Bi-local Approach to Higher Spin Gravity

by

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Dedicated to my family

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Preface

It is a great opportunity to do research at High Energy Group at Brown University. Although it is not an easy task however, it is a fun and meaningful endeavor. I would feel really happy if I can contribute to this field even for just a little bit.

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Abstract of "Bi-local Approach to Higher Spin Gravity"

by Qibin Ye, Ph.D., Brown University, May 2013

We review a field theory approach to Higher Spin Gravity in 4 dimensional Anti de Sitter (AdS) space within the framework of the AdS/CFT correspondence. Based on large N collective field theory of vector type models, we develop a bi-local dipole picture of Higher Spin theory. We also describe a geometric (Kahler space) framework for the bi-local theory which applies to $Sp(2N)$ fermions and potentially to the de Sitter (dS) correspondence. We discuss in this framework the structure and size of the bi-local Hilbert space and the implementation of (finite N) exclusion principle. For the correspondence based on free CFTs we first discuss the transformation for $O(N)$ collective field and the Higher spin filed, and then the nature of bulk $1/N$ interactions through an S-matrix which by the Coleman-Mandula theorem is argued to be equal to 1.

CHAPTER 1

Introduction

The AdS/CFT correspondence [**1**] represents a major tool in our understanding of non-perturbative phenomena in gauge theory (and other related systems). Insight into the mechanism behind this duality was obtained through various different tools, such as large N expansion, D-branes and higher symmetries. The explicit construction, even in the simplest models, has not been achieved yet (except in special limits or sub-sectors of the full theory such as the 1/2 BPS case). What characterizes the correspondence is the emergence of AdS spacetime (and of extra Kaluza-Klein dimensions) and even more remarkably of gravitational and stringy degrees of freedom.

Recently a very simple model has been studied (the $O(N)$ vector model) with its duality [**2, 17**] to AdS higher-spin gravity of Vasiliev [**3, 5**]. This proposal, which identifies the critical points of the 3 dimensional $O(N)$ vector model with two versions of the 4 dimensional Vasiliev theory, has received solid support and definite degree of understanding [**17, 18, 20, 28, 29, 30, 39, 40, 41, 44, 53, 54, 45**]. Equally interesting is the correspondence between 2d minimal CFT's and 3d Chern-Simons Higher Spin Gravity [**21, 57, 58, 59, 60, 67**]. These large N dualities involve quantum field theories that have been thought to be understood for some time and a relatively novel version of HS Gravity built on a single Regge trajectory. These theories feature many properties that have been unreachable in String Theory, in particular the structure and explicit form of the higher spin gauge symmetry group. They also offer a potentially solvable framework for studies of black hole formation and de Sitter theory itself [**78, 52**].

In the case of three dimensional $O(N)$ vector field theory, one has two conformally invariant fixed points, the UV and the IR one. The HS duals are given by the same Vasiliev theory [**3, 4, 68, 69, 70, 71, 72**] but with different boundary conditions on the scalar field [**17**]. This provides a simple relationship between a (HS) theory dual to a free N-component scalar field (UV) and the nontrivial dual corresponding to the IR CFT. The correspondence provided by the free $O(N)$ scalar field theory is then of central interest. This theory is characterized by an infinite sequence of conserved currents which are themselves boundary duals of higher spin fields and whose correlation functions represent a point of comparison [**28, 29**] between the two descriptions.

This thesis is about the bi-local approach to higher spin gravity. In chapter two, we discuss the correspondence in the quantum mechanics level which involves the collective dipole model and higher spin particle, which demonstrates that $d + 1$ dimensional AdS spacetime and higher spins are generated in terms of the d dimensional collective dipole. We first introduce the collective dipole as a two-body system with constraints and discuss gauge fixing to time-like or light-cone gauges. Following that, we introduce the higher spin particle in AdS. In the end of this chapter, we give a one-to-one map which shows how the two systems are transformed into one another.

In chapter three, we move on to discuss the higher spin theory correspondence with large N vector model both in the frame work of AdS/CFT and dS/CFT. We briefly describe the AdS/CFT in the example of Higher Spin theory/$O(N)$ vector model by summarizing the work in [**20, 30**]. Then we switch to dS/CFT case. We start by introducing the $Sp(N)$ vector model and then construct a collective field theory of the *Lorentzian* $Sp(2N)$ model which captures the singlet state dynamics of the $Sp(2N)$ vector model. We further establish the bi-local theory as the bulk spacetime representation of de Sitter higher spin gravity by double analytic continuation. We also describe a geometric (pseudo-spin) version of the collective theory which will be seen to incorporate the Grassmannian origin of the field operators. In the end of this chapter, we also give an interpretation of phase transition from a different perspective than shown by Shenker and Yin.

In chapter four, we apply Coleman-Mandula theorem in higher spin theory and introduce S matrix for higher spin theory and show that $S = 1$. We first discuss the differences between "boundary S-matrix" and "collective S-matrix" that we propose. In particular we give an LSZ formula for the S-matrix and evaluate the associated three- and four-point amplitudes using the cubic and quartic vertices of the $1/N$ theory demonstrating the result $S = 1$. In the end of this chapter, we present a construction of a nonlinear bi-local field transformation that linearizes the theory.

CHAPTER 2

Collective Dipole Picture of Higher Spin Theory

1. Overview

In this chapter, we study a discrete bi-particle system which we call *the collective dipole*. A dipole picture was originally contemplated by Fronsdal and Flato [**7**] in the study of Rac representations of the conformal group [**8**]. It was also identified in studies of high energy scattering in QCD [**9**]. It has also appeared in studies of noncommutative field theory in [**10**]. In what follows we describe and study its classical dynamics and work out the details of its map to AdS higher spin particle. This we do in any dimension d showing the reconstruction of higher-spin system in AdS through a canonical transform. As such the collective dipole offers possibly the simplest system for a deeper understanding of the emergence of extra spacetime and higher-spin degrees of freedom.

The content of this chapter is as follows: in section two we describe the collective dipole as a two-body system with constraints and discuss gauge fixing to time-like or light-cone gauges. In section three we describe the system representing a higher spin particle in AdS. In section four we explain how the two systems are transformed into one another through a one-to-one map. This completes the demonstration that $d+1$ dimensional AdS spacetime and higher spins are generated in terms of the d dimensional collective dipole.

2. The Collective Dipole

The large N quantum field theory of the $O(N)$ vector model

$$L = \int d^d x \frac{1}{2}(\partial_\mu \phi^i)(\partial^\mu \phi^i) + V(\phi \cdot \phi), \qquad i = 1, ..., N \tag{2.1}$$

represents a relatively simple field theory for critical phenomena and more recently as a model of AdS_{d+1}/CFT_d correspondence. In three dimensions, besides the free field theory UV fixed point, one also has a nontrivial IR fixed point (for detailed studies see [**11**]). The AdS/CFT duality with Vasiliev's higher spin theory for both fixed points was understood in [**17**] and subsequent more recent work. In any dimension d, the free theory in the large N limit exhibits a duality with a theory of higher spin in $d+1$ dimensions. The origin of higher spins and of the emergence of the extra radial AdS spatial dimension was given [**20**] in terms of bi-local (collective) fields

$$\Psi(x_1^\mu, x_2^\nu) = \sum_{i=1}^{N} \phi^i(x_1^\mu)\phi^i(x_2^\nu), \tag{2.2}$$

where $\mu, \nu = 0, 1, \cdots, d-1$ with the metric diag$(-, +, \cdots, +)$. These fields close a set of Schwinger-Dyson equations with an effective action that leads to a systematic $1/N$ expansion [**16**]. It was argued in [**20, 30**] that this provides a bulk description of the AdS_4 dual higher-spin gravity (for the two conformal fixed points of the three dimensional field theory). This picture was sharpened in the time-like or null-plane quantization scheme, where the bi-local field involves a single time

$$\Psi(t, \vec{x}_1, \vec{x}_2) = \sum_{i=1}^{N} \phi^i(t, \vec{x}_1)\phi^i(t, \vec{x}_2). \tag{2.3}$$

In this case a precise one-to-one map was formulated in [**30**] relating the light-cone higher spin field in AdS_4 and the collective bi-local field

$$\begin{aligned}\Phi(x^-, x, z, \theta) \ &= \int dp^+ dp^x dp^z e^{i(x^- p^+ + x p^x + z p^z)} \\ &\cdot \int dp_1^+ dp_2^+ dp_1 dp_2 \delta(p_1^+ + p_2^+ - p^+)\delta(p_1 + p_2 - p^x) \\ &\cdot \delta\Big(p_1\sqrt{p_2^+/p_1^+} - p_2\sqrt{p_1^+/p_2^+} - p^z\Big) \\ &\cdot \delta\big(2\arctan\sqrt{p_2^+/p_1^+} - \theta\big)\tilde{\Psi}(p_1^+, p_2^+, p_1, p_2),\end{aligned} \tag{2.4}$$

where $\tilde{\Psi}(p_1^+, p_2^+, p_1, p_2)$ is the Fourier transform of the field $\Psi(x_1^-, x_2^-, x_1, x_2)$.

The physical basis of the correspondence can then be identified by a bi-particle system of a collective dipole which through a canonical transformation maps into the first quantization version of the higher spin system. In what follows, we discuss and study this dipole construction in full detail. Our goal is to establish a first quantized or rather a world-sheet description of the AdS/CFT construction developed in [**20, 30**].

Let us start with the two-particle system in d-dimensional Minkowski space-time with the action

$$S = \int d\tau_1 m\sqrt{|\dot{x}_1^2(\tau_1)|} + \int d\tau_2 m\sqrt{|\dot{x}_2^2(\tau_2)|} \tag{2.5}$$

which leads to the constraints

$$p_1^2 + m^2 = 0, \tag{2.6}$$

$$p_2^2 + m^2 = 0. \tag{2.7}$$

Switching to the center-of-mass variables

$$P = p_1 + p_2, \qquad X = \frac{1}{2}(x_1 + x_2), \tag{2.8}$$

$$p = p_1 - p_2, \qquad x = \frac{1}{2}(x_1 - x_2), \tag{2.9}$$

the constraints (2.6-2.7) become

$$T_1 = P^2 + p^2 + 4m^2 = 0, \tag{2.10}$$

$$T_2 = P \cdot p = 0. \tag{2.11}$$

This system, written in the above covariant form, is described with two time coordinates. It, therefore, can potentially have problems with unitarity and the appearance of ghosts as discussed in the investigations of [**31**]. In the present simple system one has the existence of a canonical gauge in which one can eliminate (gauge fix) the relative time and obtain a physical picture with a single time. This is analogous to the (second-quantized) collective field theory where one also had a covariant and a canonical, equal-time representation [**32**].

Let us describe the details of such gauge fixing procedure; it was given some time ago [**33**] in connection with the investigation of Yukawa's bi-local field theory. One introduces the condition

$$T_3 = P \cdot x = 0. \tag{2.12}$$

Then the constraints (2.11, 2.12) become second-class while (2.10) remains first-class. If one considers the interacting problem with $m = m(x^2)$, then the above condition arises from the the Poisson commutation of (2.10) and (2.11).

Next, taking P_μ to be time-like, we can explicitly solve the second class constraints and eliminate the relative time coordinate. First, one makes a canonical transformation

$$
\begin{aligned}
P_\mu &= P_\mu, &(2.13)\\
X^\mu &= \tilde{X}^\mu + u^L b^{\mu s}\pi_s - \frac{\pi_L}{P^2} b_s{}^\mu u^s - u^r \pi_s b_r{}^\nu \frac{\partial b_\nu{}^s}{\partial P_\mu} + \frac{u^L \pi_L}{P^2} P^\mu, &(2.14)\\
p_\mu &= \frac{P_\mu}{P^2}\pi_L + b_\mu{}^r \pi_r, &(2.15)\\
x^\mu &= P^\mu u^L + b_r{}^\mu u^r, &(2.16)
\end{aligned}
$$

with $r, s = 1, ..., d-1$ and $b_r{}^\mu$ satisfying

$$
\begin{aligned}
b_r{}^\mu P_\mu &= 0, &(2.17)\\
b_{\mu r} b_s{}^\mu &= g_{rs} = (+, \cdots, +), &(2.18)\\
b^\mu{}_r b_\nu{}^r &= g^\mu{}_\nu - \frac{P^\mu P_\nu}{P^2}. &(2.19)
\end{aligned}
$$

One can easily see that u^L, π_L are the components parallel to P_μ while u^r, π_r are normal to P_μ. Then the constraints (2.11) and (2.12) lead to $u^L = \pi_L = 0$. Therefore the system can be described using only the center-of-mass coordinates $(\tilde{X}^\mu, P_\mu)$ and the relative (spatial) coordinates $(\vec{u}, \vec{\pi})$. The canonical transformation (2.13-2.16) is simplified to be

$$
\begin{aligned}
P_\mu &= P_\mu, &(2.20)\\
X^\mu &= \tilde{X}^\mu - u^r \pi_s b_r{}^\nu \frac{\partial b_\nu{}^s}{\partial P_\mu}, &(2.21)\\
p_\mu &= b_\mu{}^r \pi_r, &(2.22)\\
x^\mu &= b_r{}^\mu u^r. &(2.23)
\end{aligned}
$$

For the massless case where $m = 0$, the conformal generators of the bi-particle system are given by

$$\hat{P}^\mu = p_1^\mu + p_2^\mu, \tag{2.24}$$

$$\hat{M}^{\mu\nu} = x_1^\mu p_1^\nu - x_1^\nu p_1^\mu + x_2^\mu p_2^\nu - x_2^\nu p_2^\mu, \tag{2.25}$$

$$\hat{D} = x_1^\mu p_{1\mu} + x_2^\mu p_{2\mu}, \tag{2.26}$$

$$\hat{K}^\mu = (x_1^\nu p_{1\nu}) x_1^\mu - \frac{1}{2}(x_1^\nu x_{1\nu}) p_1^\mu + (x_2^\nu p_{2\nu}) x_2^\mu - \frac{1}{2}(x_2^\nu x_{2\nu}) p_2^\mu, \tag{2.27}$$

where we have neglected the scaling constant term for simplicity. It is instructive to find the explicit form of the conformal generators. Choosing a solution satisfied by (2.17-2.19) as follows

$$b_{0r} = \frac{P_r}{\sqrt{|P^2|}}, \qquad b_{ir} = \delta_{ir} - \frac{P_i P_r}{P^2 + P^0\sqrt{|P^2|}}, \qquad \mu = (0, i) \tag{2.28}$$

one achieves a single time ($X^0 = \tilde{X}^0 = t$) formulation of the conformal generators

$$\hat{P}^0 = P^0 = \sqrt{\vec{P}^2 + \vec{\pi}^2}, \tag{2.29}$$

$$\hat{P}^i = P^i, \tag{2.30}$$

$$\hat{M}^{0i} = tP^i - \tilde{X}^i P^0 + \frac{1}{\sqrt{|P^2|} - P^0}(u^i P^s \pi_s - \pi^i P_r u^r), \tag{2.31}$$

$$\hat{M}^{ij} = \tilde{X}^i P^j - \tilde{X}^j P^i + u^i \pi^j - u^j \pi^i, \tag{2.32}$$

$$\hat{D} = -tP^0 + \tilde{X}^i P_i + u^i \pi_i, \tag{2.33}$$

$$\begin{aligned} \hat{K}^0 = & -\frac{1}{2}t^2 P^0 + t(\tilde{X}^i P_i + u^i \pi_i) + \frac{1}{\sqrt{|P^2|} - P^0}(\tilde{X}^i u_i P^s \pi_s - \tilde{X}^i \pi_i P_r u^r) \\ & -\frac{1}{2}P^0[\tilde{X}^i \tilde{X}_i + u^i u_i] + \frac{1}{2P^2}\frac{2\sqrt{|P^2|} - P^0}{(\sqrt{|P^2|} - P^0)^2}(u^i P^s \pi_s - \pi^i P_r u^r)^2, \end{aligned} \tag{2.34}$$

$$\begin{aligned} \hat{K}^i = & \frac{1}{2}t^2 P^i + t[-P^0 \tilde{X}^i + \frac{1}{\sqrt{|P^2|} - P^0}(u^i P^s \pi_s - \pi^i P_r u^r)] \\ & + \tilde{X}^i[\tilde{X}^j P_j + u^j \pi_j] - \pi^i[\tilde{X}^j u_j + \frac{1}{P^2 + P^0\sqrt{|P^2|}} u^j u_j P^s \pi_s] \\ & + u^i[\tilde{X}^j \pi_j + \frac{1}{P^2 + P^0\sqrt{|P^2|}}(2u^j \pi_j P^s \pi_s - \pi^j \pi_j P_r u^r)] \\ & -\frac{1}{2}P^i\big[\tilde{X}^j \tilde{X}_j + u^j u_j + \frac{1}{P^2(\sqrt{|P^2|} - P^0)^2}(u^i P^s \pi_s - \pi^i P_r u^r)^2\big]. \end{aligned} \tag{2.35}$$

Now recall the canonical, equal-time ($x_1^0 = x_2^0 = t$) collective field version of the bi-particle system, where the conformal generators are given by

$$\hat{P}^0 = p_1^0 + p_2^0 = \sqrt{\vec{p_1}^2} + \sqrt{\vec{p_2}^2}, \tag{2.36}$$

$$\hat{P}^i = p_1^i + p_2^i, \tag{2.37}$$

$$\hat{M}^{0i} = t(p_1^i + p_2^i) - x_1^i p_1^0 - x_2^i p_2^0, \tag{2.38}$$

$$\hat{M}^{ij} = x_1^i p_1^j - x_1^j p_1^i + x_2^i p_2^j - x_2^j p_2^i, \tag{2.39}$$

$$\hat{D} = -t(p_1^0 + p_2^0) + x_1^i p_1^i + x_2^i p_2^i, \tag{2.40}$$

$$\begin{aligned}\hat{K}^0 &= -\frac{1}{2}t^2(p_1^0 + p_2^0) + t(x_1^i p_1^i + x_2^i p_2^i) \\ &\quad -\frac{1}{2}x_1^i x_1^i p_1^0 - \frac{1}{2}x_2^i x_2^i p_2^0,\end{aligned} \tag{2.41}$$

$$\begin{aligned}\hat{K}^i &= \frac{1}{2}t^2(p_1^i + p_2^i) - t(x_1^i p_1^0 + x_2^i p_2^0) \\ &\quad +x_1^j p_1^j x_1^i + x_2^j p_2^j x_2^i - \frac{1}{2}x_1^j x_1^j p_1^i - \frac{1}{2}x_2^j x_2^j p_2^i.\end{aligned} \tag{2.42}$$

There is a simple canonical transformation between the phase space $(\tilde{X}^i, P^i; u^i, \pi^i)$ and the bi-particle phase pace $(x_1^i, p_1^i; x_2^i, p_2^i)$, which transforms the generators (2.29-2.35) to (2.36-2.42). It is given by

$$P^i = p_1^i + p_2^i, \tag{2.43}$$

$$\begin{aligned}\tilde{X}^i &= \frac{x_1^i p_1^0 + x_2^i p_2^0}{p_1^0 + p_2^0} + \frac{1}{P^0(P^2 + P^0\sqrt{|P^2|})} \\ &\quad \times[(x_1^i - x_2^i)(p_1^j p_2^0 - p_2^j p_1^0)(p_1^j + p_2^j) \\ &\quad -(p_1^i p_2^0 - p_2^i p_1^0)(x_1^j - x_2^j)(p_1^j + p_2^j)],\end{aligned} \tag{2.44}$$

$$\pi^i = -\frac{\sqrt{|P^2|} - 2p_2^0}{\sqrt{|P^2|} - P^0} p_1^i + \frac{\sqrt{|P^2|} - 2p_1^0}{\sqrt{|P^2|} - P^0} p_2^i, \tag{2.45}$$

$$\begin{aligned}u^i &= -\frac{1}{2}(x_1^i - x_2^i) - \frac{p_1^0 - p_2^0}{(P^0)^2 P^2}(x_1^j - x_2^j)(p_1^j + p_2^j)(p_1^i p_2^0 - p_2^i p_1^0) \\ &\quad +\frac{2p_1^0 p_2^0}{(P^0)^2(P^2 + P^0\sqrt{|P^2|})}(x_1^j - x_2^j)(p_1^j + p_2^j)(p_1^i + p_2^i).\end{aligned} \tag{2.46}$$

We have in the above described the canonical structure of the composite, two particle "collective" dipole system. It was constructed to describe the singlet subspace of the vector model CFT. Since the CFT has two collective field representations (one covariant with an associated action and another equal-time with a related Hamiltonian) it was important to demonstrate the existence of a single time gauge. We have also seen that in this gauge the dipole system exhibits an identical canonical structure to the collective field theory one. This structure is characterized by an additive contribution to the symmetry generators which we established. In the next section we will review the field theory of higher spins in AdS and describe its first quantized description given by particles with spin moving in AdS spacetime.

3. Higher Spin Theory in AdS

We now switch to a discussion of higher spin theory in AdS_{d+1} spacetime. From the field theoretic description of this theory we will deduce a first-quantized AdS particle system (with spin). We will then demonstrate in section four that the AdS spin particle system emerges through a canonical change of variables from the d-dimensional collective dipole system.

3.1. Higher Spin Fields. There are two formalisms for describing higher spin fields, one being the frame-like formulation in terms of generalized vielbeins and spin connections, the other the metric-like formulation due to Fronsdal [**34**], which employs higher tensor fields with arbitrary rank and symmetry properties. Here, we will use the second formulation. One has a spin s field represented by a symmetric and double traceless tensor of rank s: $h_{\mu_1\ldots\mu_s}(x^\mu)$, which obeys the equations of motion [**35**]

$$\begin{aligned}\nabla_\rho\nabla^\rho h_{\mu_1\ldots\mu_s} - s\nabla_\rho\nabla_{\mu_1}h^\rho{}_{\mu_2\ldots\mu_s} + \frac{1}{2}s(s-1)\nabla_{\mu_1}\nabla_{\mu_2}h^\rho{}_{\rho\mu_3\ldots\mu_s}\\ +2(s-1)(s+d-2)h_{\mu_1\ldots\mu_s} = 0.\end{aligned} \tag{2.47}$$

The gauge transformation is given by

$$\delta_\Lambda h^{\mu_1\ldots\mu_s} = \nabla^{\mu_1}\Lambda^{\mu_2\ldots\mu_s}, \qquad g_{\mu_2\mu_3}\Lambda^{\mu_2\ldots\mu_s} = 0. \tag{2.48}$$

A covariant gauge can be specified with the gauge conditions

$$\nabla^\rho h_{\rho\mu_2\ldots\mu_s} = 0, \qquad g^{\rho\sigma}h_{\rho\sigma\mu_3\ldots\mu_s} = 0. \tag{2.49}$$

Then the equation of motion (2.47) reduces to

$$(\Box + m^2)h_{\mu_1\ldots\mu_s} = 0, \tag{2.50}$$

with the effective mass $m^2 = s^2 + (d-5)s - 2(d-2)$.

It is useful to embed the $d+1$ dimensional AdS spacetime x^μ into $d+2$ dimensional hyperboloid x^α with the metric diag$(-,+,...,+,-)$. The higher spin field $h_{\mu_1\ldots\mu_s}(x^\mu)$ is related to the field $k_{\alpha_1\ldots\alpha_s}(x^\alpha)$ by

$$h_{\mu_1\ldots\mu_s}(x^\mu) = x_{\mu_1}^{\ \alpha_1}\cdots x_{\mu_s}^{\ \alpha_s}k_{\alpha_1\ldots\alpha_s}(x^\alpha), \tag{2.51}$$

where $x_\mu^{\ \alpha} = \partial x^\alpha/\partial x^\mu$. Introducing an internal set of coordinates y^α spacetime, one forms the field with all spins

$$K(x^\alpha, y^\alpha) \equiv \sum_s k_{\alpha_1\ldots\alpha_s}(x^\alpha)y^{\alpha_1}\cdots y^{\alpha_s}. \tag{2.52}$$

In this notation the constraints implied by embedding, the covariant gauge conditions as well as the equations of motion become the following system of equations [**8**]

$$\partial_x^2 K(x,y) = 0, \tag{2.53}$$

$$\partial_y^2 K(x,y) = 0, \tag{2.54}$$

$$\partial_x \cdot \partial_y K(x,y) = 0, \tag{2.55}$$

$$(x\cdot\partial_x + y\cdot\partial_y + 1)K(x,y) = 0, \tag{2.56}$$

$$x\cdot\partial_y K(x,y) = 0. \tag{2.57}$$

It is easy to check that the constraints (2.53-2.57) are all first-class constraints. We should also point out that $\Phi(x^-, x, z, \theta)$ in (2.4) is the light-cone form of $K(x^\alpha, y^\alpha)$ in AdS_4.

In the above representation one has an asymmetry between the spacetime coordinates x and the internal spin coordinates y due to (2.57). One can through a series of canonical transformations achieve a totally symmetric description. The transformation takes the form

$$\Phi(p,q) = (FK)(x,y) \tag{2.58}$$

where $p = (x+y)/2$, $q = (x-y)/2$ and the kernel for a particular spin s is given by

$$F_s = \sum_k (4^k k!)^{-1} (y \cdot \partial_x)^{2k} / (\hat{n}+1)(\hat{n}+2)\cdots(\hat{n}+k) \tag{2.59}$$

with $\hat{n} = y \cdot \partial_y$. After the mapping (2.59), as well as a Fourier transformation

$$\Phi(u,v) = \int dpdq \; e^{ip\cdot u} e^{iq\cdot v} \Phi(p,q), \tag{2.60}$$

one finds the symmetric version

$$(u \cdot \partial_u + 1/2)\Phi(u,v) = 0, \tag{2.61}$$

$$(v \cdot \partial_v + 1/2)\Phi(u,v) = 0, \tag{2.62}$$

$$u^2 = 0, \tag{2.63}$$

$$v^2 = 0, \tag{2.64}$$

$$u \cdot v = 0. \tag{2.65}$$

Next we show that it is possible to reduce the system by solving the first four constraints (2.61-2.64) which are decoupled into two sets of constraints involving separately u and v. Parameterizing the cone $u^2 = 0$ as

$$u_0 = U \sin t, \qquad u_{d+1} = U \cos t, \qquad \vec{u} = U\hat{u}, \qquad \hat{u}^2 = 1, \tag{2.66}$$

we find the constraint (2.61) becomes $\partial/\partial U + 1/2$. Consequently the dependence on the variable U can be factored out

$$\phi(u) = U^{-1/2}\phi(t,\hat{u}), \tag{2.67}$$

and the remaining degrees of freedom are the coordinates $(t, \hat{u})$ (and its conjugates). Similarly, this reduction works for the v system. Therefore, by solving the first four constraints, we reduced the bi-local field $\Phi(u,v)$ with $2(d+2)$ variables to $2d$ variables. This agrees precisely with the bi-local collective field $\Phi(x_1^\mu, x_2^\mu)$ in d dimensions. However, in this formulation, we need to interpret (2.65) as the equation of motion, which does not take the form the collective equation of motion [**20**]. In order to make contact with the collective field equation, one can replace (2.65) with a new constraint

$$\partial_u^2 \partial_v^2 \Phi(u,v) = 0, \tag{2.68}$$

which does not commute with (2.65). As shown in [**20**], this is the equation of motion for the collective field after a field redefinition. This shows that the bi-local collective field theory of [**20**] corresponds to another gauge choice when compared with the Fronsdal's covariant gauge of higher spin theory.

3.2. Higher Spin Particles in AdS$_4$. To describe particles in AdS with spin, one uses the spacetime coordinate x and an internal spin coordinate y. For simplicity, we will mainly discuss the AdS$_4$ case (only in this subsection), which corresponds to the isometry group $SO(2,3)$. The system requires constraints expressing strong conservation of the phase space counterparts of the second- and fourth-order Casimir operators of $SO(2,3)$. We have the generators

$$J_{AB} = x_A p^x_B - x_B p^x_A + y_A p^y_B - y_B p^y_A \tag{2.69}$$

where x^A and y^A represent two separate objects and $A, B = 0,1,2,3,5$ with the metric $\eta_{AB} = \mathrm{diag}(-,+,+,+,-)$. The second- and fourth-order Casimir operators are given by

$$\begin{aligned} \Omega_1 &= \frac{1}{2} J_{AB} J^{AB} \\ &= x^2 p_x^2 - (x \cdot p_x)^2 + y^2 p_y^2 - (y \cdot p_y)^2 \\ &\quad + 2(x \cdot y)(p_x \cdot p_y) - 2(x \cdot p_y)(y \cdot p_x), \end{aligned} \tag{2.70}$$

$$\begin{aligned} \Omega_2 &= \frac{1}{4} J_{AB} J^B{}_C J^C{}_D J^{DA} - \frac{1}{2}\Big(\frac{1}{2} J_{AB} J^{AB}\Big)^2 \\ &= x^2 (p_y^2 (y p_x)^2 + p_x^2 (y p_y)^2 - 2(p_x p_y)(y p_x)(y p_y)) \\ &\quad + y^2 (p_y^2 (x p_x)^2 + p_x^2 (x p_y)^2 - 2(p_x p_y)(x p_x)(x p_y)) \\ &\quad + x^2 y^2 ((p_x p_y)^2 - p_x^2 p_y^2) + (xy)^2 (p_x^2 p_y^2 - (p_x p_y)^2) \\ &\quad - (x p_y)^2 (y p_x)^2 - (x p_x)^2 (y p_y)^2 + 2(x p_x)(x p_y)(y p_x)(y p_y) \\ &\quad + 2(p_x p_y)(x p_x)(xy)(y p_y) + 2(p_x p_y)(x p_y)(xy)(y p_x) \\ &\quad - 2 p_x^2 (x p_y)(xy)(y p_y) - 2 p_y^2 (x p_x)(xy)(y p_x). \end{aligned} \tag{2.71}$$

They are constrained by

$$\Omega_1 + E_0^2 + s^2 = 0, \tag{2.72}$$

$$\Omega_2 + E_0^2 s^2 = 0. \tag{2.73}$$

One solution to the constraints leads to

$$x \cdot p_x = -E_0, \tag{2.74}$$

$$x \cdot p_y = 0, \tag{2.75}$$

$$y \cdot p_y = s, \tag{2.76}$$

$$p_x^2 = 0, \tag{2.77}$$

$$p_y^2 = 0, \tag{2.78}$$

$$p_x \cdot p_y = 0. \tag{2.79}$$

The massless higher spin particle corresponds to the special case $E_0 = s + 1$. These constraints are seen to agree with Fronsdal's covariant formulation of higher-spin theory (2.53-2.57). Another canonical representation of the higher-spin particle system solving the constraints (2.72-2.73) was given in [**36**]

$$x^2 + r^2 = 0, \tag{2.80}$$

$$x \cdot p_x = 0, \tag{2.81}$$

$$x \cdot y = 0, \tag{2.82}$$

$$x \cdot p_y = 0, \tag{2.83}$$

$$y \cdot p_y = 0, \tag{2.84}$$

$$p_y^2 = 0, \tag{2.85}$$

$$p_x^2 = \frac{E_0^2 + s^2}{r^2}, \tag{2.86}$$

$$p_x \cdot p_y = \left(\frac{E_0^2 s^2}{r^2 y^2}\right)^{1/2}, \tag{2.87}$$

where r is the radius of the AdS spacetime and (2.80, 2.82) are gauge conditions for the first-class constraints (2.81, 2.83) respectively.

4. AdS$_{d+1}$ from d-dimensional dipole

We now come to the main part of our construction. We will show (in the framework of the light-cone gauge) that d-dimensional relativistic bi-particle system of section two can be mapped into the higher spin AdS$_{d+1}$ particle system that we have just described. This map will be accomplished by an explicit canonical transformation between the respective phase space variables. In the process, we will be able to map the collective field version of the generators of the conformal group to the generators that can be constructed in AdS spacetime.

For specifying the light-cone gauge of higher spin theory in AdS$_{d+1}$ one starts, following [**38**], with the covariant (and gauge invariant) description with the AdS and internal coordinates denoted by $(x_{\hat{\mu}}, p^{\hat{\mu}}, \bar{\alpha}_{\hat{\mu}}, \alpha^{\hat{\mu}})$, $\hat{\mu} = 0, 1, 2, .., d$. One can parametrize the AdS$_{d+1}$ space with the Poincaré coordinates

$$dx_{\hat{\mu}} dx^{\hat{\mu}} = \frac{1}{z^2}(-dt^2 + dx_i^2 + dz^2 + dx_d^2), \qquad i = 1, ..., d-2. \tag{2.88}$$

The light-cone variables and transverse coordinates are denoted as

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^d \pm x^0), \qquad x^I = (x^i, z). \tag{2.89}$$

The light-cone gauge [**38**], is now fully specified by the conditions

$$\bar{\alpha}^+ = 0, \tag{2.90}$$

$$\alpha^I \bar{\alpha}^I = s, \tag{2.91}$$

$$\bar{\alpha}^I \bar{\alpha}^I = 0, \tag{2.92}$$

$$\bar{\alpha}^- = -\frac{p^I}{p^+}\bar{\alpha}^I + \frac{s+d-1}{p^+}\bar{\alpha}^z - \frac{2(p^+ - \alpha^+\bar{\alpha}^z)}{p^+(p^+ - 2\alpha^+\bar{\alpha}^z)}\bar{\alpha}^z, \tag{2.93}$$

$$\begin{aligned}&(p^{\hat{\mu}} - \alpha^{\hat{\mu}}\bar{\alpha}^z)^2 - (2\alpha^z - \alpha^{\hat{\mu}}p^{\hat{\mu}} - \alpha^z\alpha^{\hat{\mu}}\bar{\alpha}^{\hat{\mu}} + \alpha^2\bar{\alpha}^z)\frac{2(p^+ - \alpha^+\bar{\alpha}^z)}{p^+(p^+ - 2\alpha^+\bar{\alpha}^z)}\bar{\alpha}^z \\ &-d(p^z - \alpha^z\bar{\alpha}^z) - s^2 + (4-d)s + 2d - 4 = 0.\end{aligned} \tag{2.94}$$

Here (2.90) represents the light cone gauge condition, and the constraints (2.91, 2.92, 2.94) are analogous to (2.76, 2.78) and (2.77) in our particle description respectively. From the Lorentz condition (2.79), one can solve for $\bar{\alpha}^-$ (2.93). For more detailed studies of light-cone higher spin theory in AdS spacetime the reader should consult [**38**].

Our construction of the canonical relationship between the two sets of variables will come from the comparison of two different representations of the generators of the conformal group: one corresponding to the d-dimensional dipole and the other to the $d+1$ dimensional higher-spin AdS particle. For this we first recall the light-cone form of generators in AdS given by [**38**]

$$\hat{P}^- = -\frac{p_I^2}{2p^+} - \frac{1}{2z^2p^+}(\frac{1}{2}m_{ij}^2 - \frac{1}{4}(d-3)(d-5)), \quad (2.95)$$

$$\hat{P}^+ = p^+, \quad (2.96)$$

$$\hat{P}^i = p^i, \quad (2.97)$$

$$\hat{J}^{+-} = t\hat{P}^- - x^-p^+, \quad (2.98)$$

$$\hat{J}^{+i} = tp^i - x^ip^+, \quad (2.99)$$

$$\hat{J}^{-i} = x^-p^i - x^i\hat{P}^- + m^{iJ}\frac{p^J}{p^+} - \frac{1}{2zp^+}\{m^{zj}, m^{ji}\}, \quad (2.100)$$

$$\hat{J}^{ij} = x^ip^j - x^jp^i + m^{ij}, \quad (2.101)$$

$$\hat{D} = t\hat{P}^- + x^-p^+ + x^Ip^I + \frac{d-1}{2}, \quad (2.102)$$

$$\hat{K}^- = -\frac{1}{2}x_I^2\hat{P}^- + x^-(x^-p^+ + x^Ip^I + \frac{d-1}{2}) + \frac{1}{p^+}x^Ip^Jm^{IJ} - \frac{x^I}{2zp^+}\{m^{zJ}, m^{JI}\}, \quad (2.103)$$

$$\hat{K}^+ = t^2\hat{P}^- + t(x^Ip^I + \frac{d-1}{2}) - \frac{1}{2}x_I^2p^+, \quad (2.104)$$

$$\hat{K}^i = t(x^i\hat{P}^- - x^-p^i - m^{iJ}\frac{p^J}{p^+} + \frac{1}{2zp^+}\{m^{zj}, m^{ji}\}) - \frac{1}{2}x_J^2p^i + x^i(x^-p^+ + x^Ip^I + \frac{d-1}{2}) + m^{iI}x^I. \quad (2.105)$$

These generators are to be compared with the bi-local CFT$_d$ transformations. In the light-cone gauge ($x_1^+ = x_2^+ = t$), one has the bi-local collective field

$$\Psi(x_1^\mu, x_2^\nu) \mapsto \Psi(t; x_1^-, x_1^i; x_2^-, x_2^j). \quad (2.106)$$

The conformal generators take the form

$$\hat{P}^- = p_1^- + p_2^- = -\Big(\frac{p_1^i p_1^i}{2p_1^+} + \frac{p_2^i p_2^i}{2p_2^+}\Big), \tag{2.107}$$

$$\hat{P}^+ = p_1^+ + p_2^+, \tag{2.108}$$

$$\hat{P}^i = p_1^i + p_2^i, \tag{2.109}$$

$$\hat{J}^{+-} = t\hat{P}^- - x_1^- p_1^+ - x_2^- p_2^+, \tag{2.110}$$

$$\hat{J}^{+i} = t\hat{P}^i - x_1^i p_1^+ - x_2^i p_2^+, \tag{2.111}$$

$$\hat{J}^{-i} = x_1^- p_1^i + x_2^- p_2^i + x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+}, \tag{2.112}$$

$$\hat{J}^{ij} = x_1^i p_1^j - x_1^j p_1^i + x_2^i p_2^j - x_2^j p_2^i, \tag{2.113}$$

$$\hat{D} = t\hat{P}^- + x_1^- p_1^+ + x_2^- p_2^+ + x_1^i p_1^i + x_2^i p_2^i + 2d_\phi, \tag{2.114}$$

$$\begin{aligned} \hat{K}^- = {} & x_1^i x_1^i \frac{p_1^j p_1^j}{4p_1^+} + x_2^i x_2^i \frac{p_2^j p_2^j}{4p_2^+} + x_1^-(x_1^- p_1^+ + x_1^i p_1^i + d_\phi) \\ & + x_2^-(x_2^- p_2^+ + x_2^i p_2^i + d_\phi), \end{aligned} \tag{2.115}$$

$$\hat{K}^+ = t^2 \hat{P}^- + t(x_1^i p_1^i + x_2^i p_2^i + 2d_\phi) - \frac{1}{2} x_1^i x_1^i p_1^+ - \frac{1}{2} x_2^i x_2^i p_2^+, \tag{2.116}$$

$$\begin{aligned} \hat{K}^i = {} & -t\Big(x_1^i \frac{p_1^j p_1^j}{2p_1^+} + x_2^i \frac{p_2^j p_2^j}{2p_2^+} + x_1^- p_1^i + x_2^- p_2^i\Big) - \frac{1}{2} x_1^j x_1^j p_1^i - \frac{1}{2} x_2^j x_2^j p_2^i \\ & + x_1^i(x_1^- p_1^+ + x_1^j p_1^j + d_\phi) + x_2^i(x_2^- p_2^+ + x_2^j p_2^j + d_\phi). \end{aligned} \tag{2.117}$$

For simplicity, we will again neglect the scale dimension terms on both sides in the following discussion, which can be added at the quantum level. Furthermore, the Poisson bracket $\{m^{zj}, m^{ji}\}$ and $\{m^{zJ}, m^{JI}\}$ can be simplified as $2m^{zj}m^{ji}$ and $2m^{zJ}m^{JI}$ respectively.

The phase space on the two sides are $(x^-, x^i, z, \theta^{IJ}; p^+, p^i, p^z, m^{IJ})$ and $(x_1^-, x_1^i, x_2^-, x_2^i; p_1^+, p_1^i, p_2^+, p_2^i)$. The canonical transformation is found by comparing the higher-spin generators (2.95-2.105) with the collective dipole generators (2.107-2.117). The

AdS coordinates and conjugate momenta are given by

$$x^- = \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+}, \tag{2.118}$$

$$p^+ = p_1^+ + p_2^+, \tag{2.119}$$

$$x^i = \frac{x_1^i p_1^+ + x_2^i p_2^+}{p_1^+ + p_2^+}, \tag{2.120}$$

$$p^i = p_1^i + p_2^i, \tag{2.121}$$

$$z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}\sqrt{(x_1^i - x_2^i)^2}, \tag{2.122}$$

$$p^z = \frac{x_1^j - x_2^j}{\sqrt{(x_1^i - x_2^i)^2}}\Big(p_1^j\sqrt{\frac{p_2^+}{p_1^+}} - p_2^j\sqrt{\frac{p_1^+}{p_2^+}}\Big), \tag{2.123}$$

and the angular momenta are given by

$$m^{ij} = \frac{1}{p_1^+ + p_2^+}[(x_1^i - x_2^i)(p_1^j p_2^+ - p_2^j p_1^+) - (x_1^j - x_2^j)(p_1^i p_2^+ - p_2^i p_1^+)], \tag{2.124}$$

$$\begin{aligned} m^{iz} &= \frac{x_1^i - x_2^i}{\sqrt{(x_1^j - x_2^j)^2}}\Big[\sqrt{p_1^+ p_2^+}(x_1^- - x_2^-) + \frac{\big((p_1^+)^2 p_2^j + (p_2^+)^2 p_1^j\big)(x_1^j - x_2^j)}{(p_1^+ + p_2^+)\sqrt{p_1^+ p_2^+}}\Big] \\ &+ \frac{1}{2}\frac{p_1^+ - p_2^+}{p_1^+ + p_2^+}\sqrt{(x_1^j - x_2^j)^2}\Big(p_1^i\sqrt{\frac{p_2^+}{p_1^+}} - p_2^i\sqrt{\frac{p_1^+}{p_2^+}}\Big). \end{aligned} \tag{2.125}$$

One can verify by using the Poisson brackets that this is a canonical transformation.

$$\{x^-, p^+\} = 1, \qquad \{x^i, p^j\} = \delta^{ij}, \qquad \{z, p^z\} = 1, \tag{2.126}$$

$$\{m^{IJ}, m^{KL}\} = \delta^{JK}m^{IL} + \delta^{IL}m^{JK} - \delta^{JL}m^{IK} - \delta^{IK}m^{JL}, \tag{2.127}$$

with all others vanishing.

In summary, we have established a one-to-one map between the phase space coordinates of the collective dipole and the phase space of the higher spin AdS particle. This map generalizes the earlier construction established in [**30**] for $d = 3$ to any dimension. The map provides a simple explicit model of the AdS/CFT correspondence. In the light-cone gauge that we have used, the map reconstructs the AdS theory in the bulk. Issues of locality in the AdS spacetime have been studied recently in [**95**]. This construction demonstrates how a non-local (bi-particle space) is transformed into the local AdS space-time with higher spins.

CHAPTER 3

Higher Spin Theory From Large N Vector Model

1. Overview

Collective fields extend the space of (conformal) operators (and conserved currents) that are usually used for "holographic" comparisons of correlators and amplitudes at the boundary. In the specific case of the $O(N)$ ($Sp(N)$) vector models these fields are given by bi-local invariants representing scalar products of basic local fields. The correspondence of higher spin theory with $O(N)$ ($Sp(N)$) vector model has been studied both in the framework of AdS/CFT (dS/CFT).

In section two, we summarized the work in [**20, 30**]. It was demonstrated in [**30**] in the example of 3d free CFT that the bi-local field contains fully the additional (radial) AdS dimension and also the infinite sequence of fields with growing spins. This construction (done in the light-cone gauge) provides a full one-to-one map between (fields) observables of the field theory and fields of the higher-spin gravity.

In contrast to AdS/CFT correspondence, any dS/CFT correspondence [**50**] involves an emergent holographic direction which is *timelike*. It is then of interest to understand how a *timelike* dimension is generated from the large-N degrees of freedom. Recently, Anninos, Hartman and Strominger [**78**] put forward a conjecture that the *Euclidean* $Sp(2N)$ vector model in three dimensions is dual to Vasiliev higher spin theory in four dimensional de Sitter space.

In section three, we introduced the $Sp(2N)$ vector model. Then in section four we construct a collective field theory of the *Lorentzian* $Sp(2N)$ model which captures the singlet state dynamics of the $Sp(2N)$ vector model. Using the results of [**20**] and [**30**] we then in section five argue that a natural interpretation of the resulting action is by double analytic continuation which makes the emergent direction time-like, relating this to higher spin theory in dS_4, in a way reminiscent of the way the Louiville mode in worldsheet string theory has to be interpreted as a time beyond critical dimensions [**56**]. Our map establishes the bi-local theory as the bulk space-time representation of de Sitter higher spin gravity.

Since the bilocal collective field is a composite of two Grassmann variables therefore it might not appear to be a genuine bosonic field. In particular, for finite N, a sufficiently large power of the field operator vanishes, reflecting its Grassmannian origin [1]. This is further reflected on the size of its Hilbert space.

In section six we will describe a geometric (pseudo-spin) version of the collective theory which will be seen to incorporate these effects. For dS/CFT, this implies that the true number of degrees of freedom in the dual higher spin theory in dS is, in this framework, reduced from what is seen perturbatively (with $G = R_{dS}^2/N$ being the coupling constant squared).

In the last section we will give an interpretation of phase transition from a different perspective than shown by Shenker and Yin [**41**].

2. Bi-local representation of O(N) CFT$_3$

The $O(N)$/Higher Spin duality is based on a three dimensional N-component scalar field theory

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi^a \partial^\mu \phi^a + \frac{g}{4}(\phi \cdot \phi)^2 \ , \qquad a = 1, \cdots, N \tag{3.1}$$

where $\phi^a = \phi^a(t, \vec{x}) = \phi^a(x^+, x^-, x^\perp)$; $\mu = 0, 1, 2$. This theory features two critical points with conformal symmetry: the UV fixed point at zero coupling ($g = 0$) and the nontrivial IR fixed point at nonzero coupling constant ($g \neq 0$). The latter can be evaluated in the large N limit and serves as the classic example of critical phenomena in 3d.

For the correspondence with higher spin fields, a central role is played by the sequence of traceless and symmetric higher spin currents

$$J_{\mu_1\mu_2\cdots\mu_s} \quad = \quad \sum_{k=0}^{s}(-1)^k \binom{s-1/2}{k}\binom{s-1/2}{s-k}$$

[1]This property of higher spin currents has already been recognized in [**52**]

$$\times \partial_{\mu_1} \cdots \partial_{\mu_k} \phi^a \ \partial_{\mu_{k+1}} \cdots \partial_{\mu_s} \phi^a - \text{traces} \tag{3.2}$$

which are exactly conserved in the free case. These operators can be summarized in the semi bi-local form by the generating functional

$$\mathcal{O}(x, \epsilon) = \phi^a(x - \epsilon) \sum_{n=0}^{\infty} \frac{1}{(2n)!} \left(2\epsilon^2 \overleftarrow{\partial_x} \cdot \overrightarrow{\partial_x} - 4(\epsilon \cdot \overleftarrow{\partial_x})(\epsilon \cdot \overrightarrow{\partial_x}) \right)^n \phi^a(x + \epsilon) \tag{3.3}$$

where $\epsilon^2 = 0$ to satisfy the traceless condition. As a result, ϵ represents a cone with a two dimensional coordinate and altogether $\mathcal{O}(x, \epsilon)$ is a five dimensional semi bi-local field. The currents that it generates represent boundary duals of AdS_4 higher spin fields

$$J_{\mu_1 \mu_2 \cdots \mu_s}(x) \leftrightarrow \mathcal{H}_{\hat{\mu}_1 \hat{\mu}_2 \cdots \hat{\mu}_s}(x, z \to 0) \tag{3.4}$$

where $ds^2 = \frac{dx^2 + dz^2}{z^2}$ is the AdS_4 metric.

In the AdS/CFT correspondence, correlation functions of currents are to match up with the boundary transition amplitudes (sometimes referred to as the boundary S-matrix) of the higher dimensional AdS theory. A successful demonstration of this was accomplished in the three-point case by Giombi and Yin [**28, 29**] who were able to match the two critical points of the vector model with two versions of Vasiliev's Higher Spin Gravity in AdS_4. The trivial and nontrivial fixed points are seen as conjectured by Klebanov and Polyakov [**17**] to correspond to different boundary conditions involving the lowest spin ($s = 0$) field.

A constructive approach for this AdS_4/CFT_3 correspondence, given in [**20**], is based on the notion of collective fields. These are described by bi-local invariants of the $O(N)$ field theory

$$\Phi(x, y) \equiv \phi(x) \cdot \phi(y) = \sum_{a=1}^{N} \phi^a(x) \cdot \phi^a(y) \tag{3.5}$$

that close under the Large N Schwinger-Dyson equations. These operators represent a more general set than the conformal fields $\mathcal{O}(x, \epsilon)$ since there is no restriction to a cone. The collective action evaluates the complete $O(N)$ invariant partition function

$$Z = \int [d\phi^a(x)] e^{-S[\phi]} = \int \prod_{x,y} [d\Phi(x, y)] \mu(\Phi) e^{-S_c[\Phi]} \tag{3.6}$$

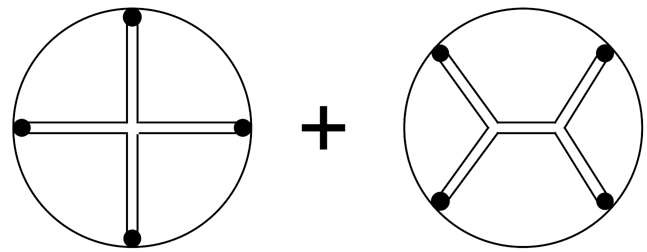


FIGURE 1. Illustration of the four-point collective field diagrams.

where the measure is given by $\mu(\Phi) = (\det\Phi)^{V_x V_p}$ with $V_x = L^3$ the volume of space and $V_p = \Lambda^3$ the volume of momentum space with Λ being the momentum cutoff. Explicitly one has the collective action

$$S_c[\Phi] = \mathrm{Tr}\left[-(\partial_x^2 + \partial_y^2)\Phi(x,y) + V\right] + \frac{N}{2}\mathrm{Tr}\,\ln\Phi(x,y) \tag{3.7}$$

where the trace is defined as $\mathrm{Tr}\,B = \int d^3x\, B(x,x)$. This collective action is nonlinear, with $1/N$ appearing as the expansion parameter. Through the identification of $1/N$ with G_N (the coupling constant of higher spin gravity), this collective field representation provides a bulk description of the dual AdS theory. One also has a natural (star) product defined as $(\Psi \star \Phi)(x,y) = \int dz\,\Psi(x,z)\Phi(z,y)$.

The perturbative expansion is defined in this (bi-local) space. The nonlinear equation of motion specified by S_c gives the background in the expansion: $\Phi = \Phi_0 + \frac{1}{\sqrt{N}}\eta$. Expanding about the background gives us an infinite number of interaction vertices [**16**]

$$S_c[\Phi] = S[\Phi_0] + \mathrm{Tr}[\Phi_0^{-1}\eta\Phi_0^{-1}\eta] + \frac{g}{4}\eta^2 + \sum_{n\geq 3} N^{1-n/2}\,\mathrm{Tr}B^n\ , \tag{3.8}$$

where $B \equiv \Phi_0^{-1}\eta$. The nonlinearities built into S_c are precisely such that all invariant correlators: $\langle\phi(x_1)\cdot\phi(y_1)\cdots\phi(x_n)\cdot\phi(y_n)\rangle$ are now reproduced through the Witten (Feynman) diagrams with $1/N$ vertices. The four-point example is shown in Figure 1. We stress that this nonlinear structure is there for both the free and the interacting fixed point.

This bi-local theory is expected to represent a (covariant) gauge fixing of Vasiliev's gauge invariant theory. An attempt at a gauge invariant formalism is given in [**39**].

A one-to-one relationship between bi-local and AdS higher spin fields can be demonstrated in a physical (single-time) picture. The existence of such a gauge and the discussion of the collective dipole underlying the collective construction is given in [**40**].

The single-time formulation that we will follow involves the equal time bi-local operators

$$\Psi(t,\vec{x},\vec{y}) = \sum_a \phi^a(t,\vec{x})\phi^a(t,\vec{y}) \tag{3.9}$$

and its conjugate momenta: $\Pi(\vec{x},\vec{y}) = -i\frac{\delta}{\delta\Psi(\vec{x},\vec{y})}$ with a Hamiltonian of the form

$$H = 2\mathrm{Tr}(\Pi\Psi\Pi) + \frac{1}{2}\int d\vec{x}[-\nabla^2_{\vec{x}}\Psi(\tilde{x},\tilde{y})|_{\tilde{x}=\tilde{y}}] + \frac{N^2}{8}\mathrm{Tr}\,\Psi^{-1}, \tag{3.10}$$

where we have set the coupling constant $g = 0$. This Hamiltonian again has a natural $1/N$ expansion, after a background shift

$$\Psi = \Psi_0 + \frac{1}{\sqrt{N}}\eta\ , \qquad \Pi = \sqrt{N}\pi\ , \tag{3.11}$$

with $\Psi^0_{\vec{x}\vec{y}} = \int d\vec{k} e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\frac{1}{2\sqrt{\vec{k}^2}}$, one gets a quadratic Hamiltonian

$$H^{(2)} = 2\mathrm{Tr}(\pi\Psi_0\pi) + \frac{1}{8}\mathrm{Tr}(\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1})\ , \tag{3.12}$$

which in momentum space reads

$$H^{(2)} = \frac{1}{2}\int d\vec{k}_1 d\vec{k}_2\ \pi_{\vec{k}_1\vec{k}_2}\pi_{\vec{k}_1\vec{k}_2} + \frac{1}{8}\int d\vec{k}_1 d\vec{k}_2\ \eta_{\vec{k}_1\vec{k}_2}\left(\psi^{0\ -1}_{\vec{k}_1} + \psi^{0\ -1}_{\vec{k}_2}\right)^2 \eta_{\vec{k}_1\vec{k}_2} \tag{3.13}$$

producing the (singlet) spectrum $\omega_{\vec{k}_1\vec{k}_2} = \sqrt{\vec{k}_1^2} + \sqrt{\vec{k}_2^2}$ of the $O(N)$ theory. A sequence of $1/N$ vertices representing interactions can be found similarly and the cubic and quartic interactions are given explicitly as

$$H^{(3)} = \frac{2}{\sqrt{N}}\mathrm{Tr}(\pi\eta\pi) - \frac{1}{8\sqrt{N}}\mathrm{Tr}\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\ , \tag{3.14}$$

$$H^{(4)} = \frac{1}{8N}\mathrm{Tr}\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\eta\Psi_0^{-1}\ . \tag{3.15}$$

We note that the form of these vertices is the same for both the free (UV) and the interacting (IR) conformal theories (the only difference is induced by the different background shifts in these two cases).

One also has a null-plane version of this construction which would correspond to light-cone gauge higher spin theory. This was used in [**30**] to demonstrate the one-to-one map between the two descriptions: the null-plane bi-locals $\Psi(x^+; x_1^-, x_2^-; x_1, x_2)$ and the higher spin fields $\mathcal{H}(x^+; x^-, x, z; \theta)$ in AdS$_4$ (with θ denoting a coordinate that generates the sequence of higher spins). Both fields have same number of dimensions $1+2+2 = 1+3+1$, the same representation of the conformal group, and the same number of degrees of freedom.

The bi-local to AdS canonical transformation given in [**30**] reads

$$x^- = \frac{x_1^- p_1^+ + x_2^- p_2^+}{p_1^+ + p_2^+} \, , \qquad x = \frac{x_1 p_1^+ + x_2 p_2^+}{p_1^+ + p_2^+} \, , \tag{3.16}$$

$$z = \frac{\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+}(x_1 - x_2) \, , \qquad \theta = 2 \arctan \sqrt{p_2^+/p_1^+} \, , \tag{3.17}$$

where p_i^+ are the conjugate momenta of x_i^-. The map going from the bi-local field to the higher spin field is given by an integral transformation

$$\begin{aligned}
&\mathcal{H}(x^-, x, z, \theta) = \int dp^+ dp^x dp^z \, e^{i(x^- p^+ + x p^x + z p^z)} \int dp_1^+ dp_1 dp_2^+ dp_2 \\
&\delta(p_1^+ + p_2^+ - p^+) \delta(p_1 + p_2 - p^x) \delta(p_1 \sqrt{p_2^+/p_1^+} - p_2 \sqrt{p_1^+/p_2^+} - p^z) \\
&\delta(2 \arctan \sqrt{p_2^+/p_1^+} - \theta) \tilde{\Psi}(p_1^+, p_2^+, p_1, p_2) \, ,
\end{aligned} \tag{3.18}$$

where $\tilde{\Psi}(p_1^+, p_2^+, p_1, p_2)$ is the Fourier transform of the bi-local field $\Psi(x_1^-, x_2^-, x_1, x_2)$.

It was shown in [**30**] that under this transformation all the generators of the bi-local theory map into the generators of light-cone Higher Spin Gravity in the form given by Metsaev [**38**]. In particular, the quadratic bi-local Hamiltonian

$$P^{-(2)} = \int dx_1^- dx_1 dx_2^- dx_2 \, \Psi^\dagger \left(-\frac{\nabla_1^2}{2\partial_{x_1^-}} - \frac{\nabla_2^2}{2\partial_{x_2^-}} \right) \Psi \tag{3.19}$$

takes an AdS$_4$ form

$$P^{-(2)} = \int dx^- dx dz d\theta \, \mathcal{H}^\dagger \left(-\frac{\partial_x^2 + \partial_z^2}{2\partial_{x^-}} \right) \mathcal{H} \, . \tag{3.20}$$

This establishes, at the quadratic level, that the bi-local representation is identical to the local AdS$_4$ higher spin representation. One should note that the $1/N$ vertices do not become local in AdS spacetime. In fact the light-cone gauge fixing of Vasiliev's theory has not been established yet and based on the collective map one can expect that it takes nonlocal form.

Another important check regarding the identification of the "extra" AdS coordinate z can be seen by taking the $z \to 0$ limit. Evaluating the bi-local field at $z = 0$ gives the following "boundary" form

$$\begin{aligned} \mathcal{H}(x^+, x^-, x, \theta) &= \int dp_1^+ dp_2^+ e^{ix^-(p_1^+ + p_2^+)} \\ &\quad \cdot \delta(\theta - 2\tan^{-1}\sqrt{p_2^+/p_1^+})\tilde{\Psi}(p_1^+, p_2^+; x, x) \ . \end{aligned} \tag{3.21}$$

Expanding the kernel in the above transformation intoits Fourier series, for a fixed even spin s, one finds agreement with conformal operators of a fixed spin s which are explicitly given in [**73, 74**] by

$$\mathcal{O}^s = \sum_{k=0}^{s} \frac{(-1)^k\, \Gamma(s+\frac{1}{2})\Gamma(s+\frac{1}{2})}{k!(s-k)!\, \Gamma(s-k+\frac{1}{2})\Gamma(k+\frac{1}{2})} (\partial_+)^k \phi\, (\partial_+)^{s-k}\phi \ . \tag{3.22}$$

As a result, in the bi-local picture one has a clear definition of the boundary $z = 0$ and the notion of boundary amplitudes (boundary S-matrix). Due to the construction through collective field theory, one is guaranteed to reproduce the boundary correlators in full agreement with the $O(N)$ model. The bulk/bi-local theory is nonlinear with nonlinearities governed by $1/N = G_N$. All this provides a nontrivial check of the collective picture and the proposal that bi-local fields provide a bulk representation of AdS_4 higher spin fields.

3. The $Sp(2N)$ vector model

The $Sp(2N)$ vector model in d spacetime dimensions is defined by the action

$$S = i\int dt d^{d-1}x \ \left[\{\partial_t\phi_1^i\partial_t\phi_2^i - \nabla\phi_1^i\nabla\phi_2^i\} - V(i\phi_1^i\phi_2^i)\right] \tag{3.23}$$

where ϕ_1^i, ϕ_2^i with $i = 1\cdots N$ are N pairs of Grassmann fields. This is of course a model of ghosts.

In this section we will quantize this model following [**79**] and [**80**]. In this quantization, the fields ϕ_1^i and ϕ_2^i are Hermitian operators, while the canonically conjugate momenta

$$P_1^i = i\partial_t \phi_2^i \ , \ P_2^i = -i\partial_t \phi_1^i \tag{3.24}$$

are anti-Hermitian. The Hamiltonian H is Hermitian

$$H = i \int d^{d-1}x \left[P_2^i P_1^i + \nabla\phi_1^i \nabla\phi_2^i + V(i\phi_1^i \phi_2^i) \right] \tag{3.25}$$

The (equal time) canonical anticommutation relations are

$$\begin{aligned} \{\phi_i^a(\vec{x}), P_j^b(\vec{y})\} &= -i\delta_{ij}\delta^{ab}\delta^{d-1}(\vec{x}-\vec{x}') \\ \{\phi_a^i(\vec{x}), \phi_b^j(\vec{y})\} &= \{P_a^i(\vec{x}), P_b^j(\vec{y})\} = 0 \ , \ (a,b=1,2) \end{aligned} \tag{3.26}$$

with all other anticommutators vanishing. With these anticommutators, the equations of motion for the corresponding Heisenberg picture operators

$$\partial_t^2 \phi_a^i - \nabla^2 \phi_a^i + V' = 0 \tag{3.27}$$

follow. The operator relations (3.26) allow a representation of the operators are given by

$$\phi_i^a(\vec{x}) \rightarrow \phi_i^a(\vec{x}) \ , \ P_i^a \rightarrow -i\frac{\delta}{\delta\phi_i^a(\vec{x})} \tag{3.28}$$

where ϕ_a^i are now Grassmann fields.

For the free theory, the solution to the equation of motion is

$$\phi_a^i(\vec{x},t) = \int \frac{d^{d-1}k}{(2\pi)^{d-1}\sqrt{2|k|}} \left[\alpha_a^i(\vec{k}) e^{-i(|k|t-\vec{k}\cdot\vec{x})} + \alpha_a^{i\dagger}(\vec{k}) e^{i(|k|t-\vec{k}\cdot\vec{x})} \right] \tag{3.29}$$

where the operators α_a^i satisfy

$$\{\alpha_1^i(\vec{k}), \alpha_2^{\dagger j}(\vec{k}')\} = i\delta^{ij}\delta(\vec{k}-\vec{k}') \ , \ \{\alpha_1^{\dagger i}(\vec{k}), \alpha_2^j(\vec{k}')\} = -i\delta^{ij}\delta(\vec{k}-\vec{k}') \tag{3.30}$$

with all the other anticommutators vanishing. The Hamiltonian is given by

$$H = i \int [d\vec{k}] \ |\vec{k}| \left[\alpha_1(\vec{k})^\dagger \alpha_2(\vec{k}) - \alpha_2(\vec{k})^\dagger \alpha_1(\vec{k}) \right] \tag{3.31}$$

The basic commutators lead to

$$[H, \alpha_a^i(k)] = -k\alpha_a^i(k) \ , \ [H, \alpha_a^{i\dagger}] = k\alpha_a^{i\dagger}(k) \tag{3.32}$$

To discuss the quantization of the free theory it is useful to review the quantization of the $Sp(2N)$ oscillator, following [**80**] [2]. The Hamiltonian is

$$H = i\left(-\frac{\partial^2}{\partial\phi_2^i\partial\phi_1^i} + k^2\phi_1^i\phi_2^i\right) \tag{3.33}$$

where ϕ_1^i, ϕ_2^i are N pairs of Grassmann numbers. Due to the Grassmann nature of the variables, the spectrum of the theory is bounded both from below and from above. In the Schrodinger picture, the oscillators are defined by:

$$\phi_a^i = \frac{1}{\sqrt{2k}}[\alpha_a^i + \alpha_a^{i\dagger}] \tag{3.34}$$

while the momenta are

$$P_a^i = \epsilon_{ab}\sqrt{\frac{k}{2}}(\alpha_b^i - \alpha_b^{i\dagger}) \tag{3.35}$$

The ground state $|0\rangle$ and the highest state $|2N\rangle$ are then defined by the conditions

$$\alpha_a^i|0\rangle = 0 \ , \ \ \alpha_a^{i\dagger}|2N\rangle = 0 \tag{3.36}$$

with the wavefunctions

$$\Psi_0 = \exp[-ik\phi_1^i\phi_2^i] \ , \ \ \Psi_{2N} = \exp[ik\phi_1^i\phi_2^i] \tag{3.37}$$

and the energy spectrum is given by

$$E_n = k[n-N] \ , \ \ n = 0, 1, \cdots, 2N \tag{3.38}$$

Finally, the Feynman correlator of the Grassmann coordinates is easily seen to be

$$\langle 0|T[\phi_1^i(t)\phi_2^j(t')]|0\rangle = \frac{i\delta^{ij}}{2k}\ e^{-ik|t-t'|} \tag{3.39}$$

Extension of these results to the free field theory is straight forward: for each momentum $\vec{k}$, we have a fock space with a finite number of states.

[2]Note that our notation is different from that of [**80**]

4. Collective Field Theory for the $Sp(2N)$ model

In the representation (3.28), a general wavefunctional is given by $\Psi[\phi^i_a(\vec{x}), t]$. Our aim is to obtain a description of the singlet sector of the theory, i.e. wavefunctionals that are invariant under the $Sp(2N)$ rotations of the fields $\phi^i_a(\vec{x})$. All the invariants in field space are functions of the bi-local collective fields

$$\rho(\vec{x}, \vec{y}) \equiv i\epsilon^{ab}\phi^i_a(\vec{x})\phi^i_b(\vec{y}) \tag{3.40}$$

We have defined this collective field to be Hermitian (which is why there is a i in the definition). Notice that clearly $\rho(\vec{x}, \vec{y}) = \rho(\vec{y}, \vec{x})$. The aim now is to rewrite the theory in terms of a Hamiltonian that is a functional of $\rho(\vec{x}, \vec{y})$ and its canonical conjugate $-i\frac{\delta}{\delta\rho(\vec{x},\vec{y})}$ that acts on wavefunctionals, which are in turn functionals of $\rho(\vec{x}, \vec{y})$.

It is important to remember that $\rho(\vec{x}, \vec{y})$ is not a genuine bosonic field. This will have important consequences at finite N. In a perturbative expansion in $1/N$, however, there is no problem in treating $\rho(\vec{x}, \vec{y})$ as a bosonic field [**16**].

Before dealing with the $Sp(2N)$ field theory, it is useful to review some aspects of the collective theory for the usual $O(N)$ model, starting with the $O(N)$ oscillator.

4.1. Collective fields for the $O(N)$ theory. In this section we review the bi-local collective field theory construction for the $O(N)$ field theory, starting with the $O(N)$ oscillator. This has a Hamiltonian

$$H = \frac{1}{2}[P^iP^i + k^2X^iX^i] \tag{3.41}$$

The collective variable is the square of the radial coordinate $\sigma = X^iX^i$ and the Jacobian for transformation from X^i to σ and the angles is

$$J(\sigma) = \frac{1}{2}t\sigma^{(N-2)/2}\Omega_{N-1} \tag{3.42}$$

where Ω_{N-1} is the volume of unit S^{N-1}. The idea is to find the Hamiltonian $H(\sigma, \frac{\partial}{\partial\sigma})$ which acts on wavefunctions $[J(\sigma)]^{1/2}\Psi(\sigma)$. The key observation of [**32**] is that this can also be obtained by requiring that $H(\sigma, \frac{\partial}{\partial\sigma})$ acting on wavefunctions $[J(\sigma)]^{1/2}\Psi(\sigma)$

is Hermitian with the trivial measure $d\sigma$. This determines both the Jacobian and the Hamiltonian and the technique generalizes to higher dimensional field theory. The final result is well known,

$$H_{coll} = -2\frac{\partial}{\partial\sigma}\sigma\frac{\partial}{\partial\sigma} + \frac{(N-2)^2}{8\sigma} + \frac{1}{2}k^2\sigma \tag{3.43}$$

The large-N expansion then proceeds as usual by expanding around the saddle point solution σ_0 which minimizes the potential [3],

$$\sigma_0^2 = \frac{N^2}{4k^2} \tag{3.44}$$

Clearly, we have to choose the positive sign since in this case σ is a *positive* real quantity,

$$\sigma_0 = \frac{N}{2k} \tag{3.45}$$

which reproduces the coincident time two point function $\langle 0|X^i(t)X^i(t)|0\rangle$ and the correct ground state energy, $E_0 = \frac{N}{2}k$. The subleading contributions are then obtained by expanding around the saddle point,

$$\sigma = \sigma_0 + \sqrt{\frac{2N}{k}}\eta \ , \ \Pi_\sigma = \sqrt{\frac{k}{2N}}\pi_\eta \tag{3.46}$$

The quadratic part of the Hamiltonian becomes

$$H^{(2)} = \frac{1}{2}\left[\pi_\eta^2 + 4k^2\eta^2\right] \tag{3.47}$$

This leads to the excitation spectrum to $O(1)$, $E_n = 2nk$ with $n = 0, 1, \cdots, \infty$. The Hamiltonian of course contains all powers of η. *Terms with even number of the fluctuations* (π_η, η) *come with odd factors of* σ_0. This fact will play a key role in the following.

In the following it will be necessary to consider wavefunctions. It follows directly from (3.41) that the ground state wavefunction is given by (up to a normalization

[3] To see why the saddle point approximation is valid, rescale $\sigma \to N\sigma$ and $\Pi_\sigma \to \frac{1}{N}\Pi_\sigma$ so that there is an overall factor of N in front of the potential energy term. We will, however, stick to the unrescaled fields.

which is not important for our purposes)

$$\Psi_0(X^i) = \exp[-\frac{k}{2}\sigma] \sim \exp[-\sqrt{\frac{Nk}{2}}\eta] \tag{3.48}$$

where we have expanded σ as in (3.46), used (3.45) and ignored an overall constant. We should get the same result from the collective theory. Recalling that the collective wavefunction is related to the original wavefunction by a Jacobian factor, the ground state wavefunction follows from (3.47)

$$\Psi'_0(\eta) = [J(\sigma)]^{-\frac{1}{2}} \exp[-k\eta^2] \tag{3.49}$$

The presence of the Jacobian is crucial in obtaining agreement with (3.48) [**48**]. Expanding the argument in the Jacobian in powers of η according to (3.46) it is easy to see that the quadratic term in η, coming from the Jacobian, exactly cancels the explicit quadratic term in (3.49) and similarly the linear term in η is in exact agreement with (3.48). The expression (3.49) contains all powers of η in the exponentiated - these should also cancel once one takes into account the cubic and higher order terms in the collective Hamiltonian as well as finite N corrections which we have ignored to begin with. The above formalism can be easily generalized to an additional invariant potential, since the latter is a function of σ.

The collective theory for $O(N)$ field theory can be constructed following similar argument. We reproduce the relevant formulae from [**32**] that are direct generalizations of the formulae for the oscillator. The $O(N)$ model has the Hamiltonian

$$H = \frac{1}{2}\int d^{d-1}x \left[-\frac{\delta^2}{\delta\phi^i(\vec{x})\delta\phi^i(\vec{x})} + \nabla\phi^i(\vec{x})\nabla\phi^i(\vec{x}) + U[\phi^i(\vec{x})\phi^i(\vec{x})] \right] \tag{3.50}$$

The singlet sector Hamiltonian in terms of the bi-local collective field $\sigma(\vec{x},\vec{y}) = \phi^i(\vec{x})\phi^i(\vec{y})$ and its canonically conjugate momentum $\Pi_\sigma(\vec{x},\vec{y})$ is, to leading order in $1/N$ [4]

$$H^{O(N)}_{coll} = 2\mathrm{Tr}\left[(\Pi_\sigma \sigma \Pi_\sigma) + \frac{N^2}{16}\sigma^{-1} \right] - \frac{1}{2}\int d\vec{x} \nabla^2_x \sigma(\vec{x},\vec{y})|_{\vec{y}=\vec{x}} + U(\sigma(\vec{x},\vec{x})) \tag{3.51}$$

[4]To subleading order there are singular terms which are crucial for reproducing the correct $1/N$ contributions.

where the spatial coordinates are treated as matrix indices.

So far our considerations are valid for an arbitrary interaction potential U. Let us now restrict ourselves to the free theory, $U = 0$ to discuss the large-N solution explicitly. In momentum space the saddle point solution is

$$\sigma(\vec{k}_1, \vec{k}_2) = \frac{N}{2|\vec{k}_1|}\delta(\vec{k}_1 - \vec{k}_2) \tag{3.52}$$

Once again we have chosen the positive sign in the solution of the saddle point equation, and the saddle point value of the collective field agrees with the two point correlation function of the basic vector field, which should be positive. The $1/N$ expansion is generated in a fashion identical to the single oscillator,

$$\sigma(\vec{k}_1, \vec{k}_2) = \sigma_0(\vec{k}_1, \vec{k}_2) + \left(\frac{|\vec{k}_1||\vec{k}_2|}{N(|\vec{k}_1| + |\vec{k}_2|)}\right)^{-\frac{1}{2}} \eta(\vec{k}_1, \vec{k}_2)\ ,\ \Pi_\sigma = \left(\frac{|\vec{k}_1||\vec{k}_2|}{N(|\vec{k}_1| + |\vec{k}_2|)}\right)^{\frac{1}{2}} \pi_\eta(\vec{k}_1, \vec{k}_2) \tag{3.53}$$

the quadratic piece becomes

$$H^{(2)} = \frac{1}{2}\int d\vec{k}_1 d\vec{k}_2 \left[\pi_\eta(\vec{k}_1, \vec{k}_2)\pi_\eta(\vec{k}_1, \vec{k}_2) + (|\vec{k}_1| + |\vec{k}_2|)^2 \eta(\vec{k}_1, \vec{k}_2)\eta(\vec{k}_1, \vec{k}_2)\right] \tag{3.54}$$

so that the energy spectrum is given by

$$E(\vec{k}_1, \vec{k}_2) = |\vec{k}_1| + |\vec{k}_2| \tag{3.55}$$

as it should be. It is easy to check that the unequal time two point function of the fluctuations reproduces the connected part of the two point function of the full collective field as calculated from the free field theory. A nontrivial U can be reinstated easily (see e.g. the treatment of the $(\vec{\phi}^2)^2$ model in [**20**], which discusses the RG flow to the nontrivial IR fixed point).

4.2. Collective theory for the $Sp(2N)$ oscillator. Since there is a representation of the field operator and the conjugate momentum operator of the $Sp(2N)$ theory in terms of Grassmann fields, (3.28), it is clear that the derivation of the collective field theory of the $Sp(2N)$ model closely parallels that of the $O(N)$ theory. In this subsection we consider the $Sp(2N)$ oscillator. The Hamiltonian is given by (3.33).

The collective variable is

$$\rho = i\epsilon^{ab}\phi_a^i\phi_b^i \tag{3.56}$$

The fully connected correlators of this collective variable have a simple relationship with those of the $O(2N)$ harmonic oscillator,

$$\langle\rho(t_1)\rho(t_2)\cdots\rho(t_n)\rangle_{Sp(2N)}^{conn} = -\langle\sigma(t_1)\sigma(t_2)\cdots\sigma(t_n)\rangle_{SO(2N)}^{conn} \tag{3.57}$$

This result follows from (3.39) and the application of Wick's theorem for Grassmann variables.

The collective variable ρ is a Grassmann even variable - it is not an usual bosonic variable. This key fact is intimately related to the finite number of states of the $Sp(2N)$ oscillator. In this section we will show that in a $1/N$ expansion we can nevertheless proceed, defering a proper discussion of this point to a later section.

The Hamiltonian for the collective theory is obtained by the same method used to obtain the collective theory in the bosonic case, with various negative sign coming from the Grassmann nature of the variables. Using the chain rule and taking care of negative signs coming because of Grassmann numbers, one gets the Jacobian $J'(\rho)$ (determined by requiring the hermicity of $J^{-1/2}HJ^{1/2}$)

$$J'(\rho) = A'\ \rho^{-(N+1)} \tag{3.58}$$

where A' is a constant. The negative power of ρ of course reflects the Grassmann nature of the variables. [5] Despite this difference, the final collective Hamiltonian is in fact *identical* to the $O(2N)$ oscillator collective Hamiltonian

$$H_{coll}^{Sp(2N)} = -2\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{N^2}{2\rho} + \frac{1}{2}k^2\rho \tag{3.59}$$

This leads to the same saddle point equation, and the solutions satisfy the same equation as (3.44) with $N \to 2N$.

[5] This ρ dependence of the Jacobian follows from a direct calculation $J'(\rho) = \int d\phi_1^i d\phi_2^i \delta(\rho - i\phi_1^i\phi_2^i) = \int d\lambda e^{i\lambda\rho}\int d\phi_1^i d\phi_2^i\ e^{-i\lambda\phi_1^i\phi_2^i} \sim \rho^{-(N+1)}$

In the $O(2N)$ oscillator, we had to choose the positive sign, since σ is by definition a real *positive* variable. In this case, there is no reason for ρ to be positive. In fact we need to choose the negative sign, since (3.57) requires that the one point function of ρ must be the negative of the one point function of σ.

$$\rho_0 = -\frac{N}{k} \tag{3.60}$$

It is interesting that the singlet sectors of the $O(2N)$ and $Sp(2N)$ models are described by two different solutions of the *same* collective theory.

The leading order ground state energy is the Hamiltonian evaluated on the saddle point,

$$E_{gs} = -Nk \tag{3.61}$$

in agreement with (3.38). The fluctuation Hamiltonian is obtained as usual by expanding

$$\rho = \rho_0 + \sqrt{\frac{4N}{k}}\xi \ , \ \Pi_\rho = \sqrt{\frac{k}{4N}}\pi_\xi \tag{3.62}$$

The quadratic Hamiltonian is now *negative*, essentially because of the negative sign in the saddle point,

$$H_\xi^{(2)} = -\frac{1}{2}\left[\pi_\xi^2 + 4k^2\xi^2\right] \tag{3.63}$$

A standard quantization of this theory leads to a spectrum which is unbounded from below. We will now argue that we need to quantize this theory rather differently, in a way similar to the treatment of [**49**]. This involves defining annihilation and creation operators $a_\xi, a_\xi^\dagger$

$$\xi = \frac{1}{\sqrt{4k}}[a_\xi + a_\xi^\dagger] \ , \ \pi_\xi = i\sqrt{k}[a_\xi - a_\xi^\dagger] \tag{3.64}$$

which now satisfy

$$[a_\xi, a_\xi^\dagger] = -1 \ , \ [H, a_\xi] = -2ka_\xi \ , \ [H, a_\xi^\dagger] = 2ka_\xi^\dagger \tag{3.65}$$

Because of the negative sign of the first commutator in (3.65) a standard quantization will lead to a *highest energy* state annihilated by $a_\xi^\dagger$, and then the action of powers of

a_ξ leads to an infinite tower of states with lower and lower energies. The highest state has a normalizable wavefunction of the standard form $e^{-k\xi^2}$ (Note that the expression for π_ξ has a negative sign compared to the usual harmonic oscillator). It is easy to see that this standard quantization does not reproduce the correct two-point function of the $Sp(2N)$ theory, does not lead to the correct spectrum (3.38) and, as shown below, does not lead to the correct wavefunction.

All this happens because ρ and hence ξ is not really a bosonic variable, and this allows other possibilities. Consider now a state $|0\rangle_\xi$ which is annihilated by the annihilation operator a_ξ. This leads to a wavefunction $\exp[k\xi^2]$, which is inadmissible if ξ is really a bosonic variable since it would be non-normalizable. However the true integration is over the Grassmann partons of these collective fields, and in terms of Grassmann integration this wavefunction is perfectly fine. This is in fact the state which has to be identified with the ground state of the $Sp(2N)$ oscillator. Including the factor of the Jacobian, the full wavefunction is (at large N)

$$\Psi'_{0\xi}[\xi] = [J'(\rho)]^{-1/2}\exp[k\xi^2] = [-\frac{N}{k} + 2\sqrt{\frac{N}{k}}\xi]^{N/2}\exp[k\xi^2] \tag{3.66}$$

Expanding the Jacobian factor in powers of ξ one now sees that the term which is quadratic in ξ cancels exactly, leaving with

$$\Psi'_{0\xi}[\xi] = \exp[-\sqrt{Nk}\xi + O(\xi^3)] \tag{3.67}$$

This is easily seen to exactly agree with Ψ_0 in (3.37)

$$\Psi_0 \sim \exp[-\frac{1}{2}k\rho] \sim \exp[-\sqrt{Nk}\xi] \tag{3.68}$$

up to a constant. Once again we need to take into account the interaction terms in the collective Hamiltonian to check that the $O(\xi^3)$ terms cancel. It can be easily verified that the propagator of fluctuations ξ will now be *negative* of the usual harmonic oscillator propagator. Furthermore the action of $a^\dagger_\xi$ now generates a tower of states with the energies (3.38) - except that the integer n is not bounded by N.

The fact that we get an unbounded (from above) spectrum from the collective theory is not a surprise. This is an expansion around $N = \infty$ and at $N = \infty$

the spectrum of $Sp(2N)$ is also unbounded. At finite N a change of variables to ρ is not useful because of the constraints coming from the Grassmann origin of ρ. Nevertheless, even in the $1/N$ expansion, the Grassmann origin allows us to consider wavefunctions which would be otherwise considered inadmissible.

The negative propagator ensures that the relationship (3.57) is satisfied for the 2 point functions. Once this choice is made, the relationship (3.57) holds for all m-point functions to the leading order in the large-N limit. As commented earlier, a term with even number of π_ξ or ξ would have an odd number of factors of ρ_0. Therefore a n-point vertex in the theory will differ from the corresponding n-point vertex of the $O(N)$ theory by a factor of $(-1)^{n+1}$. The connected correlator which appears in (3.57) is the sum of all connected tree diagrams with n external legs. The collective theory gives us the following Feynman rules

1 Every propagator contributes a negative sign.

2 A p point vertex has a factor of $(-1)^{p+1}$

We now argue that these rules ensure the validity of the basic relation (3.56). We do it by the following simple diagrammatic method:

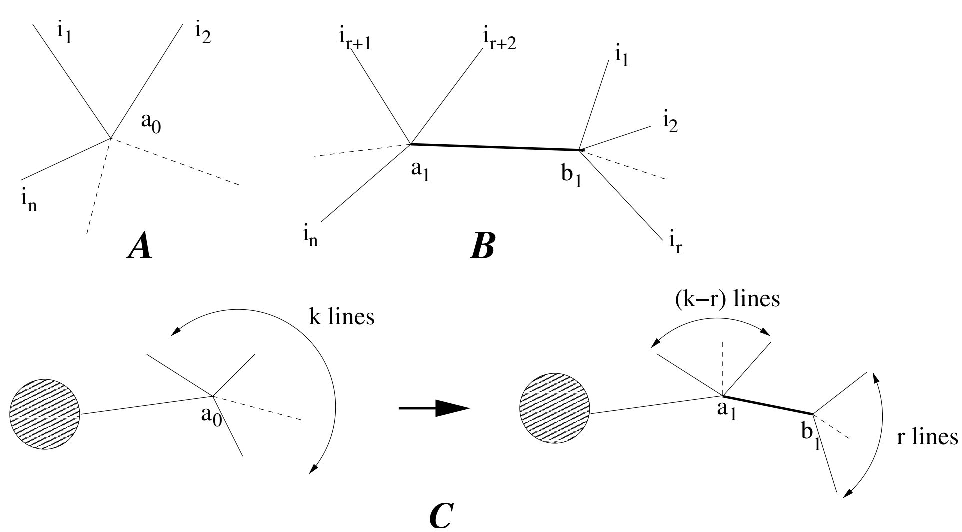


FIGURE 2. Connected tree level correlators of the collective theory

Consider first the simplest diagram for a n-point function, figure A, which is a star graph. The net sign of the diagram is $(-1)^{n+1} \times (-1)^n = -1$, where the first factor is from the vertex a_0 and the second one from the number of lines. Now we proceed to construct all other tree level diagrams from A, by pulling 'r' lines resulting in figure B, which now has vertices, a_1 and b_1 joined by a new line. It is easy to see, that the sign of figure A is not changed by this operation. The net sign of figure B is $(-1)^{(n-r+1)+1} \times (-1)^{(r+1)+1} \times (-1)^{(n+1)} = -1$, where the 3 factors are from a_1, b_1 and the number of lines respectively. In figure C we repeat this method for the substar diagrams until we exhaust all possibilities. It is easy to see that the sign stays invariant. Assigning a sign α to the blob, we first find the net sign of the left diagram in figure C. It turns out to be, $\alpha \times (-1)^{(k+1)+1} \times (-1)^{k+1} = -\alpha$. After the "pulling" operation we get $\alpha \times (-1)^{(k-r+2)+1} \times (-1)^{(r+1)+1} \times (-1)^{k+1+1} = -\alpha$. Thus it is proved that in every move the sign is preserved. This proves the relationship (3.57) for all correlation functions.

4.3. $Sp(2N)$ **Correlators.** Our discussion of the bosonic $O(N)$ collective field theory shows that the $Sp(2N)$ collective field theory in momentum space is a straightforward generalization. In this subsection we discuss the relevant features of the collective theory for the free $Sp(2N)$ model.

The collective Hamiltonian is again exactly the same as in the $O(N)$ theory, given by (3.51) with $\sigma \to \rho$. Since the connected correlators of the collective fields satisfy

$$\begin{aligned} &\langle \rho(\vec{k}_1, \vec{k}'_1, t_1)\rho(\vec{k}_2, \vec{k}'_2, t_2) \cdots \rho(\vec{k}_n, \vec{k}'_n, t_n)\rangle^{conn}_{Sp(2N)} \\ &= -\langle \sigma(\vec{k}_1, \vec{k}'_1, t_1)\sigma(\vec{k}_2, \vec{k}'_2, t_2) \cdots \sigma(\vec{k}_n, \vec{k}'_n, t_n)\rangle^{conn}_{SO(2N)} \end{aligned} \tag{3.69}$$

we now need to choose the negative saddle point,

$$\rho_0(\vec{k}, \vec{k}', t) = -\frac{N}{|\vec{k}|}\delta(\vec{k} - \vec{k}') \tag{3.70}$$

The fluctuation Hamiltonian once again has a factor of $(-1)^{n+1}$ for the n-point vertex. In particular, the propagator of the collective field is negative of that of the $O(N)$ collective field - the quadratic Hamiltonian has an overall negative sign! This is required - the diagramatic argument for the $Sp(2N)$ oscillator generalizes in a straightforward fashion, ensuring that (3.69) holds.

5. Bulk Dual of the $Sp(2N)$ model

In [**20**], it was proposed that the collective field theory for the d dimensional free $O(N)$ theory is in fact Vasiliev's higher spin theory in AdS$_{d+1}$. It is easy to see that the collective field has the right collection of fields. Consider for example $d = 3$. The field depends on four spatial variables, which may be reorganized as three spatial coordinates one of which is restricted to be positive and an angle. A fourier series in the angle then gives rise to a set of fields $\chi_{\pm n}$ which depend on three spatial variables, with the integer n denoting the conjugate to the angle. Symmetry under interchange of the arguments of the collective field then requires n to be even integers. But this is precisely the content of a theory of massless even spin fields in four space-time dimensions, with n labelling the spin and the two signs corresponding to the two helicities. (Recall that in four space-time dimensions massless fields with any spin have just two helicity states).

The precise relationship between collective fields and higher spin fields in AdS was found in [**30**] which we now summarize for $d = 3$. The correspondence is formulated in the light cone quantization. Denote the usual Minkowski coordinates on the space-time on which the $O(N)$ fields live by t, y, x and define light cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}}(t \pm y) \tag{3.71}$$

The conjugate momenta to x^+, x^- are denoted by p^-, p^+. Then in light front quantization where x^+ is treated as time, the Schrodinger picture fields are $\phi^i(x^-, x)$ while the momentum space fields are given by $\phi^i(p^+, p)$. The corresponding collective field is then defined as

$$\sigma(p_1^+, p_1; p_2^+, p_2) = \phi^i(p_1^+, p_1)\phi^i(p_2^+, p_2) \tag{3.72}$$

The fluctuation of this field around the saddle point is denoted by $\Psi(p_1^+, p_1; p_2^+, p_2)$. Now define the following bilocal field

$$\begin{aligned}\Phi(p^+, p^x, z, \theta) \;\; = & \int dp^z dp_1^+ dp_2^+ dp_1 dp_2 \; K(p^+, p^x, z, \theta; p_1^+, p_1, p_2^+, p_2) \\ & \cdot\Psi(p_1^+, p_1; p_2^+, p_2)\end{aligned} \tag{3.73}$$

where the kernel is given by

$$\begin{aligned} &K(p^+, p^x, z, \theta; p_1^+, p_1, p_2^+, p_2) \\ &= z\ e^{izp_z}\ \delta(p_1^+ + p_2^+ - p^+)\ \delta(p_1 + p_2 - p) \\ &\cdot\delta(p_1\sqrt{\frac{p_2^+}{p_1^+}} - p_2\sqrt{\frac{p_1^+}{p_2^+}} - p^z)\ \delta(2\tan^{-1}\sqrt{\frac{p_2^+}{p_1^+}} - \theta) \end{aligned} \tag{3.74}$$

In [**30**] it was shown that the Fourier transforms of the field $\Phi(p^+, p^x, z, \theta)$ with respect to θ satisfy the same linearized equation of motion as the physical helicity modes of higher spin gauge fields in AdS_4 in light cone gauge. The metric of this AdS_4 is given by the standard Poincare form

$$ds^2 = \frac{1}{z^2}[-2dx^+dx^- + dx^2 + dz^2] = \frac{1}{z^2}[-dt^2 + dy^2 + dx^2 + dz^2] \tag{3.75}$$

The momenta p^+, p are conjugate to x^-, x. The additional dimension generated from the large-N degrees of freedom is z, which is canonically conjugate to p^z and is given in terms of the phase space coordinate of the bi-locals by

$$z = \frac{(x_1 - x_2)\sqrt{p_1^+ p_2^+}}{p_1^+ + p_2^+} \tag{3.76}$$

In particular, the linearized equation for the spin zero field, $\varphi(x^-, x, z)$, follows from the quadratic action

$$S = \frac{1}{2}\int dx^+dx^-dzdx\ \left[\frac{1}{z^2}\left(-2\partial_+\varphi\partial_-\varphi - (\partial_x\varphi)^2 - (\partial_z\varphi)^2\right) + \frac{2}{z^4}\varphi^2\right] \tag{3.77}$$

which is of course the action of a conformally coupled scalar in the AdS_4 with coordinates given by (3.76). The actions for the spin-$2s$ fields can be similarly written down. Even though these actions are derived using light cone coordinates, they can be covariantized easily since these are free actions. In terms of the coordinates t, y, x, z the scalar action is given by

$$S = \frac{1}{2}\int dtdzdxdy\ \left[\frac{1}{z^2}\left((\partial_t\varphi)^2 - (\partial_y\varphi)^2 - (\partial_x\varphi)^2 - (\partial_z\varphi)^2\right) + \frac{2}{z^4}\varphi^2\right] \tag{3.78}$$

Let us now turn to the $Sp(2N)$ collective theory. One can define once again the fields as in (3.73) and (3.74). The coordinates (x^+, x^-, x, z) will continue to transform appropriately under AdS isometries. However, we saw earlier that the quadratic part of the Hamiltonian, and therefore the quadratic part of the action will have an overall *negative* sign.

A negative kinetic term signifies a pathology. Indeed we derived this theory with the Lorentzian signature $Sp(2N)$ model, which has negative norm states. The negative kinetic term of the collective theory is possibly intimately related to this lack of unitarity.

However, the form of the action (3.78) cries out for a *analytic continuation*

$$z = i\tau \ , \ t = -iw \tag{3.79}$$

Under this continuation the action, S becomes

$$S' = \frac{1}{2}\int d\tau dw dx dy \ \left[\frac{1}{\tau^2}\left((\partial_\tau\varphi)^2 - (\partial_y\varphi)^2 - (\partial_x\varphi)^2 - (\partial_w\varphi)^2\right) - \frac{2}{\tau^4}\varphi^2\right] \tag{3.80}$$

The sign of the mass term has not changed in this analytic continuation, and this action has become the action of a conformally coupled scalar field in de Sitter space with the metric

$$ds^2 = \frac{1}{\tau^2}[-d\tau^2 + dx^2 + dy^2 + dw^2] \tag{3.81}$$

This mechanism works for all *even* higher spin fields at the quadratic level.

To summarize, the collective field theory of the three dimensional Lorentzian $Sp(2N)$ model can be written as a theory of massless even spin fields in AdS_4, but with negative kinetic terms. Under a double analytic continuation this becomes the action in dS_4 with positive kinetic terms. This is consistent with the conjecture of [**78**] that the *Euclidean* $Sp(N)$ model is dual to Vasiliev theory in dS_4. It is interesting to note that the way an emergent holographic direction is similar to the way the Liouville mode has to be interpeted as a time dimension in worldsheet supercritical string theory [**56**]. In this latter case, the sign of the kinetic term for the Liouville mode is negative for $d > d_{cr}$.

Even for the $O(N)$ model, the collective field is an represents seemingly an overcomplete description, since for a finite number of points in space K, one replaces at most NK variables by K^2 variables, which is much larger in the thermodynamic and continuum limit. However, in the perturbative $1/N$ expansion this is not an issue and the collective theory is known to reproduce the standard results of the $O(N)$ model. The issue becomes of significance at finite N level. The relevance of incorporating for such features has been noted in [**52, 41**].

For the fermionic $Sp(2N)$ model, there appears potentially an even more important redundancy related to the Grassmannian origin of the construction. Consequently the fields are to obey nontrivial constraint relationships and the Hilbert space is subject to a cutoff of highly excited states. This 'exclusion principle' was noted already in the AdS correspondence involving S_N orbifolds[**75, 76, 77**].

In an expansion around $N = \infty$ most effects of this are invisible. Our discussion shows that this can be regarded as a theory of higher spin fields in dS that is insensitive to these effects. However, as we saw above, the Grassmannian origin was already of importance in choosing the correct saddle point and the correct quantization of the quadratic Hamiltonian. In the next section we will address the question of finite N and the Hilbert space of the bi-local theory. In the framework of a geometric (pseudo-spin) representation, we will give evidence that the bi-local theory is non-perturbatively satisfactory at the finite N level.

6. Geometric Representation and The Hilbert Space

The bi-local collective field representation is seen to give a bulk description dS space and the Higher Spin fields. It provides an interacting theory with vertices governed by $G = 1/N$ as the coupling constant. We will now show that the collective theory has an equivalent geometric (Pseudo-spin) variable description appropriate for nonperturbative considerations. The essence of this (geometric) description is in reinterpreting the bi-local collective fields (and their canonical conjugates) as matrix variables (of infinite dimensionality) endowed with a Kähler structure.

This geometric description will provide a tractable framework for quantization and non-perturbative definition of the bi-local and HS de Sitter theory. It will be seen capable to incorporate non-perturbative features related to the Grassmannian origin of bi-local fields and its Hilbert space. Pseudo-spin collective variables represent all $Sp(2N)$ invariant variables of the theory (both commuting and non-commuting). These close a compact algebra and at large N are constrained by the corresponding Casimir operator. One therefore has an algebraic pseudo-spin system whose non-linearity is governed by the coupling constant $G = 1/N$. As such they have been employed earlier for developing a large N expansion [**90**] and as a model for quantization [**91**]. This version of the theory is in its perturbative $(1/N)$ expansion identical to the bi-local collective representation. It therefore has the same map to and correspondence with Higher Spin dS_4 at perturbative level. We will see however that the geometric representation becomes of use for defining (and evaluating) the Hilbert space and its quantization.

To describe the pseudo-spin description of the $Sp(2N)$ theory we will follow the quantization procedure of [**93**]. In this approach one starts from the action:

$$S = \int d^d x \; dt (\partial^\mu \eta_1^i \partial_\mu \eta_2^i) \tag{3.82}$$

and deduces the canonical anti-commutation relations

$$\{\eta_1^i(x,t)\partial_t \eta_2^j(x',t)\} = -\{\eta_2^i(x,t)\partial_t \eta_1^j(x',t)\} = i\delta^d(x-x')\delta^{ij} \tag{3.83}$$

The quantization based on the mode expansion

$$\begin{aligned}
\eta_1^i(x) &= \int \frac{d^d k}{(2\pi)^{d/2}\sqrt{2\omega_k}} (a_{k+}^{i\dagger} e^{-ikx} + a_{k-}^i e^{ikx}) \\
\eta_2^i(x) &= \int \frac{d^d k}{(2\pi)^{d/2}\sqrt{2\omega_k}} (-a_{k-}^{i\dagger} e^{-ikx} + a_{k+}^i e^{ikx})
\end{aligned} \tag{3.84}$$

with

$$\{a_{k-}^i, a_{k'-}^{j\dagger}\} = \{a_{k+}^i, a_{k'+}^{j\dagger}\} = \delta^d(k-k')\delta^{ij} \tag{3.85}$$

Note that in this approach the operators η_a^i are not Hermitian, but pseudo-Hermitian in the sense of [**83**].

Pseudo-spin bi-local variables will be introduced based on $Sp(2N)$ invariance, we have the vectors:

$$\begin{aligned}
\eta &= (\eta_1^1, \eta_2^1, \eta_1^2, \eta_2^2, \cdots, \eta_1^N, \eta_2^N) \\
a(k) &= (a_{k-}^1, a_{k+}^1, a_{k-}^2, a_{k+}^2, \cdots, a_{k-}^N, a_{k+}^N) \\
\tilde{a}(k) &= (a_{k+}^{1\dagger}, -a_{k-}^{1\dagger}, a_{k+}^{2\dagger}, -a_{k-}^{2\dagger}, \cdots, a_{k\dagger}^{N+}, -a_{k-}^{N\dagger}) \qquad (3.86)
\end{aligned}$$

and the notation:

$$\eta(x) = \int \frac{d^d k}{(2\pi)^{d/2}\sqrt{2\omega_k}} (\tilde{a}(k) e^{-ikx} + a(k) e^{ikx}) \qquad (3.87)$$

so that a complete set of $Sp(2N)$ invariant operators now follows:

$$\begin{aligned}
S(p_1, p_2) &= \frac{-i}{2\sqrt{N}} a^T(p_1) \epsilon_N a(p_2) = \frac{i}{2\sqrt{N}} \sum_{i=1}^{N} (a_{p_1+}^i a_{p_2-}^i + a_{p_2+}^i a_{p_1-}^i) \\
S^\dagger(p_1, p_2) &= \frac{-i}{2\sqrt{N}} \tilde{a}^T(p_1) \epsilon_N \tilde{a}(p_2) = \frac{i}{2\sqrt{N}} \sum_{i=1}^{N} (a_{p_1+}^{i\dagger} a_{p_2-}^{i\dagger} + a_{p_2+}^{i\dagger} a_{p_1-}^{i\dagger}) \\
B(p_1, p_2) &= \tilde{a}^T(p_1) \epsilon_N a(p_2) = \sum_{i=1}^{N} a_{p_1+}^{i\dagger} a_{p_2+}^i + a_{p_1-}^{i\dagger} a_{p_2-}^i \qquad (3.88)
\end{aligned}$$

and $\epsilon_N = \epsilon \otimes \mathbb{I}_N$, $\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

These invariant operators close an invariant algebra. The commutation relations are found to equal:

$$\begin{aligned}
\left[S(\vec{p}_1, \vec{p}_2), S^\dagger(\vec{p}_3, \vec{p}_4)\right] = & \ \frac{1}{2} (\delta_{\vec{p}_2,\vec{p}_3} \delta_{\vec{p}_4,\vec{p}_1} + \delta_{\vec{p}_2,\vec{p}_4} \delta_{\vec{p}_3,\vec{p}_1}) - \frac{1}{4N} [\delta_{\vec{p}_2,\vec{p}_3} B(\vec{p}_4, \vec{p}_1) + \delta_{\vec{p}_2,\vec{p}_4} B(\vec{p}_3, \vec{p}_1) \\
& + \delta_{\vec{p}_1,\vec{p}_3} B(\vec{p}_4, \vec{p}_2) + \delta_{\vec{p}_1,\vec{p}_4} B(\vec{p}_3, \vec{p}_2)] \\
\left[B(\vec{p}_1, \vec{p}_2), S^\dagger(\vec{p}_3, \vec{p}_4)\right] = & \ \delta_{\vec{p}_2,\vec{p}_3} S^\dagger(\vec{p}_1, \vec{p}_4) + \delta_{\vec{p}_2,\vec{p}_4} S^\dagger(\vec{p}_1, \vec{p}_3) \\
\left[B(\vec{p}_1, \vec{p}_2), S(\vec{p}_3, \vec{p}_4)\right] = & \ -\delta_{\vec{p}_1,\vec{p}_3} S(\vec{p}_2, \vec{p}_4) - \delta_{\vec{p}_1,\vec{p}_4} S(\vec{p}_2, \vec{p}_3) \qquad (3.89)
\end{aligned}$$

The singlet sector of the original $Sp(2N)$ theory is characterized by a further constraint. This constraint is is associated with the Casimir operator of of the algebra and can be shown to take the form:

$$\frac{4}{N} S^\dagger \star S + (1 - \frac{1}{N} B) \star (1 - \frac{1}{N} B) = \mathbb{I} \qquad (3.90)$$

Here we have used the matrix star product notation: $\star$ product as: with $A \star B = \int d\vec{p}_2 A(\vec{p}_1\vec{p}_2)B(\vec{p}_2\vec{p}_3)$.

The form of the Casimir, which commutes with the above pseudo-spin fields points to the compact nature of the bi-local pseudo-spin algebra associated with the $Sp(2N)$ theory. This will have major consequences which we will highlight later.

Indeed it is interesting to compare the algebra with the bosonic case, where we have:

$$\begin{aligned} S(p_1,p_2) &= \frac{1}{2\sqrt{N}}\sum_{i=1}^{2N} a_i(p_1)a_i(p_2) \\ S^\dagger(p_1,p_2) &= \frac{1}{2\sqrt{N}}\sum_{i=1}^{2N} a_i^\dagger(p_1)a_i^\dagger(p_2) \\ B(p_1,p_2) &= \sum_{i=1}^{2N} a_i^\dagger(p_1)a_i(p_2) \end{aligned} \tag{3.91}$$

with the commutation relations:

$$\begin{aligned} \left[S(\vec{p}_1,\vec{p}_2),S^\dagger(\vec{p}_3,\vec{p}_4)\right] = & \ \frac{1}{2}\left(\delta_{\vec{p}_2,\vec{p}_3}\delta_{\vec{p}_4,\vec{p}_1}+\delta_{\vec{p}_2,\vec{p}_4}\delta_{\vec{p}_3,\vec{p}_1}\right)+\frac{1}{4N}\left[\delta_{\vec{p}_2,\vec{p}_3}B(\vec{p}_4,\vec{p}_1)+\delta_{\vec{p}_2,\vec{p}_4}B(\vec{p}_3,\vec{p}_1)\right. \\ & \left.+\,\delta_{\vec{p}_1,\vec{p}_3}B(\vec{p}_4,\vec{p}_2)+\delta_{\vec{p}_1,\vec{p}_4}B(\vec{p}_3,\vec{p}_2)\right] \\ \left[B(\vec{p}_1,\vec{p}_2),S^\dagger(\vec{p}_3,\vec{p}_4)\right] = & \ \delta_{\vec{p}_2,\vec{p}_3}S^\dagger(\vec{p}_1,\vec{p}_4)+\delta_{\vec{p}_2,\vec{p}_4}S^\dagger(\vec{p}_1,\vec{p}_3) \\ \left[B(\vec{p}_1,\vec{p}_2),S(\vec{p}_3,\vec{p}_4)\right] = & \ -\delta_{\vec{p}_1,\vec{p}_3}S(\vec{p}_2,\vec{p}_4)-\delta_{\vec{p}_1,\vec{p}_4}S(\vec{p}_2,\vec{p}_3) \end{aligned} \tag{3.92}$$

In this case the Casimir constraint is found to equal:

$$-\frac{4}{N}S^\dagger \star S + (1+\frac{1}{N}B)\star(1+\frac{1}{N}B) = \mathbb{I} \tag{3.93}$$

featuring the non-compact nature of the bosonic problem.

We can therefore see that the singlet sectors of the fermionic $Sp(2N)$ theory and the bosonic $O(2N)$ theory can be described in analogy to a bi-local pseudo-spin algebraic formulations with a quadratic Casimir taking the form:

$$4\gamma S^\dagger \star S + (1-\gamma B)\star(1-\gamma B) = \mathbb{I} \tag{3.94}$$

the difference being that with $\gamma = \frac{1}{N}(-\frac{1}{N})$ for the fermionic (bosonic) case respectively. This signifies the compact versus the non-compact nature of the algebra, but also exhibits the relationship obtained through the $N \leftrightarrow -N$ switch that was central in the argument for de Sitter correspondence in [**78**].

From this algebraic bi-local formulation one can easily see the the collective field representation(s) that we have discussed in sections 2 and 3. Very simply, the Casimir constraints can be solved, and the algebra implemented in terms of a canonical pair of bi-local fields:

$$
\begin{aligned}
S(p_1p_2) &= \frac{\sqrt{-\gamma}}{2}\int dy_1dy_2e^{-i(p_1y_2+p_2y_2)}\{-\frac{2}{\kappa_{p_1}\kappa_{p_2}}\Pi\star\Psi\star\Pi(y_1y_2) - \frac{1}{2\gamma^2\kappa_{p_1}\kappa_{p_2}}\frac{1}{\Psi}(y_1y_2) \\
&\quad +\frac{\kappa_{p_1}\kappa_{p_2}}{2}\Psi(y_1y_2) - i\frac{\kappa_{p_1}}{\kappa_{p_2}}\Psi\star\Pi(y_1y_2) - i\frac{\kappa_{p_2}}{\kappa_{p_1}}\Pi\star\Psi(y_1y_2)\} \\
S^\dagger(p_1p_2) &= \frac{\sqrt{-\gamma}}{2}\int dy_1dy_2e^{-i(p_1y_2+p_2y_2)}\{-\frac{2}{\kappa_{p_1}\kappa_{p_2}}\Pi\star\Psi\star\Pi(y_1y_2) - \frac{1}{2\gamma^2\kappa_{p_1}\kappa_{p_2}}\frac{1}{\Psi}(y_1y_2) \\
&\quad +\frac{\kappa_{p_1}\kappa_{p_2}}{2}\Psi(y_1y_2) + i\frac{\kappa_{p_1}}{\kappa_{p_2}}\Psi\star\Pi(y_1y_2) + i\frac{\kappa_{p_2}}{\kappa_{p_1}}\Pi\star\Psi(y_1y_2)\} \\
B(p_1p_2) &= \frac{1}{\gamma} + \int dy_1dy_2e^{-i(p_1y_2+p_2y_2)}\{\frac{2}{\kappa_{p_1}\kappa_{p_2}}\Pi\star\Psi\star\Pi(y_1y_2) + \frac{1}{2\gamma^2\kappa_{p_1}\kappa_{p_2}}\frac{1}{\Psi}(y_1y_2) \\
&\quad +\frac{\kappa_{p_1}\kappa_{p_2}}{2}\Psi(y_1y_2) - i\frac{\kappa_{p_1}}{\kappa_{p_2}}\Psi\star\Pi(y_1y_2) + i\frac{\kappa_{p_2}}{\kappa_{p_1}}\Pi\star\Psi(y_1y_2)\}
\end{aligned}
\tag{3.95}
$$

where $\kappa_p = \sqrt{\omega_p}$.

Recalling that the Hamiltonian is given in terms of B, we now see that its bi-local form is the same in the fermionic and the bosonic cases. This explains the feature that we have established before by direct construction. While the bi-local field representation of B is the same in the fermionic and bosonic cases, the difference is seen in the representations of operators S and $S^\dagger$. These operators create singlet states in the Hilbert space and the difference contained in the sign of γ implies the opposite shifts for the background fields that we have identified. The algebraic pseudo spin reformulation is therefore seen to account for all the perturbative $(1/N)$ features of the the bi-local theory that we have identified. However, we would like to emphasize that, the algebraic formulation provides a proper framework for defining the bi-local Hilbert space.

6.1. Quantization and the Hilbert Space. The bi-local pseudo-spin algebra has several equivalent representations that turn out to be useful. Beside that collective representation that we have explained above, one has the simple oscillator representation:

$$
\begin{aligned}
S(p_1,p_2) &= \alpha\star(1-\frac{1}{N}\alpha^\dagger\star\alpha)^{\frac{1}{2}}(p_1,p_2)\\
S^\dagger(p_1,p_2) &= (1-\frac{1}{N}\alpha^\dagger\star\alpha)^{\frac{1}{2}}\star\alpha^\dagger(p_1,p_2)\\
B(p_1,p_2) &= 2\,\alpha^\dagger\star\alpha(p_1,p_2)
\end{aligned}
\tag{3.96}
$$

with standard canonical canonical commutators (or Poisson brackets).

A more relevant geometric representation is obtained through a change:

$$
\begin{aligned}
\alpha &= Z(1+\frac{1}{N}\bar{Z}Z)^{-\frac{1}{2}}\\
\alpha^\dagger &= (1+\frac{1}{N}\bar{Z}Z)^{-\frac{1}{2}}\bar{Z}
\end{aligned}
\tag{3.97}
$$

The pseudo-spins in the Z representation are given by:

$$
\begin{aligned}
S(p_1,p_2) &= Z\star(1+\frac{1}{N}\bar{Z}\star Z)^{-1}(p_1,p_2)\\
S^\dagger(p_1,p_2) &= (1+\frac{1}{N}\bar{Z}\star Z)^{-1}\star\bar{Z}(p_1,p_2)\\
B(p_1,p_2) &= 2\,Z\star(1+\frac{1}{N}\bar{Z}\star Z)^{-1}\star\bar{Z}(p_1,p_2)
\end{aligned}
\tag{3.98}
$$

It is easy to see that these satisfy the Casimir constraint: $\frac{4}{N}S^\dagger\star S+(1-\frac{1}{N}B)^2=1$.

One can write the Lagrangian in this Z representation as:

$$
\mathcal{L}=i\int dt\ \mathrm{tr}[Z(1+\frac{1}{N}\bar{Z}Z)^{-1}\dot{\bar{Z}}-\dot{Z}(1+\frac{1}{N}\bar{Z}Z)^{-1}\bar{Z}]-\mathcal{H}
\tag{3.99}
$$

For regularization purposes, it is useful to consider putting $\vec{x}$ in a box and limiting the momenta by a cutoff Λ: this makes the bi-local fields into finite dimensional matrices (which we will take to be a size K). For $Sp(2N)$ one deals with a $K\times K$ dimensional complex matrix Z, where we have obtained in the above a compact symmetric (Kähler) space :

$$
ds^2=\mathrm{tr}[dZ(1-\bar{Z}Z)^{-1}d\bar{Z}(1-Z\bar{Z})^{-1}]
\tag{3.100}
$$

According to the classification of [**92**], this would correspond to manifold $M_I(K,K)$. We note that the standard fermionic problem which was considered in detail in [**91**] corresponds to manifold $M_{III}(K,K)$ of complex antisymmetric matrices.

Quantization on Kähler manifolds in general has been formulated in detail by Berezin [**91**]. We also note that the usefulness of Kähler quantization for discretizing de Sitter space was pointed out by A. Volovich in a quantum mechanical scenario[**25**]. In the present quantization, we are dealing with a field theory with infinitely many degrees of freedom and an infinite number of Khaler matrix variables. We will now summarize some of the results of quantization which are directly relevant to the $Sp(2N)$ bi-local collective fields theory. Commutation relations of this system follow from the Poisson brackets associated with the Lagrangian $\mathcal{L}(\bar{Z}, Z)$. States in the Hilbert space are represented by (holomorphic) functions (functionals) of the bi-locals $Z(k,l)$. A Kähler scalar product defining the bi-local Hilbert space reads:

$$(F_1, F_2) = C(N,K) \int d\mu(\bar{Z}, Z) F_1(Z) F_2(\bar{Z}) \det[1 + \bar{Z}Z]^{-N} \tag{3.101}$$

with the (Kähler) integration measure:

$$d\mu = \det[1 + \bar{Z}Z]^{-2K} d\bar{Z} dZ \tag{3.102}$$

The normalization constant is found from requiring $(F_1, F_1) = 1$ for $F = 1$. Let:

$$a(N,K) = \frac{1}{C(N,K)} = \int d\mu(\bar{Z}, Z) \det[1 + \bar{Z}Z]^{-N} \tag{3.103}$$

This leads to the matrix integral (complex Penner Model)

$$a(N,K) = \frac{1}{C(N,K)} = \int \prod_{k,l=1}^{K} d\bar{Z}(k,l) dZ(k,l) \det[1 + \bar{Z}Z]^{-2K-N} \tag{3.104}$$

which determines $C(N,K)$.

The following results on quantization of this type of Kähler system are of note: First, the parameter N: much like for ordinary spin, one can show that N (and therefore G in Higher Spin Theory) can only take integer values, i.e. $N = 0, 1, 2, 3, \cdots$. Next, one has question about the total number of states in the above Hilbert space. Naively, the bi-local theory would seem to grossly overcount the number of states of the original fermionic theory. Originally one essentially had $2NK$ fermionic degrees of freedom with a finite Hilbert space. The bi-local description is based on (complex) bosonic variables of dimensions K^2 and the corresponding Hilbert space would appear to be much larger. However, due to the compact nature of the phase space, the number of states is in fact much smaller.

We will now evaluate this number (at finite N and K) for the present case of $Sp(2N)$ (in [**91**] ordinary fermions were studied) and show that the exact dimension of the bi-local Hilbert space in geometric (Kähler) quantization agrees with the dimension of the singlet Hilbert space of the $Sp(2N)$ fermionic theory.

The dimension of quantized Hilbert space is found as follows: considering the operator $\hat{O} = I$ one has that:

$$\mathrm{Tr}(I) = C(N,K) \int \prod_{k,l=1}^{K} d\bar{Z}(k,l) dZ(k,l) \det[1 + \bar{Z}Z]^{-2K} \tag{3.105}$$

Consequently the dimension of the bi-local Hilbert space is given by:

$$\mathrm{Dim}\ \mathcal{H}_B = \frac{C(N,K)}{C(0,K)} = \frac{a(0,K)}{a(N,K)} \tag{3.106}$$

The evaluation of the matrix (Penner) integral therefore also determines the dimension of the bi-local Hilbert space. Since this evaluation is a little bit involved, we present it in the following. Evaluation of matrix integrals (for real matrices) is given in [**81**] the extension to the complex case was considered in [**82**].

We will use results of [**92**], whereby every (complex) matrix can be reduced through (symmetry) transformations to a diagonal form:

$$Z(k,l) \rightarrow \begin{bmatrix} \omega_1 & & & & \\ & \omega_2 & & 0 & \\ & & \omega_3 & & \\ & 0 & & \ddots & \\ & & & & \omega_K \end{bmatrix} \tag{3.107}$$

and the matrix integration measure becomes:

$$[d\bar{Z} dZ] = |\Delta(\omega)|^2 \prod_{l=1}^{K} d\omega_l d\Omega \tag{3.108}$$

where $d\Omega$ denotes "angular" parts of the integration and $\Delta(x_1, \cdots, x_K) = \prod_{k<l}(x_k - x_l)$ is a Vandermonde determinant, with $x_i = \omega_i^2$. Consequently the matrix integral for $a(N, K)$ (and $C(N, K)$) becomes:

$$a(N,K) = \frac{\text{Vol } \Omega}{K!} \int \Delta(x_1, \cdots, x_K)^2 \prod_l (1+\omega_l^2)^{-2K-N} \prod_l d\omega_l \tag{3.109}$$

Through a change of variables, $x_i = -\frac{y_i}{1-y_i}$, we get:

$$a(N,K) = \frac{\text{Vol } \Omega}{2^K K!} \int_0^\Lambda \prod_i^K dy_i \Delta(y_1, \cdots, y_K)^2 \prod_i (1-y_i)^N \tag{3.110}$$

This integral can be evaluated exactly. It belongs to a class of integrals evaluated by Selberg in 1944 [**84**]:

$$\begin{aligned} I(\alpha,\beta,\gamma,n) &= \int_0^1 dx_1 \cdots \int_0^1 dx_n |\Delta(x)|^{2\gamma} \prod_{j=1}^n x_j^{\alpha-1}(1-x_j)^{\beta-1} \\ &= \prod_{j=0}^{n-1} \frac{\Gamma(1+\gamma+j\gamma)\Gamma(\alpha+j\gamma)\Gamma(\beta+j\gamma)}{\Gamma(1+\gamma)\Gamma(\alpha+\beta+(n+j-1)\gamma)} \end{aligned} \tag{3.111}$$

We have the case with $\alpha = 1, \ \ \beta = N+1, \ \ \gamma = 1, \ \ n = K$ and

$$I(1, N+1, 1, K) = \prod_{j=0}^{K-1} \frac{\Gamma(2+j)\Gamma(1+j)\Gamma(N+1+j)}{\Gamma(2)\Gamma(N+K+j+1)} \tag{3.112}$$

We therefore obtain the following formula for the number of states in our bi-local $Sp(2N)$ Hilbert space:

$$\text{Dim } \mathcal{H}_B = \prod_{j=0}^{K-1} \frac{\Gamma(j+1)\Gamma(N+K+j+1)}{\Gamma(K+j+1)\Gamma(N+j+1)} \tag{3.113}$$

We have compared this number with explicit enumeration of $Sp(2N)$ invariant states in the fermionic Hilbert space (for low values of N and K) and found complete agreement. It is probably not that difficult to prove agreement for all N, K. This settles however the potential problem of overcompletness of the bi-local representation. Since the $Sp(2N)$ counting uses the fermionic nature of creation operators and features exclusion when occupation numbers grow above certain limit it is seen that bi-local geometric quantization elegantly incorporates these effects. The compact nature of the associated infinite dimensional Kähler manifold secures the correct dimensionality of the the singlet Hilbert space. By using Stirling's approximation for the number of states in the bi-local Hilbert space (3.113), we see the dimension growing linearly in N (with $K \gg N$):

$$\ln(\text{Dim } \mathcal{H}_B) \sim 2NK \;\ln 2 \qquad \text{at leading order} \tag{3.114}$$

This is a clear demonstration of the presence of an N-dependent cutoff in agreement with the fermionic nature of the original $Sp(2N)$ Hilbert space. So in the nonlinear bi-local theory with $G = 1/N$ as coupling constant, we have the desired effect that the Hilbert space is cutoff through $1/G$ effects. Consequently we conclude that the geometric bi-local representation with infinite dimensional matrices $Z(k,l)$ provides a complete framework for quantization of the bi-local theory and of de Sitter HS Gravity.

The following further results on quantization of this type of Kähler systems have direct relevance to Higher Spin duality. First, the parameter N (and therefore G in Higher Spin Theory) can only take integer values, i.e. $N = 0, 1, 2, 3, \cdots$. This feature might appear to be very puzzling from Vasiliev's theory itself, but the fact that there exists a geometric (Kähler manifold) representation of the theory provides the explanation. We therefore expect that Vasiliev's theory when suitably canonically quantized takes the form of the above geometric Kähler system.

We also mention a very recent study of finite $N \to N + 1$ deformation in these theories [**94**]. This can possibly also be investigated by the present Hilbert space method as well.

7. Phase Transition

It was shown by Shenker and Yin [**41**] that the N-component vector model undergoes a phase transition at high temperature. The transition occurs at temperature of order $\sqrt{N}$ where $1/N = G$ plays the role of coupling constant. This is an important non-perturbative effect that characterizes Higher Spin theories. The argument in [**41**] is based on the exact analysis of $O(N)$ vector model partition function. Here we will show how the transition (and the presence of two phases) can be understood from the bulk field theoretic viewpoint.

We have already described two versions of bi-local field theory: the covariant one and a canonical (time-like) gauge one. The canonical gauge version (with the Hamiltonian in (3.10)) represents the singlet spectrum of the theory, and then the partition function is simply

$$Z(\beta) = \mathrm{Tr}\left(e^{-\beta H_{(2)}}\right) \tag{3.115}$$

giving in the large-N limit the answer

$$F(\beta) = \sum_{\vec{k}_1,\vec{k}_2} \ln\left(1 - e^{-\beta\omega(\vec{k}_1,\vec{k}_2)}\right) \tag{3.116}$$

corresponding to the singlet bi-local spectrum with $E_{\vec{k}_1,\vec{k}_2} = \omega(\vec{k}_1,\vec{k}_2) = |\vec{k}_1| + |\vec{k}_2|$. This leads to the $O(1)$ result:

$$F_1(\beta) \sim V\zeta(5)T^4 \tag{3.117}$$

where the power (and the argument of the ζ-function) features the dimensionality $D = 4$ of the bi-local space: $(\vec{x}_1, \vec{x}_2)$. This recovers the lower phase of [**41**].

The upper phase can be seen through a stationary point of the bi-local action as it was given in [**20**]. Namely, the covariant collective action (3.7) at finite temperature (with periodic boundary conditions in Euclidean time) has the following stationary-point solution:

$$\Phi_\beta(x,y) = \sum_{k_\mu} \frac{e^{ik\cdot(x-y)}}{k_0^2 - \vec{k}^2} \tag{3.118}$$

which is quantized as $k_0 = \frac{2\pi n}{\beta}$. Evaluation of the action leads to the $O(N)$ partition function:

$$F_N \equiv S_c(\Phi_\beta) = -\frac{N}{2}\sum_{n,\vec{k}} \ln\left(\vec{k}^2 + (\frac{2\pi n}{\beta})^2\right) \tag{3.119}$$

giving the upper phase result

$$F^N(\beta) \sim -NV\zeta(3)T^2 \tag{3.120}$$

as stated, this result is proportional to N characterizing the N-component vector model. In this and also lower phase case the volume is inherited from the special volume of the CFT.

An interpretation of this phase transition was suggested by Shenker and Yin in terms of an increase/decrease of number of degrees of freedom, namely from bi-locals to N-component partons. From the bi-local field theory viewpoint, we would like to offer an additional interpretation. In terms of the collective dipole (much like in the case of a string) the upper temperature phase is associated with condensation of extra ("winding") modes, an effect which gives a classical result of order $N = 1/G$. The covariant gauge bi-local field (used in exhibiting the upper phase) indeed contains such an extra mode whose relevance comes at finite temperature. It will be interesting to investigate this scenario and in general the physics of this interesting phase transition further.

CHAPTER 4

Coleman-Mandula Theorem in Higher Spin Theory

1. Overview

Since conserved currents imply the existence of an infinite sequence of conserved charges and higher symmetries, one is faced with the question regarding the implementation (and implication) of the Coleman-Mandula theorem. This question was raised and addressed in the recent work of Maldacena and Zhiboedov [**44, 45**] who were able to show the existence of conserved currents (charges) implies that the correlation functions are built in terms of free fields. This demonstrates the simplicity of the corresponding Vasiliev theory. One still, however, has the question regarding the triviality of the theory in the bulk. In standard field theories this question is addressed (and answered) through the S-matrix. The Coleman-Mandula theorem in particular would imply $S = 1$ for theories with higher symmetries. Due to the equivalence theorem (under field transformations) this means that there exists a field redefinition which linearizes the field equations. In the AdS/CFT framework one sometimes think of the correlators as taking the role analogous to an S-matrix. A proposal along this line, offered by Mack [**86, 87**], has been nicely implemented in recent works [**97, 98, 99, 100**]. If this analogy is taken at face value, one has the puzzling fact that this "boundary" S-matrix is non-trivial, even for the correspondence based on free theory.

We have in [**20, 30**] formulated a constructive approach to bulk AdS duality and HS Gravity in terms of bi-locals. It leads to a nonlinear, interacting theory (with $1/N$ as the coupling constant) which was seen to possess all the properties of the dual AdS theory. This theory reproduces arbitrary-point correlation functions and provides a construction of HS theory (in various gauges) based on CFT [**40**]. The construction also, as we will explain, offers a framework for defining and calculating an S-matrix and addressing the implementation of the Coleman-Mandula theorem in the nonlinear bulk framework. The construction is based on the two-particle collective dipole and its interactions in the large N limit. It has been known since the early work that nontrivial collective phenomena can appear "even" for free theories (for example excitations near the large N fermion surface). In the present case we are led to consider the S-matrix for collective dipoles; the corresponding LSZ reduction formula is easily stated as a limit of bi-local correlators. Its evaluation will produce the result $S = 1$ as claimed in the title.[1]

$S = 1$ implies triviality, namely that interactions can be removed by a nonlinear transformation of fields (by this we mean the $1/N$ interactions which equal G_N interactions in Vasiliev's theory). We demonstrate this for the nonlinear dipole representation, where we establish a construction of a nonlinear field transformation that linearizes the effective large N field theory. This transformation is analogous to a construction of the so-called master field [**101, 102**].

The content of this chapter is as follows. In section 2 we discuss the differences between "boundary S-matrix" and "collective S-matrix" that we propose. In particular we give an LSZ formula for the S-matrix and evaluate the associated three- and four-point amplitudes using the cubic and quartic vertices of the $1/N$ theory demonstrating the result $S = 1$. In section 3 we present a construction of a nonlinear bi-local field transformation that linearizes the theory.

2. Coleman-Mandula Theorem in $\mathbf{AdS_4/CFT_3}$

Our concern is the simplest case of the correspondence which involves the UV fixed point CFT of noninteracting N-component bosonic or fermionic fields and the corresponding Vasiliev theory. These theories are characterized by the existence of an infinite sequence of higher spin currents that are conserved. Consequently one has

[1] We mention that this is analogous to an earlier situation involving the $c = 1$ matrix model with 2d string correspondence where one had the statement "$S = 1$ for $c = 1$" demonstrated in [**105**]. The only difference is that the collective boson (representing fluctuations above the fermion surface) is now replaced by the collective dipole.

a higher symmetry and an infinite sequence of generators

$$Q^s = \int d\vec{x} J_{0\mu_1\mu_2\cdots\mu_s} \ . \tag{4.1}$$

In such a theory, the Coleman-Mandula theorem implies that the S-matrix should be 1. The relevance and implications of the Coleman-Mandula theorem in $\mathrm{AdS}_4/\mathrm{CFT}_3$ was recently considered in the work of Maldacena and Zhiboedov [**44, 45**]. Using the light cone charges

$$Q^s = \int dx^- dx J_{---\cdots-} \ , \tag{4.2}$$

they demonstrate that the existence of this infinite sequence implies that the correlators are given by free fields, establishing in this sense that the theory can be categorized as simple.

The recovered correlators $C_n = \langle \mathcal{O}_1 \mathcal{O}_2 \cdots \mathcal{O}_n \rangle$ are nonetheless nonzero for all n. They describe a nonlinear bulk theory, with nonlinearities governed by $1/N = G_N$. The question then concerns the fate of these nonlinearities characterizing the AdS_4 HS theory.

Boundary correlators are sometimes described in the literature as a "boundary S-matrix" of the AdS theory. In fact Mack [**86, 87**] has put forward arguments whereby CFT correlation functions themselves possess a structure equivalent to an S-matrix. He argued that they can be in general written in an integral form (the Mellin representation), which then implies various properties (crossing, duality, etc.) in support of their S-matrix interpretation. This interpretation was strengthened by the AdS calculation [**97**]. Nevertheless, this "boundary S-matrix" lacks some of the key features of a genuine scattering matrix.

Based on the collective construction we would like to put forward (and evaluate) another more direct S-matrix which we will base on the physical picture of (collective) dipoles that underlie the CFT_3/Higher Spin Holography. In bi-local field theory this would corresponds to amplitudes of "mesons". Following this picture we first identify an appropriate on-shell relation (specified as always by the quadratic Hamiltonian) and then define the S-matrix through a standard reduction formula where external leg poles are amputated.

2.1. An example. Before proceeding with the details, we describe an analogous example that features a simple (free) theory duality: the old $d = 1$ Matrix Model / $2d$ non-critical string theory correspondence [**103**]. One has the matrix Hamiltonian corresponding to N^2 decoupled harmonic oscillators

$$H = -\frac{1}{2}\sum_{\alpha=1}^{N^2}\left(\frac{\partial^2}{\partial M_\alpha^2} - M_\alpha^2\right). \tag{4.3}$$

In this model one also had an infinite sequence of higher charges: $Q_s = \mathrm{Tr}[(P^2 - M^2)^s]$ and an infinite $\mathcal{W}_\infty$ symmetry. In the basic matrix theory representation, there is clearly no scattering and no visible S-matrix. A spacetime interpretation of the model (and an S-matrix) is found through the collective (Fermi-Droplet) representation represented by the large N collective Hamiltonian [**103**]

$$H_c = \int dx\left(\frac{1}{2}\partial_x\Pi(x)\phi(x)\partial_x\Pi(x) + \frac{\pi^2}{6}\phi^3 - \frac{x^2}{2}\phi\right) \tag{4.4}$$

where $\phi(x)$ and $\Pi(x)$ obey the canonical commutation relations $[\phi(x), \Pi(y)] = i\delta(x - y)$. This collective Hamiltonian correctly reproduces all the correlators $\langle\mathcal{O}_{n_1}\mathcal{O}_{n_2}\cdots\mathcal{O}_{n_k}\rangle$ for the most general invariant operators $\mathcal{O}_n = \mathrm{Tr}(M^n) = \int dx\, x^n\phi$.

Small fluctuations of this (collective) theory $\phi = \phi_0 + \partial_x\psi$, $\Pi = -\partial_x^{-1}\dot{\psi}$ features a 2d massless boson [**104**]

$$H^{(2)} = \int d\sigma\left(\frac{1}{2}\dot{\psi}^2(t,\sigma) + \frac{\pi^2}{2}\psi'^2(t,\sigma)\right) , \tag{4.5}$$

where the prime is the derivative with respect to the Liouville coordinate defined by

$$\sigma = \frac{1}{\pi}\int_0^x \frac{dy}{\phi_0(y)} . \tag{4.6}$$

Consequently one is led to consider the scattering of collective massless bosons [**105**] with an on-shell condition: $K_\mu = (E, K)$ and $E^2 - K^2 = 0$. Evaluation of the corresponding scattering amplitudes gives the S-matrix. For the three-point scattering amplitude, one has

$$\begin{aligned} S_3(E_1,E_2,E_3) &= 2\pi\delta(E_1+E_2+E_3)\left[\prod_{i=1}^{3}(E_i-K_i)-\prod_{i=1}^{3}(E_i+K_i)\right] \\ &= 2\pi\delta(E_1+E_2+E_3)\left[\prod_{i=1}^{3}\big(E_i-|E_i|\big)-\prod_{i=1}^{3}\big(E_i+|E_i|\big)\right] \quad (4.7) \end{aligned}$$

where we have used $K_i = |E_i|$ (corresponding to Liouville as time). For the scattering of incoming (outgoing) particles, we have $E_1, E_2 > 0$, $E_3 < 0$, so that

$$S_3(+,+,-) = 0 \ . \quad (4.8)$$

In the same way one can show the result $S_{n\geq 4} = 0$ due to Gross and Klebanov. A change of boundary conditions (in particular Dirichlet), gives however a non-trivial result $S_n \neq 0$ which was then compared with the string scattering amplitudes.

2.2. Evaluation of the three- and four-point amplitudes. Let us now return to the bi-local theory and consider therefore the S-matrix for scattering of "collective dipoles". In a time-like gauge (single-time), one has the on-shell relation: $E^2 - (|\vec{k}_1| + |\vec{k}_2|)^2 = 0$, and the S-matrix can be defined by the LSZ-type reduction formula

$$S = \lim \prod_i (E_i^2 - (|\vec{k}_i| + |\vec{k}_{i'}|)^2)\langle \tilde{\Psi}(E_1, \vec{k}_1, \vec{k}_{1'})\tilde{\Psi}(E_2, \vec{k}_2, \vec{k}_{2'})\cdots\rangle \quad (4.9)$$

where the $\tilde{\Psi}$ operators denote energy-momentum transforms of the bi-local fields (3.9). The limit implies the on-shell specification for the energies of the dipoles. In the light-cone gauge, (4.9) would correspond to

$$\lim \prod_i (P_i^- - \frac{p_i^2}{2p_i^+} - \frac{p_{i'}^2}{2p_{i'}^+})\langle \tilde{\Psi}(P_1^-; p_1^+, p_1, p_{1'}^+, p_{1'})\tilde{\Psi}(P_2^-; p_2^+, p_2, p_{2'}^+, p_{2'})\cdots\rangle \ .$$

We note that the correlation functions appearing in this construction are not the correlation functions of conformal current operators $J_{--\ldots-}$. As Maldacena and Zhiboedov have discussed, the Ward identities based on currents provide a reconstruction of correlation functions for bi-local operators of the form $\mathcal{B}(x^+; (x_1^-, x_2^-); x_1 = x_2)$. Since these are bi-local in x but local in the other coordinates one is not in a position to consider the above defined S-matrix.

Our evaluation of the S-matrix proceeds as follows. Using the time-like quantization we will evaluate the 3 and 4-point scattering amplitude corresponding to associated Witten diagrams. In momentum space, in terms of creation-annihilation operators, the cubic (3.14) and quartic (3.15) interaction potentials take the form

$$\begin{aligned} H^{(3)} &= \frac{\sqrt{2}}{\sqrt{N}} \int \prod_{i=1}^{3} d\vec{k}_i \Big[-\frac{\omega_{k_1 k_2 k_3}}{3} \alpha_{\vec{k}_1 \vec{k}_2} \alpha_{-\vec{k}_2 \vec{k}_3} \alpha_{-\vec{k}_3 -\vec{k}_1} \\ &\quad + \omega_{k_2} \alpha_{\vec{k}_1 \vec{k}_2} \alpha_{-\vec{k}_2 \vec{k}_3} \alpha^{\dagger}_{\vec{k}_3 \vec{k}_1} + h.c. \Big] \qquad (4.10) \\ H^{(4)} &= \frac{1}{N} \int \prod_{i=1}^{4} d\vec{k}_i \, \frac{\omega_{k_1 k_2 k_3 k_4}}{4} \Big[\alpha_{\vec{k}_1 \vec{k}_2} \alpha_{-\vec{k}_2 \vec{k}_3} \alpha_{-\vec{k}_3 \vec{k}_4} \alpha_{-\vec{k}_4 -\vec{k}_1} \\ &\quad + 4 \alpha_{\vec{k}_1 \vec{k}_2} \alpha_{-\vec{k}_2 \vec{k}_3} \alpha_{-\vec{k}_3 \vec{k}_4} \alpha^{\dagger}_{\vec{k}_4 \vec{k}_1} + h.c. \\ &\quad + 4 \alpha_{\vec{k}_1 \vec{k}_2} \alpha_{-\vec{k}_2 \vec{k}_3} \alpha^{\dagger}_{\vec{k}_3 \vec{k}_4} \alpha^{\dagger}_{-\vec{k}_4 \vec{k}_1} + 2 \alpha_{\vec{k}_1 \vec{k}_2} \alpha^{\dagger}_{\vec{k}_2 \vec{k}_3} \alpha_{\vec{k}_3 \vec{k}_4} \alpha^{\dagger}_{\vec{k}_4 \vec{k}_1} \Big] \qquad (4.11) \end{aligned}$$

where we used the notation $\omega_{k_1 k_2 \cdots k_i} \equiv \omega_{k_1} + \omega_{k_2} + \cdots + \omega_{k_i}$ and $h.c.$ means taking the Hermitian conjugate of *only* the terms before.

We first evaluate the three-point correlation function at order $\frac{1}{\sqrt{N}}$:

$$\langle \mathrm{T}(\eta(t_1; x_1, y_1) \eta(t_2; x_2, y_2) \eta(t_3; x_3, y_3)) \rangle \ . \qquad (4.12)$$

The propagator is given by

$$\begin{aligned} &\langle 0 | \mathrm{T}(\eta(t_1; x_1, y_1) \eta(t_2; x_2, y_2)) | 0 \rangle = \int d\vec{k}_1 d\vec{k}_2 dE e^{-iE(t_1 - t_2)} \\ &\times e^{i\vec{k}_1 \cdot (\vec{x}_1 - \vec{x}_2)} e^{i\vec{k}_2 \cdot (\vec{y}_1 - \vec{y}_2)} \frac{\omega_{k_1} + \omega_{k_2}}{\omega_{k_1} \omega_{k_2}} \frac{i}{E^2 - (\omega_{k_1} + \omega_{k_2})^2 + i\epsilon} \ . \qquad (4.13) \end{aligned}$$

The corresponding Feynman diagram is shown in Figure 1(a). The vertices follow from (3.14), working in momentum space, one has

$$\begin{aligned} &\langle 0 | \mathrm{T}(\eta(E_1; p_1, p_{1'}) \eta(E_2; p_2, p_{2'}) \eta(E_3; p_3, p_{3'})) | 0 \rangle \\ &= \frac{4}{\sqrt{N}} \delta(E_1 + E_2 + E_3) \delta(\vec{p}_1 - \vec{p}_3) \delta(\vec{p}_{2'} - \vec{p}_{3'}) \delta(\vec{p}_{1'} + \vec{p}_2)) \\ &\quad \times \frac{1}{E_1^2 - (\omega_{p_1} + \omega_{p_{1'}})^2 + i\epsilon} \frac{1}{E_2^2 - (\omega_{p_2} + \omega_{p_{2'}})^2 + i\epsilon} \frac{1}{E_3^2 - (\omega_{p_3} + \omega_{p_{3'}})^2 + i\epsilon} \\ &\quad \times \Big\{ \frac{\omega_{p_2} + \omega_{p_3}}{\omega_{p_2} \omega_{p_3}} [E_2 E_3 + (\omega_{p_2} + \omega_{p_{2'}})(\omega_{p_3} + \omega_{p_{3'}})] \\ &\quad + \frac{\omega_{p_{1'}} + \omega_{p_{3'}}}{\omega_{p_{1'}} \omega_{p_{3'}}} [E_1 E_3 + (\omega_{p_1} + \omega_{p_{1'}})(\omega_{p_3} + \omega_{p_{3'}})] \\ &\quad + \frac{\omega_{p_1} + \omega_{p_{2'}}}{\omega_{p_1} \omega_{p_{2'}}} [E_1 E_2 + (\omega_{p_1} + \omega_{p_{1'}})(\omega_{p_2} + \omega_{p_{2'}})] \Big\} \ . \qquad (4.14) \end{aligned}$$

The on-shell three-point $(1+2\to 3)$ scattering amplitude is obtained by amputating the leg poles and putting the external states on-shell leading to

$$\begin{aligned} S(1+2\to 3) &= -\frac{\sqrt{2}}{8\sqrt{N}}(E_1+E_2-E_3)\,\delta(E_1+E_2-E_3) \\ &\quad \{\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_{2'}-\vec{p}_{3'})\delta(\vec{p}_{1'}+\vec{p}_2)+7\text{ more terms}\} \end{aligned} \tag{4.15}$$

where we have used energy conservation and the delta functions. The result $S_3=0$ is now manifest.

Next for the four-dipole scattering $(1+2\to 3+4)$, we use the interaction picture and also the creation-annihilation basis as given in (4.10, 4.11). The $1/N$ contributions to the S_4 scattering amplitude are collected as follows

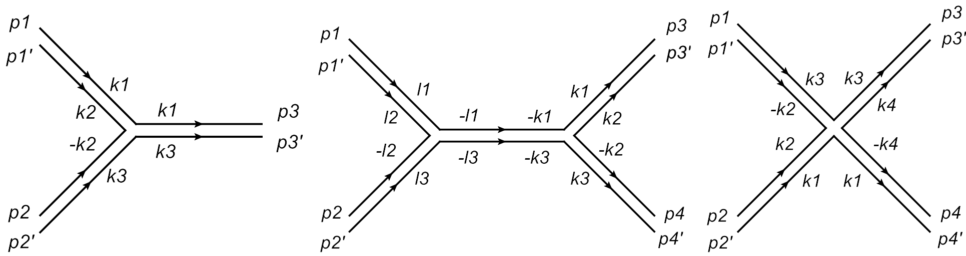
$$\begin{aligned} &-\frac{2}{9N}\int d\vec{k}_i d\vec{l}_j\,\omega_{k_1k_2k_3}\omega_{l_1l_2l_3}\langle 0|\alpha_{\vec{p}_3\vec{p}_{3'}}\alpha_{\vec{p}_4\vec{p}_{4'}}\alpha_{\vec{k}_1\vec{k}_2}\alpha_{-\vec{k}_2\vec{k}_3}\alpha_{-\vec{k}_3-\vec{k}_1}\alpha^\dagger_{\vec{l}_1\vec{l}_2} \\ &\qquad \alpha^\dagger_{-\vec{l}_2\vec{l}_3}\alpha^\dagger_{-\vec{l}_3-\vec{l}_1}\alpha^\dagger_{\vec{p}_1\vec{p}_{1'}}\alpha^\dagger_{\vec{p}_2\vec{p}_{2'}}|0\rangle \\ &-\frac{2}{N}\int d\vec{k}_i d\vec{l}_j\,\omega_{k_2}\omega_{l_2}\langle 0|\alpha_{\vec{p}_3\vec{p}_{3'}}\alpha_{\vec{p}_4\vec{p}_{4'}}\alpha_{\vec{k}_1\vec{k}_2}\alpha_{-\vec{k}_2\vec{k}_3}\alpha^\dagger_{\vec{k}_3\vec{k}_1} \\ &\qquad \alpha^\dagger_{\vec{l}_1\vec{l}_2}\alpha^\dagger_{-\vec{l}_2\vec{l}_3}\alpha_{\vec{l}_3\vec{l}_1}\alpha^\dagger_{\vec{p}_1\vec{p}_{1'}}\alpha^\dagger_{\vec{p}_2\vec{p}_{2'}}|0\rangle \\ &-\frac{i}{N}\int d\vec{k}_i\,\omega_{k_1k_2k_3k_4}\langle 0|\alpha_{\vec{p}_3\vec{p}_{3'}}\alpha_{\vec{p}_4\vec{p}_{4'}}\alpha_{\vec{k}_1\vec{k}_2}\alpha_{-\vec{k}_2\vec{k}_3} \\ &\qquad \alpha^\dagger_{\vec{k}_3\vec{k}_4}\alpha^\dagger_{-\vec{k}_4\vec{k}_1}\alpha^\dagger_{\vec{p}_1\vec{p}_{1'}}\alpha^\dagger_{\vec{p}_2\vec{p}_{2'}}|0\rangle \\ &-\frac{i}{2N}\int d\vec{k}_i\,\omega_{k_1k_2k_3k_4}\langle 0|\alpha_{\vec{p}_3\vec{p}_{3'}}\alpha_{\vec{p}_4\vec{p}_{4'}}\alpha_{\vec{k}_1\vec{k}_2}\alpha^\dagger_{\vec{k}_2\vec{k}_3} \\ &\qquad \alpha_{\vec{k}_3\vec{k}_4}\alpha^\dagger_{\vec{k}_4\vec{k}_1}\alpha^\dagger_{\vec{p}_1\vec{p}_{1'}}\alpha^\dagger_{\vec{p}_2\vec{p}_{2'}}|0\rangle\;. \end{aligned} \tag{4.16}$$

The relevant bi-local propagator symmetrized over the momenta is

$$\begin{aligned} \langle 0|T\alpha_{\vec{p}_1\vec{p}_{1'}}(t_1)\alpha^\dagger_{\vec{p}_2\vec{p}_{2'}}(t_2)|0\rangle &= \int dE\,\frac{ie^{-iE(t_1-t_2)}}{E-\omega_{p_1}-\omega_{p_{1'}}} \\ &\times\frac{1}{2}[\delta(\vec{p}_1-\vec{p}_2)\delta(\vec{p}_{1'}-\vec{p}_{2'})+\delta(\vec{p}_1-\vec{p}_{2'})\delta(\vec{p}_{1'}-\vec{p}_2)]\;. \end{aligned} \tag{4.17}$$

The first term of (4.16) has only s-channel contributions shown in Figure 1(b), while the second term of (4.16) has all s,t,u-channel contributions. The s-channel diagrams and their twisted ones (due to the symmetrization of propagators) are summed to be

$$\begin{aligned} &\tfrac{i}{8N}\delta(E_1+E_2-E_3-E_4)\times \\ \Big[\quad &\omega_{p_{2'}p_3}\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_{1'}+\vec{p}_2)\delta(\vec{p}_{2'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15\text{ similar terms} \\ +\quad &\omega_{p_{1'}p_3}\delta(\vec{p}_2-\vec{p}_3)\delta(\vec{p}_1+\vec{p}_{2'})\delta(\vec{p}_{1'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15\text{ similar terms}\Big]\;. \end{aligned} \tag{4.18}$$



(a) Three-dipole diagram; (b) The s-channel diagram; (c) The cross-shaped diagram.

FIGURE 1. Scattering of three and four dipoles.

It is also convenient to calculate the t, u-channel diagrams together, with their twisted diagrams, they are summed to be

$$
\begin{aligned}
& \frac{i}{8N}\delta(E_1+E_2-E_3-E_4)\times \\
\Big[\quad & \omega_{p_{1'}p_2}\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_{1'}+\vec{p}_2)\delta(\vec{p}_{2'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15\text{ terms} \\
+ \quad & \omega_{p_1p_{2'}}\delta(\vec{p}_2-\vec{p}_3)\delta(\vec{p}_1+\vec{p}_{2'})\delta(\vec{p}_{1'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15\text{ terms}\Big] \qquad (4.19) \\
+ \quad & \frac{i}{16N}\delta(E_1+E_2-E_3-E_4)\times \\
\Big[\quad & \omega_{p_1p_{1'}p_2p_{2'}}\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_{1'}-\vec{p}_4)\delta(\vec{p}_{2'}-\vec{p}_{4'})\delta(\vec{p}_2-\vec{p}_{3'})+15\text{ terms} \\
+ \quad & \omega_{p_1p_{1'}p_2p_{2'}}\delta(\vec{p}_2-\vec{p}_3)\delta(\vec{p}_{2'}-\vec{p}_4)\delta(\vec{p}_{1'}-\vec{p}_{4'})\delta(\vec{p}_1-\vec{p}_{3'})+15\text{ terms}\Big]\,. \qquad (4.20)
\end{aligned}
$$

The third term of (4.16) is the cross-shaped diagram shown in Figure 1(c), which gives the result

$$
\begin{aligned}
- \quad & \frac{i}{8N}\delta(E_1+E_2-E_3-E_4)\times \\
\Big[\quad & \omega_{p_{1'}p_{2'}p_3p_4}\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_{1'}+\vec{p}_2)\delta(\vec{p}_{2'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15\text{ terms} \\
+ \quad & \omega_{p_1p_2p_{3'}p_{4'}}\delta(\vec{p}_2-\vec{p}_3)\delta(\vec{p}_1+\vec{p}_{2'})\delta(\vec{p}_{1'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15\text{ terms}\Big]\,. \qquad (4.21)
\end{aligned}
$$

The calculation of the fourth term is similar to the third one, which gives the result

$$
\begin{aligned}
- \quad & \frac{i}{16N}\delta(E_1+E_2-E_3-E_4)\times \\
\Big[\quad & \omega_{p_1p_{1'}p_2p_{2'}}\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_{1'}-\vec{p}_4)\delta(\vec{p}_{2'}-\vec{p}_{4'})\delta(\vec{p}_2-\vec{p}_{3'})+15\text{ terms} \\
+ \quad & \omega_{p_1p_{1'}p_2p_{2'}}\delta(\vec{p}_2-\vec{p}_3)\delta(\vec{p}_{2'}-\vec{p}_4)\delta(\vec{p}_{1'}-\vec{p}_{4'})\delta(\vec{p}_1-\vec{p}_{3'})+15\text{ terms}\Big]\,. \qquad (4.22)
\end{aligned}
$$

Summing all the diagrams, it is easy to see (4.20) and (4.22) cancel each other, while the rest diagrams give the final result

$$\begin{aligned} S(1+2\to 3+4) = & \tfrac{i}{16N}(E_1+E_2-E_3-E_4)\delta(E_1+E_2-E_3-E_4) \\ & \times\big[\delta(\vec{p}_1-\vec{p}_3)\delta(\vec{p}_{1'}+\vec{p}_2)\delta(\vec{p}_{2'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15 \text{ more terms} \\ & +\delta(\vec{p}_2-\vec{p}_3)\delta(\vec{p}_1+\vec{p}_{2'})\delta(\vec{p}_{1'}-\vec{p}_{4'})\delta(\vec{p}_{3'}+\vec{p}_4)+15 \text{ more terms}\big]\ , \end{aligned} \tag{4.23}$$

which implies $S_4 = 0$.

It is clear that the direct evaluation can be continued to higher point scattering with the conjectured result $S_{n\geq 5} = 0$. One can describe the nonlinear collective field theory in the following way: its nonlinearity, and higher point vertices are precisely such that they reproduce the boundary correlators through bi-local (Witten) diagrams. These same diagrams however give vanishing results in the on-shell evaluation as described above. We also mention that in the framework of BCFW recursions for higher spin interactions, the relevance of extended observables was noted in [**88, 89**].

In general quantum field theory, one has the equivalence theorem. A vanishing S-matrix implies that there should exist a (nonlinear) field transformation which linearizes the theory. For the present case this concerns the linearization of bulk $G_N = 1/N$ interactions. We will in the next section describe such a field transformation.

Since we view the collective construction to represent a gauge fixed description of Vasiliev's HS theory, analogous statements are expected to hold there. Finally it is also clear that one can expect that any change of boundary conditions will result in non-trivial S-matrix.

3. Field Transformation

We have concluded in the previous section that the S-matrix equals 1 for the bi-local theory of the free UV fixed point. The theory is nonlinear with a sequence of $1/N$ vertices which are needed to reproduce arbitrary n-point correlators (and the "boundary S-matrix"). By correspondence Vasiliev's HS theory has the same properties. As suggested in section 2, this implies that there should be a field transformation that linearizes the $G_N = 1/N$ interactions. We will now describe such a procedure for deducing the transformation. The procedure is based on considering an algebraic description of the bi-local system. We will be able to show that the bi-local pseudo-spin algebra has among other two representations: one equalling the nonlinear collective field theory and another in which the Hamiltonian becomes quadratic.

For the free theory in question one has exact creation operators for the singlet sector of the theory. They are given by the bi-local operators

$$A(\vec{p}_1,\vec{p}_2) = \frac{1}{\sqrt{2N}}\sum_i a^i(\vec{p}_1)a^i(\vec{p}_2) \ , \tag{4.24}$$

$$A^\dagger(\vec{p}_1,\vec{p}_2) = \frac{1}{\sqrt{2N}}\sum_i a^{i\,\dagger}(\vec{p}_1)a^{i\,\dagger}(\vec{p}_2) \ , \tag{4.25}$$

$$B(\vec{p}_1,\vec{p}_2) = \frac{1}{2}\sum_i a^{i\,\dagger}(\vec{p}_1)a^i(\vec{p}_2) \ . \tag{4.26}$$

In terms of these collective variables the Hamiltonian is

$$H = \int d\vec{p}\,\mathcal{H}(\vec{p},\vec{p}) \ , \qquad \mathcal{H}(\vec{p},\vec{p}) = 2\omega_{\vec{p}}\,B(\vec{p},\vec{p}) + \frac{N}{2}\omega_{\vec{p}}\,\delta(\vec{0}) \ . \tag{4.27}$$

The above operators (representing bi-local pseudo-spin variables) close an algebra

$$\begin{aligned}\left[A(\vec{p}_1,\vec{p}_2),A^\dagger(\vec{p}_3,\vec{p}_4)\right] &= \frac{1}{2}\left(\delta_{\vec{p}_2,\vec{p}_3}\delta_{\vec{p}_4,\vec{p}_1}+\delta_{\vec{p}_2,\vec{p}_4}\delta_{\vec{p}_3,\vec{p}_1}\right)+\frac{1}{N}\big[\delta_{\vec{p}_2,\vec{p}_3}B(\vec{p}_4,\vec{p}_1)\\ &\quad +\delta_{\vec{p}_2,\vec{p}_4}B(\vec{p}_3,\vec{p}_1)+\delta_{\vec{p}_1,\vec{p}_3}B(\vec{p}_4,\vec{p}_2)+\delta_{\vec{p}_1,\vec{p}_4}B(\vec{p}_3,\vec{p}_2)\big] \ ,\end{aligned} \tag{4.28}$$

$$\left[B(\vec{p}_1,\vec{p}_2),A^\dagger(\vec{p}_3,\vec{p}_4)\right] = \frac{1}{2}\big(\delta_{\vec{p}_2,\vec{p}_3}A^\dagger(\vec{p}_1,\vec{p}_4)+\delta_{\vec{p}_2,\vec{p}_4}A^\dagger(\vec{p}_1,\vec{p}_3)\big) \ , \tag{4.29}$$

$$\left[B(\vec{p}_1,\vec{p}_2),A(\vec{p}_3,\vec{p}_4)\right] = -\frac{1}{2}\big(\delta_{\vec{p}_1,\vec{p}_3}A(\vec{p}_2,\vec{p}_4)+\delta_{\vec{p}_1,\vec{p}_4}A(\vec{p}_2,\vec{p}_3)\big) \ . \tag{4.30}$$

We note that the theory based on this algebra was studied in detail by Berezin [**91**]. In the $O(N)$ case one finds the quadratic (Casimir) constraint

$$-\frac{8}{N}A^\dagger\star A+\left(1+\frac{4}{N}B\right)\star\left(1+\frac{4}{N}B\right)=\mathbb{I} \ . \tag{4.31}$$

The importance of the Casimir constraint is that it implies that the above non-commuting set of bi-local operators is not independent. In particular the bi-local pseudo-spin algebra has representations in terms of canonical pairs of variables.

The canonical collective theory based on the equal-time bi-local field and its conjugate provides one specific representation of the above algebra. Explicitly, one can show

$$\begin{aligned} A(\vec{x}_1,\vec{x}_2) &= \int d\vec{p}_1 d\vec{p}_2 d\vec{y}_1 d\vec{y}_2 e^{i\vec{p}_1\cdot(\vec{x}_1-\vec{y}_1)} e^{i\vec{p}_2\cdot(\vec{x}_2-\vec{y}_2)} \\ & \Big[\frac{-2}{\sqrt{\omega_{p_1}\omega_{p_2}}}\Pi(\vec{y}_1,\vec{z}_1)\star\Psi(\vec{z}_1,\vec{z}_2)\star\Pi(\vec{z}_2,\vec{y}_2) \\ & -i\sqrt{N}\sqrt{\frac{\omega_{p_2}}{\omega_{p_1}}}\Psi(\vec{y}_2,\vec{z}_1)\star\Pi(\vec{y}_1,\vec{z}_1) \\ & -i\sqrt{N}\sqrt{\frac{\omega_{p_1}}{\omega_{p_2}}}\Psi(\vec{y}_1,\vec{z}_1)\star\Pi(\vec{y}_2,\vec{z}_1) \\ & -\frac{N}{8}\frac{1}{\sqrt{\omega_{p_1}\omega_{p_2}}}\frac{1}{\Psi}(\vec{y}_1,\vec{y}_2) + \frac{N\sqrt{\omega_{p_1}\omega_{p_2}}}{2}\Psi(\vec{y}_1,\vec{y}_2)\Big] \ . \end{aligned} \tag{4.32}$$

Transforming it to momentum space and expanding in $1/N$ we generate an infinite series

$$\begin{aligned} A(\vec{k}_1,\vec{k}_2) &= \alpha_{\vec{k}_1\vec{k}_2} - \frac{1}{\sqrt{2N}}\Big[\alpha_{\vec{k}_1\vec{k}_3}\star\alpha_{-\vec{k}_3\vec{k}_2} - \alpha^{\dagger}_{\vec{k}_1\vec{k}_3}\star\alpha^{\dagger}_{-\vec{k}_3\vec{k}_2} \\ & -\alpha_{\vec{k}_1\vec{k}_3}\star\alpha^{\dagger}_{\vec{k}_3\vec{k}_2} - \alpha_{\vec{k}_1\vec{k}_3}\star\alpha^{\dagger}_{\vec{k}_3\vec{k}_2}\Big] + O(\alpha^3) \ , \end{aligned} \tag{4.33}$$

$$\begin{aligned} B(\vec{k}_1,\vec{k}_2) &= \frac{1}{2}\Big[\alpha_{\vec{k}_1\vec{k}_3}\star\alpha^{\dagger}_{\vec{k}_3\vec{k}_2} + \alpha^{\dagger}_{\vec{k}_1\vec{k}_3}\star\alpha_{\vec{k}_3\vec{k}_2}\Big] + \sqrt{\frac{2}{N}}\Big[\alpha_{\vec{k}_1\vec{k}_3}\star\alpha_{-\vec{k}_3\vec{k}_4}\star\alpha_{-\vec{k}_4\vec{k}_2} \\ & +\alpha_{\vec{k}_1\vec{k}_3}\star\alpha^{\dagger}_{\vec{k}_3\vec{k}_4}\star\alpha_{\vec{k}_4\vec{k}_2} - \alpha^{\dagger}_{\vec{k}_1\vec{k}_3}\star\alpha_{\vec{k}_3\vec{k}_4}\star\alpha^{\dagger}_{\vec{k}_4\vec{k}_2} \\ & -\alpha^{\dagger}_{\vec{k}_1\vec{k}_3}\star\alpha^{\dagger}_{-\vec{k}_3\vec{k}_4}\star\alpha^{\dagger}_{-\vec{k}_4\vec{k}_2}\Big] + O(\alpha^4) \ . \end{aligned} \tag{4.34}$$

The key to our arguments is the fact that one can write another realization of the algebra in terms of an oscillator $\beta(\vec{p}_1,\vec{p}_2)$ obeying

$$\beta(\vec{p}_1,\vec{p}_2) = \left(1+\frac{2}{N}B\right)^{-\frac{1}{2}}(\vec{p}_1,\vec{p})\star A(\vec{p},\vec{p}_2) \tag{4.35}$$

$$\beta^{\dagger}(\vec{p}_1,\vec{p}_2) = A^{\dagger}(\vec{p}_1,\vec{p})\star\left(1+\frac{2}{N}B\right)^{-\frac{1}{2}}(\vec{p},\vec{p}_2) \tag{4.36}$$

which has two important properties that

$$B(\vec{p}_1,\vec{p}_2) = \beta^{\dagger}(\vec{p}_1,\vec{p})\star\beta(\vec{p},\vec{p}_2) \tag{4.37}$$

$$\left[\beta(\vec{p}_1,\vec{p}_2),\beta^{\dagger}(\vec{p}_3,\vec{p}_4)\right] = \delta_{\vec{p}_1,\vec{p}_4}\delta_{\vec{p}_2,\vec{p}_3} \ . \tag{4.38}$$

We see that in this realization the Hamiltonian is quadratic due to (4.27). Furthermore, using (4.35) one can generate the transformation between the fields

$$\beta(\vec{k}_1,\vec{k}_2) = \quad \alpha_{\vec{k}_1\vec{k}_2} - \tfrac{1}{\sqrt{2N}}\Big[\alpha_{\vec{k}_1\vec{k}_3}\star\alpha_{-\vec{k}_3\vec{k}_2} - \alpha^{\dagger}_{\vec{k}_1\vec{k}_3}\star\alpha^{\dagger}_{-\vec{k}_3\vec{k}_2}$$

$$-\alpha_{\vec{k}_1\vec{k}_3} \star \alpha^{\dagger}_{\vec{k}_3\vec{k}_2} - \alpha_{\vec{k}_1\vec{k}_3} \star \alpha^{\dagger}_{\vec{k}_3\vec{k}_2}\Big] + O(\alpha^3) \ . \tag{4.39}$$

In conclusion we have presented a construction of the field transformation (in bi-local space) that linearizes the nonlinear $1/N$ Hamiltonian. Under this transformation the correlation functions change but the S-matrix does not. This represents the working of the Coleman-Mandula theorem in the large N dual associated with the free field CFT. As such it complements the Maldacena-Zhiboedov argument for these theories.

CHAPTER 5

Conclusion

We have in this thesis described the collective dipole picture of AdS/CFT correspondence. This picture was extracted from the bi-local field representation of a conformally invariant $O(N)$ vector model. These fields which fully describe the $O(N)$ singlet sector of the theory were seen to contain the full interacting bulk AdS theory with higher spins. A first quantized description represents a bi-particle system which we called the collective dipole.

We have studied the structure of constraints and the gauge fixing of the dipole system. This issue itself is nontrivial as we are dealing with a fully relativistic system with two time coordinates. Consequently various issues related to unitarity and absence of ghosts have to be addressed. We have following earlier work discussed in detail the issue of gauge fixing to a physical time-like or light-cone gauge. Using a gauge condition which leads to elimination of the relative time, we have exhibited the existence of a unitary, ghost free representation of the dipole system. This gauge also establishes contact with the equal time Hamiltonian bi-local field theory.

Using the light-cone frame we have then demonstrated the correspondence with the higher-spin particle in AdS space-time. This correspondence is constructed in terms of an explicit one-to-one canonical map relating the d-dimensional collective dipole with the $d+1$ dimensional higher spin particle in AdS. The map gives an explicit reconstruction of the extra (radial) AdS space dimension and of the infinite sequence of higher spin states. As such it likely represents the simplest system where the AdS/CFT correspondence is established in the bulk.

For higher spin theory the relevance of the dipole picture lies in the following. It provides a first quantized world sheet description of the theory and also has the promise to lead to a BRST quantization of the the system. The BRST approach has been extremely relevant in the case of string theory [**96**] (and of course gauge theory [**107**]), however, even though there have been various attempts there is not as yet a complete BRST description of Vasiliev's Higher Spin gauge theory.

Further, we have motivated the use of double analytic continuation and hence the connection between the $Sp(2N)$ model and de Sitter higher field theory for the quadratic action for the collective field. To establish this connection one needs to establish this for the interaction terms. This is highly nontrivial, and in fact the connection between the collective theory for the $O(N)$ model and the AdS higher spin theory is only beginning to be understood. We believe that once this is understood well, one can address the question for the $Sp(2N)$-dS connection.

In this thesis we have dealt mostly with the *free* $Sp(2N)$ vector model. As the parallel $O(N)$/AdS case, this theory is characterized with an infinite sequence of conserved higher spin currents and associated conserved charges. The question regarding the implementation of the Coleman-Mandula theorem then arises, this question was discussed recently in [**44, 45, 46**]. One can expect that identical conclusions hold for the present $Sp(2N)$ case. The bi-local collective field theory technqiue is trivially extendible to the linear sigma model based on $Sp(2N)$, as commented in section (4.2). Of particular interest is the IR behavior of the theory which presumably takes the theory from the Gaussian fixed point to a nontrivial fixed point.

It is well known that dS/CFT correspondence is quite different from AdS/CFT correspondence, particularly in the interpretation of bulk correlation functions [**50, 51**]. We have not addressed these issues in this thesis. Recently it has been proposed that the $Sp(2N)$/dS connection can be used to understand subtle points about dS/CFT [**52**]. We hope that an explicit construction as described in this thesis will be valuable for a deeper understanding of these issues.

The bi-local formulation that we have presented was cast in a geometric, pseudo-spin framework. We have suggested that this representation offers the best framework for quantization of the bi-local theory and consequently the Hilbert space in dS/CFT.

We have demonstrated through counting of the size of the Hilbert space that it incorporates finite N effects through a cutoff which depends on the coupling constant of the theory: $G = 1/N$. Most importantly it incorporates the finite N exclusion principle and provides an explanation on the quantization of $G = 1/N$ from the bulk point of view. These features are obviously of definite relevance for understanding quantization of Gravity in de Sitter space-time. Nevertheless the question of understanding de Sitter entropy from this 3 dimensional CFT remains an interesting and challenging problem.

It would be interesting to consider the analogues of $Sp(2N)$/dS correspondence in the CFT_2/Chern-Simons version [**58, 21, 57**], as well as to three dimensional conformal theories which have a line of fixed points, as in [**43**]. Finally higher spin theories arise as limits of string theory in several contexts, e.g. [**18**] and [**43**]. It would be interesting to see if these models can be modified to realize a dS/CFT correspondence in string theory.

Finally we discussed some features of the Higher Spin AdS correspondence involving free $O(N)$ fields, especially the existence of an (infinite) sequence of higher symmetries in these theories, which raises the question regarding the implementation of the Coleman-Mandula theorem. Our focus was the question regarding the nonlinear $1/N$ theory which reproduces the (boundary) correlators. We argued that in these theories we are able to define a genuine S-matrix representing the scattering of collective dipoles. The S-matrix is specified with the standard LSZ procedure as an on-shell limit of (bi-local) correlation functions.

For the theory based on the free correspondence i.e. the UV fixed point of the vector model we have evaluated the S-matrix showing the result $S = 1$. This represents the consequence of the Coleman-Mandula theorem for the associated Higher Spin theory and complements the results of Maldacena and Zhiboedov. As we have discussed it implies that the nonlinear Higher Spin theory can be linearized through nonlinear field transformations. We have explicitly constructed such a transformation in the bi-local framework. We have also emphasized that a change of boundary conditions will change the above conclusion, namely one expects a nontrivial S-matrix. Based on the present results and the earlier $c = 1$ case, it is plausible to conclude that these features will characterize any large N correspondence based on free fields.

To conclude, the bi-local field representation possesses several relevant features which have implications on the nature of higher spin theory. The correspondence between $O(N)$ vector models and higher spin AdS gravity itself demonstrates a very interesting example of AdS/CFT correspondence.

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