scientific reports

OPEN

Generalization to *d*-dimensions of a fermionic path integral for exact enumeration of polygons on hypercubic lattices

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The generating function for polygons on the square lattice has been known for many decades and is closely related to the path integral formulation of a free fermion model. On the cubic and hypercubic lattices the generating function is still unknown and the problem remains open. It has been conjectured that the three-dimensional (3D) and higher dimensional problems are not solvable—or, to be more precise, that there are no differentiably finite (*D*-finite) solutions. In this context, very recently a Berezin integral of an exponentiated Grassmann action was found for the polygon generating function on the cubic lattice, making explicit the connection between 3D polygons and a model of interacting fermions. Here we address the problem of how to generalize the 3D result to higher dimensions. We derive a Grassmann representation in terms of a Berezin integral for the generating function of polygons on *d*-dimensional hypercubic lattices. On the one hand, this new result admittedly brings us no closer to the problem of finding an explicit analytic expression for the desired generating function for polygons. On the other hand, however, the significant advance reported here precisely quantifies the remarkable mathematical difficulty of finding the explicit generating function. Indeed, the non-quadratic functional form of the Grassmann action that we derive here provides a very clear picture of the formidable mathematical obstruction that would need to be overcome. Specifically, in *d* dimensions, the Grassmann action contains terms of degree $2(d-1)$, so the model describes interacting rather than free fermions. It is an open problem whether or not these models of interacting fermions can in principle be free fermionized through some still undiscovered algebraic method, but it is widely believed that this goal is mathematically impossible.

Understanding the intricate patterns and configurations of objects, such as polygons, polyominoes, and polycubes of fixed size on regular lattices, has long been a topic of interest in statistical mechanics and enumerative combinatorics[1]. Here we focus on multipolygons, defined as connected or disconnected undirected graphs on regular lattices, with each edge linking precisely 2 nearest-neighbor lattice sites, and all nodes having an even degree. Hence, informally, multipolygons are a collection of possibly overlapping polygons that can share nodes but not edges. The generating function for 2D multipolygons on the square lattice can be expressed in terms of Onsager's solution of the 2D Ising model[2]. Hence, very efficient hypergeometric formulas are known for counting 2D multipolygons, via the known expressions for calculating series expansions of the Ising model to arbitrary order[3].

The connection of these combinatorial problems to the physics of fermions was only slowly unveiled, over a period of decades. The subtle incorporation of fermionic principles within the structure of the 2D Ising model was already implicit in the seminal contributions of Onsager[2] and Kaufmann[4], evident in their adept exploitation of quaternion algebra and the manipulation of generators associated with the Pauli spin matrices, respectively. Subsequently, in 1964, Schultz, Mattis, and Lieb rigorously established the connection of the 2D Ising problem to a model of free fermions[5], via fermionic creation and annihilation operators satisfying canonical anticommutation relations and via application of the Jordan-Wigner transformation.

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A key historical development in the field (that, surprisingly, has gone somewhat unrecognized) was the seminal contribution of Felix A. Berezin in his 1969 paper[6] on the exact solution of the 2D Ising model via a path integral over fully anticommuting (Grassmann) variables. Berezin is widely credited with laying the foundation for path integrals of Grassmann variables, now known as Berezin integrals[7]. Grassmann variables and Berezin integrals have important applications in quantum field theory and supersymmetry, leading to deep insights that continue to resonate across various branches of mathematical physics and that even led to an entirely new subfield of mathematics, namely, supermathematics. Notwithstanding Berezin's 1969 paper, it was not until 1980 that the powerful fermionic path integral formulation of the 2D Ising model became more generally known.

Indeed, the Grassmann variable approach to the Ising model became widely disseminated via the seminal trilogy of papers by Samuel[8–10], presenting the fermionic path integral formulation of classical lattice spin models (via a Grassmann action slightly different from the one used by Berezin in Ref.[6] for the 2D case). Following these developments numerous advances have been made[11–23]. However, for a period of more than four decades since the 1980 papers of Samuel, a Grassmann variable approach to counting multipolygons on 3D lattices remained open. Very recently, partial progress was made on this problem whereby a quartic Grassmann action was obtained that correctly counts multipolygons on simple cubic lattices of any finite size[24]. Here we make similar progress in 4 and higher dimensions.

In this article, we address the question of how to generalize the 2D and 3D Grassmann representations to d-dimensional hypercubic lattices. This problem is interesting because (i) it has never before been studied, (ii) the answer should reveal how the (in)tractability of the combinatorial calculations depends on the dimension and (iii) there is a direct and well-known connection to the d-dimensional Ising model, which is considered by many to be one of the most important problems in equilibrium statistical mechanics[25]. Our main result is an expression for the Grassmann action that counts polygons in hypercubic lattices.

In "The Grassmann action for the domain-Wall generating function in d dimensions", we review the Grassmann action for the domain-wall generating function in d dimensions, following Samuel[10]. Subsequently, in "The Grassmann action for the multipolygon generating function in d dimensions", our focus shifts towards deriving the multipolygon generating function. Finally, in "Conclusion" we conclude with a brief discussion of the significance of our results, in the context of the crucial question of whether or not it may be possible to free-fermionize the problem – and if so, the expected complexity of the ensuing mathematical challenge.

The Grassmann action for the domain-Wall generating function in *d* dimensions

In this section, we review the Grassmann action for the generating function for domain walls in d dimensions. The action was written down by Samuel in 1980[10]. In d-dimensions, to each spin we associate q Grassmann variables, where

$$q = 2d(d-1). \tag{1}$$

The quantities $\boldsymbol{\eta}$ represent the Grassmann (or anticommuting) variables, and the superscript (w) denotes the wall term for the component action. Following a notation similar to that of Ref.[10], we first write the Grassmann action for the counting closed hypersurfaces of given surface area as

$$S(\boldsymbol{\eta}, s) = s^2 S^{(w)}_{2(d-1)}(\boldsymbol{\eta}) + S_2(\boldsymbol{\eta}), \tag{2}$$

where $S^{(w)}_{2(d-1)}(\boldsymbol{\eta})$ is the wall-term involving the products of $2(d-1)$ Grassmann variables:

$$\begin{aligned} S^{(w)}_{2(d-1)}(\boldsymbol{\eta}) = \sum_{\boldsymbol{x}} \sum_{\substack{i_1, i_2, \ldots, i_d = 1 \\ \text{cyclic}}} & \eta^{i_2\dagger}\left(\boldsymbol{x} + \boldsymbol{v}_{i_1,i_2}\right) \eta^{i_2}\left(\boldsymbol{x} + \boldsymbol{u}_{i_1,i_2}\right) \eta^{i_3\dagger}\left(\boldsymbol{x} + \boldsymbol{v}_{i_1,i_3}\right) \eta^{i_3}\left(\boldsymbol{x} + \boldsymbol{u}_{i_1,i_3}\right) \cdots \\ \times \qquad & \eta^{i_d\dagger}\left(\boldsymbol{x} + \boldsymbol{v}_{i_1,i_2}\right) \eta^{i_d}\left(\boldsymbol{x} + \boldsymbol{u}_{i_1,i_2}\right). \end{aligned} \tag{3}$$

The variable s plays the role of a Boltzmann weight, whereby each wall hyper-plaquette has weight s^2. Here the vectors $\boldsymbol{u}_{i,j}$ and $\boldsymbol{v}_{i,j}$ are defined as

$$\boldsymbol{u}_{i,j} = \frac{\boldsymbol{e}_i + \boldsymbol{e}_j}{2}, \tag{4}$$

$$\boldsymbol{v}_{i,j} = \frac{\boldsymbol{e}_i - \boldsymbol{e}_j}{2}, \tag{5}$$

with the vectors $\boldsymbol{e}_i$ being the d unitary vectors of the lattice. Note that, unlike the notation used in Ref.[8] for the $2d$ case, with the present notation the point $\boldsymbol{x}$ represents the actual position of the spin on the lattice, whereas points such as $\boldsymbol{x} + \boldsymbol{v}_{i,j}$ or $\boldsymbol{x} + \boldsymbol{u}_{i,j}$ represent points on the dual lattice (delimiting the domain walls). Note that this is also a different notation for the Grassman variables compared to the notation used in Refs.[24,26].

Here, $S_2(\eta)$ is a purely quadratic term composed of the corner and monomer terms according to

$$S_2(\boldsymbol{\eta}) = S_{\text{Corner}}(\boldsymbol{\eta}) + S_{\text{Monomer}}(\boldsymbol{\eta}), \tag{6}$$

where

$$S_{\text{Corner}}(\boldsymbol{\eta}) = \sum_{\boldsymbol{x}} \sum_{\substack{i,j=1 \\ i<j}}^{d} [\eta^{i\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\eta^{j}(\boldsymbol{x}+\boldsymbol{u}_{i,j}) + \eta^{j\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\eta^{i\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j}) \\ \eta^{i}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\eta^{j\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j}) + \eta^{i}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\eta^{j}(\boldsymbol{x}+\boldsymbol{u}_{i,j})], \tag{7}$$

and

$$S_{\text{Monomer}}(\boldsymbol{\eta}) = \sum_{\boldsymbol{x}} \sum_{\substack{i,j=1 \\ i<j}} \left[\eta^{i}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\eta^{i\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j}) + \eta^{j\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\eta^{j\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\right]. \tag{8}$$

Let $\mathrm{d}\boldsymbol{\eta}^{\dagger}\mathrm{d}\boldsymbol{\eta}$ be shorthand for the product of all the qN Grassmann variables, N being the number of spins:

$$\mathrm{d}\boldsymbol{\eta}^{\dagger}\mathrm{d}\boldsymbol{\eta} = \prod_{\boldsymbol{x}} \prod_{i<j} \mathrm{d}\eta^{i\dagger}(\boldsymbol{x}+\boldsymbol{u}_{i,j})\mathrm{d}\eta^{i}(\boldsymbol{x}+\boldsymbol{u}_{i,j}). \tag{9}$$

Then the generating function for the domain walls is given by

$$\Xi_N(s) = \int \mathrm{d}\boldsymbol{\eta}^{\dagger}\mathrm{d}\boldsymbol{\eta}\ \exp\left[S(\boldsymbol{\eta}, s)\right]. \tag{10}$$

The action $S(\boldsymbol{\eta}, s)$ in Eq. (2) contains products of $2(d-1)$ Grassmann variables so that, for $d \geq 3$, it is not quadratic. As a consequence, for $d \geq 3$, Eq. (10) does not correspond to a model of free fermions, but rather to a model of interacting fermions. See Fig. 1 for the arrangement of the Grassmann variables for the 3D case.

All the results above are well known and none are new or controversial. They are, however, the starting point for further calculations (see below) that lead to new results.

The Grassmann action for the multipolygon generating function in *d* dimensions

In this section, we present our main new result, namely, a Grassmann action that can enumerate polygons on hypercubic lattices in any dimension. Specifically, we derive a fermionic path-integral representation of the generating function for multipolygons in d dimensions.

For convenience, we define a new variable t related to s as follows:

$$t = \frac{1-s^2}{1+s^2} \iff s = \sqrt{\frac{1-t}{1+t}}. \tag{11}$$

This pair of variables corresponds to the low- and high-temperature variables for series expansions of the partition function of the Ising model, but for our purposes they are simply variables for certain generating functions.

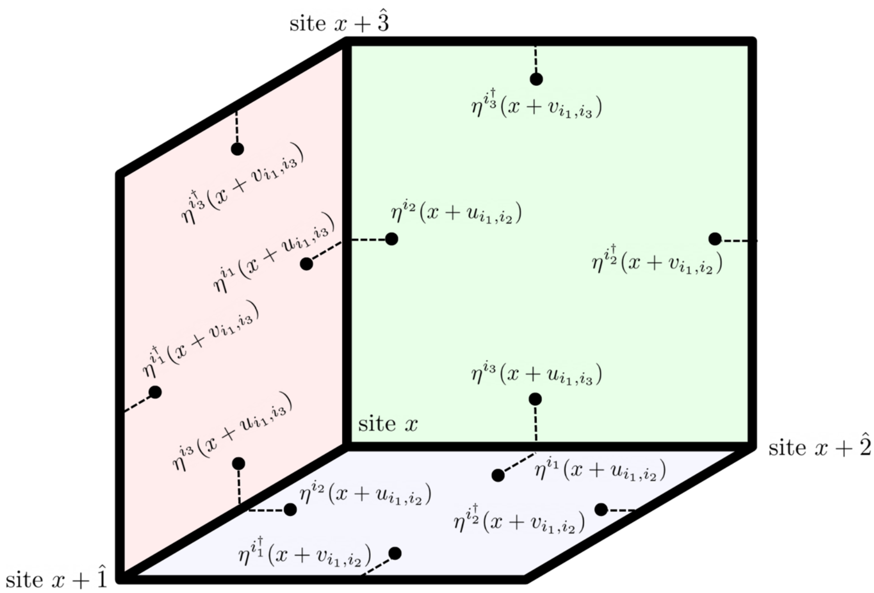

Figure 1. Arrangement of the Grassmann variables on the simple cubic lattice for $d = 3$. On a hypercubic lattice of d-dimensions, the faces have $2(d-1)$ Grassmann variables. Since there are d orientations for the faces, there are a total $2d(d-1)$ variables in d dimensions. By insisting that the Berezin integral be saturated, only closed hypersurfaces are allowed to contribute to the integral. This allows the counting of multipolygons on the hypercubic lattice. Compare this figure to Figure 1 of Ref.[24] and to Figure 6 of Ref.[26].

We saw above that s is the variable for counting closed hypersurfaces. The variable t will be used to count multipolygons.

Next, consider the following known equation expressing the generating function of the multipolygons, $\Lambda_N(t)$, in terms of the generating function of the domain-walls, $\Xi_N(s)$,

$$\Lambda_N(t) = 2^{1-N}(1+t)^{Nd}\,\Xi_N(s). \tag{12}$$

A derivation of this known result can be found for Eq. (6) of Ref.[24]. However, this relation is usually presented in the context of $d = 2$ or $d = 3$, but not for general d. Therefore, for the convenience of the reader the full derivation is given here for completeness.

Consider the Ising model on a d-dimensional hypercubic lattice with N sites, with a Hamiltonian given by

$$H = -J\sum_{\langle i,j\rangle}\sigma_i\sigma_j\,. \tag{13}$$

Here, $\langle i,j\rangle$ is the collection of all nearest neighbor sites. The system's configuration space consists of 2^N states, as each spin σ_i can take on one of two possible values (± 1). Thus, the canonical partition function Z can be expressed as

$$Z_N = \sum_{\sigma_1=\pm 1}\sum_{\sigma_2=\pm 1}\cdots\sum_{\sigma_N=\pm 1}\exp\left[-\beta H(\sigma)\right] = \sum_{\sigma_1\ldots\sigma_N}\exp\left(\beta J\sum_{\langle i,j\rangle}\sigma_i\sigma_j\right) = \sum_{\sigma_1\ldots\sigma_N}\prod_{\langle i,j\rangle}\exp\left(\beta J\sigma_i\sigma_j\right). \tag{14}$$

Let $t = \tanh(\beta J)$. Then Euler's formula gives us

$$\prod_{\langle i,j\rangle}\exp\left[\beta J\left(\sigma_i\sigma_j\right)\right] = \prod_{\langle i,j\rangle}\left[\cosh(\beta J) + \sigma_i\sigma_j\sinh(\beta J)\right] = \left(1-t^2\right)^{dN/2}\prod_{\langle i,j\rangle}\left(t\sigma_i\sigma_j + 1\right). \tag{15}$$

Here, dN represents the total number of nearest-neighbor pairs, considering periodic boundary conditions. Substituting Eq. (15) back into the Eq. (14), we get

$$Z_N = \left(1-t^2\right)^{-dN/2}\sum_{\sigma_1=\pm 1}\sum_{\sigma_2=\pm 1}\cdots\sum_{\sigma_N=\pm 1}\prod_{\langle i,j\rangle}\left(t\sigma_i\sigma_j + 1\right). \tag{16}$$

Expanding the product we obtain

$$\prod_{\langle i,j\rangle}\left(t\sigma_i\sigma_j + 1\right) = \sum_{m=0}^{dN} t^m \prod_{i_m,j_m}\sigma_{i_m}\sigma_{j_m}. \tag{17}$$

Here, the summation is over all distinct sets of m nearest-neighbor pairs out of the total dN pairs. Consider a node σ_k with degree q_k in a graph with m edges.The nodes of the graph are the spin sites, and the edges are the bonds between neighboring sites. In the iterated product over all graphs with m edges, σ_k appears q_k times. When q_k is odd the sum over $\sigma_k = \pm 1$ yields $(-1) + 1 = 0$. Thus, any graph with at least one node of odd degree contributes zero to the partition function. Therefore, we only consider graphs where all nodes have even degrees (i.e., $q_k = 2, 4, \ldots, 2d$). For nodes with even degrees, we have

$$\sum_{\sigma_k=\pm 1}\sigma^{q_k} = 1 + 1 = 2. \tag{18}$$

Consequently, for graphs with V nodes, each node contributes a factor of 2, leading to a total factor of 2^V. There are $N - V$ spins not involved in the graph of m edges. The sum over these spins yields an additional factor of 2 per node, resulting in a factor of 2^{N-V}. Thus, the combined contribution from both involved and uninvolved spins is $2^{V+N-V} = 2^N$. We separate the case $m = 0$ and rewrite the product

$$\prod_{\langle i,j\rangle}\left(t\sigma_i\sigma_j + 1\right) = \sum_{m=0}^{dN} t^m \prod_{i_m,j_m}\sigma_{i_m}\sigma_{j_m} = 1 + \sum_{m=4}^{dN} t^m \sum_{i_m,j_m}\sigma_{i_m}\sigma_{j_m}\,. \tag{19}$$

When this quantity is summed over all spin states, the surviving terms correspond to graphs in which all nodes have even degrees and whose edges link nearest-neighbor sites. These graphs are, by definition, multipolygons. Let $\mathcal{A}$ denote the set of these admissible multipolygon graphs. The partition function thus becomes

$$Z_N = 2^N\left(1-t^2\right)^{-dN/2}\left[1 + \sum_{G\in\mathcal{A}} t^{m(G)}\right] \tag{20}$$

$$= 2^N\left(1-t^2\right)^{-dN/2}\Lambda_N(t), \tag{21}$$

where $m(G)$ is the number of edges in an admissible graph $G \in \mathcal{A}$ and where $\Lambda_N(t)$ is the generating function for the enumeration of multipolygons of a fixed length within a d-dimensional lattice:

$$\Lambda_N(t) = \sum_{n=0}^{Nd} a_n t^n. \tag{22}$$

Here, the coefficient a_n denotes the number of multipolygons consisting of n edges out of a total of Nd edges. Notably, $a_0 = 1$ corresponds to the unique empty graph.

The Eq. (21) is known as the high temperature series expansion of the Ising model[27]. We now consider the low temperature series expansion in the variable $s = \exp[-\beta J]$. It is easy to check that the variables s and t satisfy (11). The partition function can be expanded as a series in s by enumerating the configurations of magnetic domain walls. Let $\Xi_N(s)$ denote the generating function for domain wall configurations of fixed size, as established in Ref.[10]. In two dimensions, this corresponds to the number of multipolygons of a fixed length due to the self-dual nature of the 2D Ising model. In contrast, in three dimensions, magnetic domain walls form closed surfaces on the dual lattice. On a d-dimensional hypercubic lattice, the magnetic domain walls are closed hypersurfaces that separate regions of opposite magnetization. When all spins are aligned, the area of this hypersurface is zero. When all spins are anti-aligned, the hypersurface attains its maximum area, of dN units, because a unit hypercube has $2d$ faces, but each face belongs to 2 adjacent hypercubes. We define $\Xi_N(s)$ via the series,

$$\Xi_N(s) = \sum_{n=0}^{Nd} b_n s^{2n}, \tag{23}$$

where b_n represents the total number of possible configurations of fixed hypersurface area n of the closed hypersurfaces. If the Hamiltonian were defined with zero ground state energy, then $2\,\Xi_N$ automatically is the partition function, where the factor 2 is due to the degeneracy arising from the symmetry under the flipping of all spins, $\sigma \to -\sigma$. However, the ground state energy of our Hamiltonian (13) is not zero, instead it is $-dNJ$. Correcting for the ground state energy, we get

$$Z_N = 2\exp(\beta dNJ)\; \Xi_N(s) = 2s^{-Nd}\,\Xi_N(s). \tag{24}$$

Observe that, given Ξ_N, one can derive Λ_N (and vice versa) by equating (21) and (24). Hence,

$$\Lambda_N(t) = 2^{1-N}\left(1-t^2\right)^{Nd/2} s^{-Nd}\,\Xi_N\left(\sqrt{\frac{1-t}{1+t}}\right), \tag{25}$$

from which follows the claim (12). In two dimensions, both Ξ and Λ follow from Onsager's solution[2]. However, in higher dimensions, neither Ξ nor Λ is explicitly known.

By substituting Eq. (10) into the above equation, we get

$$\Lambda_N(t) = 2^{1-N}(1+t)^{Nd} \int \mathrm{d}\boldsymbol{\eta}^\dagger \mathrm{d}\boldsymbol{\eta}\, \exp\left[S(\boldsymbol{\eta}, s)\right]. \tag{26}$$

On rescaling each Grassmann variable η with $\eta \to \lambda\eta$, and taking into account that there is a total of Nq variables, with q given by Eq. (1), we get

$$\Lambda_N(t) = 2^{1-N}(1+t)^{Nd}\lambda^{-qN} \int \mathrm{d}\boldsymbol{\eta}^\dagger \mathrm{d}\boldsymbol{\eta}\, \exp\left[S(\lambda\boldsymbol{\eta}, s)\right]. \tag{27}$$

As a consequence, if we choose λ in such a way that the prefactor in the above equation is 1, we get

$$\Lambda_N(t) = \int \mathrm{d}\boldsymbol{\eta}^\dagger \mathrm{d}\boldsymbol{\eta}\, \exp\left[S\left(2^{1/(qN)-1/q}(1+t)^{d/q}\boldsymbol{\eta}, s\right)\right] = \int \mathrm{d}\boldsymbol{\eta}^\dagger \mathrm{d}\boldsymbol{\eta}\, \exp\left[\tilde{S}(\boldsymbol{\eta}, s)\right], \tag{28}$$

where we have introduced the renormalized action $\tilde{S}(\boldsymbol{\eta}, s)$ which, in terms of Eq. (2), is given by

$$\tilde{S}(\boldsymbol{\eta}, s) = \frac{1-t}{1+t}\left[2^{1/(qN)-1/q}(1+t)^{d/q}\right]^{2(d-1)} S^{(w)}_{2(d-1)}(\boldsymbol{\eta}) + \left[2^{1/(qN)-1/q}(1+t)^{d/q}\right]^{2} S_2(\boldsymbol{\eta}), \tag{29}$$

where we have also used $s^2 = (1-t)/(1+t)$. By using explicitly $q = 2d(d-1)$ we get

$$\tilde{S}(\boldsymbol{\eta}, s) = 2^{1/(dN)-1/d}(1-t) S^{(w)}_{2(d-1)}(\boldsymbol{\eta}) + 2^{1/[d(d-1)N]-1/[d(d-1)]}(1+t)^{1/(d-1)} S_2(\boldsymbol{\eta}). \tag{30}$$

In particular, in the thermodynamic limit the renormalized action simplifies as

$$\tilde{S}(\boldsymbol{\eta}, s) = \frac{(1-t)}{\sqrt[d]{2}} S^{(w)}_{2(d-1)}(\boldsymbol{\eta}) + \frac{(1+t)^{1/(d-1)}}{\sqrt[d(d-1)]{2}} S_2(\boldsymbol{\eta}). \tag{31}$$

Conclusion

We have addressed and solved the problem of formulating a Grassmann representation for the generating function of multipolygons on hypercubic lattices. Starting with the Berezin integral on the cubic lattice, we took a two step approach to arrive at the final result. In "The Grassmann action for the multipolygon generating function in d dimensions" we derived the Grassmann action for the multipolygon generating function on hypercubic lattices. Our main results are Eqs. (30) and (31).

Recent advances in the study of fermionic systems have provided additional tools for understanding topological phases of matter, particularly through the use of higher cup product structures on hypercubic lattices. The Grassmann integral, central to our exploration of polygon generating functions, finds a parallel in the work on higher cup products, which have been applied to lattice models of topological phases. These studies reveal connections between the algebraic structures of fermionic phases and their topological counterparts, providing insights into the mathematical complexities involved. Although this topic lies beyond the scope of the present work, the study by Gaiotto and Kasputin on spin TQFTs discusses the role of discrete spin structures in resolving ambiguities within fermionic symmetry-protected topological orders. Indeed, there is much interest on such structures in lattice models of fermions[28]. Similarly, the formalism developed by Chen and Tata, which employs higher cup products to derive explicit lattice models of topological phases, demonstrates the potential for extending such constructions to arbitrary dimensions[29]. These studies make evident the intricate relationship between algebraic topology and lattice models. Indeed, the interplay between algebraic topology and the physics of fermions provides a fertile ground for future exploration and aligns with our aim to extend the Grassmann representation to d-dimensional hypercubic lattices.

Even partial progress on being able to compute the generating function of multipolygons on the simple cubic lattice would represent a significant advance. An analogy with the 2D case is helpful. In the last decade, Siudem, Fronczak and Fronczak obtained an exact expression for the coefficients in the series expansion of the partition function of the 2D Ising model on the infinite square lattice[30]. Their expression is different from the hypergeometric formulas obtained via a different approach[3]. Although a closed-form expression for the coefficients in the 3D case is far beyond the reach of the current state of the art, it is still conceivable that efficient algorithms might be able to compute the first few dozen or even hundreds of the lowest order terms. Such an an achievement would represent a remarkable advance. The same is, in this sense, even more true for higher dimensions.

We briefly comment on the advances reported here relative to those in Ref.[24]: (i) Ref.[24] treats only the special case $d = 3$ whereas here the general d-dimensional case is solved; (ii) we have shown here that as d increases, so does the degree of the Grassmann action. Specifically, the 2D case can be treated with a quadratic action, the 3D case with a quartic action, but even a quartic action is unable to count polygons in 4D; (iii) Our results reveal how "non-free-fermion" the polygon counting problem is in higher dimensions. In fact, our results show that counting polygons is precisely as difficult as counting closed hypersurfaces.

Finally, we note that in the context of the three-dimensional case, one can study the continuum limit of the 3D Ising model through the lens of fermion string theory[31–33], as evidenced in Ref.[24]. We believe a similar connection can be made in higher dimensions, and hope that this question can be addressed and definitely answered in the future.

We end by again recalling that Grassmann actions with terms of degree higher than quadratic correspond to models of interacting fermions, in contrast to quadratic actions that describe free fermion models. There is no mathematical proof at the time of writing showing that no free fermion model is equivalent to the quartic-action model of interacting fermions that correctly enumerates 3D multipolygons[24]. However, it is widely believed that no such free fermion model can exist. Similarly, for d-dimensional multipolygons the situation is expected to be even more complicated. Nevertheless, if such free fermion models could be shown to exist, it would represent a major breakthrough.

Data availability

All data generated or analysed during this study are included in this published article.

Received: 26 April 2024; Accepted: 30 August 2024
Published online: 27 September 2024

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Acknowledgements

G. W. C. Rocha thanks J. V. F. L. Aires and L. M. P. Faria for their assistance with the Fig. 1. G. M. V. thanks J. H. H. Perk for interesting discussions. G. M. V. acknowledges support from CNPq (Grant no. 302414/2022-3). C. G. B. acknowledges support from CNPq (Grant no. 309495/2021- 0).

Competing intersts

The authors declare no competing interests.

Additional information

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