

Lévy models for collapse of the wave function

Dorje C Brody^{1,2,*}  and Lane P Hughston³ 

¹ Department of Mathematics, University of Surrey, Guildford GU2 7XH, United Kingdom

² St Petersburg National Research University of Information Technologies, Mechanics and Optics, St Petersburg 197101, Russia

³ Department of Computing, Goldsmiths University of London, New Cross, London, SE14 6NW, United Kingdom

E-mail: d.brody@surrey.ac.uk

Received 11 August 2022; revised 16 December 2022

Accepted for publication 23 February 2023

Published 6 March 2023



CrossMark

Abstract

Recently there has been much progress in the development of stochastic models for state reduction in quantum mechanics. In such models, the collapse of the wave function is a physical process, governed by a nonlinear stochastic differential equation that generalizes the Schrödinger equation. The present paper considers energy-based stochastic extensions of the Schrödinger equation. Most of the work carried out hitherto in this area has been concerned with models where the process driving the stochastic dynamics of the quantum state is Brownian motion. Here, the Brownian framework is broadened to a wider class of models where the noise process is of the Lévy type, admitting stationary and independent increments. The properties of such models are different from those of Brownian reduction models. In particular, for Lévy models the decoherence rate depends on the overall scale of the energy. Thus, in Lévy reduction models, a macroscopic quantum system will spontaneously collapse to an eigenstate even if the energy level gaps are small.

Keywords: quantum measurement, state reduction, measurement problem, stochastic master equation, Lindblad-GSK equation, Born rule, Lüders projection postulate.

* Author to whom any correspondence should be addressed.



Original Content from this work may be used under the terms of the [Creative Commons Attribution 4.0 licence](https://creativecommons.org/licenses/by/4.0/). Any further distribution of this work must maintain attribution to the author(s) and the title of the work, journal citation and DOI.

1. The stochastic Schrödinger equation

A number of authors have worked on the development of dynamical models for the collapse of the wave function [1–20]. For overviews see [21–24]. Such models have a highly nontrivial relationship with the probabilistic hypotheses of standard quantum mechanics. Progress in this area can be classified into work on (a) spontaneous localization of the state and (b) collapse of the state vector to an energy eigenstate. We are concerned with the latter here. Our goal is to show how the well-established framework for stochastic state reduction based on Brownian noise can be extended to a wider class of models based on noise processes with stationary and independent increments, so-called Lévy processes. In general, such processes have jumps. A Lévy process is continuous if and only if it is a Brownian motion. A pure jump Lévy process can be decomposed into the sum of a finite activity process and an infinite activity process. Processes of finite activity have the property that jumps occur at a finite rate. Processes of infinite activity jump infinitely often over any finite interval of time. We provide examples of state reduction models based on each of these types of Lévy processes. We argue that there is no reason *a priori* to prefer continuous processes over discontinuous processes in models for quantum state reduction.

For the dynamics of the state vector in the simplest energy-driven model with a Brownian driver, we have the following well-known stochastic differential equation of the Ito type defined on a finite dimensional Hilbert space:

$$d|\psi_t\rangle = -i\hbar^{-1}\hat{H}|\psi_t\rangle dt - \frac{1}{8}\sigma^2(\hat{H} - H_t)^2|\psi_t\rangle dt + \frac{1}{2}\sigma(\hat{H} - H_t)|\psi_t\rangle dW_t. \quad (1)$$

Here $|\psi_t\rangle$ denotes the state at time t , with initial condition $|\psi_0\rangle$, \hat{H} is the Hamiltonian, $\{W_t\}_{t\geq 0}$ is a standard one-dimensional Brownian motion, and

$$H_t = \frac{\langle\psi_t|\hat{H}|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle} \quad (2)$$

is the expectation of \hat{H} in the state $|\psi_t\rangle$. The parameter σ , which has the units

$$\sigma \sim [\text{energy}]^{-1}[\text{time}]^{-1/2}, \quad (3)$$

determines the characteristic timescale $\tau_R = 1/\sigma^2 V_0$ for the collapse of the wave function. Here V_0 denotes the initial value of the squared uncertainty of the energy. More generally, the conditional variance of the energy is defined for $t \geq 0$ by

$$V_t = \frac{\langle\psi_t|(\hat{H} - H_t)^2|\psi_t\rangle}{\langle\psi_t|\psi_t\rangle}. \quad (4)$$

The energy-conserving stochastic Schrödinger equation based on Brownian driver, originally introduced in [6], is the simplest known rigorous model for state reduction in which the Born probability rules can be derived dynamically.

The dynamics of $|\psi_t\rangle$ set out in (1) are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$. For convenience we recall some key definitions [10, 14, 16, 22]. The elements of Ω represent the possible outcomes of chance in the model under consideration. The event space \mathcal{F} is a σ -algebra of subsets of Ω . The measure \mathbb{P} assigns a probability $\mathbb{P}: A \in \mathcal{F} \mapsto \mathbb{P}(A) \in [0, 1]$ to each event A in \mathcal{F} , given by

$$\mathbb{P}(A) = \int_{\Omega} \mathbb{1}_A(\omega) \mathbb{P}(d\omega), \quad (5)$$

where $\mathbb{1}_A$ denotes the indicator function of the set $A \subset \Omega$, taking the value one if $\omega \in A$, and zero otherwise. Here $\omega \in \Omega$ denotes a typical outcome of chance. A function $X : \Omega \rightarrow \mathbb{R}$ is said to be a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ if X is \mathcal{F} -measurable, that is, if for any $x \in \mathbb{R} \cup \{\pm\infty\}$ it holds that the set $\{\omega \in \Omega : X(\omega) \leq x\}$ is an element of \mathcal{F} . The distribution of X is the function $F_X : \mathbb{R} \cup \{\pm\infty\} \rightarrow [0, 1]$ defined by $F_X(x) = \mathbb{P}(X \leq x)$. The expectation of X , which takes values in the extended real line, is then defined by the Lebesgue integral

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) \mathbb{P}(d\omega). \quad (6)$$

We note that if $\mathbb{E}[\max(X, 0)] = \infty$ and $\mathbb{E}[\min(X, 0)] = -\infty$ then $\mathbb{E}[X]$ is not defined. We say that X is integrable under \mathbb{P} if $\mathbb{E}[|X|] < \infty$. The subtle measure-theoretic definition of conditional expectation due to Kolmogorov [25] involves a construction that generalizes the elementary notion of conditional probability and adds more precision to the idea. Specifically, if \mathcal{E} is a sub- σ -algebra of the σ -algebra \mathcal{F} on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and if X is an integrable random variable, then the conditional expectation $\mathbb{E}[X|\mathcal{E}]$ of X with respect to \mathcal{E} is defined as follows. We write $Y = \mathbb{E}[X|\mathcal{E}]$ for any \mathcal{E} -measurable random variable Y with the property that for any $A \in \mathcal{E}$ it holds that

$$\int_{\Omega} \mathbb{1}_A(\omega) X(\omega) \mathbb{P}(d\omega) = \int_{\Omega} \mathbb{1}_A(\omega) Y(\omega) \mathbb{P}(d\omega). \quad (7)$$

The conditional expectation is unique modulo differences on sets of \mathbb{P} -measure zero. Any particular choice of Y from such an equivalence class is called a version of $\mathbb{E}[X|\mathcal{E}]$. Then if \mathcal{D} is a sub- σ -algebra of \mathcal{E} , and \mathcal{E} is a sub- σ -algebra of \mathcal{F} , we have the tower property of conditional expectation: $\mathbb{E}[\mathbb{E}[X|\mathcal{E}]] = \mathbb{E}[X]$. In particular, because the trivial σ -algebra $\mathcal{Z} = \{\Omega, \emptyset\}$ satisfies $\mathbb{E}[X|\mathcal{Z}] = \mathbb{E}[X]$ for any integrable random variable X , and \mathcal{Z} is a sub- σ -algebra of \mathcal{E} , we have $\mathbb{E}[\mathbb{E}[X|\mathcal{E}]] = \mathbb{E}[X]$ for any sub- σ -algebra \mathcal{E} .

The filtration $\{\mathcal{F}_t\}_{0 \leq t < \infty}$ consists of a nested family of sub- σ -algebras of \mathcal{F} such that $s \leq t$ implies that \mathcal{F}_s is a sub- σ -algebra of \mathcal{F}_t . In the case of a filtration we often use the simplifying notation $\mathbb{E}_t[X] = \mathbb{E}[X|\mathcal{F}_t]$ for the conditional expectation. Then by the tower property we have $\mathbb{E}_s[\mathbb{E}_t[X]] = \mathbb{E}_s[X]$ for $s \leq t$.

By a random process (or stochastic process) we mean a collection of random variables $\{X_t\}_{t \geq 0}$. A random process is said to be adapted to $\{\mathcal{F}_t\}$ if for each $t \geq 0$ it holds that X_t is \mathcal{F}_t -measurable. An adapted process $\{X_t\}_{t \geq 0}$ is said to be a martingale if $\mathbb{E}[|X_t|] < \infty$ for $t \geq 0$ and $\mathbb{E}_s[X_t] = X_s$ for $0 \leq s \leq t < \infty$. Thus the martingale property characterizes the dynamics of a quantity that at each step is only on average conserved. A process $\{X_t\}_{t \geq 0}$ is a supermartingale if $\mathbb{E}[|X_t|] < \infty$ for $t \geq 0$ and $\mathbb{E}_s[X_t] \leq X_s$ for all $0 \leq s \leq t < \infty$.

With these definitions at hand one finds as a consequence of (1) that the expectation of the energy is a martingale and that the variance of the energy is a supermartingale. That is to say, we have

$$\mathbb{E}_s[H_t] = H_s, \quad \mathbb{E}_s[V_t] \leq V_s. \quad (8)$$

These relations can be worked out by an application of Ito's lemma to (2) and (4), from which one infers

$$dH_t = \sigma V_t dW_t, \quad dV_t = -\sigma^2 V_t^2 dt + \sigma \kappa_t dW_t, \quad (9)$$

where $\kappa_t = \langle \psi_t | (\hat{H} - H_t)^3 | \psi_t \rangle / \langle \psi_t | \psi_t \rangle$. In particular, since the energy is bounded, the fact that $\{H_t\}$ has no drift implies that it is a martingale. Then the fact that the drift of $\{V_t\}$ is negative shows that $\{V_t\}$ is a supermartingale.

The martingale condition can thus be interpreted as the form of the energy conservation law that applies even when a system is not in a definite state of energy. The supermartingale property satisfied by $\{V_t\}$ captures the essence of what is meant by a reduction process in quantum mechanics. In particular, one can show by use of (1) that

$$\lim_{t \rightarrow \infty} \mathbb{E}[V_t] = 0, \quad (10)$$

which implies that reduction proceeds to an energy eigenstate, because only at an energy eigenstate do we have $V_t = 0$. Writing H_∞ for the random terminal value of the energy, one can prove that $H_t = \mathbb{E}_t[H_\infty]$ and $V_t = \mathbb{E}_t[(H_\infty - H_t)^2]$. That is to say, $\{H_t\}$ and $\{V_t\}$ are respectively the \mathcal{F}_t -conditional mean and variance of the terminal value of the energy after reduction. In particular, we have the relations

$$H_0 = \mathbb{E}[H_\infty], \quad V_0 = \mathbb{E}[(H_\infty - H_0)^2], \quad (11)$$

which form the basis of the statistical interpretation of quantum mechanics [10, 12, 14, 16]. The first of these shows that the so-called expectation value of the observable \hat{H} in the state $|\psi_0\rangle$ is equal to the expectation (in the probabilistic sense) of the random variable corresponding to the outcome of the measurement of the energy. Similarly, the squared uncertainty of \hat{H} in the state $|\psi_0\rangle$ is equal to the variance of the outcome of the measurement. These results all carry through to the Lévy-based models that we consider shortly.

2. Generalization to mixed states

If we take the view that the general state of a quantum system is described by a density matrix, and that state reduction prevails at the level of density matrices, then the dynamical equation for the reduction of the density matrix $\{\hat{\rho}_t\}_{t \geq 0}$ takes the form of the following stochastic master equation [20]:

$$d\hat{\rho}_t = -i\hbar^{-1}[\hat{H}, \hat{\rho}_t] dt + \frac{1}{4}\sigma^2 \mathcal{L}_{\hat{H}} \hat{\rho}_t dt + \frac{1}{2}\sigma((\hat{H} - H_t)\hat{\rho}_t + \hat{\rho}_t(\hat{H} - H_t))dW_t. \quad (12)$$

Here we write $H_t = \text{tr}(\hat{H}\hat{\rho}_t)$ for the expectation of the Hamiltonian and $\mathcal{L}_{\hat{H}}$ denotes the Lindblad-GKS super-operator [26, 27], given in the present context by

$$\mathcal{L}_{\hat{H}} \hat{\rho}_t = \hat{H}\hat{\rho}_t\hat{H} - \frac{1}{2}\hat{\rho}_t\hat{H}^2 - \frac{1}{2}\hat{H}^2\hat{\rho}_t. \quad (13)$$

One can show that the dynamical equation for $\hat{\rho}_t$ has the following properties: (a) the trace of $\hat{\rho}_t$ is preserved, (b) the positivity of $\hat{\rho}_t$ is preserved, (c) the conditional expectation of the energy has the martingale property (energy is conserved on average), and (d) the variance of the energy is a supermartingale (state reduction occurs). Additionally, we find that if $\hat{\Pi}_j$ denotes the projection operator onto the Hilbert subspace associated to the energy eigenvalue E_j , then (e) the process $\{\Pi_{jt}\}_{t \geq 0}$ defined by

$$\Pi_{jt} = \text{tr}(\hat{\rho}_t \hat{\Pi}_j) \quad (14)$$

is a martingale. This generalizes a result of Adler and Horwitz [12] and shows the Born rule can be deduced under the dynamical equation (12) in the situation where the system is in a random mixed state. One can see at a glance that the mean density matrix will satisfy the Lindblad equation, a property that is not so obvious from the dynamics of the state vector. In fact, in our dynamical model for the density matrix we find that state reduction proceeds in

accordance with the Lüders projection postulate [28] to the Lüders eigenstate associated with the eigenvalue E_j obtained as the result of the measurement:

$$\hat{\rho}_0 \rightarrow \frac{\hat{\Pi}_j \hat{\rho}_0 \hat{\Pi}_j}{\text{tr}(\hat{\rho}_0 \hat{\Pi}_j)}. \quad (15)$$

It is worth emphasizing, in this connection, that the Born rule is an *assumption* in standard quantum mechanics, part of the statistical interpretation of the theory. The same holds for the Lüders projection postulate. But the Born rule and the Lüders projection postulate both arise as *theorems* of the energy-driven reduction model, even for initially mixed states. In this respect, the energy-driven model can be regarded as superior to (a) the GRW model [1] and (b) the continuous spontaneous localization model [4, 7], neither of which exhibit this property on an exact basis.

Now, one might ask whether it even makes sense to speak of the reduction of a mixed state. After all, a mixed state is usually understood to represent an ensemble, rather than a single particle. Indeed, most of the literature of dynamic state reduction models adheres to such a view. It seems that early on, since the work of [1, 4, 7, 29], the idea that a single quantum system in isolation is necessarily represented by a pure state was somehow embedded in the thinking of most of those who were working on reduction models. It may be that this view simply echoed the thinking of the majority of physicists at the time, and it is not surprising that such a view continues to be held by many to this day. Nonetheless, it is worth noting that as early as 1947 it was recognized [30] that the general state of a quantum system is a density matrix and that the ensemble interpretation is not necessary.

It is interesting therefore to observe that stochastic reduction models can be readily formulated, as we have shown here, with the property that the status of the density matrix as representing the general state of a single isolated quantum system is sustained. In fact, the theory assumes a simpler form when it is pursued at the level of the density matrix, as we have seen, and thus forms a satisfactory basis upon which one can pursue more general classes of models such as the Lévy models we consider later.

3. Diagonalization of the density matrix

From a purely probabilistic stance one can nevertheless give a completely consistent interpretation of the mean density matrix

$$\hat{\mu}_t = \mathbb{E}[\hat{\rho}_t] \quad (16)$$

in terms of ensembles. One envisages a collection of N independent identical copies of the given quantum system, each evolving from the same initial state $\hat{\rho}_0$. In each case the evolution is governed by an equation of the form (12), but with an independent \mathbb{P} -Brownian driver. It follows by Kolmogorov's strong law of large numbers that as the number of systems gets large, the ensemble average of the density matrices converges almost surely to the expectation of the random density matrix arising in the case of a single system. Thus, if $\{\hat{\rho}_t^r\}_{r=1,\dots,N}$ are the density matrix processes of the N independent quantum systems, then

$$\mathbb{P} \left(\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{r=1}^N \hat{\rho}_t^r = \hat{\mu}_t \right) = 1. \quad (17)$$

It should be emphasized that in a stochastic model of the type we are considering, there is no metaphysics involved in giving an ensemble interpretation to the mean density matrix. Rather, as we have seen, such an interpretation follows rigorously as a consequence of the law of large numbers. Indeed, all valid statements about ensembles in the present context can be formulated in such terms. The law of large numbers is a theorem, not an intuition.

As an example, it will be useful to consider the phenomenon of decoherence from this point of view. We shall show that starting from an arbitrary initial density matrix, the mean density matrix diagonalizes in the frame of the energy projectors as t approaches infinity. First we observe, as we remarked earlier, that as a consequence of (12) we obtain the usual deterministic master equation for the mean density matrix:

$$\frac{d\hat{\mu}_t}{dt} = -i\hbar^{-1} [\hat{H}, \hat{\mu}_t] + \frac{1}{4} \sigma^2 \mathcal{L}_{\hat{H}} \hat{\mu}_t. \quad (18)$$

This can be derived by integrating equation (12) with the introduction of the initial condition, then taking the expectation of each side of the equation, then using the Fubini theorem to interchange the integrals and the expectations, then differentiating with respect to t . A calculation [14] shows that the solution of the deterministic master equation is

$$\hat{\mu}_t = \sum_j \hat{\Pi}_j \hat{\rho}_0 \hat{\Pi}_j + \sum_{j \neq k} e^{-i\hbar^{-1}(E_j - E_k)t - \frac{1}{8} \sigma^2 (E_j - E_k)^2 t} \hat{\Pi}_j \hat{\rho}_0 \hat{\Pi}_k. \quad (19)$$

One sees that the second term is exponentially damped as time passes and that asymptotically one is left with the first term alone, in which the mean density matrix is diagonalized.

Thus one can say that asymptotically the ensemble is equivalent to that of a mixture of energy eigenstates of the projector type, where the proportion belonging to eigenvalue E_j is given by $\text{tr}(\hat{\rho}_0 \hat{\Pi}_j)$. This is indeed the result one would expect on intuitive grounds in the statistical interpretation of standard quantum mechanics when $\hat{\mu}_\infty$ represents the state of a system after an energy measurement has been made but before the outcome is known. We should stress, however, that in the present context we are deriving the result directly from the stochastic dynamics of the independent elements of the ensemble, without assuming the statistical interpretation of standard quantum mechanics.

4. Signal detection and state reduction

Before we develop our theory of state reduction models based on Lévy noise, it will be useful first to outline how a solution to the dynamical equation (1) can be obtained in the case of a pure state. A corresponding solution to the stochastic mixed-state evolution equation can then be formulated by analogy. A general closed-form solution to (1) was obtained in [16] and developed in greater detail in [22]. Our purpose in this section is to review the solution methodology, which then paves the way towards Lévy generalizations.

It turns out that a highly effective approach to solving (1) is by *constructing* the solution explicitly using techniques of signal detection, rather than *solving* the given differential equation. Once the solution is constructed, it is straightforward to show that it satisfies the differential equation that we intended to solve. Hence we begin with the consideration of a classical signal detection problem, leaving aside quantum theory for the moment.

In signal detection, one is typically interested in inferring the true value of a signal, or message, given noisy observations. Let us assume that the unknown signal is represented by a

fixed, time-independent random variable H defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that H takes the value E_j with the probability p_j , where $j = 1, 2, \dots, n$. Now suppose that the signal is revealed continuously in time at a constant rate σ , but the signal is obscured by a Brownian noise $\{B_t\}_{t \geq 0}$. Then the noisy observations of the signal can be modeled by an information-providing process $\{\xi_t\}_{t \geq 0}$ that takes the form

$$\xi_t = \sigma H t + B_t. \quad (20)$$

The task in signal detection at time t is to determine the best estimate of the signal H given the observed ‘signal-plus-noise’ time series $\{\xi_s\}_{0 \leq s \leq t}$ up to that time. The notion of ‘best’ estimate evidently depends on the criterion used to judge the merits of the estimate, but for a wide range of reasonable criteria, such as minimization of the quadratic error, the best estimate of H is the conditional expectation

$$H_t = \sum_{j=1}^n E_j \mathbb{P}(H = E_j | \{\xi_s\}_{0 \leq s \leq t}). \quad (21)$$

Thus we need to work out the conditional probabilities

$$\pi_{jt} = \mathbb{P}(H = E_j | \{\xi_s\}_{0 \leq s \leq t}), \quad j = 1, 2, \dots, n. \quad (22)$$

Since the information process has the Markov property (one can prove this), and since

$$\lim_{t \rightarrow \infty} \frac{1}{\sigma t} \xi_t = H, \quad (23)$$

one finds that (22) reduces to a simpler expression, namely,

$$\pi_{jt} = \mathbb{P}(H = E_j | \xi_t), \quad (24)$$

which can be worked out explicitly by use of the following form of the Bayes formula:

$$\mathbb{P}(H = E_j | \xi_t) = \frac{\mathbb{P}(H = E_j) \rho(\xi_t | H = E_j)}{\sum_{k=1}^n \mathbb{P}(H = E_k) \rho(\xi_t | H = E_k)}. \quad (25)$$

Here $\rho(x | H = E_j)$ denotes the value at $x \in \mathbb{R}$ of the conditional density function of the random variable ξ_t . By use of $\mathbb{P}(H = E_j) = p_j$, and the fact that conditional on $H = E_j$ the random variable ξ_t is normally distributed with mean $\sigma E_j t$ and variance t , we deduce that

$$\pi_{jt} = \frac{p_j \exp(\sigma E_j \xi_t - \frac{1}{2} \sigma^2 E_j^2 t)}{\sum_{k=1}^n p_k \exp(\sigma E_k \xi_t - \frac{1}{2} \sigma^2 E_k^2 t)}, \quad (26)$$

from which the best estimate of the signal can be determined.

Let us work out the stochastic differential of the process $\{\pi_{jt}\}_{t \geq 0}$. By use of Ito’s formula one finds that

$$d\pi_{jt} = \sigma(E_j - H_t) \pi_{jt} (d\xi_t - \sigma H_t dt). \quad (27)$$

Then if we define a process $\{W_t\}_{t \geq 0}$ by setting

$$W_t = \xi_t - \sigma \int_0^t H_s ds, \quad (28)$$

it can be shown [16, 22] that $\{W_t\}$ is a standard Brownian motion under \mathbb{P} . That is to say, $\{W_t\}$ turns out to be an $\{\mathcal{F}_t\}$ -adapted Gaussian process with mean zero and autocovariance $\text{Cov}(W_s, W_t) = s$ for $0 \leq s \leq t$, with stationary and independent increments. Then we have

$$d\pi_{it} = \sigma(E_i - H_t) \pi_{it} dW_t. \quad (29)$$

Furthermore, if we consider the square-root probability processes $\sqrt{\pi_{jt}}$, for $j = 1, 2, \dots, n$, then by use of Ito's lemma and (27) we deduce that

$$d\sqrt{\pi_{jt}} = \frac{1}{2} \sigma(E_j - H_t) \sqrt{\pi_{jt}} dW_t - \frac{1}{8} \sigma^2(E_j - H_t)^2 \sqrt{\pi_{jt}} dt. \quad (30)$$

With these results at hand, let us consider a quantum system characterized by a Hamiltonian \hat{H} that may or may not be degenerate. We assume, for the moment, that the initial state of the system is pure, with state vector $|\psi_0\rangle$. As before, we let $\hat{\Pi}_j$ denote the projection operator onto the Hilbert subspace associated to the energy eigenvalue E_j . Let us denote by $|E_j\rangle$ the normalized Lüders state obtained by projecting the initial state $|\psi_0\rangle$ onto the Hilbert subspace with energy eigenvalue E_j . Thus,

$$|E_j\rangle = \frac{1}{\sqrt{p_j}} \hat{\Pi}_j |\psi_0\rangle, \quad p_j = \frac{\langle \psi_0 | \hat{\Pi}_j | \psi_0 \rangle}{\langle \psi_0 | \psi_0 \rangle}. \quad (31)$$

Next, we define a state vector process $\{|\psi_t\rangle\}_{t \geq 0}$ by setting

$$|\psi_t\rangle = \sum_j \sqrt{\pi_{jt}} e^{-i\hbar^{-1} E_j t} |E_j\rangle, \quad (32)$$

where the π_{jt} are given by (26). Then a calculation making use of (30) shows that $|\psi_t\rangle$ is a solution to the stochastic Schrödinger equation (1) with the initial condition $|\psi_0\rangle$. The advantage of the filtering method is that one can work directly with the solutions of the stochastic differential equation. The construction of (32), along with (26), only requires the computation of the conditional probabilities. In particular, no stochastic integration is required to arrive at the solution. This is the approach that we shall use shortly when we turn to look at collapse models based on Lévy information.

5. Solution to the stochastic master equation

The structure of the solution to the stochastic Schrödinger equation obtained in the previous section sheds light on foundational issues and at the same time suggests generalizations from a mathematical perspective. In particular, in the case of the stochastic master equation (12), a solution for the density matrix can be constructed by the same line of argument.

In the setting of a system based on finite-dimensional Hilbert space we regard the Hamiltonian \hat{H} (possibly degenerate) and the initial state $\hat{\rho}_0$ (possibly of low rank) as being given. As before, let us write $\hat{\Pi}_j$ ($j = 1, \dots, n$) for the projection operator onto the eigenspace of energy E_j . Then we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ upon which we define a Brownian motion $\{B_t\}_{t \geq 0}$ along with an independent random variable H taking values in the set $(E_j)_{j=1, \dots, n}$ such that

$$\mathbb{P}(H = E_j) = \text{tr}(\hat{\rho}_0 \hat{\Pi}_j). \quad (33)$$

Next, we introduce a Brownian information process of the form (20). Finally, we set

$$\hat{K}_t = \sum_{j=1}^n \hat{\Pi}_j e^{i\hbar^{-1} E_j t + \frac{1}{2} \sigma E_j \xi_t - \frac{1}{4} \sigma^2 E_j^2 t}. \quad (34)$$

Then one can show that the solution for the state process $\{\hat{\rho}_t\}_{t \geq 0}$ is given by

$$\hat{\rho}_t = \frac{\hat{K}_t^\dagger \hat{\rho}_0 \hat{K}_t}{\text{tr}(\hat{K}_t^\dagger \hat{\rho}_0 \hat{K}_t)}, \quad (35)$$

where the information process $\{\xi_t\}_{t \geq 0}$ is related to the Brownian driver $\{W_t\}_{t \geq 0}$ of (12) in accordance with (28). In particular, one can prove that $\{\hat{\rho}_t\}_{t \geq 0}$ satisfies the dynamical equation (12) with the Brownian driver $\{W_t\}_{t \geq 0}$ and the prescribed initial condition and that $\{\hat{\rho}_t\}_{t \geq 0}$ has the properties (a), (b), (c), (d), (e) stated in section 2.

6. Lévy information

We turn to a generalization of the foregoing considerations to a much wider class of processes. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we define a Lévy process $\{\xi_t\}_{t \geq 0}$. By a Lévy process we mean a random process with stationary, independent increments. Brownian motion is an example of a Lévy process and indeed it is the only example of a continuous Lévy process, i.e. a process with continuous sample paths. One can think of the different types of Lévy processes as representing different types of homogeneous noise.

We shall assume in the following that $\{\xi_t\}_{t \geq 0}$ admits exponential moments. By this we mean that there exists an open interval $S \subset \mathbb{R}$ containing the origin such that $S \subset C$ where

$$C = \{c \in \mathbb{R} : \mathbb{E}[\exp c \xi_t] < \infty\}. \quad (36)$$

It can be shown that for any Lévy process admitting exponential moments there exists a strictly convex function $\psi : C \rightarrow \mathbb{R}$ such that

$$\frac{1}{t} \log \mathbb{E}[\exp \alpha \xi_t] = \psi(\alpha) \quad (37)$$

for $\alpha \in C$. We refer to ψ as the Lévy exponent associated with the Lévy process $\{\xi_t\}_{t \geq 0}$. By the Lévy–Khinchine theorem [31, 32], which is one of the foundational results of the theory, there exists a constant p , a constant $q \geq 0$, and a Lévy measure $\nu(dx)$ such that

$$\psi(\alpha) = p\alpha + \frac{1}{2}q\alpha^2 + \int_{\mathbb{R}} (e^{\alpha z} - 1 - \alpha z \mathbb{1}_{\{|z| < 1\}}) \nu(dz), \quad (38)$$

and we refer to (38) as the Lévy–Khinchine representation. By a Lévy measure ν on \mathbb{R} we mean a σ -finite (but not necessarily finite) measure satisfying $\nu(\{0\}) = 0$ and

$$\int_{\mathbb{R}} \min(1, z^2) \nu(dz) < \infty. \quad (39)$$

We call $\{p, q, \nu(dz)\}$ the characteristic triplet of the Lévy process. Note that $\nu(A)$ is finite on any interval $A \in \mathbb{R}$ bounded away from the origin, but may be infinite on an interval that includes the origin. If a Lévy process has Lévy measure $\nu(dx)$, the rate at which jumps arrive for which the jump size is in the interval $[a, b]$ for $a < b$ with $\{0\} \notin [a, b]$ is given by

$$m[a, b] = \int_{[a, b]} \nu(dz), \quad (40)$$

which by (39) is evidently finite. If $a > 0$ and $\lim_{a \rightarrow 0} m[a, b] = \infty$ or if $b < 0$ and $\lim_{b \rightarrow 0} m[a, b] = \infty$ then we say that the Lévy process has infinite activity. In this case, the process admits infinitely many very small jumps in any finite interval of time. Otherwise, the

process has finite activity and can be represented by a compound Poisson process, in which case the normalized measure

$$p(dz) = \frac{\nu(dz)}{\int_{\mathbb{R}} \nu(dz)} \quad (41)$$

gives the probability distribution of the size of a typical jump, and jumps arrive at the rate

$$m_\nu = \int_{\mathbb{R}} \nu(dz). \quad (42)$$

We are now in a position to define Lévy information [33]. The idea of a Lévy information process is that it generalizes the information process (20) for Brownian noise to the general class of Lévy processes introduced above. In the case of a signal obscured by Brownian noise, it is natural that the signal and noise should admit an additive decomposition. This is why we often hear the phrase ‘signal plus noise’ in this context. The result is a Brownian motion $\{B_t\}$ with a linear drift, as we see in (20). One can then find a change of measure such that a drifted Brownian motion $\{\xi_t\}$ under the physical measure \mathbb{P} becomes a pure Brownian motion under another probability measure, say, \mathbb{P}^0 . This is the measure in which the observed information $\{\xi_t\}$ is content free—that is to say, free of any signal H . The change of probability measure arising in this context, concerning which we shall have more to say in section 7, is called an Esscher transformation. Conversely, once the noise type (e.g. say, a Brownian noise) is specified, one can begin with the ‘empty’ probability measure \mathbb{P}^0 in which the observation $\{\xi_t\}$ represents pure noise of the type selected, and then apply an Esscher transform to the physical measure \mathbb{P} using the signal H .

In this way, a type of information process can be created that can carry a much wider class of noise structures, not just Brownian noise. In particular, by letting $\{\xi_t\}$ be a \mathbb{P}^0 -Lévy noise, it is possible to naturally extend the theory of signal detection with Brownian noise into the general Lévy setup introduced above. This procedure, in turn, allows one to formulate models of state reduction in quantum mechanics driven by a range of different Lévy processes, each with its own special characteristics.

With these preliminaries at hand, let $\psi : C \rightarrow \mathbb{R}$ be a Lévy exponent and let X be a random variable taking values in C . Then by a Lévy information process with information X and Lévy noise type ψ we mean a process $\{\xi_t\}_{t \geq 0}$ that is conditionally Lévy with a conditional exponent of the form

$$\frac{1}{t} \log \mathbb{E}[\exp \alpha \xi_t | \mathcal{F}_X] = \psi(\alpha + X) - \psi(X), \quad (43)$$

where \mathcal{F}_X is the σ -algebra generated by the random variable X . By conditionally Lévy we mean that $\{\xi_t\}_{t \geq 0}$ has conditionally stationary and independent increments.

One can show, for example, that the Brownian information process considered earlier satisfies these conditions. In the Brownian case we have a Gaussian exponent

$$\psi(\alpha) = p\alpha + \frac{1}{2}q\alpha^2 \quad (44)$$

and $S = \mathbb{R}$, so we see that the conditional exponent (43) in this case takes the form

$$\psi(\alpha + X) - \psi(X) = \alpha X + \frac{1}{2}\alpha^2, \quad (45)$$

with a random term linear in α . In fact, one can prove that a Lévy information process can be constructed in association with any Lévy process admitting exponential moments and any integrable random variable taking values in C . In the case of the Poisson process, for example, we have

$$\psi(\alpha) = m(e^\alpha - 1), \quad (46)$$

where m is the intensity, the rate at which the events being counted occur on average. For a Poisson information process the conditional moment takes the form

$$\psi(\alpha + X) - \psi(X) = m e^X (e^\alpha - 1), \quad (47)$$

showing that the intensity is randomized (or ‘modulated’) [33, 34] and is given by $m e^X$.

Thus, in the case of Poisson noise, the information process $\{\xi_t\}$ is a Poisson process with intensity m under the ‘content free’ measure \mathbb{P}^0 . But under the physical measure \mathbb{P} the process has a randomized intensity $m e^X$. The observer therefore detects Poisson jumps, from which the task is to infer the jump intensity $m e^X$, and hence the value of the signal X . The observer is already aware of the base rate m , and thus by counting the number of jumps over some interval of time they can estimate the value of X .

The idea can be illustrated as follows. Imagine a situation where a laboratory may have been contaminated with a small amount of radioactive substance. A Geiger counter is used to measure the radiation level. If there is no contamination, the counter will click randomly at a low rate of activity, corresponding to the normal level of background radiation, but if the contaminant is present the counter will click at a higher rate. In this example, X can take two possible values, with $X = 0$ corresponding to the normal rate of background activity and $X = \log(1 + \epsilon)$ for some $\epsilon > 0$ corresponding to the case where there is contamination. Analogously, for each type of Lévy noise one can think of a class of signal detection problems, for which the available observations are represented by Lévy information processes.

7. Change of measure

To establish the existence of Lévy information processes we can use a so-called change-of-measure technique. Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P}^0)$ where \mathbb{P}^0 will be called the base measure. We assume that $(\Omega, \mathcal{F}, \mathbb{P}^0)$ supports a Lévy process $\{\xi_t\}_{t \geq 0}$ admitting exponential moments and we define the set C as in (36). Equivalently,

$$C = \left\{ c \in \mathbb{R} : \int_{\mathbb{R}} e^{cz} \nu(dz) < \infty \right\}, \quad (48)$$

where $\nu(dz)$ is the Lévy measure associated with $\{\xi_t\}$. Now let $\{\mathcal{F}_t\}_{t \geq 0}$ be the filtration generated by $\{\xi_t\}$. One can check that the process $\{\Lambda_t^\kappa\}_{t \geq 0}$ defined for $\kappa \in C$ by

$$\Lambda_t^\kappa = \exp(\kappa \xi_t - \psi(\kappa)t) \quad (49)$$

is a martingale. This follows on account of the independent increments property of Lévy processes. Then for each $t \in \mathbb{R}^+$ we can define a measure \mathbb{P}_t^κ on (Ω, \mathcal{F}_t) by setting

$$\mathbb{P}_t^\kappa(A) = \mathbb{E}^{\mathbb{P}_t^\kappa}[\mathbf{1}(A)] = \mathbb{E}^{\mathbb{P}^0}[\Lambda_t^\kappa \mathbf{1}(A)] \quad (50)$$

for any $A \in \mathcal{F}_t$. The martingale property of $\{\Lambda_s^\kappa\}_{0 \leq s \leq t}$ ensures that \mathbb{P}_t^κ is a probability measure, since $\mathbb{P}_t^\kappa(\Omega) = \mathbb{E}^{\mathbb{P}^0}[\Lambda_t^\kappa \mathbf{1}\{\Omega\}] = \Lambda_0^\kappa = 1$. We observe that if $s \leq t$ and if A is \mathcal{F}_s -measurable, then

$$\begin{aligned} \mathbb{P}_t^\kappa(A) &= \mathbb{E}^{\mathbb{P}^0} \left[\mathbb{E}^{\mathbb{P}^0} [\Lambda_t^\kappa \mathbf{1}(A) | \mathcal{F}_s] \right] \\ &= \mathbb{E}^{\mathbb{P}^0} \left[\mathbb{E}^{\mathbb{P}^0} [\Lambda_t^\kappa | \mathcal{F}_s] \mathbf{1}(A) \right] \\ &= \mathbb{E}^{\mathbb{P}^0} [\Lambda_s^\kappa \mathbf{1}(A)] \\ &= \mathbb{P}_s^\kappa(A), \end{aligned} \quad (51)$$

which shows that the measures defined on \mathcal{F}_s and \mathcal{F}_t are compatible for $s \leq t$, in the sense that if $A \in \mathcal{F}_s$ then the measure of A on $(\Omega, \mathcal{F}_s, \mathbb{P}_s^\kappa)$ is the same as its measure on $(\Omega, \mathcal{F}_t, \mathbb{P}_t^\kappa)$. With that in mind, we can ease the notation by dropping the subscript t on \mathbb{P}_t^κ .

When $\{\xi_t\}$, which is by assumption a Lévy process on $(\Omega, \mathcal{F}, \mathbb{P}^0)$, is restricted to the time frame $\{\xi_s\}_{0 \leq s \leq T}$, then it is also a Lévy process on $(\Omega, \mathcal{F}_T, \mathbb{P}^\kappa)$, for any choice of T . That is to say, it can be shown that for each $T \geq 0$ the process $\{\xi_s\}_{0 \leq s \leq T}$ has stationary and independent increments under \mathbb{P}^κ .

But when $\{\xi_s\}_{0 \leq s \leq T}$ is regarded as a process on $(\Omega, \mathcal{F}_T, \mathbb{P}^\kappa)$, its properties shift: if $\{p_0, q_0, \nu_0(dz)\}$ is the characteristic triplet of $\{\xi_t\}_{0 \leq t \leq T}$ when it is regarded as a Lévy process on $(\Omega, \mathcal{F}, \mathbb{P}^0)$, then on $(\Omega, \mathcal{F}, \mathbb{P}^\kappa)$ the process has a transformed characteristic triplet $\{p_\kappa, q_\kappa, \nu_\kappa(dz)\}$ of the form

$$p_\kappa = p_0 + \kappa q_0 + \int_{\mathbb{R}} (e^{\kappa z} - 1) \mathbf{1}(|z| < 1) z \nu(dz), \quad q_\kappa = q_0, \quad \nu_\kappa(dz) = e^{\kappa z} \nu(dz). \quad (52)$$

Such a shift is called an Esscher transformation [35]. Thus, a Lévy process, when viewed from the untransformed probability space, has different characteristics from those of the same process when it is viewed from the Esscher-transformed probability space.

The idea of a *Lévy information process* involves a similar construction, where we randomize the parameter of the Esscher transformation. Let $(\Omega, \mathcal{F}, \mathbb{P}^0)$, $\{\xi_t\}_{t \geq 0}$, and $\{\mathcal{F}_t\}_{t \geq 0}$ be as above, and let X be an integrable random variable such that $\{\xi_t\}$ and X are \mathbb{P}^0 -independent. Let $\{\mathcal{G}_t\}_{t \geq 0}$ be the filtration generated jointly by $\{\xi_t\}$ and X . Then for each $t \geq 0$ we have

$$\mathcal{G}_t = \sigma[\{\xi_s\}_{0 \leq s \leq t}, X], \quad (53)$$

and clearly it holds that $\mathcal{F}_t \subset \mathcal{G}_t$. The next step is to define a new probability measure \mathbb{P}_t^X on \mathcal{G}_t by setting

$$\mathbb{P}_t^X(A) = \mathbb{E}^{\mathbb{P}^0} [\Lambda_t^X \mathbf{1}(A)], \quad \Lambda_t^X = e^{X\xi_t - \psi(X)t}, \quad (54)$$

for any $A \in \mathcal{G}_t$. It is straightforward to check that the process $\{\Lambda_s^X\}_{0 \leq s \leq t}$ is a martingale on $(\Omega, \mathcal{G}_t, \mathbb{P}^0)$, which ensures that the measures $\{\mathbb{P}_t^X(A)\}_{t \in \mathbb{R}^+}$ are compatible for various values of t , so we can drop the t and write $\mathbb{P}^X(A)$ for the transformed measure. To proceed further we need the following formula for the conditional expectation with respect to \mathcal{F}_t .

Proposition 1. *For any integrable random variable Z on $(\Omega, \mathcal{G}_t, \mathbb{P}^X)$ it holds that*

$$\mathbb{E}^{\mathbb{P}^X} [Z | \mathcal{F}_t] = \frac{\mathbb{E}^{\mathbb{P}^0} [e^{X\xi_t - \psi(X)t} Z | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}^0} [e^{X\xi_t - \psi(X)t} | \mathcal{F}_t]}. \quad (55)$$

Proof. We recall Kolmogorov's definition of conditional expectation, given in section 1. In the present situation the role of Y is played by the right side of (55), so we must show that

$$\mathbb{E}^{\mathbb{P}^X}[Y\mathbb{1}(A)] = \mathbb{E}^{\mathbb{P}^X}[Z\mathbb{1}(A)] \quad (56)$$

for any \mathcal{F}_t -measurable set A , where

$$Y = \frac{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} Z | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} | \mathcal{F}_t]}. \quad (57)$$

But if A is \mathcal{F}_t -measurable we obtain

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^X}[Y\mathbb{1}(A)] &= \mathbb{E}^{\mathbb{P}^0}\left[e^{X\xi_t - \psi(X)t} \frac{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} Z | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} | \mathcal{F}_t]} \mathbb{1}(A)\right], \\ &= \mathbb{E}^{\mathbb{P}^0}\left[e^{X\xi_t - \psi(X)t} \frac{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} Z\mathbb{1}(A) | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} | \mathcal{F}_t]}\right], \\ &= \mathbb{E}^{\mathbb{P}^0}\left[\mathbb{E}^{\mathbb{P}^0}\left[e^{X\xi_t - \psi(X)t} \frac{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} Z\mathbb{1}(A) | \mathcal{F}_t]}{\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} | \mathcal{F}_t]} \middle| \mathcal{F}_t\right]\right] \\ &= \mathbb{E}^{\mathbb{P}^0}\left[\mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} Z\mathbb{1}(A) | \mathcal{F}_t]\right] \\ &= \mathbb{E}^{\mathbb{P}^X}[Z\mathbb{1}(A)], \end{aligned} \quad (58)$$

and thus we deduce (56) by repeated use of the tower property. \square

In fact, (55) arises naturally as a form of the so-called Kallianpur–Striebel formula [36]. A special case of (55) is particularly useful.

Proposition 2. *Let the function $f: \mathbb{R} \rightarrow \mathbb{R}$ be such that $f(X)$ is integrable. Then we have*

$$\mathbb{E}^{\mathbb{P}^X}[f(X) | \mathcal{F}_t] = \frac{\int f(x) e^{x\xi_t - \psi(x)t} \mu(dx)}{\int e^{x\xi_t - \psi(x)t} \mu(dx)}, \quad (59)$$

where the distribution of X is given by

$$\mathbb{P}(X \leq a) = \int_{-\infty}^a \mu(dx). \quad (60)$$

Now suppose that $\{\xi_t\}_{t \geq 0}$ is a Lévy process on $(\Omega, \mathcal{F}, \mathbb{P}^0)$ with Lévy exponent $\psi(\alpha)$ for $\alpha \in C$. Then, for any times t and T such that $0 \leq t \leq T$ we have

$$\begin{aligned} \mathbb{E}^{\mathbb{P}^X}[e^{\alpha\xi_t} | \mathcal{F}_X] &= \mathbb{E}^{\mathbb{P}^0}[e^{X\xi_t - \psi(X)t} e^{\alpha\xi_t} | \mathcal{F}_X] \\ &= \mathbb{E}^{\mathbb{P}^0}[e^{(X+\alpha)\xi_t - \psi(X)t} | \mathcal{F}_X] \\ &= e^{(\psi(X+\alpha) - \psi(X))t}, \end{aligned} \quad (61)$$

since X and $\{\xi_t\}_{0 \leq t \leq T}$ are independent on $(\Omega, \mathcal{F}_T, \mathbb{P}^0)$, and this gives (43). A similar calculation shows that $\{\xi_t\}_{t \geq 0}$ has \mathcal{F}_X -conditionally stationary and independent increments under \mathbb{P}^X . We thus deduce that $\{\xi_t\}_{t \geq 0}$ is a Lévy information process on $(\Omega, \mathcal{F}, \mathbb{P}^X)$ and hence that Lévy information processes can be constructed for any noise type.

8. Quantum state reduction with Lévy jumps

Suppose that we are given a quantum system defined on a finite dimensional Hilbert space with Hamiltonian \hat{H} and initial state $\hat{\rho}_0$. We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which we define a random variable H taking values in the set $\{E_j\}_{j=1, \dots, n}$ such that (33) holds. We define a Lévy information process $\{\xi_t\}_{t \geq 0}$ carrying the information of H such that

$$\frac{1}{t} \log \mathbb{E}[\exp \alpha \xi_t | \mathcal{F}_H] = \psi(\alpha + \lambda H) - \psi(\lambda H), \quad (62)$$

where λ is a model parameter with the units

$$\lambda \sim [\text{energy}]^{-1}. \quad (63)$$

Our conventions going forward are such that the Lévy information process is dimensionless, as is the argument of the Lévy exponent. Thus H has units of energy, t has units of time, and the Lévy exponent has units of inverse time.

Now consider the situation where the initial state is pure, so $\hat{\rho}_0 = |\psi_0\rangle\langle\psi_0|$ for some given initial state vector $|\psi_0\rangle$. Then for the dynamics of a Lévy-driven state vector leading to a collapse of the wave function to an energy eigenstates we generalize the approach laid out in section 4 and look at a model of the form

$$|\psi_t\rangle = \sum_{j=1}^n \sqrt{\pi_{jt}} e^{-i\hbar^{-1}E_j t} |E_j\rangle, \quad \pi_{jt} = \mathbb{E}[\mathbb{1}\{H = E_j\} | \mathcal{F}_t], \quad (64)$$

where the $|E_j\rangle$ are the normalized Lüders eigenstates given by (31) and π_{jt} is the conditional probability for reduction to a state with energy E_j , given Lévy information. Then we apply proposition 2 in the case for which the distribution of the random variable $X = \lambda H$ is

$$\mu(dx) = \sum_{j=1}^n p_j \delta_{\lambda E_j}(dx), \quad (65)$$

where $\delta_y(dx)$ denotes the Dirac measure concentrated at y , and we obtain

$$\pi_{jt} = \frac{p_j \exp(\lambda E_j \xi_t - \psi(\lambda E_j)t)}{\sum_{k=1}^n p_k \exp(\lambda E_k \xi_t - \psi(\lambda E_k)t)} \quad (66)$$

for the conditional probabilities. That gives us our model for the collapse dynamics of a state vector driven by Lévy information.

More generally, for the dynamics of a Lévy-driven density matrix leading to a reduction of the initial state to an energy eigenstate, we propose a model of the form

$$\hat{\rho}_t = \frac{e^{-i\hbar^{-1}\hat{H}t + \frac{1}{2}\lambda\hat{H}\xi_t - \frac{1}{2}\psi(\lambda\hat{H})t} \hat{\rho}_0 e^{i\hbar^{-1}\hat{H}t + \frac{1}{2}\lambda\hat{H}\xi_t - \frac{1}{2}\psi(\lambda\hat{H})t}}{\text{tr}\left(\hat{\rho}_0 e^{\lambda\hat{H}\xi_t - \psi(\lambda\hat{H})t}\right)}. \quad (67)$$

It is straightforward to check that when $\hat{\rho}_0$ is pure, our model for the dynamics of the density matrix is consistent with the state vector dynamics considered above. One can also check that the associated mean density matrix satisfies a deterministic dynamical equation of the Lindblad type. This is far from obvious but the proof will be given in section 9. Then in section 10 we show that the model leads to reduction of the state to an energy eigenstate of the Lüders type satisfying the Born rule.

One can see at a glance that (67) reduces to an expression of the form (35) in the Brownian case, but the precise relation may not be immediately obvious, since the units of the parameter σ are not the same as those of λ . In the Brownian situation it is convenient to choose units such

that B_t has dimensions of square-root time, in which case the information process defined by $\sigma Ht + B_t$ likewise has units of square-root time, which implies that the units of the parameter σ are given by (3). In the Brownian case such choices are convenient and widely used.

If, however, one wishes to treat Brownian motion as a species of Lévy process alongside and in additive combination with other Lévy processes, one needs a single convention that traverses the Lévy category. This means that Lévy processes should be made dimensionless.

After all, there are many examples of Lévy processes that take the form of counting processes, which are dimensionless in their natural setting. The dimensionless Brownian motion with drift associated with the Gaussian exponent (44) takes the form $pt + q^{1/2}B_t$, where $\{B_t\}_{t \geq 0}$ is a standard Brownian motion (with units of square-root time), and the corresponding dimensionless Lévy information process $\{\xi_t\}_{t \geq 0}$ with conditional exponent (62) is given by

$$\xi_t = q\lambda Ht + q^{1/2}B_t. \quad (68)$$

The Brownian information process introduced in section 4 is obtained by dividing (68) by $q^{1/2}$ and setting $\sigma = q^{1/2}\lambda$, which gives (3). This may seem little complicated, but there is a clash between the conventions of physicists, who treat Brownian motion as if it has the physical units of square-root time, and probability theorists, who regard Brownian motion (and time itself) as dimensionless. As long as we work with Brownian motion alone, either convention will do, but once general Lévy processes are brought into play, we find that the probabilistic conventions work better.

A compromise can be reached by choosing the second as the unit of time and treating q as a model parameter. Then if we choose $q = 1$ Hz we recover the formulae that we used earlier for a standard Brownian information process with parameter σ . But in general we need a flexible value for q , since it characterizes the weighting of the Gaussian component of a Lévy process relative to its other components.

In fact, once adjustments are made to take into account the units, one can show that the main results mentioned earlier in connection with the Brownian model go through for a Lévy model. Hence from a physical point of view, we reach the important conclusion that there is no obvious reason *a priori* to prefer a Brownian model over any other Lévy model.

9. Calculation of the decoherence rate

Now consider the mean density matrix $\hat{\mu}_t = \mathbb{E}[\hat{\rho}_t]$. Here the expectation is calculated under the physical measure and it can be worked out explicitly using proposition 1 and a change-of-measure trick. In particular, writing \mathbb{P}^0 for the base measure, we use the tower property and the fact that ξ_t is F_t -measurable to show that

$$\begin{aligned} \mathbb{E}^{\mathbb{P}}[\hat{\rho}_t] &= \mathbb{E}^{\mathbb{P}^0} \left[e^{\lambda H \xi_t - \psi(\lambda H)t} \hat{\rho}_t \right] \\ &= \mathbb{E}^{\mathbb{P}^0} \left[\mathbb{E}^{\mathbb{P}^0} \left[e^{\lambda H \xi_t - \psi(\lambda H)t} \hat{\rho}_t \mid \mathcal{F}_t \right] \right] \\ &= \mathbb{E}^{\mathbb{P}^0} \left[\int_{\mathbb{R}} e^{\lambda x \xi_t - \psi(\lambda x)t} \mu(dx) \hat{\rho}_t \right], \end{aligned} \quad (69)$$

where $\mu(dx)$ is the probability measure of H . Now, by (33) and (65) we have

$$\int_{\mathbb{R}} e^{\lambda x \xi_t - \psi(\lambda x)t} \mu(dx) = \sum_{k=1}^n p_k \exp(\lambda E_k \xi_t - \psi(\lambda E_k)t) = \text{tr} \left(\hat{\rho}_0 e^{\lambda \hat{H} \xi_t - \psi(\lambda \hat{H})t} \right), \quad (70)$$

since $\hat{H} = \sum_{j=1}^n E_j \hat{\Pi}_j$. The term in the denominator of (67) gets cancelled and we obtain

$$\mathbb{E}^{\mathbb{P}}[\hat{\rho}_t] = \mathbb{E}^{\mathbb{P}^0} \left[e^{-i\hbar^{-1}\hat{H}t + \frac{1}{2}\lambda\hat{H}\xi_t - \frac{1}{2}\psi(\lambda\hat{H})t} \hat{\rho}_0 e^{i\hbar^{-1}\hat{H}t + \frac{1}{2}\lambda\hat{H}\xi_t - \frac{1}{2}\psi(\lambda\hat{H})t} \right]. \quad (71)$$

Because the exponential moments of ξ_t under \mathbb{P}^0 can be expressed in terms of the Lévy exponent, we are able to work out the following exact expression for the matrix elements of $\hat{\rho}_t$ with respect to the energy eigenspace projectors:

$$\hat{\Pi}_m \hat{\rho}_t \hat{\Pi}_n = e^{-i\hbar^{-1}(E_m - E_n)t + \psi\left(\frac{1}{2}\lambda(E_m + E_n)\right)t - \frac{1}{2}\psi(\lambda E_m)t - \frac{1}{2}\psi(\lambda E_n)t} \hat{\Pi}_m \hat{\rho}_0 \hat{\Pi}_n. \quad (72)$$

Thus, more succinctly, we can write

$$\hat{\Pi}_m \hat{\rho}_t \hat{\Pi}_n = \exp\left(-i\hbar^{-1}(E_m - E_n)t - \Gamma_{mn}t\right) \hat{\Pi}_m \hat{\rho}_0 \hat{\Pi}_n, \quad (73)$$

where

$$\Gamma_{mn} = \frac{1}{2}\psi(\lambda E_m) + \frac{1}{2}\psi(\lambda E_n) - \psi\left(\frac{1}{2}\lambda(E_m + E_n)\right). \quad (74)$$

Then as a consequence of the Lévy–Khinchine representation (38) we deduce that

$$\Gamma_{mn} = \frac{1}{8}q\lambda^2(E_m - E_n)^2 + \frac{1}{2} \int_{\mathbb{R}} \left(e^{\frac{1}{2}\lambda E_m z} - e^{\frac{1}{2}\lambda E_n z} \right)^2 \nu(dz). \quad (75)$$

The key point is that Γ_{mn} vanishes along the diagonal and is strictly positive for $m \neq n$. This positivity can also be seen to follow from the fact that the Lévy exponent is strictly convex. We conclude that *the mean density matrix diagonalizes as t gets large*. This is the decoherence effect induced by the reduction process.

The dynamical equation satisfied by the mean density matrix can be worked out by differentiating (73) and takes the form

$$\begin{aligned} \frac{d\hat{\rho}_t}{dt} = & i[\hat{H}, \hat{\rho}_t] + \frac{1}{4}q\lambda^2 \left(\hat{H}\hat{\rho}_t\hat{H} - \frac{1}{2}\hat{\rho}_t\hat{H}^2 - \frac{1}{2}\hat{H}^2\hat{\rho}_t \right) \\ & + \int_{-\infty}^{\infty} \left(\hat{L}(z)\hat{\rho}_t\hat{L}(z) - \frac{1}{2}\hat{L}^2(z)\hat{\rho}_t - \frac{1}{2}\hat{\rho}_t\hat{L}^2(z) \right) \nu(dz), \end{aligned} \quad (76)$$

which is evidently of the Lindblad type, where

$$\hat{L}(z) = e^{\frac{1}{2}\lambda\hat{H}z}. \quad (77)$$

This is consistent with the idea that if a stochastic modification of standard quantum mechanics is to avoid causality violation it must lead to Lindblad-type dynamics for the mean density matrix [6, 19, 29, 37, 38], and we see that this condition is satisfied by the class of models presently under consideration.

Indeed, as [38] puts the matter, the overall structure of an objective reduction model is fixed by two natural physical requirements: ‘The first is the requirement of state vector normalization—the unit norm of the state vector should be maintained in time. The second is the requirement that there should be no faster than light signaling—the density matrix averaged over the noise should satisfy a linear evolution equation of Lindblad form.’

Some insight into the nature of the decoherence process can be gained if we analyze the terms of (75). Let us consider first the Brownian case, which just involves the first term of (75). In this case we can set $\sigma^2 = q\lambda^2$ and we find that the decoherence rate is given by

$$\Gamma_{mn} = \frac{1}{8}\sigma^2(E_m - E_n)^2 \quad (78)$$

for the matrix element corresponding to a typical pair of energy levels, where σ has units of the form (3). This result for the decoherence rate is consistent with rule of thumb that the reduction time scale in the Brownian case goes like $\tau_R \sim 1/\sigma^2 V_0$, as we mentioned earlier in section 1, where V_0 denotes the initial value of the squared uncertainty in the energy. For clearly, V_0 can be expressed as a weighted sum of squares of differences of energy eigenvalues. For example, for a two-level system one has $V_0 = p_1 p_2 (E_1 - E_2)^2$, in the case of a three-level systems one has $V_0 = p_1 p_2 (E_1 - E_2)^2 + p_1 p_3 (E_1 - E_3)^2 + p_2 p_3 (E_2 - E_3)^2$, and so on.

The phenomenological case for energy-based reduction with a Brownian noise has been investigated extensively [10, 12, 15, 17, 38, 39]. If one takes the view that state reduction is linked in some way to gravitation, as many have, then a reasonable guess for σ based on dimensional analysis is that it is given by a relation of the form

$$\sigma^2 \sim M_p^{-2} T_p^{-1} = 2.8 \text{ MeV}^{-2} \text{ s}^{-1}. \quad (79)$$

An interesting feature of this conjecture is that the large numbers of the Planck mass M_p and the Planck time T_p cancel, and one is left with a laboratory-scale value for the reduction parameter [10]. The data show that such a value for σ is not unreasonable, in the sense that none of the many situations that have been analyzed in detail rule it out decidedly, although there is little by way of direct evidence in favor of it, at least as matters stand.

As an example we can look at the framework proposed by Weinberg [40] in his analysis of the decoherence timescales associated with experiments involving atomic clocks. He derives a Lindblad equation based on his approach and argues that the resulting decoherence rate Γ coming from objective reduction must satisfy a bound of the form $\Gamma T < 1$ where T is the Ramsey time associated with the clock. He concludes that, ‘Unfortunately we have no idea of what target of Γ we should aim at, or even how Γ might vary from one transition to another.’ Weinberg then proceeds to examine two extreme cases, namely where Γ is constant and where Γ scales like $E_e - E_g$ where E_e and E_g denote the excited and ground energy levels of the stable states of the clock. In the case of a $^{133}\text{Cs}^+$ ion, for example, the hyperfine transition frequency is known to great accuracy and is given by $\Delta\nu_{\text{Cs}} = 9.19263177 \text{ GHz}$. This is of course the definition of the Hertz. The corresponding energy difference is then $\Delta E_{\text{Cs}} = h\Delta\nu_{\text{Cs}} \approx 6.091 \times 10^{-17} \text{ erg}$ or equivalently $3.801 \times 10^{-5} \text{ eV}$. The Ramsey time varies according to the type of clock but is typically of the order of a few seconds although Weinberg points to a case involving $^{171}\text{Yb}^+$ for which $T > 600 \text{ s}$. Arguing on this basis he concludes that $\Gamma < 10^{-18} \text{ eV}$ if Γ does not depend on the transition frequency.

In the case of an energy-based reduction model with Brownian noise one can take matters a step further, because the precise dependence of Γ on the transition frequency is given by equation (75) in that model. This allows us to work out a bound on σ . To get a feeling for the numbers involved, let us consider the $^{133}\text{Cs}^+$ hyperfine frequency and use a Ramsey time of 1 s. Then by (75) we have

$$\Gamma_{\text{eg}} = \frac{1}{8} \sigma^2 (\Delta E_{\text{Cs}})^2 < 1 \text{ s}^{-1} \quad (80)$$

and hence $\sigma^2 < 8 (\Delta E_{\text{Cs}})^{-2} \text{ s}^{-1}$ which gives $\sigma^2 < 0.5537 \times 10^{22} \text{ MeV}^{-2} \text{ s}^{-1}$. Thus we obtain an upper bound on σ^2 from the atomic clock data, though not a particularly stringent one, and certainly the Planckian value for σ^2 that we considered in (79) is well within it.

This example illustrates the fact that at the current level of technology it is not easy to identify decisive tests that would rule out energy-based reduction models based on Brownian noise, even though such tests are clearly possible in principle. Let us now turn to the Lévy case. Here there are some surprises. Suppose for simplicity we consider pure jump models for which $|\lambda E_j z| \ll 1$ for all jump sizes in the support of the Lévy measure. In that case we can

make a Taylor expansion of the exponential terms inside the integral with respect to the Lévy measure in our formula (75) for the decoherence rate, neglecting the Brownian terms, and we obtain

$$\Gamma_{mn} = \frac{1}{8} \lambda^2 (E_m - E_n)^2 \int_{\mathbb{R}} z^2 \nu(dz). \quad (81)$$

The surprising thing here is that when this approximation is valid (i.e. for small jumps, small energies, small λ) the expression for the decoherence rate is of the same form as that of the Brownian model, with the second moment of the Lévy measure playing the role of the q parameter. Thus, as long as the effects of the Lévy model are in some sense perturbative, they do not qualitatively change the conclusions of the Brownian reduction models.

We proceed to another surprise. Weinberg [40] appears in his analysis of atomic clocks simply to have assumed that decoherence rate depends on the transition frequencies, but not the overall levels of the energy eigenvalues. This is in certain respects a plausible assumption, and indeed it holds in Brownian reduction models. But in the Lévy reduction models the situation is different. In general, the decoherence rate depends on the overall levels of the energy eigenvalues as well as on their differences. This can be seen if we write (75) as

$$\Gamma_{mn} = 2 \int_{\mathbb{R}} e^{\frac{1}{2} \lambda (E_m + E_n) z} \sinh^2 \left(\frac{1}{4} \lambda (E_m - E_n) z \right) \nu(dz), \quad (82)$$

again neglecting the Brownian terms. For simplicity, suppose we consider the case of a spectrally positive Lévy process (positive jumps) in the situation where the energy levels are non-negative. In that case, the decoherence rate is clearly an increasing function of $E_m + E_n$. Then even in the situation where the energy gaps are small, i.e. such that $\lambda |E_m - E_n| z \ll 1$, one will get a high rate of decoherence if the overall energy levels are high, satisfying $\lambda (E_m + E_n) z \gg 1$. Thus, large systems will decohere quickly, even if the transition frequencies are small, whereas small systems will not decohere.

In this respect the Lévy models differ fundamentally from their Brownian counterparts. In particular, noting that $\sinh x \sim x$ for $x \ll 1$ we see that (82) reduces to

$$\Gamma_{mn} \approx \frac{1}{8} \lambda^2 (E_m - E_n)^2 \int_{\mathbb{R}} e^{\frac{1}{2} \lambda (E_m + E_n) z} z^2 \nu(dz), \quad (83)$$

similar to the Gaussian case in its dependence on the transition frequency, but the effect of the large energies is to enhance the effective value of the q parameter. For example, in the case of a Poisson process of intensity m_ν , for which the jumps are of size unity, the Lévy measure is given by $\nu(dz) = m_\nu \delta_1(dz)$ and we obtain

$$\Gamma_{mn} \approx \frac{1}{8} m_\nu e^{\frac{1}{2} \lambda (E_m + E_n)} \lambda^2 (E_m - E_n)^2, \quad (84)$$

for which the effective q factor takes the form

$$\tilde{q}_{mn} = m_\nu e^{\frac{1}{2} \lambda (E_m + E_n)}. \quad (85)$$

This example shows how the decoherence rate increases exponentially in the Poisson model when the overall energy scale is increased.

In fact, the integral on the right hand side of equation (83) can be computed explicitly in terms of the Lévy exponent and we get

$$\Gamma_{mn} \approx \frac{1}{8} \lambda^2 (E_m - E_n)^2 \psi'' \left(\frac{1}{2} \lambda (E_m + E_n) \right), \quad (86)$$

which is positive on account of the convexity of the Lévy exponent. Thus, we have

$$\tilde{q}_{mn} = \psi''(\frac{1}{2}\lambda(E_m + E_n)), \quad (87)$$

which can be worked out explicitly in the case of various examples. Note that $\psi''(\alpha)$ is an increasing function of its argument if the Lévy process is spectrally positive.

In the case of a compound Poisson process, for which $\psi(\alpha) = m_\nu(\phi(\alpha) - 1)$, where $\phi(\alpha)$ is the moment generating function of the random jump size, we obtain $\tilde{q} = m_\nu \phi''(\frac{1}{2}\lambda(E_m + E_n))$. As a specific example of such a process, with spectral positivity, we look at the situation for which the jumps are exponentially distributed, with probability density $\mathbb{P}[Y \in dy] = \beta e^{-\beta y} dy$ for $y > 0$, with $\beta > 0$. The moment generating function is

$$\mathbb{E}[e^{\alpha Y}] = \frac{1}{(1 - \beta^{-1}\alpha)}, \quad (88)$$

with $\alpha < \beta$. Thus, for $\frac{1}{2}\lambda(E_m + E_n) < \beta$ we deduce that the effective q factor takes the form

$$\tilde{q}_{mn} = m_\nu \frac{\beta}{(\beta - \frac{1}{2}\lambda(E_m + E_n))^3} \quad (89)$$

in the case of a compound Poisson process with exponentially distributed jumps. As another example, we can consider the case of a gamma information process, which is applicable in the situation in which the energy eigenvalues satisfy $0 < \lambda E_m < 1$, the Lévy exponent is given by the expression $\psi(\alpha) = -m_\nu \log(1 - \alpha)$, for $0 < \alpha < 1$, and therefore

$$\tilde{q}_{mn} = m_\nu \frac{1}{(1 - \frac{1}{2}\lambda(E_m + E_n))^2}. \quad (90)$$

Finally, we remark on the implications of this analysis for the measurement problem. If measurement takes the form of making an entanglement of the measuring apparatus with the system being measured, then in the case of a spectrally positive Lévy model there is no need to invoke the idea that large-scale macroscopic superpositions are required for the outcome of the measuring apparatus. The coupling between the system and the apparatus can be such that the different possible outcomes for the system are linked to rather small differences in the overall energy of the apparatus. The state of the apparatus will collapse nonetheless, despite the energy differences being small, on account of the amplification effect we have just discussed, bringing with it a collapse of the state of the small system to which it is coupled. This may explain why, within the Copenhagen interpretation of standard quantum mechanics, the mere act of measuring the energy of a small system forces it into an eigenstate. Likewise our approach may explain why the formation of a latent image in a photographic emulsion may be sufficient to collapse the state of the system being measured even if the image is developed at a later time [8, 41].

10. Proof of reduction

To show that the Lévy-information based models we have introduced in the previous sections have the reduction property, we establish a result that characterizes the asymptotic properties of the exponential martingale associated with a Lévy process.

Proposition 3. *Let $\{\xi_t\}_{t \geq 0}$ be a Lévy process that admits exponential moments. Let S denote the largest open set in \mathbb{R} such that*

$$\mathbb{E}[e^{\kappa \xi_t}] < \infty \quad (91)$$

for $\kappa \in S$. Then for any $\epsilon > 0$ and any $\kappa \in S$ such that $\kappa \neq 0$ it holds that

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(e^{\kappa \xi_t - \psi(\kappa)t} > \epsilon \right) = 0. \quad (92)$$

Proof. We require Cantelli's inequality [42, 43], which is a strengthened version of Chebyshev's inequality holding in the case of a one-sided probability distribution that says that for any square-integrable random variable $Z: \Omega \rightarrow \mathbb{R}$ and any $b \geq 0$ it holds that

$$\mathbb{P}(Z - \mathbb{E}[Z] \geq b) \leq \frac{\text{Var}[Z]}{\text{Var}[Z] + b^2}. \quad (93)$$

Now, for any Lévy process admitting exponential moments we have $\mathbb{E}[\xi_t] = \psi'(0)t$ and $\text{Var}[\xi_t] = \psi''(0)t$, from which by use of Cantelli's inequality we obtain

$$\begin{aligned} \mathbb{P} \left(e^{\kappa \xi_t - \psi(\kappa)t} > \epsilon \right) &= \mathbb{P} \left(\xi_t > \kappa^{-1} [\log \epsilon + \psi(\kappa)t] \right) \\ &= \mathbb{P} \left(\xi_t - \mathbb{E}[\xi_t] > \kappa^{-1} [\log \epsilon + (\psi(\kappa) - \kappa \psi'(0))t] \right) \\ &\leq \frac{\psi''(0)t}{\psi''(0)t + \kappa^{-2} [\log \epsilon + (\psi(\kappa) - \kappa \psi'(0))t]^2}. \end{aligned} \quad (94)$$

But the convexity of the Lévy exponent implies that $\psi(\kappa) > \kappa \psi'(0)$ for all $\kappa \in S - \{0\}$, and the claimed result follows immediately. \square

Then to show that the dynamical state process (67) reduces to an energy eigenstate it suffices to establish the following.

Proposition 4. *If the outcome of chance $\omega \in \Omega$ is such that $H(\omega) = E_j$ for some particular value of j , then for any $\epsilon > 0$ it holds that*

$$\lim_{t \rightarrow \infty} \mathbb{P} \left(1 - \text{tr}(\hat{\Pi}_j \hat{\rho}_t) > \epsilon \right) = 0. \quad (95)$$

Proof. It follows as a consequence of (67) and the cyclic property of the trace that

$$\text{tr}(\hat{\Pi}_j \hat{\rho}_t) = \frac{e^{\lambda E_j \xi_t - \psi(\lambda E_j)t} \text{tr}(\hat{\Pi}_j \hat{\rho}_0)}{\sum_i e^{\lambda E_i \xi_t - \psi(\lambda E_i)t} \text{tr}(\hat{\Pi}_i \hat{\rho}_0)}, \quad (96)$$

and hence

$$\text{tr}(\hat{\Pi}_j \hat{\rho}_t) = \frac{\text{tr}(\hat{\Pi}_j \hat{\rho}_0)}{\text{tr}(\hat{\Pi}_j \hat{\rho}_0) + \sum_{i \neq j} e^{\lambda(E_i - E_j)\xi_t - \psi(\lambda E_i)t + \psi(\lambda E_j)t} \text{tr}(\hat{\Pi}_i \hat{\rho}_0)}. \quad (97)$$

Now, conditional on information $H = E_j$, the process $\{\xi_t\}_{t \geq 0}$ is Lévy, with Lévy exponent $\psi(\alpha + \lambda E_j) - \psi(\lambda E_j)$. It follows that the process $\{M_{ijt}\}_{t \geq 0}$ defined for $i \neq j$ by

$$M_{ijt} = e^{\lambda(E_i - E_j)\xi_t - \psi(\lambda E_i)t + \psi(\lambda E_j)t} \quad (98)$$

is an exponential martingale. We know therefore that $\{M_{ijt}\}$ converges to zero by proposition 3, from which we deduce that $\text{tr} \hat{\Pi}_j \hat{\rho}_t$ converges to unity. \square

Thus we have shown that the dynamical model defined by (67) is a state reduction process that carries the initial state to an energy eigenstate in such a way that the associated energy expectation process is a martingale. Indeed, the processes obtained by transvecting the state with any of the energy projection operators and taking the trace are likewise martingales. It follows that the actual probability of collapse to a Lüders eigenstate of energy E_j agrees with the probability calculated via the Born rule. Here we refer to the extended form of the Born rule

fully applicable to density matrices and a possibly degenerate spectrum for the Hamiltonian. Results of this type have been known for some time in the case of Brownian noise [6, 7, 10, 12, 14, 16, 20, 22], both for dynamics of state vectors and the dynamics of density matrices, but the extension to the Lévy class is new.

The new degrees of freedom that can be expressed in a Lévy model are embodied in the structure of the Lévy exponent for the underlying noise, or equivalently the parameter q together with the Lévy measure. In a number of situations one can construct explicit models of Lévy information processes [33]. In such cases we can go further and use the model as a basis for simulation studies. In particular, in addition to those based on Brownian noise and Poisson noise, explicit models can be constructed for information processes based on Lévy processes with infinite activity, including various examples that are well known in the theory of finance and insurance, such as (a) the gamma process [44–46] and (b) the variance gamma process [47, 48]. Such infinite activity information processes are different in character from their Brownian and Poisson counterparts.

In closing, we comment that our explicit formula for the decoherence rate (75) marks a clear distinction between quantum state reduction models based on Brownian noise and the more general category of reduction models based on Lévy noise, and paves the way towards possible applications of such models, some of which we have touched on in the present paper. The idea that changing the nature of the underlying noise might give some new insights into the measurement problem comes as a surprise and we hope to pursue the topic further.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

We are grateful for comments from participants at the conference General Relativity, Quantum Mechanics and Everything in Between, Celebrating 92 Springs of Professor Lawrence Paul Horwitz, April 2022, Ariel University, Israel, where this work was presented. We thank Christopher Fuchs, Blake Stacey, and other seminar participants at the Department of Physics, University of Massachusetts Boston, for stimulating discussions in May 2022. We also thank Lajos Diósi, Philip Pearle and the anonymous referees for their comments. D C B acknowledges support from the Russian Science Foundation (Grant 20-11-20226) and the Templeton Foundation (Grant 62210). The opinions expressed in this publication are those of the authors and do not necessarily reflect the views of the Templeton Foundation.

Appendix. Examples of Lévy reduction models

It may be worthwhile if we explain things in more detail with a few concrete examples of Lévy reduction models to show the occurrence of wave function collapse systematically. We look at reduction models for three canonical examples of Lévy processes: (a) Brownian motion, (b) the Poisson process, and (c) the gamma process, these being representative of the continuous case, the finite activity case, and the infinite activity case. The Brownian model is of course well studied in the literature, but by presenting this example in the same notation and in parallel with the two other cases, we hope that the models based on pure jump Lévy processes will be

clearer. For simplicity we consider a quantum system with only two energy levels E_1 and E_2 . The generalization to situations with more than two energy levels is straightforward and can be left as an exercise.

Throughout the discussion that follows we are given a quantum system based on a finite dimensional Hilbert space with initial state $\hat{\rho}_0$ and a Hamiltonian operator with two energy levels. For the projectors on to the Hilbert subspaces with energies E_1 and E_2 respectively we write $\hat{\Pi}_1$ and $\hat{\Pi}_2$. We consider a system for which reduction is driven by a Lévy information process $\{\xi_t\}_{t \geq 0}$ based on an underlying Lévy noise with Lévy exponent $\psi(\alpha)$, satisfying

$$\frac{1}{t} \log \mathbb{E}[\exp \alpha \xi_t | \mathcal{F}_H] = \psi(\alpha + \lambda H) - \psi(\lambda H), \quad (99)$$

where λ is a model parameter and H is a random variable taking the value E_1 with probability $p_1 = \text{tr}(\hat{\Pi}_1 \hat{\rho}_0)$ and the value E_2 with probability $p_2 = \text{tr}(\hat{\Pi}_2 \hat{\rho}_0)$.

Brownian noise

We consider a dimensionless information process of the form $\xi_t = q\lambda Ht + q^{1/2}B_t$, where the parameter q has dimensions of inverse time and $\{B_t\}_{t \geq 0}$ is a standard Brownian motion, with vanishing drift and variance t . For simplicity we set $q = 1$ Hz and then the information process takes the form

$$\xi_t = \lambda Ht + B_t. \quad (100)$$

The Lévy exponent is given in the Brownian case by

$$\psi(\alpha) = \frac{1}{2} \alpha^2 \quad (101)$$

for $\alpha \in \mathbb{R}$. A straightforward calculation shows that for the conditional Lévy exponent in this case we have

$$\frac{1}{t} \log \mathbb{E}[\exp \alpha(\lambda Ht + B_t) | \mathcal{F}_H] = \alpha \lambda H + \frac{1}{2} \alpha^2 = \frac{1}{2} (\alpha + \lambda H)^2 - \frac{1}{2} (\lambda H)^2, \quad (102)$$

confirming that (100) is an information process, satisfying (99). The exponential martingale $\{M_t^\kappa\}_{t \geq 0}$ with parameter κ associated with a standard Brownian motion takes the form

$$M_t^\kappa = e^{\kappa B_t - \frac{1}{2} \kappa^2 t}. \quad (103)$$

Such processes are widely used in the theory of finance to model the random fluctuations of share prices. Since B_t is normally distributed with mean 0 and variance t , then for any value of $t > 0$, no matter how large, clearly we have

$$\mathbb{E} \left[e^{\kappa B_t - \frac{1}{2} \kappa^2 t} \right] = 1. \quad (104)$$

It may then come as a surprise that for any $\epsilon > 0$, no matter how small, we have

$$\lim_{t \rightarrow \infty} \mathbb{P} \left[e^{\kappa B_t - \frac{1}{2} \kappa^2 t} > \epsilon \right] = 0. \quad (105)$$

That is to say, the exponential Brownian motion process converges to zero in probability. This can be shown with an application of Cantelli's inequality. For those who have any doubts, we can check the result directly by use of an old-school probability calculation. Writing $N(x)$, $x \in \mathbb{R}$, for the standard normal distribution function, we have

$$\begin{aligned}\mathbb{P}\left[e^{\kappa B_t - \frac{1}{2}\kappa^2 t} > \epsilon\right] &= \mathbb{P}\left[\frac{1}{t^{1/2}}B_t > \frac{1}{\kappa t^{1/2}}\log \epsilon + \frac{1}{2}\kappa t^{1/2}\right] \\ &= 1 - N\left(\frac{1}{\kappa t^{1/2}}\log \epsilon + \frac{1}{2}\kappa t^{1/2}\right),\end{aligned}\quad (106)$$

since the random variable $t^{-1/2}B_t$ is normally distributed with mean zero and variance unity. But $\lim_{x \rightarrow \infty} N(x) = 1$ and the claimed result follows immediately. Similar results hold for the exponential martingales associated with other Lévy processes.

Now we are in a position to look at the conditional probability for the outcome of the reduction process to be an eigenstate with energy E_1 (the corresponding calculations for E_2 follow by symmetry). According to the general theory outlined in section 8 we have

$$\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) = \frac{p_1 e^{\lambda E_1 \xi_t - \frac{1}{2}\lambda^2 E_1^2 t}}{p_1 e^{\lambda E_1 \xi_t - \frac{1}{2}\lambda^2 E_1^2 t} + p_2 e^{\lambda E_2 \xi_t - \frac{1}{2}\lambda^2 E_2^2 t}}, \quad (107)$$

by virtue of equation (96), and hence

$$\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) = \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1)\xi_t - \frac{1}{2}\lambda^2 E_2^2 t + \frac{1}{2}\lambda^2 E_1^2 t}}. \quad (108)$$

To show that reduction occurs, we need to prove that if the outcome of chance is that $H = E_1$, then $\lim_{t \rightarrow \infty} \text{tr}(\hat{\Pi}_1 \hat{\rho}_t) = 1$, whereas if the outcome of chance is $H = E_2$ then $\lim_{t \rightarrow \infty} \text{tr}(\hat{\Pi}_1 \hat{\rho}_t) = 0$. This may not be obvious on a casual glance at (108). But the point is that if $H = E_1$ then $\xi_t = \lambda E_1 t + B_t$. Substituting this into (108) we get

$$\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) \Big|_{H=E_1} = \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1)(\lambda E_1 t + B_t) - \frac{1}{2}\lambda^2 E_2^2 t + \frac{1}{2}\lambda^2 E_1^2 t}}, \quad (109)$$

and hence after some rearrangement, we have

$$\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) \Big|_{H=E_1} = \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1)B_t - \frac{1}{2}\lambda^2(E_2 - E_1)^2 t}}. \quad (110)$$

We observe that an exponential martingale of the type we were considering earlier appears in the denominator. This converges to zero in probability, and it follows that for any $\epsilon > 0$ it holds that

$$\lim_{t \rightarrow \infty} \mathbb{P}\left[\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) \Big|_{H=E_1} < 1 - \epsilon\right] = 0, \quad (111)$$

which signifies that reduction to a state of energy E_1 has taken place. A similar calculation then shows that

$$\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) \Big|_{H=E_2} = \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1)B_t + \frac{1}{2}\lambda^2(E_2 - E_1)^2 t}}, \quad (112)$$

and hence that

$$\lim_{t \rightarrow \infty} \mathbb{P}\left[\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) \Big|_{H=E_2} > \epsilon\right] = 0. \quad (113)$$

Poisson noise

The Poisson process $\{N_t\}_{t \geq 0}$ with intensity $m > 0$ is a nondecreasing jump process with unit jumps at the rate m . The Lévy exponent is given by

$$\psi(\alpha) = m(e^\alpha - 1) \quad (114)$$

and the corresponding Lévy measure takes the form

$$\nu(dz) = m \delta_1(dz), \quad (115)$$

where $\delta_1(dz)$ denotes the Dirac measure concentrated at jump size unity. The Poisson process takes values in the nonnegative integers, whose distribution at time t is

$$\mathbb{P}[N_t = n] = e^{-mt} \frac{(mt)^n}{n!}. \quad (116)$$

For the corresponding exponential martingale with parameter κ we have

$$M_t^\kappa = e^{\kappa N_t - m(e^\kappa - 1)t}. \quad (117)$$

A calculation gives $\mathbb{E}[N_t] = mt$ and $\text{Var}[N_t] = mt$. Cantelli's inequality then tells us that

$$\mathbb{P}\left(e^{\kappa \xi_t - \psi(\kappa)t} > \epsilon\right) \leq \frac{mt}{mt + \kappa^{-2} [\log \epsilon + m(e^\kappa - 1 - \kappa)t]^2}. \quad (118)$$

One can check that $\inf_{\kappa \in \mathbb{R}} (e^\kappa - 1 - \kappa) = 0$ and hence that for any $\kappa > 0$ the exponential martingale (117) converges to zero in probability as t grows large.

A Poisson information process with parameter m can be modeled by letting $\{N(t)\}_{t \geq 0}$ be a standard Poisson process with parameter m as described above and setting

$$\xi_t = N(e^{\lambda H} t). \quad (119)$$

For the conditional Lévy exponent in this case we have

$$\frac{1}{t} \log \mathbb{E}[\exp \alpha N(e^{\lambda H} t) | \mathcal{F}_H] = m e^{\lambda H} (e^\alpha - 1) = m (e^{\alpha + \lambda H} - 1) - m (e^{\lambda H} - 1), \quad (120)$$

which shows that (119) satisfies (99) and thus is indeed an information process. The conditional probability for reduction to an eigenstate with eigenvalue E_1 is

$$\begin{aligned} \text{tr}(\hat{\Pi}_1 \hat{\rho}_t) &= \frac{p_1 e^{\lambda E_1 \xi_t - m(e^{\lambda E_1} - 1)t}}{p_1 e^{\lambda E_1 \xi_t - m(e^{\lambda E_1} - 1)t} + p_2 e^{\lambda E_2 \xi_t - m(e^{\lambda E_2} - 1)t}} \\ &= \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1) \xi_t - m e^{\lambda E_1} (e^{\lambda(E_2 - E_1)} - 1)t}}. \end{aligned} \quad (121)$$

Therefore,

$$\text{tr}(\hat{\Pi}_1 \hat{\rho}_t) \Big|_{H=E_1} = \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1) \xi_t |_{H=E_1} - m e^{\lambda E_1} (e^{\lambda(E_2 - E_1)} - 1)t}}. \quad (122)$$

However, $\xi_t |_{H=E_1}$ is a standard Poisson process with rate $m e^{\lambda E_1}$, so an exponential martingale appears in the expression above, from which we get (111) and hence reduction.

Gamma noise

Physicists are familiar with Brownian motion and the Poisson process, both of which appear frequently in the literature. The gamma process is a newer idea, dating from the 1950s, exhibiting interesting features with which physicists are perhaps less familiar. Most important among these is the phenomenon of *infinite activity*. By a gamma process with rate m and scale φ we mean a Lévy process $\{\gamma_t\}_{t \geq 0}$ with Lévy exponent

$$\psi(\alpha) = -m \log(1 - \varphi \alpha), \quad (123)$$

where $\alpha < \varphi^{-1}$. The probability density for γ_t is that of the gamma distribution, given by

$$\mathbb{P}(\gamma_t \in dx) = \frac{\varphi^{-mt} x^{mt-1} e^{-x/\varphi}}{\Gamma[mt]} dx \quad (124)$$

for $x > 0$, and zero otherwise, where $\Gamma[a]$ is the gamma function. A calculation using the identity $\Gamma[a+1] = a\Gamma[a]$ then shows that $\mathbb{E}[\gamma_t] = m\varphi t$ and $\text{Var}[\gamma_t] = m\varphi^2 t$. Note that the mean and variance determine the rate and scale. If $\varphi = 1$ then $\{\gamma_t\}$ is called a *standard* gamma process with rate m . If $\varphi \neq 1$ we say that $\{\gamma_t\}$ is a *scaled* gamma process. The Lévy measure associated with $\{\gamma_t\}_{t \geq 0}$ is given by

$$\nu(dz) = m \frac{1}{z} \exp(-\varphi z) dz \quad (125)$$

for $z > 0$, and zero otherwise. It follows that $\nu(\mathbb{R}) = \infty$ and hence that the gamma process has infinite activity. Thus, the jumps are all positive and the number of jumps in any finite interval of time is infinite.

Let $\{\gamma_t\}$ be a standard gamma process with rate m and let the parameter $\kappa \in \mathbb{R}$ be such that $\kappa < 1$. Then the process $\{M_t^\kappa\}$ defined by

$$M_t^\kappa = (1 - \kappa)^{mt} e^{\kappa \gamma_t} \quad (126)$$

is a martingale and by Cantelli's inequality we have

$$\mathbb{P}((1 - \kappa)^{mt} e^{\kappa \gamma_t} > \epsilon) \leq \frac{mt}{mt + \kappa^{-2} [\log \epsilon + m(-\log(1 - \kappa) - \kappa)t]^2}. \quad (127)$$

One can check that $-\log(1 - \kappa) - \kappa > 0$ for all $\kappa < 1$. This follows from the basic logarithmic inequality $\log x \leq x - 1$ for all $x \geq 0$ with equality at $x = 1$. Hence for any $\kappa < 1$ the exponential gamma martingale (126) converges to zero in probability as t grows large.

If we let $\{M_t^\kappa\}$ act as a change of measure density for the transformation $\mathbb{P}^0 \rightarrow \mathbb{P}^\kappa$, then $\{\gamma_t\}$ is a *scaled* gamma process under \mathbb{P}^κ , with rate m and scale $1/(1 - \kappa)$. Thus, the effect of an Esscher transformation on a gamma process is to alter its scale.

Now let $\{\gamma_t\}$ be a standard gamma process with rate m and let the independent random variable X satisfy $X < 1$ almost surely. Then the process $\{\xi_t\}$ defined by

$$\xi_t = \frac{1}{1 - X} \gamma_t \quad (128)$$

is a Lévy information process with signal X and gamma noise. Thus $\{\xi_t\}$ is conditionally a scaled gamma process. Then as a consequence of (123) and (128) we have

$$\frac{1}{t} \ln \mathbb{E}^\mathbb{P} [\exp(\alpha \xi_t) | \mathcal{F}^X] = \frac{1}{t} \ln \mathbb{E}^\mathbb{P} \left[\exp \left(\frac{\alpha \gamma_t}{1 - X} \right) \middle| \mathcal{F}^X \right] = \log \left(1 - \frac{\alpha}{1 - X} \right)^{-m}. \quad (129)$$

Next, we observe that

$$-m \ln \left(1 - \frac{\alpha}{1 - X} \right) = -m \ln(1 - (X + \alpha)) + m \ln(1 - X). \quad (130)$$

It follows that the conditional exponent of $\{\xi_t\}$ takes the form $\psi(\alpha + X) - \psi(X)$, which shows that $\{\xi_t\}$ is an information process.

We proceed to look at the conditional probability for reduction to an energy eigenstate with energy E_1 in the context of a gamma information model. In this case the parameter λ is chosen to satisfy $\max(\lambda E_1, \lambda E_2) < 1$ and we have

$$\begin{aligned}\mathrm{tr}(\hat{\Pi}_1 \hat{\rho}_t) &= \frac{p_1 e^{\lambda E_1 \xi_t} (1 - \lambda E_1)^{mt}}{p_1 e^{\lambda E_1 \xi_t} (1 - \lambda E_1)^{mt} + p_2 e^{\lambda E_2 \xi_t} (1 - \lambda E_2)^{mt}} \\ &= \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1)\xi_t} (1 - \lambda E_2)^{mt} (1 - \lambda E_1)^{-mt}},\end{aligned}\quad (131)$$

and therefore

$$\begin{aligned}\mathrm{tr}(\hat{\Pi}_1 \hat{\rho}_t) \Big|_{H=E_1} &= \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1)\xi_t} \Big|_{H=E_1} (1 - \lambda E_2)^{mt} (1 - \lambda E_1)^{-mt}} \\ &= \frac{p_1}{p_1 + p_2 e^{\lambda(E_2 - E_1)(1 - \lambda E_1)^{-1} \gamma_t} (1 - \lambda E_2)^{mt} (1 - \lambda E_1)^{-mt}}.\end{aligned}\quad (132)$$

However,

$$\left(1 - \frac{\lambda(E_2 - E_1)}{1 - \lambda E_1}\right)^{mt} = (1 - \lambda E_2)^{mt} (1 - \lambda E_1)^{-mt}, \quad (133)$$

and therefore the second term in the denominator on the right side of (132) is an exponential gamma martingale, which converges to zero in probability for large t . Thus we have (111).

ORCID iDs

Dorje C Brody  <https://orcid.org/0000-0002-9242-9554>

Lane P Hughston  <https://orcid.org/0000-0003-1185-934X>

References

- [1] Ghirardi G, Rimini A and Weber T 1986 Unified dynamics for microscopic and macroscopic systems *Phys. Rev. D* **34** 470–91
- [2] Diósi L 1988a Quantum stochastic processes as models for state vector reduction *J. Phys. A: Math. Gen.* **21** 2885–97
- [3] Diósi L 1988b Continuous quantum measurement and Ito formalism *Phys. Lett. A* **129** 419–23
- [4] Pearle P 1989 Combining stochastic dynamical state-vector reduction with spontaneous localization *Phys. Rev. A* **39** 2277–89
- [5] Diósi L 1989 Models for universal reduction of macroscopic quantum fluctuations *Phys. Rev. A* **40** 1165–74
- [6] Gisin N 1989 Stochastic quantum dynamics and relativity *Helv. Phys. Acta* **62** 363–71
- [7] Ghirardi G, Pearle P and Rimini A 1990 Markov processes in Hilbert space and continuous spontaneous localisation of systems of identical particles *Phys. Rev. A* **42** 78–89
- [8] Gisin N and Percival I C 1993 The quantum state diffusion picture of physical processes *J. Phys. A: Math. Gen.* **26** 2245
- [9] Percival I C 1994 Primary state diffusion *Proc. R. Soc. A* **447** 189
- [10] Hughston L P 1996 Geometry of stochastic state vector reduction *Proc. R. Soc. A* **452** 953–79
- [11] Pearle P 2000 Wave function collapse and conservation laws *Found. Phys.* **30** 1145–60
- [12] Adler S L and Horwitz L P 2000 Structure and properties of Hughston's stochastic extension of the Schrödinger equation *J. Math. Phys.* **41** 2485–99
- [13] Adler S L and Brun T A 2001 Generalized stochastic Schrödinger equations for state vector collapse *J. Phys. A: Math. Gen.* **34** 4797–809
- [14] Adler S L, Brody D C, Brun T A and Hughston L P 2001 Martingale models for quantum state reduction *J. Phys. A: Math. Gen.* **34** 8795–820
- [15] Adler S L 2002 Environmental influence on the measurement process in stochastic reduction models *J. Phys. A: Math. Gen.* **35** 841–58
- [16] Brody D C and Hughston L P 2002 Efficient simulation of quantum state reduction *J. Math. Phys.* **43** 5254–61
- [17] Adler S L 2003 Weisskopf–Wigner decay theory for the energy-driven stochastic Schrödinger equation *Phys. Rev. D* **67** 025007

- [18] Brody D C and Hughston L P 2005 Finite-time stochastic reduction models *J. Math. Phys.* **46** 1–7
- [19] Weinberg S 2012 Collapse of the state vector *Phys. Rev. A* **85** 062116
- [20] Brody D C and Hughston L P 2018 Quantum state reduction *Collapse of the Wave Function* ed S Gao (Cambridge: Cambridge University Press) pp 47–74
- [21] Bassi A and Ghirardi G C 2003 Dynamical reduction models *Phys. Rep.* **379** 257–426
- [22] Brody D C and Hughston L P 2006 Quantum noise and stochastic reduction *J. Phys. A: Math. Gen.* **39** 833–76
- [23] Bassi A, Lochan K, Satin S, Singh T P and Ulbricht H 2013 *Rev. Mod. Phys.* **85** 471–562
- [24] Guo S (ed) 2018 *Collapse of the Wave Function* (Cambridge: Cambridge University Press) (<https://doi.org/10.1017/9781316995457>)
- [25] Kolmogorov A N 1956 *Foundations of the Theory of Probability Theory* (New York: Chelsea Publishing Company) Kolmogorov A N 1933 *Grundbegriffe der Wahrscheinlichkeitsrechnung (Ergebnisse Der Mathematik)* Nathan Morrison, translator (Engl. transl.)
- [26] Lindblad G 1976 On the generators of quantum dynamical semigroups *Commun. Math. Phys.* **48** 119–30
- [27] Gorini V, Kossakowski A and Sudarshan E C G 1976 Completely positive dynamical semigroups of N -level systems *J. Math. Phys.* **17** 821–5
- [28] Lüders G 1951 Über die Zustandsänderung durch den Messprozess *Ann. Phys., Lpz.* **8** 322–8
- [29] Gisin N 1990 Weinberg’s non-linear quantum mechanics and supraluminal communications *Phys. Lett. A* **143** 1–2
- [30] Segal I E 1947 Postulates for general quantum mechanics *Ann. Math.* **48** 930–48
- [31] Applebaum D 2009 *Lévy Processes and Stochastic Calculus* 2nd edn (Cambridge: Cambridge University Press)
- [32] Sato K 1999 *Lévy Processes and Infinitely Divisible Distributions* (Cambridge: Cambridge University Press)
- [33] Brody D C, Hughston L P and Yang X 2013 Signal processing with Lévy information *Proc. R. Soc. A* **469** 1–23
- [34] Segall A, Davis M H A and Kailath T 1975 Nonlinear filtering with counting observations *IEEE Trans. Inf. Theory* **21** 143–9
- [35] Esscher F 1932 On the probability function in the collective theory of risk *Scand. Actuar. J.* **15** 175–95
- [36] Kallianpur G and Striebel C 1968 Estimation of stochastic systems: arbitrary system process with additive white noise observation errors *Ann. Math. Stat.* **39** 785–801
- [37] Polchinski J 1990 Weinberg’s nonlinear quantum mechanics and the Einstein–Podolsky–Rosen paradox *Phys. Rev. Lett.* **66** 397–400
- [38] Adler S L 2016 Gravitation and the noise needed in objective reduction models *Quantum Nonlocality and Reality: 50 Years of Bell’s Theorem* ed S Gao (Cambridge: Cambridge University Press) pp 390–9
- [39] Pearle P 2004 Problems and aspects of energy-driven wave-function collapse models *Phys. Rev. A* **69** 042106
- [40] Weinberg S 2016 Lindblad decoherence in atomic clocks *Phys. Rev. A* **94** 042117
- [41] Adler S L 2018 Connecting the dots: Mott for emulsions, collapse models, colored noise, frame dependence of measurements, evasion of the “Free Will Theorem” *Found. Phys.* **48** 1557–67
- [42] Cantelli F P 1928 Sui confini della probabilità *Atti del Congresso Internazionale del Matematici (Bologna)* vol 6 pp 47–59
- [43] Ghosh B K 2002 Probability inequalities related to Markov’s theorem *Am. Stat.* **56** 186–90
- [44] Dickson D C M and Waters H R 1993 Gamma processes and finite-time survival probabilities *ASTIN Bull.* **23** 259–72
- [45] Yor M 2007 Some remarkable properties of gamma processes *Advances in Mathematical Finance* ed R Elliott, M Fu, R Jarrow and J-Y Yen (Basel: Birkhäuser) (https://doi.org/10.1007/978-0-8176-4545-8_3)
- [46] Brody D C, Hughston L P and Macrina A 2008 Dam rain and cumulative gain *Proc. R. Soc. A* **464** 1801–22
- [47] Brody D C, Hughston L P and Mackie E 2012 General theory of geometric Lévy models for dynamic asset pricing *Proc. R. Soc. A* **468** 1778–98
- [48] Madan D and Milne F 1991 Option pricing with VG martingale components *Math. Financ.* **1** 39–55