aspects from QCD, in particular because of a different gauge group. Therefore it is highly desirable to develop our understanding of gauge/gravity duality and to construct a gravity dual to the pure $SU(M)$ $\mathcal{N} = 1$ SYM – a closest supersymmetric cousin of the non-SUSY gauge theory. This would provide the description to the phenomenon of confinement in $SU(N)$ YM which is a crucial step toward understanding QCD.

Unfortunately this goal cannot be achieved at the present level of understanding. This is because pure $SU(N)$ $\mathcal{N} = 1$ SUSY gauge theory may correspond to the highly curved background. Indeed $SU(M)$ theory can be achieved by taking $g_S M \to 0$ limit and sending the scale of the last step of the cascade $SU(2M) \times SU(M)$ to infinity. This is opposite to the limit $g_S M \to \infty$ which makes the curvature small. Therefore all stringy corrections in α' are important for small $g_S M$. One cannot rely on gauge/gravity duality and has to incorporate an infinite tower of stringy corrections. This task is very difficult and can not be done with available techniques. Therefore even if the dual background would be somehow constructed it still may be of not practical use. We therefore return to the cascading theory with large $g_s M$.

The confining $SU((k+1)M) \times SU(kM)$ theory has a non-trivial continuous space of supersymmetric vacua known as *baryonic branch*. Although continuous moduli space are typical for $\mathcal{N} = 1$ theories with unbroken SUSY, this example is special because its gravity dual is known. On the gravity side the moduli space corresponds to the continuous family of supergravity backgrounds, sharing the same behavior in the UV region. This family was recently constructed [13] using the newly developed $SU(3)$ structure method. Although each particular solution on the branch is an ordinary example of gauge/gravity duality, the continuous family of solutions poses some new interesting questions. For a given supergravity solution free parameters like the asymptotic value of dilaton can be arbitrarily changed without violating the duality. For a family of solutions this change should be "uniform" to preserve the same UV universality class of gauge theory. In other words, the requirement that the family of gravity solutions describes the same microscopic theory in different IR phases specifies the boundary condition at $r \to \infty$. The leading asymptotic of all solutions from the family should share the same behavior in the UV region[14].

The gauge/gravity duality we have discussed so far is a powerful String Theory method to study gauge theory dynamics. It was noted in the beginning that in order to decouple the filed theory from the gravity, string theory should be considered on an infinite warped throat like $h^{1/2}(dr^2 + \mathrm{d}s_5^2)$ of (1.4). An intriguing idea is to apply the results of gauge/gravity duality toward compactifications of string theory. This can be done by considering a special compactification manifold $\mathcal{M}$ (usually of Calabi-Yau type) with a region resembling the throat geometry [15]. One can start with a compact Calabi-Yau with singularity, similar to the conifold singularity (1.3). Then the D3 brane placed near the singular locus will be described by field theory from above. The separation of scales between gauge theory and gravity (Planck scale) is related to the "length" of the throat stretching from the bulk of Calabi-Yau. This scenario has several advantages. First, it admits an elegant solution to the hierarchy problem through the geometrical parameters of the manifold $\mathcal{M}$. There is some evidence that singularities like (1.3) are typical features of a generic compact Calabi-Yau [16]. Therefore this scenario may be natural from the stringy landscape point of view [17]. Finally, the physics below the Planck scale is governed by the dynamics in the throat. Since the geometry in the throat is usually known explicitly and in general is much simpler than that in the bulk of Calabi-Yau one can effectively use theoretical tools of gauge/string duality to study the low energy dynamics in very detail. In other words, the warped throat scenarios have an advantage of being controllable and have predicting power unlike many other compactification scenarios of String Theory.

It is convenient to divide the low-energy phenomena into two groups – originating in the throat and in the bulk. The former are controllable, while the latter can be analyzed only with some uncertainty. Even if the underlying mechanism is clear, few explicit predictions can be made about the phenomena from the second class. Again, this is because the detailed information about the geometry and fluxes in the bulk are usually not known.

To construct a realistic compactification of String Theory one has to avoid unnaturally light modes – the moduli of the background. Fixing these moduli is a crucial step in model building [18]. There are a few typical scenarios which allow all moduli to be fixed dynamically. The one we are focused on in this work is an orientifold of type IIB theory with D3 and D7-branes. As will be discussed in more detail in chapter 4, all the moduli are fixed in this setup dynamically with help of flux and nonperturbative superpotentials. Although the main features of this mechanism are already well-understood, they originate in the bulk and the detailed prediction of fixed moduli values are not possible.

Large scale isotropy of our Universe together with the recent studies of the Cosmic Microwave Background have solidified inflation as a successful scenario of the early Universe. Precise measurements of the CMB anisotropy provide a very restrictive check of theoretical models. To match cosmological predictions, the stringy models of the early Universe require carefully designed fine tuning of the parameters. The ambiguity in values of fixed moduli and an excessively large number of stringy flux compactifications favor the ad-hoc logic that the compactification with necessary values of parameters always exists. This logic is usually applied to the various models of stringy inflation as the flat inflational potential is difficult to achieve. The main theoretical problem is then to show that a given model can sustain inflation at least for a certain choice of parameters.

One of the most popular scenarios of stringy inflation is based on the dynamics of D3 moving along the throat down to the tip [18]. The effective mass of D3 is expected to be much lighter than the Planck scale and that is why the location of D3 is a promising candidate for inflaton field. In a general model proposed and studied in [18], in addition to the force coming from the nonperturbative superpotential due to gluino condensation on D7, the D3 is also a subject to force from anti-D3 located at the tip. The latter is required to uplift the potential to a positive value to match observations of the cosmological constant. The potential generated by anti-D3 at the tip is not flat enough by itself to support inflation. A crucial question is whether the contribution of nonperturbative superpotential can flatten it enough at least for some choice of parameters. We provide evidence for this using a specific embedding of D7-branes.

1.1 Outline

This thesis is devoted to certain examples of gauge/gravity duality and their applications to cosmology along the lines outlined in the introduction. Our main example is the theory on conifold singularity (1.3). As was discussed above, the theory with M extra colors $SU((k+1)M) \times SU(kM)$ has continuous moduli space (baryonic branch), which corresponds to the family of gravity backgrounds. We review the gauge/gravity duality for this theory in chapter 2.

We start with a review of the conformal $SU(N) \times SU(N)$ theory on conical singularity in section (2.1.1) and proceed with the detailed discussion of dual geometry (1.4) in section (2.1.2). Section (2.2.1) generalizes our consideration to the field theory with M extra colors. The dual geometry of deformed conifold (1.5) is discussed in detail in section (2.2.2).

Section (2.3) is devoted to the family of supergravity backgrounds, BGMPZ solutions of [13], dual to the gauge theory on baryonic branch. We review the geometrical properties of the solutions and present explicit formulae for the metric and fluxes whenever possible. We also discuss proper choice of boundary conditions reflecting the UV universality of gauge theory. The section concludes with a discussion of the κ-symmetry condition for a D-brane placed on the conifold at an arbitrary point on the branch.

The main focus of chapter 3 is the gauge/gravity duality along the branch. Section 3.1 is devoted to the BPS domain wall which separates isomorphic vacua with different values of gluino condensate. Gauge theory analysis suggests the tension of such an object to be moduli independent. We study D5-brane which is gravity dual to the domain wall in question and confirm this result by use of κ-symmetry and geometry of BGMPZ solutions.

We proceed with another example of gauge/gravity duality, the string theory description of baryon operators, in section 3.2. The baryon operator is dual to a D5-brane wrapping the base of the conifold. This can be used to measure baryonic condensate on the gravity side. It is given by the DBI action of the Euclidean D5-brane covering entire six-dimensional internal space. Using this prescription we found the relation between the parameter along baryonic branch in gauge theory, $\langle \mathcal{A} \rangle$, and the corresponding parameter on the gravity side. We have also reproduced the quantum constraint [19] along the branch

$$\langle \mathcal{A}\mathcal{B} \rangle = \text{const} \ . \tag{1.6}$$

The two examples of chapter 3 confirm that the family of BGMPZ solutions provides a correct description of $SU((k+1)M) \times SU(kM)$ gauge theory on baryonic branch on moduli space.

Chapter 4 develops the ideas of warped throat compactification presented in the introduction. Namely, we calculate nonperturbative contributions to the superpotential of D3-D7 system placed on the throat, which is assumed to be a part of a compact Calabi-Yau manifold. The nonperturbative superpotential in question governs the dynamics of mobile D3 as it depends on the D3's location on conifold z_α, $\alpha = 1, 2, 3$. For the case of $N7$ D7-branes wrapping a four-cycle Σ_4 in conifold (1.3), defined by $f(z_\alpha) = 0$, the non-trivial part of superpotential turns out to be

$$W_{np} \propto f(z)^{1/N_7} \ . \tag{1.7}$$

This result was a missing ingredient in understanding the dynamics of the D3-D7 system. It allows a detailed study of D3 rolling down to the tip.

Chapter 5 is devoted to a string inflation model based on this setup. The location of D3 plays the role of an inflaton. The inflation occurs when the potential for moving D3 is sufficiently flat. Our analysis reveals that in general the potential is too steep to support inflation near the tip. Nevertheless with appropriate fine tuning of parameters, the potential has an inflection point where inflation can occur. The cosmological predictions of such model are highly sensitive to initial conditions and the model itself requires unexpectedly large amount of fine-tuning to support enough e-fold of expansion. Our findings clarify the status of such models and propose new directions of study.

Chapter 6 concludes the thesis with a discussion of the results.

Chapter 2

The warped deformed conifold and the dual gauge theory

2.1 D-branes at conical singularities and conformal gauge theories

The simplest example of gauge/string duality refers to the stack of D3-branes on a smooth manifold. In the planar limit $g_{YM}^2 N$ - fixed, $N \to \infty$ the D-brane dynamics reduces to the superconformal $\mathcal{N} = 4$ gauge theory on the world-volume. The same physical system can be described via string theory on $AdS_5 \times S^5$. The observation that gauge theory in planar limit can be described via string theory on special AdS_5 background constitutes the main idea of AdS/CFT correspondence [1, 2, 3].

Similarly, Klebanov and Witten [9] suggested that N D-branes at the singularity $z_i = 0$ of the conifold

$$\sum_i^4 z_i^2 = 0 \ , \tag{2.1}$$

will result in "conifold" field theory – certain $\mathcal{N} = 1$ superconformal field theory dual to the string theory on $AdS_5 \times X$, where X was identified as Einstein manifold $T^{1,1}$. The introduction to the "conifold" field theory below is followed by a detailed discussion of the dual geometry.

2.1.1 The "conifold" field theory

D-branes on conifold singularity

We start with identifying the field content of the effective theory on the stack of N D3-branes placed on the conical singularity (2.1). Following Klebanov and Witten [9] we start with only one brane placed on the cone. Its moduli space is described by (2.1), which can be "solved" in terms of 4 independent complex numbers A_i, B_j, with $i, j = 1, 2$ subject to "gauge symmetry" $A_i \to \lambda A_i$, $B_j \to \lambda^{-1} B_j$

$$W_{ij} = A_i B_j \ . \tag{2.2}$$

The complex matrix W_{ij} is related to z_i via (2.14). The $SO(4)$ symmetry of the geometry (2.1) is a group of global symmetries of the gauge theory. This suggests the $SU(2) \times SU(2)$ doublets A_i, B_j are chiral superfields and the constraint (2.1) $\det W = 0$ should follow from dynamics.

In the case of N D3-branes the abelian gauge group becomes $SU(N) \times SU(N)$ with A_i and B_j in the $(N, \overline{N})$ and $(\overline{N}, N)$ representation correspondingly [9]. The $U(1)$ factors of $U(N) \times U(N)$ decouple when theory flows in the IR to a line of fixed points.

In addition to the $SU(2) \times SU(2)$ symmetry there is anomaly-free $U(1)_R$ R-symmetry which shifts arguments of A_i, B_j. It acts on geometry (2.1) by shifting arguments of z_i. Both A_i and B_j has 1/2 charge under $U(1)_R$ and the most general superpotential respecting $SU(2) \times SU(2) \times U(1)_R$ is [9]

$$W_0 = \frac{\lambda}{2} \epsilon^{ii'} \epsilon^{jj'} \mathrm{Tr}(A_i B_j A_{i'} B_{j'}) \ . \tag{2.3}$$

There is another anomaly-free abelian symmetry $U(1)_{baryon}$ which shifts A_i, B_j is opposite directions

$$A_i \to A_i e^{i\varphi} \ , \quad B_j \to B_j e^{-i\varphi} \ . \tag{2.4}$$

.

At the classical level the superpotential (2.3) describes symmetric product of N points on the conifold (2.1). This can be seen by considering diagonal A_i and B_j.

Klebanov and Witten argued that the theory in question flows to the superconformal point in the IR. They conjectured that the resulting planar CFT is dual to the string theory on $AdS_5 \times T^{1,1}$.

String theory on $AdS_5 \times T^{1,1}$

The solution of string theory on $AdS \times T^{1,1}$ is specified by the warp factor H_{KW}

$$ds_{10}^2 = H_{KW}^{-1/2} dx_{3,1}^2 + H_{KW}^{1/2}(dr^2 + r^2 ds_{T^{1,1}}^2) \ ,$$
$$C_4 = dx^0 \wedge ...dx^3 H_{KW}^{-1} \ ,$$
$$H_{KW} = \frac{L^4}{r^4} \ , \qquad L^4 = 4\pi g_s N (\alpha')^2 \ . \tag{2.5}$$

This background is different from $AdS \times S^5$ of [1, 2, 3] by the substitution of $ds^2_{T^{1,1}}$ instead of $ds^2_{S^5}$. The dual field theory was identified in the previous subsection through the analysis of global symmetries. Here we follow [20] to give a supporting argument which goes beyond simple symmetry analysis. Let us consider a $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ SYM which breaks supersymmetry to $\mathcal{N} = 2$. The orbifold group acts by changing sign of 4 directions in $\mathbb{R}^6 \supset S^5$ i.e. 4 chiral real fields Φ_I of gauge theory. These fields will be denoted as A_i, B_j, while the invariant fields are Φ and $\tilde{\Phi}$. The orbifold breaks the gauge group $U(2N)$ to $U(N) \times U(N)$ and the cubic superpotential of $\mathcal{N} = 2$ in new notations is

$$g\text{Tr}\Phi(A_1B_1 + A_2B_2) + g\text{Tr}\tilde{\Phi}(B_1A_1 + B_2A_2) \ . \tag{2.6}$$

If we perturb the theory by a $\mathbb{Z}_2$ odd operator

$$\frac{m}{2}\text{Tr}(\Phi^2 - \tilde{\Phi}^2) \ , \tag{2.7}$$

the $\mathcal{N} = 2$ supersymmetry will be broken down to $\mathcal{N} = 1$, and conformal symmetry will be broken by m^2. The field theory will flow to the IR fixed point. By integrating out massive fields $\Phi, \tilde{\Phi}$ we recover the superpotential (2.3)

$$\frac{g^2}{2m} \mathrm{Tr}\left(A_1 B_1 A_2 B_2 - B_1 A_1 B_2 A_2\right) . \tag{2.8}$$

Therefore we conclude that this theory flows to the "conifold" CFT discussed above. It can be shown that the dual geometry of $T^{1,1}$ emerges from $S^5/\mathbb{Z}_2$ via blowing-up of orbifold singularity of $S^5/\mathbb{Z}_2$. It can be shown that this mechanism is dual to the RG flow in gauge theory [9]. The detailed discussion of blow-up is quite lengthy and we substitute it by another observation confirming $T^{1,1}$ as dual geometry for field theory with superpotential (2.3). We compare the evolution of central charge along RG flow in both gauge theory and string theory on $AdS_5 \times X$. In the conformal case the matrix of fermion R-charges is traceless $\mathrm{Tr}R = 0$ and the central charge c [21, 22]

$$c = \frac{9}{32} \mathrm{Tr} R^3 , \tag{2.9}$$

is given in terms of dual geometry [23]

$$c = \frac{\pi^3 N^2}{4 \mathrm{Vol}(X)} . \tag{2.10}$$

The Einstein manifold X is normalized such that $R_{IJ} = 4 g_{IJ}$ i.e. in our case it is either S^5 of unit radius or $T^{1,1}$ with metric (2.35). The matrix R is diagonal in both cases. In the case of $\mathbb{Z}_2$ orbifold of $\mathcal{N} = 4$ there are 3 pairs of chiral $N \times N$ superfields A_i, B_j, and $(\Phi, \tilde{\Phi})$, each has R-charge $2/3$. Hence the fermion components have charge $-1/3$. The fourth pair of fermions χ_1, χ_2 from vector multiplets have charge 1 and therefore $N^{-2}\mathrm{Tr}R = 6(-1/3) + 2 = 0$. The central charge

$$\frac{c}{N^2} = \frac{9}{32}\left(6(-1/27) + 2\right) = \frac{1}{2} , \tag{2.11}$$

in coincidence with $\mathrm{Vol}(S^3/\mathbb{Z}_2) = \pi^3/2$. In the case of "conifold" filed theory the fields $\Phi, \tilde{\Phi}$ are integrated out and the R-charge of remaining fields A_i, B_j is $1/2$ as follows from the quartic form of superpotential. The R-charge of vector multiplet remaines the same. Then $N^{-2}\mathrm{Tr}R = (4(-1/2)+2) = 0$ and

$$\frac{c}{N^2} = \frac{9}{32}\,(4(-1/8)+2) = \frac{27}{64}\ . \tag{2.12}$$

This confirms the choice of $T^{1,1}$ as of dual geometry since $\mathrm{Vol}(T^{1,1}) = \frac{16\pi^3}{27}$. The latter follows from (2.35) and can be easily verified.

2.1.2 Geometry of singular conifold

This subsection is devoted to the geometry of singular conifold – a cone over $T^{1,1}$. We start with introducing the notation in the next subsection and then proceed with a derivation of Ricci-flat metric on singular conifold.

Geometry of Singular Conifold

The singular conifold $\mathcal{C}_0$ is a complex 3-dimensional subspace in $\mathbb{C}^4$ defined by the equation [24]

$$\sum_{i=1}^{4} z_i^2 = 0\,, \tag{2.13}$$

where $\{z_i; i = 1,2,3,4\}$ are complex coordinates in $\mathbb{C}^4$. The conifold constraint (2.13) may be formulated with help of complex matrix W

$$\mathrm{det}W = 0\,, \quad W \equiv \frac{1}{\sqrt{2}}(z_\alpha\sigma^\alpha + iz_4\mathbf{1}) = \frac{1}{\sqrt{2}}\begin{pmatrix} z_3+iz_4 & z_1-iz_2 \\ z_1+iz_2 & -z_3+iz_4 \end{pmatrix}\,, \tag{2.14}$$

where $\{\sigma_\alpha; \alpha = 1,2,3\}$ are Pauli matrices satisfying $\sigma_\alpha\sigma_\beta = \delta_{\alpha\beta}\mathbf{1} + i\epsilon_{\alpha\beta\gamma}\sigma_\gamma$. The radial coordinate of the conifold, $\hat{r}$, is defined by [24]

$$r^3 \equiv \left(\frac{2}{3}\right)^{3/2}\hat{r}^3 \equiv \mathrm{Tr}(WW^\dagger) = \sum_{i=1}^{4}|z_i|^2\ . \tag{2.15}$$

Equation (2.13) defines complex structure on the conifold. The metric can be specified through Kähler potential $k(z_i,\bar{z}_j)$. To preserve $SO(4)$ symmetry of (2.13) we focus on the potential of the form $k(r^3)$. In this case the metric $g_{\alpha\overline{\beta}} \equiv \partial_\alpha \partial_{\overline{\beta}} k$ is given by

$$
\begin{aligned}
ds^2 &= \partial_\alpha \partial_{\overline{\beta}} k \, du^\alpha \overline{du^\beta} \\
&= k'' \, |\mathrm{Tr}(W^\dagger dW)|^2 + k' \, \mathrm{Tr}(dW dW^\dagger) \ ,
\end{aligned}
\tag{2.16}
$$

where prime stands for derivative with respect to r^3 and $\{u_\alpha\}$ are some complex coordinates on the conifold, say $u_\alpha = z_\alpha$. These variables may not be independent coordinates. If u^α satisfy some constraint $f(u^\alpha) = 0$, one just needs to impose a constraint $du^\alpha \partial_\alpha f = 0$ on differentials du^α in (2.16).

Singular conifold is a Calabi-Yau manifold and admits Ricci-flat metric. Any Kähler potential

$$
k' = \left(\frac{1}{r^3} + \frac{c}{r^9} \right)^{1/3} \ , \tag{2.17}
$$

leads to a Ricci-flat metric through (2.16) [24, 25]. In the special case $c = 0$ the metric (2.16) has the form of conic geometry $ds^2 \simeq dr^2 + r^2 ds^2_{T^{1,1}}$, where the base $T^{1,1}$ is r-independent. Compact space $T^{1,1}$ will be discussed in more detail later in this section. At this point it is convenient to introduce unconstrained real coordinates on conifold. To keep the description explicitly $SU(2) \times SU(2)$ invariant we start with introducing angles θ, ϕ, ψ on $SU(2)$

$$
L(\theta, \phi, \psi) = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \in SU(2) \quad , \tag{2.18}
$$

$$
a = e^{\frac{i}{2}(\psi + \phi)} \cos \frac{\theta}{2} \ , \quad b = e^{-\frac{i}{2}(\psi - \phi)} \sin \frac{\theta}{2} \ , \tag{2.19}
$$

and express matrix W through a pair $L_1(\theta_1, \phi_1, \psi_1), \ L_2(\theta_2, \phi_2, \psi_2)$

$$
(L_1, L_2) \in SU(2) \times SU(2) \ , \tag{2.20}
$$

$$
W = L_1 Z L_2^+ \quad , \tag{2.21}
$$

$$
Z = \begin{pmatrix} 0 & r^{3/2} \\ 0 & 0 \end{pmatrix} . \tag{2.22}
$$

Obviously, equation $\det W = 0$ is invariant under $W \to U_1 W U_2^+$, where $(U_1, U_2) \in SU(2) \times SU(2)$. To complete the construction we need to get rid of one extra angular variable as the total number of real coordinates on conifold is six. Explicit check confirms that W depends on ψ_1 and ψ_2 only through the combination $\psi = \psi_1 + \psi_2 \in [0..4\pi]$. We choose ψ to be a new independent coordinate in addition to θ_i, ϕ_i and r.

The coordinates z_i can be expressed through angular variables θ_i, ϕ_i, ψ and r as follows

$$w_1 = 2^{-1/2}(-z_1 - iz_2) = r^{3/2} e^{\frac{i}{2}(\psi - \phi_1 - \phi_2)} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} , \tag{2.23}$$

$$w_2 = 2^{-1/2}(z_1 - iz_2) = r^{3/2} e^{\frac{i}{2}(\psi + \phi_1 + \phi_2)} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} , \tag{2.24}$$

$$w_3 = 2^{-1/2}(z_3 + iz_4) = r^{3/2} e^{\frac{i}{2}(\psi + \phi_1 - \phi_2)} \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2} , \tag{2.25}$$

$$w_4 = 2^{-1/2}(z_3 - iz_4) = r^{3/2} e^{\frac{i}{2}(\psi - \phi_1 + \phi_2)} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} . \tag{2.26}$$

Here we also introduce another set of coordinates on $\mathbb{C}^4$, w_i, which is a subject to constraint

$$w_1 w_2 - w_3 w_4 = 0 . \tag{2.27}$$

Before we return to the discussion of the metric, let us define a complete (together with dr) set of one-forms ϵ_I^α via

$$i\epsilon_I^\alpha = \mathrm{Tr}\left(L_I^+ dL_I \sigma^\alpha\right) , \quad I = 1,2 , \quad \alpha = 1,2,3 , \tag{2.28}$$

$$\epsilon_I^1 = \sin\psi_I \sin\theta_I d\phi_I + \cos\psi_I d\theta_I ,$$

$$\epsilon_I^2 = \cos\psi_I \sin\theta_I d\phi_I - \sin\psi_I d\theta_I ,$$

$$\epsilon_I^3 = d\psi_I + \cos\theta_I d\phi_I . \tag{2.29}$$

Each one-form ϵ_I^α is obviously invariant under $SU(2) \times SU(2)$. Therefore the form

$$g_5 = \epsilon_1^3 + \epsilon_2^3 = d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2 , \tag{2.30}$$

is also invariant. We will discuss how to construct a general $SU(2)\times SU(2)$ invariant $(0,2)$ tensor later in the section (2.2.2). It is sufficient for now that any combination of $\epsilon_I^1, \epsilon_I^2, g_5$ is invariant by construction.

Now we are ready to return back to the metric (2.16). The Kähler potential [24, 25]

$$k(z_i,\overline{z_i}) = \frac{3}{2}\left(\sum_{i=1}^{4}|z_i|^2\right)^{2/3} = \frac{3}{2}r^2 = \hat{r}^2\,, \tag{2.31}$$

leades to the Ricci-flat conic geometry. Using the explicit form of w_i (2.23-2.26) we find

$$\mathrm{Tr}(dWdW^\dagger) = \sum_i |dw_i|^2 = \frac{9r}{4}dr^2 + \frac{r^3}{4}\left[g_5^2 + (\epsilon_1^1)^2 + (\epsilon_1^2)^2 + (\epsilon_2^1)^2 + (\epsilon_2^2)^2\right]\,, \tag{2.32}$$

and

$$\mathrm{Tr}(W^+dW) = \sum_i \overline{w}_i dw_i = \frac{r^2}{2}\left(3dr + irg_5\right)\,. \tag{2.33}$$

Eventually we find metric on conifold to be

$$ds^2 = d\hat{r}^2 + \hat{r}^2 ds^2_{T^{1,1}}\,, \tag{2.34}$$

with

$$ds^2_{T^{1,1}} = \frac{1}{9}\Big(d\psi + \sum_{i=1}^{2}\cos\theta_i d\phi_i\Big)^2 + \frac{1}{6}\sum_{i=1}^{2}(d\theta_i + \sin^2\theta_i d\phi_i^2)\,. \tag{2.35}$$

The metric above defines Einstein space $T^{1,1}$. The real coordinates have the range $\{r\in[0,\infty], \theta_i\in[0,\pi], \phi_i\in[0,2\pi], \psi\in[0,4\pi]\}$. To find the symmetries of $T^{1,1}$ we fix radius $r^3 = \sum_i |z_i|^2$ and describe points on $T^{1,1}$ through pair $(L_1,L_2)\in SU(2)\times SU(2)$ via $r^{-3/2}L_1 Z L_2^+$. As we already mentioned before, W depends on ψ_i only through $\psi_1+\psi_2$. This means that the map $r^{-3/2}W : SU(2)\times SU(2)\to T^{1,1}$ is degenerate. It maps an orbit of $U(1)$ which shifts ψ_1 and ψ_2 in opposite directions into the same point on $T^{1,1}$: $(L_1, L_2) \sim (L_1 U, L_2 U^{-1})$

$$L_1 Z L_2^+ = (L_1 U) Z (L_2 U^{-1})^+ \ , \tag{2.36}$$

$$U = \begin{pmatrix} e^{i\Psi} & 0 \\ 0 & e^{-i\Psi} \end{pmatrix} \in U(1) \ . \tag{2.37}$$

Therefore $T^{1,1}$ can be defined as

$$T^{1,1} = \frac{SU(2) \times SU(2)}{U(1)} \ . \tag{2.38}$$

We conclude this section by noting that besides being invariant under $SU(2) \times SU(2)$, $T^{1,1}$ has additional symmetry $U(1)_R$, which acts by shifting ψ,

$$(L_1, L_2) \to (L_1 U_R, L_2 U_R) \ . \tag{2.39}$$

The matrix U_R given by (2.37). This follows either from (2.35) or, in the case of more general Kähler potential, from the invariance of (2.13) as well as (2.32,2.33) under (2.39). This symmetry is dual to $U(1)$ R-symmetry in gauge theory and plays an important role in establishing gauge/string duality as we have seen above.

2.2 Cascading gauge theory and deformed conifold

This section is devoted to the confining $SU(N+M) \times SU(N)$ gauge theory and its dual description in terms of IIB SUGRA. Firstly we review the properties of field theory including classical and quantum moduli space in section (2.2.1) and then proceed with a detailed discussion of dual geometry in section (2.2.2).

2.2.1 Cascading gauge theory

We start with $SU(N) \times SU(N)$ "conifold" gauge theory of section (2.1.1) and add M colors to one of the gauge groups $SU(N+M) \times SU(N)$. The field content and superpotential (2.3) remain the same. Extra M colors break conformal invariance and the combination of couplings

$$\frac{8\pi^2}{g_1^2} - \frac{8\pi^2}{g_2^2} = 6M\log(\Lambda/\mu)\left(1 + \mathcal{O}\left((M/N)^2\right)\right) \ , \tag{2.40}$$

runs with the scale, although the combination

$$\frac{8\pi^2}{g_1^2} + \frac{8\pi^2}{g_2^2} \tag{2.41}$$

remains scale invariant [12]. Equation (2.40) suggests that the coupling g_1 of $SU(N+M)$ diverges as the theory flows from UV into IR. At this point the old microscopic description is not valid anymore and one has to switch to a Seiberg-dual description of the theory [12, 11]. The gauge group $SU(N+M)$ has $2N$ flavors and thus becomes $SU(2N-(N+M)) = SU(N-M)$ in the Seiberg-dual description. In addition to the existing superpotential (2.3)

$$W_0 = \frac{\lambda}{2}\epsilon^{ii'}\epsilon^{jj'}\mathrm{Tr}\mathcal{M}_{ij}\mathcal{M}_{i'j'} \ , \tag{2.42}$$

rewritten through the meson matrix $\mathcal{M}_{ij} = A_i B_j$, the dual theory acquires extra term

$$W = W_0 + \mu \mathrm{Tr}\mathcal{M}_{ij}A'_i B'_j \ . \tag{2.43}$$

Here A'_i, B'_j are the bi-fundamental fields in $SU(N-M)\times SU(N)$ theory. The meson field $\mathcal{M}_{ij}$ is massive an can be integrated out leaving superpotential (2.3) with renormalized coupling constant λ'

$$W = \frac{\lambda'}{2}\epsilon^{ii'}\epsilon^{jj'}\mathrm{Tr}(\mathbb{A}_i\mathbb{B}_j\mathbb{A}_{i'}\mathbb{B}_{j'}) \ . \tag{2.44}$$

Therefore the dual theory has $SU(N-M)\times SU(N)$ gauge group and the superpotential (2.3) with the new coupling constant λ'. This is essentially the same gauge theory with the number of colors N shifted by M. The behavior when effective number of colors N runs with the scale is called duality cascade (see [26] for a review).

The last step of the cascade depends on $\tilde{p} = N \bmod M$. In this thesis we focus on $N = kM$, when the last step is described by $SU(2M) \times SU(M) \to SU(M)$. The gauge theory is confining in this case and its gravity dual has small curvature everywhere. Hence SUGRA approximation is consistent. Otherwise, if $\tilde{p} \neq 0$ the confinement doe not occur. If $\tilde{p}$ is large enough to produce back reaction captured by dual geometry i.e. $\tilde{p}/M = \text{const}$ in the planar limit $M \to \infty$ the absence of confinement will be reflected by the IR behavior of the warp factor $H(0) \to \infty$.

Adding fractional D5 branes

The extra M colors introduced above have simple meaning in terms of dual geometry. These are M fractional D5 branes wrapping non-compact 2-cycle of singular conifold [27]. The fractional D5-branes create M units of flux through the 3-cycle and the effective number of colors N is given by an integral over base of the cone

$$\frac{1}{4\pi^2\alpha'} \int_{S^3} F_3 = M \ , \quad \frac{1}{(4\pi^2\alpha')^2} \int_{T^{1,1}} F_5 = N \ . \tag{2.45}$$

Unlike the three-form, $dF_3 = 0$, five-form is not closed $dF_5 = H_3 \wedge F_3$ and its integral over the base of the cone $\int_{T^{1,1}} F_5$ depends on radius r of the conifold. According to the general gauge/gravity duality, radius r is associated with the energy scale of gauge theory μ. Therefore the dependence of effective number of colors N on radius is a gravity dual of cascade behavior in gauge theory. In fact this can be confirmed by comparing the logarithmic running of coupling constants (2.40) obtained from gauge theory with the result of calculation in IIB SUGRA [12].

Non-vanishing flux through S^3 leads to a singular energy density if S^3 shrinks near the tip. Klebanov and Strassler suggested that the S^3 at the tip should be blown-up to a finite size to avoid singularity. They proposed the topology of *deformed conifold* $\mathcal{C}_\varepsilon$

$$\sum_i z_i^2 = -\frac{\varepsilon^2}{2} \ , \tag{2.46}$$

as the candidate for the gravity-dual of $SU(N+M) \times SU(M)$ gauge theory. This choice does not affect UV dynamics as the deformed conifold resembles the singular one for large radius i.e. for large energy scale. But the non-trivial deformation $\varepsilon \neq 0$ prevents S^3 from shrinking at the tip and keeps solution smooth everywhere.

There are a number of ways to justify the geometry (2.46). The most straightforward way is to see how it emerges directly from field-theory analysis. Following Klebanov and Strassler we consider a theory with an extra color $SU((k+1)M + 1) \times SU(kM+1)$. The idea is that the additional degrees of freedom corresponding to extra color will "probe" the geometry of moduli space. The theory at the bottom of the cascade has gauge group $SU(M+1)$ and fields A_i, B_j in $M+1, \overline{M+1}$ representation correspondingly. The gauge-invariant meson matrix $\mathcal{M}_{ij} = A_i B_j$ enters classical superpotential $W_0 = \lambda \mathrm{Det}\mathcal{M}_{ij}$ and leads to the classical moduli space $\mathrm{Det}\mathcal{M} = 0$ i.e. to the singular conifold $\mathcal{C}_0$. In the far IR this theory develops non-perturbative Affleck-Dine-Seiberg superpotential [28] which is responsible for chiral symmetry breaking

$$W = \lambda \mathrm{Det}\mathcal{M} + (M-1)\left[\frac{2\Lambda^{3M+1}}{\mathrm{Det}N}\right]^{\frac{1}{M-1}} . \tag{2.47}$$

The supersymmetric vacua are given by

$$\mathrm{Det}\mathcal{M}_{ij} = \left[\frac{2\Lambda^{3M+1}}{(2\lambda)^{M-1}}\right]^{\frac{1}{M}} . \tag{2.48}$$

Notice that the geometry of (2.48) coincides with (2.46).

The R-symmetry is broken by (2.48) to Z_2 and there are M distinct solutions related to each other by $Z_M \subset U(1)_R$. These M branches of moduli space are characterized by M different values of gluino condensate $\langle\lambda\lambda\rangle^M \sim \Lambda^{3M}$. Eventually we find M copies of deformed conifold

$$\oplus_{r=1}^{M} \mathcal{C}_\varepsilon \tag{2.49}$$

labelled by the phase of gluino condensate $e^{\frac{2\pi r}{M}}$ to be the moduli space of the probe.

In the following we will recover the same result for moduli space without introducing probe branes.

Classical flat directions

We start with classical flat directions of $SU(N+M)\times SU(N)$ theory. At the classical level, we have the F and D-flatness conditions. The latter is

$$\begin{aligned} \sum_i A_i A_i^\dagger - \sum_j B_j^\dagger B_j &= \tfrac{\mathcal{U}}{N}\mathbb{1}_N \ , \\ \sum_i A_i^\dagger A_i - \sum_j B_j B_j^\dagger &= \tfrac{\mathcal{U}}{M+N}\mathbb{1}_{M+N} \ , \end{aligned} \tag{2.50}$$

where $\mathbb{1}_N$ and $\mathbb{1}_{M+N}$ are $N \times N$ and $(M+N)\times(M+N)$ unit matrices. Real constant

$$\mathcal{U} = \mathrm{Tr}\left(\sum_i A_i A_i^\dagger - \sum_j B_j^\dagger B_j\right) \ , \tag{2.51}$$

parameterizes the family of solutions and plays the role of flat parameter. In the quantum theory, $\mathcal{U}$ is an operator (2.129), whose expectation value labels different ground states.

The solutions of these equations for the case of interest $N = kM$ can be divided into two groups – mesonic and baryonic.

Mesonic flat direction

The mesonic flat directions correspond to the non-zero meson matrix $\mathcal{M}_{ij} = A_i B_j$. In the general case it can be diagonalized and the solution has the form

$$A_i = \begin{pmatrix} A_{i1}^1 & & & & \\ & A_{i2}^2 & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & A_{\alpha N}^N \end{pmatrix}$$

$$B_j = \begin{pmatrix} B^1_{j1} & & & & \\ & B^2_{j2} & & & \\ & & . & & \\ & & & . & \\ & & & & B^N_{jN} \end{pmatrix}$$

$$\forall a \quad \sum_i |A^a_{ia}|^2 - \sum_j |B^a_{ja}|^2 = 0 \qquad . \tag{2.52}$$

This solution breaks gauge symmetry to $SU(M) \times U(1)^{N-1}$ and the moduli space is characterized by N sets of coordinates $\mathcal{M}^a_{ij} = A^a_{ia} B^a_{ja}$ with $\mathrm{Det}_{ij}(\mathcal{M}^a) = 0$ up to permutations over the index a. This is a symmetric product of N copies of the (singular) conifold $\mathcal{C}_0$ [14]

$$Sym_N(\mathcal{C}_0) \ , \tag{2.53}$$

which resembles conformal field theory with $M = 0$.

Baryonic flat directions

The baryonic flat direction of confining $SU(N+M) \times SU(N)$ theory with $N = kM$ is given by [14]

$$A_{\alpha=1} \;=\; C \begin{pmatrix} \sqrt{k} & 0 & 0 & . & 0 & 0 \\ 0 & \sqrt{k-1} & 0 & . & 0 & 0 \\ 0 & 0 & \sqrt{k-2} & . & 0 & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & . & 1 & 0 \end{pmatrix} \otimes \mathbb{1}_M \ ,$$

$$A_{\alpha=2} \;=\; C \begin{pmatrix} 0 & 1 & 0 & . & 0 & 0 \\ 0 & 0 & \sqrt{2} & . & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & . & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & \sqrt{k} \end{pmatrix} \otimes \mathbb{1}_M \ ,$$

$$\begin{aligned} B_{\dot\alpha=1} &= 0 \,, \\ B_{\dot\alpha=2} &= 0 \,. \end{aligned} \tag{2.54}$$

The arbitrary complex number C is related to the modulus $\mathcal{U}$ of (2.51) via $\mathcal{U} = k(k+1)M|C|^2$. Non-zero C breaks $U(1)_{baryon}$ which gives the name to the flat direction. There is gauge and $SU(2)\times SU(2)$ invariant baryonic operator $\mathcal{A} \sim (A_1 A_2)^{k(k+1)M/2}$, with appropriate contraction of indexes [29], which has non-zero expectation value along the branch $\langle \mathcal{A} \rangle \sim C^{k(k+1)M}$. It is equivalent to the combination of real parameter $\mathcal{U}$ and the charge under $U(1)_{baryon}$.

There is another classical baryonic branch isomorphic with (2.52) under the $\mathbb{Z}_2$ symmetry $\mathcal{I}$ exchanging $A \longleftrightarrow B$, accompanied by complex conjugation. In this case $\mathcal{U} = -k(k+1)M|C|^2$ and expectation of baryon $\mathcal{B} \sim (B_1 B_2)^{k(k+1)M/2}$ (also called anti-baryon) serves as the module.

Each of these branches has one complex dimension and is parameterized by $C^{k(k+1)M}$. They touch each other at the origin, $C = 0$. On quantum level these branches merge into a single smooth branch as will be discussed below.

Quantum moduli space and gluino condensate

Both mesonic and baryonic branches discussed above preserve $SU(M)$ gauge symmetry on classical level. In case of mesonic branch this $SU(M)$ is a part of $SU(N+M)$, and in the case of baryonic branch $SU(M) \subset SU(M)_1 \times SU(M)_2$ where $SU(M)_1 \subset SU(M)^{k+1} \subset SU((k+1)M)$ and $SU(M)_2 \subset SU(M)^k \subset SU(kM)$. The unbroken $SU(M)$ is confining and this leads to the well-known gluino condensation phenomenon. Namely, the classical moduli space $\mathcal{C}_{cl}$ is multiplied into a sum of isomorphic branches parameterized by the value of gluino condensate $\langle \lambda\lambda \rangle \sim \Lambda^3 e^{\frac{2\pi i r}{M}}$

$$\oplus_{l=1}^{M} \mathcal{C}_{qm} \,. \tag{2.55}$$

The gluino condensate breaks non-anomalous subgroup $\mathbb{Z}_{2M} \subset U(1)_R$ down to $\mathbb{Z}_2$. The discussion above is somewhat schematic because classical moduli space $\mathcal{C}_{cl}$ is different from its quantum analog $\mathcal{C}_{qm}$. In fact the mesonic branch $Sym_N(\mathcal{C}_0)$ is changed by $\oplus_{l=0}^{k} Sym_{N-lM}(\mathcal{C}_{\varepsilon_l})$ on quantum level [14]. We proceed with a detailed discussion of the quantum moduli space $\mathcal{C}_{qm}$ for the bottom of the cascade $k = 1$ below.

Bottom of the cascade $SU(2M) \times SU(M)$

The $SU(2M)\times SU(M)$ theory at the bottom of the cascade has baryons [12, 29, 30]

$$\begin{aligned} \mathcal{A} &= \epsilon_{\alpha_1\alpha_2\ldots\alpha_{2M}}(A_1)_1^{\alpha_1}(A_1)_2^{\alpha_2}\ldots(A_1)_M^{\alpha_M}(A_2)_1^{\alpha_{M+1}}(A_2)_2^{\alpha_{M+2}}\ldots(A_1)_M^{\alpha_{2M}}, \\ \mathcal{B} &= \epsilon^{\alpha_1\alpha_2\ldots\alpha_{2M}}(B_1)^1_{\alpha_1}(B_1)^2_{\alpha_2}\ldots(B_1)^M_{\alpha_M}(B_2)^1_{\alpha_{M+1}}(B_2)^2_{\alpha_{M+2}}\ldots(B_1)^M_{\alpha_{2M}}. \end{aligned} \tag{2.56}$$

and mesons $\mathcal{M}_{ija}^{b} = (A_i)_a^c(B_j)_c^b$. The baryons are singlets under gauge groups and global symmetry $SU(2)\times SU(2)$ while the mesons are charged under $SU(2)\times SU(2)$ and $SU(M)$. It follows from the definitions above that the fields $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ are not independent: on classical level $\mathrm{Det}_{ijab}\mathcal{M} - \mathcal{A}\mathcal{B} = 0$. At the quantum level, this constraint is modified by nonperturbative quantum corrections [19]

$$\mathrm{Det}_{ijab}\mathcal{M} - \mathcal{A}\mathcal{B} = \Lambda_{2M}^{4M}\ , \tag{2.57}$$

which follows from the effective superpotential [19]

$$W_{eff} = W_0 + L\left(\mathrm{Det}_{ijab}\mathcal{M} - \mathcal{A}\mathcal{B} - \Lambda_{2M}^{4M}\right)\ . \tag{2.58}$$

The field L is a Lagrange multiplier and has no kinetic term i.e. it is infinitely massive. The superpotential (2.58) is applicable only at zero energy, not a low energy. It describes moduli space but not the low-energy dynamics. It includes massive fields like L and one massive component of $\mathcal{M}$, $\mathcal{A}$ or $\mathcal{B}$ which are not associated with any massive particles in the spectrum. Instead, they should be interpreted as auxiliary fields in the low energy theory.

The theory with superpotentail (2.58) is IR free and its moduli space can be easily analyzed. There are two branches at the quantum level – mesonic and baryonic, which are related to the classical ones.

The mesonic branch is characterized by $\mathcal{A} = \mathcal{B} = 0$ and $\mathcal{M}$ constrained by $\mathrm{Det}_{ijab}\mathcal{M} = \Lambda_{2M}^{4M}$. Together with D-term constraint this leads to the moduli space $\oplus_r^M Sym_M(\mathcal{C}_\varepsilon)$ with the deformation parameter of conifold $\varepsilon \sim \Lambda_{2M}^3$.

The baryonic branch has $\mathcal{M} = 0$ and $\mathcal{AB} = \Lambda_{2M}^{4M}$. The two classical branches with $\mathcal{AB} = 0$ are combined into a single smooth one complex dimensional branch parameterized by the parameter ζ

$$\mathcal{A} = i\Lambda_{2M}^{2M}\zeta \quad ,$$
$$\mathcal{B} = -i\Lambda_{2M}^{2M}\zeta^{-1} \ . \tag{2.59}$$

The symmetry $\mathcal{I}$ exchanges $A \leftrightarrow B^+$ and inverts ζ:

$$\mathcal{I} : \zeta \to \frac{1}{\zeta^*} \ . \tag{2.60}$$

The low energy theory includes $SU(M)$ gauge sector which enhances the moduli space into M distinct but isomorphic branches via gluino condensation.

We have already stated that the expectation values of the mesons are interpreted as D3-branes in the bulk of the deformed conifold $\mathcal{M} = \mathcal{C}_\varepsilon$.

2.2.2 Geometry of deformed conifold

The deformed conifold $\mathcal{C}_\varepsilon$ is defined similarly to the singular conifold $\mathcal{C}_0$ of (2.13) by imposing an equation in $\mathbb{C}^4$

$$\sum_i z_i^2 = -\frac{\varepsilon^2}{2} \ . \tag{2.61}$$

This constraint can be rewritten with help of complex matrix W similarly to (2.14)

$$\det W = -\frac{\varepsilon^2}{2} \ , \tag{2.62}$$

where

$$W = L_1 Z_d L_2^+ \quad , \tag{2.63}$$

$$Z_d = \frac{\varepsilon}{\sqrt{2}} \begin{pmatrix} 0 & e^{t/2} \\ e^{-t/2} & 0 \end{pmatrix} \ . \tag{2.64}$$

New radial variable $t \in [0..\infty)$ can be matached with r at large t (UV) region

$$r^3 = \varepsilon^2 \cosh(t) \to \varepsilon^2 e^t/2 \ . \tag{2.65}$$

Although Z_d is different from Z of (2.22), analog of equation (2.36) is satisfied $UZ_d(U^{-1})^+ = Z_d$ and W depends on ψ_i only through $\psi = \psi_1 + \psi_2$. Nevertheless the metric is not invariant under $U(1)_R : Z_d \to U_R Z_d U_R^+$ and the group of symmetries reduces to $SU(2) \times SU(2)$. The explicit expressions for $z_i(\theta_i, \phi_i, \psi, r)$ can be obtained from (2.14) and (2.63). These formulae are quite bulky and we will not write them here. Instead we calculate $\mathrm{Tr}(dW^+dW)$ and $\mathrm{Tr}(W^+dW)$ which are the building blocks of Kähler metric (2.16). These expressions are obviously $SU(2) \times SU(2)$ invariant, although it may be tricky to see that once they are written through θ_i, ϕ_i and ψ. One way to prove invariance is to express everything in terms of ϵ_i^α and check that ψ_i enter only through $\psi = \psi_1 + \psi_2$ in the resulting expression. Here we use slightly different approach and following [25] we introduce new set of one-forms[1]

$$\begin{aligned} e_1 = d\theta_1 \ , \quad e_2 = -\sin\theta_1 d\phi_1 \ , \\ \epsilon_\alpha = \epsilon_2^\alpha|_{\psi_2=\psi} \ , \quad \alpha = 1, 2 \ . \end{aligned} \tag{2.66}$$

Next, we would like to show that the combination $e_1\epsilon_1 + e_2\epsilon_2$ is invariant under $SU(2) \times SU(2)$[2]. To make the logic transparent we label the $SU(2)$'s as follows $SU(2)_1 \times SU(2)_2$ and notice that forms e_1, e_2 are invariant under $SU(2)_2$. This is because e_1, e_2 are one-forms on $SU(2)_1$ and thus not affected by $SU(2)_2$. The forms ϵ_1, ϵ_2 are also invariant under $SU(2)_2$ as follows from (2.28). Therefore the whole expression $e_1\epsilon_1 + e_2\epsilon_2$ is invariant under $SU(2)_2$.

To show that $e_1\epsilon_1 + e_2\epsilon_2$ is also invariant under $SU(2)_1$ we introduce yet another set of one-forms

$$\begin{aligned} \hat{e}_1 = d\theta_2 \ , \quad \hat{e}_2 = -\sin\theta_2 d\phi_2 \ , \\ \hat{\epsilon}_\alpha = \epsilon_1^\alpha|_{\psi_1=\psi} \ , \quad \alpha = 1, 2 \ . \end{aligned} \tag{2.67}$$

[1] This is equivalent to fixing the "gauge" $\psi_1 = 0, \psi = \psi_2$.

[2] Alternatively one can check that $\sum_\alpha \epsilon_1^\alpha \epsilon_2^\alpha$ depends on ψ_i only through ψ.

which is different from (2.66) only by interchange of index $1 \leftrightarrow 2$. Therefore $\hat{e}_1\hat{\epsilon}_1 + \hat{e}_2\hat{\epsilon}_2$ is invariant under $SU(2)_1$. A straightforward check shows that

$$e_1\epsilon_1 + e_2\epsilon_2 = \hat{e}_1\hat{\epsilon}_1 + \hat{e}_2\hat{\epsilon}_2 \ , \tag{2.68}$$

which completes the proof. Expression above is invariant under $SU(2)_1$ and under $SU(2)_2$ and therefore it is $SU(2) \times SU(2)$ invariant.

Now we are ready to proceed with Kähler metric (2.16)

$$\mathrm{Tr}(dW^+dW) = \sum |dz_i|^2 = \frac{\varepsilon^2}{4}\cosh(t)\left[g_5^2 + (\epsilon_1^1)^2 + (\epsilon_1^2)^2 + (\epsilon_2^1)^2 + (\epsilon_2^2)^2\right] + \\ +\frac{\varepsilon^2}{2}\left(e_1\epsilon_1 + e_2\epsilon_2\right) \ , \tag{2.69}$$

$$\mathrm{Tr}(W^+dW) = \sum_i \overline{z}_i dz_i = \frac{\varepsilon^2}{2}\sinh(t)(dt + ig_5) \ . \tag{2.70}$$

Notice that (2.69) is not invariant under U_R as was mentioned before.

Again we focus on Kähler potential of the form $k = k(t)$ to preserve explicit $SO(4)$ invariance. Similarly to the singular conifold, the deformed conifold is Calabi-Yau and admits Ricci-flat metric. The corresponding one-dimensional family of Kähler potentials is

$$\frac{dk}{\varepsilon^2 d\cosh(t)} \equiv k(t)' = \frac{1}{\varepsilon^2\cosh(t)}\left(\frac{3}{2}\varepsilon^4\left[\cosh(t)\sinh(t) - t\right] + c\right)^{1/3} \ . \tag{2.71}$$

The Kähler potential (2.17) can be recovered by taking $\epsilon \to 0$ while keeping $\epsilon\cosh(t)$ fixed.

A particular choice of $c = 0$ leads to the metric on deformed conifold, used in Klebanov-Strassler solution

$$ds^2_{\mathcal{M}} = \frac{\varepsilon^{4/3}K(t)}{2}\left[\sinh^2\left(\frac{t}{2}\right)\left(g_1^2 + g_2^2\right) + \cosh^2\left(\frac{t}{2}\right)\left(g_3^2 + g_4^2\right) + \frac{1}{3K(t)^3}\left(dt^2 + g_5^2\right)\right] \ , \tag{2.72}$$

with

$$K(t) \equiv \frac{(\sinh(t)\cosh(t) - t)^{1/3}}{\sinh(t)} \ . \tag{2.73}$$

To make a connection with original paper by Klebanov and Strassler [12] we have used the set of one-forms g_i which makes metric diagonal. They are related to the forms e_i, ϵ_i as follows

$$\begin{aligned} g_1 &\equiv \frac{e_2 - \epsilon_2}{\sqrt{2}} , \quad g_2 \equiv \frac{e_1 - \epsilon_1}{\sqrt{2}} , \\ g_3 &\equiv \frac{e_2 + \epsilon_2}{\sqrt{2}} , \quad g_4 \equiv \frac{e_1 + \epsilon_1}{\sqrt{2}} . \end{aligned} \tag{2.74}$$

Deformed conifold at the tip

The metric (2.72) is not singular and smooth everywhere unlike (2.34) which has conical singularity at the tip $r = 0$. To investigate the behavior of (2.72) at the tip we take $t = 0$ and rewrite deformed conifold constraint $\det W = -\frac{\varepsilon^2}{2}$ as

$$|X|^2 + |Y|^2 = \frac{\varepsilon^2}{2} , \tag{2.75}$$

where X and Y are

$$\begin{aligned} X &= \frac{\varepsilon}{\sqrt{2}} e^{\frac{i}{2}(\phi_1+\phi_2)} \left(e^{i\psi/2} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} - e^{-i\psi/2} \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} \right) , \\ Y &= \frac{\varepsilon}{\sqrt{2}} e^{\frac{i}{2}(\phi_1-\phi_2)} \left(e^{i\psi/2} \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2} - e^{-i\psi/2} \sin\frac{\theta_1}{2} \cos\frac{\theta_2}{2} \right) . \end{aligned} \tag{2.76}$$

Two complex numbers $X, Y(\theta_i, \phi_i, \psi)$ parametrize the 3-sphere through the constraint (2.75). The metric of confiold reduces to the metric of Euclidean S^3 as well. To see that we write the metric on a S^3 of unit radius

$$ds^2_{S^3} = \frac{2}{\varepsilon^2}(|dX|^2 + |dY|^2) = \frac{1}{4}\left(g_5^2 + 2g_3^2 + 2g_4^2\right) , \tag{2.77}$$

and compare it with small t expansion of $ds^2_{\mathcal{M}}$ [31]

$$ds^2_{\mathcal{M}} \simeq \left(\frac{2\varepsilon^4}{3}\right)^{1/3} \frac{1}{4}\left(g_5^2 + 2g_3^2 + 2g_4^2\right) + \frac{1}{8}\left(\frac{2\varepsilon^4}{3}\right)^{1/3} t^2 \left(g_1^2 + g_2^2\right) + \mathcal{O}(t^3) . \tag{2.78}$$

We find that near the tip the deformed conifold degenerates into Euclidean 3-sphere of finite radius $\left(\frac{2\varepsilon^4}{3}\right)^{1/6}$.

Holomorphic $(0,3)$ and Kähler forms on deformed conifold

To make the description of geometry complete we would like to present here the expressions for the closed holomorphic 3-form $\tilde{\Omega}_{KS}$ and Kähler form J_{KS}

$$\tilde{\Omega}_{KS} = \frac{\epsilon^2}{96}(dt + ig_5) \wedge [(e_1 \wedge e_2 + \epsilon_1 \wedge \epsilon_2) + \\ +i \sinh(t)(e_1 \wedge \epsilon_1 + e_2 \wedge \epsilon_2) + \cosh(t)(e_1 \wedge \epsilon_2 + \epsilon_1 \wedge e_2)] \ , \tag{2.79}$$

$$\tilde{J}_{KS} = f_{KS}\,(e_1 \wedge e_2 - \epsilon_1 \wedge \epsilon_2) + df_{kS} \wedge g_5 \ , \\ f_{KS} = \frac{\epsilon^{4/3}}{4}(\cosh(t)\sinh(t) - t)^{1/3} \ . \tag{2.80}$$

Klebanov-Strassler solution

The geometry of Klebanov-Strassler (KS) [12] solution is a warped product of deformed conifold (2.72) and flat Minkowski space

$$ds^2 = H_{KS}(t)^{-1/2} dx_{3,1}^2 + H_{KS}(t)^{1/2} ds_{\mathcal{M}}^2 \ . \tag{2.81}$$

with warp factor $H_{KS}(t)$

$$\begin{aligned} H_{KS}(t) &= (g_s M \alpha')^2 2\varepsilon^{-8/3} I(t) \ , \\ I(t) &\equiv \int_t^\infty dx \frac{x\coth(x) - 1}{\sinh^2(x)} (\sinh(x)\cosh(x) - x)^{1/3} \ . \end{aligned} \tag{2.82}$$

This integral cannot be performed analytically. Therefore we present here some numerical results about $I(t)$ near $t = 0$ and at $t \to \infty$. In the small t region $I(t)$ approaches constant value

$$I(t) = 0.5699 - 2^{-2/3} 3^{-2} t^2 + \mathcal{O}(t^4) \ . \tag{2.83}$$

In the UV region $I(t)$ can be approximated by

$$I(t) = 2^{-8/3} 3(4t - 1)e^{-4t/3} - 2^{7/3} 5^{-3} \left(25t^2 - 85t + 12\right) e^{-10t/3} + \mathcal{O}(e^{-16t/3}) \ . \tag{2.84}$$

The fact that H_{KS} is finite at $t = 0$ indicates that the dual gauge theory is in the confining phase.

KS solution has nontrivial NS-NS 3-form $H = dB_2$,

$$B_2 = \frac{g_s M\alpha'}{2} \frac{t\coth(t) - 1}{\sinh(t)} \left[\sinh^2\left(\frac{t}{2}\right) g^1 \wedge g^2 + \cosh^2\left(\frac{t}{2}\right) g^3 \wedge g^4\right], \tag{2.85}$$

and the R-R fluxes, which can be compactly written as

$$F_3 = \frac{M\alpha'}{2}\left\{g^3 \wedge g^4 \wedge g^5 + d\left[\frac{\sinh(t) - t}{2\sinh(t)}(g^1 \wedge g^3 + g^2 \wedge g^4)\right]\right\}, \tag{2.86}$$

$$\tilde{F}_5 = dC_4 + B_2 \wedge F_3 = (1 + *)\,(B_2 \wedge F_3)\ . \tag{2.87}$$

It is also useful to to write down corresponding R-R potentials:

$$C_2 = \frac{M\alpha'}{2}\Big[\frac{\psi}{2}(g^1 \wedge g^2 + g^3 \wedge g^4) - \frac{1}{2}\cos\theta_1 \cos\theta_2\, d\phi_1 \wedge d\phi_2 - \frac{t}{2\sinh(t)}(g^1 \wedge g^3 + g^2 \wedge g^4)\Big]\ , \tag{2.88}$$

$$C_4 = \frac{1}{g_s H_{KS}(t)}\, dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3\ . \tag{2.89}$$

The R-R 2-form is not well-defined as it does not preserve $\psi \to \psi + 4\pi$ symmetry. This reflects the fact that F_3 contains non-exact piece $g_3 \wedge g_4 \wedge g_5$ responsible for the flux through S^3. Explicit ψ dependence in (2.88) corresponds to the gravity dual mechanism of chiral-symmetry breaking [32].

The complex form $G_3 = H_3 + ig_sF_3$ is imaginary self dual $*_6 G_3 = iG_3$ with respect to the six-dimensional metric (2.72). This implies constant dilaton $\phi = 0$.

The forms (2.85,2.86) are invariant under $SU(2) \times SU(2)$. Although this is not obvious from the expressions above this can be easily established. One way is to represent B_2 and $G = H_3 + ig_sF_3$ through z_i and dz_i [31]. Another approach is to use e_i, ϵ_i basis instead of g_i. This method will be employed in the next section where we discuss $SU(2) \times SU(2)$ invariant ansatz for metric and fluxes.

There is an additional $\mathbb{Z}_2$ symmetry of KS solution, $\mathcal{I}$, which exchanges (θ_1, ϕ_1) with (θ_2, ϕ_2) accompanied by the action of $-I$ of SL(2, $\mathbb{Z}$) which changes sign of H_3 and F_3. This symmetry plays an important role in identifying KS solution with $\mathbb{Z}_2$ invariant point on moduli space of gauge theory in section (3.2.1).

From here on we set the deformation parameter ε to unity for notational simplicity, and also choose $M\alpha' = 2$ and $g_s = 1$, unless they are written explicitly.

Klebanov-Tseytlin limit of the Klebanov-Strassler solution

The Klebanov-Tseytlin (KT) solution [33] is dual to the $SU(N+M) \times SU(N)$ theory with N not necessarily proportional to M. In that sense it is more general than KS. In fact, the KT solution is singular at IR and thus provides a reliable description for gauge theory only in UV. Therefore, the KT can be understood as an intermediate step between the conformal KW solution dual to $SU(N) \times SU(N)$ theory and singularity-free KS with $N \sim M$. KT solution was constructed before KS and it is simpler than KS because it captures physics only in UV region. That is why we present KT as a certain simplifying limit of KS solution unlike traditional approach when KT precedes KS.

The KT solution is a UV limit of KS and hence can be obtained from KS by taking the limit $t \to \infty$. It is convenient to use radial variable r which is (2.65) which is

$$r^3 = \frac{\varepsilon^2}{2} e^t \ . \tag{2.90}$$

Then the metric (2.81) reduces to

$$ds_{10}^2 = H_{KT}^{-1/2} dx_{3,1}^2 + H_{KT}^{1/2} \left(dr^2 + r^2 ds_{T^{1,1},}^2 \right) \ , \tag{2.91}$$

with warp factor $H_{KT}(r)$ [33]

$$H_{KT} = \frac{27\pi(\alpha')^2 \left(2\pi g_s N + 3(g_s M)^2 \log(r/r_0) + 3(g_s M)^2/4\right)}{8\pi r^4} \ . \tag{2.92}$$

Instead of size of deformation ε we have "minimal radius" r_0 where the naked singularity occurs. Warp factor H_{KT} also contains N in addition to M as was discussed above. It is clear that only a combination of N and r_0 is meaningful.

The R-R fluxes and B_2 field are also simplified in KT limit. Since the manifold $\mathcal{M}$ is simply a cone over $T^{1,1} \cong S^2 \times S^3$ the fluxes can be represented through the volume forms of the 2 and 3-cycles

$$w_2 = \frac{1}{2}\left(e_1 \wedge e_2 + \epsilon_1 \wedge \epsilon_2\right) \ , \qquad w_3 = g_5 \wedge w_2 \ , \tag{2.93}$$

$$\int_{S^2} w_2 = 4\pi \ , \qquad \int_{S^3} w_3 = 8\pi^2 \ , \tag{2.94}$$

$$w_2 \wedge w_3 = 54 \mathrm{Vol}(T^{1,1}) \ . \tag{2.95}$$

Namely [33]

$$B_2 = \frac{3\alpha' g_s M}{2}\log(r/r_0) w_2 \ , \quad H_3 = \frac{3\alpha' g_s M}{2r} dr \wedge w_2 \ , \tag{2.96}$$

$$F_3 = \frac{M\alpha'}{2} w_3 \ , \qquad \mathcal{F}_5 = 27\pi(\alpha')^2 N_{eff}(r) \mathrm{Vol}(T^{1,1}) \ , \tag{2.97}$$

$$N_{eff} = N + \frac{3}{2\pi} g_s M^2 \log(r/r_0) \ . \tag{2.98}$$

The effective number of colors N_{eff} runs with energy scale r according to the cascade behavior [20].

As we will see in the next section, the BGMPZ solutions dual to the gauge theory on baryonic branch share the same behavior in the UV region. Therefore the formulae above provide a simple description for the geometry far away from the tip, not only for KS but for the whole BGMPZ family. Many applications of gauge/gravity duality are not sensitive to the IR physics. In this case, KT solution is preferable as it simplifies the calculation. Thus in chapter 4 we calculate superpotential on the D3-brane placed in the throat together with D7, assuming that neither brane is close to the tip. This calculation is quite lengthy and usage of KT geometry rather than KS or BGMPZ is a valuable advantage.

2.3 BGMPZ family of solutions and baryonic branch of the gauge theory

In this section we are going to review the BGMPZ family of solutions [13]. These solutions preserve $\mathcal{N} = 1$ SUSY and global $SU(2) \times SU(2)$ symmetry. They were found with the help of the PT ansatz [25] and $SU(3)$ structure method [34, 35]. We proceed by reviewing the PT ansatz in the next section and later we briefly explain the main idea of the $SU(3)$ structure method. We refer the reader interested in more details to the original papers.

2.3.1 The PT ansatz and method of SU(3) structure

Papadopoulos-Tseytlin ansatz

In section (2.2.2) we discussed how to show that the symmetric tensor on conifold

$$e_1\epsilon_1 + e_2\epsilon_2 \tag{2.99}$$

is $SU(2) \times SU(2)$ invariant. The main idea was to represent (2.99) through a dual basis (2.67)

$$e_1\epsilon_1 + e_2\epsilon_2 = \hat{e}_1\hat{\epsilon}_1 + \hat{e}_2\hat{\epsilon}_2 \ . \tag{2.100}$$

Besides (2.99) we also have invariant combinations

$$e_1^2 + e_2^2 = \hat{\epsilon}_1^2 + \hat{\epsilon}_2^2 \ , \tag{2.101}$$

$$\epsilon_1^2 + \epsilon_2^2 = \hat{e}_1^2 + \hat{e}_2^2 \ , \tag{2.102}$$

as well as dt and g_5. The PT ansatz for ten-dimensional metric

$$ds^2 = e^{2A}dx_{3,1}^2 + d\tilde{s}_{\mathcal{M}}^2 = e^{2A}dx_{3,1}^2 + \sum_{i=1}^{6} G_i^2 \ , \tag{2.103}$$

is a warped product of flat Minkowski space and a conifold $\mathcal{M}$, where the warped metric on conifold $d\tilde{S}_{\mathcal{M}}^2$ is a combination of the invariant pieces above

$$d\tilde{S}_{\mathcal{M}}^2 = \sum_{i=1}^{6} G_i^2 = e^x v^{-1}(dt^2 + g_5^2) +$$
$$+ e^{x-g}\left[(e^{2g} + a^2)(e_1^2 + e_2^2) + (\epsilon_1^2 + \epsilon_2^2) - 2a(e_1\epsilon_1 + e_2\epsilon_2)\right] \ . \tag{2.104}$$

The $\mathbb{Z}_2$ symmetry which exchange (θ_1, ϕ_1) with (θ_2, ϕ_2) is broken unless $e^{2g} + a^2 = 1$.

The choice of vielbeins G_i is not unique. Our choice below is dictated by a requirement that three complex forms $\mathbb{G}_I = (G_{2I-1} + iG_{2I})$

$$\begin{aligned} G_1 &\equiv e^{(x+g)/2}\, e_1 \ , & G_2 &\equiv \frac{\cosh(t)+a}{\sinh(t)}\, e^{(x+g)/2}\, e_2 + \frac{e^g}{\sinh(t)}\, e^{(x-g)/2}\, (\epsilon_2 - a e_2) \ , \\ G_3 &\equiv e^{(x-g)/2}\, (\epsilon_1 - a e_1) \ , & G_4 &\equiv \frac{e^g}{\sinh(t)}\, e^{(x+g)/2}\, e_2 - \frac{\cosh(t)+a}{\sinh(t)}\, e^{(x-g)/2}\, (\epsilon_2 - a e_2) \ , \\ G_5 &\equiv e^{x/2}\, v^{-1/2} dt \ , & G_6 &\equiv e^{x/2}\, v^{-1/2} g_5 \ , \end{aligned} \tag{2.105}$$

are holomorphic, i.e. the eigenvectors of the complex structure. While in the KS case there was a single warp factor $h(t)$, now we find five functions $A(t), x(t), g(t), a(t), v(t)$.

In terms of these one-forms the warped "holomorphic" $(3,0)$ form is

$$\Omega = (G_1 + iG_2) \wedge (G_3 + iG_4) \wedge (G_5 + iG_6) \ , \tag{2.106}$$

and the warped fundamental $(1,1)$ form is

$$J = \frac{i}{2}\Big[(G_1+iG_2)\wedge(G_1-iG_2)+(G_3+iG_4)\wedge(G_3-iG_4)+(G_5+iG_6)\wedge(G_5-iG_6)\Big] \ . \tag{2.107}$$

If manifold $\mathcal{M}$ is a Calabi-Yau with Ricci-flat metric as it is in the KS case, Ω and J are closed holomorphic and Kähler forms multiplied by the warp factors $H^{3/2}$ and H respectively. This result can be generalized to the non Ricci flat metric, provided the background preserves $\mathcal{N} = 1$ SUSY. In the IIB theory the $SU(3)$ structure manifold $\mathcal{M}$ is complex, the pseudo-Kähler form $e^{2A-\phi}J$ is not necessarily closed, but the 3-form is closed $d(e^{3A-\phi}\Omega) = 0$. In the case of IIA theory the $SU(3)$ structure manifold is Kähler i.e. $e^{2A-\phi}J$ is closed, but $e^{3A-\phi}\Omega$ is not (see, for example, [35]).

To write down PT ansatz for R-R and NS-NS forms we need to find antisymmetric analog of (2.99). Here we list such combinations together with their representations via dual basis[3]

$$
\begin{aligned}
e_1 \wedge \epsilon_1 + e_2 \wedge \epsilon_2 &= \hat{\epsilon}_1 \wedge \hat{e}_1 + \hat{\epsilon}_2 \wedge \hat{e}_2 \ , \\
e_1 \wedge \epsilon_2 - e_2 \wedge \epsilon_1 &= -\hat{e}_1 \wedge \hat{\epsilon}_2 + \hat{e}_2 \wedge \hat{\epsilon}_1 \ , \\
e_1 \wedge e_2 = -\hat{\epsilon}_1 \wedge \hat{\epsilon}_2 \ , \ \epsilon_1 &\wedge \epsilon_2 = -\hat{e}_1 \wedge \hat{e}_2 \ .
\end{aligned}
\tag{2.108}
$$

The PT ansatz contains 4 functions $h_1, h_2, \chi, b(t)$ in the flux sector and one constant $P = -\left(\frac{M\alpha'}{4}\right)$, which is $P = -1/2$ in our notations

$$
\begin{aligned}
B_2 &= h_1 \left(\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2\right) + \chi \left(e_1 \wedge e_2 - \epsilon_1 \wedge \epsilon_2\right) + h_2 \left(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1\right) \ , \\
F_3 &= P g_5 \wedge \left[\epsilon_1 \wedge \epsilon_2 + e_1 \wedge e_2 - b \left(\epsilon_1 \wedge e_2 - \epsilon_2 \wedge e_1\right)\right] + P \, dt \wedge \left[b' \left(\epsilon_1 \wedge e_1 + \epsilon_2 \wedge e_2\right)\right] \ , \\
\tilde{F}_5 &= \mathcal{F}_5 + *_{10}\mathcal{F}_5 \ , \qquad \mathcal{F}_5 = 2P(h_1 + b h_2) \, e_1 \wedge e_2 \wedge \epsilon_1 \wedge \epsilon_2 \wedge g_5 \ .
\end{aligned}
\tag{2.109}
$$

The R-R 3-form F_3 has the same non-vanishing flux through S^3 as in KS case (2.86). The exact part of F_3 is parameterized by $b(T)$ which turns out to be the same as in the KS case.

The R-R scalar vanishes $C = 0$, but the dilaton $\phi(t)$ may depend on radial coordinate t, as the background is not imaginary self dual. This completes our discussion of the PT ansatz and we proceed with a brief discussion of the method of $SU(3)$ structure.

Method of $SU(3)$ structure

The method of $SU(3)$ structure is an approach to classify classical supersymmetric solutions of supergravity. To be supersymmetric the background must be invariant under algebra of supersymmetry transformations. In the case of classical bosonic

[3] Again, invariance of (2.108) can be demonstrated by expressing them through ϵ_I^α and checking that ψ_i appears only through the combination $\psi = \psi_1 + \psi_2$.

background the only non-trivial transformations are those of fermion fields [36] (here we assumed R-R scalar is zero)

$$\begin{aligned} \delta\lambda &= \frac{i}{2}\partial_A\phi\Gamma^A\Psi^* - \frac{i}{24}(G_3)_{A_1A_2A_3}\Gamma^{A_1A_2A_3}\Psi = 0 \quad , \\ \delta\psi_A &= D_A\,\Psi \,+\, \frac{i}{1920}\,(F_5)_{A_1\cdots A_5}\,\Gamma^{A_1\cdots A_5}\Gamma_A\Psi \,+ \\ &+ \frac{1}{96}(F_3)_{A_1A_2A_3}\left(\Gamma_A^{\;A_1A_2A_3} \,-\, 9\delta_A^{A_1}\;\Gamma^{A_2A_3}\right)\Psi^* = 0\;. \end{aligned} \tag{2.110}$$

Here the Killing spinor Ψ is a parameter of supersymmetry transformation and Ψ^* denotes its charge conjugate $B\Psi^*$. We do not write charge conjugation matrix B explicitly assuming Majorana representation of gamma-algebra with $B = 1$.

For the background based on warped product of flat Minkowski space and six-dimensional manifold $\mathcal{M}$ it is useful to represent Ψ via four and six-dimensional parts ζ^-,η^-

$$\begin{aligned} &\Psi = a\,\zeta^- \otimes \eta^- + b\,\zeta^+ \otimes \eta^+ \;, \\ &\eta^+ = (\eta^-)^*, \qquad \zeta^+ = (\zeta^-)^*\;. \end{aligned} \tag{2.111}$$

In the IIB case the spinors ζ^-,η^- have definite chirality in four and six dimensions

$$\begin{aligned} &\Gamma_7 = \Gamma_{1..6}\;, \Gamma_\pm = \frac{1}{2}(1 \pm \Gamma_7)\;, \\ &\psi = \zeta^- \otimes \eta^-\;, \qquad \Gamma_7\psi = i\psi\;, \end{aligned} \tag{2.112}$$

such that Ψ is ten-dimensional chiral spinor $\Gamma_{x_0..x_3 1..6}\Psi = -\Psi$. Any chiral spinor η^- in six dimensions is a pure spinor i.e. it is annihilated by half of gamma-algebra

$$(\Gamma_1 - i\Gamma_2)\psi = (\Gamma_3 - i\Gamma_4)\psi = (\Gamma_5 - i\Gamma_6)\psi = 0\;, \tag{2.113}$$

with appropriated choice of $\Gamma_1, .., \Gamma_6$. Therefore there is $SU(3)$ which acts on complexified tangent space leaving η^- invariant. In that sense pure six-dimensional spinor η^- specifies $SU(3)$ structure on manifold $\mathcal{M}$. The idea of the $SU(3)$ structure method is to decompose equations (2.110) into the representations of $SU(3)$.

This is an elegant way of dealing with tensor equations. The same result can be achieved by multiplying (2.110) by all possible combination of gamma matrixes and $(\eta^-)^T$ or $(\eta^-)^+$. The unknown Killing spinor Ψ will disappear and the resulting equations can be rewritten through "holomorphic" (3,0)-form

$$\Omega_{ABC} = \psi^T \Gamma_{ABC} \psi \ , \tag{2.114}$$

and pseudo-Kähler $(1,1)$ form

$$J_{AB} = i\psi^+ \Gamma_{AB} \psi \ . \tag{2.115}$$

If Ω and J are specified through an ansatz like (2.106,2.107) the equations (2.110) provide a set of first order differential equations on the ansatz functions and a, b from (2.111). If the choice of the ansatz for forms and vielbeins was correct the resulting system of coupled differential equations can be solved and hence the classical supergravity solution can be found.

The agenda above was fulfilled for the choice of veilbeins (2.105) and PT-ansatz for the forms (2.109) in [13] by Butti, Grana, Minasian, Petrini, and Zaffaroni. They assumed that a is real when b is pure imaginary and succeeded in solving resulting set of differential equations. This step involves a lot of technicalities and is quite complicated. Therefore we will not discuss it here. In the next subsection we present the result of their calculation and discuss the family of classical solutions they found.

2.3.2 The BGMPZ family of solutions and boundary conditions

The family of solutions found in [13] also known as BGMPZ family of solutions is formulated through a system of coupled first order differential equations for functions

$a(t)$ and $v(t)$

$$
\begin{aligned}
a' &= -\frac{\sqrt{-1-a^2-2\,a\,\cosh t}\,\left(1+a\,\cosh t\right)}{v\,\sinh t} - \frac{a\,\sinh t\,\left(t+a\,\sinh t\right)}{t\,\cosh t-\sinh t},\\
v' &= \frac{-3\,a\,\sinh t}{\sqrt{-1-a^2-2\,a\,\cosh t}} + \\
&+v\,\Big[-a^2\cosh^3 t+2\,a\,t\coth t+a\cosh^2 t\,(2-4\,t\coth t)+\cosh t\,\big(1+2\,a^2\\
&-\left(2+a^2\right)t\coth t\big)+\frac{t}{\sinh t}\Big]\,/\,\Big[\big(1+a^2+2a\cosh t\big)\left(t\cosh t-\sinh t\right)\Big]\ . \qquad (2.116)
\end{aligned}
$$

This equations are highly non-lineal and its analytical solutions is known only in the KS and MN [37] cases. The system above has two-dimensional family of solutions. Nevertheless only one-dimensional subfamily is of interest as other solutions are singular at $t = 0$ [13]. Small t expansion of (2.116) suggests that regular solution has asymptotic $a \to -1$ and $v \to 0$ and can be found near $t = 0$ by Taylor expansion [13]

$$
\begin{aligned}
a &= -1+\left(\frac{1}{2}+\frac{y}{3}\right)t^2+\mathcal{O}(t^4)\ ,\\
v &= t+\left(-\frac{2}{5}+\frac{7}{9}y^2\right)t^3+\mathcal{O}(t^5)\ . \qquad (2.117)
\end{aligned}
$$

The integration constant $y \in (-1..1)$ parameterizes subfamily of regular solutions. The solutions (2.117) share leading asymptotic in UV

$$
\begin{aligned}
a &= -2e^{-1}+U(t-1)e^{-5t/3}+\mathcal{O}(e^{-7t/3})\ ,\\
v &= \frac{3}{2}+\frac{9}{16}U^2(6-4t+t^2)e^{-4t/3}+\mathcal{O}(e^{-2t})\ , \qquad (2.118)
\end{aligned}
$$

where the integration constant $U(y) \in (-\infty..\infty)$ specifies the behavior at $t \to \infty$ and can be determined through y. It is more convenient to use U rather than y to parameterize the family because the behavior in UV region admits simple interpretation via gauge/gravity duality [14].

Some functions are unambiguously determined in terms of a, v and t or even known explicitly for the whole BGMPZ family

$$
\begin{aligned}
b &= -\frac{t}{\sinh(t)}\ ,\\
e^{2g} &= -1-a^2-2a\cosh(t)\ . \qquad (2.119)
\end{aligned}
$$

The next step after $a, v(t)$ are known is to integrate the equation for dilaton

$$\phi' = \frac{(C-b)\,(a\,C-1)^2}{(b\,C-1)\,S}\,e^{-2\,g}\ , \qquad C \equiv -\cosh(t),\ \ S \equiv -\sinh(t)\ . \tag{2.120}$$

This equation is obviously invariant under the shift of dilaton $\phi \to \phi + \mathrm{cosnt}$. For the given solution this in nothing else as rescaling of coupling constant g_s and is perfectly permissible. In the case of family of solutions we need to be more careful. For the solutions to describe different IR vacua of the same gauge theory these solutions should lie in the same UV universality class and share the same coupling constant. Therefore to describe gravity dual of baryonic branch we require the UV asymptotic value of dilaton to be U-independent [14]. It is convenient to choose it to be zero

$$\forall\, U \quad \lim_{t\to\infty} \phi(t) \to 0\ . \tag{2.121}$$

In this case the UV expansion for dilaton is

$$\phi = -\frac{3}{64}U^2(4t-1)e^{-4t/3} + \mathcal{O}(U^4 e^{-8t/3})\ . \tag{2.122}$$

It turn out that the (2.120) can be integrated. This is done in later section (2.3.2).

Once ϕ is determined all other functions can be expressed through a, v, ϕ and t. The additional integration constant η [13] appears in the process of integration. Its meaning can be understand by considering equation for warp factor $A' = A'(a, v, \phi, t, \eta)$, which can be integrated [14]

$$e^{-4A} = \left(e^{-2\phi} - \eta^2\right) e^{-4A_0}\ . \tag{2.123}$$

To decouple gravity and make possible interpretation of supergravity solution as of dual to a gauge theory, the warp factor (2.123) should approach zero in UV and therefore $\eta = e^{-\phi(t=\infty)} = 1$. Further for the solutions to lie in the same UV universality class we reuqire the (sub)leading asymptotic to be universal i.e. U-independent. This is archived through a particular choice of integration constant A_0

$$e^{-4A} = U^{-2}\left(e^{-2\phi} - 1\right) \to \frac{3}{32}(4t-1)e^{-4t/3} + \mathcal{O}(e^{-10t/3}) \ . \tag{2.124}$$

This expression means that the warp factor e^{-4A} is U-independent at infinity and can be substituted by H_{KS} in certain UV calculations.

All other functions x, h_1, h_2, χ can be expressed determined a, v and ϕ through the relations

$$e^{2x} = \frac{(bC-1)^2}{4(aC-1)^2}e^{2g+2\phi}(1-\eta^2 e^{2\phi}) \ , \qquad h_1 = -h_2\, C \ ,$$
$$\chi' = a(b-C)(aC-1)e^{2(\phi-g)} \ , \qquad h_2 = \frac{\eta e^{2\phi}(bC-1)}{2S} \ . \tag{2.125}$$

Here we assume boundary conditions (2.121) and (2.124). The solutions with these specific boundary conditions are dual to $SU((k+1)M)\times SU(kM)$ theory on the baryonic branch of moduli space [13, 14]. Therefore from now on we will denote this family as baryonic branch, although one need to have in mind that this is not an accurate definition. The baryonic branch itself is a part of gauge theory moduli space when the solutions in question is gravity dual description to it.

The KS solution corresponds to

$$a_{KS} = -\frac{1}{\cosh(t)} \ ,$$
$$v_{KS} = \frac{3}{2}\frac{\cosh(t)\sinh(t)-t}{\sinh(t)^2} \ , \tag{2.126}$$

and hence $y = U = 0$.

The $\mathbb{Z}_2$ symmetry $\mathcal{I}$ exchanges θ_1, ϕ_1 and θ_2, ϕ_2 and hence exchanges e^{-g} and $e^g + e^{-g}a^2$ in the metric (2.104). It can be defined through the action on PT ansatz

$$a \to -\frac{a}{1+2a\cosh(t)} \ , \tag{2.127}$$

with ϕ,v and other fields except g stay invariant. Actually it is easy to show that ae^{-g} also stays invariant while $(1+a\cosh(t))e^{-g}$ changes sign. Large t expansion of (2.127) gives that $\mathcal{I}$ changes sign of U

$$\mathcal{I}:\ U \to -U\ , \tag{2.128}$$

i.e. non-zero U leads to $\mathcal{I}$ breaking. This helps to clarify the gauge theory interpretation of U as a dual parameter to the expectation value of the $\mathbb{Z}_2$ breaking operator

$$\mathcal{U} = \mathrm{Tr}\left(\sum_{\alpha} A_\alpha A_\alpha^\dagger - \sum_{\dot\alpha} B_{\dot\alpha}^\dagger B_{\dot\alpha}\right)\ . \tag{2.129}$$

Indeed the $\mathbb{Z}_2$ breaking occurs through a difference in the radii of two S^2 formed by θ_i, ϕ_i. In the UV limit it is

$$e^{x-g}\left(e^{2g}+a^2-1\right) \simeq U t^{3/2} e^{-2t/3} + ... \tag{2.130}$$

This is in agreement with (2.129) having dimension 2 [14, 38]. Consequently we identify

$$\langle \mathcal{U} \rangle_{QFT} \sim U\ . \tag{2.131}$$

Closed holomorphic 3-form and expression for dilaton

The method of $SU(3)$ structure guarantees that the six-dimensional manifold of compactification is a complex manifold [35]. In the case of baryonic branch the solutions share the same complex structure on deformed conifold, inherited from $\mathbb{C}^4$ via (2.61). Actually this complex structure on deformed conifold is unique for fixed value of ε. For a compact Calabi-Yau 3-fold the space of $(2,1)$ cohomologies $\mathcal{H}^{2,1}$ can be identified with a tangent space in the space of complex structures. $\mathcal{H}^{2,1}$ is not empty in our case: the non-vanishing RR flux through non-shrinking S^3 is exactly of $(2,1)$ type [13]. But because of non-compact geometry of conifold the corresponding deformation of complex structure is trivial and equivalent to an infinitesimal change of coordinate system.

The metric (2.72) is Ricci-flat and corresponding closed holomorphic 3-form $\tilde{\Omega}_{KS}$ which satisfies Det$g_{KS} = \frac{i}{8}\tilde{\Omega}_{KS} \wedge \overline{\tilde{\Omega}}_{KS}$ was given in (2.79). For all other BGMPZ solutions there also should be a closed holomorphic form $\tilde{\Omega}$ since these solutions are equivalent as complex manifolds. Thus we conclude that $\tilde{\Omega}$ is equal to $\tilde{\Omega}_{KS}$ multiplied by some constant. Notice that such $\tilde{\Omega}$ does not necessarily satisfy Det$g = \frac{i}{8}\tilde{\Omega} \wedge \overline{\tilde{\Omega}}$, where g_{IJ} is the unwarped metric on the deformed conifold. From another side equation (2.106) defines holomorphic form which satisfies Det$G = \frac{i}{8}\Omega \wedge \overline{\Omega}$, but it is not closed. It is clear that Ω is proportional to $\tilde{\Omega}$ although the non-holomorphic proportionality function can not be fixed by holomorphic properties of geometry. The method of $SU(3)$ structure explicitly predicts this function and guarantees that

$$\tilde{\Omega} = e^{3A-\phi}\Omega \ , \tag{2.132}$$

is closed $d\tilde{\Omega} = 0$ [35]. This equation, together with (2.106) leads to the following expression for $\tilde{\Omega}$ in the BGMPZ case

$$\tilde{\Omega} = e^{3A-\phi+3x/2}v^{-1/2}(dt + ig_5) \wedge \left(-\frac{i}{\sinh(t)}(e_1 \wedge e_2 + \epsilon_1 \wedge \epsilon_2) + \right. \\ \left. +(e_1 \wedge \epsilon_1 + e_2 \wedge \epsilon_2) - i\frac{\cosh(t)}{\sinh(t)}(e_1 \wedge \epsilon_2 + \epsilon_1 \wedge e_2) \right) \ . \tag{2.133}$$

A straightforward check confirms that $\tilde{\Omega}$ is closed if

$$e^{3A-\phi+3x/2}v^{-1/2} \sim \sinh(t) \ . \tag{2.134}$$

The proportionality coefficient is obviously not fixed by $d\tilde{\Omega} = 0$ and is reflected in (2.132) as an ambiguity in definition of ϕ and A. Nevertheless since A is known through a, v and ϕ (2.123) up to an additive constant equation (2.134) can be used to find ϕ [39]

$$e^{4\phi} = -\frac{64v(a\cosh(t)+1)^3\sinh(t)^5}{3U^3(-1-a^2-2a\cosh(t))^{3/2}(t\cosh(t)-\sinh(t))^3}e^{4\phi_{UV}} \ . \tag{2.135}$$

This expression does not depend on η or choice of boundary condition for A and can be checked by differentiating and substituting it into equation for ϕ' (2.120). The boundary condition for dilaton at $t \to \infty$ is specified by ϕ_{UV}.

From now on we put $\phi_{UV} = 0$ according to the discussion in the previous section. This choice immediately implies that $\tilde{\Omega} = \tilde{\Omega}_{KS}$ for all solutions on the baryonic branch i.e. the proportionality coefficient from (2.134) is U-independent. To show that one can take t to infinity and notice that Ω has to have U-independent leading asymptotic to satisfy $\mathrm{Det}G = \Omega \wedge \Omega$. Since the leading asymptotics of A and ϕ are also U-independent $\tilde{\Omega}$ should be U-independent for large t as well. It means $\tilde{\Omega}(U) = \tilde{\Omega}_{KS}$ for large t and hence everywhere.

2.3.3 D-Branes on the conifold and kappa-symmetry

A Dirichlet brane with p spatially extended dimensions is described by the sum of Dirac-Born-Infeld and Chern-Simons action. The former is a generalization of "geometrical" Nambu action when the latter describes the interaction of D-brane with the R-R fields [40, 41, 42]

$$S = S_{DBI} + S_{CS} = -\int_\Sigma \mathrm{d}^{p+1}\sigma e^{-\phi}\sqrt{-\det(g+\mathcal{F})} + \int_\Sigma e^{\mathcal{F}} \wedge C \ . \qquad (2.136)$$

The worldvolume of the brane Σ has induced metric g and the brane tension T_p is set to unity. There is induced gauge filed A_1 on Σ which enters the action through the combination $\mathcal{F} = F_2 + P[B_2]$ with $F_2 = dA_1$. Finally $C = \sum_i C_i$ is the formal sum of the R-R potentials.

If D-brane is supersymmetric the action (2.136) is invariant under κ-symmetry on-shell [43, 44, 45]. Kappa-symemtry provides first-order Bogomolny-type equation

$$\Gamma_\kappa \epsilon = \epsilon \ , \qquad (2.137)$$

which is easier to deal than equations of motion. Here spinor ϵ is a generator of the supersymmetry transformation and Γ_κ is specified below.

Any ϵ satisfying (2.137) guarantees world-volume supersymmetry in the probe aproximation. If further ϵ coincides with the generator of supersymmetry in the bulk Ψ the supersymmetry of the brane is compatible with that one of background.

In type IIB theory the ten-dimensional spinor ϵ is a pair of Majoran-Weyl spinors

$$\epsilon = \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \; , \qquad \epsilon_1 = (\epsilon + \epsilon^*)/2 \; , \qquad \epsilon_2 = (\epsilon - \epsilon^*)/2i \; . \tag{2.138}$$

For the $(9, 1)$ signature the κ-symmetry operator Γ_κ is

$$\Gamma_\kappa = \frac{\sqrt{-\det g}}{\sqrt{-\det(g + \mathcal{F})}} \sum_{n=0}^{\infty} (-1)^n \not{\mathcal{F}}^n \Gamma_{(p+1)} \otimes (\sigma_3)^{n + \frac{p-3}{2}} i\sigma_2 \; , \qquad \Gamma_{(p+1)} \equiv \frac{1}{(p+1)!\sqrt{-\det g}} \, \epsilon^{\mu_1 \ldots \mu_{p+1}} \gamma_{\mu_1 \ldots \mu_{p+1}} \; , \qquad \not{\mathcal{F}}^n \equiv \frac{1}{2^n n!} \gamma_{\nu_1 \ldots \nu_{2n}} \mathcal{F}_{\sigma_1 \sigma_2} \ldots \mathcal{F}_{\sigma_{2n-1} \sigma_{2n}} g^{\nu_1 \sigma_1} \ldots g^{\nu_{2n} \sigma_{2n}} \; , \tag{2.139}$$

where Pauli matrixes σ_α act on the doublet (2.138). The Greeks are the indexes for the worldvolume coordinates and Γ_μ are the "pull-back" of gamma-algebra from ten dimensions. We use $x_0 \ldots x_3$ for the directions in Minkowski space and indexes $1, 2 \ldots 6$ for the veilbeins (2.105) along $\mathcal{M}$.

For the BGMPZ family, including KS solution the Killing spinor Ψ is given by (2.111) with [13, 14]

$$a = \frac{e^{\phi/4}(1 + e^\phi)^{3/8}}{(1 - e^\phi)^{1/8}} \; , \qquad b = i \frac{e^{\phi/4}(1 - e^\phi)^{3/8}}{(1 + e^\phi)^{1/8}} \; , \tag{2.140}$$

(this expression for b is for $U > 0$; b changes sign when U does). The corresponding Majorana-Weyl spinors used in the κ-symmetry equation (2.137) are

$$\epsilon_1 = \frac{1}{2} \left((a + b^*) \zeta^- \otimes \eta^- + (a^* + b) \zeta^+ \otimes \eta^+ \right) \; , \qquad \epsilon_2 = \frac{1}{2i} \left((a - b^*) \zeta^- \otimes \eta^- - (a^* - b) \zeta^+ \otimes \eta^+ \right) \; . \tag{2.141}$$

Chapter 3

Gravity-dual description of low-energy dynamics: probe branes in the throat

3.1 BPS domain wall and D5 brane

3.1.1 Domain wall in gauge theory and supergravity

BPS domain wall in the gauge theory

In this section we examine BPS domain walls separating different vacua in field theory. Namely we consider two isomorphic copies of baryonic branch, different by the value of gluino condensate l (2.55) but identical otherwise. These branches are transformed one into each other by the action of $Z_M \subset Z_{2M} \subset U(1)_R$ [46, 47, 48].

It is easy to see that the tension of this domain wall should be moduli independent. Indeed the tension of the domain wall separating two supersymmetric vacua characterized by the parameters l, l' and the parameters U and U' along the branch. It is given by the difference of superpotentials $T = |W_l(U) - W_{l'}(U')|$. Since the branch is flat, $W_l(U)$ does not depend on U . Therefore tension is independent on both U and U' and depends only on the quantum numbers l, l'.

In the case of baryonic branch $W_l(U) = M\Lambda^3(M,k)e^{\frac{2\pi l}{M}}$ and [14]

$$T \sim M\left|\Lambda(M,k)^3(e^{\frac{2\pi il}{M}} - e^{\frac{2\pi il'}{M}})\right| \ . \tag{3.1}$$

For large M this becomes

$$M\left|\Lambda(M,k)^3(e^{\frac{2\pi il}{M}} - e^{\frac{2\pi il'}{M}})\right| \to 2\pi\left|\Lambda(M,k)^3(l-l')\right| \ . \tag{3.2}$$

Standard large M counting has $\Lambda(M,k)^3 \sim M$ [46], and the tension of the domain wall is of order M. Therefore, in the ‘t Hooft limit, this scales as a D-brane tension [46]. Indeed, in the string theory dual of our gauge theory these domain walls are the D5-branes wrapping the S^3 at the bottom of the deformed conifold $l - l'$ times [12, 37, 49].

Domain wall and dual geometry

We have identified D5-brane wrapping S^3 at the tip of the conifold as the BPS domain wall separating two vacua with different value of gluino condensate l. Later in this section we will show that the tension of this brane is independent on the baryonic branch modulus U, in agreement with the field theory consideration above. Therefore, in order to calculate the tension of the wrapped D5-brane, we will work at the $\mathbb{Z}_2$ symmetric locus on the baryonic branch, $|\mathcal{A}| = |\mathcal{B}|$, described by the KS solution [12]. Recall that the KS metric is

$$ds_{10}^2 = H_{KS}^{-1/2}(t)dx^2 + H_{KS}^{1/2}(t)ds_6^{\mathcal{M}} \ , \tag{3.3}$$

where $ds_6^{\mathcal{M}}$ is the Calabi-Yau metric on the deformed conifold $\mathcal{C}_\varepsilon$ (2.72). At the tip $t = 0$ one finds a 3-sphere of radius $\varepsilon^{2/3}(2/3)^{1/6}$ (2.78). Hence, its volume is $2\pi^2\varepsilon^2\sqrt{2/3}$ and the tension of the domain wall is

$$T = \varepsilon^2\frac{\sqrt{2/3}}{16\pi^3 g_s(\alpha')^3} \ . \tag{3.4}$$

Note that powers of $H_{KS}(0)$ cancel in this calculation, since the D5-brane has three directions within $\mathbb{R}^{3,1}$ and three within the deformed conifold.

To match the string and field theory parameters, we set (3.4) equal to the field theory result,

$$\Lambda(M,k)^3 \sim M\frac{\varepsilon^2}{g_s M(\alpha')^3} \ . \tag{3.5}$$

Since both ε and $g_s M$ are held fixed in the 't Hooft limit, we see that $\Lambda(M,k)^3$ is of order M [46].Thus, the IR scale kept fixed in the large M limit is

$$\tilde{\Lambda}(M,k) = M^{-1/3}\Lambda(M,k) \ , \tag{3.6}$$

and we find

$$\frac{\varepsilon^2}{(\alpha')^3} \sim g_s M \tilde{\Lambda}(M,k)^3 \ . \tag{3.7}$$

3.1.2 Domain wall along the baryonic branch

In this subsection we follow [39] to show that the D5 brane wrapping minimal S^3 at the tip of the conifold is BPS saturated and its tension is constant along the baryonic branch. First, we reformulate the kappa-symmetry equation in the form of calibration condition [50, 51]. Then we demonstrate that the D5 brane saturates the calibration condition and hence it is BPS. The U-independence of the tension will follow from the fact that calibration form is independent on the moduli.

Kappa-Symmetry

We start with a general kappa-symmetry equation (2.137) applied to the case of $D5$ brane stretched along three directions in Minkowski space and wrapping a 3-cycle Σ on the conifold $\mathcal{M}$. We reserve the Greek indices for the directions along Σ while the directions in Minkowski space will be denoted as x_0, x_1, x_2. The pull-back of the NS-NS form B_2 and the induced gauge field $F_2 = dA_1$ are not extended into Minskowski directions, $M_{\mu\nu} = P[B_2]_{\mu\nu} + (F_2)_{\mu\nu}$. Consequently we can use gamma-algebra identity

$$\frac{1}{2!}M_{\mu\nu}g^{\mu\mu'}g^{\nu\nu'}\frac{1}{3!}\gamma_{\mu'\nu'}\epsilon^{\rho\sigma\lambda}\gamma_{\rho\sigma\lambda} = -\frac{1}{2}\epsilon^{\mu\nu\rho}M_{\mu\nu}\gamma_\rho \ , \tag{3.8}$$

to simplify κ-symmetry equation

$$\Gamma_\kappa \Psi = \frac{i\gamma_{x_0x_1x_2}}{\sqrt{\mathrm{Det}(\mathrm{g}+\mathrm{M})}}(\sigma_3 i\sigma_2 \frac{\epsilon^{\mu\nu\rho}}{3}\gamma_{\mu\nu\rho} + i\sigma_2 \frac{\epsilon^{\mu\nu\rho}}{2!}M_{\mu\nu}\gamma_\rho)\Psi = \Psi \ . \tag{3.9}$$

The expression in parenthesis can be split into two linearly independent parts, linear and cubic into gamma-matrixes Γ_A, $A = 0, .., 9$. Note, that this is not the same as splitting (3.9) into terms with and without $M_{\mu\nu}$. Using the identity

$$\Gamma_{[ABC]}\psi = \bar{\Omega}_{ABC}\Gamma_{135}\psi - i\left(J_{AB}\Gamma_C + J_{CA}\Gamma_B + J_{BC}\Gamma_A\right)\psi \ , \tag{3.10}$$

we express κ-symmetry operator in the form

$$\Gamma_\kappa \Psi = \frac{i\gamma_{x_0x_1x_2}}{\sqrt{\mathrm{Det}(\mathrm{g}+\mathrm{M})}}\frac{\epsilon^{\mu\nu\rho}}{2}\begin{pmatrix} 0 & iJ_{\mu\nu}\Gamma_7 + M_{\mu\nu} \\ iJ_{\mu\nu}\Gamma_7 - M_{\mu\nu} & 0 \end{pmatrix}\gamma_\rho\Psi +$$
$$+\frac{i\gamma_{x_0x_1x_2}}{\sqrt{\mathrm{Det}(\mathrm{g}+\mathrm{M})}}\begin{pmatrix} 0 & P_\epsilon[\Omega]\Gamma_+ + P_\epsilon[\bar{\Omega}]\Gamma_- \\ P_\epsilon[\Omega]\Gamma_+ + P_\epsilon[\bar{\Omega}]\Gamma_- & 0 \end{pmatrix}\Gamma_{135}\Psi = \Psi \ .$$

The chiral projectors $\Gamma_{7,\pm}$ are defined in (2.112) and we also introduce concise notation for the contraction

$$P_\epsilon[\Omega] \equiv \frac{\epsilon^{\mu\nu\lambda}}{3!}P[\Omega]_{\mu\nu\lambda} \ . \tag{3.11}$$

Since $\Gamma_A\psi$ are linearly independent over $\mathbb{R}$ and can not be expressed through $\psi, \psi^*, \Gamma_{135}\psi, \Gamma_{135}\psi^*$ and $\Gamma_A\psi^*$ we have

$$\begin{aligned} \frac{a-b^*}{2i}\frac{1}{2}\epsilon^{\mu\nu\rho}\left(-iJ_{\mu\nu} + M_{\mu\nu}\right)\gamma_\rho\psi = 0 \ , \\ \frac{a+b^*}{2}\frac{1}{2}\epsilon^{\mu\nu\rho}\left(+iJ_{\mu\nu} + M_{\mu\nu}\right)\gamma_\rho\psi = 0 \ . \end{aligned} \tag{3.12}$$

Now, since $a \pm ib \neq 0$ we have $P[J] \pm iM = 0$ or

$$P[J]_{\mu\nu} = M_{\mu\nu} = 0 \ . \tag{3.13}$$

This is the first condition of κ-symmetry: the magnetic field should vanish and the 3-cycle Σ is a special Lagrangian submanifold.

Now the κ-symmetry equation simplifies as follows

$$\Gamma_\kappa \Psi = \frac{i\gamma_{x_0x_1x_2}}{\sqrt{\mathrm{Det}g}} \begin{pmatrix} 0 & P_\epsilon[\Omega]\Gamma_+ + P_\epsilon[\bar{\Omega}]\Gamma_- \\ P_\epsilon[\Omega]\Gamma_+ + P_\epsilon[\bar{\Omega}]\Gamma_- & 0 \end{pmatrix} \Gamma_{123}\Psi = \Psi \ . \tag{3.14}$$

Both chiral components of Ψ lead to the same equation

$$\frac{P_\epsilon[\bar{\Omega}]}{\sqrt{\mathrm{Detg}}} i\gamma_{x_0x_1x_2}\Gamma_{135}\psi = i\psi^* \ . \tag{3.15}$$

We have also used here that the coefficients a and b are real and pure imaginary respectively. This equation leads to the following constraints

$$|P_\epsilon[\Omega]| = \sqrt{\mathrm{Detg}} \ , \tag{3.16}$$

$$\gamma_{x_0x_1x_2}\Gamma_{123}\psi = e^{i\phi}\psi^* \ , \tag{3.17}$$

with phase ϕ being related to the argument of $P_\epsilon[\Omega]$. The latter constraint (3.17) should be understood in the following way: it is an equation on ζ^-, a four-dimensional part of the spinor ψ, which specifies SUSY generator in gauge theory. This equation can be solve for any ϕ preserving the half of unbroken SUSY. Since ζ^- is constant so should be ϕ i.e for the D5 to be BPS the argument of pull-backed holomorphic form $P_\epsilon[\Omega]$ should be constant along the 3-cycle Σ.

Now we are ready to summarize the BPS for D5 brane (compare with BPS condition for Euclidean D2 wrapping 3-cycle in IIA theory [52]). The magnetic field and pull-back of pseudo-Kähler form should vanish (special Lagrangian condition 3.13). Induced volume should be equal to the modulus of the pulled-back holomorphic form (3.16). The pull-back of holomorphic form should have constant phase on Σ (3.17). It turns out that these constraints can be formulated in an elegant form of calibration condition which is discussed in the next subsection.

Calibration condition

We would like to formulate calibration condition which would coincide with kappa-symmetry constraints upon saturation.

First, we start with inequality

$$\sqrt{\mathrm{Det}(g+M)} \geq \sqrt{\mathrm{Det}g} \ , \tag{3.18}$$

and notice that it is saturated if and only if $M_{\mu\nu} = 0$. This is because

$$\begin{aligned} \mathrm{Det}(g+M) = \mathrm{Det}g + g_{\mu\nu}M^{\mu}M^{\nu} \ , \\ M^{\mu} = \frac{1}{2}\epsilon^{\mu\nu\rho}M_{\nu\rho} \ , \end{aligned} \tag{3.19}$$

and induced metric $g_{\mu\nu}$ is positively defined.

Second, we would like to prove that

$$\sqrt{\mathrm{Detg}} \geq \left| \frac{\epsilon^{\mu\nu\lambda}}{3!} P[\Omega]_{\mu\nu\lambda} \right| \ , \tag{3.20}$$

and saturation requires $J_{\mu\nu} = 0$. It is convenient to work with complex veilbeins $\mathbb{G}_I = G_{2I-1} + iG_I$, $I = 1, 2, 3$, which diagonalize metric, Ω, and J (2.106,2.107)

$$\Omega = \mathbb{G}_1 \wedge \mathbb{G}_2 \wedge \mathbb{G}_3 \ , \tag{3.21}$$

$$J = \frac{i}{2}\sum_I \mathbb{G}_I \wedge \overline{\mathbb{G}}_I \ . \tag{3.22}$$

We can use the freedom of choosing special coordinate system $\varphi^I(\sigma^\mu)$ on the part of D5 world-volume wrapping Σ such that the induced metric

$$g_{\mu\nu} = \sum_I \partial_{(\mu}\varphi^I \partial_{\nu)}\overline{\varphi}^I \ , \tag{3.23}$$

is diagonal $g_{\mu\nu} = \delta_{\mu\nu}$ in a given point. It is convenient to think about $\partial_\mu \varphi^I$ as complex vector in $\mathbb{C}^3$. Let us introduce three vectors X^I, Y^I, Z^I as follows

$$\partial_1\varphi^I = X^I \ , \quad \partial_2\varphi^I = Y^I \ , \quad \partial_3\varphi^I = Z^I \ . \tag{3.24}$$

In these terms the pullback of Ω is given by the determinant

$$P_\epsilon[\Omega] = \frac{\epsilon^{\mu\nu\rho}}{3!}P[\Omega]_{\mu\nu\rho} = \mathrm{Det}\hat{\Omega} \ , \tag{3.25}$$

$$\hat{\Omega} = \begin{pmatrix} X^1 & Y^1 & Z^1 \\ X^2 & Y^2 & Z^2 \\ X^3 & Y^3 & Z^3 \end{pmatrix} \ , \tag{3.26}$$

and the condition $g_{\mu\nu} = \delta_{\mu\nu}$ is

$$|X|^2 = |Y|^2 = |Z|^2 = 1 \ , \tag{3.27}$$

$$\Re(Y^+X) = \Re(Z^+Y) = \Re(X^+Z) = 0 \ . \tag{3.28}$$

Now we can use $SU(3)$ which acts on index I and leaves metric, J, and Ω invariant and bring X^I to the form $|X^1| = 1,\ X^2 = X^3 = 0$. Then the "unbroken" $SU(2)$ which preserves X^I can be used to cast Y^I in the form $|Y^1|^2 + |Y^2|^2 = 1\ , Y^3 = 0$. This will simplify the form of $\mathrm{Det}\hat{\Omega} = X^1Y^2Z^3$ and

$$|P_\epsilon[\Omega]| = \left|X^1Y^2Z^3\right| \leq 1 = \sqrt{\mathrm{Det}g} \ . \tag{3.29}$$

The inequality (3.20) is proven. The saturation condition $|Y^2| = |Z^3| = 1$ requires $Y^1 = Z^1 = Z^2 = 0$. This condition can be written as

$$\Im(Y^+X) = \Im(Z^+Y) = \Im(X^+Z) = 0 \ . \tag{3.30}$$

Together with (3.28) the equation (3.30) in covariant notations is nothing else but the special Lagrangian condition $P[J]_{\mu\nu} = 0$.

At the last step we need to accommodate the constancy of phase ϕ via a saturation of inequality. This is easy to do by taking integral of Ω over Σ

$$\int_\Sigma |\Omega| \geq \left|\int_\Sigma \Omega\right| \ . \tag{3.31}$$

The same result can be archived by multiplying Ω in (3.31) by any real-valued function. By choosing this function to be $e^{3A-\phi}$ we make right-hand-side independent on Σ as it depends on its cohomology class only (see (2.132)).

Eventually we have that the tension of D5 wrapped over Σ

$$T_{D5} = \int_\Sigma e^{3A-\phi}\mathrm{Det}(g+M) \geq \left|\int_\Sigma \tilde{\Omega}\right| \tag{3.32}$$

is calibrated by the closed holomorphic form (here we also neglected overall coefficient in front of DBI action). Hence it does not depend on embedding cycle Σ, but only on its cohomology class. This is exactly what one would expect for the tension of BPS object. Later we discuss a similar result for SUSY D7 wrapping 4-cycle Σ_4 in $\mathcal{M}$ in section (4.2).

Tension of D5 wrapping S^3 at the tip

The calibration condition (3.32) derived in above suggests that only the smallest 3-cycle potentially gives rise to the BPS embedding. In the case of deformed conifold geometry the smallest nontrivial 3-cycle is the S^3 at the tip. Now we are going to show that the calibration condition is saturated by this choice for all values of U. It will also imply that the tension is constant along the branch as the form $\tilde{\Omega}$ in (3.32) is U-independent.

Since the NS-NS field B_2 (2.85) produces no flux through any 3-cycle $\int_\Sigma H_3 = 0$ one can always choose induced gauge field to vanish $M = P[B] + dA_1$. In the case of S^3 at the tip this is even easier to do since $B_2 = 0$ at $t = 0$ and A_1 vanishes as well.

Now, the tension of D5 is given by $e^{3A-\phi}(t=0)$ multiplied by a geometrical size of S^3. Expanding metric (3.3) near the tip

$$dS^2_{\mathcal{M}} = v^{-1}e^x\big|_{t=0}\left(g_5^2 + (e_1+\epsilon_1)^2 + (e_2+\epsilon_2)^2\right) + \mathcal{O}(t^2) \tag{3.33}$$

we recover S^3 with radius $R = 2\,v^{-1/2}e^{x/2}\big|_{t=0}$ (compare with (2.78)). The corresponding volume $\mathrm{Vol}(S^3) = 2\pi^2 R^3$ and the tension

$$T_{D5} = 8\,v^{-3/2}e^{3A-\phi+3x/2}\big|_{t=0}\,2\pi^2\ . \tag{3.34}$$

To integrate $\tilde{\Omega}$ over S^3 (2.75) we need to fix two of five angular coordinates. A convenient choice $\theta_2 = \phi_2 = 0$ reduces $\tilde{\Omega}$ to

$$\tilde{\Omega} = v^{-1/2}e^{3A-\phi+3/2x}\frac{1}{\sinh(t)}\bigg|_{t=0} d\psi\wedge d\theta_1\wedge\sin\theta_1 d\phi_1\ , \tag{3.35}$$

and the integral

$$\int_{S^3}\tilde{\Omega} = v^{-3/2}e^{3A-\phi+3x/2}\bigg|_{t=0} 4\pi\times 2\times 2\pi\ = T_{D5}\ , \tag{3.36}$$

because $\left.\frac{v}{\sinh(t)}\right|_{t=0} = 1$. Now we substitute expression for ϕ (2.135) and expand the result near $t = 0$

$$T_{D5} \propto 6^{-3/2} 16\pi^2 \ . \tag{3.37}$$

The answer is U-independent as it already follows from (3.36).

3.2 Baryonic condensate and Euclidean D5-brane

In this section we consider Euclidean D5-brane dual to the baryon operator in gauge theory and calculate the baryonic condensates. This section is based on paper [53], written in collaboration with M. Benna and I. R. Klebanov.

3.2.1 Euclidean D5-branes and baryon operators

Baryonic operator and gauge/string duality

In section (2.3.2) we have reviewed the BGMPZ family of solutions and discuss its duality to the baryonic branch of the gauge theory. This conjecture was supported by consideration of the BPS domain wall in section (3.1.2). In this section we would like to elaborate on the duality and establish the relation between the moduli in field theory, ζ, (2.59) and parameter of BGMPZ solutions U from the section (2.3.2). One can use $U(1)_{baryon}$ to set the phase of ζ to zero, and from now on we assume $\zeta = |\zeta|$ is real. Then the $\mathbb{Z}_2$ symmetry $\mathcal{I}$ can be used to identify the "origin" of the branch, the KS solution, on both sides of the duality. As follows from (2.60,2.128) the KS solution corresponds to $\zeta = 1 \Leftrightarrow U = 0$. The semiclassical consideration of operator $\mathcal{U}$ (2.129) suggests the naive relation [14]

$$U \sim \log|\zeta| \sim k(k+1)M(|C|^2 - |C|^{-2}) \sim k(k+1)M(|\zeta|^{\frac{2}{k(k+1)M}} - |\zeta|^{\frac{-2}{k(k+1)M}}) \; . \tag{3.38}$$

This relation is based on classical form of A_i, B_j (2.54) and does not include quantum corrections. To find quantum analog of (3.38) we consider a baryon vertex dual to the baryon operators $\mathcal{A}, \mathcal{B}$ and calculate expectation value ζ on gravity side.

Unlike the di-baryons of conformal $SU(N) \times SU(N)$ [9, 54] the baryons of in the cascading $N \sim kM$ theory are singlets under $SU(2) \times SU(2)$. Therefore, the natural candidate for the string theory dual of baryon operators (2.56) is the D5-brane wrapping the base of the conifold at large radius r [29].

The calculation of expectation value $\langle \mathcal{A} \rangle$ may be done in the following way. First, we calculate two-point correlation function $\langle \mathcal{A}(x_1)\mathcal{A}(x_2) \rangle$ and then factorize correlator by separating the points $|x_1 - x_2| \to \infty$. In the semi-classical approach to the AdS/CFT correspondence the two-point function is given by the Euclidean D5 world-volume stretching between base of the conifold at large r at x_1 and x_2. If x_1 and x_2 are sufficiently separated from each other the D5-brane in the middle tends to the region of small size which is located at small radius r. After separating x_1 and x_2 by infinite distance the factorization occurs $\langle \mathcal{A}(x_1)\mathcal{A}(x_2) \rangle \sim \langle \mathcal{A} \rangle^2$ and we expect the solution to consist of two pieces, each interpolating between the base of the cone at large radius and smoothly wrapping the conifold at the tip. Therefore the expectation value of the baryon operator can be measured by an Euclidean D5-brane with world volume wrapping six-dimensional conifold $\mathcal{M}$ and which is point-like in Minkowski space [55]. This object has a single $T^{1,1}$ boundary at large r, corresponding to the insertion of just one baryon operator. The non-zero expectation value of baryon operator does not break supersymmetry. Hence the D5-brane in question also should be SUSY and satisfy appropriate κ-symmetry condition. This requires non-trivial induced gauge field A_1 on the D5-brane i.e. the D5 will have D3-branes dissolved in it [29].

The geometry of embedding is fixed as the Euclidean D5 completely covers the deformed conifold $\mathcal{M}$. Therefore, the only uncertainty is related to the induced gauge field A_1. Since the baryon vertex has to be $SU(2) \times SU(2)$ invariant, so should be A_1. This leaves us with a very restrictive ansatz

$$A_1 = \zeta(t) g_5 \ , \tag{3.39}$$

with only one unknown function $\zeta(t)$. We will find two solutions $\zeta = \zeta_{\mathcal{A},\mathcal{B}}$ with appropriate behavior at infinity, which correspond to the two baryon operators $\mathcal{A}, \mathcal{B}$. The Chern-Simons term is pure imaginary after turning into Euclidean space, and the corresponding equation of motion should be satisfied independently of the DBI part. The ansatz (3.39) extremises and actually vanishes the Chern-Simons term for any $\zeta(t)$. Therefore CS term is not important for our consideration and we drop it from our analysis from now on. In fact, it can be shown that the CS term describes the coupling of baryon vertex to the Goldstone boson of spontaneously broken $U(1)_{baryon}$. In this way it is responsible for the $U(1)_{baryon}$ charge of the baryon vertex. The anti-baryons $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ are given by inverting orientation of the D5-brane. This changes the sign of CS term and inverses the charge under $U(1)_{baryon}$ [53].

According to the AdS/CFT correspondence the expectation value of an operator $\mathcal{O}$ is given by the coefficient $\Phi_{\mathcal{O}}$ in the expansion of dual field O near the boundary

$$O(r) = \phi_{\mathcal{O}}\, r^{\Delta_{\mathcal{O}}-4} + \Phi_{\mathcal{O}}\, r^{-\Delta_{\mathcal{O}}}\ . \tag{3.40}$$

The source term $\phi_{\mathcal{O}}$ is the coefficient of leading asymptotic which diverges according to the dimension of operator $\Delta_{\mathcal{O}}$. To calculate expectation value $\Phi_{\mathcal{O}}$ one needs to subtract the divergence. In the cascading theory, which is near-AdS in the UV, the same formulae hold modulo powers of $\ln r$ [56, 57]. The baryon vertex is a brane and the corresponding field O at the semi-classical level is given by the classical DBI action $S_{D5}(r)$ of the D5 ending at radius r

$$O_{\mathcal{A}}(r) \sim e^{-S_{D5}[\zeta_{\mathcal{A}},r]}\ . \tag{3.41}$$

We will find action $S_{D5}(r)$ being divergent at large r providing the information about dimension $\Delta_{\mathcal{A}}$. After subtracting the divergent part we will be able to calculate $\langle\mathcal{A}\rangle$ as a function of U.

3.2.2 Bogomolny equation for Euclidean D5-Brane

Now we would like to formulate the κ-symmetry condition for the Euclidean D5-brane. The original κ-symmetry projector (2.139) was derived for the Lorentzian brane in (9,1) signature spacetime. Therefore, it is not immediately clear how to apply it to the Euclidean objects like D5 in question. The naive prescription is to Wick-rotate the κ-symmetry projector by introducing factor $-i$ in (2.139) such that Γ_κ is hermitian and $\Gamma_\kappa^2 = 1$ holds.

The κ-symmetry condition for Euclidean D5-brane is then given by

$$\begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} = \Gamma_\kappa \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} \sim \left[-(\not{\mathcal{F}} + \not{\mathcal{F}}^3) + (1 + \not{\mathcal{F}}^2)\, \sigma_3 \right] \sigma_2\, \Gamma_{123456} \begin{pmatrix} \epsilon_1 \\ \epsilon_2 \end{pmatrix} . \tag{3.42}$$

The analysis of this equation can be simplified by noting that $\Gamma_{1..6}\psi^\pm = \mp i\psi^\pm$ and that the spinors $\psi^\pm$ are in fact eigenvectors of $\not{\mathcal{F}}^n$

$$\not{\mathcal{F}}\psi^\pm = \pm i\psi^\pm \left(\mathcal{F}_{12} + \mathcal{F}_{34} + \mathcal{F}_{56} \right) , \tag{3.43}$$

$$\not{\mathcal{F}}^2\psi^\pm = -\psi^\pm \left(\mathcal{F}_{12}\mathcal{F}_{34} + \mathcal{F}_{14}\mathcal{F}_{23} + \mathcal{F}_{12}\mathcal{F}_{56} + \mathcal{F}_{34}\mathcal{F}_{56} \right) , \tag{3.44}$$

$$\not{\mathcal{F}}^3\psi^\pm = \mp i\psi^\pm \left(\mathcal{F}_{12}\mathcal{F}_{34}\mathcal{F}_{56} + \mathcal{F}_{14}\mathcal{F}_{23}\mathcal{F}_{56} \right) , \tag{3.45}$$

where the indices refer to the basis one-forms (2.105). Using these expressions and the ansatz (3.39) for the gauge field, the two terms in (3.42) can be written in a simple form

$$\begin{aligned} \left[1 + \not{\mathcal{F}}^2 \right] \psi^\pm &= \left[\mathfrak{a} + v e^{-x} \mathfrak{b} \xi' \right] \psi^\pm , \\ \left[\not{\mathcal{F}} + \not{\mathcal{F}}^3 \right] \psi^\pm &= \pm i \left[-\mathfrak{b} + v e^{-x} \mathfrak{a} \xi' \right] \psi^\pm , \end{aligned} \tag{3.46}$$

with

$$\begin{aligned} \mathfrak{a}(\xi, t) &\equiv e^{-2x}[e^{2x} + h_2^2 \sinh^2(t) - (\xi + \chi)^2] , \\ \mathfrak{b}(\xi, t) &\equiv 2e^{-x-g} \sinh(t)[a(\xi + \chi) - h_2(1 + a\cosh(t))] . \end{aligned} \tag{3.47}$$

Using the expression (2.141) for Killing spinor we find

$$\xi' = \frac{e^x \mathfrak{b}}{v \mathfrak{a}} . \tag{3.48}$$

The calculation above can be done in an elegant way without extensive use of gamma-algebra. It is just enough to notice that the equations (3.43-3.45) can be used to rewrite κ-symmetry condition (3.42) in the geometrical terms

$$\frac{1}{2!}J\wedge J\wedge\mathcal{F}-\frac{1}{3!}\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F}=\mathfrak{g}\left(\frac{1}{3!}J\wedge J\wedge J-\frac{1}{2!}J\wedge\mathcal{F}\wedge\mathcal{F}\right)\,,$$
$$\mathcal{F}^{2,0}=0\,. \tag{3.49}$$

The dependence on Killing spinor here is accommodated through the fundamental form J (2.107) and function $\mathfrak{g}$ which can be expressed through a, b.

Now, the gauge field $\mathcal{F}=B_2+dA_1$ from the ansatz (3.39)

$$\begin{aligned}\mathcal{F} &= \frac{ie^{-x}}{2\sinh(t)}\times \qquad (3.50)\\ &\Big[e^{-g}\Big[\tilde{\xi}(\cosh(t)+2a+a^2\cosh(t))+h_2\sinh^2(t)(1-a^2)\Big](G_1+iG_2)\wedge(G_1-iG_2)\\ &+e^{g}\Big[\tilde{\xi}\cosh(t)-h_2\sinh^2(t)\Big](G_3+iG_4)\wedge(G_3-iG_4)\\ &+\xi' v\sinh(t)(G_5+iG_6)\wedge(G_5-iG_6)+\Big[\tilde{\xi}(1+a\cosh(t))-h_2 a\sinh^2(t)\Big]\\ &\big((G_1+iG_2)\wedge(G_3-iG_4)+(G_3+iG_4)\wedge(G_1-iG_2)\big)\Big]\,,\end{aligned}$$

is obviously of $(1,1)$ type and second condition of (3.49) is satisfied. The relations (3.46) admit geometrical formulation

$$\begin{aligned}\frac{1}{3!}J\wedge J\wedge J-\frac{1}{2!}J\wedge\mathcal{F}\wedge\mathcal{F} &= (\mathfrak{a}+ve^{-x}\mathfrak{b}\,\xi')\,\mathrm{vol}_6\,,\\ \frac{1}{2!}J\wedge J\wedge\mathcal{F}-\frac{1}{3!}\mathcal{F}\wedge\mathcal{F}\wedge\mathcal{F} &= (-\mathfrak{b}+ve^{-x}\mathfrak{a}\,\xi')\,\mathrm{vol}_6\,,\\ \frac{1}{3!}J\wedge J\wedge J &\equiv \mathrm{vol}_6\,,\end{aligned} \tag{3.51}$$

and together with (3.49) this immediately leads to the equation for ξ'

$$\xi'=\frac{e^{x}(\mathfrak{g}\mathfrak{a}+\mathfrak{b})}{v(\mathfrak{a}-\mathfrak{g}\mathfrak{b})}\,. \tag{3.52}$$

For the Euclidean D5-brane $\mathfrak{g}=0$ and we return to (3.48).

κ-symmetry and equation of motion

The κ-symmetry equation (3.52) has meaning of Bogomolny equation i.e. it should solve the equation of motion coming from DBI action (we have dropped trivial angle dependence)

$$S_{DBI} = \int_{\mathcal{M}} e^{-\phi} \sqrt{\mathrm{Det}\,(g + \mathcal{F})}\ , \tag{3.53}$$

$$\mathrm{Det}\,(g + \mathcal{F}) = v^{-2} e^{6x} (1 + \xi'^2 v^2 e^{-2x}) (\mathfrak{a}^2 + \mathfrak{b}^2)\ . \tag{3.54}$$

The equation of motion can be simplified by use of (3.52)

$$\begin{aligned} &\frac{\delta}{\delta\xi}\left[e^{-\phi}\sqrt{\det(G+F+B)}\right] = 0 = \\ &\frac{2e^{-\phi}e^{2x}\sqrt{1+\mathfrak{g}^2}}{v(\mathfrak{a}-\mathfrak{g}\mathfrak{b})}\left[-(\xi+\chi)e^{-x}\mathfrak{a} + e^{-g} a \sinh(t)\mathfrak{b}\right] - \frac{d}{dt}\left[\frac{e^{-\phi}e^{2x}(\mathfrak{g}\mathfrak{a}+\mathfrak{b})}{\sqrt{1+\mathfrak{g}^2}}\right] \end{aligned} \tag{3.55}$$

One can use (3.52) once again after differentiating last term in (3.55). Then the equation of motion reduces to some third-order polynomial in ξ which should vanish. Hence each of four coefficients in front of $1, .., \xi^3$ should be zero. This does not happen for $\mathfrak{g} = 0$ and we have to conclude that the naive prescription for the "Euclidean" κ-symmetry does not work. In fact it can be shown that the equation (3.52) with $\mathfrak{g} = 0$ solves the equation of motion for D7-brane with DBI action modified by an extra $e^{-\phi}$ multiplier. It will be interesting to better understand this relation.

Nevertheless there is another candidate for κ-symmetry condition for the Euclidean D5. It is the conventional κ-symmetry condition for Lorentzian D9-brane covering both Minkowski space and conifold $\mathcal{M}$[1]. Extra four dimensions in Minkowski space add $\Gamma_{x_0..x_3}$ to (3.42). This does not affect the form of (3.49) but changes $\mathfrak{g}$. The new $\mathfrak{g}$ is given by

$$\mathfrak{g} = \mathfrak{g}_5 = i\frac{a^2+b^2}{2ab} = \frac{e^{\phi}}{\sqrt{1-e^{2\phi}}}\ . \tag{3.56}$$

The new $\mathfrak{g}$ solves equation of motion (3.55) and confirms that (3.52) with (3.56) is the Bogomolny equation for the Euclidean D5-brane. We proceed with the analysis of equation (3.52) with $\mathfrak{g} = \mathfrak{g}_5$ in the next section.

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[1] Author is grateful to L. Martucci for suggesting this.

3.2.3 Calculation of baryonic condensates

Euclidean D5-Brane along Baryonic Branch

In this subsection we are calculating the dependence of the baryon expectation value using supergravity solutions. All supergravity backgrounds dual to the baryonic branch have the same asymptotic [14] and we will see that all divergent terms (cubic, quadratic and linear in t) in the asymptotic expansion of the action are U-independent. This implies that the scaling dimension of the baryon operator does not depend on U, in agreement with the field theory expectation. However, the finite term in the asymptotic expansion of the brane action does depend on U. This provides a map from the one-parameter family of supergravity solutions labelled by U to the family of field theory vacua with labelled by baryon expectation value $\zeta \sim \langle A \rangle$.

Solving for the Gauge Field and Integrating the Action

We proceed with the expression for ξ' (3.52) and g (3.56)

$$\xi' = \frac{e^x(\mathfrak{g}\mathfrak{a} + \mathfrak{b})}{v(\mathfrak{a} - \mathfrak{g}\mathfrak{b})} \ , \qquad \mathfrak{g} = \frac{e^{\phi}}{\sqrt{1 - e^{2\phi}}} \ . \tag{3.57}$$

This equation admits integrated form

$$\frac{d}{dt}\left[-\frac{1}{3}\xi^3 + \left(\frac{a h_2 \sinh^2(t)}{1 + a\cosh(t)} - \chi \right) \xi^2 + \left(e^{2x} - h_2^2 \sinh^2(t) - \chi^2 + \frac{2a h_2 \sinh^2(t)}{1 + a\cosh(t)}\chi \right) \xi \right]$$
$$= -\frac{h_2 \sinh(t) e^g}{v(1 + a\cosh(t))}[e^{2x} + h_2^2 \sinh^2(t) - \chi^2] + \frac{2e^{2x}\sinh(t)}{v e^g}[a\chi - h_2(1 + a\cosh(t))] \ . \tag{3.58}$$

For notational convenience we define

$$\begin{aligned}
\tilde{\xi} &\equiv \xi + \chi \ , && (3.59)\\
\mathfrak{A}(t) &\equiv \frac{a h_2 \sinh^2(t)}{1 + a\cosh(t)} \ , && (3.60)\\
\mathfrak{B}(t) &\equiv e^{2x} - h_2^2 \sinh^2(t) \ , && (3.61)\\
\rho(t) &\equiv \int_0^t \Bigg[\frac{h_2 \sinh(t) e^g}{v(1 + a\cosh(t))} [e^{2x} + h_2^2 \sinh^2(t)] \\
&\quad + \frac{2e^{2x} h_2 \sinh(t)(1 + a\cosh(t))}{v e^g} - [e^{2x} - h_2^2 \sinh^2(t)]\chi' \Bigg] dt \ , && (3.62)
\end{aligned}$$

which allows us to write (3.58) more compactly

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big[-\frac{1}{3}\tilde{\xi}^3 + \mathfrak{A}(t)\tilde{\xi}^2 + \mathfrak{B}(t)\tilde{\xi} + \rho(t) \Big] = 0 \ . \tag{3.63}$$

Thus the solutions for the shifted field $\tilde{\xi}$ are given by the roots of the third order polynomial

$$-\frac{1}{3}\tilde{\xi}^3 + \mathfrak{A}(t)\tilde{\xi}^2 + \mathfrak{B}(t)\tilde{\xi} + \rho(t) = C \ , \tag{3.64}$$

where C is the integration constant.[2] To fix it, we consider the small t expansion, which is valid for any U

$$\begin{aligned}
\mathfrak{A} &\sim t + \mathcal{O}(t^3) \ , && (3.65)\\
\mathfrak{B} &\sim t^2 + \mathcal{O}(t^4) \ , && (3.66)\\
\rho &\sim t^3 + \mathcal{O}(t^4) \ . && (3.67)
\end{aligned}$$

Note that at $t = 0$ all coefficients in (3.64) vanish, except the first one; therefore, the integration constant C has to be zero for this cubic to admit more than one real solution. Then we find that $\tilde{\xi} = 0$ at $t = 0$ for any solution on the baryonic branch.

Let us examine the cubic equation (3.64) more closely in the KS limit $U \to 0$. We see that $a \to -\frac{1}{\cosh(t)}$ and therefore $(1 + a\cosh(t))$ vanishes. For small U [30, 14, 13]

$$(1 + a\cosh(t)) = 2^{-5/3} U Z(t) + \mathcal{O}(U^2) \ , \tag{3.68}$$

$$Z(t) \equiv \frac{(t - \tanh(t))}{(\sinh(t)\cosh(t) - 1)^{1/3}} \ . \tag{3.69}$$

[2]This equation is quite general; it does not assume $\eta = 1$ that characterize the baryonic branch as discussed in section (2.3.2).

In this case $\mathfrak{A}$ and the first term in ρ diverge as U^{-1}. All other terms can be dropped and we have instead of (3.63)

$$\tilde{\xi}^2 \frac{a h_2 \sinh^2(t)}{Z(t)} + \int_0^t dt \, \frac{h_2 \sinh(t) e^g}{v Z(t)} [e^{2x} + h_2^2 \sinh(t)^2] = 0 \ , \tag{3.70}$$

or infinite ξ. After substituting the KS values for a, v, h_2, x we find

$$\xi^2 = (\sinh(t)\cosh(t) - t)^{-1/3} J(t) \ , \tag{3.71}$$

where

$$J(t) = \int_0^t \left(\frac{\sinh^2(x)\, h(x)}{24} + \frac{\sinh^2(x)(x \coth(x) - 1)^2}{6\,(\sinh(x)\cosh(x) - x)^{2/3}} \right) dx \ . \tag{3.72}$$

While it would be desirable to obtain a closed form expression for the integral $\rho(t)$ in order to evaluate ξ explicitly, this appears to be impossible, since even in the KS case we cannot perform the corresponding integral $J(t)$.

Evaluating the DBI Lagrangian on-shell using (3.57) we find

$$e^{-\phi}\sqrt{\det(G+\mathcal{F})} = \frac{e^{-\phi} e^{3x} \sqrt{1+\mathfrak{g}^2}\,(\mathfrak{a}^2+\mathfrak{b}^2)}{v|\mathfrak{a}-\mathfrak{g}\mathfrak{b}|} \ , \tag{3.73}$$

where we have taken the absolute value since the sign of $\mathfrak{a} - \mathfrak{g}\mathfrak{b}$ will turn out to depend on which root of equation (3.64) we pick.

For the baryonic branch backgrounds we can show that the action is a total derivative. First note that the DBI Lagrangian (3.73) can be rewritten in the form

$$\begin{aligned} e^{-\phi}\sqrt{\det(G+\mathcal{F})} &= \frac{e^{-\phi}e^{3x}}{v\sqrt{1+\mathfrak{g}^2}} \frac{(\mathfrak{g}\mathfrak{a}+\mathfrak{b})^2 + (\mathfrak{a}-\mathfrak{g}\mathfrak{b})^2}{|\mathfrak{a}-\mathfrak{g}\mathfrak{b}|} \\ &= \left| \frac{e^{4x}(1+a\cosh(t))}{v h_2 \sinh(t) e^g} [v e^{-x} \xi' (\mathfrak{g}\mathfrak{a}+\mathfrak{b}) + (\mathfrak{a}-\mathfrak{g}\mathfrak{b})] \right| \ , \end{aligned} \tag{3.74}$$

where the right hand side is now cubic in ξ (and its derivative) much like the differential equation (3.57). In fact, substituting for $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{g} = \mathfrak{g}_5$ this equation can be integrated in the same manner, which results in the action

$$S = \left| -\frac{1}{3}\tilde{\xi}^3 + \mathfrak{C}(t)\tilde{\xi}^2 + \mathfrak{D}(t)\tilde{\xi} + \sigma(t) \right| \ , \tag{3.75}$$

with $\mathfrak{C}, \mathfrak{D}, \sigma$ defined as

$$\begin{aligned}
\mathfrak{C} &= -\frac{e^{2x} a\,(1 + a\cosh(t))}{h_2 e^{2g}} \ , && (3.76)\\
\mathfrak{D} &= [e^{2x} + h_2^2 \sinh^2(t) + 2e^{2x}(1 + a\cosh(t))^2 e^{-2g}] \ , && (3.77)\\
\sigma &= -\int_0^t \left[\frac{e^{2x}(1 + a\cosh(t))}{v h_2 \sinh(t) e^g} [e^{2x} - h_2^2 \sinh^2(t)] + \right. && (3.78)\\
& \qquad \left. [e^{2x} + h_2^2 \sinh^2(t) + 2e^{2x}(1 + a\cosh(t))^2 e^{-2g}]\chi' \right] dt \ . && (3.79)
\end{aligned}$$

Again the ξ-independent term is an integral, that we denoted by $\sigma(t)$. Thus we have a fairly explicit expression for the action involving two integrals: $\rho(t)$, which appears in the equation for $\tilde{\xi}$, and $\sigma(t)$.

Although the leading UV asymptotic of $\tilde{\xi}$ and $\mathfrak{C}, \mathfrak{D}, \sigma$ may depend on U the t-dependence is universal. Therefore the rate of UV divergence of action (3.75) is the same for any solution along the branch. The two solutions with asymptotic of (3.71) correspond to the baryons $\mathcal{A}$ and $\mathcal{B}$. Although the action diverges, the divergence $\log^3(r) \sim t^3$ is logarithmical and can be interpreted in terms of holographic renormalization group [53]. We will return to this point later in the next subsection.

The third solution of (3.63), which is divergent in the KS case, produces a badly divergent action and is therefore unacceptable. Restoring the $-\tilde{\xi}^3/3$ term in (3.70) we see that in the GHK region $U \to 0$ the third solution is simply

$$\xi = -\frac{2^{2/3} 3}{U} \left(\cosh(t)\sinh(t) - t\right)^{1/3} + \mathcal{O}(U) \ . \qquad (3.80)$$

The value of the Lagrangian in this case is

$$\sqrt{\det(G + \mathcal{F})} = \frac{36}{U^3} \sinh^2(t) + \mathcal{O}(U^{-2}) \ . \qquad (3.81)$$

This expression can be used to extract the leading UV asymptotics of the Lagrangian for any U as the UV behavior is universal for all U:

$$\sqrt{\det(G + \mathcal{F})} \to \frac{9}{U^3} e^{2t} \ . \qquad (3.82)$$

Since the action for the third solution diverges exponentially at large t it does not seem possible to interpret this solution as the dual of an operator in the same sense as we do for the other two solutions.

Baryonic condensates

We shall now study the D5-brane action (3.75) in more detail. First we develop an asymptotic expansion of the action (3.75) as a function of the cut-off. This expansion is useful because the divergent terms give the scaling dimension of the baryon operator, while the finite term encodes its expectation value.[3] Then we present a perturbative treatment of small U region followed by a numerical analysis of the whole baryonic branch. The main result of this section will be an expression for the expectation value ζ as a function of U.

To calculate the baryonic condensates we need asymptotic behavior of $\mathfrak{A}, \mathfrak{B}, \rho$ and $\mathfrak{C}, \mathfrak{D}$ for large t. Notice that since for any U the solution approaches the KS solution at large t, the terms divergent at $U = 0$ are UV divergent as well:

$$\mathfrak{A} \to \frac{e^{2t/3}}{U} + \mathcal{O}(e^{-2t/3}) \;, \tag{3.83}$$

$$\mathfrak{B} \to \mathcal{O}(t^2) \;, \tag{3.84}$$

$$\rho \to -\frac{e^{2t/3}}{U}\left(\frac{1}{4}t^2 - \frac{7}{8}t + \frac{47}{32}\right) + \mathcal{O}(1) \;, \tag{3.85}$$

$$\mathfrak{C} \to \mathcal{O}(e^{-2t/3}) \;, \tag{3.86}$$

$$\mathfrak{D} \to \left(\frac{1}{4}t^2 - \frac{t}{8} + \frac{5}{32}\right) + \mathcal{O}(e^{-4t/3}) \;. \tag{3.87}$$

From the expansion for $\mathfrak{A}, \mathfrak{B}, \rho$ we find that at large t the gauge field $\tilde{\xi}$ grows linearly with t and approaches the KS value with exponential precision

$$\tilde{\xi}(t, U) \to \pm\left(\frac{1}{4}t^2 - \frac{7}{8}t + \frac{47}{32}\right)^{1/2} + \mathcal{O}(e^{-2t/3}) \;. \tag{3.88}$$

It is crucial that the dependence on U in (3.88) is exponentially suppressed.

---

[3]A systematic procedure for isolating the finite terms is holographic renormalization [58, 59]. Here we employ a naive approach and leaving a rigorous justification for the future work.

Since $\mathfrak{C}$ is exponentially small and the leading term in $\mathfrak{D}$ is U-independent we can explicitly express the action (3.75) in terms of σ:

$$S_{\pm}(U,t) = \Delta(t) \pm \sigma(U,t) + \mathcal{O}(e^{-2t/3}) \ , \tag{3.89}$$

where Δ is given by

$$\Delta(t) \ = \ \frac{1}{6}(t^2+t-2)\left(\frac{1}{4}t^2 - \frac{7}{8}t + \frac{47}{32}\right)^{1/2} \ , \tag{3.90}$$

and encodes the UV divergent part of the action

$$\left| -\frac{1}{3}\tilde{\xi}^3 + \mathfrak{D}(t)\tilde{\xi} \right| \ = \ \Delta(t) + \mathcal{O}(e^{-2t/3}) \ . \tag{3.91}$$

The power divergence of action (3.90) has clear interpretation in terms of holographic renormalization group. The dimensions of operators $\Delta_{\mathcal{A},\mathcal{B}}(r)$ are related to the divergent action $S(t)$ in the UV

$$\Delta_{\mathcal{A},\mathcal{B}}(r) = \frac{dS_{\pm}}{d\log r} \ , \tag{3.92}$$

with r related to t in (2.65). After restoring α' and $g_s M$, and taking into account the prefactors, the action $S_{\pm}$ can be rewritten as

$$S_{\pm} = \frac{9g_s^2M^3}{16\pi^2}\log^3(r) + \mathcal{O}(\log^2(r)) \ . \tag{3.93}$$

After differentiation in (3.92) and matching radius r to the k-th step of Seiberg duality $r(k) = r_0 \exp\left(\frac{2\pi k}{3g_s M}\right)$ we recover the answer

$$\Delta_{\mathcal{A},\mathcal{B}} = \frac{3}{4}Mk(k+1) \ , \tag{3.94}$$

which also follows from a naive field-theory analysis [53]. This confirms that our construction of baryon vertex indeed describes the baryon operators.

Now we proceed with (3.89) and argue that the two signs stand for the two baryons $\mathcal{A}$ and $\mathcal{B}$. Actually we will show that the baryonic branch constraint, $\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle$ = const, follows from this interpretation. As was mentioned in section

(2.3.2), the $\mathcal{I}$-symmetry which exchanges the $\mathcal{A}$ and $\mathcal{B}$ baryons is equivalent to the changing of sign $U \to -U$. Our explicit expression (3.89) confirms that an exponential precision

$$\begin{aligned} S_+(U,t) &= S_-(-U,t) \ , \qquad (3.95)\\ S_-(U,t) &= S_+(-U,t) \ , \qquad (3.96) \end{aligned}$$

since $\sigma(U,t)$ is antisymmetric in U according to the arguments presented around (2.127). In order to find the expectation value of the baryons we evaluate the action (3.75) on these solutions and remove the divergence by subtracting the KS value. The expectation values hence are given by $\exp[-\lim_{t\to\infty} S_f(\xi_{1,2})]$, where S_f denotes the finite part of the action. It is simplest to work with the product (normalized to the KS value) and ratio of the expectation values. The former is given by

$$\frac{\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle}{\langle\mathcal{A}\rangle_{KS}\langle\mathcal{B}\rangle_{KS}} = \lim_{t\to\infty} \exp\left[S_+(U,t) + S_-(U,t) - 2S(0,t)\right] \ , \qquad (3.97)$$

where we have used the fact that the two solutions coincide in the KS case because $\sigma = 0$. It follows from (3.97) that

$$\langle\mathcal{A}\rangle\langle\mathcal{B}\rangle = \langle\mathcal{A}\rangle_{KS}\langle\mathcal{B}\rangle_{KS} \ , \qquad (3.98)$$

which corresponds to the constraint $\mathcal{AB} = -\Lambda_{2M}^{4M}$ in the gauge theory. The ratio of the baryon condensates is given by

$$\frac{\langle\mathcal{A}\rangle}{\langle\mathcal{B}\rangle} = \lim_{t\to\infty} \exp\left[S_+(U,t) - S_-(U,t)\right] = \lim_{t\to\infty} e^{2\sigma} \ , \qquad (3.99)$$

or

$$\log\langle\mathcal{A}\rangle \simeq \lim_{t\to\infty} \sigma(t) \ . \qquad (3.100)$$

Unfortunately σ can not be calculated analytically. However, this integral can be evaluated numerically. In the small U region of GHK [30] the answer is linear in U

$$\lim_{t\to\infty} \sigma(t) \simeq 3.3773U + \mathcal{O}(U^3) \ , \qquad (3.101)$$

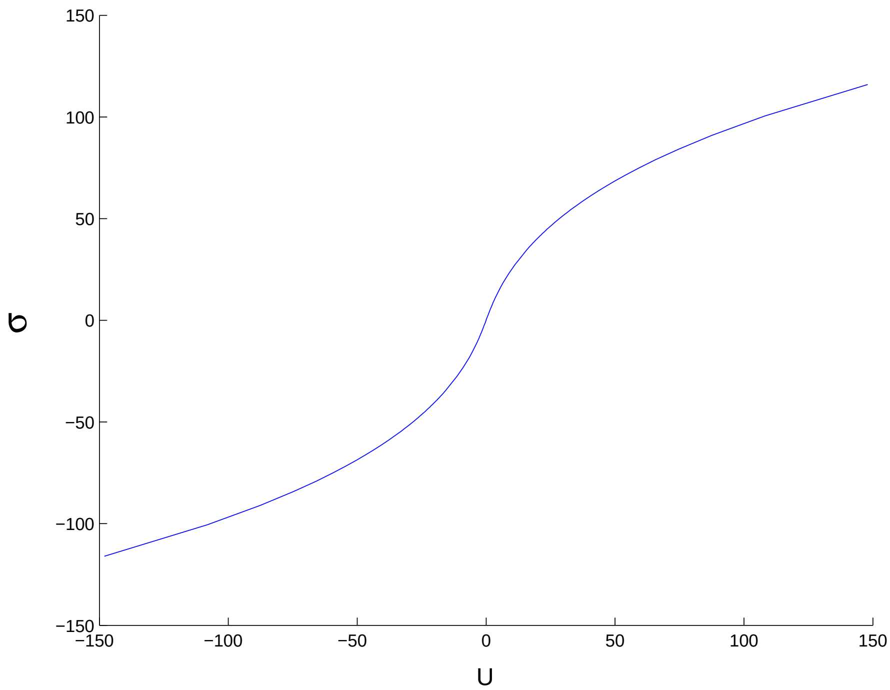

Figure 3.1: Plot of numerical results for the $\mathcal{O}(t^0)$ term in the asymptotic expansion of the action versus U. The slope at $U = 0$ matches the value calculated from (3.101). The baryon expectation value $\langle \mathcal{A} \rangle \sim \langle \mathcal{B} \rangle^{-1}$ in units of Λ_{2M}^{2M} is given by the exponential of this function.

The numerical result for the rest of the baryonic branch is more complicated . We present it in the form of the plot shown in Figure 1. Since $\langle \mathcal{A} \rangle \sim \zeta$ this plot provides a map from the SUGRA modulus U to the field theory modulus ζ.

Chapter 4

Nonperturbative superpotential in the D3-D7 system

This chapter is devoted to the calculation of nonperturbative superpotential on a D3 brane due to gluino condensate on a stack of D7-branes. The main results, presented in this chapter were initially obtained in the work [68], written in collaboration with D. Baumann, I. R. Klebanov, J. Maldacena, L. McAllister, and A. Murugan.

4.1 Warped throats and moduli stabilization

Warped throat compactifications in String Theory

The warped throat compactifications provide an appealing mechanism to introduce the techniques of gauge/gravity duality into the scenario of string compactifications. The idea is to consider a compact Calabi-Yau manifold with some conic singularity and internal fluxes. Then, in the vicinity of the singularity the background will not be far from the infinite throat solutions discussed in Chapter 2 and 3. At high energies, however, the gauge theory on the stack of branes will feel the bulk of the Calabi-Yau. This corresponds to the coupling of the low-energy effective field theory to four-dimensional gravity modes. Such a warped throat scenario provides a convenient mechanism of splitting the field theory and gravity scales and resolving the hierarchy problem.

There is evidence that singularities like (1.3) generally appear when the moduli of the Calabi-Yau are varied [16]. Therefore, the field theory on the D3-brane placed at the singularity can be a natural prediction of string theory.

Another strong advantage of warped throat compactifications is that the dynamics of string theory on the throat is controllable. Unlike the metric in the bulk of the Calabi-Yau the metric in the throat is known explicitly. Moreover, for sufficiently large flux the curvature is small everywhere on the throat, thus providing reliable supergravity approximation.

Nonperturbative volume stabilization

The key issue for the compactification scenarios is to assure that all massless moduli are fixed dynamically . For that reason one needs the non-trivial fluxes to generate a moduli-fixing potential.

All moduli may be divided into three major groups: the Kähler moduli ρ, responsible for the "sizes" of the Calabi-Yau; the complex moduli χ, responsible for the complex structure, and the dilaton-axion modulus.

The Gukov-Vafa-Witten flux induced superpotential [60]

$$W_{\text{flux}}(\chi_\star) = \int G_3 \wedge \tilde{\Omega} \equiv W_0 \;, \tag{4.1}$$

stabilizes the complex structure and the dilaton-axion. Nevertheless the Kähler moduli are not fixed in this way. This problem can be solved by embedding D3-D7-branes into an orientifold of the IIB theory. The gauge theory on D7 develops nonperturbative superpotential through gluino condensation. It is dependent on Kähler moduli and hence can lead to their stabilization [61].

For simplicity let us consider a model with one Kähler modulus ρ. Then the nonperturbative superpotential is expected to be of the form [61]

$$W_{\text{np}}(\rho) = A(\chi) e^{-a\rho} \,. \tag{4.2}$$

The pre-exponent factor $A(\chi)$ is a holomorphic function of the complex structure moduli $\chi \equiv \{\chi_1, \ldots, \chi_{h^{2,1}}\}$. Later we will see that it also depends on the details of the D3-D7 system such as location of the D3-brane. The factor A arises from the one-loop correction to the nonperturbative superpotential. It is a threshold correction to the gauge coupling on the D7-branes. The a is such that $a\rho$ is volume of the four-cycle wrapped by D7-brane.

KKLT scenario

Now we are going to discuss a simple scenario of Kähler moduli stabilization [61] in some detail. To simplify our considerations we either assume that the D3 brane is fixed or consider the system without mobile D3 branes. In this case the full superpotential W is the sum of the constant flux term W_0 at fixed complex structure $\chi_\star$ and the nonperturbative term $W_{\rm np}$

$$W = W_0 + Ae^{-a\rho} \, . \tag{4.3}$$

The Kähler modulus ρ is fixed dynamically through the minimization of F-term potential V_F

$$V_F = e^{\kappa_4^2 \mathcal{K}} \left[\mathcal{K}^{\rho\bar{\rho}} D_\rho W \overline{D_\rho W} - 3\kappa_4^2 |W|^2 \right] \, , \tag{4.4}$$

where the Kähler potential $\mathcal{K}$ is

$$\kappa_4^2 \mathcal{K} = -3 \log\left(\rho + \bar{\rho}\right) \, . \tag{4.5}$$

The minimum of the potential (4.4) is determined through the equation [61]

$$\left.\frac{\partial V_F}{\partial \rho}\right|_{\rho^*} = 0 \;\Leftrightarrow\; \frac{|W_0|}{A} e^{a\rho_F} = \frac{2}{3} a\rho_F + 1 \, , \tag{4.6}$$

and the value of the potential at the minimum is negative

$$V_{KKLT} = -2 \frac{e^{-2a\rho_F}}{a\rho_F} \, . \tag{4.7}$$

To match positive vacuum energy observations, the KKLT potential (4.7) should be uplifted. One particular way to do that is to place an anti-D3 at the tip of the cone [18], although it can be done in a variety of ways.

Mobile D3 brane

In the discussion above the D3 brane was assumed to be fixed. Nevertheless the location of the D3 X is not fixed and actually enters the effective potential of the theory on the D7. In fact the mobile D3 is not a difficulty but a big advantage of this setup. Typically after adding the mobile D3-brane the Kähler moduli remain stabilized. At the same time, the location of D3, X, is not massless yet the mass is generally much smaller than the Planck scale. Therefore the location of the D3 is a promising candidate for the inflaton field [18]. We elaborate on this scenario in chapter 5. In this chapter we merely focus on the nonperturbative superpotential (4.2) and discuss how it depends of the D3-brane location X. The answer we yet have to derive can be written in the form $A(X)$. We find $A(X)$ explicitly in this chapter.

The model

Our model consists of N_7 D7 branes wrapping a cycle Σ_4 in the compact Calabi-Yau and a mobile D3-brane. The D7-branes are embedded supersymmetrically and their location as well as the holomorphic moduli are fixed due to flux induced potential. The Kähler potential is fixed dynamically according to KKLT scenario as outlined above. To make the dynamics controllable and to work at energies well below the Planck scale we assume that the D3 is located in the warped throat, which is a part of the Calabi-Yau. The warped throat in our consideration will be approximated by a warped deformed, warped singular conifold or any other known non-compact conic Calabi-Yau. This assures that the Kähler potential $k(X, \bar{X})$ is known. The probe D3 will move along the conifold toward the tip, i.e. the low-energy region.

This goes along with the interpretation of its position X as the inflaton field.

To be able to calculate $A(X)$ we assume the cycle Σ_4 significantly stretches inside the throat. The D3 backreacts on the D7 via a small deformation of the geometry. This deformation rapidly decreases with the distance. Therefore the D7 placed in the bulk would not feel the D3 and we would return to the original KKLT proposal. The same argument suggests that if the throat is long enough the X dependent part of W_{np} will come solely from the throat region. That is why the problem of calculating $A(X)$ admits an explicit solution. To this end, we need to consider an infinite throat solution like those from chapter 2 and calculate $A(X)$ in this case. This answer is an exact answer for the non-compact scenario[1]. The embedding of the D7 should be SUSY such that the D7 does not experience any force and is immobile in the infinite-throat approximation. This implies that the associated four-cycle Σ_4 is holomorphic. Other restrictions on Σ_4 will be discussed in the next section.

4.2 Nonperturbative superpotential and Green's function method

Warped volumes and the superpotential

The nonperturbative superpotential W_{np} discussed in the previous section (4.1) depends exponentially on the warped volume of the associated four-cycle Σ_4. It governs the gauge coupling of the gauge theory on D7-branes. To see this, consider a warped product of Minkowski space with the throat $\mathcal{M}$

$$\mathrm{d}s^2 = h^{-1/2}(Y)\eta_{ab}\mathrm{d}x^a\mathrm{d}x^b + h^{1/2}(Y)g_{IJ}\mathrm{d}Y^I\mathrm{d}Y^J\,. \tag{4.8}$$

Here Y^I and g_{IJ} are six coordinates and the unwarped metric on $\mathcal{M}$.

[1]There is a subtlety related to holomorphicity of $A(X)$ in the non-compact case. It is briefly discussed below in section (4.2.1). More details can be found in [68].

The Yang-Mills coupling g_7 of the $7+1$ dimensional gauge theory living on a stack of D7-branes is given by

$$g_7^2 \equiv 2(2\pi)^5 g_s(\alpha')^2 \,. \tag{4.9}$$

The full D7-brane action (2.136) is

$$S = \frac{1}{g_7^2} \int\limits_{\mathbb{R}^{3,1}\times\Sigma_4} \mathrm{d}^4x\ \mathrm{d}^4\xi e^{-\phi}\sqrt{-\mathrm{Det}(G^{ind}+\mathcal{F})} + \frac{1}{g_7^2} \int\limits_{\mathbb{R}^{3,1}\times\Sigma_4} e^{\mathcal{F}} \wedge C \ . \tag{4.10}$$

The magnetic field $\mathcal{F}$ is a sum of the pull-back of NS-NS form $P[B_2]_{\mu\nu}$ on Σ_4 and the induced gauge field dA_1 along Σ_4, $F_{\mu\nu}$ and along Minkowski space F_{ab}. The induced metric G^{ind} consists of two parts: the metric $h^{-1/2}\eta_{ab}$ along $\mathbb{R}^{3,1}$ and the induced metric $h^{-1/2}g^{ind}_{\mu\nu}$ on Σ_4.

In the absence of the magnetic field along Σ_4, $\mathcal{F}_{\mu\nu} = P[B_2]_{\mu\nu} + F_{\mu\nu} = 0$, the action can be significantly simplified. The Cherns-Simons term vanishes and the DBI term can be decomposed into two corresponding to Σ_4 and to Minkowski space. The latter

$$\int\limits_{\mathbb{R}^{3,1}} \mathrm{d}^4x\ \sqrt{h^{-1/2}\eta_{ab} + F_{ab}} \ , \tag{4.11}$$

can be expanded in powers of F_{ab} leading to the following effective action for the gauge fields on D7-branes

$$S_{YM} = \frac{1}{2g_7^2} \int_{\Sigma_4} \mathrm{d}^4\xi \sqrt{g^{ind}}\, h(Y) \cdot \int\limits_{\mathbb{R}^{3,1}} \mathrm{d}^4x \mathrm{Tr}\, F^2 \,. \tag{4.12}$$

The key point here is the appearance of a single power of $h(Y)$ [62]. Defining the warped volume of Σ_4,

$$V^w_{\Sigma_4} \equiv \int_{\Sigma_4} \mathrm{d}^4\xi \sqrt{g^{ind}}\, h(Y) \tag{4.13}$$

and recalling the D3-brane tension

$$T_3 \equiv \frac{1}{(2\pi)^3 g_s(\alpha')^2} \ , \tag{4.14}$$

we read off the gauge coupling of the four-dimensional theory from (4.12):

$$\frac{1}{g^2} = \frac{V^w_{\Sigma_4}}{g_7^2} = \frac{T_3 V^w_{\Sigma_4}}{8\pi^2} \,. \tag{4.15}$$

In $\mathcal{N} = 1$ super-YM theory, the Wilsonian gauge coupling is the real part of a holomorphic function which receives one-loop corrections, but no higher perturbative corrections [63, 64, 65]. The modulus of the gaugino condensate superpotential in $SU(N_{D7})$ super-YM with ultraviolet cutoff $\Lambda_{\rm UV}$ is given by

$$|W_{\rm np}| = 16\pi^2 \Lambda^3_{\rm UV} \exp\Big(-\frac{1}{N_{D7}}\frac{8\pi^2}{g^2}\Big) \propto \exp\Big(-\frac{T_3 V^w_{\Sigma_4}}{N_{D7}}\Big) \,. \tag{4.16}$$

The mobile D3-brane adds a flavor to the $SU(N_{D7})$ gauge theory, whose mass m is a holomorphic function of the D3-brane coordinates. In particular, the mass vanishes when the D3-brane coincides with the D7-brane. In such a gauge theory, the superpotential is proportional to $m^{1/N_{D7}}$ [66]. Our explicit closed-string channel calculations will confirm this form of the superpotential.

Corrections to the Warped Volumes of Four-Cycles

The displacement of a D3-brane in $\mathcal{M}$ creates a slight distortion of the warp factor $\delta h(Y)$ which now becomes dependent on the location of the D3-brane X

$$h(Y) \to h(Y) + \delta h(X;Y) \,. \tag{4.17}$$

At leading order the metric and other fields remain unchanged [62]. The correction to the warp factor affects the warped volumes of the four-cycle

$$\delta V^w_{\Sigma_4} \equiv \int_{\Sigma_4} \mathrm{d}^4\xi \sqrt{g^{ind}(\xi)}\, \delta h(X;Y(\xi)) \,. \tag{4.18}$$

By computing this change in volume we will extract the dependence of the superpotential on X. In the non-compact throat approximation, we will calculate $\delta V^w_{\Sigma_4}$ explicitly, and find that it is the real part of a holomorphic function $\zeta(X)$. Its imaginary part can be determined by the integral of the Ramond-Ramond four-form perturbation δC_4 over Σ_4 although we are not doing this calculation here. In the conical examples considered in this thesis the holomorphic function $\zeta(X)$ can be deduced from its real part (4.18).

In the compact case, $\delta V^w_{\Sigma_4}$ is no longer the real part of any holomorphic function. Instead it acquires a non-holomorphic piece, which can be combined with the gauge-invariant Kähler modulus such that the full answer is the real part of holomorphic function. This observation solves the 'rho-problem' of [67] and confirms that the X-dependence of the superpotential (4.16) in the compact case coincides with the non-compact result, provided that the D3 is located far from the compactification region [68].

The nonperturbative superpotential (4.16) generated by the gaugino condensation can be rewritten in the following form [61, 18]

$$W_{np} = A(X)e^{-a\rho} = A_0 \exp\Big(-\frac{T_3\,\zeta(X)}{N_{D7}}\Big)e^{-a\rho}\,. \tag{4.19}$$

The unknown constant A_0 depends on the values at which the complex structure moduli are stabilized, but is independent of the D3-brane position. The Kähler modulus ρ depends on the unwarped volume of Σ_4 and is fixed dynamically.

Effects of induced magnetic field

Our result (4.19) for the X-dependent part of the superpotential (4.16) is based on the assumption that the magnetic field $\mathcal{F}_{\mu\nu}$ along Σ_4 vanishes in (4.10). Now we will show that this result is actually correct for the supersymmetric D7-brane even if $\mathcal{F}_{\mu\nu} \neq 0$ [69]. For that reason we need to restore $\mathcal{F}_{\mu\nu}$ in (4.12)

$$\mathrm{Det}\left(h^{1/2}g^{ind}_{\mu\nu} + \mathcal{F}_{\mu\nu}\right) = \left(h\sqrt{g^{ind}} - \mathrm{Pf}\mathcal{F}\right)^2 + h(P[\tilde{J}]\wedge\mathcal{F})^2_{1234}\,, \tag{4.20}$$

and in the CS term

$$\int\limits_{\mathbb{R}^{3,1}\times\Sigma_4} e^{\mathcal{F}}\wedge C = \frac{1}{2}\int\limits_{\Sigma_4} h^{-1}(Y)\mathcal{F}\wedge\mathcal{F}\int\limits_{\times\mathbb{R}^{3,1}} \mathrm{d}^4x\,. \tag{4.21}$$

Here $\tilde{J} \equiv h^{-1} J$ is the Kähler form (2.107), and the Pfaffian $\text{Pf}\mathcal{F}$ of a two-form $\mathcal{F}$ on Σ_4 is defined as follows

$$\text{Pf}\mathcal{F} = \frac{1}{2}\left(\mathcal{F} \wedge \mathcal{F}\right)_{1234} = \frac{1}{4!2}\epsilon^{\mu\nu\rho\sigma}\left(\mathcal{F} \wedge \mathcal{F}\right)_{\mu\nu\rho\sigma} \ . \tag{4.22}$$

In the expression above (4.20) we have used the fact that the D7-brane is supersymmetric. This condition is necessary to cancel the forces on the D7 and to fix it inside the throat. Namely, we have used that $\mathcal{F}^{2,0} = 0$ and $\mathcal{F}$ is of type $(1,1)$. In addition to the constraint that Σ_4 is holomorphic, κ-symmetry also requires

$$h^{\frac{1}{2}} P[\tilde{J}] \wedge \mathcal{F} = \frac{1}{2}\tanh\theta\left(P[\tilde{J} \wedge \tilde{J}] - \mathcal{F} \wedge \mathcal{F}\right) \ , \tag{4.23}$$

where the angle θ is related to the geometry of the background and in the case of the baryonic branch solutions is given by $\cos\theta = e^{\phi}$. It is zero in the case of warped Calabi-Yau, like KS solution. Adding the DBI and CS pieces together we notice that the terms with magnetic field cancel each other from the D7 tension

$$S_{D7} = \frac{1}{g_7}\int_{\Sigma_4} P[\tilde{J} \wedge \tilde{J}] \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x + S_{YM} \ , \tag{4.24}$$

and that the effective action for the gauge field is modified by $\mathcal{F}$

$$S_{YM} = \frac{1}{2g_7^2}\int_{\Sigma_4} \mathrm{d}^4\xi \left(\sqrt{g^{ind}}\, h(Y) - \text{Pf}\mathcal{F}\right) \cdot \int_{\mathbb{R}^{3,1}} \mathrm{d}^4 x \ \text{Tr}\, F^2 \, . \tag{4.25}$$

Since $\tilde{J}$ is closed the tension (4.24) is independent of Σ_4 and depends only on its cohomology class. A similar result was found for a D5 brane wrapping S^3 at the tip of the cone in section (3.1.2).

Since the location of the D3-brane enters (4.25) only through δh the extra term $\text{Pf}\mathcal{F}$ does not cause any difference between (4.25) and (4.12) at the level of the correction (4.18). Therefore the X-dependent part $\zeta(X)$ of the superpotential (4.19) remains the same with and without a magnetic field along Σ_4 [69].

4.2.1 Method of calculating the backreaction

The Green's function method

A D3-brane located at some position X in a six-dimensional space with coordinates Y acts as a point source for a perturbation δh of the geometry:

$$-\nabla_Y^2 \delta h(X;Y) = \mathcal{C}\left[\frac{\delta^{(6)}(X-Y)}{\sqrt{g(Y)}} - \rho_{bg}(Y)\right] . \tag{4.26}$$

That is, the perturbation δh is a Green's function for the Laplace problem on the background of interest. Here $\mathcal{C} \equiv 2\kappa_{10}^2 T_3 = (2\pi)^4 g_s(\alpha')^2$ ensures the correct normalization of a single D3-brane source term relative to the four-dimensional Einstein-Hilbert action. A consistent flux compactification contains a background charge density $\rho_{bg}(Y)$ which satisfies

$$\int \mathrm{d}^6Y \sqrt{g}\,\rho_{bg}(Y) = 1 \tag{4.27}$$

to account for the Gauss's law constraint on the compact space [15].

To solve (4.26), we first solve

$$-\nabla_{Y'}^2 \Phi(Y;Y') = -\nabla_Y^2 \Phi(Y;Y') = \frac{\delta^{(6)}(Y-Y')}{\sqrt{g}} - \frac{1}{V_6}\,, \tag{4.28}$$

where $V_6 \equiv \int \mathrm{d}^6Y \sqrt{g}$. The solution to (4.26) is then

$$\delta h(X;Y) = \mathcal{C}\left[\Phi(X;Y) - \int \mathrm{d}^6Y' \sqrt{g}\,\Phi(Y;Y')\rho_{bg}(Y')\right] . \tag{4.29}$$

In the non-compact case V_6 is infinite and Φ is proportional to the Green's function G. The last term in (4.29) is X-independent and can be dropped in the calculation of $\zeta(X)$. In the general case

$$-\nabla_X^2 \delta h(X;Y) = \mathcal{C}\left[\frac{\delta^{(6)}(X-Y)}{\sqrt{g(X)}} - \frac{1}{V_6}\right] , \tag{4.30}$$

and this expression is independent of the background charge ρ_{bg}. Again in the non-compact case the last term vanishes and we have

$$\delta h(X;Y) = \mathcal{C} G(X,Y) . \tag{4.31}$$

To compute $A(X)$ from (4.19), we simply solve for the Green's function $G(X,Y)$ obeying (4.30) with zero $\frac{1}{V_6}$ and then integrate δh over the four-cycle of interest, according to (4.18).

Green's function on conic geometries

The D3-branes that we consider in this paper are point sources in the six-dimensional internal space. The backreaction they induce on the background geometry can therefore be related to the Green's functions for the Laplace problem on conical geometries $\mathbb{R}^+ \times \mathbb{X}$ (see section (4.2.1))

$$-\nabla_X^2 G(X;X') = \frac{\delta^{(6)}(X-X')}{\sqrt{g(X)}} \,. \tag{4.32}$$

In the following we present explicit results for the Green's function on the singular conifold. In the large r-limit, far from the tip, the Green's functions for the resolved and deformed conifold reduce to those of the singular conifold.

In the singular conifold geometry $dr^2 + r^2 ds^2_{T^{1,1}}$, the defining equation (4.32) for the Green's function becomes

$$\frac{1}{r^5}\frac{\partial}{\partial r}\Big(r^5 \frac{\partial}{\partial r} G\Big) + \frac{1}{r^2}\nabla_\Psi^2 G = -\frac{1}{r^5}\delta(r-r')\delta_{\mathbb{X}}(\Psi-\Psi') \,, \tag{4.33}$$

where ∇^2_Ψ and $\delta_{\mathbb{X}}(\Psi-\Psi')$ are the Laplacian and the normalized delta function on $\mathbb{X}$, respectively. Ψ stands collectively for the five angular coordinates of the base and $X \equiv (r,\Psi)$. An explicit solution for the Green's function is obtained by a series expansion of the form

$$G(X;X') = \sum_L Y_L^*(\Psi')Y_L(\Psi)H_L(r;r') \,. \tag{4.34}$$

The Y_L's are eigenfunctions of the angular Laplacian,

$$\nabla_\Psi^2 Y_L(\Psi) = -\Lambda_L Y_L(\Psi) \,, \tag{4.35}$$

where the multi-index L represents the set of discrete quantum numbers related to the symmetries of the base of the cone. The angular eigenproblem is worked out in detail in section (4.3). If the angular wavefunctions are normalized as

$$\int \mathrm{d}^5\Psi \, \sqrt{g_{\mathbb{X}}} \, Y_L^*(\Psi) Y_{L'}(\Psi) = \delta_{LL'} \,, \tag{4.36}$$

then

$$\sum_L Y_L^*(\Psi') Y_L(\Psi) = \delta_{T^{1,1}}(\Psi - \Psi') \,, \tag{4.37}$$

and equation (4.33) reduces to the radial equation

$$\frac{1}{r^5}\frac{\partial}{\partial r}\left(r^5 \frac{\partial}{\partial r} H_L\right) - \frac{\Lambda_L}{r^2} H_L = -\frac{1}{r^5}\delta(r - r') \,, \tag{4.38}$$

whose solution away from $r = r'$ is

$$H_L(r; r') = A_\pm(r') r^{c_L^\pm} \,, \qquad c_L^\pm \equiv -2 \pm \sqrt{\Lambda_L + 4} \,. \tag{4.39}$$

The constants $A_\pm$ are uniquely determined by integrating equation (4.38) across $r = r'$. The Green's function on the singular conifold is

$$G(X; X') = \sum_L \frac{1}{2\sqrt{\Lambda_L + 4}} \times Y_L^*(\Psi') Y_L(\Psi) \times \begin{cases} \frac{1}{r'^4}\left(\frac{r}{r'}\right)^{c_L^+} & r \leq r' \,, \\ \frac{1}{r^4}\left(\frac{r'}{r}\right)^{c_L^+} & r \geq r' \,, \end{cases} \tag{4.40}$$

where the angular eigenfunctions $Y_L(\Psi)$ are given explicitly in section (4.3).

Gauge theory interpretation of Green's function method

The calculation of the correction to the superpotential (4.18) on conic geometries with δh given by (4.31) and Green's function given by (4.40) has a simple interpretation in terms of gauge theory. Having in mind the conformal "conifold" field theory of section (2.1.1) dual to supergravity on singular conifold we can interpret the $\frac{1}{r}$ expansion of δh

$$\delta h = \frac{27\pi g_s (\alpha')^2}{4r^4} \left[1 + \sum_i \frac{c_i f_i(\theta_1, \theta_2, \phi_1, \phi_2, \psi)}{r^{\Delta_i}} \right] \,, \tag{4.41}$$

via the AdS/CFT correspondence. Each term in (4.41) corresponds to a gauge-invariant operator $\mathcal{O}_i$ in the gauge theory with dimension Δ_i and c_i is proportional to the expectation values $\langle \mathcal{O}_i \rangle$ determined by the position of the D3-brane [38]. The angle-dependent part $f_i(\theta_1, \theta_2, \phi_1, \phi_2, \psi)$ is a wave-function of the Laplacian on $T^{1,1}$ and can be rewritten through w_i of (2.23-2.26) which makes an explicit connection with a gauge-theory operator via (2.2).

There is a set of chiral operators $\mathrm{Tr}[A_{i_1} B_{j_1} A_{i_2} B_{j_2} \ldots A_{i_k} B_{j_k}]$ symmetric in both i and j indexes. They have integer R-charge k and dimension $\Delta_i^{\rm chiral} = 3k/2$ and transform as $(k/2, k/2)$ under $SU(2) \times SU(2)$. These operators correspond to the spherical harmonics on $T^{1,1}$, which transforms as $(k/2, k/2)$ under $SU(2) \times SU(2)$. All these terms will have non-zero c_i i.e. they will contribute to $\zeta(X)$ after integration over Σ_4 in (4.31).

All other terms in (4.41), which will be refereed as "non-chiral" give no contribution after integration in (4.31). These two sets of "chiral" and "non-chiral" harmonics will be considered separately in the next section.

4.3 Computation of backreaction on the conifold

Eigenfunctions of the Laplacian on $T^{1,1}$

In this section we complete the calculation of the Green's function on the singular conifold (4.40) by constructing the eigenfunctions of the Laplacian on $T^{1,1}$

$$\begin{aligned} \nabla^2_\Psi Y_L &= \frac{1}{\sqrt{g}} \partial_m (g^{mn} \sqrt{g} \partial_n Y_L) = (6\nabla_1^2 + 6\nabla_2^2 + 9\nabla_R^2) Y_L \\ &= -\Lambda_L Y_L \,, \end{aligned} \tag{4.42}$$

where

$$\nabla_i^2 Y_L \equiv \frac{1}{\sin\theta_i} \partial_{\theta_i} (\sin\theta_i \partial_{\theta_i} Y_L) + \left(\frac{1}{\sin\theta_i} \partial_{\phi_i} - \cot\theta_i \partial_\psi \right)^2 Y_L \,, \tag{4.43}$$

$$\nabla_R^2 Y_L \equiv \partial_\psi^2 Y_L \,. \tag{4.44}$$

The solution to equation (4.42) is obtained through separation of variables

$$Y_L(\Psi) = J_{l_1,m_1,R}(\theta_1) J_{l_2,m_2,R}(\theta_2) e^{im_1\phi_1+im_2\phi_2} e^{\frac{i}{2}R\psi} , \tag{4.45}$$

where

$$\frac{1}{\sin\theta_i}\partial_{\theta_i}(\sin\theta_i \partial_{\theta_i} J_{l_i,m_i,R}(\theta_i)) - \Big(\frac{m_i}{\sin\theta_i} - \frac{R}{2}\cot\theta_i\Big)^2 J_{l_i,m_i,R}(\theta_i) = -\Lambda_{l_i,R} J_{l_i,m_i,R}(\theta_i) . \tag{4.46}$$

The eigenvalues are $\Lambda_{l_i,R} \equiv l_i(l_i+1) - \frac{R^2}{4}$. Explicit solutions for equation (4.46) are given in terms of hypergeometric functions ${}_2F_1(a,b,c;x)$

$$\begin{aligned} J^{\Upsilon}_{l_i,m_i,R}(\theta_i) &= N^{\Upsilon}_L (\sin\theta_i)^{m_i}\Big(\cot\frac{\theta_i}{2}\Big)^{R/2} \times \\ &\quad {}_2F_1\Big(-l_i+m_i, 1+l_i+m_i, 1+m_i-\frac{R}{2}; \sin^2\frac{\theta_i}{2}\Big) , \end{aligned} \tag{4.47}$$

$$\begin{aligned} J^{\Omega}_{l_i,m_i,R}(\theta_i) &= N^{\Omega}_L (\sin\theta_i)^{R/2}\Big(\cot\frac{\theta_i}{2}\Big)^{m_i} \times \\ &\quad {}_2F_1\Big(-l_i+\frac{R}{2}, 1+l_i+\frac{R}{2}, 1-m_i+\frac{R}{2}; \sin^2\frac{\theta_i}{2}\Big) , \end{aligned} \tag{4.48}$$

where N^{Υ}_L and N^{Ω}_L are determined by the normalization condition (4.36). If $m_i \geq R/2$, solution Υ is non-singular. If $m_i \leq R/2$, solution Ω is non-singular. The full wavefunction corresponds to the spectrum

$$\Lambda_L = 6\Big(l_1(l_1+1) + l_2(l_2+1) - \frac{R^2}{8}\Big) . \tag{4.49}$$

The eigenfunctions transform under $SU(2)_1 \times SU(2)_2$ as the spin (l_1, l_2) representation and under the $U(1)_R$ with charge R. The multi-index L has the data:

$$L \equiv (l_1,l_2),(m_1,m_2),R .$$

The following restrictions on the quantum numbers correspond to the existence of single-valued regular solutions:

- l_1 and l_2 are both integers or both half-integers.
- $m_1 \in \{-l_1, \cdots, l_1\}$ and $m_2 \in \{-l_2, \cdots, l_2\}$.

- $R \in \mathbb{Z}$ with $\frac{R}{2} \in \{-l_1, \cdots, l_1\}$ and $\frac{R}{2} \in \{-l_2, \cdots, l_2\}$.

As discussed in section (4.2.1), chiral operators in the dual gauge theory correspond to $l_1 = \frac{R}{2} = l_2$.

Supersymmetric four-cycles in the conifold

The κ-symmetry specifies the set of conditions for the D7-brane to be supersymmetric. In the absence of NS-NS field, as in the case of the singular conifold of section (2.1.1) the induced gauge field A_1 can be set to zero. The only constraint left implies that the D7 has to be embedded along a holomorphic four-cycle Σ_4. For the set of holomorphic cycles

$$f(w_i) \equiv \prod_{i=1}^{4} w_i^{p_i} - \mu^P = 0 \,. \tag{4.50}$$

the κ-symmetry condition was checked explicitly [70]. Here $p_i \in \mathbb{Z}$, $P \equiv \sum_{i=1}^{4} p_i$, and $\mu \in \mathbb{C}$ are constants defining the embedding of the D7-branes. In real coordinates $\phi_i, \theta_i, \psi, r$ of section (2.1.1) the embedding condition (4.50) becomes

$$\psi(\phi_1, \phi_2) = n_1\phi_1 + n_2\phi_2 + \psi_s \,, \tag{4.51}$$

$$r(\theta_1, \theta_2) = r_{\rm min}\left[x^{1+n_1}(1-x)^{1-n_1}y^{1+n_2}(1-y)^{1-n_2}\right]^{-1/6} \,, \tag{4.52}$$

where

$$r_{\rm min}^{3/2} \equiv |\mu| \,, \tag{4.53}$$

$$\frac{1}{2}\psi_s \equiv \arg(\mu) + \frac{2\pi s}{P} \,, \quad s \in \{0, 1, \ldots, P-1\} \,. \tag{4.54}$$

Here we choose $\phi_1, \phi_2, \theta_1, \theta_2$ to be the coordinates on the four-cycle. It is convenient to define new coordinates x, y

$$x \equiv \sin^2\frac{\theta_1}{2} \,, \quad y \equiv \sin^2\frac{\theta_2}{2} \tag{4.55}$$

and the rational winding numbers

$$n_1 \equiv \frac{p_1 - p_2 - p_3 + p_4}{P} \,, \quad n_2 \equiv \frac{p_1 - p_2 + p_3 - p_4}{P} \,. \tag{4.56}$$

To compute the integral over the four-cycle we will need the induced volume form on the wrapped D7-brane. By substituting the embedding equations (4.51,4.52) into the metric of singular conifold (2.34,2.35) we obtain

$$\mathrm{d}\theta_1\mathrm{d}\theta_2\mathrm{d}\phi_1\mathrm{d}\phi_2\,\sqrt{g^{ind}} = \frac{V_{T^{1,1}}}{16\pi^3}\,r^4\,\mathcal{G}(x,y)\,\mathrm{d}x\mathrm{d}y\mathrm{d}\phi_1\mathrm{d}\phi_2\,, \tag{4.57}$$

where

$$\begin{aligned}\mathcal{G}(x,y) \;&\equiv\; \frac{(1+n_1)^2}{2}\frac{1}{x(1-x)} - 2n_1\frac{1}{1-x}\\ &+\; \frac{(1+n_2)^2}{2}\frac{1}{y(1-y)} - 2n_2\frac{1}{1-y} - 1\,.\end{aligned} \tag{4.58}$$

The volume of $T^{1,1}$ defined in (4.57) is

$$V_{T^{1,1}} \equiv \int \mathrm{d}^5\Psi\sqrt{g_{T^{1,1}}} = \frac{16\pi^3}{27}\,, \tag{4.59}$$

with Ψ standing for all five angular coordinates on $T^{1,1}$.

Embedding, induced metric and a selection rule

Equation (4.51) and the form of the angular eigenfunctions of the Green's function (see section (4.3)) imply that the correction to the warped volume

$$\delta V^w_{\Sigma_4} = \mathrm{Re}(\zeta(X')) = \int_{\Sigma_4} \mathrm{d}^4X\sqrt{g^{ind}(X)}\,\delta h(X;X')\,, \tag{4.60}$$

is proportional to

$$\frac{e^{\frac{i}{2}R\psi_s}}{(2\pi)^2}\int_0^{2\pi}\mathrm{d}\phi_1\,e^{i(m_1+\frac{R}{2}n_1)\phi_1}\int_0^{2\pi}\mathrm{d}\phi_2\,e^{i(m_2+\frac{R}{2}n_2)\phi_2} = e^{\frac{i}{2}R\psi_s}\delta_{m_1,-\frac{R}{2}n_1}\cdot\delta_{m_2,-\frac{R}{2}n_2}\,. \tag{4.61}$$

We may therefore restrict the computation to values of the R-charge that satisfy

$$m_1 = -\frac{R}{2}n_1\,,\quad m_2 = -\frac{R}{2}n_2\,. \tag{4.62}$$

The winding numbers n_i (4.56) are rational numbers of the form

$$n_i \equiv \frac{\tilde{n}_i}{q}\,,\quad \tilde{n}_i\in\mathbb{Z}\,, \tag{4.63}$$

where $\tilde{n}_i$ and q do not have a common divisor. Therefore the requirement that the magnetic quantum numbers m_i be integer or half-integer leads to the following selection rule for the R-charge

$$R = q \cdot k \,, \quad k \in \mathbb{Z} \,. \tag{4.64}$$

Green's function and reduced angular eigenfunctions

The Green's function on the conifold from section (4.2.1) is

$$G(X;X') = \sum_L Y_L^*(\Psi')Y_L(\Psi)H_L(r;r') \,, \tag{4.65}$$

where it is important that the angular eigenfunctions from section (4.3) are normalized correctly on $T^{1,1}$

$$\int \mathrm{d}^5\Psi \sqrt{g_{T^{1,1}}}|Y_L|^2 = 1 \,, \tag{4.66}$$

or

$$V_{T^{1,1}} \int_0^1 \mathrm{d}x\, [J_{l_1,m_1,R}(x)]^2 \int_0^1 \mathrm{d}y\, [J_{l_2,m_2,R}(y)]^2 = 1 \,. \tag{4.67}$$

The coordinates x and y are defined in (4.55). Next, we show that the hypergeometric angular eigenfunctions reduce to Jacobi polynomials if we define

$$l_1 \equiv \frac{R}{2} + L_1 \,, \quad l_2 \equiv \frac{R}{2} + L_2 \,, \quad L_1, L_2 \in \mathbb{Z} \,. \tag{4.68}$$

This parameterization is convenient because chiral terms are easily identified by $L_1 = 0 = L_2$. Non-chiral terms correspond to non-zero L_1 and/or L_2. Without loss of generality we define chiral terms to have $R > 0$ and anti-chiral terms to have $R < 0$. With these restrictions the angular eigenfunctions of section (4.3) simplify to

$$J_{\frac{R}{2}+L_1,-\frac{R}{2}n_1,R}(x) = x^{\frac{R}{4}(1+n_1)}(1-x)^{\frac{R}{4}(1-n_1)}\, P_{L_1,R,n_1}(x) \,, \tag{4.69}$$

$$J_{\frac{R}{2}+L_2,-\frac{R}{2}n_2,R}(y) = y^{\frac{R}{4}(1+n_2)}(1-y)^{\frac{R}{4}(1-n_2)}\, P_{L_2,R,n_2}(y) \,, \tag{4.70}$$

where

$$P_{L_1,R,n_1}(x) \equiv N_{L_1,R,n_1} P_{L_1}^{\frac{R}{2}(1+n_1),\frac{R}{2}(1-n_1)}(1-2x)\,, \tag{4.71}$$

$$P_{L_2,R,n_2}(y) \equiv N_{L_2,R,n_2} P_{L_2}^{\frac{R}{2}(1+n_2),\frac{R}{2}(1-n_2)}(1-2y)\,. \tag{4.72}$$

The $P_N^{\alpha,\beta}$ are Jacobi polynomials and the normalization constants N_{L_1,R,n_1} and N_{L_2,R,n_2} can be determined from (4.67).

Main integral

Assembling the ingredients of the previous subsections (induced metric, embedding constraint, Green's function) we find that (4.60) may be expressed as

$$\begin{aligned} T_3\ \delta V^w_{\Sigma_4} &= (2\pi)^3 \int_0^1 \mathrm{d}x\mathrm{d}y\,\sqrt{g^{ind}(x,y)}\sum_{L,\psi_s} Y^*_L(x',y')Y_L(x,y)H_L(r;r') \\ &= \frac{V_{T^{1,1}}}{2}\sum_{L,\psi_s} Y^*_L\,(r')^{c^+_L}\times e^{\frac{i}{2}R\psi'_s} r_{\min}^{-c^+_L}\times\frac{I^n_K(Q^+_L)}{\sqrt{\Lambda_L+4}}\,, \end{aligned} \tag{4.73}$$

where

$$I^n_K(Q^+_L) \equiv \int_0^1 \mathrm{d}x\mathrm{d}y\,\mathcal{G}(x,y)\left(\frac{r(x,y)}{r_{\min}}\right)^{-6Q^+_L} P_{L_1,R,n_1}(x)P_{L_2,R,n_2}(y)\,. \tag{4.74}$$

Here $K\equiv(L_1,L_2,R)$, $n\equiv(n_1,n_2)$ and

$$Q^\pm_L \equiv \frac{c^\pm_L}{6}+\frac{R}{4}\,,\qquad c^\pm_L\equiv -2\pm\sqrt{\Lambda_L+4}\,. \tag{4.75}$$

The sum in equation (4.73) is restricted by the selection rules (4.62) and (4.64). Equation (4.74) is the main result of this section. In the following we will show that the integral vanishes for all non-chiral terms and reduces to a simple expression for (anti)chiral terms.

4.3.1 Non-chiral contributions

In this section we prove that

$$
\begin{aligned}
I_K^n(Q) \;\equiv\; & \int_0^1 \mathrm{d}x\mathrm{d}y \; P_{L_1,R,n_1}(x) P_{L_2,R,n_2}(y) \times \\
& \times x^{Q(1+n_1)}(1-x)^{Q(1-n_1)} y^{Q(1+n_2)}(1-y)^{Q(1-n_2)} \times \\
& \times \Big[\frac{(1+n_1)^2}{2}\frac{1}{x(1-x)} - 2n_1 \frac{1}{1-x} \\
& + \frac{(1+n_2)^2}{2}\frac{1}{y(1-y)} - 2n_2 \frac{1}{1-y} - 1 \Big]
\end{aligned}
\tag{4.76}
$$

vanishes for $Q \to Q_L^+$ iff $L_1 \neq 0$ or $L_2 \neq 0$. This proves that non-chiral terms do not contribute to the perturbation $\delta V_{\Sigma_4}^w$ to the warped four-cycle volume.

The Jacobi polynomial $P_N^{\alpha,\beta}(x)$ satisfies the following differential equation

$$
\begin{aligned}
& -N(N+\alpha+\beta+1) P_N^{\alpha,\beta}(1-2x) = \\
& \quad = x^{-\alpha}(1-x)^{-\beta} \frac{d}{dx}\left(x^{1+\alpha}(1-x)^{1+\beta} \frac{d}{dx} P_N^{\alpha,\beta}(1-2x) \right) .
\end{aligned}
\tag{4.77}
$$

Multiplying both sides by $x^{q_\alpha}(1-x)^{q_\beta}$ and integrating over x gives

$$
\begin{aligned}
& -N(N+\alpha+\beta+1) \int_0^1 \mathrm{d}x \, P_N^{\alpha,\beta}(1-2x) x^{q_\alpha}(1-x)^{q_\beta} = \\
& \quad = \int_0^1 \mathrm{d}x \, P_N^{\alpha,\beta}(1-2x) x^{q_\alpha}(1-x)^{q_\beta} \times \\
& \quad \times \Big[(q_\alpha + q_\beta + 1)(\alpha + \beta - q_\alpha - q_\beta) + \frac{q_\alpha(\alpha - q_\alpha) - q_\beta(\beta - q_\beta)}{(1-x)} + \frac{q_\alpha(q_\alpha - \alpha)}{x(1-x)} \Big] ,
\end{aligned}
\tag{4.78}
$$

where we have used integration by parts. In the case of interest, (4.76), we make the following identifications: $N \equiv L_1$, $\alpha \equiv \frac{R}{2}(1+n_1)$, $\beta \equiv \frac{R}{2}(1-n_1)$, $q_\alpha \equiv Q(1+n_1)$, $q_\beta \equiv Q(1-n_1)$. This gives

$$
\begin{aligned}
& \int_0^1 \mathrm{d}x \, P_{L_1}^{\frac{R}{2}(1+n_1),\frac{R}{2}(1-n_1)}(1-2x)\, x^{Q(1+n_1)}(1-x)^{Q(1-n_1)} \times \left(\frac{(1+n_1)^2}{2x(1-x)} - \frac{2n_1}{(1-x)} \right) = \\
& = X_{L_1,R,Q} \int_0^1 \mathrm{d}x \, P_{L_1}^{\frac{R}{2}(1+n_1),\frac{R}{2}(1-n_1)}(1-2x)\; x^{Q(1+n_1)}(1-x)^{Q(1-n_1)} ,
\end{aligned}
\tag{4.79}
$$

where

$$X_{L_1,R,Q} \equiv \frac{(2Q+4Q^2-L_1^2-L_1R-R-2L_1-2RQ)}{Q(2Q-R)}\,.$$

The corresponding identity for the y-integral follows from the above expression and the replacements $L_1 \to L_2$ and $n_1 \to n_2$. We then notice that the integral (4.76) is

$$\begin{aligned} I_K^n(Q) &= (X_{L_1,R,Q}+Y_{L_2,R,Q}-1)\times\Lambda_{L_1,R,n_1,Q}\,\Lambda_{L_2,R,n_2,Q} \\ &= \frac{6(Q-Q_L^+)(Q-Q_L^-)}{Q(2Q-R)}\times\Lambda_{L_1,R,n_1,Q}\,\Lambda_{L_2,R,n_2,Q}\,, \end{aligned} \tag{4.80}$$

where

$$\Lambda_{L_1,R,n_1,Q} \equiv \int_0^1 \mathrm{d}x\,P_{L_1,R,n_1}(x)\;x^{Q(1+n_1)}(1-x)^{Q(1-n_1)}\,, \tag{4.81}$$

$$\Lambda_{L_2,R,n_2,Q} \equiv \int_0^1 \mathrm{d}y\,P_{L_2,R,n_2}(y)\;y^{Q(1+n_2)}(1-y)^{Q(1-n_2)}\,. \tag{4.82}$$

Since $I_K^n(Q) \propto (Q-Q_L^+)$ it just remain to observe that the integrals (4.81) and (4.82) are finite to conclude that

$$\lim_{Q\to Q_L^+} I_K^n = 0 \quad \text{iff} \quad Q_L^+ \neq \frac{R}{2}\,. \tag{4.83}$$

This proves that non-chiral terms do not contribute corrections to the warped volume of any holomorphic four-cycle of the form (4.50).

4.3.2 Chiral contributions

Finally, let us consider the special case $Q_L^+ = \frac{R}{2}$ which corresponds to chiral operators ($L_1 = L_2 = 0$) in the dual gauge theory. In this case,

$$I_R^{\text{chiral}} \equiv \lim_{Q\to\frac{R}{2}} I_K^n = \frac{3R+4}{2}\frac{1}{R}\times\Lambda_{0,R,n_1,\frac{R}{2}}\times\Lambda_{0,R,n_2,\frac{R}{2}}\,, \tag{4.84}$$

where

$$\Lambda_{0,R,n_1,\frac{R}{2}} \equiv \int_0^1 \mathrm{d}x\,P_{0,R,n_1}(x)\;x^{\frac{R}{2}(1+n_1)}(1-x)^{\frac{R}{2}(1-n_1)}\,, \tag{4.85}$$

$$\Lambda_{0,R,n_2,\frac{R}{2}} \equiv \int_0^1 \mathrm{d}y\,P_{0,R,n_2}(y)\;y^{\frac{R}{2}(1+n_2)}(1-y)^{\frac{R}{2}(1-n_2)}\,. \tag{4.86}$$

Notice that $P_{0,R,n_i} = N_{0,R,n_i} = (N_{0,R,n_i})^{-1}(P_{0,R,n_i})^2$. Hence,

$$\Lambda_{0,R,n_1,\frac{R}{2}} \equiv (N_{0,R,n_1})^{-1} \int_0^1 \mathrm{d}x \left(P_{0,R,n_1}(x) \left[x^{(1+n_1)}(1-x)^{(1-n_1)} \right]^{R/4} \right)^2$$
$$\Lambda_{0,R,n_2,\frac{R}{2}} \equiv (N_{0,R,n_2})^{-1} \int_0^1 \mathrm{d}y \left(P_{0,R,n_2}(y) \left[y^{(1+n_2)}(1-y)^{(1-n_2)} \right]^{R/4} \right)^2$$

and

$$\Lambda_{0,R,n_1,\frac{R}{2}} \times \Lambda_{0,R,n_2,\frac{R}{2}} = \frac{1}{V_{T^{1,1}} N_{0,R,n_1} N_{0,R,n_2}} \tag{4.87}$$

by the normalization condition (4.67) on the angular wave function. Therefore, we get the simple result

$$\frac{I_R^{\mathrm{chiral}}}{\sqrt{\Lambda_R^{\mathrm{chiral}} + 4}} = \frac{1}{V_{T^{1,1}} N_{0,R,n_1} N_{0,R,n_2}} \times \frac{1}{R} \,. \tag{4.88}$$

We substitute this into equation (4.73) and get

$$T_3 \; (\delta V_{\Sigma_4}^w)_{\mathrm{chiral}} = \frac{1}{2} \sum_s \sum_{R=q\cdot k} \frac{1}{R} \times \left(\prod_i (\bar{w}_i')^{p_i} \right)^{R/P} \times \frac{1}{\bar{\mu}^R} \times e^{i\frac{R}{P} 2\pi s} \,, \tag{4.89}$$

where we used

$$(r')^{3R/2} \frac{Y_R^*(\Psi')}{N_{0,R,n_1} N_{0,R,n_2}} = \left(\prod_i (\bar{w}_i')^{p_i} \right)^{R/P} \tag{4.90}$$

and

$$e^{i\arg(\mu)R} r_{\mathrm{min}}^{-3R/2} = \frac{1}{\bar{\mu}^R} \,. \tag{4.91}$$

The sum over s in (4.89) counts the P different roots of equation (4.50):

$$\sum_{s=0}^{P-1} e^{\frac{q\cdot k}{P} 2\pi s} = P \, \delta_{\frac{q\cdot k}{P}, j} \,, \quad j \in \mathbb{Z} \,. \tag{4.92}$$

Dropping primes, we therefore arrive at the following sum

$$T_3 \; (\delta V_{\Sigma_4}^w)_{\mathrm{chiral}} = \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j} \times \left(\prod_i \bar{w}_i^{p_i} \right)^j \times \frac{1}{\bar{\mu}^{P\cdot j}} \,, \tag{4.93}$$

which gives

$$T_3\ (\delta V^w_{\Sigma_4})_{\rm chiral} = -\frac{1}{2}\,\log\left[1-\frac{\prod_i \bar{w}_i^{p_i}}{\bar{\mu}^P}\right]\,. \tag{4.94}$$

For the anti-chiral terms ($R < 0$) an equivalent computation gives the complex conjugate of this result.

The $R = 0$ term formally gives a divergent contribution that needs to be regularized by introducing a UV cutoff at the end of the throat. Alternatively, as discussed in section (4.2.1), this term does not appear if we define δh as the solution of (4.26) with $\sqrt{g}\,\rho_{bg}(Y) = \delta^{(6)}(Y - X_0)$. This choice amounts to evaluating the change in the warp factor, δh, created by moving the D3-brane from some reference point X_0 to X. We may choose the reference point X_0 to be at the tip of the cone, $r = 0$, and thereby remove the divergent zero mode.

Result for singular conifold

The total change in the warped volume of the four-cycle is therefore

$$\delta V^w_{\Sigma_4} = (\delta V^w_{\Sigma_4})_{\rm chiral} + (\delta V^w_{\Sigma_4})_{\rm anti-chiral} \tag{4.95}$$

and

$$T_3\,{\rm Re}(\zeta) = T_3\,\delta V^w_{\Sigma_4} = -{\rm Re}\Big(\log\left[\frac{\mu^P - \prod_i w_i^{p_i}}{\mu^P}\right]\Big)\,. \tag{4.96}$$

Finally, the prefactor of the nonperturbative superpotential is

$$A(w_i) = A_0\,e^{-T_3\zeta/n} = A_0\,\Big(\frac{\mu^P - \prod_i w_i^{p_i}}{\mu^P}\Big)^{1/N_7}\,. \tag{4.97}$$

The simple form of the result is not unexpected. It resembles a similar result for Euclidean D3 brane, obtained in F-theory [71].

4.4 Computation of backreaction on the $Y^{p,q}$ cones

4.4.1 Setup

Metric and Coordinates on $Y^{p,q}$

Cones over $Y^{p,q}$ manifolds have the metric

$$\mathrm{d}s^2 = \mathrm{d}r^2 + r^2 \mathrm{d}s^2_{Y^{p,q}}\,, \tag{4.98}$$

where the Sasaki-Einstein metric on the $Y^{p,q}$ base is given by [73, 72]

$$\begin{aligned} \mathrm{d}s^2_{Y^{p,q}} &= \frac{1-y}{6}\,(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \frac{1}{v(y)w(y)}\mathrm{d}y^2 + \frac{v(y)}{9}(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)^2 \\ &\quad + w(y)\big[\mathrm{d}\alpha + f(y)\,(\mathrm{d}\psi + \cos\theta\,\mathrm{d}\phi)\,\big]^2\,. \end{aligned} \tag{4.99}$$

The following functions have been defined:

$$v(y) \equiv \frac{b - 3y^2 + 2y^3}{b - y^2}\,, \quad w(y) \equiv \frac{2(b-y^2)}{1-y}\,, \quad f(y) \equiv \frac{b - 2y + y^2}{6(b-y^2)}\,, \tag{4.100}$$

with

$$b = \frac{1}{2} - \frac{p^2 - 3q^2}{4p^3}\,\sqrt{4p^2 - 3q^2}\,. \tag{4.101}$$

The parameters p and q are two coprime positive integers. The zeros of $v(y)$ are

$$y_{1,2} \equiv \frac{1}{4p}\left(2p \mp 3q - \sqrt{4p^2 - 3q^2}\right), \qquad y_3 \equiv \frac{3}{2} - (y_1 + y_2)\,. \tag{4.102}$$

It is also convenient to introduce

$$x = \frac{y - y_1}{y_2 - y_1}\,. \tag{4.103}$$

The angular coordinates θ, ϕ, ψ, x, and α span the ranges:

$$\begin{aligned} &0 \le \theta \le \pi\,, \quad 0 < \phi \le 2\pi\,, \quad 0 < \psi \le 2\pi\,, \\ &0 \le x \le 1\,, \quad 0 < \alpha \le 2\pi\ell\,, \end{aligned} \tag{4.104}$$

where $\ell \equiv -\frac{q}{4p^2 y_1 y_2}$.

Green's function

The Green's function on the $Y^{p,q}$ cone is

$$G(X;X') = \sum_L \frac{1}{4(\lambda+1)} \times Y_L^*(\Psi')Y_L(\Psi) \times \begin{cases} \frac{1}{r'^4}\left(\frac{r}{r'}\right)^{2\lambda} & r \le r' \,, \\ \frac{1}{r^4}\left(\frac{r'}{r}\right)^{2\lambda} & r \ge r' \,. \end{cases} \tag{4.105}$$

Here L is again a complete set of quantum numbers and Ψ represents the set of angular coordinates $(\theta, \phi, \psi, x, \alpha)$. The eigenvalue of the angular Laplacian is $\Lambda_L \equiv 4\lambda(\lambda+2)$. The spectrum of the scalar Laplacian on $Y^{p,q}$, as well as the eigenfunctions $Y_L(\Psi)$, were calculated in [74, 75]. We do not review this treatment here, but simply present an explicit form of $Y_L(\Psi)$

$$Y_L(\Psi) = N_L\, e^{i(m\phi + n_\psi \psi + \frac{n_\alpha}{\ell}\alpha)} J_{l,m,2n_\psi}(\theta) R_{n_\alpha,n_\psi,l,\lambda}(x)\,, \tag{4.106}$$

where

$$R_{n_\alpha,n_\psi,l,\lambda}(x) = x^{\alpha_1}(1-x)^{\alpha_2}(a-x)^{\alpha_3}h(x)\,, \quad a \equiv \frac{y_1 - y_3}{y_1 - y_2}\,. \tag{4.107}$$

The parameters α_i depend on n_ψ, n_α (see [75]), and the function $h(x)$ satisfies the following differential equation

$$\left[\frac{d^2}{dx^2} + \left(\frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a}\right)\frac{d}{dx} + \frac{\alpha\beta x - k}{x(1-x)(a-x)}\right] h(x) = 0\,. \tag{4.108}$$

The parameters α, β, γ, δ, ϵ, k depend on p, q and on the quantum numbers of the $Y^{p,q}$ base. Explicit expressions may be found in [75].

Finally, we introduce the normalization condition that fixes N_L in (4.106). If we define $z \equiv \sin^2\frac{\theta}{2}$ then the normalization condition

$$\int \mathrm{d}^5\Psi \sqrt{g_{Y^{p,q}}}|Y_L|^2 = 1 \tag{4.109}$$

becomes

$$N_L^2 \int_0^1 \mathrm{d}z\, \mathrm{d}x\, \sqrt{g(x,z)}\, J^2 R^2 = \frac{1}{(2\pi)^3 \ell}\,, \tag{4.110}$$

where

$$\sqrt{g(x,z)} = \sqrt{g(x)} = \frac{q(2p + 3q + \sqrt{4p^2 - 3q^2} - 6qx)}{24p^2}\,. \tag{4.111}$$

Embedding, induced metric and a selection rule

The holomorphic embedding of four-cycles in $Y^{p,q}$ cones is described by the algebraic equation [76]

$$\prod_{i=1}^{3} w_i^{p_i} = \mu^{2p_3} , \tag{4.112}$$

where

$$\begin{aligned} w_1 &\equiv \tan\frac{\theta}{2}\, e^{-i\phi} , && (4.113)\\ w_2 &\equiv \frac{1}{2}\sin\theta\, x^{\frac{1}{2y_1}}(1-x)^{\frac{1}{2y_2}}(a-x)^{\frac{1}{2y_3}}\, e^{i(\psi+6\alpha)} , && (4.114)\\ w_3 &\equiv \frac{1}{2} r^3 \sin\theta\, [x(1-x)(a-x)]^{1/2} e^{i\psi} . && (4.115) \end{aligned}$$

This results in the following embedding equations in terms of the real coordinates

$$\begin{aligned} \psi &= \frac{1}{1+n_2}(n_1\phi - 6n_2\alpha) - \psi_s , && (4.116)\\ r &= r_{\min}\left[z^{1+n_1+n_2}(1-z)^{1-n_1+n_2}\right]^{-1/6}\left[x^{2e_1}(1-x)^{2e_2}(a-x)^{2e_3}\right]^{-1/6} \\ &\equiv r_{\min} r_z r_x , && (4.117) \end{aligned}$$

where

$$\begin{aligned} \psi_s &\equiv \arg(\mu) + \frac{2\pi s}{p_2+p_3} , \quad s \in \{0,1,\ldots,(p_2+p_3)-1\} && (4.118)\\ r_{\min}^{3/2} &\equiv |\mu| , && (4.119) \end{aligned}$$

and

$$\begin{aligned} e_i &\equiv \frac{1}{2}\left(1+\frac{n_2}{y_i}\right) , && (4.120)\\ n_1 &\equiv \frac{p_1}{p_3} , && (4.121)\\ n_2 &\equiv \frac{p_2}{p_3} . && (4.122) \end{aligned}$$

Integration over ϕ and α together with the embedding equation (4.116) dictates the following selection rules for the quantum numbers of the angular eigenfunctions

(4.106),

$$m = -\frac{n_1}{2} Q_R \ , \quad n_\alpha = 3\ell n_2 Q_R \ , \quad n_\psi = \frac{1+n_2}{2} Q_R \, , \tag{4.123}$$

where Q_R is the R-charge defined as $Q_R \equiv 2n_\psi - \frac{1}{3\ell} n_\alpha$. In this case $\alpha_i = e_i \frac{Q_R}{2}$.

Finally, we need the determinant of the induced metric on the four-cycle

$$\mathrm{d}\theta \mathrm{d}x \sqrt{g^{ind}} = \frac{r^4}{z(1-z)x(1-x)(a-x)} \mathcal{G}(x,z) \, \mathrm{d}z \mathrm{d}x \, . \tag{4.124}$$

The function $\mathcal{G}$ is too involved to be written out explicitly here, but is available upon request. It is a polynomial of order 3 in x and of order 2 in z.

Main integral

The main integral (the analog of (4.74)) is therefore given by

$$I_L = \int \frac{\mathrm{d}x \mathrm{d}z \, \mathcal{G}(x,z) \, N_L^2}{z(1-z)x(1-x)(a-x)} \left(\frac{r}{r_{\min}}\right)^{-6Q_L^+} P^{a,b}_{A=l-n_\psi}(1-2z) h_L(x) \, , \tag{4.125}$$

with $a \equiv (1+n_1+n_2)\frac{Q_R}{2}$, $b \equiv (1-n_1+n_2)\frac{Q_R}{2}$ and $6Q_L^+ \equiv 2\lambda + \frac{3}{2}Q_R$. We will calculate this integral for a general $6Q_L^+ = 2w + \frac{3}{2}Q_R$ and then take the limit $w \to \lambda$.

First we compute the integral over z in complete analogy to the singular conifold case of section (4.3). The Jacobi polynomial satisfies

$$r_z^{3Q_R} \frac{d}{dz}\left(r_z^{-3Q_R} z(1-z) \frac{d}{dz} P^{a,b}_A(1-2z)\right) + A(A+1+a+b) P^{a,b}_A(1-2z) = 0 \tag{4.126}$$

Let us multiply this equation by $r_z^{-(2w+\frac{3}{2}Q_R)}$ and integrate over z. It can be shown that there is a third order polynomial $\mathbb{G}(x)$ which is implicitly defined by the following relation

$$\begin{aligned} \frac{\mathcal{G}(x,z)}{z(1-z)} - \mathbb{G}(x) &= \frac{\mathcal{G}(x,z=0)}{(1+n_1+n_2)^2 \left(\frac{w^2}{9^2} - \frac{Q_R^2}{16}\right)} \times \\ &\times \left[r_z^{2w+\frac{3}{2}Q_r} \frac{d}{dz}\left(z(1-z) r_z^{-3Q_R} \frac{d}{dz}\left(r_z^{\frac{3}{2}Q_R - 2w}\right)\right) + A(A+1+a+b)\right] \end{aligned} \tag{4.127}$$

The right-hand side vanishes after multiplying by $r_z^{-6Q_L^+} P_A^{a,b}(1-2z)$ and integrating, and we get

$$I_L = \int \frac{\mathrm{d}x\, \mathbb{G}(x)\, N_L^2}{x(1-x)(a-x)}\, r_x^{-6Q_L^+} h_L(x) \int \mathrm{d}z\, r_z^{-6Q_L^+} P_A^{a,b}(1-2z)\ . \tag{4.128}$$

4.4.2 Non-chiral contributions

To evaluate (4.128) we make use of the differential equation (4.108). We multiply (4.108) by $r_x^{-2w-\frac{3}{2}Q_R}$ and integrate over x. There exists a first order polynomial $M\sqrt{g(x)}$ such that

$$\begin{aligned} & \frac{\mathbb{G}(x)}{x(1-x)(a-x)} - M\sqrt{g(x)} = \\ = \; & \frac{144\,\mathbb{G}(x=0)}{(1-n_2)(3Q_R+4\lambda)(18Q_Rn_2+8\lambda n_2-9Q_R-4\lambda-24)} \times \Big[(\alpha\beta x-k) - \\ & -r_x^{2w+\frac{3}{2}Q_R}\frac{d}{dx}\Big(r_x^{-2w-\frac{3}{2}Q_R}(\gamma(1-x)(a-x)+\delta x(x-a)+\epsilon x(x-1))\Big) \\ & +r_x^{2w+\frac{3}{2}Q_R}\frac{d^2}{dx^2}\Big(x(1-x)(a-x)r_x^{-2w-\frac{3}{2}Q_R}\Big)\Big]\,, \end{aligned} \tag{4.129}$$

where we defined

$$M \equiv \frac{48(\lambda-w)(\lambda+w+2)}{(1+n_2)(16w^2-9Q_R^2)}\ . \tag{4.130}$$

After multiplying by $r_x^{-6Q_L^+}h(x)$ and integrating over x, the right-hand side vanishes and we have

$$I_L = MN_L^2\int \mathrm{d}x\mathrm{d}z\,\sqrt{g(x,z)}\left(\frac{r}{r_{\min}}\right)^{-6Q_L^+} P_A^{a,b}(1-2z)h(x) \tag{4.131}$$

$$= MN_L\int \mathrm{d}z\mathrm{d}x\,\sqrt{g}\left(\frac{r}{r_{\min}}\right)^{-2\lambda} JR\,. \tag{4.132}$$

Since $\lim_{w\to\lambda} M = 0$, this immediately implies that $\lim_{w\to\lambda} I_L = 0$ 'on-shell', *i.e.* for all operators except for the chiral ones. Just as for the singular conifold case, we have therefore proven that non-chiral terms do not contribute to the perturbation to the warped four-cycle volume.

4.4.3 Chiral contributions

For the chiral operators one finds

$$\lambda = \frac{3}{4} Q_R \tag{4.133}$$

and both the numerator and the denominator of M (4.130) vanish. Chiral operators also require $A = l - n_\psi$ to be equal to zero. Taking the chiral limit we therefore find

$$\begin{aligned} I_L &= \frac{(3Q_R + 4)}{(1+n_2)Q_R} N_L^2 \int \frac{\mathrm{d}x\, q(2p + 3q + \sqrt{4p^2 - 3q^2} - 6qx)}{24p^2} \left(\frac{r}{r_{\min}}\right)^{-3Q_R} && (4.134)\\ &= \frac{(3Q_R + 4)}{(1+n_2)Q_R} \frac{1}{(2\pi)^3 \ell}\,, && (4.135)\end{aligned}$$

since $A = 0$ implies $P_A^{a,b}(1-2z) = 1$ and $h(x) = 1$. The integral in (4.134) reduces to the normalization condition (4.110). Finally, we use the identity for chiral wave-functions $r^{\frac{3}{2}Q_R} Y_L(\Psi) = (w_1^{n_1} w_2^{n_2} w_3)^{\frac{Q_R}{2}}$ and the relation between $T_3(\delta V^w_{\Sigma_4})_{\text{chiral}}$ and I_L (an analog of (4.73)). Note that the $(2\pi)^3$ in (4.73) should be changed to $(2\pi)^3 \ell$ as α runs from 0 to $2\pi\ell$. We hence arrive at the analog of (4.89)

$$T_3(\delta V^w_{\Sigma_4})_{\text{chiral}} = \frac{1}{2} \sum_{Q_R, s} \frac{2}{(1+n_2)Q_R} (\bar{w}_1^{n_1} \bar{w}_2^{n_2} \bar{w}_3)^{\frac{Q_R}{2}}\, e^{i \frac{(1+n_2)}{2} Q_R \psi_s}\,, \tag{4.136}$$

where we recall that $\psi_s = \frac{2\pi s}{p_2 + p_3}$. The summation over s effectively picks out $n_\psi = \frac{(1+n_2)}{2} Q_R$ to be of the form $(p_2 + p_3)s'$ with natural s', or $Q_R = 2p_3 s'$. After summation over s' we have

$$T_3(\delta V^w_{\Sigma_4})_{\text{chiral}} = -\frac{1}{2} \log \left[\frac{\bar{\mu}^{2p_3} - \prod_i \bar{w}_i^{p_i}}{\bar{\mu}^{2p_3}}\right]. \tag{4.137}$$

A similar calculation for the anti-chiral contributions gives the complex conjugate of (4.137).

Result for the cones over $Y^{p,q}$

The final result for the perturbation of the warped volume of four-cycles in cones over $Y^{p,q}$ manifolds is then

$$T_3\, \delta V^w_{\Sigma_4} = -\text{Re}\left(\log \left[\frac{\mu^{2p_3} - \prod_i w_i^{p_i}}{\mu^{2p_3}}\right]\right), \tag{4.138}$$

so that

$$A(w_i) = A_0 \left(\frac{\mu^{2p_3} - \prod_i w_i^{p_i}}{\mu^{2p_3}} \right)^{1/N_7} . \qquad (4.139)$$

Chapter 5

Applications to cosmology

In this chapter we study a particular string theoretical model of inflation based on a D3-brane moving along the warped throat in the presence of a stack of D7-branes. Our analysis exploits the nonperturbative superpotential derived in chapter 4 and follows papers [77, 69], written in collaboration with D. Baumann, I. R. Klebanov, L. McAllister, and P. Steinhardt.

5.1 Model of D-brane inflation

Inflation and string theory

In this chapter we discuss a particular model of stringy inflation based on the dynamics of a mobile D3-brane. Our interest in this topic is due to the growing role of cosmology and physics of the early Universe in contemporary high energy physics. Observational cosmology provides us with a whole new set of experimental observations related to the physics of the early Universe. This data can serve as a restrictive test of any proposed fundamental theory or model. Recent studies of the CMB spectra have solidified inflation as a successful scenario of the early Universe [78]. More precise observations have sharpened the set of allowed parameters excluding many inflation models. This is why we need string theoretic models of inflation capable of matching the experimental data.

One of the main problems in building such a model is that stringy models of the early Universe are often implicit. They are formulated in the effective field theory and do not bear any predictive power. Our objective is to consider a model of inflation originating in string theory with as much rigor as possible at the moment.

A model: mobile D3 on the throat

Our model is a development of the setup proposed in [18]. We consider a warped throat compactification based on the KS solution of section (2.2). The moduli of the compactficication are fixed by the flux-induced superpotential, and the Kähler modulus is fixed because of the nonperturbative gluino condensation due to the stack of n D7-branes. This is essentially the same setup we discussed in section (4.1). The mobile D3 located in the throat is moving toward the tip causing inflation and the location of D3 plays the role of the inflaton field(s). Our aim is to calculate the effective potential for the D3 and check if it is suitable for supporting inflation.

As was discussed in chapter 4, the full potential must have at least two terms. The first is the F-term (4.4)

$$V_F = e^{\kappa^2 \mathcal{K}}\Big[D_\Sigma W \mathcal{K}^{\Sigma\overline{\Omega}}\overline{D_\Omega W} - 3\kappa^2 W\overline{W}\Big]\,, \qquad \kappa^2 = M_P^{-2} \equiv 8\pi G\,, \tag{5.1}$$

with the superpotential

$$W = W_0 + A(z_\alpha)e^{-a\rho}\;, \qquad a = \frac{2\pi}{n}\;, \tag{5.2}$$

given by (4.3) and (4.19). Here we introduced three complex coordinates z_α to parameterize the throat. The F-term potential (5.1) includes an inverse metric $\mathcal{K}^{\Sigma\overline{\Omega}}$ on the moduli space (ρ, z_α). It is obtained from the Kähler potential [79]

$$\kappa^2\mathcal{K}(\rho,\overline{\rho},z_\alpha,\overline{z_\alpha}) = -3\log[\rho+\bar{\rho}-\gamma k(z_\alpha,\overline{z_\alpha})] \equiv -3\log U\,, \tag{5.3}$$

with the constant $\gamma \equiv \frac{\sigma_0}{3}\frac{T_3}{M_P^2}$ being related to the stabilized vacuum value of the Kähler modulus with the D3-brane sitting at the tip [69] $2\sigma_0 \equiv \rho_*(0) + \bar{\rho}^*(0)$.

The Kähler potential on the throat $k(z_\alpha, \bar{z}_\alpha)$ was discussed in case of the conifold geometry in chapter 2.

As follows from a simple analysis [61] outlined in section (4.1) the vacuum value of V_F is negative. To obtain positive or zero cosmological constant this potential must be uplifted. We follow [18] and consider an anti-D3 brane placed at the tip of the conifold producing the potential due to the Coulomb interaction with the D3-brane [18]

$$V_D(\rho, r) = \frac{D(r)}{U^2(\rho, r)}\,, \qquad D(r) \equiv D\left[1 - \frac{3D}{16\pi^2}\frac{1}{(T_3 r^2)^2}\right] \approx D\,. \tag{5.4}$$

Far away from the tip, the correction $1/r^4$ in $D(r)$ is small and in many cases $D(r)$ can be approximated by a constant D.

We are focused on energies much lower than the Planck scale but well above the scale of physics at the bottom of the throat estimated to be 10^{13-14}GeV[18]. This means that the D3 is located sufficiently far from the bulk of Calabi-Yau and at the same time not very close to the tip of the throat. This choice is dictated by both the experimental data suggesting inflation below the Planck scale and our desire to construct a controllable model. As was outlined in section (4.1) we also assume that the D7 embedding preserves SUSY and that the D7 stretches sufficiently far inside the throat. The first requirement is reminiscent of the condition that the D7 is fixed. The latter assures that the nonperturbative potential is not very small and capable of Kähler moduli stabilization. We also assume that the D7 does not fall to the tip which may be the case after throat is compactified. Similarly we neglect possible interaction between the anti-D3 and the D7 assuming it is small enough because of their large separation. This is a subtle point as the anti-D3 breaks SUSY and may therefore influence the D7. These questions definitely merit further study which we leave for the future.

Since we are working at energies much larger than the field theory scale, neither the D3 nor the D7 feels the deformation of the cone ε (2.61). Therefore in our calculations below we assume the geometry is that of the KT solution from section (2.2.2) [33] i.e. the throat is a warped singular conifold.

5.2 D3-brane potential in presence of D7 and anti-D3 branes

5.2.1 Calculation of the potential

F-term potential in homogenous coordinates

Our first task is to calculate the potential (5.1) explicitly. For that reason it is convenient to write the F-term potential (5.1) in terms of the four homogeneous coordinates z_i of the embedding space $\mathbb{C}^4$ which makes the action of $SO(4)$ symmetry transparent. For that reason we define a new metric $\widehat{\mathcal{K}}^{A\overline{B}}$ which depends on z_i in such a way that for any function $W(z_i)$ the following identity is satisfied

$$D_A W \widehat{\mathcal{K}}^{A\overline{B}} \overline{D_B W} \equiv D_\Sigma W \mathcal{K}^{\Sigma\overline{\Omega}} \overline{D_\Omega W} \,, \tag{5.5}$$

where $\{Z^A\} \equiv \{\rho, z_i; i = 1,2,3,4\}$ and $\{Z^\Sigma\} \equiv \{\rho, z_\alpha; \alpha = 1,2,3\}$. In this equation the conifold constraint, $z_4^2 = z_4^2(z_\alpha) = -\sum_{\alpha=1}^{3} z_\alpha^2$, is substituted *after* differentiation on the left-hand side and *before* differentiation on the right-hand side. The metric $\widehat{\mathcal{K}}^{A\overline{B}}(z_i)$ defined through (5.5) is not unique and the choice of one over another is a matter of convenience. We construct $\widehat{\mathcal{K}}^{A\overline{B}}$ with the help of the auxiliary matrix ${J^A}_\Sigma$

$$\widehat{\mathcal{K}}^{A\overline{B}} = {J^A}_\Sigma \, \mathcal{K}^{\Sigma\overline{\Omega}} \, {J^{\overline{B}}}_{\overline{\Omega}} \,, \tag{5.6}$$

where ${J^A}_\Sigma$ is defined as follows

$$D_\Sigma W = \frac{\partial Z^A}{\partial Z^\Sigma} D_A W \equiv {J^A}_\Sigma \, D_A W \,, \qquad {J^A}_\Sigma = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & \delta_{i\alpha} \\ 0 & \frac{-z_\alpha}{\sqrt{-\sum_{\gamma=1}^{3} z_\gamma^2}} \end{array} \right) . \tag{5.7}$$

This gives $\widehat{\mathcal{K}}^{A\overline{B}}$ as a function of z_α. To find it as a function of z_i guess a $\widehat{\mathcal{K}}^{A\overline{B}}(z_i)$ such that it reduces to $\widehat{\mathcal{K}}^{A\overline{B}}(z_\alpha)$ after substituting the conifold constraint. This step and hence $\widehat{\mathcal{K}}^{A\overline{B}}(z_i)$ is not unique. Nevertheless finding an $SO(4)$–invariant $\widehat{\mathcal{K}}^{A\overline{B}}(z_i)$ is not difficult, *e.g.* replacing $\left(-\sum_{\gamma=1}^{3} z_\gamma^2\right)^{1/2}$ by z_4 everywhere in $\widehat{\mathcal{K}}^{A\overline{B}}(z_\alpha)$ and $J^A{}_\Sigma$ we find

$$\widehat{\mathcal{K}}^{A\overline{B}} = \frac{\kappa^2 U}{3}\left(\begin{array}{c|c} U+\gamma k_l \hat{k}^{l\overline{m}} k_{\overline{m}} & k_l \hat{k}^{l\overline{\jmath}} \\ \hline \hat{k}^{i\overline{m}} k_{\overline{m}} & \frac{1}{\gamma}\hat{k}^{i\overline{\jmath}} \end{array}\right), \tag{5.8}$$

where

$$k_i = \frac{\overline{z_i}}{r}, \tag{5.9}$$

and

$$\hat{k}^{i\overline{\jmath}} = J^i{}_\alpha\, k^{\alpha\overline{\beta}}\, J^{\overline{\jmath}}{}_{\overline{\beta}} = r\left[\delta^{i\overline{\jmath}} + \frac{1}{2}\frac{z_i\overline{z_j}}{r^3} - \frac{\overline{z_i}z_j}{r^3}\right]. \tag{5.10}$$

Notice that $\hat{k}^{i\overline{\jmath}}$ is *not* the inverse of $k_{i\overline{\jmath}} = \frac{1}{r}\left[\delta_{i\overline{\jmath}} - \frac{1}{3}\frac{\overline{z_i}z_j}{r^3}\right]$, which is $k^{i\overline{\jmath}} = r\left[\delta^{i\overline{\jmath}} + \frac{1}{2}\frac{z_i\overline{z_j}}{r^3}\right]$. From (5.9) and (5.10) one then finds

$$k_l\hat{k}^{l\overline{\jmath}} = \frac{3}{2}\overline{z^j}, \qquad k_l\hat{k}^{l\overline{m}}k_{\overline{m}} = \frac{3}{2}r^2 = \hat{r}^2 = k, \tag{5.11}$$

and hence,

$$\widehat{\mathcal{K}}^{A\overline{B}} = \frac{\kappa^2 U}{3}\left(\begin{array}{c|c} \rho+\overline{\rho} & \frac{3}{2}\overline{z_j} \\ \hline \frac{3}{2}z_i & \frac{r}{\gamma}\left[\delta^{i\overline{\jmath}} + \frac{1}{2}\frac{z_i\overline{z_j}}{r^3} - \frac{\overline{z_i}z_j}{r^3}\right] \end{array}\right). \tag{5.12}$$

Using the above results we arrive at the F-term potential

$$V_F = \frac{\kappa^2}{3U^2}\left[(\rho+\overline{\rho})|W_{,\rho}|^2 - 3(\overline{W}W_{,\rho} + c.c.) + \frac{3}{2}(\overline{W_{,\rho}}z^i W_{,i} + c.c.) + \frac{1}{\gamma}\hat{k}^{i\overline{\jmath}}W_{,i}\overline{W_{,j}}\right] \tag{5.13}$$

The result (5.13) is essential for the our analysis. It can be rewritten in terms of the w_i-coordinates is

$$V_F = \frac{\kappa^2}{3U^2}\left[(\rho+\overline{\rho})|W_{,\rho}|^2 - 3(\overline{W}W_{,\rho} + c.c.) + \frac{3}{2}(\overline{W_{,\rho}}w^i W_{,i} + c.c.) + \frac{1}{\gamma}\hat{k}_w^{i\overline{\jmath}}W_{,i}\overline{W_{,j}}\right] \tag{5.14}$$

where

$$\hat{k}_w^{i\bar{j}} = r\left[\delta^{i\bar{j}} + \frac{1}{2}\frac{w_i\overline{w_j}}{r^3} - \frac{c_i^{i'}c_j^{j'}\overline{w_{i'}}w_{j'}}{r^3}\right] . \tag{5.15}$$

The matrix $c_j^{i'}$ has only four non-zero elements $c_2^1 = c_1^2 = 1$ and $c_4^3 = c_3^4 = -1$.

The last two terms in the parentheses in (5.13) and (5.14) vanish if A, and hence W, is coordinate independent $\partial_{z_i}A = 0$. In this sense the last two terms are the correction ΔV_F to the KKLT result (4.4). Let us also make a remark that the first two terms coincide with the KKLT potential only if $\partial_{z_i}A = 0$.

Integrating out the imaginary part of ρ

The imaginary part of ρ

$$\rho = \sigma + i\tau \ , \tag{5.16}$$

can be integrated explicitly for any superpotential (4.3)

$$W = W_0 + A(z_i)e^{-a\rho} \ . \tag{5.17}$$

Indeed ρ will combine into $\rho + \bar{\rho} = 2\sigma$ everywhere in (5.13,5.14) except the term

$$-3(\bar{W}W_{,\rho} + c.c.) = 6a|A|^2e^{-2\sigma} + 6ae^{-2\sigma}\underline{\underline{\mathrm{Re}(W_0\bar{A}e^{-ia\tau})}} \ . \tag{5.18}$$

The second term, the only one with τ dependence, can be easily minimized with respect to τ

$$-6a|W_0||A|e^{-2a\sigma} \ . \tag{5.19}$$

From now on we assume that the imaginary part of ρ is integrated out in all expressions for V_F. The real part, σ, is more difficult to integrate out. There is no analytical expression for $\sigma(z_i, r)$ and minimization with respect to σ should be done numerically. Some times it is also convenient to find an approximate expression for $\sigma(r)$ along a radial trajectory in expanding in r and $\frac{1}{a\sigma_F}$, where σ_F is the KKLT stabilized value with $A = A(z = 0)$ [69].

Uplifting the potential

The full potential governing the motion of the D3-brane is the F-term (5.1) and the Coulomb interaction between the D3 and anti-D3-brane (5.4) $V = V_D + V_F$. The relative magnitude of these terms is a parameter of the model which may be fine-tuned. It is convenient to introduce the variable

$$s \equiv \frac{V_D(0,\sigma_F)}{|V_F(0,\sigma_F)|} , \tag{5.20}$$

which is a ratio of the D-term and the F-term *before* uplifting. A non-zero vacuum value of the potential requires $1 < s \lesssim \mathcal{O}(3)$. Since the minimum with respect to σ changes after nontrivial V_D is introduced we need once again solve the equation

$$\left.\frac{\partial V}{\partial \sigma}\right|_{\sigma_0} = 0 . \tag{5.21}$$

Expanding in the small parameter $\frac{1}{a\sigma_F}$ we find the new minimum for D3 sitting at the tip $r = 0$

$$\sigma_0 = \sigma_F + \frac{s}{a^2\sigma_F} + \mathcal{O}\left(\frac{1}{a^2\sigma_F^2}\right) . \tag{5.22}$$

The difference is indeed quite small as $a\sigma_F$ is of order 10.

5.2.2 Choosing a trajectory

Our next step is to consider the two simplest supersymmetric embeddings of D7 due to Kuperstein [80] and Ouyang [81] and choose the one most suitable for creating flat inflaton potential. Our choice above is not only a matter of simplicity. The embeddings in question are linear in the homogenous coordinates. It can be shown that higher degree embeddings, like the ACR embeddings of [70], lead to a higher power of the leading term in the Taylor expansion in the inflaton of the effecive F-term potential [69]. Therefore the embedding of lowest degree is the most natural candidate to cancel the unwanted mass term from V_D.

Our logic below is the following. For each embedding we first investigate the set of extremal radial trajectories i.e. the trajectories along the radius with fixed position on $T^{1,1}$. We require that the potential is extremal under angular perturbations. This property does not guarantee that the motion is *stable* under a perturbation of the angular coordinates. After identifying the set of extremal radial trajectories for Kuperstein and Ouyang embeddings we study the stability of motion along these trajectories. As a result we identify the particular trajectory $z_1 = -\frac{r^{3/2}}{\sqrt{2}}$ for Kuperstein embedding as the most promising scenario. It is studied in detail in the next section.

Kuperstein embedding

We start our consideration with an embedding $z_1 = \mu$ suggested by Kuperstein in [80]. In this case the superpotential (5.2) is given by (4.97)

$$W = W_0 + A_0 \left(1 - \frac{z_1}{\mu}\right)^{1/n} . \tag{5.23}$$

From this it follows that the potential $V = V_F + V_D$ depends on ρ, r and z_1 in the combinations $z_1 + \bar{z}_1$ and $|z_1|^2$. For the potential to be extremal under the perturbation of angular variables Ψ_i for all radii r, the variations

$$\frac{\partial |z_1|^2}{\partial \Psi_i} = \frac{\partial (z_1 + \overline{z_1})}{\partial \Psi_i} = 0 \tag{5.24}$$

should vanish. We examine (5.24) by introducing local coordinates in the vicinity of a fiducial point $\mathbf{z}_0 \equiv (z_1', z_2', z_3', z_4')$. The coordinates around this point are given by the five generators of $SO(4)$ acting nontrivially on $\mathbf{z}_0$

$$\mathbf{z}(r, \Psi_i) = \exp(\mathsf{T})\, \mathbf{z}_0 \,. \tag{5.25}$$

The Kuperstein embedding, $z_1 = \mu$, breaks the global $SO(4)$ symmetry of the conifold down to $SO(3)$, and the D3-brane potential preserves this $SO(3)$ symmetry. We will find that the actual trajectory breaks this $SO(3)$ down to $SO(2)$, which we take to act on z_3 and z_4. The coordinates that make this $SO(2)$ stability group manifest are given by

$$\mathsf{T} \equiv \left(\begin{array}{cc|cc} 0 & \alpha_2 & \alpha_3 & \alpha_4 \\ -\alpha_2 & 0 & \beta_3 & \beta_4 \\ \hline -\alpha_3 & -\beta_3 & 0 & 0 \\ -\alpha_4 & -\beta_4 & 0 & 0 \end{array}\right), \tag{5.26}$$

where $\Psi_i \equiv \{\alpha_i, \beta_i\} \in \mathbb{R}$ are the local coordinates of the base of the cone. We aim to find $\mathbf{z}_0$ such that the potential $V(z_1 + \overline{z_1}, |z_1|^2)$ is extremal along $\mathbf{z}_0$. We here find trajectories along which the linear variation of $z_1 + \overline{z_1}$ and $|z_1|^2$ vanishes. First, we observe from (5.25) and (5.26) that for arbitrary $\mathbf{z}_0$ we have

$$\delta z_1 = \sum_{i=2}^{4} \alpha_i z_i' \,, \qquad \alpha_i \in \mathbb{R} \,. \tag{5.27}$$

and, hence,

$$\delta |z_1|^2 = \sum_{i=2}^{4} \alpha_i (z_i' \bar{z}_1' + z_1' \bar{z}_i') \equiv 0 \,. \tag{5.28}$$

To satisfy (5.28) for all α_i one requires

$$z_i' = i\rho_i z_1' \,, \qquad \rho_i \in \mathbb{R} \,. \tag{5.29}$$

We may use $SO(3)$ to set $\rho_3 = \rho_4 = 0$, while keeping ρ_2 finite. The conifold constraint, $z_1^2 + z_2^2 = 0$, then implies $\rho_2 = \pm 1$, while the requirement

$$\delta(z_1' + \bar{z}_1') = a_2(z_2' + \bar{z}_2') = 0 \,, \tag{5.30}$$

makes z_2' purely imaginary and z_1' real. This proves that the following are the extremal trajectories of the brane potential for the Kuperstein potential

$$z_1' = \pm \frac{1}{\sqrt{2}} r^{3/2} \,, \qquad z_2' = \pm i z_1' \,. \tag{5.31}$$

Ouyang embedding

For the Ouyang embedding, $w_1 = \mu$, the superpotential (5.2) is

$$W = W_0 + A_0 \left(1 - \frac{w_1}{\mu}\right)^{1/n} , \tag{5.32}$$

and the brane potential depends on $w_1 + \overline{w_1}$, $|w_1|^2$ and $|w_2|^2$. The latter comes from $\hat{k}_w^{1\bar{1}}$ of (5.15). To find extremal trajectories of the potential we therefore require

$$\frac{\partial |w_1|^2}{\partial \Psi_i} = \frac{\partial |w_2|^2}{\partial \Psi_i} = \frac{\partial (w_1 + \overline{w_1})}{\partial \Psi_i} = 0 \,. \tag{5.33}$$

We introduce local coordinates by applying generators of $SU(2)$ to the generic point W_0

$$W = e^{i\mathsf{T}_1} W_0 e^{-i\mathsf{T}_2} \,, \qquad W_0 \equiv \begin{pmatrix} w_3' & w_2' \\ w_1' & w_4' \end{pmatrix} , \tag{5.34}$$

where

$$\mathsf{T}_i \equiv \begin{pmatrix} \alpha_i & \beta_i + i\gamma_i \\ \beta_i - i\gamma_i & -\alpha_i \end{pmatrix} . \tag{5.35}$$

This implies

$$\delta w_1 = -i(\alpha_1 + \alpha_2) w_1' + (-\beta_1 + i\gamma_1) w_3' + (\beta_2 - i\gamma_2) w_4' + \cdots \tag{5.36}$$

and $\delta(w_1 + \bar{w}_1) = 0$ gives $w_1' \in \mathbb{R}$, $w_3' = w_4' = 0$. We find that $\delta |w_1|^2 = 0$ and $\delta |w_2|^2 = 0$ if $w_2' \in \mathbb{R}$. The conifold constraint $w_1' w_2' = 0$ then restricts the solution to the following two options:

$$w_1' = 0 \,, \qquad |w_2'| = r^{3/2} \,, \qquad \Leftrightarrow \qquad \theta_1 = \theta_2 = 0 \,, \tag{5.37}$$

or

$$w_1' = \pm r^{3/2} \,, \qquad w_2' = 0 \,, \qquad \Leftrightarrow \qquad \theta_1 = \theta_2 = \pi \,. \tag{5.38}$$

For the trajectory $w'_1 = 0$, $|w_2|^2 = r^3$ the contribution of all terms with the derivatives of A vanish. Therefore the effective potential along such a trajectory is that of KKLT. It is called a *delta-flat* potential [77, 69]. The inflation along this trajectory is impossible, as it was in the KKLMMT case [18]. This result for the trajectory in question was first obtained in [82].

For the trajectories $w'_1 = \pm r^{3/2}$ the correction to the KKLT potential ΔV_F does not vanish. In fact, the effective potential along these trajectories is identical to the potential for the Kuperstein case (5.31). To see that one need to express the potential in both cases through the radius r and σ, $V = V(r, \sigma)$, and the coefficient

$$c \equiv \frac{1}{4\pi\gamma r_\mu^2} \ . \tag{5.39}$$

The relation between r_μ and μ depends on the embedding. We define r_μ as the value of radius $r = r_\mu$ when the D3 moving along the extremal trajectory (with appropriate sign) hits the D7. For the Kuperstein embedding this implies the definition $r_\mu^3 = 2\mu^2$ when for Ouyang embedding $r_\mu^3 = \mu^2$. This definition of r_μ is not only natural from geometrical point of view. In fact this definition is suggested by a normalization of the kinetic term of the inflaton field [69]. Therefore the profile of the potentials coincide if expressed in physical units. This is already enough to conclude that the Ouyang embedding has no advantage over the Kuperstein one. In fact the Ouyang embedding has a disadvantage because the motion along the $w'_1 = \pm r^{3/2}$ trajectory is unstable for small r.

Stability for small r

To investigate the stability of the trajectories (5.31) and (5.38) it is convenient to write the "correction" to the KKLT potential explicitly for both cases. For the Kuperstein embedding

$$\Delta V_F = \frac{\kappa^2}{3U^2}\left[\frac{3}{2}(\overline{W_{,\rho}} z_1 W_{,z_1} + c.c.) + \frac{1}{\gamma}\hat{k}^{1\bar{1}} W_{,z_1}\overline{W_{,z_1}}\right] , \tag{5.40}$$

where

$$\hat{k}^{1\bar{1}} = r\left(1 - \frac{1}{2}\frac{|z_1|^2}{r^3}\right), \tag{5.41}$$

and for the Ouyang embedding

$$\Delta V_F = \frac{\kappa^2}{3U^2}\left[\frac{3}{2}(\overline{W_{,\rho}} w_1 W_{,w_1} + c.c.) + \frac{1}{\gamma}\hat{k}_w^{1\bar{1}} W_{,w_1}\overline{W_{,w_1}}\right], \tag{5.42}$$

with

$$\hat{k}_w^{1\bar{1}} = r\left(1 + \frac{|w_1|^2}{2r^3} - \frac{|w_2|^2}{r^3}\right). \tag{5.43}$$

In both cases, stability near the tip $r \to 0$ is controlled by the term with $\hat{k}^{1\bar{1}}$ ($\hat{k}_w^{1\bar{1}}$). To see that we consider the Kuperstein embedding. The term with $\hat{k}^{1\bar{1}}$ contains r^{-3}, and its contribution to the second derivative of the potential with respect to an angular variable Ψ_i, $\frac{\partial^2 V}{\partial \Psi_i^2}$, grows as r. All other terms grow at least as $r^{3/2}$ (this follows from $\frac{\partial}{\partial \Psi} = \frac{\partial z_i}{\partial \Psi}\frac{\partial}{\partial z_i} + c.c.$ and $\frac{\partial z_i}{\partial \Psi} \sim r^{3/2}$). A parallel consideration confirms that $\hat{k}_w^{1\bar{1}}$ is responsible for the leading contribution to the stability analysis in the case of the Ouyang embedding as well.

Now, the trajectories $z_1 = \pm\frac{r^{3/2}}{\sqrt{2}}$ maximize $|z_1|^2$ for a given r, and any variation of angles may only increase $k^{1\bar{1}} = r\left(1 - \frac{|z_1|^2}{2r^3}\right)$. Hence the trajectories in question are stable at small r under fluctuations of any angles that affect $|z_1|^2$. So far, this analysis does not include the phase of z_1, which of course leaves $|z_1|^2$ invariant. The leading correction to the potential from fluctuations of this phase comes not through $k^{1\bar{1}}$ but through terms in V proportional to $z_1 + \overline{z_1}$. These terms change sign when z_1 does; thus, one of the signs in $z_1 = \pm\frac{r^{3/2}}{\sqrt{2}}$ corresponds to the stable trajectory, while the other sign corresponds to an unstable trajectory. It can be shown that if the shift of stabilized value of σ (5.22) is taken into account, the potential is stable for negative $z_1 = -r^{3/2}/\sqrt{2}$ [69].

The analysis for the Ouyang embedding is very similar. The delta-flat trajectory $|w_2|^2 = r^3$ ($\theta_1 = \theta_2 = 0$) maximizes the ratio $\frac{|w_2|^2}{r^3}$. Thus, any angular fluctuation can only decrease the ratio $\frac{|w_2|^2}{r^3}$, without affecting $\frac{|w_1|^2}{r^3}$ to second order. This is easily checked with the help of the angular coordinates θ_i (2.23). On the other hand, the trajectory $w_1 = \pm r^{3/2}$ ($\theta_1 = \theta_2 = \pi$) maximizes $\frac{|w_1|^2}{r^3}$, and angular fluctuations away from this trajectory decrease the ratio $\frac{|w_1|^2}{r^3}$, without affecting $\frac{|w_2|^2}{r^3}$ to second order. As a result, $\hat{k}_w^{1\bar{1}} = r\left(1 + \frac{|w_1|^2}{2r^3} - \frac{|w_2|^2}{r^3}\right)$ cannot decrease in the case of the delta-flat trajectory $|w_2|^2 = r^3$, but necessarily has a negative mode along the non-delta-flat trajectory $w_1 = \pm r^{3/2}$. Hence, the non-delta-flat trajectory is unstable for small r. No further consideration is needed to show that the delta-flat trajectory $|w_2|^2 = r^3$ is stable. Since angular fluctuations around $w_1 = 0$ cannot affect the term involving $w_1 + \overline{w_1}$, the leading contribution always comes from $\hat{k}_w^{1\bar{1}}$.

We have therefore demonstrated that near the tip, the trajectory $z_1 = -\frac{r^{3/2}}{\sqrt{2}}$ is stable for the Kuperstein embedding, whereas the trajectory $w_1 = \pm r^{3/2}$ in the Ouyang embedding is unstable. We investigate the possibility of inflation in the case of Kuperstein embedding in the next section.

5.2.3 Effective potential for Kuperstein embedding

Effective potential

In this section we analyze the potential for the D3-brane in presence of n D7-branes wrapping the cycle $z_1 - \mu = 0$. In the previous section we identified the extremal (and in fact stable) under angular fluctuations radial trajectory $z_1 = -r^{3/2}/\sqrt{2}$. The effective potential along this trajectory expressed through the canonically normalized inflaton field $\phi \propto r$ is

$$\begin{aligned}\mathbb{V}(\phi) \;=\; & \frac{\kappa^2 a|A_0|^2}{3}\frac{e^{-2a\sigma}}{U^2(\phi)}g(\phi)^{2/n}\left[2a\sigma + 6 - \; 6e^{a\sigma}\frac{|W_0|}{|A_0|}\frac{1}{g(\phi)^{1/n}}\right. \\ & \left. + \frac{3c}{n}\frac{\phi}{\phi_\mu}\frac{1}{g(\phi)^2} - \frac{3}{n}\frac{1}{g(\phi)}\frac{\phi^{3/2}}{\phi_\mu^{3/2}}\right] + \frac{D(\phi)}{U^2(\phi)}\,. \end{aligned} \tag{5.44}$$

Here we have introduced notations to make (5.44) concise. Thus ϕ_μ is the reminiscent of the minimal radial coordinate of D7 $\phi_\mu^2 \propto (2\mu^2)^{2/3}$ and g is $A(z)/A(0)$,

$g(\phi) \equiv 1 + \left(\frac{\phi}{\phi_\mu}\right)^{3/2}$.

The normalization procedure for the inflaton $\phi = T \propto r$ should be clarified. The Kähler modulus σ changes insignificantly when r does. Although this change should be taken into account when one calculates the effective potential, it is negligible in the kinetic term. Therefore the normalization of the inflaton field ϕ comes from the kinetic term of r only. The latter follows from the Kähler potential (5.3). Calculating the kinetic term at the tip where $\sigma = \sigma_0$ (5.22) and requiring that it is equal to $\dot{\phi}^2/M_{Planck}^2$, we find that

$$\gamma k(z_\alpha, \bar{z}_\alpha) r^{-2} = \frac{3}{2}\gamma = \frac{\sigma_0}{3} \ . \tag{5.45}$$

This leads to the $U(\phi)$ from (5.44)

$$U = 2\sigma - \frac{\sigma_0}{3}\phi^2/M_{Planck}^2 \ . \tag{5.46}$$

The effective potential (5.44) depends on two variables $\phi \propto r$ and σ. The massive Kähler modulus σ should be integrated out by minimizing the potential for a given ϕ

$$\left.\frac{\partial V}{\partial \sigma}\right|_{\sigma=\sigma_*(\phi)} = 0 \ . \tag{5.47}$$

We have mentioned before that $\sigma_*(\phi)$ can not be found in analytical form. Nevertheless one can find it by doing a Taylor expansion in ϕ if the D3 is close enough to the tip.

Inflation at small ϕ ?

The Taylor expansion of $\sigma_*(\phi)$ is helpful for analyzing the possibility of inflation at small ϕ. Let us clarify here that small ϕ stands for the radius r much smaller than the minimal radial coordinate of the D7 $r \ll r_\mu$, but much bigger than $\varepsilon^{2/3}$. In our setup the typical value of M_{D7} specified by r_μ is of order $M_{Planck} = 10^{19}$GeV and the field theory scale $\varepsilon^{2/3}$ is usually taken to be 10^{12-13}GeV.

A straightforward calculation gives at leading order

$$\sigma_*(\phi) = \sigma_0 + \frac{1}{2\pi}\left(\frac{\phi}{\phi_\mu}\right)^{3/2} . \tag{5.48}$$

It is remarkable that the leading term scales as $\phi^{3/2}$ (the term with ϕ^2 has a very small coefficient and thus is negligible). This implies that the correction from (5.44) to the KKLT potential uplifted by the D-term caused by the coordinate dependence of $A(z_\alpha)$ does not include a quadratic term ϕ^2. Therefore the coefficient in front of ϕ^2 is the same as in the KKLMMT case [18]

$$\frac{V(\phi)}{V(0)} = 1 + \frac{1}{3}\phi^2/M^2_{Planck} + v(\phi) . \tag{5.49}$$

This makes inflation for small ϕ impossible. The correction $v(\phi)$ is a polynomial in $\phi^{1/2}$ but does not have a ϕ^2 term. Therefore it can not cancel $\frac{1}{3}\phi^2/M^2_{Planck}$ even with an infinite amount of fine-tuning.

This is an interesting result as we have shown that at least in certain cases the flat potential for a given range of the field cannot be achieved for any set of parameters.

Inflation near inflection point

To investigate the potential away from the tip one needs to integrate $\sigma_*(\phi)$ numerically. For various parameters the profile of the potential drastically changes from monotonic to a potential with a local minimum and maximum. It is possible to fine-tune the parameters such that the potential has an inflection point. Here we give an example of such a set of parameters

$$\begin{aligned} &n = 8 , \quad \phi_\mu = \frac{1}{4} , \quad A_0 = 1 , \\ &D = 1.2 \times 10^{-8} \Leftrightarrow a\sigma_F \approx 10.1 . \end{aligned} \tag{5.50}$$

The sketch of the corresponding potential can be found on the next page, in Fig. 5.1.

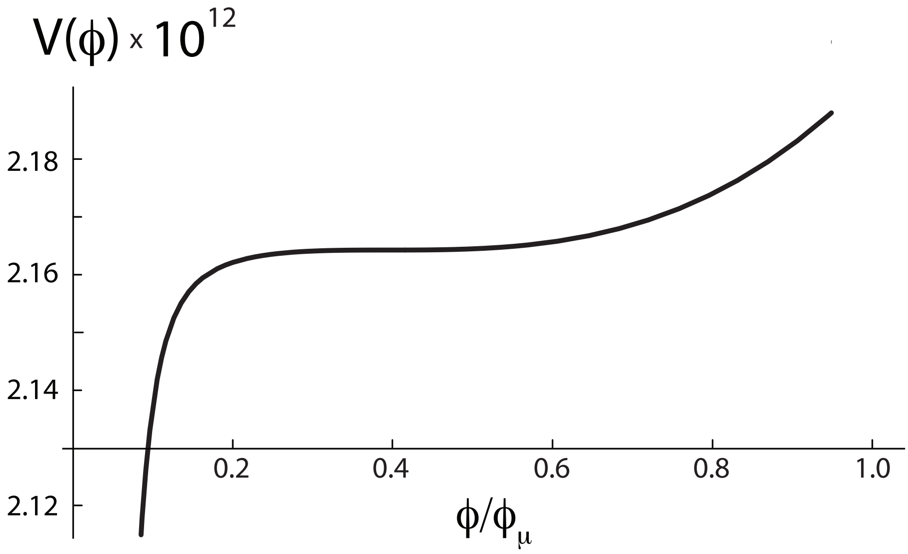

Figure 5.1: Potential with an inflection point $V(\phi)$.

The potential with an inflection point is a promising setup for inflation. If the ratio V'/V calculated at the inflection point is sufficiently small the inflation can cause a large number of e-folds of expansion. The weakness of this scenario is that the cosmological predictions of such a model are highly sensitive to the change of parameters and initial conditions [77, 69].

Chapter 6

Discussion

This thesis has been focused on models of gauge/gravity duality and their applications to cosmology. We have studied the cascading $SU((k+1)M) \times SU(kM)$ theory on the baryonic branch of moduli space and the corresponding family of supergravity backgrounds. Besides the KS solution corresponding to a particular locus of the branch, all backgrounds in question are based on generalized Calabi-Yau manifolds. This is one of the few known examples where such manifolds are constructed explicitly. The metric, fluxes and dilaton are unambiguously determined in terms of two functions $a, v(t)$ which satisfy the first order system

$$\dot{a} = \dot{a}(a, v, t) \ , \quad \dot{v} = \dot{v}(a, v, t) \ , \tag{6.1}$$

and have certain behavior at the boundary $t \to \infty$. It is worth mentioning that this system does not admit a known analytical solution except for the special cases of the KS and MN solutions. For the KS and MN solutions the system (6.1) can be derived from an effective superpotential [25]. The attempts to find such a superpotential for the rest of the baryonic branch have not been successful so far.

One of our main goals was to verify and develop the duality between the family of supergravity solutions above and the corresponding field theory. For that reason we have studied various D-branes dual to the objects in field theory. Although we did not find analytic solutions to the system (6.1), we found that the supergravity analysis confirms the *exact* relations previously derived in field theory. Thus the unbroken SUSY in field theory guarantees that the domain wall separating two isomorphic branches of moduli space is BPS and its tension is moduli-independent. Our consideration of a dual object, a D5-brane wrapping the minimal three-cycle Σ on the conifold, confirms this result. We did not need to know the background functions a, v explicitly to show that the tension T_{D5} is the same for all solutions along the branch. Instead we have used the calibration condition

$$T_{D5} \geq \int_{\Sigma} \tilde{\Omega} \ , \tag{6.2}$$

where $\tilde{\Omega}$ is the closed holomorphic three-form. All the solutions along the branch share the same complex structure but have different metric and fluxes. This is enough to conclude that $\tilde{\Omega}$ is the same for all solutions on the branch, and so is the tension of the BPS D5-branes, saturating the inequality (6.2).

Another interesting example where we were able to confirm an exact gauge theory relation is the baryonic branch constraint

$$\langle \mathcal{A}\mathcal{B} \rangle = \text{const} \ . \tag{6.3}$$

In field theory this relation originates from the quantum constraint $\mathcal{A}\mathcal{B} = -\Lambda_{2M}^{4M}$, which defines the baryonic branch of moduli space. To reproduce this relation on gravity side we have to measure the baryonic condensate. To this end we constructed the object dual to the baryonic operator, a D5 wrapping the base of the conifold at large radius. Then the condensate is given by the value of the DBI action of the Euclidean D5 covering the whole conifold. To preserve supersymmetry, the D5 should be accompanied by D3-branes dissolved in it. This is equivalent to the gauge field induced on the D5's world-volume. The equation for the gauge field cannot be solved analytically, but can be presented in a form resembling (6.1)

$$\dot{\xi} = \dot{\xi}(\xi, a, v, t) \ . \tag{6.4}$$

This equation admits serval solutions corresponding to different baryon operators $\mathcal{A}, \mathcal{B}$. The field theory in question has a $\mathbb{Z}_2$ symmetry $\mathcal{I}$ which exchanges the baryons. This symmetry acts nontrivially on the equations (6.1) and (6.4). It turns out that the DBI action $S_{D5}[\xi]$ calculated on the world-volume stretching to the cut-off radius t can be split into two parts with an exponential precision $S_{D5} = \Delta(t) + \sigma + \mathcal{O}(e^{-2t/3})$. The first part Δ is common to both solutions and diverges. It corresponds to the dimension of the operator and is irrelevant for the calculation of the condensate. The finite part σ, responsible for the condensate, changes its sign under the action of $\mathcal{I}$. Therefore the renormalized value of the action $S^R_{D5} = \pm\sigma(U)$ differs by a sign for the baryons $\mathcal{A}, \mathcal{B}$. Since the expectation value of the baryon operator is proportional to $e^{-S^R_{D5}}$ we immediately find that the relation (6.3) holds exactly along the branch. This construction also provides a connection between the parameter in field theory, $\langle \mathcal{A} \rangle$, and the one on the gravity side U.

There is another "exact" quantity related to the baryon operator. It is the charge of a baryon operator under $U(1)_{baryon}$. Obviously it should be constant along the branch. On the gravity side the charge of the baryon operator is given by the coupling of the D5-brane to the Goldstone boson of the spontaneously broken $U(1)_{baryon}$. This coupling comes from the Chern-Simons term and can be easily calculated, provided the wave-function for the axion is known. Then, according to the logic above, the value of the coupling should be constant along the branch. At the moment the axion wave-function is known only at the KS point ($U = 0$). It will be very interesting to construct the wave function of the axion for all values of U and verify that it couples to the D5-brane with a U-independent coefficient. This is an interesting problem for the future.

Besides being important for the gauge/gravity duality, the warped throat solutions can be successfully applied towards stringy cosmological models. Among the desirable features of these models is that they are calculable because of controllable dynamics in the throat. As an illustration of this idea, we have explicitly calculated the effect on a D3-brane of the nonperturbative superpotential due to gluino condensation on a stack of D7-branes. Although the calculation is quite lengthy the answer is very simple. The latter probably can be anticipated because of its holomorphic structure. All the terms potentially contributing to the answer can be labelled by the corresponding operators in field theory. It turns out that the terms corresponding to the non-chiral operators vanish and only the "chiral" terms contribute. Besides a general understanding that this has to be related to the unbroken SUSY of the D3-D7, the cancellation of the non-chiral terms was proved by direct calculation. The calculation presented in this thesis deals with the conic geometries over $T^{1,1}$ and $Y^{p,q}$ and may be generalized to any conic geometry. Nevertheless it is not clear if this result can be extended to more complicated backgrounds. Perhaps a clear argument why the non-chiral terms should not contribute can also shed some light on the derivation of the non-perturbative superpotential for non-conic backgrounds.

The superpotential discussed in the paragraph above is an essential ingredient of many stringy models of inflation with the location of the D3 playing the role of the inflaton. The superpotential governs the dynamics of the probe D3 when it moves along the throat. The resulting effective potential on the six-dimensional conifold has a complicated shape and admits many different scenarios of D3 motion. We have analyzed the simplest scenario of a radial descent toward the tip and found that the effective potential is generically too steep to support inflation. Although there is freedom in specifying the parameters of the model and the character of the motion, we found that, although it is possible to find inflation in this scenario, it is very difficult to achieve.

Our result is based on some assumptions which are not rigorously justified. One of them is the belief that the flux-induced and nonperturbative dynamics will fix the lowest point of D7, μ, somewhere is the middle of the throat. However it could be that the D7 will either "fall" down to the tip $\mu \to 0$ or be "pulled" to the bulk of Calabi-Yau $\mu \sim M_{Planck}$. In both cases the character of the dynamics will be drastically changed and would have to be studied independently. We hope that these and other related questions will be addressed in the near future.

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