TOWARDS THE QUANTIZATION OF INTEGRABLE NON-LINEAR SIGMA MODELS

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ABSTRACT OF THE DISSERTATION

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The class of integrable Non-Linear Sigma Models (NLSM) have many interesting applications in Quantum Field Theory, Condensed Matter Physics and String Theory. However, despite being integrable, their study still presents many challenges. In this thesis the quantization of integrable NLSM is considered within the framework of the Quantum Inverse Scattering Method (QISM). The main focus are the O(3) and O(4) NLSM and their integrable deformations. On these examples, we will encounter and discuss the long-standing conceptual challenges of quantizing NLSM in general. A key technical tool, that will allow us to make progress, is the so-called ODE/IQFT correspondence. Among the results presented in this thesis is a new approach to the problem with non-ultralocality; a study of the integrable structures in the $O(3)$ model as well as its deformation; and a remarkable relation between the Casimir energies of the deformed $O(4)$ model and certain solutions of the modified sinh-Gordon equation.

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Dedication

To my little brothers Lev, Matvei and Niki

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Chapter 1
General Introduction

The development of Quantum Field Theory (QFT) has been closely tied to the study of the fundamental forces. Its early progress was motivated by the problem of quantizing the electromagnetic field that would describe the quantum interactions of light with matter. Despite that a perturbative treatement was mainly used, the approach resulted in remarkable success. The agreement of the theoretical predictions for the anomalous magnetic moment and the Lamb shift with experiments became a gold standard for physics [1].

The subsequent discovery of non-Abelian gauge theories laid the foundation for the Standard Model. However, unlike quantum electrodynamics, such theories can exhibit asymptotic freedom. This happens in quantum chromodynamics – the theory believed to describe the strong interactions. For asymptotically free theories, though perturbation theory works well for high energies, the most interesting low energy physics lies outside of its scope. This motivated the development of alternative methods such as the large-N expansion [2], instanton calculus [3], lattice simulations [4], etc. However their applicability has been tough to assess and justify in the case of realistic 4D non-Abelian gauge theories. This has led to a strong interest in studying simplified models, where new methods can be understood and tested in full details.

Another perspective on QFT came from Condensed Matter Physics. Many important concepts such as the renormalization group, spontaneous symmetry breaking and asymptotic freedom were understood and developed independently in this context.

As an example, asymptotic freedom in Condensed Matter Physics was encountered almost ten years before the famous 1973 works of Gross & Wilczek [5] and Politzer [6] on this phenomena in non-Abelian gauge theories. It was found by Kondo in his study of the anomalous behaviour of the low temperature resistance in certain metals, now known as the Kondo effect [7]. Remarkably, one of the most powerful and fundamental concepts in QFT – the renormalization group – was first applied in full to the Kondo problem by Wilson [8].

Despite that the Kondo effect occurs in three dimensions, the essential physics is captured by a one dimensional model. The latter turns out to be a quantum integrable system, which was solved exactly in [9, 10]. The solution is obtained via the Bethe ansatz approach, pioneered in the 1930's by Hans Bethe in his study of the Heisenberg spin chain [11]. Since then, a large variety of $1 + 1$ dimensional models have been solved using this method. Like the Kondo model, many of them describe interesting physical phenomena in real systems. This includes the Lieb-Liniger Bosé gas, which was recently realized in ultra-cold ^{87}Rb atoms confined to a 1D optical trap [12, 13]; and the Thirring model that describes electrons inside a one-dimensional conductor (the Tomonaga-Luttinger liquid). The latter was experimentally realized using carbon nanotubes at low temperatures [14].

The theory of quantum integrable systems has profited much from the deep connection between QFT and statistical mechanics. Many of its fundamental concepts arose from the study of exactly solvable 2D lattice models. The latter culminated with the seminal works in the 60's and 70's [15, 16, 17, 18] that led to the discovery of new mathematical structures now collectively known as the Yang-Baxter algebras. Their study gave rise to the theory of quantum groups [19, 20, 21] and lead to remarkable developments in many traditional areas of mathematics including representation theory [22], geometry and topology [23, 24], combinatorics [25], etc.

Until recently, the theory of quantum integrable systems itself could have been regarded as a relatively isolated area of physics, though with interesting applications to Condensed Matter Physics, but of limited applicability to realistic problems of High Energy Physics. This situation is changing, however, with the discovery of a remarkable series of links between supersymmetric gauge theories and quantum integrability [26, 27, 28, 29, 30]. One area where these developments have been keenly felt is in the study of the AdS/CFT correspondence. On the CFT side it was observed that for every order in perturbation theory the computation of the scaling dimensions in $\mathcal{N} = 4$ super Yang-Mills theory in the t'Hooft limit could be reduced to the eigenvalue problem of a certain integrable spin chain [31]. By adapting the Bethe ansatz approach, the exact computation of the scaling dimensions was achieved for all values of the t'Hooft coupling (see [32] for a review). On the AdS side, the dual description involves type IIB superstring theory in the $AdS^5 \times S^5$ background. In the planar limit, the theory describes the propagation of a free string in this background. The latter is essentially a $1+1$ dimensional Non-Linear Sigma Model (NLSM) on $AdS^5 \times S^5$. This sigma model was shown to be classically and likely quantum integrable [33]. However, the study of the quantum model has so far proven difficult.

NLSM in $1+1$ dimensions are an especially interesting class of theories, whose applications range beyond the study of the AdS/CFT correspondence. In their original setting, they were used as laboratories for better understanding aspects of non-Abelian gauge theories in a simpler context including asymptotic freedom, confinement [34], instanton counting [35] etc. Supersymmetric NLSM have important applications in Condensed Matter Physics, where they are used to model disordered electronic systems like the integer quantum Hall effect [36]. Classical and quantum NLSM are also studied by mathematicians as they provide a natural framework for a large variety of geometrical problems such as the harmonic map [37] and geometric

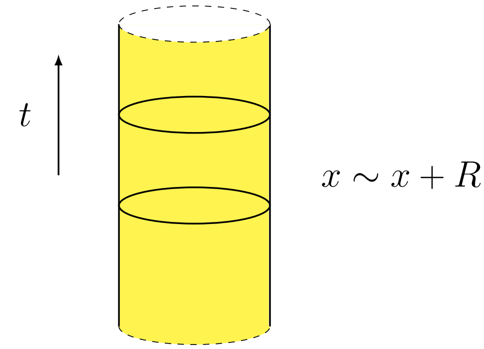


Figure 1.1: The integration contour for the Wilson loop can be moved freely along the space-time cylinder.

flows (see e.g. [38]). Because of the many applications of NLSM, the integrable cases, where a detailed study is possible, have attracted a great deal of attention.

A unified and systematic approach to quantum integrability is the so-called Quantum Inverse Scattering Method (QISM) [39, 40]. Its roots can be traced back to the study of classically integrable 1+1 dimensional Partial Differential Equations (PDEs) that typically occur in the theory of solitons. The most famous of these is the Korteweg de Vries equation that describes waves propagating in shallow water [41]. An ingenious method for its solution was proposed in [42] that is based on the study of the scattering problem for the 1D stationary Schrodinger equation. The generalization of this approach to other integrable PDEs came following the work of Lax [43] and became known as the inverse scattering transform method. In the contemporary formulation of this method, the key ingredient is a Lie algebra-valued world sheet connection whose flatness condition (Zero-Curvature Representation) is equivalent to the classical equations of motion. For the flat connection the Wilson loops of the form

$$T = \mathrm{Tr}\,\mathcal{P}\exp\int_C \boldsymbol{A} \tag{1.1}$$

remain unchanged under continuous deformations of the integration contour (see fig. 1.1) and generate an infinite family of conserved quantities. The latter can be used to solve the field theory within the framework of the inverse scattering method [44].

The bringing together of ideas from the inverse scattering transform and the Yang-Baxter algebras triggered the development of the QISM. The approach is based on the study of the common spectrum of the transfer-matrices (T-operators) – the quantum counterpart of the classical Wilson loops. The original formulation of the QISM was restricted to the so-called "ultralocal" models. In this case, the elementary transport matrices $\boldsymbol{M}_n = \overleftarrow{\mathcal{P}} \exp \int_{x_n}^{x_{n+1}} \boldsymbol{A}$ commute for different segments of the discretized path, while for the same segment they form a Yang-Baxter algebra. The most studied class of integrable models is the one where the Yang-Baxter algebra of the ultralocal operators $\boldsymbol{M}_n$ admits a finite-dimensional representation. In this case the discretized quantum system can be interpreted as an exactly soluble lattice model whose solution can be obtained by means of the Bethe ansatz method. The solution of the continuous QFT is achieved by taking a proper scaling limit. An archetype of this scenario is the sine-Gordon model, while the corresponding statistical system is known as the inhomogeneous 6-vertex model [45].

In spite of its success, the QISM failed when it was applied to classically integrable NLSM, including the simplest cases of the $O(3)$ and $O(4)$ models. The Zero-Curvature Representation has been known for these theories since the seventies [46]. Nevertheless, it turned out to be problematic to trace the classical counterpart to the Yang-Baxter algebras, which is a crucial ingredient for the quantization in the framework of the QISM. The technical obstacle is that the elementary transport matrices $\boldsymbol{M}_n$ are non-ultralocal, i.e., they no longer commute for different segments. These are symptoms of the broader difficulty tied to the UV behaviour of the quantum model, which exhibits effects such as asymptotic freedom and dimensional transmutation that have no direct analogues in the classical counterpart.

For the $O(4)$ model, an attempt to bypass the non-ultralocality problem was made by Polyakov & Wiegmann [47] and later Faddeev & Reshetikhin [48]. In both works, the NLSM was replaced by a different model, which was free from the non-ultralocality issue. Polyakov and Wiegmann considered a model with N_f fermion flavors, whereas Faddeev and Reshetikhin focused on a certain spin-S chain. They studied the thermodynamics using the Bethe ansatz technique and gained valuable results for the $O(4)$ sigma model through the large N_f and $S \to \infty$ limit, respectively. Both limiting procedures yielded the same system of so-called thermodynamic Bethe ansatz equations, which was then justified by a comparison with perturbative calculations and the exact results from the S-matrix bootstrap [49]. Since that time, a number of impressive results have been achieved for some particular NLSM. However, a deeper understanding as well as a systematic approach to the quantization of NLSM so far do not exist.

In the series of works [50, 51, 52], Bazhanov, Lukyanov and Zamolodchikov proposed an alternative approach to the transfermatrices in integrable QFT (the so-called BLZ approach). In the case of integrable Conformal Field Theories (CFT), it was demonstrated that the T-operators can be constructed without any discretization procedure. Later it was observed that many deep properties of representations of Yang-Baxter algebras in integrable CFT can be encoded in the monodromies of certain linear Ordinary Differential Equations (ODE) [53, 54, 55]. These results were extended to massive Integrable Quantum Field Theories (IQFT) [56]. The general relation is referred to as the ODE/IQFT correspondence.

The ODE/IQFT correspondence means that, for a given integrable QFT there exists a certain classically integrable field theory such that the eigenvalues of the quantum T-operators coincide with the on-shell values of the Wilson loops calculated on certain solutions of the classical equations of motion. For example, the classical counterpart of the quantum sine-Gordon model is given by the so-called Modified Sinh-Gordon (MShG) equation. The MShG equation is a classically integrable PDE which admits a Zero-Curvature Representation. According to the ODE/IQFT correspondence, spectra of the quantum transfermatrices in the sine-Gordon model coincide with a set of values of the Wilson loops calculated on a certain class of solutions of the MShG equation [56].

Thus, the ODE/IQFT correspondence reduces the calculation of the spectrum of quantum transfermatrices to a certain problem in the theory of classically integrable equations. The latter can be effectively treated by the inverse scattering transform method. This makes the ODE/IQFT correspondence a very powerful tool. In particular, it gives a practical way to make progress in the conceptual long standing problem of the quantization of integrable NLSM.

This thesis is devoted to the study of integrable NLSM within the BLZ approach. The main focus is on the $O(3)$ and $O(4)$ NLSM as well as their integrable deformations. The key technical tool, that has allowed progress to be made, is the ODE/IQFT correspondence. The plan of this dissertation is as follows:

- Chapter 2 is preliminary. It gives a short review of NLSM in $1+1$ dimensions. First we consider the analogue of the $O(4)$ model in classical mechanics. Next, the NLSM action is introduced and its renormalizability in $1+1$ dimensions is discussed. The last part contains some specific examples. Among these are the deformed $O(3)$ (the so-called 2D sausage model) and $O(4)$ NLSM (3D sausage) that form the subject-matter of this thesis.

- In Chapter 3 some aspects of classical integrability for a $1+1$ dimensional field theory are discussed. This includes the Zero-Curvature Representation for the classical equations of motion, which implies the existence of an infinite family of conserved quantities. As an illustration, the Zero-Curvature Representation is given for the Principal Chiral Field and its two parameter deformation, the so-called Klimcik model. The later part of the chapter considers the Poisson commutativity condition of the conserved charges. For its proof, the central rôle of the classical Yang-Baxter algebra, or equivalently the Sklyanin exchange relations, are emphasized. Next, it is explained how the non-ultralocality problem creates difficulties with the derivation of the Sklyanin exchange relations. It is shown how to bypass the problem for the case of the $O(3)$ model and its one parameter deformation.

- Chapter 4 deals with the problem of non-ultralocality in full and is based on the work [57]. First the quantum Yang-Baxter algebras are introduced. To get some intuition, we illustrate how they arise in a statistical mechanics system – the six vertex model. It is discussed how the Yang-Baxter algebras can be considered as the quantum version of the Sklyanin exchange relations. Based on this "correspondence principle", we formulate a strategy for recovering the Sklyanin exchange relations in a non-ultralocal theory. This is first demonstrated for a non-ultralocal system based on the $U(1)$ current algebra that is related to the Korteweg de Vries equation. Then we carry this out for a more complicated case, where the non-ultralocal system is based on the $SU(2)$ current algebra. It is explained how this last example is relevant to the 3D sausage model (two parameter deformation of $O(4)$ NLSM). With these results, we argue that the Poisson commutativity condition of the conserved charges is satisfied in the 3D sausage.

- Chapters 5, 6 and 7 consider the quantization of the 2D sausage model and closely follow the work [58].

  (a) In Chapter 5 we consider the CFT underlying the UV fixed point of the 2D sausage, the so-called cigar NLSM. Its quantum transfer-matrices are constructed and their basic properties are outlined. Next we discuss some facts about the quantum cigar including its Hilbert space. The dual description of the cigar is presented. The chapter ends with a discussion of the local integrals of motion in the theory.

(b) In Chapter 6, the quantum transfer-matrices are considered in the parameter domain that is unphysical for the cigar NLSM. It turns out that these operators can be interpreted as the transfermatrices for the $\mathbb{Z}_n$ parafermionic models [59] that describe the scaling limit of the Fateev-Zamolodchikov $\mathbb{Z}_n$ spin chain. The transfermatrix is explicitly constructed on the lattice and its scaling limit is discussed. The computation of the spectrum is achieved via the ODE/IQFT correspondence. Using this correspondence, the eigenvalues of the transfermatrix are expressed in terms of certain connection coefficients for a certain class of ODEs. These relations are extended to the parameter domain that is physical for the cigar NLSM. Finally, a system of non-linear integral equations are derived for the computation of the vacuum eigenvalue of the transfermatrix for both the cigar NLSM and the $\mathbb{Z}_n$ parafermionic models.

(c) Chapter 7 is focused on the 2D sausage model. Its starts by recounting some basic facts, including its dual description, the structure of its Hilbert space, and its UV/IR behaviour. The non-linear integral equations for the cigar obtained in the previous chapter are generalized to the 2D sausage and a variety of checks are made to certify their validity. Next, the algebraic structures in the sausage model are discussed and their main properties are listed. The chapter ends by presenting the ODE/IQFT correspondence for the sausage NLSM.

- Chapter 8 is based on the work [60]. It is devoted to a demonstration of the

ODE/IQFT correspondence for the 3D sausage model. The correspondence predicts that the vacuum energies of the theory in finite volume can be expressed in terms of certain solutions of the classical MShG equation. To provide support to this conjecture, numerical data for the vacuum energy obtained via the solution of the PDE is compared with the UV and IR asymptotics taken from field theory computations. Excellent agreement was found.

- Chapter 9 is devoted to a discussion.

The dissertation is based on the published works [57, 58, 60] and the recent preprint [61].

Chapter 2
Non-Linear Sigma Models

2.1 The free top: a classical mechanics analogue of an NLSM

Before introducing the NLSM action, it is useful to gain some intuition by discussing its equivalent in classical mechanics. A NLSM in "$0+1$ dimensions" is a mechanical system that has no potential energy term and where the non-trivial dynamics comes from the presence of constraints on the motion. A famous physical example is the free top, where the constraints correspond to the condition that it is rigid. Below we will make a short digression into the theory of tops, which will help give us an intuitive picture for a NLSM.

For the case of the free top, the motion of the center of mass is trivial. The remaining rotational degrees of freedom can be described using a 3×3 rotation matrix $\boldsymbol{U}(t) \in SO(3)$ that relates the orientation of the fixed frame to the laboratory

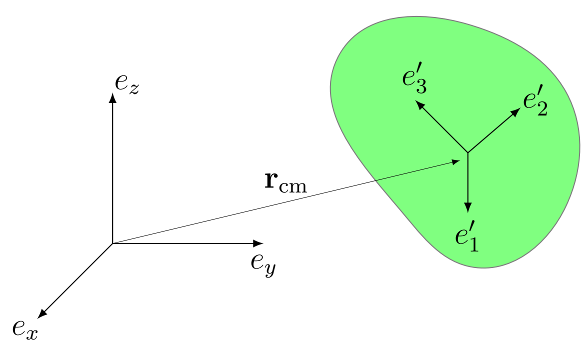


Figure 2.1: In analyzing the motion of a rigid body it is convenient to work in the co-ordinate system $\{e'_1, e'_2, e'_3\}$ which is fixed to the body and where the axes coincide with the principal axes of inertia.

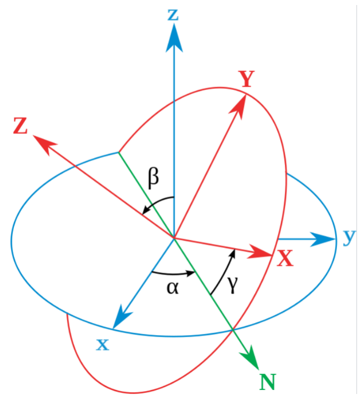


Figure 2.2: Any rotation matrix $\boldsymbol{U}$ can be parameterized using the Euler angles (α, β, γ).

frame (see fig. 2.1). The Lagrangian of the free top coincides with its kinetic energy and is given by

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^{3} I_a \, \omega_a^2 \,. \tag{2.1}$$

Here I_a are the principal moments of inertia and ω_a are the components of the angular momenta about the principal axes (see fig. 2.1). Note that the latter can be written in terms of $\boldsymbol{U}(t)$ by using the infinitesimal rotation matrices $\mathtt{t}_a$:

$$\omega_a = \frac{1}{2\mathrm{i}} \operatorname{Tr}\left(\boldsymbol{U}^{-1} \dot{\boldsymbol{U}} \, \mathtt{t}_a \right) \qquad (a = 1, 2, 3) \,. \tag{2.2}$$

In the simplest case of the spherical top, where all the moments of inertia are equal, the Lagrangian (2.1) takes the form

$$\mathcal{L} = -\frac{1}{2 f^2} \left\langle \boldsymbol{U}^{-1} \dot{\boldsymbol{U}}, \boldsymbol{U}^{-1} \dot{\boldsymbol{U}} \right\rangle \,, \tag{2.3}$$

where $f^{-2} \equiv I_1 = I_2 = I_3$ and $\langle \mathtt{a}, \mathtt{b} \rangle = \operatorname{Tr}(\mathtt{ab})$. In fact, in this thesis, the field theory version of a slightly different model will be important to us. We will take the matrix $\boldsymbol{U}$ in (2.3) to be a 2×2 unitary matrix, rather than an element of $SO(3)$. Since the combination $\boldsymbol{U}^{-1} \dot{\boldsymbol{U}}$ lies in the Lie algebra, which is equivalent for $SU(2)$ and $SO(3)$, the models are identical at the level of the Lagrangian. The difference is only in the global aspects of the motion.

Any 2×2 unitary matrix can be written in the form

$$\boldsymbol{U} = \begin{pmatrix} n_4 + \mathrm{i}\, n_3 & n_2 + \mathrm{i}\, n_1 \\ -n_2 + \mathrm{i} n_1 & n_4 - \mathrm{i} n_3 \end{pmatrix} , \qquad n_1^2 + n_2^2 + n_3^2 + n_4^2 = 1 \ , \tag{2.4}$$

so that the group manifold of $SU(2)$ is a three dimensional sphere. To write the explicit form of the Lagrangian, it is useful to parameterize the unit vector $\mathbf{n} = (n_1, n_2, n_3, n_4)$ by the three angles $X^\alpha = \{\theta, \chi_1, \chi_2\}$ as

$$n_1 \pm \mathrm{i} n_2 = \mathrm{e}^{\pm \mathrm{i}\chi_1} \cos(\theta)\,, \qquad n_3 \pm \mathrm{i} n_4 = \mathrm{e}^{\pm\mathrm{i}\chi_2} \sin(\theta)\ .$$

In terms of these, the Lagrangian takes the form

$$\mathcal{L} = \frac{1}{2}\, G_{\alpha\beta}\, \dot{X}^\alpha\, \dot{X}^\beta\,, \tag{2.5}$$

where

$$G_{\alpha\beta}\, \mathrm{d}X^\alpha\, \mathrm{d}X^\beta = \frac{4}{f^2}\, \Big((\mathrm{d}\theta)^2 + \cos^2(\theta)\, (\mathrm{d}\chi_1)^2 + \sin^2(\theta)\, (\mathrm{d}\chi_2)^2 \Big)\,. \tag{2.6}$$

The latter is immediately recognized to be the standard sphere metric. This way, the Lagrangian (2.5) with $\boldsymbol{U} \in SU(2)$ coincides with that of a free particle moving on the round three-sphere.

As was already mentioned, the Lagrangian (2.3) with $\boldsymbol{U} \in SO(3)$ is identical with that when $\boldsymbol{U} \in SU(2)$. Indeed, using the Euler angles (see fig. 2.2) and taking $\theta + \frac{\pi-\beta}{2}$, $\chi_1 = \frac{1}{2}(\alpha - \gamma)$, $\chi_2 = \frac{1}{2}(\alpha+\gamma)$, one arrives at eqs. (2.5) and (2.6) for the spherical top Lagrangian. However, the problem of the spherical top and the problem of a free particle moving on the three-sphere correspond to different compactification conditions. Namely,

$$\text{Spherical top} \ : \ \chi_1 \pm \chi_2 \sim \chi_1 \pm \chi_2 + 2\pi$$

$$\text{Particle on three sphere} \ : \ \chi_1 \sim \chi_1 + 2\pi\,, \quad \chi_2 \sim \chi_2 + 2\pi\,.$$

The Euler top, i.e., eq. (2.1) with all the I_a different, is a deformation of the spherical top. The Lagrangian still has the same form (2.5) and the model can be regarded as a free particle moving on a manifold that is topologically the same as the three-sphere. However, the metric is no longer that of the round sphere but is a certain deformation of eq. (2.6).

2.2 Principal Chiral Field and general NLSM

The first NLSM to appear was the four dimensional field theory version of the spherical top (2.3). The theory was proposed in the 1960's in a paper by Gell-Mann and Levy [62] to describe the low energy physics of the strong interactions in the chiral limit. For this reason models of this type have become known as the Principal Chiral Field (PCF). The phenomenological approach to the strong force, initiated by Gell-Mann and Levy, has culminated in the development of chiral perturbation theory (see e.g. [63]).

In $d+1$ space-time dimensions the PCF action takes the form

$$\mathcal{A} = -\frac{1}{2f^2}\int_{\mathcal{W}} \mathrm{d}t\, \mathrm{d}^d\mathbf{x}\, \sqrt{-\eta}\, \left\langle \boldsymbol{U}^{-1}\,\partial_\mu \boldsymbol{U},\, \boldsymbol{U}^{-1}\,\partial^\mu \boldsymbol{U} \right\rangle . \tag{2.7}$$

Here $\boldsymbol{U}(t,\mathbf{x})$ should be considered to be a map from the Minkowski space world-sheet to the group manifold. Notice that this formula makes sense for $\boldsymbol{U}$ an element of any Lie group $\mathfrak{G}$. Since $\boldsymbol{U}^{-1}\,\partial_\mu \boldsymbol{U}$ takes values in the Lie algebra, the angular brackets $\langle\cdot\,,\cdot\rangle$ should be interpreted to be the Killing form.

A NLSM is the field theory generalization of the mechanical system (2.5). Its configuration space consists of maps from the world sheet to a Riemannian manifold known as the target space. In a co-ordinate frame the maps are given by a set of functions $\{X^a(t,\mathbf{x})\}$. The NLSM action, in the simplest set-up, reads as

$$\mathcal{A} = \frac{1}{2}\int_{\mathcal{W}} \mathrm{d}t\, \mathrm{d}^d\mathbf{x}\, \sqrt{-\eta}\; G_{\alpha\beta}(X)\,\partial_\mu X^\alpha\,\partial^\mu X^\beta\,. \tag{2.8}$$

Field configurations that minimize this action are harmonic maps, that satisfy a generalized version of the Laplace equation

$$\partial_\mu \partial^\mu X^\gamma + \Gamma^\gamma{}_{\alpha\beta}\, \partial_\mu X^\alpha\, \partial^\mu X^\beta = 0\,. \tag{2.9}$$

2.3 Renormalizability of NLSM in $1+1$D

Four dimensional NLSM are non-renormalizable and can only be treated as effective field theories. It turns out that renormalizability occurs for NLSM only in the case of $1+1$ space-time dimensions. Tied to this are many interesting quantum phenomena such as dimensional transmutation and asymptotic freedom. Below we'll start by considering the renormalizability of the PCF using perturbation theory. It will mainly follow the discussion given in [64].

2.3.1 One-loop renormalizability for the PCF

The starting point is the action functional for the PCF. Since we will be doing explicit perturbation theory computations, it is convenient to work in the Euclidean picture with t substituted by the imaginary time $x_2 = \mathrm{i}t$. The Euclidean action is given by

$$\mathcal{A}\big[\boldsymbol{U}(x)\big] = -\frac{1}{2f_0^2}\int \mathrm{d}^2x\, \Big\langle \boldsymbol{U}^{-1}\partial_\mu \boldsymbol{U},\, \boldsymbol{U}^{-1}\partial_\mu \boldsymbol{U}\Big\rangle\,, \tag{2.10}$$

where f_0 stands for the bare coupling.

Our focus is the wave functional within the one loop approximation:

$$\Psi\big[\boldsymbol{U}(x_1)\big] = \int_{\boldsymbol{U}(x)|_{x_2=0}=\boldsymbol{U}(x_1)} \mathcal{D}\boldsymbol{U}(x)\; \mathrm{e}^{-\mathcal{A}[\boldsymbol{U}(x)]}\,. \tag{2.11}$$

In the zeroeth order, which corresponds to the classical limit, one takes into account the path that minimizes the action (2.11). This path is the solution to the classical equations of motion,

$$\partial_\mu \boldsymbol{J}_\mu = 0\ , \qquad\qquad \boldsymbol{J}_\mu = \boldsymbol{U}^{-1}\partial_\mu \boldsymbol{U} \tag{2.12}$$

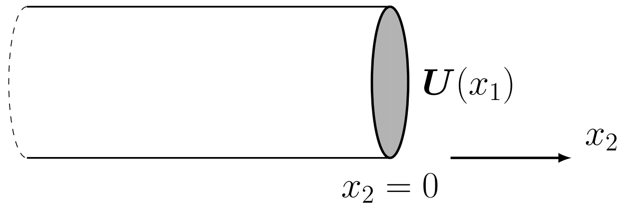


Figure 2.3: A depiction of the boundary conditions entering into the definition of the wave functional (2.11). The space co-ordinate x_1 in this figure has been compactified.

that obeys the Dirichlet boundary condition $\boldsymbol{U}(x)|_{x_2=0} = \boldsymbol{U}(x_1)$ (see fig. 2.3). We will denote this classical solution by $\boldsymbol{U}_0$. The quantum fluctuations can be included by writting $\boldsymbol{U}$ as

$$\boldsymbol{U} = \boldsymbol{U}_0\, \boldsymbol{h} \tag{2.13}$$

and integrating w.r.t. the field $\boldsymbol{h}$. Clearly the boundary conditions should be such that

$$\boldsymbol{h}(x)|_{x_2=0} = \mathbf{1}\ . \tag{2.14}$$

Considering only small fluctuations about the classical solution, one can expand $\boldsymbol{h}$ in terms of the infinitesimal field $\boldsymbol{\phi}$

$$\boldsymbol{h} = \mathrm{e}^{\mathrm{i}\boldsymbol{\phi}} = \mathbf{1} + \mathrm{i}\,\boldsymbol{\phi} - \frac{1}{2}\boldsymbol{\phi}^2 + \dots\ . \tag{2.15}$$

To express the action functional in terms of $\boldsymbol{U}_0$ and $\boldsymbol{\phi}$, a useful formula is that

$$\big\langle \boldsymbol{J}_\mu,\, \boldsymbol{J}_\mu \big\rangle = \big\langle \boldsymbol{J}^0_\mu,\, \boldsymbol{J}^0_\mu \big\rangle + \big\langle \boldsymbol{h}^{-1}\partial_\mu \boldsymbol{h},\, \boldsymbol{h}^{-1}\partial_\mu \boldsymbol{h} \big\rangle + 2\,\big\langle \boldsymbol{J}^0_\mu,\, \boldsymbol{h}^{-1}\partial_\mu \boldsymbol{h} \big\rangle\,, \tag{2.16}$$

where $\boldsymbol{J}^0_\mu = \boldsymbol{U}_0^{-1}\partial_\mu \boldsymbol{U}_0$. Substituting the expression (2.16) into the action functional yields that

$$\mathcal{A}\big[\boldsymbol{U}(x)\big] = \mathcal{A}\big[\boldsymbol{U}_0(x)\big] + \frac{1}{2f_0^2}\int \mathrm{d}^2x\,\langle \partial_\mu \boldsymbol{\phi},\, \partial_\mu \boldsymbol{\phi}\rangle + \frac{1}{2f_0^2}\int \mathrm{d}^2x\,\big\langle \boldsymbol{J}^0_\mu,\, \big[\partial^\mu \boldsymbol{\phi},\, \boldsymbol{\phi}\big]\big\rangle + O(\boldsymbol{\phi}^3)\,.$$

Here the term containing $\boldsymbol{J}^0_\mu\,\partial_\mu \boldsymbol{\phi}$ has not been included since it vanishes due to the equations of motion (2.12). Integration over the field $\boldsymbol{\phi}$ leads to a term quadratic in

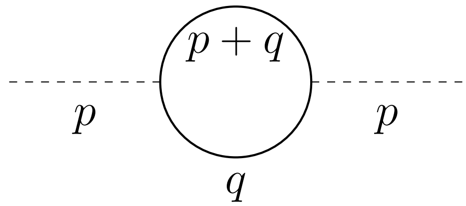


Figure 2.4: The Feynman diagram that gives the contribution to $\delta\mathcal{A}$ in eq. (2.17)

the currents $\boldsymbol{J}^0_\mu$ that is given by the Feynman diagram depicted in fig. 2.4. For our purposes, only the divergent part of this expression is required:

$$\begin{aligned}
\delta\mathcal{A} &= \int \frac{\mathrm{d}^2 p}{(2\pi)^2}\, J^a_\mu(p)\, J^b_\mu(-p) \times \frac{1}{4}\, f_{ac}{}^{d}\, f_{bd}{}^{c} \int \frac{\mathrm{d}^2 q}{(2\pi)^2}\, \frac{q_\mu q_\nu}{2q^4} + \text{finite} \\
&= \frac{1}{4\pi}\log(\Lambda) \int \mathrm{d}^2 x\, \frac{1}{2}\left\langle \boldsymbol{J}^0_\mu,\, \boldsymbol{J}^0_\mu \right\rangle + \text{finite}\,,
\end{aligned} \tag{2.17}$$

where Λ is the UV cut-off. Here we have used the notation $\boldsymbol{J}^0_\mu = \mathrm{i}\, J^a_\mu\, \mathtt{t}_a$, where $\mathtt{t}_a$ are a basis of the Lie algebra satisfying the commutation relations $[\mathtt{t}_a,\, \mathtt{t}_b] = \mathrm{i}\, f_{ab}{}^{c}\, \mathtt{t}_c$. Also, we have chosen the normalization for the Killing form to be such that $\langle \mathtt{t}_a,\, \mathtt{t}_b\rangle = -\frac{1}{4} f_{ac}{}^{d}\, f_{bd}{}^{c}$. Eq. (2.17) implies that if the renormalized coupling f is introduced via

$$\frac{1}{f^2} = \frac{1}{f_0^2} - \frac{1}{4\pi}\log(\Lambda/E) \tag{2.18}$$

the effective action remains finite at the one-loop order. Here E is some typical energy scale that has been included in order to make the argument of the logarithm finite.

The above formula requires some explanations. The analogous expression in [64] involves the group dependent factor $C_2(\mathfrak{G})$, the value of the quadratic Casimir in the adjoint representation. Eq. (2.18) does not contain any group dependent terms due to our different normalization for the trace. The Killing form is usually understood to be the matrix trace over the fundamental representation such that $\mathrm{Tr}(\mathtt{t}_a \mathtt{t}_b) = \frac{1}{2}\delta_{ab}$. This is related to our definition as $\langle \mathfrak{a}, \mathfrak{b}\rangle = \frac{1}{2}\, C_2(\mathfrak{G})\, \mathrm{Tr}(\mathfrak{a}\mathfrak{b})$.

It is illuminating to re-write eq. (2.18) in the form

$$\frac{2}{f^2} = -\frac{1}{4\pi}\, \log(E_*/E)\,, \tag{2.19}$$

where

$$E_* = \Lambda \, \mathrm{e}^{-\frac{4\pi}{f_0^2}} \, .$$

For a consistent removal of the cut-off Λ, the bare coupling must be given a cut-off dependence $f_0 = f_0(\Lambda)$ such that E_* is held fixed in the $\Lambda \to \infty$ limit. The parameter E_* has the dimensions of mass and sets an RG scale for the problem. Thus, although the classical Lagrangian is scale invariant, the quantum fluctuations cause the bare coupling to transmute into an energy scale E_*. This mechanism of dimensional transmutation also occurs in gauge theories.

Equation (2.19) implies that for a compact Lie group the renormalized coupling tends to zero when the typical energy scales become large. Hence, as the curvature of the target space is proportional to f^2 (see e.g. eq. (2.6) for the case $\mathfrak{G} = SU(2)$), the theory asympotically approaches a free theory on flat space in the high energy limit. Conversely, at low energies, the PCF becomes strongly interacting and perturbation theory breaks down. This is the hallmark of asymptotic freedom.

2.3.2 One-loop renormalizability of a general $1+1$D NLSM and Ricci flow equations

We have just discussed renormalization in the PCF. What about a general NLSM of the form (2.8)? The renormalizability of these field theories was studied by Friedan in [65]. He found that the class of NLSM is closed under the RG flow. Further he computed the RG flow equations up to second order in perturbation theory. To the lowest order they read as

$$\partial_\tau G_{\alpha\beta} = -\hbar \, R_{\alpha\beta} + O(\hbar^2) \, , \tag{2.20}$$

where $\tau \equiv -\frac{1}{2\pi} \log \Lambda$ stands for the RG time and R_{ab} denotes the Ricci tensor built from the metric. Note that for the PCF the metric and Ricci tensors coincide up to an overall constant factor so that (2.20) reduces to a differential equation for the bare coupling, which can be integrated to yield (2.18).

The formula (2.20) is somewhat famous. In string theory it has the interpretation that the conformal (Weyl) invariance of the string is equivalent to the vanishing of the Ricci tensor. The latter is none other than Einstein's gravitational equations in the vacuum. According to string theory, quantum corrections to these gravitational equations can be obtained by computing higher order terms in the r.h.s. of (2.20).

In mathematics, eq. (2.20) is identical to the Ricci flow, a sort of heat equation that tends to make the geometry more smooth and symmetric. It was the main tool that was used by Perelman to prove the geometrization conjecture. Its corollary the Poincaré conjecture, that every simply connected closed three manifold is homeomorphic to the three-sphere, was unsolved for over 100 years.

2.4 Examples

2.4.1 Anisotropic $SU(2)$ PCF

The anisotropic $SU(2)$ PCF is the field theory version of the symmetric top; the model defined through eq. (2.1) with $I_1 = I_2 \neq I_3$. Introduce the notation $J^a_\mu = \frac{1}{2\mathrm{i}}\,\mathrm{Tr}\,(\boldsymbol{U}^{-1}\partial_\mu \boldsymbol{U}\,\sigma^a)$ where $\boldsymbol{U}$ is a 2×2 special unitary matrix and σ^a are the usual Pauli matrices. Then the Lagrangian density is given by

$$\mathcal{L} = \frac{2}{f_\perp^2}\left((J^1_\mu)^2 + (J^2_\mu)^2\right) + \frac{2}{f_\parallel^2}\,(J^3_\mu)^2\,. \tag{2.21}$$

Notice that (2.21) breaks the $SU(2)\times SU(2)$ symmetry of the original PCF Lagrangian down to $U(1)\times SU(2)$. This model can be interpreted as an NLSM on the three sphere with a one-parameter deformed metric. Remarkably, the metric satisfies the RG flow equations (2.20). The dependence of the bare coupling constants $f_\perp$ and $f_\parallel$ on the cut-off is given by

$$\partial_\tau f_\perp = \hbar \, \frac{f_\perp^3}{4 f_\parallel^2} \left(2 f_\parallel^2 - f_\perp^2\right) + O(\hbar^2)$$
$$\partial_\tau f_\parallel = \hbar \, \frac{f_\perp^4}{4 \, f_\parallel} + O(\hbar^2) \; . \tag{2.22}$$

2.4.2 $O(N)$ model

Recall that any 2×2 unitary matrix $\boldsymbol{U}$ can be expressed as

$$\boldsymbol{U} = n_0 \, \mathbf{1} + \mathrm{i} \, n_1 \, \sigma^1 + \mathrm{i} \, n_2 \, \sigma^2 + \mathrm{i} \, n_3 \, \sigma^3 \; , \qquad n_1^2 + n_2^2 + n_3^2 + n_4^2 = 1 \; . \tag{2.23}$$

Setting $n_0 = 0$ identically and substituting $\boldsymbol{U}$ into the PCF Lagrangian yields that

$$\mathcal{L} = \frac{1}{2 \, f^2} \, \partial_\mu \mathbf{n} \cdot \partial^\mu \mathbf{n} \; , \tag{2.24}$$

where the vector $\mathbf{n} = (n_1, n_2, n_3)$ is constrained to lie on the two dimensional sphere $\mathbf{n} \cdot \mathbf{n} = 1$. Resolving the constraint $n_1 = \sin\theta \cos w \, , \; n_2 = \sin\theta \sin w \, , \; n_3 = \cos\theta$, eq. (2.24) takes the form

$$\mathcal{L} = \frac{1}{2 \, f^2} \left(\partial_\mu \theta \, \partial^\mu \theta + \sin^2(\theta) \, \partial_\mu w \, \partial^\mu w\right) \; . \tag{2.25}$$

The resulting NLSM has target space the round two sphere. It is known as the $O(3)$ model since it possesses global $O(3)$ symmetry. The $O(3)$ NLSM satisfies the one-loop RG flow equations and the Λ-dependence of the bare coupling reads as

$$\partial_\tau f = \frac{\hbar \, f^3}{2} + O(\hbar^2) \; . \tag{2.26}$$

The Lagrangian (2.24) can also be considered for the case when $\mathbf{n} = (n_1, \ldots n_N)$ is an N-dimensional unit vector. The corresponding NLSM is the $O(N)$ model whose target space is the $N-1$ sphere. The RG flow equations for the bare coupling still take the form (2.26). Note that the $O(4)$ model and the $SU(2)$ PCF coincide.

2.4.3 2D sausage model

A fruitful approach to constructing deformations of NLSM has been to study the RG flow equations (2.20) directly. These constitute a complicated system of non-linear differential equations whose solutions typically develop singularities. However, it turns out that they possess so-called ancient solutions that exist for the RG time $\tau \equiv -2\pi \log(\Lambda) \to -\infty$ and where the curvature remains small everywhere up to $\tau = -\infty$. The target space metric corresponding to an ancient solution can be used to define an NLSM, at least perturbatively, through the action (2.8). The short distance physics is captured entirely by perturbation theory. However, at large scales corresponding to $\tau \to +\infty$ the ancient solutions typically develop singularities where the curvature blows up so that the perturbative approach is no longer valid.

For a two-dimensional target space the RG flow equations were studied in [66]. In this case one can always choose a set of conformal co-ordinates $\{X^\alpha\}$, at least locally, for which $G_{\alpha\beta} = \mathrm{e}^{\Phi}\,\delta_{\alpha\beta}$. Then eq. (2.20) becomes a non-linear PDE for the single function Φ:

$$\partial_\tau\, \mathrm{e}^{\Phi} = \frac{1}{4\pi}\left(\frac{\partial}{\partial X^\alpha}\right)^2 \Phi\ . \tag{2.27}$$

In [66] a family of solutions was found. The corresponding metric is given by

$$G_{\alpha\beta}\,\mathrm{d}X^\alpha \mathrm{d}X^\beta = \frac{2\left((\mathrm{d}\phi)^2 + (\mathrm{d}w)^2\right)}{\kappa^{-1} + \kappa + (\kappa^{-1} - \kappa)\cosh(2\phi)}\ , \tag{2.28}$$

where $0 \leq w < 2\pi$ is an angular co-ordinate and $-\infty < \phi < +\infty$. The coupling κ here lies in the interval $\kappa \in (0, 1)$. Its dependence on the cut-off is given by:

$$\partial_\tau\, \kappa = \hbar\,(1 - \kappa^2) + O(\hbar^2)\ . \tag{2.29}$$

The target space of the NLSM corresponding to (2.28) is topologically the two-sphere. To get a better understanding of its geometry, it is instructive to make the change of variables from ϕ to the co-ordinate u, defined through the relation

$$\frac{\mathrm{cn}(u|\kappa)}{\mathrm{dn}(u|\kappa)} = \tanh\phi\ . \tag{2.30}$$

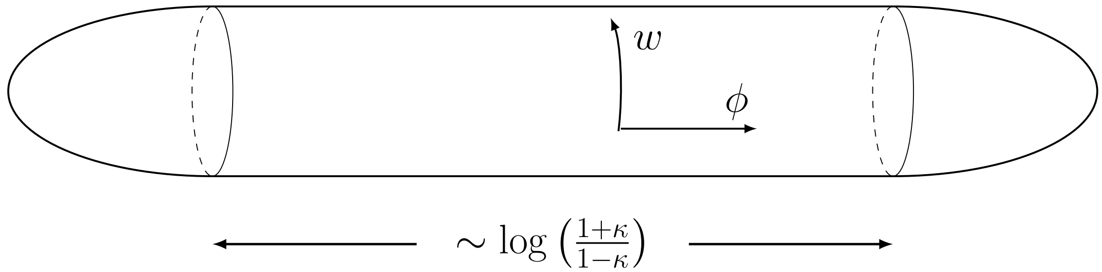


Figure 2.5: A depiction of the target manifold of the NLSM (2.31) for $1-\kappa \ll 1$.

In this thesis, the functions $\mathrm{sn}(u|\kappa)$, $\mathrm{cn}(u|\kappa)$ and $\mathrm{dn}(u|\kappa)$ will always denote the standard Jacobi elliptic functions with κ being the modulus ($\kappa^2\,\mathrm{sn}^2(u|\kappa)+\mathrm{dn}^2(u|\kappa) = 1$). In terms of the variable u, the Lagrangian is equal to

$$\mathcal{L} = \frac{\kappa}{2}\left(\partial_\mu u\,\partial^\mu u + \mathrm{sn}^2(u|\kappa)\,\partial_\mu w\,\partial^\mu w\right)\ . \tag{2.31}$$

In the limit $\kappa \to 0$ the function $\mathrm{sn}(u|\kappa)$ becomes the regular sine function so that the Lagrangian (2.31) becomes the $O(3)$ model Lagrangian (2.25) up to an overall multiplicative constant. Hence, the model is a one parameter deformation of the $O(3)$ sigma model. It is colloquially known as the "sausage model" since for $\kappa \to 1^-$ the target manifold can be pictured as a long sausage with length $\propto \log(\frac{1+\kappa}{1-\kappa})$ (see fig. 2.5).

Since the co-ordinate transformation (2.30) depends on the running parameter $\kappa = \kappa(\Lambda)$, the metric (2.31) does not satisfy the RG flow equations (2.20). An infinitesimal, coupling dependent reparametrization of the fields leads to a change in the metric

$$\delta G_{\alpha\beta} = \left(\nabla_\alpha V_\beta + \nabla_\beta V_\alpha\right)\,\delta\tau$$

with some vector V_α. Thus the general form of the RG flow equations, that admits the possibility of a coupling dependent co-ordinate transformation, reads as

$$\partial_\tau G_{\alpha\beta} = -\hbar\left(R_{\alpha\beta} + \nabla_\alpha V_\beta + \nabla_\beta V_\alpha\right) + O(\hbar^2)\ . \tag{2.32}$$

2.4.4 Cigar NLSM

In the high energy limit, it follows from eq. (2.29) that the coupling constant $\kappa \to 1^-$. In this limit most of the sausage asymptotically approaches the flat cylinder, while the curvature becomes concentrated at the tips corresponding to $\phi = \pm\infty$ (see fig. 2.5).

Near, say, the right tip the metric can be obtained from the sausage one (2.28) by shifting the field $\phi \to \phi + \frac{1}{2}\log\left(\frac{1+\kappa}{1-\kappa}\right)$ and then taking the limit $\kappa \to 1$. The Lagrangian of the corresponding NLSM is given by

$$\mathcal{L} = \frac{1}{2\,(1+\mathrm{e}^{2\phi})}\,\left(\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu w\,\partial^\mu w\right)\,. \tag{2.33}$$

Notice that for $\phi \to -\infty$, eq. (4.66) becomes the free Lagrangian describing the fields in the asymptotically flat domain.

The target space of the NLSM (4.66) coincides with the Hamilton's cigar [67]. Clearly, the sausage target space for $1-\kappa \ll 1$ can be approximated as two of these cigars glued together, separated by a distance $\propto \log(\frac{1+\kappa}{1-\kappa})$. Thus, in the $\kappa \to 1$ limit the sausage model breaks down into two independent copies of the cigar NLSM.

The cigar model is a scale invariant theory. Its target space metric satisfies the RG flow equations (2.32) with the l.h.s. set identically to zero, i.e.,

$$R_{\alpha\beta} + \nabla_\alpha V_\beta + \nabla_\beta V_\alpha = 0\,. \tag{2.34}$$

It turns out that the vector V_a can be expressed as the gradient

$$V_\alpha = \partial_\alpha\Psi, \qquad \text{with} \qquad \Psi = -\frac{1}{2}\log\left(1+\mathrm{e}^{2\phi}\right)\,. \tag{2.35}$$

2.4.5 3D sausage model

The anisotropic $SU(2)$ PCF, the analogy of the symmetric top, explicitly breaks the $SU(2)\times SU(2)$ global symmetry of the PCF down to $U(1)\times SU(2)$. Is it possible to further break this symmetry down to $U(1)\times U(1)$ similar to the Euler top? Naively introducing an extra parameter into the Lagrangian (2.21) so that the coefficients of

$(J^1_\mu)^2$ and $(J^2_\mu)^2$ differ results in a QFT that is not closed under the RG flow equations. This makes the deformation not particularly interesting.

A suitable deformation was found by Fateev in the work [68]. The approach was to substitute an explicit ansatz for the metric containing free parameters into both sides of eq. (2.20) and then to choose the parameters such that the RG flow equation is satisfied. The resulting NLSM is a two parameter deformation of the $SU(2)$ PCF known as the 3D sausage, in analogy to the 2D sausage previously discussed. The action reads as:

$$\mathcal{A} = \int \mathrm{d}^2 x \, \frac{u\,\mathrm{Tr}(\partial_\mu \boldsymbol{U}\,\partial^\mu \boldsymbol{U}^{-1}) + 2l\,(L^3_\mu)^2 + 2r\,(R^3_\mu)^2}{4(u+r)(u+l) - rl\,\big(\mathrm{Tr}(\boldsymbol{U}\,\sigma_3\,\boldsymbol{U}^{-1}\,\sigma_3)\big)^2} \ , \tag{2.36}$$

where (u, l, r) are the parameters of the model, while L^3_μ, R^3_μ stand for the σ^3 components of the left and right currents

$$L^3_\mu = \frac{1}{2\mathrm{i}}\,\mathrm{Tr}\big(\partial_\mu \boldsymbol{U}\,\boldsymbol{U}^{-1}\sigma^3\big)\,, \qquad R^3_\mu = \frac{1}{2\mathrm{i}}\,\mathrm{Tr}\big(\boldsymbol{U}^{-1}\partial_\mu \boldsymbol{U}\sigma^3\big)\ .$$

Under the RG group flow the following combinations of the parameters turn out to be RG invariants:

$$a_1,\ a_2 > 0\ : \qquad a_1\,a_2 = \frac{\pi^2}{4\sqrt{(u+r)(u+l)rl}}\ , \qquad a_1^2 - a_2^2 = \frac{\pi^2}{4}\,\frac{u(r-l)}{(u+r)(u+l)rl} \tag{2.37}$$

Then, the cut-off dependence of the couplings is described by

$$\partial_\tau u = 2\,\hbar\,(u + r + \ell)^2 + O(\hbar^2)\ . \tag{2.38}$$

2.4.6 Klimčík model

The Klimčík model is a two parameter deformation of the PCF for any group $\mathfrak{G}$ [69] that contains all of the models discussed previously as special cases. Its construction uses the so-called Yang-Baxter operator $\hat{\boldsymbol{R}}$. This is a linear operator acting in $\mathfrak{g}$ that satisfies a skew-symmetry condition

$$\big\langle \mathtt{a},\, \hat{\boldsymbol{R}}(\mathtt{b})\big\rangle = -\big\langle \hat{\boldsymbol{R}}(\mathtt{a}),\, \mathtt{b}\big\rangle \tag{2.39}$$

as well as the so-called modified Yang-Baxter equation

$$[\hat{\boldsymbol{R}}(\mathtt{a}),\, \hat{\boldsymbol{R}}(\mathtt{b})] = \hat{\boldsymbol{R}}\big([\hat{\boldsymbol{R}}(\mathtt{a}),\, \mathtt{b}] + [\mathtt{a},\, \hat{\boldsymbol{R}}(\mathtt{b})]\big) + [\mathtt{a},\, \mathtt{b}]\ , \qquad \mathtt{a},\, \mathtt{b} \in \mathfrak{g}\ . \tag{2.40}$$

In this thesis we will take the operator $\hat{\boldsymbol{R}}$ to be as follows. Using the root decomposition of the Lie algebra w.r.t. a Cartan subalgebra $\mathfrak{h}$, any element of $\mathfrak{g}$ can be written as a sum of $\mathtt{e}_\pm \in \mathfrak{n}_\pm$ from the nilpotent subalgebras and $\mathtt{h}$ lying in the Cartan. Then, the action of $\hat{\boldsymbol{R}}$ is defined by the relations $\hat{\boldsymbol{R}}\big(\mathtt{e}_\pm\big) = \mp\mathrm{i}\, \mathtt{e}_\pm$ and $\hat{\boldsymbol{R}}(\mathtt{h}) = 0$.

The Lagrangian of the Klimčík model with deformation parameters $\varepsilon_1, \varepsilon_2$ is given by

$$\mathcal{L} = -\frac{2}{f^2}\, \Big\langle \boldsymbol{U}^{-1}\partial_+\boldsymbol{U},\, \big(\hat{\mathbf{1}} - \mathrm{i}\varepsilon_1\, \hat{\boldsymbol{R}}_{\boldsymbol{U}} - \mathrm{i}\varepsilon_2\, \hat{\boldsymbol{R}}\big)^{-1}\big(\boldsymbol{U}^{-1}\partial_-\boldsymbol{U}\big)\Big\rangle\,, \tag{2.41}$$

where the action of $\hat{\boldsymbol{R}}_{\boldsymbol{U}}$ is defined as

$$\hat{\boldsymbol{R}}_{\boldsymbol{U}}(\mathtt{a}) = \boldsymbol{U}^{-1}\, \hat{\boldsymbol{R}}\big(\boldsymbol{U}\, \mathtt{a}\, \boldsymbol{U}^{-1}\big)\, \boldsymbol{U} \qquad \text{for} \quad \forall\, \mathtt{a} \in \mathfrak{g} \tag{2.42}$$

(the symbol $\boldsymbol{U}\,(\ldots)\,\boldsymbol{U}^{-1}$ denotes the adjoint action of the group element $\boldsymbol{U}$ on $\mathfrak{g}$).

Due to the skew-symmetry of $\hat{\boldsymbol{R}}$ (2.39) the Lagrangian density (2.41) is not parity invariant. In a co-ordinate system $\{X^\alpha\}$ it has the general form

$$\mathcal{L} \,=\, \tfrac{1}{2}\Big(G_{\alpha\beta}(X)\,\partial_\mu X^\alpha\,\partial^\mu X^\beta + B_{\alpha\beta}(X)\,\epsilon^{\mu\nu}\,\partial_\mu X^\alpha\,\partial_\nu X^\beta\Big)\,, \tag{2.43}$$

where $\epsilon^{01} = -\epsilon^{10} = 1$ and $\epsilon^{00} = \epsilon^{11} = 0$. The tensor $B_{\alpha\beta}$ is known as the torsion potential. Notice that the addition of the total derivative term $\partial_t\big[W_\beta(X)\,(\partial_x X - \partial_t X)\big]$ to the Lagrangian density, which can have no effect on the equations of motion, leads to a change in the B-field as $B_{\alpha\beta} \mapsto B_{\alpha\beta} + \partial_\alpha\, W_\beta - \partial_\beta\, W_\alpha$. Hence, the torsion potential is a gauge dependent term. The torsion tensor, however, which is defined as

$$H_{\alpha\beta\gamma} = \partial_\gamma B_{\alpha\beta} + \partial_\alpha B_{\beta\gamma} + \partial_\gamma B_{\beta\alpha}\ , \tag{2.44}$$

is gauge independent. For the case $\mathfrak{G} = SU(2)$ the B-field is a total derivative that can be ignored and the model coincides with the 3D sausage [70].

The RG equations are also known to two loops for the general NLSM with B-field (2.43). They form the following system of coupled PDEs [71, 72]

$$\begin{aligned} \partial_\tau G_{\alpha\beta} &= -\hbar\,\Big(R_{\alpha\beta} - \frac{1}{4} H_\alpha{}^{\gamma\eta}\, H_{\gamma\eta\beta} + \nabla_\alpha\, V_\beta + \nabla_\beta\, V_\alpha\Big) + O(\hbar^2) \\ \partial_\tau B_{\alpha\beta} &= -\hbar\,\Big(-\frac{1}{2}\,\nabla_\gamma\, H^\gamma{}_{\alpha\beta} + V_\gamma H^\gamma{}_{\alpha\beta} + \partial_\alpha\, W_\beta - \partial_\beta\, W_\alpha\Big) + O(\hbar^2)\,. \end{aligned} \tag{2.45}$$

Here the vector V_α takes into account a coupling dependent co-ordinate transformation of the metric. Similarly W_α corresponds to a coupling dependent gauge transformation of $B_{\alpha\beta}$.

The one-loop renormalizability for a general class of field theories that contain the Klimčík model as a special case was demonstrated in the work [73]. It turns out that the RG flow equations for the Klimčík model couplings depend very little on the group. In fact, they practically coincide with those derived by Fateev for the 3D sausage in the much earlier work [68]. The RG flow equations describing the cutoff dependence of the bare coupling constants are given by [74] (see also Appendix B for some details)

$$\begin{aligned} &\partial_\tau\varepsilon_1 = -\tfrac{1}{2}\,\hbar\, f^2\varepsilon_1\,\big(1-(\varepsilon_1-\varepsilon_2)^2\big)\big(1-(\varepsilon_1+\varepsilon_2)^2\,\big) + O(\hbar^2) \\ &\partial_\tau(\varepsilon_2/\varepsilon_1) = O(\hbar^2) \\ &\partial_\tau(\mathrm{g}^2\varepsilon_1) = O(\hbar^2)\ . \end{aligned} \tag{2.46}$$

The second equation in (2.46) shows that

$$\nu^2 = \frac{\varepsilon_2}{\varepsilon_1} \tag{2.47}$$

is an RG invariant and the third equation is fulfilled if we choose

$$f^2 = \left|\frac{\varepsilon_1+\varepsilon_2}{\varepsilon_1\varepsilon_2}\right|. \tag{2.48}$$

This way in the quantum theory there is only one Λ-dependent bare coupling. Within the domain

$$0<\varepsilon_1,\varepsilon_2<1$$

which will be considered in this thesis, it is convenient to use the parameterization

$$\varepsilon_1 = \frac{1}{\sqrt{(1+\kappa^{-1}\,\nu^2)(1+\kappa\nu^2)}}\,, \qquad \varepsilon_2 = \frac{\nu^2}{\sqrt{(1+\kappa^{-1}\,\nu^2)(1+\kappa\nu^2)}} \tag{2.49}$$

where $\nu^2 > 0$ and

$$\kappa = \kappa(\Lambda) \;\; : \qquad 0 < \kappa < 1\,. \tag{2.50}$$

It follows from the RG flow equations (2.46) that a consistent removal of the UV cutoff Λ requires that

$$\lim_{\Lambda\to\infty} \kappa(\Lambda) = 1^-\,. \tag{2.51}$$

Thus in the high energy limit the renormalized running coupling will tend to one from below.

Chapter 3
Classical Integrability

It turns out that all of the $1+1$ dimensional NLSM discussed before are classically, and likely quantum integrable models. Here we will discuss their classical integrability in the context of the inverse scattering method.

3.1 Zero-Curvature Representation

The solving of concrete mechanical models was a major preoccupation in 18^{th} and 19^{th} century physics. Obtaining a solution to a new and non-trivial problem was considered an achievement of applied mathematics and was usually associated with the development of a new mathematical technique. At the end of the 19^{th} century it was realized that solvability was connected with the presence of a sufficient number of Integrals of Motion (IM) in the theory that allow one to solve the differential equations. This gave rise to the notion of a Liouville integrable system in which the number of isolating and Poisson commuting IM is equal to the number of degrees of freedom. For such a system, it was proved that the equations of motion can be solved in quadratures, that is, the solution is expressible in terms of integrals over elementary functions.

In the context of $1+1$ dimensional field theory, where the number of degrees of freedom is infinite, a suitable paradigm of integrability was discovered in the mid 20^{th} century. The key ingredient in this case is a Lie algebra-valued world sheet connection that depends on an analytic spectral parameter such that the Zero-Curvature

Representation (ZCR)

$$\left[\partial_x - \boldsymbol{A}_x(\lambda),\, \partial_t - \boldsymbol{A}_t(\lambda)\right] = 0 \tag{3.1}$$

is equivalent to the classical equations of motion. Since the Wilson loops

$$T(\lambda) = \mathrm{Tr}\,\overleftarrow{\mathcal{P}}\exp\int_C \mathrm{d}x^\mu\, \boldsymbol{A}_\mu(\lambda) \tag{3.2}$$

remain unchanged under continuous deformations of the integration contour (see fig. 1.1), they generate an infinite family of conserved quantities. These can be used to solve the field theory within the framework of the inverse scattering method.

Let's consider the ZCR for some of the integrable NLSM discussed in the previous chapter.

3.1.1 PCF

The Lagrangian of the PCF in 1+1 space-time dimensions can be conveniently written using the light cone co-ordinates

$$\mathcal{L} = -\frac{2}{f^2}\left\langle \boldsymbol{U}^{-1}\partial_+\boldsymbol{U},\, \boldsymbol{U}^{-1}\partial_-\boldsymbol{U}\right\rangle\ , \tag{3.3}$$

where $\partial_\pm = \frac{1}{2}(\partial_t \pm \partial_x)$. The equations of motion coincide with the continuity equation for the currents $\boldsymbol{J}_\pm = \boldsymbol{U}^{-1}\,\partial_\pm\boldsymbol{U}$, i.e.,

$$\partial_-\boldsymbol{J}_+ + \partial_+\boldsymbol{J}_- = 0\ . \tag{3.4}$$

In addition, these currents satisfy a set of Bianchi type identities

$$\partial_-\boldsymbol{J}_+ - \partial_+\boldsymbol{J}_- + \left[\boldsymbol{J}_+,\, \boldsymbol{J}_-\right] = 0\ , \tag{3.5}$$

which are purely kinematic relations that do not make use of the equations of motion. Combining eqs. (3.4) and (3.5) enables one to express the derivatives of $\boldsymbol{J}_\pm$ in terms of their commutators

$$\partial_-\boldsymbol{J}_+ = -\tfrac{1}{2}\left[\boldsymbol{J}_+,\, \boldsymbol{J}_-\right], \qquad \partial_+\boldsymbol{J}_- = +\tfrac{1}{2}\left[\boldsymbol{J}_+,\, \boldsymbol{J}_-\right]\ . \tag{3.6}$$

Recall that $\boldsymbol{J}_\pm$ take values in the Lie algebra. A natural guess for the world sheet connection is

$$\boldsymbol{A}_\pm = \lambda_\pm \, \boldsymbol{J}_\pm \, , \tag{3.7}$$

with some constants $\lambda_\pm$. A simple computation of the ZCR (3.1) using eq. (3.6) yields that

$$\big[\partial_+ - \boldsymbol{A}_+, \, \partial_- - \boldsymbol{A}_-\big] = \Big(\lambda_+\lambda_- - \frac{\lambda_-}{2} - \frac{\lambda_+}{2}\Big) \, \big[\boldsymbol{J}_+, \, \boldsymbol{J}_-\big] \ , \tag{3.8}$$

The r.h.s. of this equation can be made to vanish by choosing the parameters λ_+ and λ_- to satisfy the constraint

$$\frac{1}{\lambda_+} + \frac{1}{\lambda_-} = 2 \ , \qquad \text{i.e.,} \qquad \lambda_\pm = \frac{1}{1 \pm \lambda} \ . \tag{3.9}$$

This flat connection (3.7), (3.9) was originally obtained in the work [46].

Apart from the local integrability condition – the zero curvature representation – proper global requirements need to be specified. In this thesis, we will consider the spacetime to be a cylinder with the space co-ordinate compactified $x \sim x + R$ (see fig. 1.1). A natural choice for the boundary conditions for the PCF are periodic boundary condtions, so that

$$J(t, x + R) = J(t, x) \ . \tag{3.10}$$

Choosing some matrix representation $\mathcal{R}$ for the Lie algebra, one can introduce the monodromy matrix at the time slice t_0 as

$$\boldsymbol{M}_\mathcal{R}(\lambda) = \pi_\mathcal{R}\Big[\overleftarrow{\mathcal{P}} \exp\Big(\int_0^R \mathrm{d}x \ \boldsymbol{A}_x(t_0, x)\Big)\Big] \ , \tag{3.11}$$

where $\boldsymbol{A}_x = \boldsymbol{A}_+ - \boldsymbol{A}_-$. It follows from the ZCR that

$$\partial_t \, \boldsymbol{M}_\mathcal{R} = \big[\pi_\mathcal{R}\big(\boldsymbol{A}_t(t_0, 0)\big), \, \boldsymbol{M}_\mathcal{R}\big] \, , \tag{3.12}$$

so that the trace $T_\mathcal{R}(\lambda) = \mathrm{Tr}\big(\boldsymbol{M}_\mathcal{R}(\lambda)\big)$ is a conserved quantity. The dependence on the arbitrary variable λ ensures that $T_\mathcal{R}(\lambda)$ generates an infinite family of IM and not just a single conserved charge.

3.1.2 Klimcik model

We will now turn to the construction of the flat connection for the Klimčík model. The latter contains all of the integrable NLSM discussed before as special cases. Hence, the flat connection for any of these models can be obtained from the Klimčík one via a specialization of the parameters.

The currents that are the analogue of $\boldsymbol{J}_\pm$ in the PCF are given by

$$\boldsymbol{I}_\pm = -2\mathrm{i}\, \big(\hat{\mathbf{1}} \pm \mathrm{i}\varepsilon_1\, \hat{\boldsymbol{R}}_{\boldsymbol{U}} \pm \mathrm{i}\varepsilon_2\, \hat{\boldsymbol{R}}\big)^{-1} \big(\boldsymbol{U}^{-1}\partial_\pm \boldsymbol{U}\big)\,. \tag{3.13}$$

Similar to eq. (3.6), the equations of motion together with the Bianchi type identities imply the following relations for $\boldsymbol{I}_\pm$:

$$\begin{aligned}
\partial_+\boldsymbol{I}_- \;&=\; +\frac{\mathrm{i}}{2}\,\varepsilon_2\,\big[\hat{\boldsymbol{R}}(\boldsymbol{I}_+),\, \boldsymbol{I}_-\big] + \frac{1}{4}\,(1-\varepsilon_1^2+\varepsilon_2^2)\,[\boldsymbol{I}_+,\, \boldsymbol{I}_-] \\
\partial_-\boldsymbol{I}_+ \;&=\; -\frac{\mathrm{i}}{2}\,\varepsilon_2\,\big[\hat{\boldsymbol{R}}(\boldsymbol{I}_-),\, \boldsymbol{I}_+\big] + \frac{1}{4}\,(1-\varepsilon_1^2+\varepsilon_2^2)\,[\boldsymbol{I}_-,\, \boldsymbol{I}_+]\;.
\end{aligned}$$

To write down the explicit formula for the connection, it is convenient to use the root decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$ and express the currents in the form

$$\boldsymbol{I}_\sigma(x) = \boldsymbol{I}^+_\sigma(x) + \boldsymbol{I}^0_\sigma(x) + \boldsymbol{I}^-_\sigma(x) \;\;:\quad \boldsymbol{I}^\pm_\sigma(x) \in \mathfrak{n}_\pm\,,\qquad \boldsymbol{I}^0_\sigma(x) \in \mathfrak{h}\,. \tag{3.14}$$

With this notation, the connection components for the Klimčík model read explicitly as

$$\boldsymbol{A}_\sigma = -\,\frac{\mathrm{i}\,\varepsilon_2}{1-\rho_\sigma^2}\,\Big((\rho_\sigma)^{1-\sigma}\,\boldsymbol{I}^+_\sigma + (\rho_\sigma)^{1+\sigma}\,\boldsymbol{I}^-_\sigma + \tfrac{1}{2}\,\big(1+\rho_\sigma^2\big)\,\boldsymbol{I}^0_\sigma\Big) \qquad (\sigma=\pm)\;, \tag{3.15}$$

where the auxiliary parameters $\rho_\pm^2$ are subject to the single constraint[1]

$$(\rho_+\rho_-)^2 = \frac{(1+\varepsilon_1-\varepsilon_2)(1-\varepsilon_1-\varepsilon_2)}{(1-\varepsilon_1+\varepsilon_2)(1+\varepsilon_1+\varepsilon_2)}\;. \tag{3.16}$$

[1] Eq. (20) from ref.[69] is equivalent to (3.15) with $\boldsymbol{L}^{\alpha,\beta}_\pm(\zeta) = \boldsymbol{A}_\pm$ provided the following identifications are made $\alpha = \mathrm{i}\,\varepsilon_1$, $\beta = \mathrm{i}\,\varepsilon_2$ and the spectral parameter $\zeta = \frac{\rho_+^2+\rho_-^{-2}-2}{\rho_+^2-\rho_-^{-2}}$.

In this thesis, we will always take $\rho \equiv \rho_+$ as the spectral parameter, and consider ρ_- to be expressed in terms of ρ_+ through eq. (3.16).

Having discussed the connection, let's turn to the construction of the Wilson loops (3.2). The Klimčík model Lagrangian is invariant w.r.t. the left and right rotations by constant elements of the Cartan subgroup $\boldsymbol{U} \mapsto \boldsymbol{H}_1\boldsymbol{U}\boldsymbol{H}_2$. Hence, a natural choice for the boundary conditions is

$$\boldsymbol{U}(t, x+R) = \boldsymbol{H}_1\boldsymbol{U}(t,x)\boldsymbol{H}_2\,, \tag{3.17}$$

With these conditions, the flat connection (3.15) becomes a quasiperiodic 1-form:

$$\boldsymbol{A}_\sigma(t, x+R) = \boldsymbol{H}_2^{-1}\,\boldsymbol{A}_\sigma(t,x)\,\boldsymbol{H}_2\,. \tag{3.18}$$

The monodromy matrix is defined similarly to before via eq. (3.11). However, due to the quasiperiodicity (3.17), the explicit computation of its time derivative yields that

$$\partial_t \boldsymbol{M}_{\mathcal{R}} = \pi_{\mathcal{R}}\big(\boldsymbol{H}_2^{-1}\big)\left[\pi_{\mathcal{R}}\big(\boldsymbol{A}_x(t_0,0)\big),\, \pi_{\mathcal{R}}\big(\boldsymbol{H}_2\big)\boldsymbol{M}_{\mathcal{R}}\right]\,. \tag{3.19}$$

The infinite family of conserved chrages for the Klimcik model is introduced via a slight modification of eq. (3.2):

$$T_{\mathcal{R}}(\rho) = \mathrm{Tr}\big[\pi_{\mathcal{R}}\big(\boldsymbol{H}_2\big)\,\boldsymbol{M}_{\mathcal{R}}(\rho)\big]\,. \tag{3.20}$$

3.1.3 3D sausage

The flat connection for the 3D sausage was originally found in [75]. In fact, this work constructs a more general classically integrable NLSM with torsion that is a four parameter deformation of the $SU(2)$ PCF and contains the 3D sausage as a two parameter sub-family. Its flat connection is explicitly given using a parameterization in terms of elliptic theta functions. The formulae are too complicated to be reproduced here, however, the specialization of this connection to the 3D sausage is given in appendix C.

Another connection for the 3D sausage can be obtained by setting $\boldsymbol{U} \in SU(2)$ in the Klimčík connection (3.15). It was shown as a result of work done for this thesis that this connection is equivalent to the one found by Lukyanov in [75] specialized to the 3D sausage. The explicit relation between Lukyanov's elliptic parameterization and the parameters $\{\varepsilon_1,\, \varepsilon_2,\, \rho_\pm\}$, together with the matrix $\boldsymbol{S}$ entering into the gauge transformation

$$\boldsymbol{A}_\pm \mapsto \boldsymbol{S}^{-1}\boldsymbol{A}_\pm\,\boldsymbol{S} - \boldsymbol{S}^{-1}\partial_\pm \boldsymbol{S} \tag{3.21}$$

is described in the appendix C.

3.2 Sklyanin exchange relations

In the previous section we discussed the rôle of the Zero-Curvature Representation in a classically integrable field theory. Provided that suitable boundary conditions are imposed, the ZCR implies that the trace of the monodromy $T(\lambda) = \mathrm{Tr}\,\overleftarrow{\mathcal{P}}\exp\oint \mathrm{d}x\, \boldsymbol{A}_x(\lambda)$ is a conserved quantity in the theory, i.e., $\partial_t\, T(\lambda) = 0$. The dependence on the auxiliary parameter λ ensures that $T(\lambda)$ generates an infinite family of IM. As in Liouville integrability, the next question regards the Poisson commutativity of these conserved charges. Using the canonical Poisson structure induced from the Lagrangian, one must show the following condition

$$\{T(\lambda),\, T(\mu)\} = 0\ . \tag{3.22}$$

The proof of (3.22) requires the study of the Poisson brackets of the connection components $\boldsymbol{A}_x$. It was found by Sklyanin, by considering specific examples such as the non-linear Schrodinger equation and the sine-Gordon model, that in certain classically integrable field theories these Poisson brackets obey the general structure [76]

$$\big\{\boldsymbol{A}_x(x|\lambda_1) \underset{,}{\otimes} \boldsymbol{A}_x(y|\lambda_2)\big\} = \big[\boldsymbol{A}_x(x|\lambda_1)\otimes\mathbf{1}+\mathbf{1}\otimes\boldsymbol{A}_x(y|\lambda_2), \boldsymbol{r}(\lambda_2/\lambda_1)\big]\,\delta(x-y)\,. \tag{3.23}$$

In this formula, with a slight abuse of notation, we assume that some representation for the Lie algebra has been chosen so that $\boldsymbol{A}_x$ denotes a finite dimensional matrix. The quantity $\boldsymbol{r}$ is the so-called classical r-matrix and it lies in the tensor product of the representations $\boldsymbol{r} = \boldsymbol{r}_{12}$. As a consequence of the skew-symmetry and the Jacobi identity of the Poisson brackets, the classical r-matrix must satisfy the following conditions

$$\boldsymbol{r}_{12}(-\lambda) = -\boldsymbol{r}_{21}(\lambda)\,, \tag{3.24}$$

and

$$\big[\boldsymbol{r}_{12}(\lambda/\mu),\, \boldsymbol{r}_{13}(\lambda)+\boldsymbol{r}_{23}(\mu)\big] + \big[\boldsymbol{r}_{13}(\lambda),\, \boldsymbol{r}_{23}(\mu)\big] = 0\,. \tag{3.25}$$

The latter is known as the classical Yang-Baxter equation and plays a fundamental rôle in the Hamiltonian approach to classically integrable field theory [77]. Moreover its quantum counterpart is central to the study of quantum integrable systems and has inspired developments in many areas of mathematics, as was mentioned in the introduction.

An important feature of the Poisson structure (3.23) is that it contains only the δ function and none of its higher derivatives. Relations of this type are known as "ultralocal" to indicate that they are well behaved for vanishing $(x-y)$. With the ultra-local Poisson brackets at hand, it is possible to show by direct computation that the monodromy matrix

$$\boldsymbol{M}(\lambda) = \overleftarrow{\mathcal{P}} \exp \int_0^R \mathrm{d}x \boldsymbol{A}_x(\lambda)\,, \tag{3.26}$$

satisfies the Sklyanin exchange relations

$$\big\{ \boldsymbol{M}(\lambda_1) \underset{,}{\otimes} \boldsymbol{M}(\lambda_2) \big\} = \big[\boldsymbol{M}(\lambda_1) \otimes \boldsymbol{M}(\lambda_2), \boldsymbol{r}(\lambda_2/\lambda_1) \big]\,. \tag{3.27}$$

The Poisson commutativity of the conserved charges (3.22) immediately follows from the above by taking the trace of both sides.

To see how (3.23) leads to the Poisson bracket relations (3.27) one can discretize the path ordered integral in (3.26) onto N segments Δ_n. Then the monodromy matrix is given as the ordered product over the elementary transport matrices

$$\boldsymbol{M} = \overleftarrow{\prod_n} \boldsymbol{M}_n\,, \qquad \boldsymbol{M}_n = \mathbf{1} + \int_{\Delta_n} \mathrm{d}x\, \boldsymbol{A}_x + O(\Delta^2)\,. \tag{3.28}$$

By repeated use of the Leibniz rule, the Poisson brackets on the l.h.s. of (3.27) can be expressed in terms of $\{\boldsymbol{M}_n \underset{,}{\otimes} \boldsymbol{M}_m\}$. For the case when $n \neq m$ this gives zero as it leads to the vanishing integral $\int_{\Delta_n} \mathrm{d}x \int_{\Delta_m} \mathrm{d}y\ \delta(x-y)$. Here it is crucial that the r.h.s. of (3.23) contains only the δ-function and none of its higher derivatives. The presence of a $\delta'(x-y)$ would contribute boundary terms to the integral when Δ_n is adjacent to Δ_m. For the same segment one can show that the elementary transport matrices satisfy the relation similar to (3.27), so that

$$\left\{ \boldsymbol{M}_m(\lambda_1) \underset{,}{\otimes} \boldsymbol{M}_n(\lambda_2) \right\} = \delta_{mn} \left[\boldsymbol{M}_m(\lambda_1) \otimes \boldsymbol{M}_n(\lambda_2), \boldsymbol{r}(\lambda_1/\lambda_2) \right] + O(\Delta^2) \ . \tag{3.29}$$

With (3.29) at hand, a direct computation of the Poisson brackets of the monodromy for different values of the spectral parameter gives that

$$\begin{aligned} \left\{ \boldsymbol{M}(\lambda_1) \underset{,}{\otimes} \boldsymbol{M}(\lambda_2) \right\} = \sum_n \boldsymbol{M}_N \otimes \boldsymbol{M}'_N \dots \ \boldsymbol{M}_{n+1} \otimes \boldsymbol{M}'_{n+1} \\ \times \left[\boldsymbol{M}_n \otimes \boldsymbol{M}'_n, \boldsymbol{r}(\lambda_1/\lambda_2) \right] \boldsymbol{M}_{n-1} \otimes \boldsymbol{M}'_{n-1} \dots \boldsymbol{M}_1 \otimes \boldsymbol{M}'_1 + O(\Delta^2) \ , \end{aligned}$$

where the shortcut notations $\boldsymbol{M}_m = \boldsymbol{M}_m(\lambda_1)$ and $\boldsymbol{M}'_m = \boldsymbol{M}_m(\lambda_2)$ are being used here. It follows from this and elementary identities for the commutator that eq. (3.27) is satisfied up to corrections of order $O(N\Delta^2)$, which vanish in the $N \to \infty$ limit.

3.2.1 Non-ultralocality problem

For many models, and NLSM in particular, the Poisson brackets of the connection components do not obey the ultra-local structure (3.23). Together with $\delta(x-y)$, they contain terms proportional to the derivative of the delta function and possibly its higher derivatives as well. Such singular terms in the Poisson brackets are symptoms of the strongly divergent behaviour in the OPE of the fields in the quantum theory. These UV divergences are related to those we discussed in the computation of the effective action in the PCF (see sec. 2.3.1). In the classical theory, the non-ultralocal form of the Poisson brackets creates serious problems with the proof of the Poisson commutativity of the conserved charges for different values of the spectral parameter. In turn, this makes the quantum counterpart to $T(\lambda)$ difficult to define.

Though the conserved charges are, of course, gauge invariant quantities the Poisson brackets of the flat connection are sensitive to the gauge transformation

$$\boldsymbol{A}_\pm \mapsto \boldsymbol{S}^{-1}\boldsymbol{A}_\pm\,\boldsymbol{S} - \boldsymbol{S}^{-1}\partial_\pm \boldsymbol{S}\ . \tag{3.30}$$

Thus it is sometimes possible to recover the key relations (3.23) for a non-ultralocal flat connection by finding an appropriate gauge. This can be demonstrated on the example of the $O(3)$ model. The usual form of the connection, obtained as a reduction of the PCF one, reads as

$$\boldsymbol{A}_\pm = \frac{[\partial_\pm \check{\mathbf{n}}, \check{\mathbf{n}}\,]}{2\,(1\pm\lambda)}\ , \tag{3.31}$$

where $\check{\mathbf{n}} = n^1\sigma^1 + n^2\sigma^2 + n^3\sigma^3$. It is simple to check that the Poisson brackets of $\boldsymbol{A}_x = \boldsymbol{A}_+ - \boldsymbol{A}_-$ contains a term proportional to $\delta'(x-y)$. However, applying the gauge transformation (3.30) with $\boldsymbol{S} = \lambda\,\mathbf{1} + \check{\mathbf{n}}$ yields the connection

$$\boldsymbol{A}_\pm = -\frac{\lambda}{(1\pm\lambda)^2}\Big(\partial_\pm\,\check{\mathbf{n}} \pm \tfrac{1}{2}\big[\check{\mathbf{n}}\,,\partial_\pm\check{\mathbf{n}}\big]\Big)\,. \tag{3.32}$$

The latter has ultralocal Poisson brackets.

The ultralocal connection (3.32) and its extension to the 2D sausage was found in the work [58].[2] It arises in a certain limit of the Klimčík model flat connection defined through eqs. (3.15) (3.16). To take this limit, one should re-write the deformation parameters $\varepsilon_1, \varepsilon_2$ in terms of κ and ν using eq. (2.49) and then set $\nu \to 0$. Though the overall factor ε_2 multiplying the connection goes to zero as ν^2 (see eq. (2.49)), the result is finite and non-zero since the currents $\boldsymbol{I}_\pm$ tend to infinity.

To perform the computation, it is convenient to use the co-ordinate frame based on the Euler decomposition of $\boldsymbol{U} \in SU(2)$:

$$\boldsymbol{U} = \mathrm{e}^{-\frac{\mathrm{i}v}{2}\mathtt{h}}\ \mathrm{e}^{-\frac{\mathrm{i}\theta}{2}(\mathtt{e}_+ + \mathtt{e}_-)}\ \mathrm{e}^{-\frac{\mathrm{i}w}{2}\mathtt{h}}\,. \tag{3.33}$$

[2]It should mentioned that the connection (3.32) appeared earlier in the overlooked paper [78], The result in [58] was obtained independently from this work.

Here $\mathtt{h}, \mathtt{e}_\pm$ are the generators of the Lie algebra $\mathfrak{sl}_2$ satisfying the commutation relations

$$[\mathtt{h}, \mathtt{e}_\pm] = \pm 2\mathtt{e}_\pm \ , \qquad [\mathtt{e}_+, \mathtt{e}_-] = \mathtt{h} \ .$$

In fact, it is useful to substitute the angle $\theta \in (0, \pi)$ for $\phi \in (-\infty, \infty)$ such that

$$\tan(\tfrac{\theta}{2}) = \ \mathrm{e}^{\phi-\phi_0} \,, \qquad \mathrm{e}^{\phi_0} = \sqrt{\frac{1+\kappa}{1-\kappa}} \ , \tag{3.34}$$

which will become the ϕ from eq. (2.28) for the 2D sausage. Taking the limit of the Klimcik connection described above, yields that

$$\begin{aligned} \boldsymbol{A}_+ &= \frac{\mathrm{i}\,\Pi_+}{1-\rho_+^2}\,\Big(\rho_+\,\mathrm{e}^{+(\phi+\mathrm{i}w)}\,\mathtt{e}_+ + \rho_+\,\mathrm{e}^{-(\phi+\mathrm{i}w)}\,\mathtt{e}_- + \tfrac{\mathrm{i}}{2}\,(1+\rho_+^2)\,\mathtt{h}\Big) \\ \boldsymbol{A}_- &= \frac{\mathrm{i}\,\Pi_-}{1-\rho_-^2}\,\Big(\rho_-\,\mathrm{e}^{-(\phi-\mathrm{i}w)}\,\mathtt{e}_+ + \rho_-\,\mathrm{e}^{+(\phi-\mathrm{i}w)}\,\mathtt{e}_- - \tfrac{\mathrm{i}}{2}\,(1+\rho_-^2)\,\mathtt{h}\Big)\,, \end{aligned} \tag{3.35}$$

where

$$\begin{aligned} \Pi_+ &= \frac{2\,\big(\partial_+\phi - \mathrm{i}\,\partial_+ w\big)}{\kappa^{-1}+\kappa+(\kappa^{-1}-\kappa)\,\cosh(2\phi)} \\ \Pi_- &= \frac{2\,\big(\partial_-\phi + \mathrm{i}\,\partial_- w\big)}{\kappa^{-1}+\kappa+(\kappa^{-1}-\kappa)\,\cosh(2\phi)} \ . \end{aligned}$$

The constraint (3.16) with ν set to zero becomes

$$\rho_+\,\rho_- = \frac{1-\kappa}{1+\kappa} \ . \tag{3.36}$$

It follows from the canonical structure induced by the 2D sausage Lagrangian that

$$\{\Pi_+(x),\,\Pi_+(y)\} = \{\Pi_-(x),\,\Pi_-(y)\} = \{\Pi_+(x),\,\Pi_-(y)\} = 0 \ .$$

Hence the Poisson brackets of the connection (3.35) can not contain a $\delta'(x-y)$. A direct computation yields that

$$\begin{aligned} \big\{\boldsymbol{A}_\pm(x|\rho_1) \underset{,}{\otimes} \boldsymbol{A}_\pm(y|\rho_2)\big\} &= \pm\big[\boldsymbol{A}_\pm(x|\rho_1)\otimes\mathbf{1} + \mathbf{1}\otimes\boldsymbol{A}_\pm(y|\rho_2), \boldsymbol{r}(\rho_2/\rho_1)\big]\,\delta(x-y) \\ \big\{\boldsymbol{A}_\pm(x|\rho_1) \underset{,}{\otimes} \boldsymbol{A}_\pm(y|\rho_2)\big\} &= 0\,. \end{aligned} \tag{3.37}$$

Here the classical r-matrix turns out to be the trigonometric one, given by

$$\boldsymbol{r}(\rho) = \frac{1}{\rho - \rho^{-1}} \left(\mathtt{e}_+ \otimes \mathtt{e}_- + \mathtt{e}_- \otimes \mathtt{e}_+ + \tfrac{1}{4} (\rho + \rho^{-1})\, \mathtt{h} \otimes \mathtt{h} \right) . \tag{3.38}$$

Of course, for a given model, an "ultralocal gauge" may not exist. Nevertheless, one can ask whether the Sklyanin exchange relations may still be present even without the ultralocal structure (3.23). To give a definite answer, a straightforward approach is to compute the Poisson brackets $\left\{ \boldsymbol{M}(\lambda_1) \underset{,}{\otimes} \boldsymbol{M}(\lambda_2) \right\}$ following, say, the discretization procedure outlined before. Due to the non-ultralocal Poisson brackets, one would encounter ambiguous integrals of the type $\int_a^b \mathrm{d}x\, \delta(x-a)$. To give them meaning, some sort of regularization would need to be introduced that could precisely define the value of the delta-function at the endpoints of the integration limit. This was attempted in a number of works.

In [79] a certain "equal-point" limiting prescription was put forward to handle the ambiguities, which enabled the introduction of a commuting family of conserved charges. However this lead to a modification of the Sklyanin exchange relations and the rôle of these "new integrable canonical structures" is unclear both for the classical as well as the quantum theory. Another type of regularization was put forward in the work [80] for the case of the PCF. With this approach, it was found that the key relations (3.27) remain unchanged.

In this thesis we will follow the method of [57]. The approach is to start with an explicit realization of the quantum counterpart to the Sklyanin exchange relations and to take the classical limit. This allows one to trace the emergence of the classical Poisson structures in a non-ultralocal system. We will demonstrate this in the next section.

Chapter 4
The Yang-Baxter algebras

4.1 Introduction

4.1.1 Yang-Baxter algebras in statistical mechanics

A fundamental rôle in the theory of quantum integrable systems is played by the algebraic structures collectively known as the Yang-Baxter algebras. These take the general form

$$\boldsymbol{R}(\lambda_2/\lambda_1)\big(\boldsymbol{M}(\lambda_1)\otimes\mathbf{1}\big)\big(\mathbf{1}\otimes\boldsymbol{M}(\lambda_2)\big)=\big(\mathbf{1}\otimes\boldsymbol{M}(\lambda_2)\big)\big(\boldsymbol{M}(\lambda_1)\otimes\mathbf{1}\big)\boldsymbol{R}(\lambda_2/\lambda_1)\,, \tag{4.1}$$

where $\boldsymbol{M}$ is an operator valued matrix, while $\boldsymbol{R}=\boldsymbol{R}_{12}$ is the R-matrix that acts in the tensor product and satisfies the Yang-Baxter relation

$$\boldsymbol{R}_{12}(\lambda_2/\lambda_1)\boldsymbol{R}_{13}(\lambda_1)\boldsymbol{R}_{23}(\lambda_2)=\boldsymbol{R}_{23}(\lambda_2)\boldsymbol{R}_{13}(\lambda_1)\boldsymbol{R}_{12}(\lambda_2/\lambda_1)\ . \tag{4.2}$$

The above equations first appeared in the context of exactly soluble lattice systems [81]. Subsequently, it was realized that many integrable field theories possess Poisson bracket algebras that can be considered the classical limit of (4.1). This triggered the development of the Quantum Inverse Scattering Method (QSIM) for the quantization of an integrable model. Within its framework, eq. (4.1) plays a rôle similar to the canonical commutation relations for a quantum mechanical system. Whereas the correspondence principle prescribes the replacement of the canonical Poisson brackets with commutators, the "first principles" quantization in integrable models starts with the formal substitution of the Sklyanin exchange relations (3.27) by the quantum

Yang-Baxter algebra. The next and most difficult step is to construct a suitable representation of (4.1).

To see how (4.1), (4.2) reduce to eqs. (3.27), (3.25) in the classical limit, one should take $\hbar \to 0$ with $\boldsymbol{R}(\lambda) = \mathbf{1} + \mathrm{i}\hbar\, \boldsymbol{r}_{12}(\lambda) + O(\hbar^2)$. It is easy to see that the classical Yang-Baxter equation appears in eq. (4.2) at second order in $\hbar$. For the classical limit of eq. (4.1), one should keep in mind that the matix $\boldsymbol{M}(\lambda)$ is operator valued so that its matrix entries do not commute. Using the correspondence principle to write

$$\left[\boldsymbol{M}(\lambda_1) \otimes \mathbf{1}, \mathbf{1} \otimes \boldsymbol{M}(\lambda_2)\right] = \mathrm{i}\hbar \left\{\boldsymbol{M}(\lambda_1) \underset{,}{\otimes} \boldsymbol{M}(\lambda_2)\right\} + O(\hbar^2)$$

and equating the coefficient of $\hbar$ in eq. (4.1) yields the Sklyanin exchange relations (3.27).

In this chapter we will use this "correspondence principle" in order to investigate the Poisson structures in a non-ultralocal theory. Starting with an explicit quantum field theory realization of eq. (4.1) we will take its classical limit and trace the emergence of the monodromy matrix satisfying the Sklyanin exchange relations. However, to get some intuition, it is useful to first illustrate the Yang-Baxter algebras in the original context where they appeared, i.e., 2D exactly soluble lattice models.

4.2 Yang-Baxter algebra in the 6-vertex model

The 6-vertex model, originally introduced to describe a two dimensional sheet of ice, is a classic theory in exactly solvable lattice systems. Here we will focus on the model defined on an $N \times M$ square lattice, as in fig. 4.1. The degrees of freedom are the spins "$\pm$", which lie on the edges joining the sites and are represented by the arrows in figure 4.1. The problem is to compute the partition function, where the Boltzmann weight of each configuration of spins on the lattice is given by the product over the local Boltzmann weights at each site.

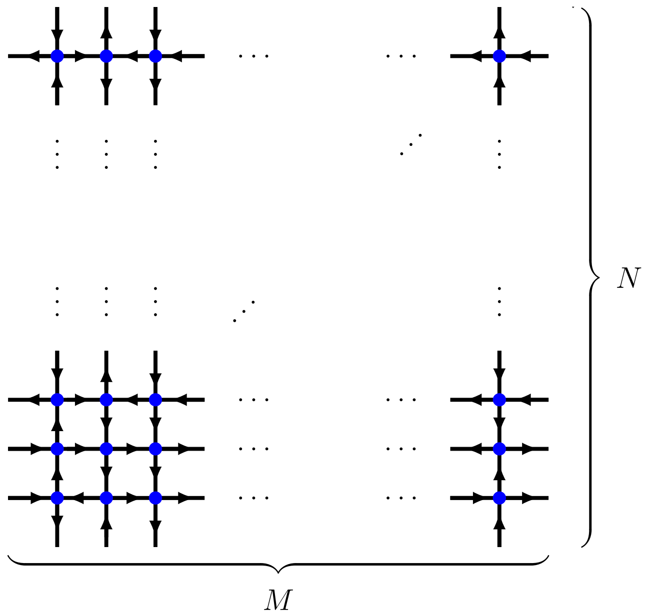


Figure 4.1: A possible configuration for the six-vertex model on an $N \times M$ square lattice that respects the toroidal boundary conditions. The spins on each edge are depicted graphically by the arrows with “+” corresponding to an up/right arrow and “−” corresponds to a down/left arrow.

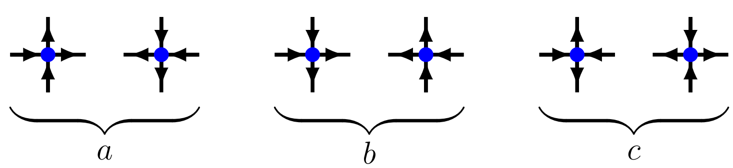
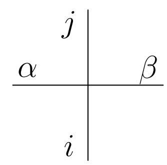


Figure 4.2: The six possible types of vertices in the six-vertex model. The parameters a, b and c label the Boltzmann weights associated to each vertex.

In the six-vertex model the spins are constrained to satisfy the so-called ice rule, that the sum of all spins around a vertex is equal to zero. This means that of the $2^4 = 16$ possible configurations of spins around a site only six are allowed, which are depicted in fig. 4.2. Assuming that the model is symmetric w.r.t. the reversal of all arrows, the six configurations are characterized by three distinct Boltzmann weights $\{a, b, c\}$ (see again fig. 4.2). Ignoring an overall normalization factor, these can be parameterized by two variables q and λ as $a = q^{-1}\lambda - q\lambda^{-1}$, $b = \lambda - \lambda^{-1}$, $c = q^{-1} - q$. It is convenient to represent them as entries of a 4×4 matrix $R_{12}(\lambda)_{\alpha i}^{\beta j}$:

$$R_{12}(\lambda)_{\alpha i}^{\beta j} =$$



The resulting matrix explicitly reads as

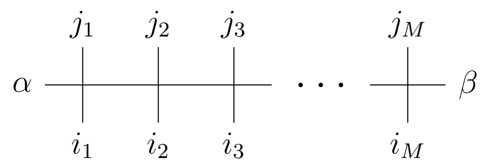
$$\boldsymbol{R}_{12}(\lambda) = \begin{pmatrix} q^{-1}\lambda - q\lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda - \lambda^{-1} & q^{-1} - q & 0 \\ 0 & q^{-1} - q & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & q^{-1}\lambda - q\lambda^{-1} \end{pmatrix} . \tag{4.3}$$

The R-matrix is usually considered for arbitrary complex values of q and λ as an operator acting in $\mathbb{C}^2 \otimes \mathbb{C}^2$. The first factor $\mathbb{C}^2$ in the tensor product is known as the auxiliary space and accounts for the spins lying on the horizontal edges, while the second factor, corresponding to the vertical edges, is called the physical space (see fig. **??**). By multiplying the R-matrices over the auxiliary space, one constructs the monodromy matrix

$$\boldsymbol{M}(\lambda) = \boldsymbol{R}_{12}(\lambda)\boldsymbol{R}_{13}(\lambda) \ldots \boldsymbol{R}_{1N}(\lambda)\,, \tag{4.4}$$

which is an operator in $\mathbb{C}^2 \otimes \mathbb{C}^{2^N}$. Schematically, the monodromy matrix can be represented as

$$M(\lambda)^{\beta\, j_1 j_2 \dots j_M}_{\alpha\, i_1 i_2 \dots i_M} =$$



where the greek indices (α, β) label the auxiliary space, while the latin ones (i_k, j_k) correspond to the physical space.

Imposing periodic boundary conditions and summing over the two possible configurations is equivalent to taking the trace over the auxiliary space. As a result, one has the transfer-matrix

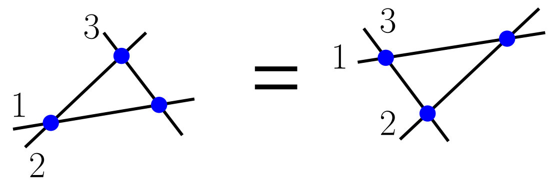
$$\boldsymbol{T}(\lambda) = \mathrm{Tr}\big(\boldsymbol{M}(\lambda)\big)\,. \tag{4.5}$$

Starting from the transfer-matrix for the bottow row, each subsequent row of the lattice is added via matrix multiplication with the transfer-matrix over the physical space. This way, the partition function of the lattice for toroidal boundary conditions is given by

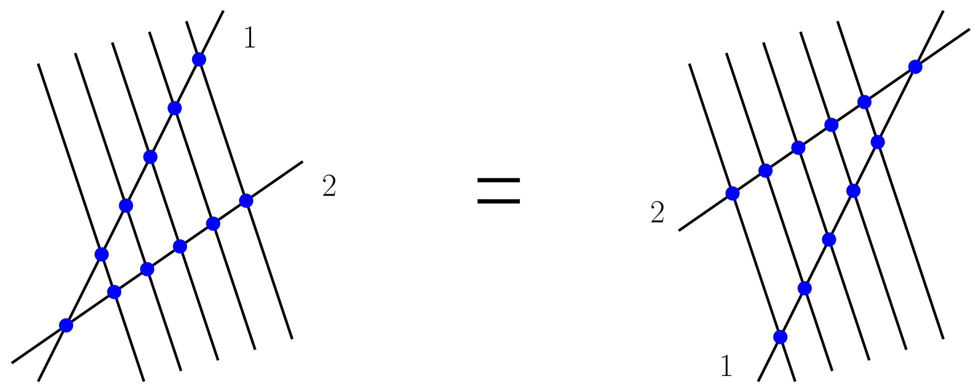
$$\mathcal{Z} = \mathrm{Tr}\big(\boldsymbol{T}^M(\lambda)\big)\ . \tag{4.6}$$

The above formula reduces the computation of the partition function to an analysis of the spectrum of $\boldsymbol{T}(\lambda)$.

The transfer-matrix is a $2^N \times 2^N$ dimensional matrix and the direct computation of its spectrum using numerical methods is impractical even for $N \sim 30$ let alone in the thermodynamic limit with $N \to \infty$. However, the six vertex model contains some underlying algebraic structures that make this calculation possible. The basic building block is the remarkable relation (4.2) satisfied by the R-matrix. In this context, the lower indices label the spaces upon which the R-matrices act, while the arguments $\lambda_{1,2}$ parametrize the Boltzmann weights. Note that q is the same for all three operators in eq. (4.2). A graphical representation of the Yang-Baxter equation is given by the following figure:



By the repeated use of the Yang-Baxter equation one can prove the relation (4.1) for the monodromy, whose graphical representation is:



In eq. (4.1) the matrix $\boldsymbol{M}(\lambda)$ is viewed as a 2×2 matrix in the auxiliary space, whose entries are $2^N \times 2^N$ dimensional matrices acting in the physical space. By taking the trace over the auxiliary space, eq. (4.1) immediately implies that the transfer matrices for different values of the parameter λ form a commuting family. This greatly simplifies the eigenvalue problem for $T(\lambda)$ and enables one to compute the partition function (4.6) in the thermodynamic limit.

4.3 From quantum universal R-matrix to $U(1)$ current algebra realization of the Sklyanin exchange relations

The algebraic structure underlying eq. (4.1) was clarified within the theory of quasi-triangular Hopf algebras by Drinfeld [20]. A basic example is when the rôle of the Hopf algebra is played by $U_q(\widehat{\mathfrak{g}})$ – the quantum deformation of the universal enveloping algebra of the affine algebra [19, 20]. In this case a crucial element is the universal

R-matrix which lies in the tensor product $U_q(\widehat{\mathfrak{g}}) \otimes U_q(\widehat{\mathfrak{g}})$ and satisfies the relation

$$\mathcal{R}^{12}\,\mathcal{R}^{13}\,\mathcal{R}^{23} \;=\; \mathcal{R}^{23}\,\mathcal{R}^{13}\,\mathcal{R}^{12}\ . \tag{4.7}$$

An important feature of $\mathcal{R}$ is that it is decomposed as $\mathcal{R} \in U_q(\widehat{\mathfrak{b}}_+) \otimes U_q(\widehat{\mathfrak{b}}_-)$ where $U_q(\widehat{\mathfrak{b}}_\pm)$ stand for the Borel subalgebras of $U_q(\widehat{\mathfrak{g}})$. If we consider now the evaluation homomorphism of $U_q(\widehat{\mathfrak{g}})$ to the loop algebra $U_q(\mathfrak{g})[\lambda, \lambda^{-1}]$ and specify an N-dimensional matrix representation π of $U_q(\mathfrak{g})$, then

$$\boldsymbol{L}(\lambda) = \big(\pi(\lambda) \otimes 1\big)[\mathcal{R}]\,, \tag{4.8}$$

is a $U_q(\widehat{\mathfrak{b}}_-)$-valued $N \times N$ matrix whose entries depend on an auxiliary parameter λ. In its turn, the formal algebraic relation (4.7) becomes the Yang-Baxter algebra (4.1) with $\boldsymbol{M}$ substituted by $\boldsymbol{L}$ while

$$\boldsymbol{R}(\lambda_2/\lambda_1) = \big(\pi(\lambda_1) \otimes \pi(\lambda_2)\big)[\mathcal{R}]\,.$$

For the purposes of this thesis we take $\mathfrak{g} = \mathfrak{sl}_2$. In this case, the Borel subalgebra $U_q(\widehat{\mathfrak{b}}_+)$ is generated by four elements, $\{y_0, y_1, h_0, h_1\}$ and its evaluation homomorphism is defined by

$$y_0 \mapsto \lambda\, q^{-\frac{\mathtt{h}}{2}}\, \mathtt{e}_+\ , \qquad y_1 \mapsto \lambda\, q^{\frac{\mathtt{h}}{2}}\, \mathtt{e}_-\ , \qquad h_0 \mapsto \mathtt{h}\ , \qquad h_1 \mapsto -\mathtt{h}\ , \tag{4.9}$$

where $\{\mathtt{h}, \mathtt{e}_\pm\}$ are the generators of $U_q(\mathfrak{sl}_2)$, subject to the commutation relations

$$[\mathtt{h},\, \mathtt{e}_\pm] = \pm\, 2\, \mathtt{e}_\pm\ , \qquad [\mathtt{e}_+,\, \mathtt{e}_-] = \frac{q^{\mathtt{h}} - q^{-\mathtt{h}}}{q - q^{-1}}\ . \tag{4.10}$$

Below, with some abuse of notation, we will not distinguish between the formal generators of $U_q(\mathfrak{sl}_2)$ and their matrices in a finite dimensional representation. Explicitly, using the formula for the universal R-matrix given in [87], one can obtain $\boldsymbol{L}(\lambda)$ as a formal series expansion in powers of the spectral parameter λ,[1]

$$
\begin{aligned}
\boldsymbol{L}(\lambda) \;=\; & \Big[1 + \lambda\,(q - q^{-1})\,(x_0\,q^{\frac{\mathtt{h}}{2}}\,\mathsf{e}_+ + x_1\,q^{-\frac{\mathtt{h}}{2}}\,\mathsf{e}_-) + \lambda^2\,\frac{(q-q^{-1})^2}{1+q^2} \\
\times\; & \Big(x_0^2\,(q^{\frac{\mathtt{h}}{2}}\,\mathsf{e}_+)^2 + x_1^2\,(q^{-\frac{\mathtt{h}}{2}}\,\mathsf{e}_-)^2 + \frac{q^2\,x_0x_1 - x_1x_0}{1-q^{-2}}\,(q^{\frac{\mathtt{h}}{2}}\,\mathsf{e}_+)(q^{-\frac{\mathtt{h}}{2}}\,\mathsf{e}_-) \\
+\; & \frac{q^2\,x_1x_0 - x_0x_1}{1-q^{-2}}\,(q^{-\frac{\mathtt{h}}{2}}\,\mathsf{e}_-)(q^{\frac{\mathtt{h}}{2}}\,\mathsf{e}_+)\Big) + \ldots\Big]\,q^{-\frac{1}{2}\mathtt{h}\,h_0}\,. \qquad (4.11)
\end{aligned}
$$

The expression in the square brackets contains only the generators $x_0, x_1 \in U_q(\widehat{\mathfrak{b}}_-)$ satisfying the Serre relations

$$
x_i^3x_j - [3]_q\,x_i^2x_jx_i + [3]_q\,x_ix_jx_i^2 - x_jx_i^3 = 0 \qquad (i,j = 0,1)\,, \qquad (4.12)
$$

where $[n]_q \equiv (q^n - q^{-n})/(q - q^{-1})$. Note that the two remaining generators h_0, h_1, which obey

$$
[h_0, x_0] = -[h_1, x_0] = -2x_0\,, \qquad [h_0, x_1] = -[h_1, x_1] = 2x_1\,, \qquad [h_0, h_1] = 0\,, \qquad (4.13)
$$

appear only in an overall factor multiplying the square bracket $[\ldots]$ in (4.11). In fact, since $h_0 + h_1$ is a central element, for our purposes and without loss of generality we have set it to be zero.

Until this point there was no need to specify a representation of $U_q(\widehat{\mathfrak{b}}_-)$ – the Yang-Baxter relation (4.1) is satisfied identically provided (4.12), (4.13) hold true. In ref.[52], a representation of $U_q(\widehat{\mathfrak{b}}_-)$ was considered in the (extended) Fock space of a single bosonic field. The Borel generators x_0, x_1 were given by the integral expressions

$$
x_0 = \frac{1}{q - q^{-1}} \int_0^R \mathrm{d}z\,V^+(z)\,, \qquad x_1 = \frac{1}{q - q^{-1}} \int_0^R \mathrm{d}z\,V^-(z)\,. \qquad (4.14)
$$

Here the vertex operators

$$
V^{\pm}(z) = \mathrm{e}^{\mp 2\mathrm{i}\beta\varphi}(z)
$$

[1] In fact, eq. (4.11) follows from an expression of the R-matrix which is equivalent to the one in [87] (and used in [52]) upon the substitution $q \mapsto q^{-1}$ (see eq. (4.18)). This is to keep with the conventions of [58].

are built from the bosonic field

$$\varphi(z) = \varphi_0 + \frac{2\pi z}{R}\,\hat{p} + \mathrm{i} \sum_{n\neq 0} \frac{a_n}{n}\,\mathrm{e}^{-\frac{2\pi \mathrm{i} n}{R} z} \tag{4.15}$$

whose Fourier coefficients satisfy the commutations relations of the Heisenberg algebra

$$[a_n, a_m] = \tfrac{n}{2}\,\delta_{n+m,0}\ , \qquad [\varphi_0, \hat{p}] = \tfrac{\mathrm{i}}{2}\ . \tag{4.16}$$

The remaining generator $h_0 = -h_1$ coincides with the zero mode momentum $\hat{p}$ up to a simple factor:

$$h_0 = \frac{2}{\beta}\,\hat{p}\ . \tag{4.17}$$

The parameter β appearing in the above formulae is related to the deformation parameter q as

$$q = \mathrm{e}^{-\mathrm{i}\pi\beta^2}\ . \tag{4.18}$$

Defining the Fock space $\mathcal{F}_p$ as the highest weight module of the Heisenberg algebra with highest weight vector $|p\rangle$: $\hat{p}\,|p\rangle = p\,|p\rangle$, it easy to see that the generators (4.14) act as

$$x_0 :\quad \mathcal{F}_p \mapsto \mathcal{F}_{p-\beta}\ , \qquad\qquad x_1 :\quad \mathcal{F}_p \mapsto \mathcal{F}_{p+\beta}$$

and hence that the matrix elements of $\boldsymbol{L}(\lambda)$ (4.11) are operators in the extended Fock space $\oplus_{n=-\infty}^{\infty}\mathcal{F}_{p+n\beta}$.

It was observed in [52] that using the commutation relations,

$$V^{\sigma_1}(z_1)\,V^{\sigma_2}(z_2) = q^{2\sigma_1\sigma_2}\,V^{\sigma_2}(z_2)\,V^{\sigma_1}(z_1)\,, \qquad z_2 > z_1 \qquad (\sigma_{1,2} = \pm) \tag{4.19}$$

the monomials built from the generators x_0, x_1 can be expressed in terms of the ordered integrals

$$J(\sigma_1, \ldots, \sigma_m) = \int\limits_{R > z_1 > z_2 > \ldots > z_m > 0} \mathrm{d}z_1 \ldots \mathrm{d}z_m\, V^{\sigma_1}(z_1) \ldots V^{\sigma_m}(z_m)\ , \tag{4.20}$$

which yields the following expression for $\boldsymbol{L}(\lambda)$

$$\boldsymbol{L}(\lambda) = \sum_{m=0}^{\infty} \lambda^m \sum_{\sigma_1 \ldots \sigma_m = \pm} \big(q^{\frac{\mathtt{h}}{2}\sigma_1} \mathtt{e}_{\sigma_1}\big) \ldots \big(q^{\frac{\mathtt{h}}{2}\sigma_m} \mathtt{e}_{\sigma_m}\big)\ J(\sigma_1, \ldots, \sigma_m)\ \mathrm{e}^{\mathrm{i}\pi\beta\,\hat{p}\,\mathtt{h}}\ . \tag{4.21}$$

The latter is recognized as the path ordered exponent

$$\boldsymbol{L}(\lambda) = \overleftarrow{\mathcal{P}}\exp\left(\lambda \int_0^R \mathrm{d}z\ \Big(V^+\, q^{\frac{\mathtt{h}}{2}}\, \mathtt{e}_+ + V^-\, q^{-\frac{\mathtt{h}}{2}}\, \mathtt{e}_-\Big)\right) \mathrm{e}^{\mathrm{i}\pi\beta\,\hat{p}\,\mathtt{h}}\ . \tag{4.22}$$

It should be emphasized that since the OPE of the vertex operators is singular,

$$V^{\pm}(z_2)\, V^{\mp}(z_1)\big|_{z_2 \to z_1 + 0} \sim (z_2 - z_1)^{-2\beta^2}\ ,$$

the ordered integrals are well defined only for $0 < \beta^2 < \frac{1}{2}$. However, each term in the formal series expansion (4.11), being expressed in terms of the basic contour integrals x_0, x_1, is well defined for all values of β except the cases when $\beta^2 = 1 - \frac{1}{2n}$ with $n = 1, 2, 3, \ldots$. In fact, the series expansion (4.11) can be thought of as an analytic regularization of the divergent path-ordered exponent (4.22) within the domain $\frac{1}{2} < \beta^2 < 1$.

Let's consider the classical limit where $\beta \to 0$ so that the deformation parameter q tends to one. The commutation relations (4.10) turn into

$$[\mathtt{h}, \mathtt{e}_{\pm}] = \pm 2\mathtt{e}_{\pm}\ , \qquad [\mathtt{e}_+, \mathtt{e}_-] = \mathtt{h}\ , \tag{4.23}$$

while $\phi \equiv \beta\,\varphi$ becomes a classical quasiperiodic field,

$$\phi(R) - \phi(0) = 2\pi P\ , \tag{4.24}$$

satisfying the Poisson bracket relations

$$\big\{\phi(z_1), \phi(z_2)\big\} = -\tfrac{1}{4}\, \epsilon(z_1 - z_2) \tag{4.25}$$

with $\epsilon(z) = 2m + 1$ for $mR < z < (m+1)R$ $(m \in \mathbb{Z})$. Since for small β there is no convergence issue the $\beta \to 0$ limit of (4.22) is straightforward, yielding the classical path-ordered exponent of the form

$$\boldsymbol{L}_{\mathrm{cl}}(\lambda) = \overleftarrow{\mathcal{P}}\exp\left(\lambda \int_0^R \mathrm{d}z\ \big(\mathrm{e}^{-2\mathrm{i}\phi}\ \mathtt{e}_+ + \mathrm{e}^{2\mathrm{i}\phi}\ \mathtt{e}_-\big)\right) \mathrm{e}^{\mathrm{i}\pi P\,\mathtt{h}}\ . \tag{4.26}$$

Here, abusing notation for the sake of readability, we denote the classical counterparts to the quantum operators by the same symbols, in particular, $\mathsf{e}_\pm$ now fulfill relations (4.23) and ϕ is the classical field satisfying (4.24), (4.25). Next we will show how $\boldsymbol{L}_{\rm cl}(\lambda)$ is related to the monodromy matrix for the classically integrable mKdV hierarchy.

4.3.1 Relation to the mKdV equation

The KdV equation was originally proposed to describe waves propagating in shallow water. Since then it has become the archetype of a classically integrable PDE, exhibiting many of their characteristic features such as solitons solutions. The KdV equation takes the form

$$\partial_t u - 6\, u\, \partial_z u + \partial_z^3 u = 0 \ . \tag{4.27}$$

In fact, for our purposes, we will be considering a closely related PDE known as the modified KdV equation

$$\partial_t j - 6 j^2\, \partial_z j + \partial_z^3 j = 0 \ . \tag{4.28}$$

The latter can be obtained from eq. (4.27) through the Miura transform [88]

$$u = j^2 + \partial_z\, j \ . \tag{4.29}$$

It turns out that the mKdV equation can be expressed in the Hamiltonian form

$$\partial_t\, j = \{H, j\}\, . \tag{4.30}$$

with

$$H = \int \mathrm{d}z \left(j^4 + \left(\partial_z\, j\right)^2 \right) , \tag{4.31}$$

while the Poisson structure is defined by

$$\{j(z_1),\, j(z_2)\} = -\delta'(z_1 - z_2)\, . \tag{4.32}$$

The components of the flat connection for the mKdV equation are given explicitly by [43]

$$\begin{aligned}\boldsymbol{A}_z &= j\,\mathtt{h} + \lambda\,(\mathtt{e}_+ + \mathtt{e}_-) \\ \boldsymbol{A}_t &= \big(2j^3 - \partial_z^2 j - 4\lambda^2\,j\big)\,\mathtt{h} + 2\lambda\,\big(j^2 + \partial_z\,j - 2\lambda^2\big)\,\mathtt{e}_+ + 2\lambda\,\big(j^2 - \partial_z\,j - 2\lambda^2\big)\,\mathtt{e}_-\end{aligned} \tag{4.33}$$

A simple computation yields that the monodromy matrix is related to the path ordered exponent from eq. (4.26) as

$$\boldsymbol{L}_{\mathrm{cl}}(\lambda)\,\mathrm{e}^{\mathrm{i}\pi P\,\mathtt{h}} = \boldsymbol{\Omega}^{-1}\,\left[\overleftarrow{\mathcal{P}}\exp\left(\int_0^R \mathrm{d}z\,\boldsymbol{A}_z(z|\lambda)\right)\right]\,\boldsymbol{\Omega} \tag{4.34}$$

As it follows from eqs.(4.32) (4.33), the connection is a non-ultralocal one so that the computation of the Poisson brackets for the monodromy is inevitably met with ambiguities in treating the contact terms. Nonetheless, the classical limit of the Yang-Baxter algebra (4.1) unambiguously yields that (3.27) is satisfied with $\boldsymbol{M}(\lambda)$ substituted by $\boldsymbol{L}_{\mathrm{cl}}(\lambda)$ from (4.26), while $\boldsymbol{r}(\lambda) = \boldsymbol{r}_-(\lambda)$, where

$$\boldsymbol{r}_-(\lambda) = -\frac{1}{\lambda - \lambda^{-1}}\,\Big(\mathtt{e}_+ \otimes \mathtt{e}_- + \mathtt{e}_- \otimes \mathtt{e}_+ + \tfrac{1}{4}(\lambda + \lambda^{-1})\,\mathtt{h}\otimes\mathtt{h}\Big)\,. \tag{4.35}$$

Thus we see that starting from an explicit realization of the quantum algebra (4.1) and taking the classical limit is a clear-cut way of obtaining the monodromy matrix satisfying the Sklyanin exchange relations for a non-ultralocal flat connection.

4.4 From quantum universal R-matrix to $SU(2)$ current algebra realization of the Sklyanin exchange relations

It is known [89, 90] that the Borel subalgebra $U_q(\widehat{\mathfrak{b}}_-) \subset U_q(\widehat{\mathfrak{sl}}_2)$ admits a realization with x_0 and x_1 given by (4.14), where the vertices $V^{\pm}$ are built from three bosonic fields φ_1, φ_2, φ_3:

$$V^{\pm} = \frac{1}{2b^2} \big(\mathrm{i}b\, \partial\varphi_1 + \alpha_2\, \partial\varphi_2 \pm \alpha_1\, \partial\varphi_3\big) \mathrm{e}^{\pm\frac{\varphi_1}{b}} \ . \tag{4.36}$$

The expansion coefficients of φ_i, defined by the formula similar to (4.15), generate three independent copies of the Heisenberg algebra (4.16). The relation (4.17) is replaced now by

$$h_0 = -h_1 = -4\mathrm{i}b\, \hat{p}_3 \ , \tag{4.37}$$

where $\hat{p}_3$ is the zero mode momentum of the field φ_3. It should be highlighted that the parameters α_1, α_2, b appearing in eq. (4.36) are subject to the constraint

$$\alpha_1^2 + \alpha_2^2 - b^2 = \tfrac{1}{2} \tag{4.38}$$

and b is related to the deformation parameter q as

$$q = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}} \qquad \text{with} \qquad \hbar = \frac{\pi}{2b^2} \ . \tag{4.39}$$

The set of generators $\{x_0, x_1, h_0, h_1\}$ defined by $(4.14), (4.36), (4.37)$ fulfill the Serre and commutation relations $(4.12), (4.13)$. In consequence, $\boldsymbol{L}(\lambda)$ (4.8) derived from the universal R-matrix by taking this realization of $U_q(\widehat{\mathfrak{b}}_-)$ satisfies the Yang-Baxter algebra (4.1). The formal power series expansion in λ (4.11) is still applicable however eq. (4.21), which expresses $\boldsymbol{L}(\lambda)$ in terms of the ordered integrals, turns out to be problematic because of an issue with convergence. Indeed, the OPE

$$V^{\sigma_2}(z_2)\, V^{\sigma_1}(z_1) \sim (z_2 - z_1)^{-2-\sigma_1\sigma_2/(2b^2)} \qquad\qquad (\sigma_{1,2} = \pm)$$

is more singular now and the ordered integrals (4.20) in general diverge. Thus the path ordered exponent expression for $\boldsymbol{L}(\lambda)$ (4.22) that was obtained from recasting the contour integrals into the ordered integrals using the commutation relations (4.19) (which are still valid) is ill defined. When taking the classical limit $b \to \infty$ it is essential to keep this in mind.

To study the classical limit, it is convenient to work with $\phi_i \equiv \varphi_i/(2b)$ which become classical quasi-periodic fields

$$\phi_i(R) - \phi_i(0) = 2\pi P_i \qquad (i = 1, 2, 3) \tag{4.40}$$

satisfying equations similar to (4.25). As it follows from (4.14), (4.36), (4.38) the classical counterparts of x_0 and x_1 are given by

$$\chi_0 = \lim_{b\to\infty} (q - q^{-1})\, x_0 = \int_0^R \mathrm{d}z\, V_{\mathrm{cl}}^{+}(z) \;, \qquad \chi_1 = \lim_{b\to\infty} (q - q^{-1})\, x_1 = \int_0^R \mathrm{d}z\, V_{\mathrm{cl}}^{-}(z) \;, \tag{4.41}$$

where

$$V_{\mathrm{cl}}^{\pm} = \left(\mathrm{i}\, \partial\phi_1 + \tfrac{1}{\sqrt{1+\nu^2}}\, \partial\phi_2 \pm \tfrac{\nu}{\sqrt{1+\nu^2}}\, \partial\phi_3\right) \mathrm{e}^{\pm 2\phi_1} \tag{4.42}$$

and

$$\nu \equiv \lim_{b\to\infty} \alpha_1/\alpha_2 \,.$$

Since the expression (4.11) for $\boldsymbol{L}(\lambda)$ does not have problems with convergence, we will use it for taking the classical limit. Each term in the series (4.11) is a polynomial w.r.t. the non-commutative variables x_0 and x_1 with coefficients depending on the deformation parameter q. To take the $\hbar \to 0$ limit one should expand q (4.39) for small $\hbar$, express the result in terms of commutators and then replace the commutators with Poisson brackets using the correspondence principle $[\,.\,,\,.\,] \mapsto \mathrm{i}\hbar\, \{\,.\,,\,.\,\}$. It is easy to see that with this procedure the first few terms shown in (4.11) become

$$\lim_{\hbar\to 0} \boldsymbol{L}(\lambda) = \Big[1 + \lambda\, (\chi_0\, \mathtt{e}_+ + \chi_1\, \mathtt{e}_-\,) + \tfrac{1}{2}\, \lambda^2 \times \tag{4.43}$$

$$\Big(\chi_0^2\; \mathtt{e}_+^2 + \chi_1^2\; \mathtt{e}_-^2 + \big(\chi_0\chi_1 + \{\chi_0, \chi_1\}\big)\; \mathtt{e}_+\mathtt{e}_- + \big(\chi_0\chi_1 + \{\chi_1, \chi_0\}\big)\; \mathtt{e}_-\mathtt{e}_+\Big) + \ldots\; \Big] \mathrm{e}^{-\pi P_3\, \mathtt{h}}$$

where $\mathtt{h}, \mathtt{e}_\pm$ satisfy the commutation relations of the $\mathfrak{sl}_2$ algebra (4.23).

The calculation for higher order coefficients quickly becomes cumbersome. For example, the formal expansion of $\mathcal{R}\, q^{\frac{h_0\otimes h_0}{2}} \in U_q(\widehat{\mathfrak{b}}_+)\otimes U_q(\widehat{\mathfrak{b}}_-)$ contains the term $y_1 y_0^2 y_1 \otimes P_4^{(1001)}(x_0,x_1)$ with

$$\begin{aligned} P_4^{(1001)}(x_0,x_1) &= \frac{q^6(q-q^{-1})^2}{[4]_q\,[2]_q}\Big(x_0^2x_1^2-[3]_q\,x_0x_1x_0x_1+x_0x_1^2x_0+[3]_q\,x_1x_0^2x_1 \\ &- [3]_q\,x_1x_0x_1x_0+x_1^2x_0^2\Big) \end{aligned}$$

which makes a fourth order contribution to the series (4.11) once the evaluation homomorphism (4.9) of y_0, y_1 is taken. Expanding q for small $\hbar$ in $P_4^{(1001)}(x_0,x_1)$ yields

$$\begin{aligned} P_4^{(1001)}(x_0,x_1) &= -\tfrac{1}{8}\,\hbar^2\Big(1+O(\hbar)\Big)\times\Big([x_0,[x_0,x_1]]x_1+x_1[x_0,[x_0,x_1]]-[x_0,x_1]^2 \\ &+ \hbar^2\big(x_0x_1x_0x_1+x_1x_0x_1x_0-x_1x_0^2x_1\big)+O(\hbar^4)\Big)\,. \end{aligned}$$

Now, replacing x_0, x_1 by their classical counterparts (4.41), using the correspondence principle and taking the limit $\hbar\to 0$ gives

$$\lim_{\hbar\to 0} P_4^{(1001)}(x_0,x_1)=\tfrac{1}{8}\left(2\chi_1\{\chi_0,\{\chi_0,\chi_1\}\}-\{\chi_0,\chi_1\}^2+\chi_0^2\chi_1^2\right).$$

For the full contribution to the fourth order of (4.43) one should take into account all sixteen polynomials $P_4^{(i_1i_2i_3i_4)}(x_0,x_1)$ with $i_1,i_2,i_3,i_4=0,1$ corresponding to the terms $y_{i_1}y_{i_2}y_{i_3}y_{i_4}\otimes P_4^{(i_1i_2i_3i_4)}(x_0,x_1)$ in the expansion of the universal R-matrix.

Our calculations to fifth order in λ support the existence of the limit

$$\lim_{\hbar\to 0}\boldsymbol{L}=\boldsymbol{L}_{\rm cl}\,. \tag{4.44}$$

By construction, $\boldsymbol{L}_{\rm cl}$ is a formal series expansion in λ whose coefficients are built from χ_0, χ_1 and their Poisson brackets.[2] To proceed further, the latter need to be

[2]Note that the elements χ_0 and χ_1 satisfy the classical analogs of the Serre relations (4.12),

$$\{\chi_i,\,\{\chi_i,\,\{\chi_i,\,\chi_j\}\}\}=\chi_i^2\,\{\chi_i,\,\chi_j\}\qquad (i,j=0,1)\,.$$

computed explicitly. This can be carried out along the following lines. Starting from the relations

$$\big\{\phi_i(z_1),\, \phi_j(z_2)\big\} = -\tfrac{1}{4}\, \delta_{ij}\, \epsilon(z_1 - z_2) \tag{4.45}$$

it is easy to show that $V^{\pm}_{\rm cl}$ (4.42) and

$$V^0_{\rm cl} \,=\, -2\,\big(\tfrac{1}{\sqrt{1+\nu^2}}\ \partial\phi_1 - {\rm i}\,\partial\phi_2\big) \tag{4.46}$$

form a closed Poisson algebra

$$\begin{aligned}
&\{V^0_{\rm cl}(z_1),\, V^0_{\rm cl}(z_2)\} = -\frac{2\nu^2}{1+\nu^2}\ \delta'(z_1 - z_2)\\
&\{V^0_{\rm cl}(z_1),\, V^{\pm}_{\rm cl}(z_2)\} = \pm\,\frac{2}{\sqrt{1+\nu^2}}\ V^{\pm}_{\rm cl}(z_1)\,\delta(z_1 - z_2)\\
&\{V^{+}_{\rm cl}(z_1),\, V^{-}_{\rm cl}(z_2)\} = -\frac{\nu^2}{1+\nu^2}\,\delta'(z_1 - z_2) + \frac{V^0_{\rm cl}(z_1)}{\sqrt{1+\nu^2}}\ \delta(z_1 - z_2) + V^{+}_{\rm cl}(z_1)\,V^{-}_{\rm cl}(z_2)\,\epsilon(z_1 - z_2)\\
&\{V^{\pm}_{\rm cl}(z_1),\, V^{\pm}_{\rm cl}(z_2)\} = -V^{\pm}_{\rm cl}(z_1)\,V^{\pm}_{\rm cl}(z_2)\,\epsilon(z_1 - z_2)\ .
\end{aligned} \tag{4.47}$$

Recall that χ_0 and χ_1 are given by integrals over the classical vertices (4.41) so that these relations are sufficient for the explicit calculation of any of the Poisson brackets occurring in the r.h.s of (4.43). However, due to the presence of the derivative of the δ-function in (4.47), ambiguous integrals occur in the computations. For instance: $\{\chi_0, \chi_1\} = c_1\, \nu^2/(1+\nu^2) + \ldots$ with

$$c_1 = -\int_0^R {\rm d}z_1\, {\rm d}z_2\, \delta'(z_1 - z_2) = \int_0^R {\rm d}z\, \big(\delta(z-R) - \delta(z)\big)\,. \tag{4.48}$$

In general, one is faced with many other sorts of integrals involving $\delta'(z_1 - z_2)$. However, they are not all independent and their number can be reduced if, before performing explicit calculations, one uses the Jacobi identity and skew-symmetry to bring the Poisson brackets to the form

$$\{\chi_{\sigma_1}, \{\chi_{\sigma_2}, \{\chi_{\sigma_3}, \{\ldots, \{\chi_{\sigma_{m-1}}, \chi_{\sigma_m}\}\ldots\} \qquad (\sigma_1, \ldots, \sigma_m = 0, 1) \tag{4.49}$$

(e.g., $\{\{\chi_0, \chi_1\}, \{\chi_0, \chi_1\}\} = \{\chi_0, \{\chi_1, \{\chi_1, \chi_0\}\}\} + \{\chi_1, \{\chi_0, \{\chi_0, \chi_1\}\}\}$). This way, in our fifth order computations we were met with only two more types of ambiguous integrals. The first is of the form

$$I_1 \;=\; \int_0^R \mathrm{d}z_1 \ldots \mathrm{d}z_4 \, \delta'(z_1 - z_3) \, \epsilon(z_2 - z_3) \, \epsilon(z_3 - z_4) \; F(z_2) \, G(z_4) \; ,$$

where F and G are some functions. Formal integration by parts w.r.t. z_3 yields

$$I_1 = c_1 \int_0^R \mathrm{d}z_1 \, \mathrm{d}z_2 \, F(z_1) \, G(z_2)$$

with c_1 as in (4.48). The other ambiguous integral is

$$I_2 \;=\; \int_0^R \mathrm{d}z_1 \mathrm{d}z_2 \mathrm{d}z_3 \, F(z_2) \, \epsilon(z_2 - z_3) \, \delta'(z_1 - z_3) \, .$$

In this case, integration by parts leads to

$$I_2 = 2\,(c_2 - 1) \int_0^R \mathrm{d}z \, F(z) \qquad \text{with} \qquad c_2 = \frac{1}{2} \int_0^R \mathrm{d}z \, \big(\delta(z - R) + \delta(z)\big) \; . \tag{4.50}$$

We explicitly computed the expansion of $\boldsymbol{L}_{\mathrm{cl}}$ to fifth order and found that all the ambiguities are absorbed in the two constants c_1 and c_2 (4.48), (4.50). Furthermore, if $c_1 = 0$ and c_2 is arbitrary, the series can be collected into a path-ordered exponent

$$\boldsymbol{L}_{\mathrm{cl}} = \overleftarrow{\mathcal{P}} \exp\left(\int_0^R \mathrm{d}z \, \boldsymbol{B} \right) \mathrm{e}^{-\pi P_1 \, \mathtt{h}} \tag{4.51}$$

with

$$\boldsymbol{B} \;=\; f \left(V_{\mathrm{cl}}^{+}(z) \; \mathtt{e}_{+} + V_{\mathrm{cl}}^{-}(z) \; \mathtt{e}_{-} \right) \; + \; \tfrac{1}{2} \, g \, V_{\mathrm{cl}}^{0}(z) \, \mathtt{h} \tag{4.52}$$

and

$$f \;=\; \lambda \sqrt{1 + \nu^2} \, \Big(1 + (1 + \nu^2 \, (c_2 - 1)) \, \lambda^2 + (1 + 4\nu^2 (c_2 - 1) + 2\nu^4 (c_2 - 1)^2) \, \lambda^4 + O(\lambda^6) \Big)$$

$$g \;=\; \lambda^2 \sqrt{1 + \nu^2} \, \Big(1 + (2\nu^2 \, (c_2 - 1) + 1) \, \lambda^2 + O(\lambda^4) \Big) \; .$$

That c_1 (4.48) vanishes seems to be a natural requirement as, in the problem at hand, the δ-function should be understood as the formal series $\frac{1}{R} \sum_{m=-\infty}^{\infty} \mathrm{e}^{\frac{2\pi i m}{R} z}$ and hence

$\delta(z-R) = \delta(z)$. Note that for the periodic δ-function the constant c_2 in (4.50) becomes

$$c_2 = \int_0^R \mathrm{d}z\, \delta(z)\,. \tag{4.53}$$

Unfortunately there is no proof that the limit (4.44) exists and can be represented by eq. (4.51) and (4.52) with some functions f and g – this has been checked perturbatively to fifth order only. However, if this is accepted as a conjecture then f and g should have the form

$$f = \frac{\rho\sqrt{1+\nu^2}}{1-\rho^2}\,, \qquad\qquad g = \frac{\rho^2\sqrt{1+\nu^2}}{1-\rho^2}\,, \tag{4.54}$$

where $\rho = \rho(\lambda)$ solves the equation

$$\lambda = \frac{\rho\,(1-\rho^2)}{1-(1+(1-c_2)\,\nu^2)\,\rho^2}\,. \tag{4.55}$$

This follows from an analysis of the simplest matrix element of $\boldsymbol{L}_{\rm cl}$ for which the series (4.43) can be obtained to all orders in λ.

To summarize, we expect that the limit (4.44) exists and results in (4.51), where $\boldsymbol{B}$ is given by

$$\boldsymbol{B}(z|\rho) = \frac{\sqrt{1+\nu^2}}{1-\rho^2}\,\Big(\rho\left(V_{\rm cl}^{+}(z)\ \mathtt{e}_+ + V_{\rm cl}^{-}(z)\ \mathtt{e}_-\right) + \tfrac{1}{2}\,\rho^2\,V_{\rm cl}^{0}(z)\ \mathtt{h}\Big) \tag{4.56}$$

and with $\rho = \rho(\lambda)$ defined through the relation (4.55). By construction $\boldsymbol{L}_{\rm cl}$ must satisfy the classical Yang-Baxter Poisson algebra,

$$\big\{\boldsymbol{L}_{\rm cl}(\rho_1) \underset{,}{\otimes} \boldsymbol{L}_{\rm cl}(\rho_2)\big\} = \big[\boldsymbol{L}_{\rm cl}(\rho_1)\otimes\boldsymbol{L}_{\rm cl}(\rho_2), \boldsymbol{r}(\lambda_1/\lambda_2)\big] \tag{4.57}$$

with $\rho_{1,2} = \rho(\lambda_{1,2})$ and[3]

$$\boldsymbol{r}(\lambda) = +\frac{1}{\lambda-\lambda^{-1}}\,\Big(\mathtt{e}_+\otimes\mathtt{e}_- + \mathtt{e}_-\otimes\mathtt{e}_+ + \tfrac{1}{4}(\lambda+\lambda^{-1})\,\mathtt{h}\otimes\mathtt{h}\Big)\,. \tag{4.58}$$

[3] Note that here the classical r-matrix differs from the one in (4.35) by an overall sign.

Eq. (4.47) implies that the Poisson brackets of $\boldsymbol{B}$ (4.52) are not local in the sense that apart from the δ-function and its derivative they contain terms with the ϵ-function. Nevertheless, a simple calculation shows that the Lie algebra valued 1-form $\boldsymbol{B}(z|\rho)$ is gauge equivalent to

$$\boldsymbol{A}(z|\rho) = \frac{\rho\sqrt{1+\nu^2}}{1-\rho^2}\big(j^+(z)\ \mathtt{e}_+ + j^-(z)\ \mathtt{e}_-\big) + \frac{1}{2}\left(\frac{\rho^2\sqrt{1+\nu^2}}{1-\rho^2} + \xi\right) j^0(z)\,\mathtt{h} \tag{4.59}$$

and the fields

$$\begin{aligned} j^\pm &= \big(\mathrm{i}\,\partial\phi_1 + \tfrac{1}{\sqrt{1+\nu^2}}\ \partial\phi_2 \pm \tfrac{\nu}{\sqrt{1+\nu^2}}\ \partial\phi_3\big)\ \mathrm{e}^{\pm 2\xi(\phi_1+\mathrm{i}\phi_2)} \\ j^0 &= -2\,\big(\tfrac{1}{\sqrt{1+\nu^2}}\,\partial\phi_1 - \mathrm{i}\,\partial\phi_2\big) \end{aligned}$$

satisfy the classical current algebra

$$\begin{aligned} \big\{j^+(z_1), j^-(z_2)\big\} &= -\frac{\nu^2}{1+\nu^2}\ \delta'(z_1-z_2) + j^0(z_1)\,\delta(z_1-z_2) \\ \big\{j^0(z_1), j^\pm(z_2)\big\} &= \pm 2\,j^\pm(z_1)\ \delta(z_1-z_2) \\ \big\{j^0(z_1), j^0(z_2)\big\} &= -\frac{2\nu^2}{1+\nu^2}\ \delta'(z_1-z_2) \\ \big\{j^\pm(z_1), j^\pm(z_2)\big\} &= 0\ . \end{aligned} \tag{4.60}$$

The constant ξ in the above formulae is given by

$$\xi = \frac{\sqrt{1+\nu^2}}{1+\sqrt{1+\nu^2}}\ .$$

It follows from eq. (4.60) that the ϵ-function is not present in the Poisson brackets of $\boldsymbol{A}$ (4.59) so they are local, although not ultralocal. In terms of the 1-form $\boldsymbol{A}$, eq.(4.51) can be re-written as

$$\boldsymbol{L}_{\mathrm{cl}}(\rho)\,\mathrm{e}^{((2\xi-1)P_1+2\mathrm{i}\xi P_2)\pi\mathtt{h}} = \boldsymbol{\Omega}^{-1}\left[\overleftarrow{\mathcal{P}}\exp\left(\int_0^R \mathrm{d}z\,\boldsymbol{A}\big(z|\rho\big)\right)\right]\boldsymbol{\Omega}\ , \tag{4.61}$$

where $\boldsymbol{\Omega} = \exp\big((\xi-1)\,\phi_3(R)\,\mathtt{h} + \mathrm{i}\,\xi\,\phi_2(R)\,\mathtt{h}\big)$ and P_i are defined by eq. (4.40). The r.h.s. of (4.61) is the monodromy matrix for the linear problem

$$\big(\partial_z - \boldsymbol{A}\big)\boldsymbol{\Psi}(z) = 0 \tag{4.62}$$

with $\boldsymbol{A}$ given by (4.59) and ρ playing the rôle of the auxiliary spectral parameter.

Despite that the Poisson brackets of the 1-form $\boldsymbol{A}$ are non-ultralocal for $\nu \neq 0$, $\boldsymbol{L}_{\mathrm{cl}}(\rho)$ in (4.61) obeys the Sklyanin exchange relations (4.57). The δ'-terms introduce an ambiguity in taking the classical limit which is manifest in the arbitrary constant c_2 (4.53). The effect of this is observed in the finite renormalization of the spectral parameter $\lambda \mapsto \rho(\lambda)$ (4.55). Notice that for the ultralocal case, i.e., $\nu = 0$, the dependence on c_2 drops out and $\rho = \lambda$.

4.4.1 The case $\nu = 0$: monodromy matrix for the cigar NLSM

For $\nu = 0$ the monodromy matrix $\boldsymbol{L}_{\mathrm{cl}}$ takes the form

$$\boldsymbol{L}_j(\rho) = \overleftarrow{\mathcal{P}} \exp\left(\frac{\mathrm{i}\rho}{1-\rho^2} \int_{t_0}^{t_0+R} \mathrm{d}z \Big(V^+ \mathtt{e}_+ + V^- \mathtt{e}_- + \mathrm{i}\rho\,(\partial\phi_1 - \mathrm{i}\,\partial\phi_2)\,\mathtt{h}\Big)\right) \mathrm{e}^{-\pi P_1 \mathtt{h}}\,, \quad (4.63)$$

where

$$V^\pm = (\partial\phi_1 - \mathrm{i}\,\partial\phi_2)\,\mathrm{e}^{\pm 2\phi_1} \quad (4.64)$$

and

$$P_1 = \frac{1}{2\pi}\,\big(\,\phi_1(t_0+R) - \phi_1(t_0)\,\big)\,. \quad (4.65)$$

Below we will discuss how this operator is related to the mondromy for the cigar NLSM.

The cigar NLSM was touched upon in section 2.4.4. Its Lagrangian is obtained from the sausage one (see (2.28)) by shifting the field $\phi \mapsto \phi + \frac{1}{2}\log\left(\frac{1+\kappa}{1-\kappa}\right)$ and then taking the limit $\kappa \to 1$. This yields that

$$\mathcal{L} = \frac{1}{2\,(1+\mathrm{e}^{2\phi})}\,\big(\partial_\mu\phi\,\partial^\mu\phi + \partial_\mu w\,\partial^\mu w\big)\,. \quad (4.66)$$

Here we consider the theory with twisted boundary conditions corresponding to (3.17) for the general Klimcik model. For the cigar, these conditions become

$$\phi(t_0, x+R) = \phi(t_0, x)\,, \qquad w(t_0, x+R) = w(t_0, x) + 2\pi k\,, \quad (4.67)$$

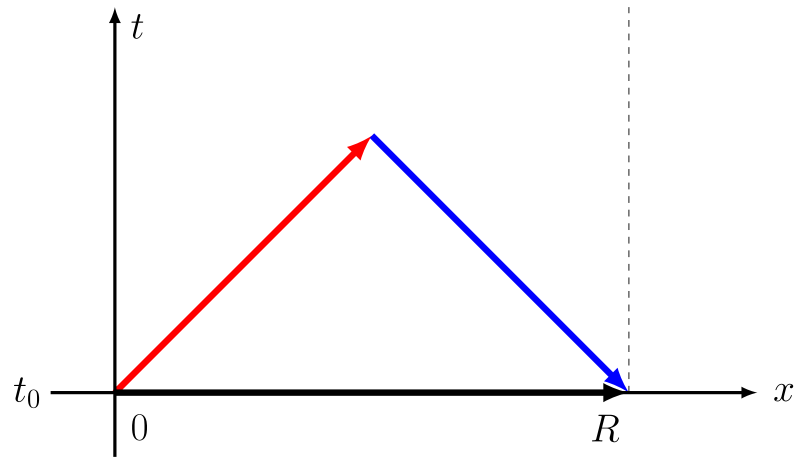


Figure 4.3: The integration contour along the time slice $t = t_0$ (black arrow) in eq (4.68) can be replaced by an integration contour along the characteristics: $x_- = t_0$ with $0 < x_+ < t_0 + R$ (red arrow) and $x_+ = t_0 + R$ with $t_0 < x_- < t_0 - R$ (blue arrow).

where k is the twist parameter such that $-\frac{1}{2} < k \leq \frac{1}{2}$.

To obtain the monodromy matrix for the cigar, we will start with the sausage one and take the limit $\kappa \to 1$. Recall that the sausage monodromy matrix is defined as

$$\boldsymbol{M}(\rho) = \overleftarrow{\mathcal{P}} \exp \int_0^R \mathrm{d}x\, \boldsymbol{A}_x(t_0, x)\,, \tag{4.68}$$

where $\boldsymbol{A}_x = \boldsymbol{A}_+ - \boldsymbol{A}_-$ and the connection components $\boldsymbol{A}_\pm$ are defined through eqs. (3.35)-(3.36). Using the magic of the ZCR, it is useful to re-express $\boldsymbol{M}(\rho)$ in terms of the light cone values of the connection. Indeed, the original integration along the time slice $t = t_0$ in (4.68) can be replaced by the path-ordered integral over the contour glued from two light-cone segments as shown in fig. 4.3. Using the notation

$$\boldsymbol{A}_+(x_+) = \boldsymbol{A}_+(t,x)|_{x_-=t_0}\,, \qquad \boldsymbol{A}_-(x_-) = \boldsymbol{A}_-(t,x)|_{x_+=t_0+R}\,, \tag{4.69}$$

one can rewrite eq.(4.68) in the form

$$\boldsymbol{M} = \overleftarrow{\mathcal{P}} \exp\left(\int_{t_0}^{t_0-R} \boldsymbol{A}_-(x_-)\,\mathrm{d}x_-\right) \overleftarrow{\mathcal{P}} \exp\left(\int_{t_0}^{t_0+R} \boldsymbol{A}_+(x_+)\,\mathrm{d}x_+\right)\,. \tag{4.70}$$

This formula is a convenient starting point for taking the limit $\kappa \to 1$. Since the product $\rho_+\,\rho_-$ vanishes as $(1-\kappa)$ for κ close to one, we will keep $\rho \equiv \rho_+$ fixed with ρ_- tending to zero. Taking the limit in this way yields that

$$\begin{aligned}
\mathrm{e}^{+\frac{1}{2}(\phi-\mathrm{i}w)\mathtt{h}}\left(\partial_+ - \boldsymbol{A}_+\right)\mathrm{e}^{-\frac{1}{2}(\phi-\mathrm{i}w)\mathtt{h}} &= \frac{\mathrm{i}\rho\left(\partial_+\phi - \mathrm{i}\,\partial_+ w\right)}{1-\rho^2}\,\left(\mathrm{e}^{+2\phi}\,\mathtt{e}_+ + \mathrm{e}^{-2\phi}\,\mathtt{e}_- + \mathrm{i}\rho\,\mathtt{h}\right)\\
\mathrm{e}^{-\frac{1}{2}(\phi+\mathrm{i}w)\mathtt{h}}\left(\partial_- - \boldsymbol{A}_-\right)\mathrm{e}^{+\frac{1}{2}(\phi+\mathrm{i}w)\mathtt{h}} &= 0\ . \qquad (4.71)
\end{aligned}$$

In taking $\kappa \to 1$ in the above formulae, we did not perform the shift $\phi \mapsto \phi + \frac{1}{2}\log\left(\frac{1+\kappa}{1-\kappa}\right)$. Hence, the fields in (4.71) take values in the asymptotically flat domain, where the cigar NLSM target-space approaches the cylinder. In this domain, the equations of motion become the d'Alembert equations, whose solution is expressed in terms of four arbitrary functions

$$\phi(t,x) = \phi_1(x_+) + \bar{\phi}_1(x_-)\,, \qquad w(t,x) = \phi_2(x_+) + \bar{\phi}_2(x_-)\,, \qquad (4.72)$$

where $x_\pm = t \pm x$. The fields $\phi_{1,2}$ and $\bar{\phi}_{1,2}$ should be understood as the asymptotic fields in the cigar NLSM.

Up to a gauge transformation, the connection component $\boldsymbol{A}_+$ can be expressed entirely in terms of the "holomorphic" components of the fields ϕ and w as:

$$\boldsymbol{S}^{-1}\left(\partial_+ - \boldsymbol{A}_+\right)\boldsymbol{S} = \frac{\mathrm{i}\rho\left(\partial\phi_1 - \mathrm{i}\,\partial\phi_2\right)}{1-\rho^2}\,\left(\mathrm{e}^{+2\phi_1}\,\mathtt{e}_+ + \mathrm{e}^{-2\phi_1}\,\mathtt{e}_- + \mathrm{i}\rho\,\mathtt{h}\right)\,, \qquad (4.73)$$

where $\boldsymbol{S} = \mathrm{e}^{\frac{1}{2}(\bar{\phi}_1 - \phi_1 + \mathrm{i}\phi_2 + \mathrm{i}\bar{\phi}_2)\,\mathtt{h}}$. Using this result, it follows that the monodromy matrix (4.70) can be brought to the form

$$\boldsymbol{M} = \boldsymbol{\Omega}^{-1}\,\overleftarrow{\mathcal{P}}\exp\left(\frac{\mathrm{i}\rho}{1-\rho^2}\int_{t_0}^{t_0+R}\mathrm{d}x_+\,\left(\partial\phi_1 - \mathrm{i}\,\partial\phi_2\right)\left(\mathrm{e}^{+2\phi_1}\,\mathtt{e}_+ + \mathrm{e}^{-2\phi_1}\,\mathtt{e}_- + \mathrm{i}\rho\,\mathtt{h}\right)\right)\mathrm{e}^{-2\pi(P_1-k)\mathtt{h}}\,\boldsymbol{\Omega}$$

with $\boldsymbol{\Omega} = \mathrm{e}^{-\frac{\mathrm{i}}{2}\omega_0\mathtt{h}}$ and

$$\omega_0 = w(t_0,R) + \mathrm{i}\phi_1(t_0+R) - \mathrm{i}\bar{\phi}_1(t_0-R)\ . \qquad (4.74)$$

Here we have used the notation

$$P_1 \;\equiv\; \tfrac{1}{2\pi}\big(\phi_1(t_0+R)-\phi_1(t_0)\big) = -\tfrac{1}{2\pi}\big(\bar\phi_1(t_0-R)-\bar\phi_1(t_0)\big)\,. \tag{4.75}$$

The equality follows from the fact that $\phi \asymp \phi_1+\bar\phi_1$ is a periodic field (see eq. (4.67)).

The asymptotically flat domain corresponds to taking ϕ to be large and negative in the cigar Lagrangian (4.66). In this case, it is easy to see that it becomes the Lagrangian of a free field theory with ϕ and w being canonically normalized fields. It follows from this that the Poisson bracket relations of the asymptotic fields ϕ_i, $\bar\phi_i$ can be chosen to be as follows

$$\{\phi_i(x_+),\, \phi_j(x'_+)\} = -\tfrac{1}{4}\,\delta_{ij}\,\epsilon(x_+-x'_+) \tag{4.76}$$

with $i,j=1,2$. This implies that

$$\boldsymbol{M}(\rho) = \boldsymbol{\Omega}^{-1}\; \boldsymbol{L}_{\rm cl}(\rho)\,\mathrm{e}^{-\pi P_1 \mathtt{h}}\,\mathrm{e}^{+2\pi\mathrm{i}k\mathtt{h}}\,\boldsymbol{\Omega}\,, \tag{4.77}$$

where $\boldsymbol{L}_{\rm cl}(\rho)$ is defined by eqs. (4.63)-(4.65).

The ultralocal structure (3.37) implies that the monodromy for the 2D sausage satisfies the Sklyanin exchange relations. Since the classical r-matrix does not depend on κ, these relations still hold true in the limit of the cigar with $\kappa\to 1$. Starting from the Sklyanin exchange relations for $\boldsymbol{M}(\rho)$, and using

$$\big\{\boldsymbol{L}_{\rm cl}(\rho),\pi P_1\big\} = \tfrac{1}{4}\,\big[\mathtt{h},\boldsymbol{L}_{\rm cl}(\rho)\big]\,,\qquad \big\{\boldsymbol{L}_{\rm cl}(\rho),\omega_0\big\} = \tfrac{\mathrm{i}}{4}\,\mathtt{h}\,\boldsymbol{L}_{\rm cl}(\rho)\,,\qquad \big\{\omega_0,\pi P_1\big\} = \tfrac{\mathrm{i}}{4}\,,$$

which follow from eqs. (4.74) (4.75) and (4.76) as well as

$$\big[1\otimes\mathtt{h}+\mathtt{h}\otimes 1,\boldsymbol{r}(\lambda)\big] = 0\,, \tag{4.78}$$

one finds that $\boldsymbol{L}_{\rm cl}$ satisfies the Poisson bracket algebra (4.57) with $\rho_{1,2}=\lambda_{1,2}$. This independently confirms the conjecture that $\boldsymbol{L}_{\rm cl}$ obeys the Sklyanin exchange relations in the case of $\nu=0$.

4.5 Monodromy matrix for the 3D sausage

The conserved charges for the 3D sausage were introduced in sec. 3.1.2. Starting from the flat connection defined through eqs. (3.15) and (3.16), we defined the monodromy as the path ordered exponent (3.11) of a matrix representation $\mathcal{R}$ of the component $\boldsymbol{A}_x = \boldsymbol{A}_+ - \boldsymbol{A}_-$. It was shown that its appropriate trace, see eq. (3.20), is a conserved quantity in the theory $\partial_t T_{\mathcal{R}}(\rho) = 0$ and hence generates an infinite family of IM. However, up till now, we have neglected to discuss the Poisson commutativity

$$\{T_{\mathcal{R}}(\rho),\, T_{\mathcal{R}'}(\rho')\} = 0 \tag{4.79}$$

of the conserved charges, which is an important ingredient for integrability. We return to this problem here.

The proof of (4.79) requires the study of the Poisson structure of the theory. It turns out that a crucial rôle in the Hamiltonian formulation of the Klimcik model is played by the currents $\boldsymbol{I}_\pm$ defined in eq. (3.13). A straightforward calculation yields that the Hamiltonian is given by

$$H = \frac{1}{2\mathrm{g}^2}\int \mathrm{d}x\,\big(\langle\, \boldsymbol{I}_+,\, \boldsymbol{I}_+\,\rangle + \langle\, \boldsymbol{I}_-,\, \boldsymbol{I}_-\,\rangle\big)\,. \tag{4.80}$$

It is more difficult to extract the Poisson structure from the Lagrangian (2.41). Nevertheless one can show that $\boldsymbol{I}_\pm$ are related by a linear transformation to the currents

$$\boldsymbol{J}_\pm(x) = \sum_a J^a_\pm(x)\,\mathtt{t}_a\,, \qquad [\mathtt{t}_a, \mathtt{t}_b] = \mathrm{i}\, f_{ab}{}^c\, \mathtt{t}_c\,, \tag{4.81}$$

which generate two independent copies of the classical current algebra:

$$\big\{J^a_\sigma(x),\, J^b_{\sigma'}(y)\big\} \;=\; \frac{1}{\mathrm{g}^2\varepsilon_1}\,\delta_{\sigma\sigma'}\,\sigma\, q^{ab}\,\delta'(x-y) + \delta_{\sigma\sigma'}\, f^{abc}\, q_{cd}\, J^d_\sigma(y)\,\delta(x-y)\,. \tag{4.82}$$

Here $\sigma, \sigma' = \pm$ and

$$q_{ab} = -\tfrac{1}{4}\, f_{ac}{}^d f_{bd}{}^c = \langle\, \mathtt{t}_a, \mathtt{t}_b\,\rangle\,. \tag{4.83}$$

For an explicit description of the linear relation between $\boldsymbol{I}_\sigma$ and $\boldsymbol{J}_\sigma$ ($\sigma = \pm$), it is convenient to use the root decomposition of the Lie algebra and represent the currents in the form

$$\boldsymbol{I}_\sigma(x) = \boldsymbol{I}^+_\sigma(x) + \boldsymbol{I}^0_\sigma(x) + \boldsymbol{I}^-_\sigma(x) \ : \quad \boldsymbol{I}^\pm_\sigma(x) \in \mathfrak{n}_\pm \ , \quad \boldsymbol{I}^0_\sigma(x) \in \mathfrak{h} \tag{4.84}$$

and similarly for $\boldsymbol{J}_\pm$. Then the relation is given in terms of three 2×2 matrices

$$\boldsymbol{I}^+_\sigma = \sum_{\sigma'=\pm} X^+_{\sigma\sigma'}\, \boldsymbol{J}^+_{\sigma'} \ , \quad \boldsymbol{I}^-_\sigma = \sum_{\sigma'=\pm} X^-_{\sigma\sigma'}\, \boldsymbol{J}^-_{\sigma'} \ , \quad \boldsymbol{I}^0_\sigma = \sum_{\sigma'=\pm} X^0_{\sigma\sigma'}\, \boldsymbol{J}^0_{\sigma'} \tag{4.85}$$

whose matrix entries $X^A_{\sigma\sigma'}$ ($A = \pm, 0$) are given in Appendix A.

It is evident from formulae (4.81)-(4.85), that the Poisson brackets relations $\{\boldsymbol{A}_x(x_1),\, \boldsymbol{A}_x(x_2)\}$ will have a complicated, non-ultralocal form. This makes the Poisson commutativity conditions (4.79) difficult to prove.

For $\varepsilon_1 = \varepsilon_2 = 0$ (which corresponds to the PCF) the computation of the Poisson brackets of the monodromy matrix was discussed in ref. [80]. In this case, the formula (3.13) for the currents becomes $\boldsymbol{I}_\pm = -2\mathrm{i}\, \boldsymbol{U}^{-1}\partial_\pm \boldsymbol{U}$. Assuming that $\rho_\pm = 1 - \varepsilon_2\, \zeta_\pm$ and $\zeta_\pm$ are kept fixed as $\varepsilon_{1,2} \to 0$, eq. (3.15) turns into the Zakharov-Mikhailov connection [46]

$$\lim_{\varepsilon_1,\varepsilon_2\to 0} \boldsymbol{A}_\pm = -\zeta_\pm^{-1}\, \boldsymbol{U}^{-1}\partial_\pm \boldsymbol{U} \ , \tag{4.86}$$

while the constraint (3.16) boils down to the relation $\zeta_+ + \zeta_- = 2$. The monodromy matrix for the PCF can be defined by taking the limit of (3.11):

$$\boldsymbol{M}^{(0)}(\zeta) = \lim_{\varepsilon_1,\varepsilon_2\to 0} \boldsymbol{M}(\rho)\big|_{\rho = 1-\varepsilon_2\zeta_+} \ , \quad \text{where} \quad \zeta_\pm \equiv 1 \pm \zeta \ . \tag{4.87}$$

In ref.[80], for overcoming the non-ultralocality problem, the authors proposed a certain formal regularization procedure which results in the Yang-Baxter Poisson algebra

$$\big\{\boldsymbol{M}^{(0)}(\zeta_1) \underset{,}{\otimes} \boldsymbol{M}^{(0)}(\zeta_2)\big\} = \big[\boldsymbol{M}^{(0)}(\zeta_1) \otimes \boldsymbol{M}^{(0)}(\zeta_2),\, \boldsymbol{r}^{(0)}(\zeta_1 - \zeta_2)\big] \tag{4.88}$$

with

$$\boldsymbol{r}^{(0)}(\zeta_1-\zeta_2) = \ -\frac{f^2}{2}\ \frac{q^{ab}\ \mathtt{t}_a\otimes\mathtt{t}_b}{\zeta_1-\zeta_2}\,. \tag{4.89}$$

Of course, eq. (4.88) complemented by $\big[\boldsymbol{H}_2\otimes\boldsymbol{H}_2,\,\boldsymbol{r}^{(0)}(\zeta)\big]=0$, immediately implies the desired commutativity conditions (4.79) specialized to the PCF. However, for the general Klimč´ik model it is uncertain whether the classical Yang-Baxter Poisson algebra emerges, even at the formal level. Below we'll try to unravel this problem for $\mathfrak{G}=SU(2)$ by using results obtained in Section 4.4.

4.5.1 Monodromy matrix for the 3D sausage

To make connection with the results of sec. 4.4, we need to take the limit $\kappa\to1$ of the Klimcik model. For this purpose, it is convenient to use the gauge $\boldsymbol{A}^{(\boldsymbol{\omega})}$, which is defined as follows. The equations of motion imply the conservation of the current $\boldsymbol{I}^0_\sigma$,[4]

$$\partial_+\boldsymbol{I}^0_-+\partial_-\boldsymbol{I}^0_+=0\,, \tag{4.90}$$

which allows one to introduce the dual field $\boldsymbol{\omega}$

$$\partial_+\boldsymbol{\omega}=-\tfrac{1}{2}\,\varepsilon_2\,\boldsymbol{I}^0_+\,,\qquad\qquad \partial_-\boldsymbol{\omega}=\tfrac{1}{2}\,\varepsilon_2\,\boldsymbol{I}^0_-\,, \tag{4.91}$$

taking values in the Cartan subalgebra $\mathfrak{h}$. Then,

$$\partial_\pm-\boldsymbol{A}^{(\omega)}_\pm=\mathrm{e}^{+\mathrm{i}\boldsymbol{\omega}}\,\big(\partial_\pm-\boldsymbol{A}_\pm\big)\,\mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}}\,. \tag{4.92}$$

To perform the $\kappa\to1$ limit, we use the co-ordinate frame defined through eqs. (3.33), (3.34). In this frame, the symmetry $\boldsymbol{U}\mapsto\boldsymbol{H}_1\boldsymbol{U}\boldsymbol{H}_2$ $(\boldsymbol{H}_1,\boldsymbol{H}_2\in\mathfrak{H})$ of

[4] In the limit $\rho_+\to\infty$ and $\rho_-\to0$ the connection (3.15) becomes upper triangular, $\boldsymbol{A}_\sigma\in\mathfrak{n}_+\oplus\mathfrak{h}$, so that eq. (4.90) immediately follows from the zero curvature representation.

the general Klimč´ik model is manifested as the invariance of the 3D sausage w.r.t. the constant shifts

$$v \mapsto v + v_0\,, \qquad w \mapsto w + w_0\,. \tag{4.93}$$

The corresponding Noether currents will be denoted by $j^{(v)}$ and $j^{(w)}$ respectively. With the continuity equations

$$\partial_+ j_-^{(A)} + \partial_- j_+^{(A)} = 0 \qquad (A = v, w) \tag{4.94}$$

one can introduce the dual fields $\tilde{v}$, $\tilde{w}$ through the relations

$$j_\pm^{(v)} = \pm\,\partial_\pm\,\tilde{v}\,, \qquad j_\pm^{(w)} = \pm\,\partial_\pm\,\tilde{w}\ . \tag{4.95}$$

It turns out that the dual field $\boldsymbol{\omega}$ defined by eq. (4.91) coincides with

$$\boldsymbol{\omega} = \frac{1}{2}\left[\sqrt{1+\nu^2}\ \tilde{w} + \frac{\mathrm{i}}{2}\log\left(\frac{\cosh(\phi_0+\phi)}{\cosh(\phi_0-\phi)}\right)\right]\mathtt{h}\,. \tag{4.96}$$

The boundary conditions (3.17) specialized for the $SU(2)$ case with

$$\boldsymbol{H}_1 = \mathrm{e}^{-\mathrm{i}\pi k_1 \mathtt{h}}\,, \qquad \boldsymbol{H}_2 = \mathrm{e}^{-\mathrm{i}\pi k_2 \mathtt{h}}\,, \tag{4.97}$$

imply the following conditions imposed on the fields (ϕ, v, w):

$$v(t, x+R) = v(t,x) + 2\pi k_1\,, \quad w(t,x+R) = w(t,x) + 2\pi k_2\,, \tag{4.98}$$

while ϕ is the periodic field

$$\phi(t, x+R) = \phi(t,x)\,. \tag{4.99}$$

Also we will focus on the neutral sector of the model, which means that the dual fields also obey the periodic boundary conditions

$$\tilde{v}(t,x+R) = \tilde{v}(t,x)\,, \qquad \tilde{w}(t,x+R) = \tilde{w}(t,x)\,. \tag{4.100}$$

Taking into account that

$$\hat{\boldsymbol{R}}(\mathtt{h}) = 0\,, \qquad \hat{\boldsymbol{R}}(\mathtt{e}_\pm) = \mp \mathrm{i}\,\mathtt{e}_\pm$$

and using the parameterization (3.33), (3.34) the Lagrangian (2.41) with f^2 as in (2.48) can be expressed in terms of three real fields (ϕ, w, v) and two real parameters κ and ν (2.49). Here there is no need to present the explicit formula, we just note that for $|\phi| \ll \phi_0$ the 3D sausage Lagrangian can be approximated by (up to a total derivative)

$$\mathcal{L} \asymp 2\left(\partial_+\phi\,\partial_-\phi + \frac{1}{1+\nu^{-2}}\,\partial_+ v\,\partial_- v + \frac{1}{1+\nu^2}\,\partial_+ w\,\partial_- w\right). \tag{4.101}$$

This implies that as $\kappa \to 1^-$, i.e., $\phi_0 \to \infty$ most of the target manifold asymptotically approaches the flat cylinder with metric $G_{\alpha\beta}\,\mathrm{d}X^\alpha \mathrm{d}X^\beta = (\mathrm{d}\phi)^2 + (1+\nu^{-2})^{-1}(\mathrm{d}v)^2 + (1+\nu^2)^{-1}(\mathrm{d}w)^2$ while the curvature is concentrated at the tips corresponding to $\phi = \pm\infty$. In the asymptotically flat domain, the general solution to the equations of motion can be expressed in terms of six arbitrary functions ϕ_i and $\bar{\phi}_i$:

$$\begin{aligned}
\phi(t,x) &\asymp \phi_1(x_+) + \bar{\phi}_1(x_-) \\
w(t,x) &\asymp \sqrt{1+\nu^{-2}}\,\big(\phi_2(x_+) + \bar{\phi}_2(x_-)\big) \\
v(t,x) &\asymp \sqrt{1+\nu^{+2}}\,\big(\phi_3(x_+) + \bar{\phi}_3(x_-)\big)\,,
\end{aligned} \tag{4.102}$$

while for the dual fields one has

$$\tilde{w}(t,x) \asymp \phi_2(x_+) - \bar{\phi}_2(x_-)\,, \qquad \tilde{v}(t,x) \asymp \phi_3(x_+) - \bar{\phi}_3(x_-)\,. \tag{4.103}$$

Having clarified the geometry of the target manifold for $\kappa \to 1^-$ one can turn to the form of the flat connection (3.15) in this limit. We assume that the co-ordinates (ϕ, w, v) are kept within the asymptotic domain where eqs. (4.102), (4.103) are valid. Also, since the product $\rho_+\rho_-$ (3.16) vanishes as $1-\kappa$, we keep ρ_+ fixed while $\rho_- \to 0$.

Then a direct calculation shows that

$$\lim_{\substack{\kappa\to 1^- \\ \rho_+-\text{fixed}}} \Big(\partial_+ - (\rho_+/\rho_-)^{+\frac{\mathtt{h}}{4}}\, \boldsymbol{A}_+^{(\omega)}\,(\rho_+/\rho_-)^{-\frac{\mathtt{h}}{4}}\Big) = \mathrm{e}^{+2\mathrm{i}\boldsymbol{\omega}_+(x_+)}\,\big(\partial_+ - \boldsymbol{B}(x_+|\rho_+)\big)\,\mathrm{e}^{-2\mathrm{i}\boldsymbol{\omega}_+(x_+)}\,, \tag{4.104}$$

where we have used the gauge $\boldsymbol{A}_+^{(\omega)}$ from eq. (4.92). The 1-form $\boldsymbol{B}$ in this equation is defined by (4.56), (4.42), (4.46) and

$$\boldsymbol{\omega}_+(x_+) = \tfrac{1}{2}\,\big(\sqrt{1+\nu^2}\;\phi_2(x_+) + \mathrm{i}\,\phi_1(x_+)\big)\,\mathtt{h}\,. \tag{4.105}$$

For the other connection component one finds

$$\lim_{\substack{\kappa\to 1^- \\ \rho_+-\text{fixed}}} (\rho_+/\rho_-)^{+\frac{\mathtt{h}}{4}}\;\boldsymbol{A}_-^{(\omega)}\,(\rho_+/\rho_-)^{-\frac{\mathtt{h}}{4}} = 0\,. \tag{4.106}$$

We now turn to the monodromy matrix that was introduced previously in (3.11). In light of eqs. (4.104), (4.106) we express $\boldsymbol{M}(\rho)$ in terms of $\boldsymbol{A}_\sigma^{(\omega)}$:

$$\boldsymbol{M}(\rho) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}(t_0,R)}\,\overleftarrow{\mathcal{P}}\exp\Big(\int_0^R \mathrm{d}x\;\boldsymbol{A}_x^{(\omega)}\Big)\Big|_{t=t_0}\;\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}(t_0,0)} \qquad (\rho\equiv\rho_+)\,. \tag{4.107}$$

Since the connection $\boldsymbol{A}_\sigma^{(\omega)}$ is flat, the integral over the segment $(0,R)$ can be transformed into the piecewise integral over the light cone segments as shown in fig. 4.3. The monodromy matrix is then expressed in terms of the light cone values of the connection as

$$\boldsymbol{M}(\rho) = \mathrm{e}^{-\mathrm{i}\boldsymbol{\omega}(t_0,R)}\,\overleftarrow{\mathcal{P}}\exp\Big(\int_{t_0}^{t_0-R} \boldsymbol{A}_-^{(\omega)}(x_-)\,\mathrm{d}x_-\Big)\,\overleftarrow{\mathcal{P}}\exp\Big(\int_{t_0}^{t_0+R} \boldsymbol{A}_+^{(\omega)}(x_+)\,\mathrm{d}x_+\Big)\,\mathrm{e}^{\mathrm{i}\boldsymbol{\omega}(t_0,0)}$$

where

$$\boldsymbol{A}_+^{(\omega)}(x_+) = \boldsymbol{A}_+^{(\omega)}(t,x)\big|_{x_-=t_0}\,, \qquad \boldsymbol{A}_-^{(\omega)}(x_-) = \boldsymbol{A}_-^{(\omega)}(t,x)\big|_{x_+=t_0+R}\,. \tag{4.108}$$

For κ close to 1 the instant t_0 can be chosen such that the values of the fields lie in the asymptotically flat region of the target manifold where formulae (4.102), (4.103) are applicable. Then with eqs. (4.104), (4.106) at hand, it is straightforward to show that the following limit exists

$$\lim_{\substack{\kappa\to 1^- \\ \rho_+-\text{fixed}}} (\rho_+/\rho_-)^{+\frac{\mathtt{h}}{4}}\,\boldsymbol{M}(\rho)\,(\rho_+/\rho_-)^{-\frac{\mathtt{h}}{4}} = \boldsymbol{M}^{(1)}(\rho)\,. \tag{4.109}$$

Explicitly, $\boldsymbol{M}^{(1)}(\rho)$ can be expressed in terms of $\boldsymbol{L}_{\rm cl}(\rho)$ previously defined in (4.51) and (4.56):

$$\boldsymbol{M}^{(1)}(\rho) = \boldsymbol{\Omega}^{-1}\ \boldsymbol{L}_{\rm cl}(\rho)\, {\rm e}^{\pi(2{\rm i}\sqrt{1+\nu^2}P_2 - P_1)\,\mathtt{h}}\ \boldsymbol{\Omega}\,. \tag{4.110}$$

Here we take into account that $\phi(t_0, x+R) = \phi(t_0, x)$, $\tilde{w}(t, x+R) = \tilde{w}(t, x)$ and use

$$\begin{aligned} P_1 &\equiv \tfrac{1}{2\pi}\big(\phi_1(t_0+R) - \phi_1(t_0)\big) = -\tfrac{1}{2\pi}\big(\bar{\phi}_1(t_0-R) - \bar{\phi}_1(t_0)\big) \\ P_2 &\equiv \tfrac{1}{2\pi}\big(\phi_2(t_0+R) - \phi_2(t_0)\big) = +\tfrac{1}{2\pi}\big(\bar{\phi}_2(t_0-R) - \bar{\phi}_2(t_0)\big) \end{aligned} \tag{4.111}$$

and

$$\boldsymbol{\Omega} = {\rm e}^{-\frac{\rm i}{2}\omega_0 \mathtt{h}}\ :\qquad \omega_0 = w(t_0, R) + {\rm i}\,\big(\phi_1(t_0+R) - \bar{\phi}_1(t_0-R)\big)\ . \tag{4.112}$$

It follows from the Lagrangian that the chiral fields ϕ_i can be chosen to satisfy the Poisson bracket relations

$$\{\phi_i(x_+),\ \phi_j(x'_+)\} = -\tfrac{1}{4}\,\delta_{ij}\,\epsilon(x_+ - x'_+) \tag{4.113}$$

and hence, using the results of the previous section, $\boldsymbol{L}_{\rm cl}(\rho)$ obeys the Sklyanin exchange relations (4.57). In the Hamiltonian picture the twisted boundary condition $w(t, x+R) = w(t, x) + 2\pi k_2$ with k_2 a non-dynamical constant is a constraint of the first kind à la Dirac which should be supplemented by a gauge fixing condition. Considering the fields in the asymptotically flat domain where formulae (4.102), (4.103) hold true leads to the relation

$$P_2 = \frac{k_2}{2\sqrt{1+\nu^2}} \tag{4.114}$$

and the gauge fixing condition can be chosen as $w(t_0, R) = 0$. This way ω_0 in (4.112) becomes $\omega_0 = {\rm i}\,\big(\phi_1(t_0+R) - \bar{\phi}_1(t_0-R)\big)$. Similarly, we supplement the periodic boundary condition $\phi(t_0, x+R) = \phi(t_0, x)$ by the constraint $\bar{\phi}_1(t_0 - R) = 0$, so that

$$\omega_0 = {\rm i}\,\phi_1(t_0+R)\,. \tag{4.115}$$

The Poisson brackets of $\boldsymbol{M}^{(1)}(\rho) = \boldsymbol{\Omega}^{-1}\ \boldsymbol{L}_{\rm cl}(\rho)\ {\rm e}^{\pi({\rm i}k_2-P_1)\,\mathtt{h}}\ \boldsymbol{\Omega}$ are obtained by using (4.57) and the simple relations

$$\big\{\boldsymbol{L}_{\rm cl}(\rho),\pi P_3\big\} = \tfrac{1}{4}\,\big[\,\mathtt{h},\boldsymbol{L}_{\rm cl}(\rho)\big]\,,\qquad \big\{\boldsymbol{L}_{\rm cl}(\rho),\omega_0\big\} = \tfrac{\rm i}{4}\,\mathtt{h}\,\boldsymbol{L}_{\rm cl}(\rho)\,,\qquad \big\{\omega_0,\pi P_1\big\} = \tfrac{\rm i}{4}\,. \tag{4.116}$$

The latter follow from eqs. (4.111), (4.113), (4.115). Also, taking into account that

$$\big[1\otimes\mathtt{h}+\mathtt{h}\otimes 1,\boldsymbol{r}(\lambda)\big] = 0\,, \tag{4.117}$$

one arrives at

$$\big\{\boldsymbol{M}^{(1)}(\rho_1)\ \underset{,}{\otimes}\ \boldsymbol{M}^{(1)}(\rho_2)\,\big\} = \big[\boldsymbol{M}^{(1)}(\rho_1)\otimes\boldsymbol{M}^{(1)}(\rho_2),\boldsymbol{r}(\lambda_1/\lambda_2)\,\big]\,, \tag{4.118}$$

where recall that $\rho_{1,2}$ depend on $\lambda_{1,2}$ via the relation (4.55).

It should be highlighted that the Poisson algebra (4.118) was obtained for a certain choice of the time slice t_0 when the fields take values in the asymptotic region. The validity of this equation for an arbitrary choice of t_0 is debatable, since the monodromy matrix itself is not a conserved quantity. However that eq. (4.118) holds true even for a particular value of t_0 is sufficient to prove the commutativity condition $\{T^{(1)}(\rho_1),T^{(1)}(\rho_2)\} = 0$ with

$$T^{(1)}(\rho) = {\rm Tr}\big[{\rm e}^{-{\rm i}\pi k_2\mathtt{h}}\boldsymbol{M}^{(1)}(\rho)\big] = \lim_{\substack{\kappa\to 1^-\\ \rho_+-{\rm fixed}}}\ {\rm Tr}\big[\,{\rm e}^{-{\rm i}\pi k_2\mathtt{h}}\,\boldsymbol{M}(\rho)\big]\,. \tag{4.119}$$

In view of the above, it makes sense to reconsider our definition of the monodromy matrix for the 3D sausage model and introduce

$$\boldsymbol{M}^{(\kappa)}(\rho) = (\rho_+/\rho_-)^{+\frac{\mathtt{h}}{4}}\,\boldsymbol{M}(\rho)\,(\rho_+/\rho_-)^{-\frac{\mathtt{h}}{4}}\qquad (\rho\equiv\rho_+)\,. \tag{4.120}$$

We've just seen that in the $\kappa\to 1^-$ limit, the matrix $\boldsymbol{M}^{(\kappa)}(\rho)$ obeys the Sklyanin exchange relations (4.118). On the other hand, the redefinition (4.120) has no effect on the monodromy matrix as $\kappa\to 0$ and both $\rho_\pm\to 1$ so that the Sklyanin exchange relations are still satisfied but in the form (4.88). Finally the case $\nu = 0$ with $\kappa \in (0,1)$ was already considered before, where it was shown that

$$\left\{ \boldsymbol{M}^{(\kappa)}(\rho_1) \underset{,}{\otimes} \boldsymbol{M}^{(\kappa)}(\rho_2) \right\} = \left[\boldsymbol{M}^{(\kappa)}(\rho_1) \otimes \boldsymbol{M}^{(\kappa)}(\rho_2), \boldsymbol{r}(\lambda_1/\lambda_2) \right] \qquad (\nu \to 0) \qquad (4.121)$$

with $\rho_{1,2} = \lambda_{1,2}$. All this suggests that the key relations (4.121) may extend to the parametric domain $\nu^2 > 0$ and $\kappa \in (0,1)$ with some function $\rho = \rho(\lambda|\nu,\kappa)$ (which is unknown in general).

Chapter 5

Transfer-matrices for the sausage model

5.1 Introduction

When faced with the problem of quantizing the 2D sausage model, one may try to follow the approach based on discretization. Due to ultralocality, the N elementary transport matrices $\pi_j\Big[\overleftarrow{\mathcal{P}}\exp\int_{x_n}^{x_{n+1}}\mathrm{d}x\,\boldsymbol{A}_x\Big]$ satisfy the same type of Poisson bracket relation as (3.27) and Poisson commute for different segments. These relations can be formally quantized leading to a certain quantum Yang-Baxter algebra. The major problem now is to construct a suitable representation of this abstract algebraic structure. In the case under consideration, the representation is, in all likelihood, infinite dimensional even for finite N. At this moment, it is not clear for us how to construct and handle such representations, let alone take the scaling limit with $N\to\infty$.

We will try to avoid discretization as much as we can and mostly follow the so-called BLZ approach – the variant of the QISM developed in the series of works [50, 51, 52]. For integrable Conformal Field Theories (CFT), it was demonstrated that the T-operators can be constructed without any discretization procedure. Later it was observed that many deep properties of representations of Yang-Baxter algebras in integrable CFT can be encoded in the monodromies of certain linear Ordinary Differential Equations (ODE) [97, 98, 53, 54, 99, 55, 100]. These results were extended to massive Integrable Quantum Field Theories (IQFT) [56] (for recent developments, see also refs.[101, 102, 103, 104, 105, 106, 107, 108]). The general relation of this type will be referred to in the paper as the ODE/IQFT correspondence.

Broadly speaking, the ODE/IQFT correspondence means that for a given IQFT the eigenvalues of the quantum T-operators are identified with certain connection coefficients for the system of equations,

$$\boldsymbol{D}(\theta)\,\boldsymbol{\Psi}=0\ ,\qquad \overline{\boldsymbol{D}}(\theta)\,\boldsymbol{\Psi}=0\ , \tag{5.1}$$

where $\boldsymbol{D}(\theta)$ and $\overline{\boldsymbol{D}}(\theta)$ stand for (singular) differential operators depending on the auxiliary parameter θ which is found to be a function of the original spectral parameter from the quantum theory. The system of ODE can be then interpreted as an auxiliary linear problem, whose compatibility condition, $[\boldsymbol{D}(\theta),\,\overline{\boldsymbol{D}}(\theta)]=0$, coincides with the zero-curvature representation for some *classically* integrable field theory. Thus the ODE/IQFT correspondence reduces the calculation of the spectrum of quantum transfer-matrices to a certain problem in the theory of classical integrable equations. The latter can be effectively treated by the inverse scattering transform method. This makes the ODE/IQFT correspondence a very powerful tool. In particular, it gives a practical way to make progress in the conceptual long standing problem of the quantization of integrable NLSM.

5.2 Chiral transfer-matrices for the cigar

The BLZ approach [50, 51, 52] begins with an analysis of the RG fixed point which controls the ultraviolet behaviour of the integrable QFT. With this in mind, let's take a quick look at the sausage NLSM. In the traditional path-integral quantization, the model should be equipped with a UV cutoff Λ. A consistent removal of the UV divergences requires that the "bare" coupling in the Lagrangian (**??**) be given a certain dependence on the cutoff momentum. To the first perturbative order the RG flow equation is given by [66]

$$\Lambda\,\frac{\partial\kappa}{\partial\Lambda}=\frac{\hbar}{2\pi}\,(1-\kappa^2)+O(\hbar^2)\ , \tag{5.2}$$

where $\hbar$ stands for the (dimensionless) Planck constant. Integrating this equation leads to

$$\frac{1-\kappa}{1+\kappa} = (E_*/\Lambda)^{\eta} \ , \tag{5.3}$$

where $\eta = \frac{\hbar}{\pi} + O(\hbar^2)$. The energy scale E_* is an RG invariant (i.e., it's kept fixed with changing Λ), so that $\kappa \to 1$ as $\Lambda \to \infty$. Having in mind the quantization of the model, this simple analysis shows that 2D sausage NLSM deserves special attention when κ is close to one.

5.2.1 Quantum transfer-matrices for the cigar NLSM

Their construction of the quantum transfermatrices for the cigar NLSM almost identically follows the steps elaborated in refs. [50, 52] in the context of the quantum KdV theory. Here we present them very briefly, referring the reader to those works for detailed explanations.

First of all we should "quantize" the Lie algebra $\mathfrak{sl}(2)$, so that $\mathtt{h}$, $\mathtt{e}_\pm$ are understood now as the generators of the quantum universal enveloping algebra $U_q\big(\mathfrak{sl}(2)\big)$:

$$[\mathtt{h},\, \mathtt{e}_\pm] = \pm 2\, \mathtt{e}_\pm \ , \qquad [\mathtt{e}_+, \mathtt{e}_-] = \frac{q^{\mathtt{h}} - q^{-\mathtt{h}}}{q - q^{-1}} \ , \tag{5.4}$$

where $q = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}}$. Consequently the symbol π_j will stand for the $(2j+1)$-dimensional representation of the quantum algebra. Instead of the Planck constant $\hbar$, for convenience we will use the parameter n:

$$\hbar \equiv \tfrac{2\pi}{n} \ , \qquad q = \mathrm{e}^{\frac{\mathrm{i}\pi}{n}} \ . \tag{5.5}$$

The quantum operator $\boldsymbol{L}_j$ is the following $(2j+1)\times(2j+1)$ operator valued matrix

$$\boldsymbol{L}_j(\lambda_+) = \pi_j \left[\overleftarrow{\mathcal{P}} \exp\left(\mathrm{i}\lambda_+ \int_{t_0}^{t_0+R} \mathrm{d}x \ \left(V^+ \, q^{\frac{\mathtt{h}}{2}}\, \mathtt{e}_+ + V^- \, q^{-\frac{\mathtt{h}}{2}}\, \mathtt{e}_- \right) \right) \mathrm{e}^{-\pi P_1 \mathtt{h}} \right] \ . \tag{5.6}$$

The vertex operators $V^{\pm}$ are defined by the set of relations:

$$V^{\pm}(x) = \left(\tfrac{1}{2}\, c^{\pm}\ \partial_x - \mathrm{i}\, \tfrac{\sqrt{n+2}}{n}\ \vartheta'_{+}(x)\right)\ \mathrm{e}^{\pm\frac{2\varphi_+}{\sqrt{n}}}(x)\ , \tag{5.7}$$

where $c^{\pm}$ are some constants and

$$\begin{aligned} \varphi_{+}(x) &= Q_1 + \frac{2\pi x}{R}\ \sqrt{n}\, P_1 + \mathrm{i} \sum_{m\neq 0} \frac{a_m}{m}\ \mathrm{e}^{-\frac{2\pi \mathrm{i} m}{R} x} \\ \vartheta_{+}(x) &= Q_2 + \frac{2\pi x}{R}\ \sqrt{n+2}\, P_2 + \mathrm{i} \sum_{m\neq 0} \frac{b_m}{m}\ \mathrm{e}^{-\frac{2\pi \mathrm{i} m}{R} x}\ , \end{aligned} \tag{5.8}$$

with

$$[\, a_m, a_l\,] = [\, b_m, b_l\,] = \tfrac{m}{2}\ \delta_{m+l,0}\ , \qquad \left[\, Q_1, \sqrt{n}\, P_1\,\right] = \left[\, Q_2, \sqrt{n+2}\, P_2\,\right] = \tfrac{\mathrm{i}}{2}\ . \tag{5.9}$$

Let $\mathcal{F}_{p_1,p_2} \equiv \mathcal{F}_{\mathbf{p}}$ ("Fock space") be the highest weight module of the Heisenberg algebra (5.9) with the highest weight vector $|\,\mathbf{p}\,\rangle$ defined by the equations

$$P_1\,|\,\mathbf{p}\,\rangle = \frac{p_1}{n}\,|\,\mathbf{p}\,\rangle\ , \qquad P_2\,|\,\mathbf{p}\,\rangle = \frac{p_2}{n+2}\ \ |\,\mathbf{p}\,\rangle\ . \tag{5.10}$$

It is easy to see that

$$V^{\pm}(x)\ \ :\quad \mathcal{F}_{p_1,p_2} \mapsto \mathcal{F}_{p_1\mp \mathrm{i},p_2}\ , \tag{5.11}$$

and therefore the matrix elements of $\boldsymbol{L}_j(\lambda)$ are operators in $\oplus_{m=-\infty}^{\infty}\mathcal{F}_{p_1+\mathrm{i}m,p_2}$. The expression (5.6) contains the ordered exponential which can be formally written in terms of a power series in λ as

$$\boldsymbol{L}_j(\lambda_+) = \pi_j \Big[\ \sum_{m=0}^{\infty} (\mathrm{i}\lambda_+)^m \int_{t_0+R>x_m>\ldots x_1>t_0} \mathrm{d}x_m \cdots \mathrm{d}x_1\ \boldsymbol{K}(x_m)\cdots \boldsymbol{K}(x_1)\, \mathrm{e}^{-\pi P_1 \mathtt{h}}\ \Big]\,, \tag{5.12}$$

where

$$\boldsymbol{K}(x) = V^{+}\ q^{\frac{\mathtt{h}}{2}}\ \mathsf{e}_{+} + V^{-}\ q^{-\frac{\mathtt{h}}{2}}\ \mathsf{e}_{-}\ . \tag{5.13}$$

However, since

$$V^{\pm}(x_2)V^{\mp}(x_1)\big|_{x_2\to x_1+0} \sim\ \frac{a}{2n^2}\ q^{-1}\ \big(x_2 - x_1\big)^{-2(1-\frac{1}{n})}\ , \tag{5.14}$$

where $a = n + 2 + (n-2)\ c^+ c^-$, the integrals in (5.12) diverge. As explained in [52], the commutation relations

$$V^{\sigma_1}(x_1) V^{\sigma_2}(x_2) = q^{2\sigma_1\sigma_2}\ V^{\sigma_2}(x_2) V^{\sigma_1}(x_1)\ , \qquad x_2 > x_1 \quad (\sigma_{1,2} = \pm) \tag{5.15}$$

allow one to re-express the integrals in (5.12) in terms of two basic contour integrals

$$\mathcal{X}_0 = \frac{1}{q - q^{-1}} \int_{t_0}^{t_0+R} \mathrm{d}x\ V^-(x)\ , \quad \mathcal{X}_1 = \frac{1}{q - q^{-1}} \int_{t_0}^{t_0+R} \mathrm{d}x\ V^+(x)\ . \tag{5.16}$$

This procedure yields an unambiguous definition of the ordered exponential in (5.6) for $n \neq 2,\, 4,\, 6 \ldots$. The case of even n needs some special attention and we will return to it later.

The operator valued matrices $\boldsymbol{L}_j$ (5.6) are designed in such a way that, for arbitrary chosen constants $c^\pm$ and t_0, they obey the quantum Yang-Baxter algebra

$$\boldsymbol{R}_{jj'}(\lambda'_+/\lambda_+) \big(\boldsymbol{L}(\lambda_+)\otimes 1\big) \big(1\otimes \boldsymbol{L}(\lambda'_+)\big) = \big(1\otimes \boldsymbol{L}(\lambda'_+)\big) \big(\boldsymbol{L}(\lambda_+)\otimes 1\big) \boldsymbol{R}_{jj'}(\lambda'_+/\lambda_+)\,, \tag{5.17}$$

where the matrix $\boldsymbol{R}_{jj'}(\lambda)$ is the trigonometric solution to the Yang-Baxter equation which acts in the space $\pi_j \otimes \pi_{j'}$. In particular

$$\boldsymbol{R}_{\frac{1}{2}\frac{1}{2}}(\lambda) = \begin{pmatrix} q^{-1}\lambda - q\,\lambda^{-1} & 0 & 0 & 0 \\ 0 & \lambda - \lambda^{-1} & q^{-1} - q & 0 \\ 0 & q^{-1} - q & \lambda - \lambda^{-1} & 0 \\ 0 & 0 & 0 & q^{-1}\lambda - q\,\lambda^{-1} \end{pmatrix}\,. \tag{5.18}$$

The proof of eq.(5.17) follows that from the work [52].

The chiral transfer-matrices, defined as

$$\tau_j(\lambda_+) = \mathrm{Tr}\Big[\, \boldsymbol{L}_j(\lambda_+)\ \mathrm{e}^{-\pi P_1 \mathtt{h}} \Big]\,, \tag{5.19}$$

satisfy the commutativity condition

$$[\,\tau_j(\lambda_+),\, \tau_{j'}(\lambda'_+)\,] = 0\,, \tag{5.20}$$

as a simple consequence of (5.17). Notice that the chiral transfer-matrices act inside a single Fock space, whereas the same is not true for an arbitrary element of $\boldsymbol{L}_j(\lambda)$. Furthermore, the Fock space $\mathcal{F}_{\mathbf{p}}$ naturally splits into the finite dimensional "level subspaces"

$$\mathcal{F}_{\mathbf{p}} = \oplus_{L=0}^{\infty} \mathcal{F}_{\mathbf{p}}^{(L)} : \quad \mathbb{L}\, \mathcal{F}_{\mathbf{p}}^{(L)} = L\, \mathcal{F}_{\mathbf{p}}^{(L)} \ , \tag{5.21}$$

where the grading operator is given by

$$\mathbb{L} = 2 \sum_{m=1}^{\infty} \big(a_{-m} a_m + b_{-m} b_m \big) \,. \tag{5.22}$$

Using the relation,

$$V^{\pm}(x+R) = q^2 \ \mathrm{e}^{\pm 4\pi P_1} \ V^{\pm}(x) \tag{5.23}$$

one can show (see Appendix C from [52]) that the $\tau_j(\lambda_+)$ commute with the grading operator, and therefore, act invariantly in each finite-dimensional level subspace:

$$\tau_j(\lambda_+) \ : \quad \mathcal{F}_{\mathbf{p}}^{(L)} \mapsto \mathcal{F}_{\mathbf{p}}^{(L)} \,. \tag{5.24}$$

The Fock space $\mathcal{F}_{\mathbf{p}}$ can be equipped with an inner product consistent with the Hermiticity conditions $a_m^{\dagger} = a_{-m}$, $b_m^{\dagger} = b_{-m}$ imposed on the Heisenberg operators (5.9). It is not difficult to show that for real p_1^2, p_2^2 and λ_+^2, $\tau(\lambda_+)$ is a Hermitian operator and

$$\big[\tau(\lambda_+)\big]^{\dagger} = \tau(\pm\lambda_+^*) \,. \tag{5.25}$$

Notice that the commutativity with the grading operators can be interpreted as the independence of the chiral transfer-matrix on the arbitrary chosen constant t_0. It turns out that they further do not depend on the constants $c^{\pm}$ appearing in the definition of the vertex operators $V^{\pm}$ (5.7). Also, a simple dimensional analysis shows that the spectral parameter λ_+ and R occur in the chiral transfer-matrix through the combination $\lambda_+^2 R^{\frac{2}{n}}$ only. It is convenient to introduce a dimensionless spectral parameter λ by means of the relation

$$\lambda^2 = \Gamma^2\big(1+\tfrac{1}{n}\big)\ \big(\tfrac{nR}{2\pi}\big)^{\frac{2}{n}}\ \lambda_+^2 \tag{5.26}$$

and treat the chiral transfer-matrices as functions of this variable rather than the dimensionful λ_+.

The chiral transfer-matrices are not independent operators for different values of $j = \frac{1}{2}, 1, \ldots$. They can be expressed through the "fundamental" transfer-matrix $\tau_{\frac{1}{2}}(\lambda)$ by the so-called fusion relation [111, 112, 113]

$$\tau_j\big(\lambda\, q^{j+\frac{1}{2}}\big)\ \tau_{\frac{1}{2}}(\lambda) = \tau_{j+\frac{1}{2}}\big(\lambda\, q^{j}\big) + \tau_{j-\frac{1}{2}}\big(\lambda\, q^{j+1}\big)\,, \tag{5.27}$$

supplemented by the condition $\tau_0 = 1$. In what follows, we will mostly focus on the fundamental transfer-matrix and use the notation $\tau \equiv \tau_{\frac{1}{2}}$. The integrable structures associated with the commuting family of operators $\tau_j(\lambda)$ were already studied in the context of the so-called paperclip model – an integrable model with boundary interaction [115]. Here for convenience we make a short summary of some basic properties of the operator $\tau(\lambda)$.

For arbitrary complex $\mathbf{p} = (p_1, p_2)$, the operator $\tau(\lambda) \in \mathrm{End}\big(\mathcal{F}_{\mathbf{p}}\big)$ is an entire function of λ^2 in the sense that all its matrix elements and eigenvalues are entire functions of this variable. Thus the power series

$$\tau(\lambda) = 2\,\cosh\big(\tfrac{2\pi p_1}{n}\big) + \sum_{m=1}^{\infty} \mathbf{t}_m\, \lambda^{2m} \tag{5.28}$$

converges in the whole complex plane of λ^2 and defines an entire function with an essential singularity at $\lambda^2 = \infty$. The asymptotic expansion near the essential singularity is of primary interest. It can be written as

$$\tau(\lambda) = \exp\bigg(\frac{2\pi}{\sin(\frac{\pi n}{2})}\,(-\lambda^2)^{\frac{n}{2}}\bigg)\ \tilde{\tau}\Big(\mathrm{i}\,(-\lambda^2)^{-\frac{n}{2(n+2)}}\Big)\,, \tag{5.29}$$

where $\tilde{\tau}$ is a formal power series of the form

$$\tilde{\tau}(\tilde{\lambda}) \asymp 2\,\cos\big(\tfrac{2\pi p_2}{n+2}\big) + \sum_{m=1}^{\infty} \tilde{\mathfrak{t}}_m\ \tilde{\lambda}^{2m}\ . \tag{5.30}$$

This asymptotic expansion can be applied for arbitrary complex $\mathbf{p} = (p_1, p_2)$ and $n \neq 2,\, 4,\, 6 \ldots$. Furthermore, in the case $n \geq 1$ it holds true for $|\arg(-\lambda^2)| < \pi$.

The expansion coefficients in (5.28) and (5.30) form two infinite sets of mutually commuting operators. Using the terminology of the work [51], we will refer to $\big\{\mathfrak{t}_m\big\}_{m=1}^{\infty}$ and $\big\{\tilde{\mathfrak{t}}_m\big\}_{m=1}^{\infty}$ as the nonlocal and dual nonlocal Integrals of Motions (IM), respectively. Remarkably, the formal power series $\tilde{\tau}(\tilde{\lambda})$ can be written in a form similar to (5.19). Namely [115],

$$\tilde{\tau}(\tilde{\lambda}) = \mathrm{Tr}\bigg[\overleftarrow{\mathcal{P}}\exp\bigg(\mathrm{i}\tilde{\lambda}_+\int_{t_0}^{t_0+R}\mathrm{d}x\ \big(\Psi^+\ \sigma_+ + \Psi^-\ \sigma_-\big)\bigg)\ \mathrm{e}^{-2\pi\mathrm{i}P_2\sigma_3}\bigg]\ , \tag{5.31}$$

where $\sigma_3,\ \sigma_\pm\ =\ \frac{1}{2}\,(\sigma_1 \pm \mathrm{i}\sigma_2)$ are the conventional Pauli matrices and the vertex operators $\Psi^\pm$ are given by

$$\Psi^\pm(x) = \big(\tfrac{\sqrt{n}}{n+2}\ \varphi_+'(x) + \tfrac{1}{2}\ \partial_x\big)\ \mathrm{e}^{\pm\frac{2\mathrm{i}\theta_+}{\sqrt{n+2}}}(x)\ .$$

The scale dimension of $\Psi^\pm$ is equal to $1 + \frac{1}{n+2}$ and we assume here that they are normalized in such a way that

$$\Psi^\pm(x_2)\,\Psi^\mp(x_1)\big|_{x_2\to x_1+0} \sim\ \tfrac{2}{(n+2)^2}\ \ \big(x_2 - x_1\big)^{-2(1+\frac{1}{n+2})}\ . \tag{5.32}$$

Because of the divergencies, the path ordered exponential in (5.31) should be understood in the same manner as (5.6), i.e., the formal expansion in a power series of $\tilde{\lambda}_+$ should be rewritten in terms of the basic contour integrals similar to (5.16). With this analytical regularization the r.h.s. of eq. (5.31) becomes a formal power series in $\tilde{\lambda}_+^2 R^{-\frac{2}{n+2}}$ with unambiguously defined expansion coefficients. Up to a factor similar to that in (5.26), this combination can be identified with $\tilde{\lambda}^2$ in eq. (5.30):

$$\tilde{\lambda}^2 = \Gamma^2\big(1 - \tfrac{1}{n+2}\big)\ \big(\tfrac{(n+2)R}{2\pi}\big)^{-\frac{2}{n+2}}\ \tilde{\lambda}_+^2\ . \tag{5.33}$$

For future reference we present here explicit formulae for the "vacuum" eigenvalues of the operators $\mathfrak{t}_1$ and $\tilde{\mathfrak{t}}_1$ corresponding to the highest weight vector $|\,\mathbf{p}\,\rangle \in \mathcal{F}_{\mathbf{p}}$ (5.10):

$$t_1(p_1,p_2) = \Big(\frac{2}{n}\Big)^{\frac{2}{n}} \ \frac{\Gamma(\frac{1}{2}+\frac{1}{n})}{\sqrt{\pi}\Gamma(1+\frac{1}{n})} \ \Big(\frac{n+2}{n-2}+\frac{4p_2^2}{1+4p_1^2}\Big) \ \frac{\pi^2}{\Gamma(\frac{1+2\mathrm{i}p_1}{n})\Gamma(\frac{1-2\mathrm{i}p_1}{n})} \tag{5.34}$$

and

$$\tilde{t}_1(p_1,p_2) = \Big(\frac{n+2}{2}\Big)^{\frac{2}{n+2}} \ \frac{\Gamma(\frac{1}{2}-\frac{1}{n+2})}{\sqrt{\pi}\Gamma(1-\frac{1}{n+2})} \ \Big(\frac{n}{n+4}-\frac{4p_1^2}{1-4p_2^2}\Big) \ \frac{\pi^2}{\Gamma(-\frac{1-2p_2}{n+2})\Gamma(-\frac{1+2p_2}{n+2})}\,. \tag{5.35}$$

An efficient integral representation for calculating the vacuum eigenvalue $\tilde{t}_2(p_1,p_2)$ can be found in Appendix A of ref.[115].

For even n, the chiral transfer-matrices require some careful handling. In this case $\tau(\lambda)$ can be defined through the limiting procedure

$$\tau(\lambda)|_{n=2l} = \lim_{\epsilon\to 0}\ \exp\big(-\tfrac{4}{\epsilon}\ \lambda^{2l}\big)\ \tau(\lambda)\big|_{n=2l+\epsilon} \qquad (l=1,\,2,\,3\ldots)\ , \tag{5.36}$$

so that the asymptotic formula (5.29) should be substituted by

$$\tau(\lambda)|_{n=2l} = \exp\Big(2\,\lambda^n\ \log(-\lambda^2)\Big)\ \tilde{\tau}\Big(\mathrm{i}\,(-\lambda^2)^{-\frac{l}{2(l+1)}}\Big)\Big|_{n=2l}\,. \tag{5.37}$$

The formulae (5.29), (5.30) are not valid for positive real λ^2. In order to describe the asymptotic behaviour for $\lambda^2\to+\infty$, it is convenient to substitute the set of dual nonlocal IM (5.30) by the set $\big\{\tilde{\mathfrak{g}}_m\big\}_{m=1}^{\infty}$ which are algebraically expressed in terms of the former through the relation

$$2\,\cos(2\pi P_2)+\sum_{m=1}^{\infty}\tilde{\mathfrak{t}}_m\ z^m = 2\,\cos(2\pi P_2)\ \exp\Big(\sum_{m=1}^{\infty}\tilde{\mathfrak{g}}_m\ z^m\Big)\,. \tag{5.38}$$

Then, for arbitrary complex $\mathbf{p}=(p_1,p_2)$, $n\neq 2,\,4,\,6\ldots$, $n\geq 1$,

$$\tau(\lambda) = 4\,\cos\big(\tfrac{2\pi p_2}{n+2}\big)\ \mathrm{e}^{H(\lambda^2)}\ \cos\big(G(\lambda^2)\big) \qquad \text{as} \qquad \lambda^2\to+\infty\ , \tag{5.39}$$

where

$$\begin{aligned} H(z) &\asymp 2\pi\cot\big(\tfrac{\pi n}{2}\big)\ z^{\frac{n}{2}}+\sum_{m=1}^{\infty}\tilde{\mathfrak{g}}_m\ \cos\big(\tfrac{2\pi m}{n+2}\big)\ z^{-\frac{nm}{n+2}}+O(z^{-\infty}) \\ G(z) &\asymp 2\pi\,z^{\frac{n}{2}}+\sum_{m=1}^{\infty}\tilde{\mathfrak{g}}_m\ \sin\big(\tfrac{2\pi m}{n+2}\big)\ z^{-\frac{nm}{n+2}}+O(z^{-\infty})\,. \end{aligned} \tag{5.40}$$

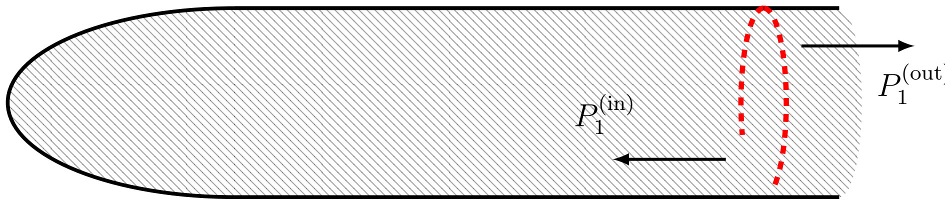


Figure 5.1: The classical scattering problem in the cigar NLSM. From the asymptotically flat domain the string approaches the tip, scatters and then escapes back to the flat region. After the scattering process the zero mode momentum changes sign.

For even n, the first term in the formal power series $H(z)$ should be replaced by $2z^{\frac{n}{2}}\ \log(z)$.

5.2.2 Basic facts about the quantum cigar

In the previous subsection, we described the formal algebraic construction of the chiral transfer-matrices. Here we briefly discuss how $\tau_j(\lambda)$ can be understood as operators in the quantum cigar NLSM (for more details on the quantum cigar see, e.g., ref.[116]).

The cigar NLSM was introduced before at the classical level by means of the Lagrangian (**??**). In the classical field theory, it is natural to consider the following scattering problem. Suppose that at $t \to -\infty$ we are given the field configuration within the asymptotically flat domain of the target manifold, i.e.,

$$\begin{aligned} \phi(t,x)|_{t\to-\infty} \ &\asymp\ \phi_0^{(\mathrm{in})} + \frac{4\pi}{R}\ P_1^{(\mathrm{in})}\,t + \sum_{m\neq 0}\frac{\mathrm{i}}{m}\,\Big(a_m^{(\mathrm{in})}\ \mathrm{e}^{-\frac{2\pi\mathrm{i}m}{R}(t+x)} + \bar{a}_m^{(\mathrm{in})}\ \mathrm{e}^{-\frac{2\pi\mathrm{i}m}{R}(t-x)}\Big) \qquad (5.41) \\ \alpha(t,x)|_{t\to-\infty} \ &\asymp\ \alpha_0^{(\mathrm{in})} + \frac{2\pi}{R}\ (kx+\tilde{k}t) + \sum_{m\neq 0}\frac{\mathrm{i}}{m}\,\Big(b_m^{(\mathrm{in})}\ \mathrm{e}^{-\frac{2\pi\mathrm{i}m}{R}(t+x)} + \bar{b}_m^{(\mathrm{in})}\ \mathrm{e}^{-\frac{2\pi\mathrm{i}m}{R}(t-x)}\Big)\ . \end{aligned}$$

In writing this equation, we took into account the boundary conditions (**??**). Also, the constant $\tilde{k}$ is the conserved charge for the Noether $U(1)$-current associated with the Lagrangian (**??**). The set, $\mathcal{A}^{(\mathrm{in})} = \{\phi_0^{(\mathrm{in})},\, P_1^{(\mathrm{in})},\, \alpha_0^{(\mathrm{in})},\, \tilde{k},\, a_m^{(\mathrm{in})},\, b_m^{(\mathrm{in})}\}$, can be interpreted as a classical "in-state" for a string propagating on the target manifold (see fig. 5.1). The nontrivial interaction occurs at some finite time when the fields take values in the vicinity of the tip of the cigar. After scattering at the tip, as $t \to +\infty$, the field configuration returns to the asymptotically flat domain and takes the same form as in the r.h.s. of (5.41) with the in-state $\mathcal{A}^{(\text{in})}$ replaced by the out-state $\mathcal{A}^{(\text{out})} = \{\phi_0^{(\text{out})},\, P_1^{(\text{out})},\; \alpha_0^{(\text{out})},\, \tilde{k},\, a_m^{(\text{out})},\, b_m^{(\text{out})}\}$.[1] The classical scattering problem can be formulated as the problem of finding the canonical transformation which maps $\mathcal{A}^{(\text{in})}$ to $\mathcal{A}^{(\text{out})}$. It turns out that the theory possesses two infinite sets of left- and right-currents [117], i.e.,

$$\partial_- W_s = 0 \ , \qquad \partial_+ \overline{W}_s = 0\, , \qquad (s = 2,\, 3, \ldots)\ , \tag{5.42}$$

so that the classical dynamics of the fields are strongly constrained. In particular, the magnitude of the zero-mode momentum remains unchanged after the scattering (see fig. 5.1),

$$P_1^{(\text{out})} = -P_1^{(\text{in})}\ . \tag{5.43}$$

Consider now the quantum theory. First of all we note that the value of the $U(1)$ charge is quantized so that $(n+2)\,\tilde{k} \,=\, m \in \mathbb{Z}$. Thus the space of states of the quantum theory splits into orthogonal subspaces $\mathcal{H}_{k,m}$ labeled by the twist parameter k and the integer m. The quantum theory still possesses the chiral currents satisfying eqs. (5.42). As a result, $\mathcal{H}_{k,m}$ can be decomposed into the highest weight irreps of the W-algebra, $\mathfrak{W}\otimes\bar{\mathfrak{W}}$ generated by the fields W_s and $\bar{W}_s$ [117]. Let $\mathcal{V}_h$ ($\bar{\mathcal{V}}_{\bar{h}}$) be the highest weight representation of the chiral W-algebra $\mathfrak{W}$ ($\bar{\mathfrak{W}}$) labeled by the highest weight h ($\bar{h}$). Then, schematically,

$$\mathcal{H}_{k,m} = \bigoplus_{\{h,\bar{h}\}} \mathcal{V}_h \otimes \overline{\mathcal{V}}_{\bar{h}}\, . \tag{5.44}$$

The highest weight h can be chosen to be a pair of numbers (Δ, w), where Δ coincides with the conformal dimensions of the highest weight vector, while w is the eigenvalue

[1] Strictly speaking, the winding number k is only conserved modulo an integer, i.e., $k^{(\text{out})} - k^{(\text{in})} \in \mathbb{Z}$. Here we ignore this and assume that $k^{(\text{out})} = k^{(\text{in})} \in (-\frac{1}{2}, +\frac{1}{2}]$.

of the dimensionless conserved charge $R^2\int_0^R \mathrm{d}x\, W_3(x)$, and similar for $\bar{h}$. Let us first focus on the "left" component $\mathcal{V}_h$ in the tensor product $\mathcal{V}_h\otimes\overline{\mathcal{V}}_{\bar{h}}$.

It should be clear that the quantum counterpart to the left components of the in-asymptotic fields (5.41) can be identified with the fields φ_+ and ϑ_+ given by (5.8). Since the quantum fields W_s are chiral currents, i.e. $W_s(t,x)=W_s(t+x)$, they can be expressed in terms of the asymptotic fields φ_+ and ϑ_+. Indeed, for given s, W_s is a certain order-s homogeneous polynomial with constant coefficients w.r.t. the fields φ'_+, ϑ'_+ and their higher derivatives (in other words, any monomials appearing within W_s contains exactly s derivative symbols). This implies that the Fock space $\mathcal{F}_{\mathbf{p}}$, which is the space of representation for the fields φ'_+, ϑ'_+, possesses the structure of the highest weight representation of the chiral W-algebra. It turns out that for real $\mathbf{p}$, the Fock space $\mathcal{F}_{\mathbf{p}}$ coincides with irrep $\mathcal{V}_h$ as a linear space, provided that $h=(\Delta,w)$ is related to $\mathbf{p}=(p_1,p_2)$ as follows

$$\begin{aligned}\Delta(p_1,p_2) &= \frac{p_1^2}{n}+\frac{p_2^2}{n+2}+\frac{1}{4n}\\ w(p_1,p_2) &= p_2\left(p_1^2+\frac{3n+2}{3(n+2)}\,p_2^2-\frac{2n+1}{12}\right).\end{aligned}\tag{5.45}$$

In fact, one can use these formulae to conveniently parameterize the highest weight h by the pair (p_1,p_2): $\mathcal{V}_h\equiv\mathcal{V}_{p_1,p_2}$. With this notation, a more accurate version of eq. (5.44) reads as

$$\mathcal{H}_{k,m}=\int_{p_1<0}^{\oplus}\mathcal{V}_{p_1,p_2}\otimes\overline{\mathcal{V}}_{p_1,\bar{p}_2}\ ,\tag{5.46}$$

where

$$p_2=\tfrac{1}{2}\big(m+(n+2)\,k\big)\ ,\qquad \bar{p}_2=\tfrac{1}{2}\big(m-(n+2)\,k\big)\ .\tag{5.47}$$

The direct integral in (5.46) does not include the domain with positive p_1, since, as follows from eqs. (5.45), $\mathcal{V}_{p_1,p_2}\equiv\mathcal{V}_{-p_1,p_2}$.

A basis of in-asymptotic states in $\mathcal{H}_{k,m}$ is formed by

$$a^{(\mathrm{in})}_{-m_1}\ldots a^{(\mathrm{in})}_{-m_N}\,\bar{a}^{(\mathrm{in})}_{-\bar{m}_1}\ldots\bar{a}^{(\mathrm{in})}_{-\bar{m}_{\bar{N}}}\,b^{(\mathrm{in})}_{-m_1}\ldots b^{(\mathrm{in})}_{-m_M}\,\bar{b}^{(\mathrm{in})}_{-\bar{m}_1}\ldots\bar{b}^{(\mathrm{in})}_{-\bar{m}_{\bar{M}}}\,|\,\mathrm{vac}\,\rangle\tag{5.48}$$

and can be identified with the states from the tensor product of the Fock space $\mathcal{F}_{p_1,p_2} \otimes \bar{\mathcal{F}}_{p_1,\bar{p}_2}$:

$$a_{-m_1} \dots a_{-m_N} \, \bar{a}_{-\bar{m}_1} \dots \bar{a}_{-\bar{m}_{\bar{N}}} \, b_{-m_1} \dots b_{-m_M} \, \bar{b}_{-\bar{m}_1} \dots \bar{b}_{-\bar{m}_{\bar{M}}} \, |\, p_1,\, p_2 \,\rangle \otimes |\, p_1,\, \bar{p}_2 \,\rangle \ . \tag{5.49}$$

Similarly for the out-states, one has

$$\begin{aligned} &a^{(\text{out})}_{-m_1} \dots a^{(\text{out})}_{-m_N} \, \bar{a}^{(\text{out})}_{-\bar{m}_1} \dots \bar{a}^{(\text{out})}_{-\bar{m}_{\bar{N}}} \, b^{(\text{out})}_{-m_1} \dots b^{(\text{out})}_{-m_M} \, \bar{b}^{(\text{out})}_{-\bar{m}_1} \dots \bar{b}^{(\text{out})}_{-\bar{m}_{\bar{M}}} \, |\, \text{vac} \,\rangle \sim \\ &a_{-m_1} \dots a_{-m_N} \, \bar{a}_{-\bar{m}_1} \dots \bar{a}_{-\bar{m}_{\bar{N}}} \, b_{-m_1} \dots b_{-m_M} \, \bar{b}_{-\bar{m}_1} \dots \bar{b}_{-\bar{m}_{\bar{M}}} \, |\, -p_1,\, p_2 \,\rangle \otimes |\, -p_1,\, \bar{p}_2 \,\rangle \ . \end{aligned} \tag{5.50}$$

Usually, the S-matrix is introduced as a unitary operator which relates the in- and out-asymptotic bases. In the case under consideration, the S-matrix can be interpreted as the intertwiner acting between the Fock spaces:

$$\hat{S} \ : \ \ \mathcal{F}_{p_1,p_2} \otimes \bar{\mathcal{F}}_{p_1,\bar{p}_2} \mapsto \mathcal{F}_{-p_1,p_2} \otimes \bar{\mathcal{F}}_{-p_1,\bar{p}_2} \ . \tag{5.51}$$

It turns out that the operator $\hat{S}$ has the following structure

$$\hat{S} = S_0(\mathbf{p}) \ \hat{S}_L \otimes \hat{S}_R \ , \tag{5.52}$$

where $\hat{S}_L$ intertwines the level subspaces, $\hat{S}_L \ : \ \mathcal{F}^{(L)}_{p_1,p_2} \mapsto \mathcal{F}^{(L)}_{-p_1,p_2}$, and is normalized by the condition $\hat{S}_L\,|\,p_1,p_2\,\rangle = |\,-p_1,\,p_2\,\rangle$, and similarly for $\hat{S}_R$. For a given level ℓ, the construction of the operators $\hat{S}_{L,R}$ is a straightforward algebraic task. The more delicate problem is finding the overall scalar factor $S_0(\mathbf{p})$. It was obtained in the minisuperspace approximation in ref. [118]. The exact form of $S_0(\mathbf{p})$ has been known since the unpublished work of the Zamolodchikov brothers [119].

Returning to the chiral transfer-matrices, let us note that these operators should act in the Hilbert space of the quantum cigar, and therefore their action should commute with the intertwiner $\hat{S}$:[2]

$$\hat{S}\,\tau(\lambda) = \tau(\lambda)\ \hat{S} \ . \tag{5.53}$$

[2]The intertwiner $\hat{S}$ should not to be confused with the so called "reflection" operator $\hat{R} : \mathcal{F}_{p_1,p_2} \otimes \bar{\mathcal{F}}_{p_1,\bar{p}_2} \mapsto \mathcal{F}_{p_1,p_2} \otimes \bar{\mathcal{F}}_{p_1,\bar{p}_2}$, and $[\hat{R}, \tau(\lambda)] = 0$. Note that $\hat{R} = \hat{\sigma} \circ \hat{S}$ where $\hat{\sigma} = \hat{\sigma}_L \otimes \hat{\sigma}_R$ and the chiral intertwiners $\hat{\sigma}_L : \mathcal{F}^{(L)}_{p_1,p_2} \mapsto \mathcal{F}^{(L)}_{-p_1,p_2}$ are defined by the conditions $\hat{\sigma}_L a_m = -a_m \hat{\sigma}_L$, $\hat{\sigma}_L b_m = +b_m \hat{\sigma}_L$, $\hat{\sigma}_L\,|\,p_1,p_2\,\rangle = |\,-p_1,p_2\,\rangle$, and similar for $\hat{\sigma}_R$ (see, e.g., [116]).

In practice, this condition implies that all matrix elements of the (dual) nonlocal IM in the basis of Fock states (5.49) are even functions of p_1 (for illustration see eqs. (5.34), (5.35)).

The quantum cigar also possesses an infinite set of the so-called local IM acting in $\mathcal{H}_{k,m}$. To get some feeling for these operators, we need to remind ourselves of an important feature of the model. Namely, it admits an equivalent "dual" description in terms of the so-called sine-Liouville model. The dual Lagrangian is given by [119]

$$\mathcal{L}^{(\mathrm{dual})} = \tfrac{1}{4\pi}\,\big((\partial_\sigma\varphi)^2 + (\partial_\sigma\vartheta)^2\big) + 2\mathcal{M}\,\mathrm{e}^{-\sqrt{n}\varphi}\,\cos\big(\sqrt{n+2}\,\vartheta\big)\,, \tag{5.54}$$

with the sine-Liouville fields satisfying the boundary conditions

$$\varphi(t, x+R) = \varphi(t,x)\ , \qquad \vartheta(t,x+R) = \vartheta(t,x) + \frac{2\pi m}{\sqrt{n+2}}\ . \tag{5.55}$$

Notice that the "coupling" $\mathcal{M}$ is a somewhat fake parameter of the Lagrangian – by an additive shift $\varphi \mapsto \varphi + const$ the value of $\mathcal{M}$ can be chosen to be any real number. Nevertheless, it is convenient to keep it unspecified.

To understand the relation between the fields in the NLSM and its dual description, let us take the "zero-mode" of the field φ

$$\varphi_0 = \int_0^R \frac{\mathrm{d}x}{R}\ \varphi(x)\ , \tag{5.56}$$

and consider the region $\varphi_0 \to +\infty$ in configuration space. In this asymptotic domain, the potential term in the action (5.54) can be neglected and $\frac{\varphi}{\sqrt{n}} \asymp \phi + const$, while $\frac{\vartheta}{\sqrt{n+2}}$ can be identified with $\tilde{\alpha}$ – the field from the cigar NLSM defined by the relation $J_\mu = \epsilon_{\mu\nu}\partial_\nu\tilde{\alpha}$, where J_μ stands for the Noether $U(1)$-current.

The twist parameter k has a natural interpretation in the dual description – it can be identified with the so-called quasimomentum. The sine-Liouville Lagrangian is invariant under the transformation $\vartheta \mapsto \vartheta + \frac{2\pi}{\sqrt{n+2}}$. Due to this periodicity, the space of states of the theory with the boundary conditions (5.55), splits on the orthogonal subspaces $\mathcal{H}_{k,m}$ such that for any state $|\,A\,\rangle \in \mathcal{H}_{k,m}$, the corresponding wave functional $\Psi_A[\,\varphi(x),\vartheta(x)\,]$ transforms as

$$\Psi_A\big[\,\varphi(x),\vartheta(x)+\tfrac{2\pi}{\sqrt{n+2}}\,\big] = \mathrm{e}^{2\pi \mathrm{i} k}\ \Psi_A[\,\varphi(x),\vartheta(x)\,]\ . \tag{5.57}$$

Let $P_s(\partial_+\varphi,\ \partial_+\vartheta,\ldots)$ be a local field of spin s, and a polynomial of $\partial_+\varphi$, $\partial_+\vartheta$ and their higher derivatives. All such fields are periodic in x, so that one can introduce the integral,

$$\mathfrak{i}_{s-1} = \Big(\frac{R}{2\pi}\Big)^{s-1}\int_0^R \frac{\mathrm{d}x}{2\pi}\ P_s(\partial_+\varphi,\ \partial_+\vartheta,\ldots)\ . \tag{5.58}$$

It turns out that for any even $s=2j$ there exists a local density (defined modulo the addition of a total derivative and an overall multiplicative constant) such that $\mathfrak{i}_{2j-1}$ is an integral of motion and satisfies the commutativity conditions

$$[\,\mathfrak{i}_{2j-1},\ \tau(\lambda)\,] = [\,\mathfrak{i}_{2j-1},\ \mathfrak{i}_{2j'-1}\,] = 0\ . \tag{5.59}$$

These operators are referred to as the (chiral) local IM. They were studied in ref.[114], where the explicit form for the first local IM and their vacuum eigenvalues, $i_{2j-1}(p_1,p_2)$ for $j=1,2,3$, can be found. Here we only note that for any $j=1,\ 2,\ \ldots$

$$P_{2j} = \sum_{l+m=j} C^{(j)}_{lm}\ (\partial_+\varphi)^{2l}(\partial_+\vartheta)^{2m}+\ldots\ , \tag{5.60}$$

where the dots stand for monomials which include higher derivatives of $\partial_+\varphi$ and $\partial_+\vartheta$ and the numerical coefficients $C^{(s)}_{lm}$ can be written as

$$C^{(j)}_{lm} = C_{2j-1}\ \frac{(-2)^{j+1}(2j-2)!}{(j+1)!}\ \frac{\big((n+2)(\frac{1}{2}-j)\big)_l\,\big((-n)(\frac{1}{2}-j)\big)_m}{l!\,m!}\ (-n)^{l-1}\,(n+2)^{m-1}\ . \tag{5.61}$$

Here $(a)_m = \prod_{i=0}^{m-1}(a+i)$ is the Pochhammer symbol. The overall normalization constant C_{2j-1} is usually set to

$$C_{2j-1} = \frac{2^{-3j}\ (j+1)!\ n(n+2)}{\big((n+2)(\frac{1}{2}-j)\big)_j\,\big((-n)(\frac{1}{2}-j)\big)_j}\ . \tag{5.62}$$

Chapter 6

Chiral transfer-matrix for $\mathbb{Z}_n$ parafermions

While quantizing the sausage model within the BLZ approach, we have run into the problem of finding the spectrum of the chiral transfer-matrices for the cigar NLSM. As it has been explained, we can consider $\tau(\lambda)$ as an operator acting in the Fock space $\mathcal{F}_{\mathbf{p}}$ with real $\mathbf{p} = (p_1, p_2)$. From the formal point of view, the same spectral problem can be posed for any complex values of $\mathbf{p}$. Notice, that for real p_2, λ^2 and pure imaginary p_1, the operator $\tau(\lambda)$ is Hermitian. The spectral problem in this case (except for $p_1 = 0$) is not directly related to the quantization of the sausage model, however for $n = 2, 3, \ldots$ and a certain discrete set of p_1 and p_2, it gives a better understanding of the interplay between the BLZ approach and that based on the discretization of the quantum system. It will be the subject of our study here.

6.1 Bosonization of $\mathbb{Z}_n$ parafermions

Let us take a closer look at the vertex operators $V^{(\pm)}$ (5.7), which appear in the construction of the chiral transfer-matrices τ_j. As it was already mentioned, the constants $c^{\pm}$ can be arbitrarily chosen. Let us set $c^{\pm} = 1$ and assume that $n \geq 2$ is a positive integer. Then, eq. (5.7) can be recognized as the bosonization relations for the Fateev-Zamolodchikov $\mathbb{Z}_n$ parafermions [120]. More precisely, as follows from the normalization condition (5.14), the chiral nonlocal fields

$$\psi^{\pm} = \sqrt{n}\; q^{-\frac{1}{2}}\; V^{\pm} \tag{6.1}$$

can be understood as canonically normalized parafermion currents,

$$\psi^{\pm}(x_2)\,\psi^{\mp}(x_1)\big|_{x_2\to x_1+0} \sim 1\times\big(x_2-x_1\big)^{-2\Delta_\psi} \tag{6.2}$$

of the conformal dimension $\Delta_\psi = 1-\frac{1}{n}$.

The chiral algebra of parafermion currents was introduced by Fateev and Zamolodchikov in ref.[121], in the construction of the $\mathbb{Z}_n$ CFT models with central charge

$$c_n = \frac{2(n-1)}{n+2}\ , \tag{6.3}$$

describing the multicritical points of the $\mathbb{Z}_n$ statistical systems (certain generalizations of the $\mathbb{Z}_2$ invariant Ising model) [59]. The chiral component of the Hilbert space of the $\mathbb{Z}_n$ CFT can be decomposed into irreps $\mathcal{V}_{\mathfrak{j}}$ of the chiral algebra. Here, the subscript $\mathfrak{j}$ stands for the highest weight of the irrep with highest weight vector $|\,\sigma_{\mathfrak{j}}\,\rangle$ having conformal dimension

$$\Delta_{\mathfrak{j}} = \frac{\mathfrak{j}\,(n-2\mathfrak{j})}{n\,(n+2)}\ . \tag{6.4}$$

The admissible values of $\mathfrak{j}$ are given by non-negative integers and half-integers restricted by the condition

$$\mathfrak{j} = 0,\ \tfrac{1}{2},\ 1,\ \tfrac{3}{2},\ldots,\ \tfrac{1}{2}\left[\tfrac{n}{2}\right]\ , \tag{6.5}$$

where $\left[\frac{n}{2}\right]$ is the integer part of $n/2$. The fundamental parafermion currents ψ^+ and ψ^- act in $\mathcal{V}_{\mathfrak{j}}$ and carry the $\mathbb{Z}_n$-charges $+2$ and -2 respectively:

$$\Omega\,\psi^{\pm}\,\Omega^{-1} = \omega^{\pm 2}\,\psi^{\pm}\ , \qquad \text{where} \qquad \omega = \mathrm{e}^{-\frac{2\pi\mathrm{i}}{n}}\ . \tag{6.6}$$

Note that $2\mathfrak{j}$ can be identified with the $\mathbb{Z}_n$-charge of the highest weight vector:[1]

$$\Omega\ |\,\sigma_{\mathfrak{j}}\,\rangle = \omega^{2\mathfrak{j}}\,|\,\sigma_{\mathfrak{j}}\,\rangle\ . \tag{6.7}$$

[1] To be more precise, the chiral component of the Hilbert space of the $\mathbb{Z}_n$ CFT contains, together with the irrep $\mathcal{V}^{(+)}_{\mathfrak{j}} \equiv \mathcal{V}_{\mathfrak{j}}$, the irrep $\mathcal{V}^{(-)}_{\mathfrak{j}}$ whose highest weight vector has the same conformal dimension (6.4) but carries the $\mathbb{Z}_n$-charge $\omega^{-2\mathfrak{j}}$. For even n, $\mathcal{V}^{(-)}_{\frac{n}{4}} = \mathcal{V}^{(+)}_{\frac{n}{4}}$. The chiral transfer-matrix (6.13) is a $\mathbb{Z}_2$-invariant operator which does not distinguish between the irreps $\mathcal{V}^{(+)}_{\mathfrak{j}}$ and $\mathcal{V}^{(-)}_{\mathfrak{j}}$.

The irrep $\mathcal{V}_{\mathfrak{j}}$ naturally splits on the subspaces $\mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})}$ characterized by a definite value of the $\mathbb{Z}_n$-charge,

$$\mathcal{V}_{\mathfrak{j}} = \Big[\oplus_{s=0}^{2\mathfrak{j}}\ \mathcal{V}_{\mathfrak{j}}^{(2\mathfrak{j}-2s)}\Big] \oplus \Big[\oplus_{s=1}^{n-2\mathfrak{j}-1}\ \mathcal{V}_{\mathfrak{j}}^{(2\mathfrak{j}+2s)}\Big]\ :\quad \Omega\,\mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})} = \omega^{\mathfrak{m}}\ \mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})}\ . \tag{6.8}$$

The lowest possible conformal dimension in the subspace $\mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})}$ is given by $\Delta_{\mathfrak{j},\mathfrak{m}}$ for $\mathfrak{m} = -2\mathfrak{j},\ -2\mathfrak{j}+2,\ \ldots, 2\mathfrak{j}$, and $\Delta_{\mathfrak{j},\mathfrak{m}} + \frac{1}{2}\,(\mathfrak{m}-2\mathfrak{j})$ for $\mathfrak{m} = 2\mathfrak{j}+2, \ldots, 2n-2\mathfrak{j}-2$. Here we use the notation

$$\Delta_{\mathfrak{j},\mathfrak{m}} = \frac{\mathfrak{j}(\mathfrak{j}+1)}{n+2} - \frac{\mathfrak{m}^2}{4n}\ . \tag{6.9}$$

In what follows $|\,\sigma_{\mathfrak{j},\mathfrak{m}}\,\rangle$ will denote the state from the subspace $\mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})}$ with $\mathfrak{m} = 2\mathfrak{j},\ 2\mathfrak{j}-2,\ \ldots, -2\mathfrak{j}$ of the lowest conformal dimension $\Delta_{\mathfrak{j},\mathfrak{m}}$.

From the mathematical point of view the bosonization of the algebra of parafermion currents implies that the subspaces $\mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})}$ with $\mathfrak{m} = 2\mathfrak{j},\ 2\mathfrak{j}-2,\ \ldots, -2\mathfrak{j}$ can be understood as a cohomology of the Fock space $\mathcal{F}_{p_1,p_2}$ where

$$p_1 = \tfrac{\mathrm{i}}{2}\ \mathfrak{m}\,, \qquad p_2 = \mathfrak{j} + \tfrac{1}{2}\ , \tag{6.10}$$

with respect to a certain BRST complex à la the Felder complex [122] involved in the bosonization of the highly reducible Verma modules over the Virasoro algebra. Among other things, the bosonization formula (6.1) leads to the following relation for the matrix elements of the parafermion currents:

$$\langle\,\sigma_{\mathfrak{j},\mathfrak{m}}\,|\prod_{m=1}^{M}\psi^{\varepsilon_m}(x_m)\,|\,\sigma_{\mathfrak{j},\mathfrak{m}}\,\rangle = (nq^{-1})^{\frac{M}{2}}\ \langle\, p_1,\, p_2\,|\prod_{m=1}^{M}V^{\varepsilon_m}(x_m)\,|\, p_1,\, p_2\,\rangle \qquad (\varepsilon_m = \pm)\ , \tag{6.11}$$

provided $\sum_{m=1}^{L}\varepsilon_m = 0$ and the pairs $(\mathfrak{j},\,\mathfrak{m})$ are related to (p_1, p_2) as in eq. (6.10). It is not difficult to see that the $\mathbb{Z}_n$-charge operator is bosonized by the relation

$$\Omega = \mathrm{e}^{4\pi P_1} \tag{6.12}$$

and the operator $\tau(\lambda)$ can be written in the form

$$\tau(\lambda) = \mathrm{Tr}\Big[\overleftarrow{\mathcal{P}}\exp\Big(\mathrm{i}\,\tfrac{\lambda_+}{\sqrt{n}}\int_{t_0}^{t_0+R}\mathrm{d}x\ \big(\psi^+\ \sigma_+ + \psi^-\ \sigma_-\big)\Big)\ \Omega^{-\frac{1}{2}\sigma_3}\Big]\ . \tag{6.13}$$

Therefore, $\tau(\lambda)$ can be understood as an operator which invariantly acts in the subspaces $\mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})}$ of the irrep $\mathcal{V}_{\mathfrak{j}}$ of the algebra of parafermion currents. We can now address the problem of the diagonalization of this operator. Notice that it is sufficient to consider $\mathfrak{m} \geq 0$, and in what follows we will always assume that

$$\begin{aligned} \mathfrak{j} &= 0,\, \tfrac{1}{2},\, 1,\, \tfrac{3}{2}, \ldots,\, \tfrac{1}{2}\left[\tfrac{n}{2}\right] \\ \mathfrak{m} &= 2\,\mathfrak{j},\, 2\,\mathfrak{j}-2,\, \ldots, 2\,\mathfrak{j}-2\,[\mathfrak{j}] \;. \end{aligned} \tag{6.14}$$

6.2 Discretization of the chiral transfer-matrix

The goal of this section is to propose a lattice version of the parafermionic chiral transfer-matrix (6.13). For this purpose we return back to the formula (**??**) for the classical conserved charges in the sausage model and follow the approach based on discretization that was mentioned in the introduction.

Split the integration contour onto N small segments of size δ and consider the elementary transport matrices in the fundamental representation:

$$\boldsymbol{M}^{(\mathrm{s})}(\mu) = \pi_{\frac{1}{2}}\left[\overleftarrow{\mathcal{P}}\exp\int_{x_{\mathrm{s}}-\delta/2}^{x_{\mathrm{s}}+\delta/2}\mathrm{d}x\,\boldsymbol{A}_x\right] \qquad (\mathrm{s}=1,\ldots,N)\;. \tag{6.15}$$

These can be expressed in terms of the elementary "light-cone" transport matrices $\boldsymbol{M}^{(\mathrm{s})}(\mu) = \bar{\boldsymbol{L}}^{(\mathrm{s})}(\mu)\,\boldsymbol{L}^{(\mathrm{s})}(\mu)$ (see fig. 6.1) and, as it follows from eq. (**??**),

$$\begin{aligned} \left\{\boldsymbol{L}^{(\mathrm{s})}(\mu) \underset{,}{\otimes} \boldsymbol{L}^{(\mathrm{s}')}(\mu')\right\} &= \left[\boldsymbol{L}^{(\mathrm{s})}(\mu)\otimes\boldsymbol{L}^{(\mathrm{s}')}(\mu')\,,\boldsymbol{r}_{\frac{1}{2}\frac{1}{2}}(\mu/\mu')\right]\,\delta_{\mathrm{ss}'} \\ \left\{\bar{\boldsymbol{L}}^{(\mathrm{s})}(\mu) \underset{,}{\otimes} \bar{\boldsymbol{L}}^{(\mathrm{s}')}(\mu')\right\} &= \left[\bar{\boldsymbol{L}}^{(\mathrm{s})}(\mu)\otimes\bar{\boldsymbol{L}}^{(\mathrm{s}')}(\mu')\,,\boldsymbol{r}_{\frac{1}{2}\frac{1}{2}}(\mu/\mu')\right]\,\delta_{\mathrm{ss}'} \\ \left\{\boldsymbol{L}^{(\mathrm{s})}(\mu) \underset{,}{\otimes} \bar{\boldsymbol{L}}^{(\mathrm{s}')}(\mu')\right\} &= 0\;, \end{aligned} \tag{6.16}$$

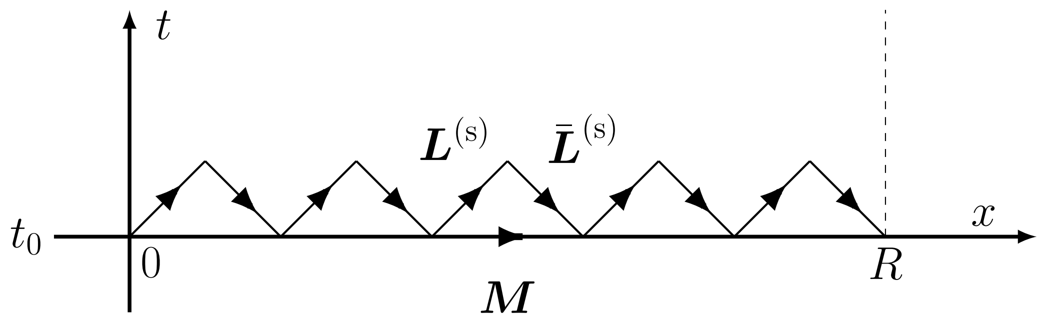


Figure 6.1: By replacing the integration over the segment at the time slice $t = t_0$ by integration over the light-cone pieces, the monodromy matrix can be expressed as a product of the elementary "light-cone" transport matrices: $\boldsymbol{M}(\mu) = \bar{\boldsymbol{L}}^{(N)}(\mu)\,\boldsymbol{L}^{(N)}(\mu)\,\ldots\,\bar{\boldsymbol{L}}^{(2)}(\mu)\,\boldsymbol{L}^{(2)}(\mu)\,\bar{\boldsymbol{L}}^{(1)}(\mu)\,\boldsymbol{L}^{(1)}(\mu)$.

where

$$\boldsymbol{r}_{\frac{1}{2}\frac{1}{2}}(\mu) = \begin{pmatrix} a(\mu) & 0 & 0 & 0 \\ 0 & 0 & c(\mu) & 0 \\ 0 & c(\mu) & 0 & 0 \\ 0 & 0 & 0 & a(\mu) \end{pmatrix} \qquad \text{with} \qquad \begin{array}{l} a(\mu) = \frac{1}{2}\,\frac{\mu+\mu^{-1}}{\mu-\mu^{-1}} \\ \\ c(\mu) = \frac{1}{\mu-\mu^{-1}} \end{array} . \tag{6.17}$$

Consider the structure of $\boldsymbol{L}^{(\mathrm{s})}(\mu) = \pi_{\frac{1}{2}}\Big[\overleftarrow{\mathcal{P}}\exp\int\limits_{x_\mathrm{s}-\delta/2}^{x_\mathrm{s}+\delta/2}\mathrm{d}x\,\boldsymbol{A}_+\Big]$. From the explicit form of $\boldsymbol{A}_+$ (**??**), one has

$$\boldsymbol{L}(\mu) = \begin{pmatrix} 1+\frac{f(\mu)}{4}\,H & \frac{\mathrm{i}}{2}\,g(\mu)\,E_- \\ \frac{\mathrm{i}}{2}\,g(\mu)\,E_+ & 1-\frac{f(\mu)}{4}\,H \end{pmatrix} + O(\delta^2)\ , \tag{6.18}$$

with

$$E_\pm = \int_{x_\mathrm{s}-\delta/2}^{x_\mathrm{s}+\delta/2}\mathrm{d}x\,\Pi_+(x)\,\mathrm{e}^{\mp Q(x)}\ , \qquad H = 2\int_{x_\mathrm{s}-\delta/2}^{x_\mathrm{s}+\delta/2}\mathrm{d}x\,\Pi_+(x)\ , \tag{6.19}$$

and, as it follows from the canonical commutation relations,

$$\{E_+, E_-\} = -H\,, \qquad \{H, E_\pm\} = \pm 2E_\pm\ . \tag{6.20}$$

Here, to simplify the notation, we have temporarily dropped the superscript "s" and are focusing on a single site. Let's look at the above formulae from a slightly different angle. Suppose we are given the matrices $\boldsymbol{L}$ of the form (6.18) with arbitrary functions $g(\mu)$ and $f(\mu)$ where H, $E_\pm$ satisfy the Poisson bracket relations (6.20). The requirement that $\boldsymbol{L}$ obeys the Yang-Baxter Poisson algebra (6.16) leads to two equations imposed on the functions f and g:

$$\begin{aligned} a(\mu/\lambda)\, g(\lambda) - c(\mu/\lambda)\, g(\mu) &= \frac{1}{2}\, g(\lambda)\, f(\mu) \\ c(\mu/\lambda) &= \frac{1}{2}\, \frac{g(\lambda) g(\mu)}{f(\lambda) - f(\mu)} \ . \end{aligned} \tag{6.21}$$

One can show that, modulo the rescaling $\mu \mapsto const\, \mu$, the most general solution to these equations is given by:

$$\begin{aligned} f(\mu) &= \frac{(1-\kappa)\, \mu + (1+\kappa)\, \mu^{-1}}{(1-\kappa)\, \mu - (1+\kappa)\, \mu^{-1}} \\ g(\mu) &= \frac{2\sqrt{1-\kappa^2}}{(1-\kappa)\, \mu - (1+\kappa)\, \mu^{-1}} \ . \end{aligned} \tag{6.22}$$

with κ an arbitrary parameter.

This simple calculation hints as to how we should proceed with the deduction of the quantum counterpart of the above formulae. Strictly speaking, there is no canonical prescription for the quantization of the Poisson brackets (6.20), however, it seems natural to substitute them by the defining relations of the $U_q(\mathfrak{sl}(2))$ quantum algebra with $q = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}}$:

$$\big[\, \mathtt{E}_+ \,, \mathtt{E}_- \,\big] = \frac{q^{\mathtt{H}} - q^{-\mathtt{H}}}{q - q^{-1}} \ , \qquad \big[\, \mathtt{H} \,, \, \mathtt{E}_\pm \,\big] = \pm 2 \ \mathtt{E}_\pm \ . \tag{6.23}$$

For the quantum version of eq. (6.18), we put forward the following ansatz

$$\boldsymbol{L}(\mu) = \begin{pmatrix} F_-(\mu)\, q^{-\frac{1}{2}\mathtt{H}} + F_+(\mu)\, q^{+\frac{1}{2}\mathtt{H}} & (q-q^{-1})\, G(\mu)\ \mathtt{E}_- \\ (q-q^{-1})\, G(\mu)\ \mathtt{E}_+ & F_-(\mu)\, q^{+\frac{1}{2}\mathtt{H}} + F_+(\mu)\, q^{-\frac{1}{2}\mathtt{H}} \end{pmatrix} \ , \tag{6.24}$$

where $F_\pm(\mu)$ and $G(\mu)$ are some functions. The classical matrix will be recovered if we assume that as $\hbar \to 0$,

$$q^{\pm\frac{1}{2}\mathtt{H}} = 1 \pm \tfrac{1}{4}\, \mathrm{i}\hbar\ \mathtt{H} + o(\delta) \ , \tag{6.25}$$

while

$$\mathtt{H} = -\mathrm{i}\,\hbar^{-1}\;H + O(\hbar^0)\;, \qquad\qquad \mathtt{E}^{(\pm)} = \hbar^{-1}\;E^{(\pm)} + O(\hbar^0)\,, \tag{6.26}$$

and also

$$\lim_{\hbar\to 0} F_\pm(\mu) = \frac{1}{2}\,\big(1 \pm f(\mu)\big)\;, \qquad \lim_{\hbar\to 0} G(\mu) = \frac{1}{2}\;g(\mu)\;. \tag{6.27}$$

It is clear that the operator valued matrix $\boldsymbol{L}$ must satisfy the Yang-Baxter algebra

$$\boldsymbol{R}_{\frac{1}{2}\frac{1}{2}}(\mu'/\mu)\,\big(\boldsymbol{L}(\mu)\otimes 1\big)\,\big(1\otimes \boldsymbol{L}(\mu')\big) = \big(1\otimes \boldsymbol{L}(\mu')\big)\big(\boldsymbol{L}(\mu)\otimes 1\big)\,\boldsymbol{R}_{\frac{1}{2}\frac{1}{2}}(\mu'/\mu)\;, \tag{6.28}$$

where $\boldsymbol{R}_{\frac{1}{2}\frac{1}{2}}$ is given by eq. (5.18). The ansatz (6.24), combined with this relation, yields

$$F_+(\mu) = +a\mu\;G(\mu)\;, \qquad F_-(\mu) = -(a\mu)^{-1}\;G(\mu)\;, \tag{6.29}$$

where a is an arbitrary constant. Consistency with eq. (6.27) and the explicit form for f and g (6.22) requires that

$$\lim_{\hbar\to 0}(a^2) = \frac{1-\kappa}{1+\kappa}\;. \tag{6.30}$$

We may now use the well known fact that the $U_q(\mathfrak{sl}(2))$ algebra with defining relations (6.23) and $q = \mathrm{e}^{\frac{\mathrm{i}\hbar}{2}}$ admits a formal realization in terms of the Heisenberg algebra

$$[\,\mathtt{Q}\,,\mathtt{P}\,] = \mathrm{i}\,\hbar\;. \tag{6.31}$$

Namely [123],

$$\begin{aligned} \mathtt{E}_+ &= \mathrm{e}^{-\frac{1}{2}\mathtt{Q}}\;\frac{\sinh(\,\frac{1}{2}\,\mathtt{P} + \hbar C\,)}{\sin(\frac{1}{2}\hbar)}\;\mathrm{e}^{-\frac{1}{2}\mathtt{Q}} \\ \mathtt{E}_- &= \mathrm{e}^{+\frac{1}{2}\mathtt{Q}}\;\frac{\sinh(\,\frac{1}{2}\,\mathtt{P} - \hbar C\,)}{\sin(\frac{1}{2}\hbar)}\;\mathrm{e}^{+\frac{1}{2}\mathtt{Q}} \\ \mathtt{H} &= -2\mathrm{i}\;\hbar^{-1}\;\mathtt{P}\;. \end{aligned} \tag{6.32}$$

It is not difficult to see that this can be thought of as the quantum counterpart of eqs. (6.19). The constant C is arbitrary and is related to the value of the quantum Casimir. In fact, it is convenient to substitute it by ℓ: $C = \mathrm{i}\,(2\ell+1)/4$, then

$$\frac{1}{2}\Big[(q+q^{-1})\left(q^{\mathtt{H}}+q^{-\mathtt{H}}\right)+(q-q^{-1})^2\ \left(\mathtt{E}_-\mathtt{E}_+ + \mathtt{E}_+\mathtt{E}_-\right)\Big] = q^{2\ell+1}+q^{-2\ell-1}\ . \quad (6.33)$$

Let us introduce the Heisenberg group generators, subject to the Weyl commutation relations

$$\mathtt{V} = \exp\left(\tfrac{1}{2}\,\mathtt{P}\right), \qquad \mathtt{U} = \exp(\,\mathtt{Q}\,)\ : \qquad \mathtt{U}\mathtt{V} = q\,\mathtt{V}\mathtt{U}\ . \quad (6.34)$$

Our analysis suggests that the 2×2 operator valued matrix $\boldsymbol{L}(\mu) = \boldsymbol{\mathcal{L}}^{(\ell)}(\mu|\mathtt{U},\mathtt{V})$, where [124, 125]

$$\boldsymbol{\mathcal{L}}^{(\ell)}(\mu|\mathtt{U},\mathtt{V}) = \begin{pmatrix} \left(\mu\ \mathtt{V}-\mu^{-1}\,\mathtt{V}^{-1}\right) & \mathrm{i}\left(q^{-\ell}\ \mathtt{V}-q^{+\ell}\ \mathtt{V}^{-1}\right)\mathtt{U} \\ \mathrm{i}\left(q^{+\ell}\ \mathtt{V}-q^{-\ell}\ \mathtt{V}^{-1}\right)\mathtt{U}^{-1} & \left(\mu\ \mathtt{V}^{-1}-\mu^{-1}\ \mathtt{V}\right) \end{pmatrix} \quad (6.35)$$

satisfies the Yang-Baxter relation (6.28). Furthermore, it is easy to see that the same properties still hold for the matrix which depends on a set of six parameters $\{a, b, c, g, r, \ell\}$:

$$\boldsymbol{\mathcal{L}}\Big[{}^{g\,r\,\ell}_{a\,b\,c}\Big](\mu\,|\,\mathtt{U},\mathtt{V}) = g\,r^{\frac{\sigma_3}{2}}\ \boldsymbol{\mathcal{L}}^{(\ell)}(a\mu\,|\,b\,\mathtt{U}, c\,\mathtt{V})\ r^{\frac{\sigma_3}{2}}\ . \quad (6.36)$$

Most of the parameters, except maybe r and ℓ, look trivial when we consider only a single site. However, for the discretized system of N sites, the possibility that the parameters may be different at different sites should be considered. In any case, one can expect that for a properly adjusted set $\{a_{\mathrm{s}}, b_{\mathrm{s}}, c_{\mathrm{s}}, g_{\mathrm{s}}, r_{\mathrm{s}}, \ell_{\mathrm{s}}\}_{\mathrm{s}=1}^{N}$ the discretized quantum transfermatrix in the fundamental representation is given by

$$\mathcal{T}^{(N)}(\mu) = (-\mu)^N\ \mathrm{Tr}\Big[\overleftarrow{\mathcal{P}}\Big(\prod_{\mathrm{s}=1}^{N}\boldsymbol{\mathcal{L}}\Big[\ {}^{g_{\mathrm{s}}\,r_{\mathrm{s}}\,\ell_{\mathrm{s}}}_{a_{\mathrm{s}}\,b_{\mathrm{s}}\,c_{\mathrm{s}}}\ \Big](\mu\,|\,\mathtt{U}_{\mathrm{s}},\mathtt{V}_{\mathrm{s}})\Big)\,(q^{d}\ \mathtt{V})^{-\sigma_3}\Big]\ , \quad (6.37)$$

where $\mathtt{V} = \prod_{\mathrm{s}=1}^{N}\mathtt{V}_{\mathrm{s}}$ is the discretized counterpart to the exponential $\mathrm{e}^{\pi P_1}$ and d is some constant. The overall factor $(-\mu)^N$ is inserted to ensure that the transfer-matrix is a polynomial in μ^2 of order N. Notice that

$$\mathcal{T}^{(N)}(0) = \mathtt{V}^{-2}\, q^{-d} \prod_{s=1}^{N} g_{\rm s}\, r_{\rm s} c_{\rm s}^{-1} + \mathtt{V}^{+2}\, q^{d} \prod_{s=1}^{N} g_{\rm s}\, r_{\rm s}^{-1} c_{\rm s} \tag{6.38}$$

is expressed in terms of integer powers of

$$\mathtt{X}_{\rm s} \equiv \mathtt{V}_{\rm s}^2 \tag{6.39}$$

rather than $\mathtt{V}_{\rm s}$. In fact, this is true for any μ, and it can be made explicit by rewriting (6.37) in the equivalent form:

$$\mathcal{T}^{(N)}(\mu) = \mathcal{C}\,\mathrm{Tr}\Big[\overleftarrow{\mathcal{P}}\Big(\prod_{j=1}^{N}\big(\boldsymbol{\mathcal{L}}_{-}^{(r_{\rm s}\,\ell_{\rm s})}\big(\mathscr{b}_{\rm s}\mathtt{U}_{\rm s},\, \mathscr{c}_{\rm s}^2\,\mathtt{X}_{\rm s}\big) - \mathscr{a}_{\rm s}{}^2\mu^2\,\boldsymbol{\mathcal{L}}_{+}^{(r_{\rm s}\,\ell_{\rm s})}\big(\mathscr{b}_{\rm s}\mathtt{U}_{\rm s},\, \mathscr{c}_{\rm s}^2\,\mathtt{X}_{\rm s}\big)\big)\Big)\begin{pmatrix} q^{-d}\,\mathtt{Z}^{-1} & 0 \\ 0 & q^{d} \end{pmatrix}\Big] \tag{6.40}$$

where $\mathcal{C}$ is a constant, $\boldsymbol{\mathcal{L}}_{\pm}$ are triangular matrices:

$$\begin{aligned} \boldsymbol{\mathcal{L}}_{-}^{(r\ell)}(\mathtt{U},\mathtt{X}) &= \begin{pmatrix} r & 0 \\ -\mathrm{i}\left(q^{-\ell-1} - q^{+\ell-1}\;\mathtt{X}\right)\mathtt{U}^{-1} & r^{-1}\,\mathtt{X} \end{pmatrix} \\ \boldsymbol{\mathcal{L}}_{+}^{(r\ell)}(\mathtt{U},\mathtt{X}) &= \begin{pmatrix} r\,\mathtt{X} & \mathrm{i}\left(q^{1+\ell} - q^{1-\ell}\,\mathtt{X}\right)\mathtt{U} \\ 0 & r^{-1} \end{pmatrix} \;, \end{aligned} \tag{6.41}$$

and

$$\mathtt{Z} = \prod_{\rm s=1}^{N} \mathtt{X}_{\rm s}\ . \tag{6.42}$$

Finally, let us note that the set of formal operators $\{\mathtt{U}_{\rm s}, \mathtt{X}_{\rm s}\}_{\rm s=1}^{N}$ satisfy the commutation relations

$$\big[\mathtt{U}_{\rm s},\, \mathtt{U}_{\rm s'}\big] = \big[\mathtt{X}_{\rm s},\, \mathtt{X}_{\rm s'}\big] = \big[\mathtt{U}_{\rm s},\, \mathtt{X}_{\rm s'}\big] = 0 \quad ({\rm s}\neq{\rm s'})\ , \qquad \mathtt{U}_{\rm s}\,\mathtt{X}_{\rm s} = q^2\,\mathtt{X}_{\rm s}\,\mathtt{U}_{\rm s}\ , \tag{6.43}$$

and also that $\mathtt{Z}$ commutes with $\mathcal{T}^{(N)}(\mu)$ for any values of the parameters in (6.40).

We are now faced with the task of specifying the parameters in (6.40). Our analysis outlined below shows that in all likelihood it is enough to consider the case with

$$\mathcal{C} = a_{\rm s} = b_{\rm s} = c_{\rm s} = r_{\rm s} = 1\,, \qquad \ell = -\tfrac{1}{2}\,, \qquad d = 2p_2 - 1\ . \tag{6.44}$$

In this case, we expect that, with a properly chosen representation of the algebra (6.43) and with a properly understood scaling limit, the operator $\mathcal{T}^{(N)}(\mu)$ can be identified with the chiral transfer-matrix $\tau(\lambda)$ defined in eq. (5.19) with $j = \frac{1}{2}$. The discretized operator should be restricted to the sector with

$$\mathtt{Z} = q^{1-2(p_2+{\rm i}p_1)}\ , \tag{6.45}$$

where, perhaps, some constraints need to be imposed on (p_1, p_2). Recall that the pair (p_1, p_2) label the Fock space $\mathcal{F}_{p_1,p_2}$ in which $\tau(\lambda)$ acts.

We came to the above conjecture through the analysis of the case of integer $n \equiv \frac{2\pi}{\hbar}$. At $n = 2,\, 3, \ldots$, the formal algebra (6.43) admits an n^N dimensional representation where the operators associated with each site are given by the $n \times n$ matrices

$$\mathtt{X}^{\alpha}_{\beta} = \delta_{\alpha+1,\beta}\ , \qquad \mathtt{U}^{\alpha}_{\beta} = \omega^{\alpha}\,\delta_{\alpha,\beta}\ , \qquad \omega = {\rm e}^{-\frac{2\pi{\rm i}}{n}}\ . \tag{6.46}$$

Here $\alpha, \beta = 0,\, \ldots\, , n-1$ and

$$\delta_{\alpha\beta} = \begin{cases} 1\ , & \alpha = \beta \pmod{n} \\ 0\ , & \alpha \neq \beta \pmod{n} \end{cases}\ . \tag{6.47}$$

Now $\mathcal{T}^{(N)}(\mu)$ is an $n^N \times n^N$ matrix. We plan to discuss the diagonalization of the operator (6.40), with the parameters $\{a_{\rm s}, b_{\rm s}, c_{\rm s}, r_{\rm s}, \ell_{\rm s}\}$ not depending on "s", in a separate publication. Here we only note that such an operator but without the diagonal matrix $\begin{pmatrix} q^{-d}\,\mathtt{Z}^{-1} & 0 \\ 0 & q^{d} \end{pmatrix}$, was studied in the works [125, 126] in the context of the chiral Potts model. The extra diagonal matrix does not significantly change the diagonalization procedure.

In the case (6.44), the diagonalization problem simplifies dramatically and can be solved within the standard Bethe ansatz framework (see next section for some details). This allows a thorough investigation of the scaling limit. An important point is that the scaling procedure requires a choice of some reference state ("vacuum") and only states whose "energy" measured from the vacuum energy remains finite as $N \to \infty$, should be taken into account. Let $\mathcal{H}_M^{(N)}$ be the subspace in the tensor product $\mathcal{H}^{(N)} \equiv \big(\mathbb{C}^n\big)^{\otimes N}$ with $\mathtt{Z} = \omega^M$. The operator $\mathcal{T}^{(N)}(\mu)$, with parameters as in eq. (6.44), restricted to the $\mathcal{H}^{(N)}_{\mathfrak{j}-\frac{\mathfrak{m}}{2}}$ subspace, with $\mathfrak{j}$ and $\mathfrak{m}$ satisfying the conditions (6.14), commutes with the Hamiltonian of the Fateev-Zamolodchikov $\mathbb{Z}_n$ spin chain

$$\mathbb{H}^{(N)} = -\frac{1}{n}\sum_{\mathtt{s}=1}^{N}\sum_{l=1}^{n-1}\frac{(\mathtt{X}_\mathtt{s})^l + \big(\mathtt{U}_\mathtt{s}\mathtt{U}_{\mathtt{s}+1}^\dagger\big)^l}{\sin\big(\frac{\pi l}{n}\big)}\Bigg|_{\mathtt{Z}=\omega^{\mathfrak{j}-\frac{\mathfrak{m}}{2}}} \tag{6.48}$$

with twisted boundary conditions

$$\mathtt{U}_{N+1} = \omega^{\mathfrak{j}+\frac{\mathfrak{m}}{2}}\ \mathtt{U}_1\ . \tag{6.49}$$

It also commutes with the lattice shift operator

$$\mathbb{P}^{(N)} = \delta^{\alpha_1}_{\beta_2}\delta^{\alpha_2}_{\beta_3}\ldots\delta^{\alpha_N}_{\beta_1+\mathfrak{j}+\frac{\mathfrak{m}}{2}}\Big|_{\mathtt{Z}=\omega^{\mathfrak{j}-\frac{\mathfrak{m}}{2}}}\ . \tag{6.50}$$

Our numerical work for the vacuum state of the Hamiltonian $\mathbb{H}^{(N)}$ in the sector $\mathcal{H}^{(N)}_{\mathfrak{j}-\frac{\mathfrak{m}}{2}}$ for different admissible values of n, $\mathfrak{j}$ and $\mathfrak{m}$ gives strong support to the following relations (see Appendix D)

$$\tau(\lambda) = \operatorname*{slim}_{N\to\infty}\ F^{(N)}(\lambda)\,\mathcal{T}^{(N)}\Big(\big(\tfrac{\pi}{N}\big)^{\frac{1}{n}}\,\lambda\Big)\ , \tag{6.51}$$

where

$$F^{(N)}(\lambda) = \begin{cases} \exp\Big(\sum_{l=1}^{[\frac{n}{2}]}\frac{\pi^{\frac{2l}{n}}}{l\cos(\frac{\pi l}{n})}\ N^{1-\frac{2l}{n}}\ \lambda^{2l}\Big) & (n \neq 2,\,4,\,\ldots) \\ \big(\frac{N\mathrm{e}}{\pi}\big)^{\frac{4}{n}\lambda^n}\quad\exp\Big(\sum_{l=1}^{[\frac{n}{2}]-1}\frac{\pi^{\frac{2l}{n}}}{l\cos(\frac{\pi l}{n})}\ N^{1-\frac{2l}{n}}\ \lambda^{2l}\Big) & (n = 2,\,4,\ldots) \end{cases}, \tag{6.52}$$

and the symbol "slim" in (6.51) stands for the scaling limit which assumes that only the low-energy states are taken into account. The operator $\tau(\lambda)$ in (6.51) should be understood as the chiral parafermionic transfer-matrix (6.13) acting in the space $\mathcal{V}_{\mathsf{j}}^{(\mathfrak{m})}$ discussed in the previous subsection.

Similarly to (6.51), one can consider the scaling limit

$$\operatorname*{slim}_{N\to\infty}\ F^{(N)}(\lambda^{-1})\,(-1)^N\ \mu^{-2N}\ \mathcal{T}^{(N)}(\mu)\,\Big|_{\mu=(\pi/N)^{\frac{1}{n}}\,\lambda^{-1}}\ .$$

This can be identified with the anti-chiral parafermionic transfer-matrix $\bar{\tau}(\lambda^{-1})$ acting in the space $\overline{\mathcal{V}}_{\mathsf{j}}^{(2\mathsf{j})}$. Therefore, in the scaling limit, (at least some of) the low energy states of $\mathcal{H}^{(N)}_{\mathsf{j}-\frac{\mathfrak{m}}{2}}$ organize into the sector $\mathcal{V}_{\mathsf{j}}^{(\mathfrak{m})}\otimes\overline{\mathcal{V}}_{\mathsf{j}}^{(2\mathsf{j})}$ of the $\mathbb{Z}_n$ CFT Hilbert space. In this sector the low energy spectrum of the Hamiltonian (6.48) and the corresponding eigenvalues of the lattice shift operator have the form

$$\begin{aligned} E^{(N)} &= e_0\,N+\tfrac{2\pi}{N}\,\big(\Delta_{\mathsf{j},\,\mathfrak{m}}+\Delta_{\mathsf{j},\,2\mathsf{j}}-\tfrac{c_n}{12}+L+\bar{L}\ \big)+o\big(N^{-1}\big)\\ P^{(N)} &= \exp\big(\tfrac{2\pi\mathrm{i}}{N}\,(\,\Delta_{\mathsf{j},\,\mathfrak{m}}-\Delta_{\mathsf{j},\,2\mathsf{j}}\,+L-\bar{L}\ )\big)\ , \end{aligned} \tag{6.53}$$

where e_0 is some constant and L and $\bar{L}$ are integers. The central charge and conformal dimensions are given by eqs. (6.3) and (6.9), respectively.

Returning to the formal operator $\mathcal{T}^{(N)}(\mu)$ (6.40)-(6.45) for arbitrary values of n, we note that the case of real (p_1,p_2) is of prime interest to the cigar NLSM. Perhaps the most promising approach to the construction of a suitable representation of the algebra (6.43) and the diagonalization of $\mathcal{T}^{(N)}(\mu)$ is based on the method of separation of variables [127].

6.3 Spectrum of the chiral transfer-matrix

In the previous sections the construction of the chiral transfer-matrices has been discussed. We are now ready to tackle the calculation of their spectrum. A powerful approach to this problem is the ODE/IQFT correspondence and here, we'll illustrate the method by calculating the vacuum eigenvalues of $\tau(\lambda)$.

6.4 Operators $\zeta_{\pm}(\theta)$

As it was already mentioned, a comprehensive discussion of the diagonalization procedure of the discretized chiral transfer-matrix $\mathcal{T}^{(N)}(\mu)$ (6.40)-(6.45) will be dealt with in a separate publication. However, it would be useful here to make a short summary of the important integrable structures which play a crucial rôle in the procedure. Namely, it is possible to explicitly construct two matrices $\mathcal{Z}_{\pm}(\mu)$ satisfying the following set of conditions:[2]

(i) **Commutativity**

$$\big[\mathcal{Z}_{\pm}(\mu), \mathcal{Z}_{\pm}(\mu')\big] = \big[\mathcal{Z}_{+}(\mu), \mathcal{Z}_{-}(\mu')\big] = 0$$

$$\big[\mathcal{Z}_{\pm}(\mu), \mathcal{T}^{(N)}(\mu')\big] = \big[\mathcal{Z}_{\pm}(\mu), \mathbb{H}^{(N)}\big] = \big[\mathcal{Z}_{\pm}(\mu), \mathbb{P}^{(N)}\big] = 0$$

(ii) **"Quantum Wronskian" type relations**

Odd n:

$$(1+\mu)^{2N}\ \mathcal{Z}_{+}(q^{-\frac{1}{2}}\mu)\, \mathcal{Z}_{+}(q^{+\frac{1}{2}}\mu) - (1-\mu)^{2N}\, \mathcal{Z}_{-}(q^{-\frac{1}{2}}\mu)\, \mathcal{Z}_{-}(q^{+\frac{1}{2}}\mu) = W(\mu)\ \mathbb{P}^{(N)}$$

Even n:

$$\mathcal{Z}_{+}(q^{-\frac{1}{2}}\mu)\, \mathcal{Z}_{+}(q^{+\frac{1}{2}}\mu) - (1-\mu^2)^{2N}\ \mathcal{Z}_{-}(q^{-\frac{1}{2}}\mu)\, \mathcal{Z}_{-}(q^{+\frac{1}{2}}\mu) = W(\mu)\ \mathbb{P}^{(N)}$$

with

$$W(\mu) = \big(1+\mu^n\big)^{2N} - \big(1-\mu^n\big)^{2N}$$

[2]All matrices appearing below are understood as operators acting invariantly in $\mathcal{H}^{(N)}_{\mathtt{j}-\frac{\mathfrak{m}}{2}}$ – the eigenspace of $\mathtt{Z}$ in $\big(\mathbb{C}^n\big)^{\otimes N}$ corresponding to the eigenvalue $\omega^{\mathtt{j}-\frac{\mathfrak{m}}{2}}$, where $\mathtt{j}$ and $\mathfrak{m}$ are restricted as in (6.14).

(iii) **"$T-Q$" type relations**

Odd n:

$$\mathcal{T}^{(N)}(\mu)\,\mathcal{Z}_{\pm}(\mu) \;=\; (1\mp q^{-\frac{1}{2}}\,\mu)^{2N}\;\;\mathcal{Z}_{\mp}(q^{-1}\mu) + (1\mp q^{+\frac{1}{2}}\,\mu)^{2N}\;\;\mathcal{Z}_{\mp}(q^{+1}\mu)$$

Even n:

$$\mathcal{T}^{(N)}(\mu)\,\mathcal{Z}_{-}(\mu) \;=\; \mathcal{Z}_{+}(q^{-1}\mu) + \mathcal{Z}_{+}(q^{+1}\mu)$$

$$\mathcal{T}^{(N)}(\mu)\,\mathcal{Z}_{+}(\mu) \;=\; (1-q^{-1}\,\mu^2)^{2N}\,\mathcal{Z}_{-}(q^{-1}\mu) + (1-q^{+1}\,\mu^2)^{2N}\,\mathcal{Z}_{-}(q^{+1}\mu)$$

(iv) **Analytical conditions**

Odd n:

$$\mathcal{Z}_{\pm}(\mu) \;=\; \mu^{\mathfrak{m}}\;\times\Big(\text{ polynomial in } \mu \text{ of degree } (n-1)\,N-2\mathfrak{j}-\mathfrak{m}\Big)$$

Even n:

$$\mathcal{Z}_{+}(\mu) \;=\; \mu^{\mathfrak{m}}\;\times\Big(\text{ polynomial in } \mu^2 \text{ of degree } \tfrac{1}{2}\,nN-\mathfrak{j}-\tfrac{1}{2}\,\mathfrak{m}\Big)$$

$$\mathcal{Z}_{-}(\mu) \;=\; \mu^{\mathfrak{m}}\;\times\Big(\text{ polynomial in } \mu^2 \text{ of degree } \tfrac{1}{2}\,(n-2)\,N-\mathfrak{j}-\tfrac{1}{2}\,\mathfrak{m}\Big)$$

(v) $\mu\to-\mu$ **symmetry**

$$\text{Odd } n: \quad \mathcal{Z}_{\pm}(-\mu) \;=\; (-1)^{\mathfrak{m}}\;\mathcal{Z}_{\mp}(\mu)$$

$$\text{Even } n: \quad \mathcal{Z}_{\pm}(-\mu) \;=\; (-1)^{\mathfrak{m}}\;\mathcal{Z}_{\pm}(\mu)$$

The scaling limit of the operators $\mathcal{Z}_{\pm}(\mu)$ is of special interest. In Appendix E we present evidence that the following scaling limits, similar to (6.51), do exist:

$$\text{Odd } n\;:\;\zeta_{\pm} = \lambda^{\mp 2\,\lambda^n}\;\underset{N\to\infty}{\mathrm{slim}}\;G^{(N)}(\pm\lambda)\;\mathcal{Z}_{\pm}\Big(\big(\tfrac{\pi}{N}\big)^{\frac{1}{n}}\,\lambda\Big) \tag{6.54}$$

$$\text{Even } n \ : \zeta_+ = \qquad \operatorname*{slim}_{N\to\infty} G_+^{(N)}(\lambda)\ \mathcal{Z}_+\Big(\big(\tfrac{\pi}{N}\big)^{\frac{1}{n}}\lambda\Big)$$

$$\zeta_- = \lambda^{4\lambda^n} \operatorname*{slim}_{N\to\infty} G_-^{(N)}(\lambda)\ \mathcal{Z}_-\Big(\big(\tfrac{\pi}{N}\big)^{\frac{1}{n}}\lambda\Big) \tag{6.55}$$

with

$$\begin{aligned} G^{(N)}(\lambda) &= \Big(\frac{\mathrm{e}N}{\pi}\Big)^{-\frac{2(n-1)}{n}\lambda^n} \exp\Big(\sum_{l=1}^{n-1}(-1)^{l+1}\frac{\pi^{\frac{l}{n}}}{l\cos(\frac{\pi l}{2n})}\ N^{1-\frac{l}{n}}\ \lambda^l\Big) \\ G_+^{(N)}(\lambda) &= \Big(\frac{\mathrm{e}N}{\pi}\Big)^{-2\lambda^n} \\ G_-^{(N)}(\lambda) &= \Big(\frac{\mathrm{e}N}{\pi}\Big)^{\frac{2(n-2)}{n}\lambda^n} \exp\Big(-\sum_{l=1}^{\frac{n}{2}-1}\frac{\pi^{\frac{2l}{n}}}{l\cos(\frac{\pi l}{n})}\ N^{1-\frac{2l}{n}}\ \lambda^{2l}\Big). \end{aligned} \tag{6.56}$$

Notice that for odd n, we include the "strange" extra factor $\lambda^{\mp 2\lambda^n}$ in the formula for $\zeta_\pm$. A similar factor $\lambda^{4\lambda^n}$ appears for ζ_- with even n. At first glance they look artificial and, furthermore, make $\zeta_\pm$ multivalued functions of λ. However, these "strange" factors allow one to write the scaling version of the quantum Wronskian type relations (ii) in a form which is applicable for both odd and even n:

$$\zeta_+\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)\,\zeta_+\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)-\zeta_-\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)\,\zeta_-\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)=2\,\sinh\big(2\pi\mathrm{e}^{\theta}\big)\ . \tag{6.57}$$

Here we have introduced θ,

$$\lambda = \mathrm{e}^{\frac{\theta}{n}} \tag{6.58}$$

and later we'll argue that $\zeta_\pm$ are single valued functions of this variable. The $T-Q$ type relations (iii) in the scaling limit also have the same form for odd and even n,

$$\tau(\lambda)\ \zeta_\pm(\theta) = \zeta_\mp(\theta-\mathrm{i}\pi)+\zeta_\mp(\theta+\mathrm{i}\pi)\ . \tag{6.59}$$

Since the operator $\tau(\lambda)$ acts in the parafermionic space $\mathcal{V}_{\mathrm{j}}^{(\mathrm{m})}$ it is natural to expect that the same holds true for $\zeta_\pm(\theta)$.

The relations (6.57) and (6.59) work for $n = 2, 3, 4\ldots$, but is it possible to extend them to non-integer n? We conjecture, that for any $n > 0$ and general values of (p_1, p_2), there exists a pair of operators which act invariantly in the Fock level subspaces

$$\zeta_\pm(\theta)\ :\quad \mathcal{F}^{(L)}_{p_1,p_2} \mapsto \mathcal{F}^{(L)}_{p_1,p_2} \tag{6.60}$$

such that

$$[\zeta_\pm(\theta),\, \zeta_\pm(\theta')] = [\zeta_+(\theta),\, \zeta_-(\theta')] = [\zeta_\pm(\theta),\, \tau(\lambda')] = 0\ , \tag{6.61}$$

satisfying the relations (6.57) and (6.59).

Unfortunately, at this moment we don't know how to explicitly construct the operators (6.60). Nevertheless, there are strong physical arguments that they do exist. In the works [114] and [115], ζ_+ and ζ_- were introduced and studied for real (p_1, p_2), as the boundary state operators in the paperclip model with topological angle equal to 0 and π, respectively.[3] Among the results of those works is the large-θ behaviour. It was proposed that the operators $\zeta_\pm$ possess the following asymptotic at $\theta \to +\infty$:

$$\begin{aligned}
\zeta_+(\theta) &\asymp \exp\Big(-\big(2\theta + \pi\cot\big(\tfrac{\pi n}{2}\big) - C\big)\,\mathrm{e}^{\theta} - \sum_{j=1}^{\infty} \mathfrak{i}_{2j-1}\,\mathrm{e}^{-\theta(2j-1)}\Big) \\
\zeta_-(\theta) &\asymp \tilde{\tau}\Big(\mathrm{e}^{-\frac{\theta}{n+2}}\Big)\ \exp\Big(\big(2\theta - \pi\cot\big(\tfrac{\pi n}{2}\big) - C\big)\,\mathrm{e}^{\theta} + \sum_{j=1}^{\infty} \mathfrak{i}_{2j-1}\ \mathrm{e}^{-\theta(2j-1)}\Big)\ .
\end{aligned} \tag{6.62}$$

Here $\tilde{\tau}(\tilde{\lambda})$ is the formal series (5.30) generating the set of dual nonlocal IM while $\{\mathfrak{i}_{2j-1}\}_{j=1}^{\infty}$ is the infinite set of local IM (5.58)-(5.62). The real constant C is non-universal and can be chosen at will. In ref.[114], it was set to $\pi\cot\big(\frac{\pi n}{2}\big)$. For our purposes it is convenient to set it to zero,

$$C = 0\ . \tag{6.63}$$

With this choice it turns out that for odd n and (p_1, p_2) restricted by the conditions (6.10),(6.14), $\zeta_\pm$ are the same functions that appear in the scaling limit (6.54). However, one can see from eq.(6.62) that the operators $\zeta_\pm$ become singular for even n.

[3] In the notations of ref. [114]: $\mathbb{B}(\kappa) = 2^{-\frac{1}{2}}\ \exp(2\theta\,\mathrm{e}^{\theta})\ \zeta_+(\theta)$, provided $\kappa = \mathrm{e}^{\theta}$.

They can be analytically regularized similar to as in eq. (5.36),

$$\zeta_{\pm}^{(\mathrm{reg})}(\theta)\big|_{n=2l} = \lim_{\epsilon\to 0}\ \exp\left(\tfrac{2}{\epsilon}\,\mathrm{e}^{\theta}\right)\ \zeta_{\pm}(\theta)\big|_{n=2l+\epsilon} \qquad (l=1,2,3\ldots)\ . \tag{6.64}$$

When the regularized operators are restricted to the parafermionic space $\mathcal{V}_{\mathtt{j}}^{(\mathtt{m})}$, they are the same as the ones on the left hand side of eq. (6.55).[4]

It is expected that the operators $\zeta_{\pm}$ are entire functions of θ (in the sense that all their matrix elements and eigenvalues are entire functions of this variable) satisfying, for real (p_1, p_2), the Hermiticity condition

$$\big[\zeta_{\pm}(\theta)\big]^{\dagger} = \zeta_{\pm}(\theta^*)\ . \tag{6.65}$$

As it was pointed out, the asymptotic formulae (6.62) are written for large positive θ, however in all likelihood, they hold true for complex values, at least in the strip $|\Im m(\theta)| < \pi$ with $\Re e(\theta) \to +\infty$.

Also, it deserves mentioning that the chiral transfer-matrices $\tau_j(\lambda)$ with $j = 0, \frac{1}{2}, 1, \ldots,$ can be expressed in terms of the operators $\zeta_{\pm}(\theta)$. Namely, for $j = \frac{1}{2}, \frac{3}{2}, \ldots$

$$\begin{aligned}\tau_j(\lambda) \ &= \ \frac{(-1)^{j+\frac{1}{2}}}{2\sinh(2\pi\mathrm{i}\,\mathrm{e}^{\theta})} \\ &\times \ \Big[\zeta_+\big(\theta - \mathrm{i}\pi(j+\tfrac{1}{2})\big)\zeta_-\big(\theta + \mathrm{i}\pi(j+\tfrac{1}{2})\big) - \zeta_-\big(\theta - \mathrm{i}\pi(j+\tfrac{1}{2})\big)\zeta_+\big(\theta + \mathrm{i}\pi(j+\tfrac{1}{2})\big)\Big]\end{aligned} \tag{6.66}$$

whereas for $j = 0,\ 1, \ldots$

$$\begin{aligned}\tau_j(\lambda) \ &= \ \frac{(-1)^{j}}{2\sinh(2\pi\mathrm{e}^{\theta})} \\ &\times \ \Big[\zeta_+\big(\theta - \mathrm{i}\pi(j+\tfrac{1}{2})\big)\zeta_+\big(\theta + \mathrm{i}\pi(j+\tfrac{1}{2})\big) - \zeta_-\big(\theta - \mathrm{i}\pi(j+\tfrac{1}{2})\big)\zeta_-\big(\theta + \mathrm{i}\pi(j+\tfrac{1}{2})\big)\Big].\end{aligned} \tag{6.67}$$

where λ and θ are related as in eq. (6.58). With these formulae the fusion relation (5.27), as well as the $T-Q$ type relations (6.59), are satisfied identically.

Finally, let us consider the "$\mu \to -\mu$ symmetry" relations (v). It is easy to see that in the scaling limit they become

[4] Here, we have abused notation because the $\zeta_{\pm}$ in (6.55) denote the continuous operators obtained by means of the lattice regularization procedure.

$$\zeta_\pm(\theta + \mathrm{i}\pi n) \;=\; (-1)^{\mathfrak{m}}\ \mathrm{e}^{\pm 2\pi \mathrm{i} \mathrm{e}^{\theta}}\ \zeta_\mp(\theta) \qquad (n - \mathrm{odd}) \tag{6.68}$$

and

$$\begin{aligned} \zeta_+(\theta + \mathrm{i}\pi n) &= (-1)^{\mathfrak{m}}\ \zeta_+(\theta) \qquad\qquad (n - \mathrm{even}) \\ \zeta_-(\theta + \mathrm{i}\pi n) &= (-1)^{\mathfrak{m}}\ \mathrm{e}^{4\pi \mathrm{i} \mathrm{e}^{\theta}}\ \zeta_-(\theta)\ . \end{aligned} \tag{6.69}$$

As we will see below, these equations are not satisfied for non-integer n and only hold for the admissible values of $\mathfrak{j}$ and $\mathfrak{m}$ (6.14). In these particular cases, they can be used to truncate the so called Y-system – the chain of relations for $Y_j(\lambda) \equiv \tau_{j-\frac{1}{2}}(\lambda)\, \tau_{j+\frac{1}{2}}(\lambda)$ (see, e.g., refs.[50, 52]). In turn, the truncated Y-system allows one to derive the set of TBA equations from which the eigenvalues of the chiral transfer-matrices can be computed. For the vacuum eigenvalues with $\mathfrak{j} = \mathfrak{m} = 0$, the TBA system can be found in ref.[115].

6.5 ODE/IQFT correspondence for the vacuum eigenvalues

Let us first consider the vacuum eigenvalues $\zeta_\pm^{(\mathrm{vac})}(\theta)$ of the operators $\zeta_\pm(\theta)$ corresponding to the real pair (p_1, p_2). The main ingredient in the ODE/IQFT correspondence for this case, is the ordinary differential equation

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}x^2} - p_1^2\ \frac{\mathrm{e}^x}{1+\mathrm{e}^x} + \left(\tfrac{1}{4} - p_2^2\right)\ \frac{\mathrm{e}^x}{(1+\mathrm{e}^x)^2} + \mathrm{e}^{2\theta}\ \left(1+\mathrm{e}^x\right)^n \right] \Psi(x) = 0\,. \tag{6.70}$$

We assume for the moment that θ is real. Eq. (6.70) has the form of a stationary zero energy Schrödinger equation with the potential $V(x)$ given by the last three terms in (6.70). The potential $V(x)$ is positive and grows fast at large positive x so that (6.70) has a solution $\Xi(x)$ decaying at $x \to +\infty$; this condition specifies $\Xi(x)$ uniquely up to normalization. To fix the normalization, we assume that

$$\Xi(x) \to \mathrm{e}^{-\frac{\theta}{2}}\ \exp\left(-\left(\tfrac{n}{4} + \mathrm{e}^{\theta}\right) x - \mathrm{e}^{\theta} \int_0^{\mathrm{e}^x} \frac{\mathrm{d}u}{u} \left((1+u)^{\frac{n}{2}} - 1\right) \right) \tag{6.71}$$

as $x \to +\infty$. On the other hand, $V(x)$ approaches the positive constant $\mathrm{e}^{2\theta}$ at large negative x. Hence eq. (6.70) has a solution which decays for large negative x; we denote this solution $\Psi_+(x)$. The condition,

$$\Psi_+(x) \to \sqrt{\pi}\, \frac{\exp\big(\mathrm{e}^{\theta}\, x\big)}{\Gamma(1+2\,\mathrm{e}^{\theta})} \qquad \text{as} \qquad x \to -\infty\ , \tag{6.72}$$

specifies the solution $\Psi_+(x)$ uniquely, including its normalization. Then

$$\zeta_+^{(\mathrm{vac})}(\theta) = \exp\big(C_+\,\mathrm{e}^{\theta}\,\big)\ W\big[\Xi, \Psi_+\big]\ , \tag{6.73}$$

where

$$C_+ = -\pi\ \cot\big(\tfrac{\pi n}{2}\big) + 2\log(2) - 2 \tag{6.74}$$

and $W[f,g]$ denotes the Wronskian $f(x)g'(x) - g(x)f'(x)$. Eq. (6.73) was proposed in the work [114].

In the paper [115] this was extended to the vacuum eigenvalue $\zeta_-^{(\mathrm{vac})}(\theta)$. The starting point was the same differential equation (6.70), but instead of Ψ_+ in (6.73), another solution which grows as $\exp(-\mathrm{e}^{\theta}x)$ at large negative x was taken. Of course, this condition alone does not define the solution uniquely since, besides the overall normalization, one can always add any amount of $\Psi_+(x)$. It is usually difficult to define a growing solution unambiguously, but in our case the following property of (6.70) helps. Let us consider x as a complex variable. The potential $V(x)$ is an analytic function of x with branch-point singularities at all points where e^x turns to -1. Let us make branch cuts starting at each of the points $x = \mathrm{i}\pi\,(2N+1)$, $N = 0, \pm1, \pm2, \ldots$ and going to $+\infty$ parallel to the real axis, and choose the branch of $V(x)$ for which $(1+\mathrm{e}^x)^n$ is real and positive on the real axis of the x-plane. Restricting attention to the domain $\Re e(x) < 0$, one finds that the potential $V(x)$ has the periodicity property

$$V(x + 2\pi\mathrm{i}) = V(x) \qquad \big(\,\Re e(x) < 0\,\big)\,. \tag{6.75}$$

Consequently, equation (6.70) has two Bloch-wave solutions ($2\mathrm{e}^{\theta} \notin \mathbb{Z}$):

$$\Psi_{\pm}(x+2\pi\mathrm{i}) = \mathrm{e}^{\pm 2\pi\mathrm{i}\,\mathrm{e}^{\theta}}\,\Psi_{\pm}(x) \qquad \big(\Re e(x)<0\big)\,, \tag{6.76}$$

where the Bloch factors can be found by taking the limit $\Re e(x) \to -\infty$. At this point we assume that $2\mathrm{e}^{\theta}$ is not an integer, so that the conditions (6.76) specify two independent solutions $\Psi_{\pm}(x)$ uniquely, up to their normalizations. Of course, the solution $\Psi_{+}(x)$ defined this way decays as $\exp(\mathrm{e}^{\theta}x)$ at $\Re e(x) \to -\infty$, and the asymptotic condition (6.72) also fixes its normalization. The solution $\Psi_{-}(x)$ grows at large negative $\Re e(x)$, and its normalization can be fixed by specifying the leading asymptotic in this domain. Thus we define $\Psi_{-}(x)$ by the conditions

$$\begin{aligned} &\Psi_{-}(x+2\pi\mathrm{i}) = \mathrm{e}^{-2\pi\mathrm{i}\,\mathrm{e}^{\theta}}\,\Psi_{-}(x) && \big(\Re e(x)<0\big) \\ &\Psi_{-}(x) \to \sqrt{\pi}\,\frac{\exp\big(-\mathrm{e}^{\theta}\,x\big)}{\Gamma(1-2\,\mathrm{e}^{\theta})} \qquad \text{as} && \Re e(x) \to -\infty\,. \end{aligned} \tag{6.77}$$

It is possible to show that both $\Psi_{+}(x)$ and $\Psi_{-}(x)$ defined by (6.72) and (6.77) are entire functions of e^{θ}, and

$$\Psi_{-}\big(x\,|\,\mathrm{e}^{\theta}\big) = \Psi_{+}\big(x\,|-\mathrm{e}^{\theta}\big)\,, \tag{6.78}$$

where we temporarily exhibited the dependence of $\Psi_{\pm}$ on the parameter e^{θ}. From the definitions (6.72) and (6.77) we have

$$W\big[\Psi_{-},\Psi_{+}\big] = \sin(2\pi\mathrm{e}^{\theta})\,. \tag{6.79}$$

The proposal in [115] was that

$$\zeta_{-}^{(\mathrm{vac})}(\theta) = \exp\big(C_{-}\,\mathrm{e}^{\theta}\big)\; W\big[\Xi,\Psi_{-}\big]\,, \tag{6.80}$$

where

$$C_{-} = -\pi\,\cot\big(\tfrac{\pi n}{2}\big) - 2\log(2) + 2 - 2\gamma_E - 2\,\psi(1+\tfrac{n}{2}) \tag{6.81}$$

with $\psi(z) = \partial_z \log\Gamma(z)$ and γ_E stands for the Euler constant.

There is much evidence to support the remarkable relations (6.73) and (6.80). Some of them are based on WKB analysis of the differential equation (6.70). Using the method of semiclassical expansion, one can systematically study the Wronskians in (6.73), (6.80) at $\theta \to +\infty$. This yields asymptotic expansions whose structures turn out to be identical to the one for $\zeta_\pm^{(\mathrm{vac})}$ following from eq. (6.62). Furthermore the WKB calculations give non-trivial predictions for the vacuum eigenvalues of the local and dual nonlocal IM. On the other hand, these vacuum eigenvalues can be directly calculated from their definitions. For example, the vacuum eigenvalue of the first dual nonlocal IM is given by eq. (5.35), while it is a simple exercise to find the vacuum eigenvalue of the first local IM $\mathfrak{i}_1$:

$$i_1(p_1,p_2) = -\frac{1}{12} + \frac{p_1^2}{n} + \frac{p_2^2}{n+2}\ . \tag{6.82}$$

It turns out that the results of the WKB analysis are in full agreement with these direct calculations.

Let us discuss now the $\theta \to -\infty$ form of the Wronskians in (6.73) and (6.80). The perturbative evaluation of the solutions $\Xi(x)$ and $\Psi_\pm(x)$ leads to the expansions [114, 115]

$$\mathrm{e}^{\pm\pi\kappa\cot(\frac{\pi n}{2})}\ \zeta_\pm^{(\mathrm{vac})}(\theta) = B_{p_1,p_2}\ \mathrm{e}^{\frac{2\mathrm{i}p_1\theta}{n}}\ F_\pm(\theta\,|\,p_1,p_2) + B_{-p_1,p_2}\ \mathrm{e}^{-\frac{2\mathrm{i}p_1\theta}{n}}\ F_\pm(\theta\,|-p_1,p_2)\,, \tag{6.83}$$

where

$$B_{p_1,p_2} = \sqrt{n}\ n^{-\frac{2\mathrm{i}p_1}{n}}\ \frac{\Gamma(-2\mathrm{i}p_1)\Gamma(1-\frac{2\mathrm{i}p_1}{n})}{\Gamma(\frac{1}{2}-p_2-\mathrm{i}p_1)\Gamma(\frac{1}{2}+p_2-\mathrm{i}p_1)} \tag{6.84}$$

and

$$F_\pm(\theta\,|p_1,p_2) = \sum_{i,j=0}^{\infty} f_{i,j}(p_1,p_2)\ (\pm 1)^i\ \mathrm{e}^{(i+\frac{2j}{n})\theta} \tag{6.85}$$

with

$$f_{0,0} = 1 \tag{6.86}$$
$$f_{0,1} = -\frac{\Gamma(\frac{1}{2}+\frac{1}{n})\Gamma(1-\frac{1}{n})}{2\sqrt{\pi}}\left(\frac{2}{n}\right)^{\frac{2}{n}-1}\left(\frac{n+2}{n-2}+\frac{4p_2^2}{1+4p_1^2}\right)\frac{\Gamma(1-\frac{1}{n}-\frac{2\mathrm{i}p_1}{n})}{\Gamma(\frac{1-2\mathrm{i}p_1}{n})}$$
$$f_{1,0} = -\psi\big(\tfrac{1}{2}-p_2-\mathrm{i}p_1\big)-\psi\big(\tfrac{1}{2}+p_2-\mathrm{i}p_1\big)+\psi\big(-\tfrac{n}{2}\big)+\gamma_E+2\,\log(2)-2\ .$$

Note that the integer powers of e^{θ} in (6.85) come from the perturbative expansion of the solution $\Psi_{\pm}(x)$. In view of (6.78), these powers in F_+ are related to the corresponding powers in F_- by a change of sign, $\mathrm{e}^{\theta}\to-\mathrm{e}^{\theta}$. At the same time, the powers of $\mathrm{e}^{\frac{2\theta}{n}}$ are the result of the expansion of $\Xi(x)$, and hence they are the same in F_- and F_+.

One of the important properties of the Wronskians on the right hand side of eqs.(6.73), (6.80), is that for a given θ, they are entire functions in both complex variables p_1^2 and p_2^2. Thus it is perfectly fine to consider them for pure imaginary values of p_1. Let's set $p_1=\frac{\mathrm{i}}{2}\,\mathfrak{m}$, $p_2=\mathfrak{j}+\frac{1}{2}$ and assume that $\mathfrak{m}\geq 0$. In this notation, the coefficient B_{p_1,p_2} (6.84) contains the factor $\Gamma(-\mathfrak{j}+\frac{\mathfrak{m}}{2})$ in the denominator, and therefore vanishes when $\mathfrak{j}-\frac{\mathfrak{m}}{2}$ is a non-negative integer. At the same time, the coefficient B_{-p_1,p_2} takes the form $B_{-p_1,p_2}=B_s(\mathfrak{m})$ with

$$B_s(\mathfrak{m})=\sqrt{n}\;n^{-\frac{\mathfrak{m}}{n}}\;\Gamma\Big(1-\frac{\mathfrak{m}}{n}\Big)\;\frac{(-1)^s}{s!}\;\frac{\Gamma(1+\mathfrak{m}+s)}{\Gamma(1+\mathfrak{m})}\qquad\big(s=\mathfrak{j}-\tfrac{1}{2}\,\mathfrak{m}=0,1,2,\dots\big) \tag{6.87}$$

which remains finite for the discrete set of $\mathfrak{j}$ and $\mathfrak{m}$ (6.14) corresponding to the bosonization of the parafermionic spaces $\mathcal{V}_{\mathfrak{j}}^{(\mathfrak{m})}$. The vanishing of the coefficient B_{p_1,p_2} does not actually mean that we can neglect the first term in the sum (6.83) – the expansion coefficients in $F_{\pm}(\theta|p_1,p_2)$ may become singular for (p_1,p_2) given by eqs. (6.10), (6.14). Nevertheless, analysis shows that such "resonances" occur only for the terms $\propto \mathrm{e}^{(-\frac{\mathfrak{m}}{n}+i+\frac{2j}{n})\theta}$ with $i\geq 0$ and $j\geq\mathfrak{m}$, i.e., having the same form as monomials in the double sum $\mathrm{e}^{\frac{\mathfrak{m}}{n}\theta}\,F_{\pm}(\theta\,|-p_1,p_2)$. Notice that for integer n, the double summations in $F_{\pm}$ can be replaced by a single one. This yields the formulae for the regularized (see eq. (6.64)) vacuum eigenvalues in the parafermionic spaces $\mathcal{V}_{\mathrm{j}}^{(\mathfrak{m})}$:

$$\zeta_{\pm}^{(\mathrm{vac})}(\theta)=\begin{cases} B_s(\mathfrak{m})\ \lambda^{\mp 2\lambda^n}\quad \lambda^{\mathfrak{m}}\ \Big(1+\ \sum_{j=1}^{\infty} f_j\ (\pm\lambda)^j\Big)\Big|_{\lambda=\mathrm{e}^{\frac{\theta}{n}}} & (n-\text{odd})\\ B_s(\mathfrak{m})\ \lambda^{(2\mp 2)\lambda^n}\ \lambda^{\mathfrak{m}}\ \Big(1+\sum_{j=1}^{\infty} f_j^{(\pm)}\ \lambda^{2j}\Big)\Big|_{\lambda=\mathrm{e}^{\frac{\theta}{n}}} & (n-\text{even})\,. \end{cases} \tag{6.88}$$

The appearance of the "strange" factors, $\lambda^{\mp 2\lambda^n}$, $\lambda^{(2\mp 2)\lambda^n}$ here can be understood as follows. The Wronskians in the l.h.s. of eqs. (6.73) and (6.80) are, of course, non-singular functions of (p_1, p_2) and $n > 0$.[5] However, the individual coefficients $f_{i,j}$ in (6.85) become singular for some values of the parameters, as it can be seen from the explicit formulae (6.86). When the values of the parameters are restricted to the parafermionic case, the singularities from the different coefficients must cancel each other giving rise to terms $\propto \theta^m \mathrm{e}^{l\theta}$ which sum up to yield the "strange" factors in (6.88). Notice also that some of the coefficients in these series expansions are known explicitly. In Appendix E their values are compared with the corresponding results obtained from the vacuum eigenvalues of the finite matrices $\mathcal{Z}_{\pm}(\mu)$.

We now return to the differential equation and consider the solution Ξ in more detail. For complex θ, the asymptotic condition (6.71) unambiguously defines $\Xi(x\,|\,\theta)$ in the strip $\big|\Im m(\theta)\big|\leq\frac{\pi}{2}$ including its boundary. The two functions $\Xi(x\,|\,\theta+\frac{\mathrm{i}\pi}{2})$ and $\Xi(x\,|\,\theta-\frac{\mathrm{i}\pi}{2})$, with real θ, form a linear basis in the space of solutions of (6.70) with $\mathrm{e}^{2\theta}$ substituted by $(-\mathrm{e}^{2\theta})$, since as it follows from (6.71),

$$W\big[\,\Xi(x\,|\,\theta+\tfrac{\mathrm{i}\pi}{2}),\,\Xi(x\,|\,\theta-\tfrac{\mathrm{i}\pi}{2})\,\big]=2\mathrm{i}\,. \tag{6.89}$$

On the other hand, formulae (6.73), (6.80) and (6.79) imply

$$\Xi(x\,|\,\theta)=\frac{\exp(\xi\,\mathrm{e}^{\theta})}{\sin(2\pi\mathrm{e}^{\theta})}\Big(\zeta_{+}^{(\mathrm{vac})}(\theta)\,\mathrm{e}^{-c\,\mathrm{e}^{\theta}}\,\Psi_{+}\big(\,x\,|-\mathrm{e}^{\theta}\,\big)-\zeta_{-}^{(\mathrm{vac})}(\theta)\,\mathrm{e}^{+c\,\mathrm{e}^{\theta}}\,\Psi_{+}\big(\,x\,|+\mathrm{e}^{\theta}\,\big)\Big) \tag{6.90}$$

[5] The only singularities at $n=2,4,\ldots$ are produced by the cotangent which shows up in the expression for the constants $C_{\pm}$ (6.74), (6.81). The reason that this term was included in the definition of $\zeta_{\pm}$ is that we would like the $T-Q$ type relations to have the simple canonical form (6.59).

which can be used to express $\Xi(x\,|\,\theta\pm\frac{\mathrm{i}\pi}{2})$ in terms of $\Psi_+(x|\pm\mathrm{i}e^{\theta})$ and then calculate the l.h.s. of (6.89). This yields the quantum Wronskian type relation (6.57) specified at the vacuum eigenvalues of $\zeta_\pm(\theta)$. Notice that in eq. (6.90) we use the constants $\xi=-\frac{1}{2}\,(C_++C_-)$ and $c=\frac{1}{2}\,(C_+-C_-)$. Of course, the factors $\exp(\pm c\,\mathrm{e}^{\theta})$ which appear in this formula can be included in the definition of the solutions $\Psi_\pm$. With the value of c determined by eqs. (6.74), (6.81), the constant C, appearing in the asymptotic expansions (6.62) for $\zeta_\pm(\theta)$, vanishes. The constant

$$\xi=-\tfrac{1}{2}\,(C_++C_-)=\pi\,\cot\left(\tfrac{\pi n}{2}\right)+\gamma_E+\psi(1+\tfrac{n}{2}) \tag{6.91}$$

can also be absorbed into the definition of the solution $\Xi(x)$.

Let us assume that the solution $\Xi(x\,|\,\theta)$ can be unambiguously continued to the whole complex plane from the strip $\left|\Im m(\theta)\right|\leq\frac{\pi}{2}$.[6] Then the function $\Xi(x\,|\,\theta+\mathrm{i}\pi(2j+\frac{1}{2}))$ with $j=0,\,\frac{1}{2},\,1,\dots$ solves the differential equation (6.70) with $\mathrm{e}^{2\theta}\mapsto(-\mathrm{e}^{2\theta})$ and can be linearly expressed in terms of the two basic solutions $\Xi(x\,|\,\theta\pm\frac{\mathrm{i}\pi}{2})$

$$\Xi\big(x\,|\,\theta+\mathrm{i}\pi(2j+\tfrac{1}{2})\big)=a_j(\theta)\;\Xi\big(x\,|\,\theta-\tfrac{\mathrm{i}\pi}{2}\big)+b_j(\theta)\;\Xi\big(x\,|\,\theta+\tfrac{\mathrm{i}\pi}{2}\big)\;. \tag{6.92}$$

With manipulations similar to those which lead to the quantum Wronskian type relations, it is straightforward to show that

$$a_j(\theta)=-b_{j-\frac{1}{2}}(\theta+\mathrm{i}\pi) \tag{6.93}$$

and the $b_j\big(\theta-\mathrm{i}\pi j\big)$ are given by the r.h.s. of (6.66), (6.67) where the operators $\zeta_\pm$ are substituted by their vacuum eigenvalues. Hence we conclude that

$$\tau_j^{(\mathrm{vac})}(\theta)=b_j(\theta-\mathrm{i}\pi j)\,. \tag{6.94}$$

6.6 ODE/IQFT correspondence for the full spectrum

In the previous subsection, our analysis was restricted to the vacuum eigenvalues. As a matter of fact, along the line of ref. [55], it can be extended to the whole spectrum

[6] It is expected that $\Xi(x\,|\,\theta)$ is an entire function of θ. However at this moment, we don't have a rigorous proof of this statement.

of the commuting families of operators. This was already done for a more general case in the work [90]. Here we give a sketch of the construction.

Let z be the the complex coordinate on $\mathbb{CP}^1\backslash\{z_1, z_2, z_3\}$, the Riemann sphere with three punctures. Consider the second order Fuchsian differential operator $-\partial_z^2+T_0(z)$, with $T_0(z)$ given by

$$T_0(z) = -\sum_{i=1}^{3}\Big(\frac{\delta_i}{(z-z_i)^2}+\frac{c_i}{z-z_i}\Big) \tag{6.95}$$

and the δ_i are regarded as independent parameters, whereas the c_i are unambiguously defined by the constraints

$$\sum_{i=1}^{3} c_i = 0\ , \qquad \sum_{i=1}^{3}(z_i\, c_i+\delta_i) = 0\ , \qquad \sum_{i=1}^{3}(z_i^2\, c_i + 2\, z_i\delta_i) = 0\ . \tag{6.96}$$

The equation

$$\big(-\partial_z^2 + T_0(z)\big)\,\psi = 0 \tag{6.97}$$

is a second-order differential equation with three regular singular points. For a generic choice of the parameters δ_i, it can be brought to the standard hypergeometric form by a change of variables. In the papers [128], a deformation of (6.97) was introduced, of the form

$$\mathcal{D}_0(\theta)\,\psi = 0\ , \qquad\qquad \mathcal{D}_0(\theta) = -\partial_z^2 + T_0(z) + \mathrm{e}^{2\theta}\,\mathcal{P}(z)\ , \tag{6.98}$$

where θ stands for an arbitrary complex parameter and

$$\mathcal{P}(z) = \frac{(z_3-z_2)^{a_1}\,(z_3-z_1)^{a_2}\,(z_2-z_1)^{a_3}}{(z-z_1)^{2-a_1}(z-z_2)^{2-a_2}(z-z_3)^{2-a_3}} \tag{6.99}$$

with

$$a_1 + a_2 + a_3 = 0\,. \tag{6.100}$$

Notice that, because of the last relation, $\mathcal{P}(z)\,(\mathrm{d}z)^2$ transforms as a quadratic differential under the $\mathbb{PSL}(2,\mathbb{C})$ group, so that the punctures z_1, z_2, z_3 on the Riemann sphere can be sent to any desirable positions.

The immediate object of our interest is a particular case of the differential equation (6.98) with

$$a_1 = -n \ , \qquad a_2 = n+2 \ , \qquad a_3 = 0 \tag{6.101}$$

and

$$\delta_1 = \tfrac{1}{4} + p_1^2 \ , \qquad \delta_2 = \tfrac{1}{4} - p_2^2 \ , \qquad \delta_3 = \tfrac{1}{4} \ . \tag{6.102}$$

Indeed, the change of variables

$$\mathrm{e}^x = \frac{z-z_3}{z-z_1}\,\frac{z_2-z_1}{z_3-z_2} \ , \qquad \Psi(x)\ (\mathrm{d}x)^{-\frac{1}{2}} = \psi(z)\,(\mathrm{d}z)^{-\frac{1}{2}} \ , \tag{6.103}$$

brings equation (6.70) to the form (6.98) specialized to this set of parameters.

As it was explained in [90], the description of the spectrum of the commuting family of transfer-matrices in the Fock level subspaces $\mathcal{F}^{(L)}_{p_1,p_2}$ is based on a differential equation of the form similar to (6.98), where the differential operator $\mathcal{D}_0(\theta)$ is substituted by

$$\mathcal{D}(\theta) = -\partial_z^2 + T_L(z) + \mathrm{e}^{2\theta}\,\mathcal{P}(z) \ , \tag{6.104}$$

which has $3+L$ singular points at $z = z_1, z_2, z_3$ and also $z = x_1, \ldots, x_L$. The form of $T_L(z)$, including the positions of the extra singularities, $z = x_1, \ldots, x_L$, is determined by the requirement that the monodromy properties of the solutions to the equation $\mathcal{D}(\theta)\,\psi(z) = 0$ are identical to those for $\mathcal{D}_0(\theta)\,\psi(z) = 0$. It turns out that this requirement can be fulfilled if the set of complex numbers $\{x_i\}_{i=1}^L$ obey a system of L algebraic equations similar to that from [55, 129, 130].

As we saw in the previous subsection, the vacuum eigenvalues of the operators $\zeta_\pm(\theta)$ and $\tau(\theta)$ can be identified with certain connection coefficients for the differential equation (6.70), or equivalently for (6.98)-(6.102).[7] Of course, similar connection

[7] By "connection coefficients" we understand the θ-dependent functions which allow one to relate different bases in the linear space of solutions of the ordinary differential equation, see eqs.(6.90) and (6.92).

coefficients can be associated to the more general differential operator (6.104). Since the singularities of the potential $T_L(z)$ at $z = x_1, \ldots, x_L$ do not affect the monodromy properties of the solutions of $\mathcal{D}(\theta)\,\psi(z) = 0$, all the relations between the connection coefficients remain unchanged. This allows one to identify them with specializations of the operator relations like (6.57), (6.59), (6.66), (6.67) and (6.62), to the eigenvalues of the commuting families of operators.

6.7 Operators $\beta_\pm(\theta)$ and $\alpha_\pm(\theta)$

With the philosophy of the ODE/IQFT correspondence in mind, let us return to the differential equation (6.70). It has three singular points at $e^x = 0,\ \infty,\ -1$, and we have already discussed the canonical bases in the space of solutions in the neighbourhood of two of them, $e^x = 0$ and ∞. We now consider the basis which is canonically defined in the vicinity of $e^x = -1$. For this purpose it is convenient to perform a change of variables

$$e^{-x} = -1 - e^{-y}\ , \qquad \Psi(x) = (1 + e^y)^{-\frac{1}{2}}\ \tilde{\Psi}(y)\ , \tag{6.105}$$

which brings (6.70) to the form

$$\left[-\frac{d^2}{dy^2} + p_2^2\ \frac{e^y}{1+e^y} + \left(\tfrac{1}{4} + p_1^2\right)\ \frac{e^y}{(1+e^y)^2} + e^{2\theta}\ \left(1 + e^y\right)^{-n-2} \right] \tilde{\Psi}(y) = 0\ . \tag{6.106}$$

For $p_2 > 0$ the differential equation (6.106) admits a unique solution such that

$$\tilde{\Theta}_+(y) \to \sqrt{\frac{\pi}{n+2}}\ (n+2)^{-\frac{2p_2}{n+2}}\ \frac{e^{-p_2 y}}{\Gamma(1 + \frac{2p_2}{n+2})} \qquad \text{as} \quad y \to +\infty\ . \tag{6.107}$$

For complex p_2, one can show that the solution is a meromorphic function, analytic in the half plane $\Re e(p_2) \geq 0$. Since the equation (6.106) is invariant w.r.t. the substitution $p_2 \mapsto -p_2$, for generic values of p_2 one can define the second linear independent solution uniquely by analytic continuation $\mathcal{A}_{p_2 \mapsto e^{\pm i\pi} p_2}$ to the half plane $\Re e(p_2) < 0$:

$$\tilde{\Theta}_-(y\,|\,e^{2\theta},\, p_2) = \mathcal{A}_{p_2 \mapsto e^{\pm i\pi} p_2}\left[\tilde{\Theta}_+(y\,|\,e^{2\theta}, p_2)\right]\ , \tag{6.108}$$

and it is easy to see that

$$W\big[\,\tilde{\Theta}_+,\,\tilde{\Theta}_-\,\big]=\sin\big(\tfrac{2\pi p_2}{n+2}\big)\ . \tag{6.109}$$

Notice that both these solutions are entire functions of the variable $\mathrm{e}^{2\theta}$ whose dependence is emphasized in the formula (6.108).

As $y\to-\infty$, eq. (6.106) admits the two linearly independent solutions

$$\tilde{\Psi}_\pm(y)\to\sqrt{\pi}\ \frac{\exp\big(\pm\,\mathrm{e}^{\theta}\,y\big)}{\Gamma(1\pm 2\,\mathrm{e}^{\theta})}\qquad\text{as}\qquad y\to-\infty\ . \tag{6.110}$$

Of course, $\tilde{\Psi}_\pm(y)$ are obtained from $\Psi_\pm(x)$ (6.72), (6.77) by means of the coordinate transformation (6.105):

$$\Psi_\pm(x)\ (\mathrm{d}x)^{-\frac{1}{2}}=\exp\big(\mp\,\mathrm{i}\pi\,\mathrm{e}^{\theta}\big)\ \ \tilde{\Psi}_\pm(y)\ (\mathrm{d}y)^{-\frac{1}{2}}\ . \tag{6.111}$$

Here the extra phase factor appears because of slightly different normalizations: $\Psi_\pm(x)$ were defined in such a way that they stay real for real x and real values of the parameters, whereas $\tilde{\Psi}_\pm(y)$ (6.110) are real for real y which is related to x as $\mathrm{e}^{-y}=-1-\mathrm{e}^{-x}$. It is important to keep in mind the presence of such phase factors, because they affect the reality conditions for the connection coefficients.

Following the philosophy of the ODE/IQFT correspondence, we consider the connection coefficients for the bases $\big\{\tilde{\Psi}_\pm(y)\big\}$ and $\big\{\tilde{\Theta}_\pm(y)\big\}$ and interpret them as vacuum eigenvalues of certain operators $\beta_\pm(\theta)$:

$$\mathrm{e}^{+c\,\mathrm{e}^{\theta}}\ \tilde{\Psi}_+(y\,|\,\mathrm{e}^{\theta})=\frac{1}{2\sin(\frac{2\pi p_2}{n+2})}\ \Big(\beta_+^{(\mathrm{vac})}(\theta)\ \tilde{\Theta}_-\big(y\,|\,\mathrm{e}^{2\theta}\big)-\beta_-^{(\mathrm{vac})}(\theta)\ \tilde{\Theta}_+\big(y\,|\,\mathrm{e}^{2\theta}\big)\Big) \tag{6.112}$$

The relation for $\mathrm{e}^{-c\,\mathrm{e}^{\theta}}\ \tilde{\Psi}_-$ is similar with $\beta_\pm^{(\mathrm{vac})}(\theta)$ substituted by $\beta_\pm^{(\mathrm{vac})}(\theta+\mathrm{i}\pi)$. Notice that it is expected that

$$\beta_\pm(\theta+\mathrm{i}\pi)=\beta_\pm(\theta-\mathrm{i}\pi)\ . \tag{6.113}$$

Further, the operators $\beta_\pm$ are entire functions of the variable e^{θ} and can be written in the form of a convergent series

$$\beta_\pm(\theta)=b_\pm\ \bigg(1+\sum_{m=1}^{\infty}\mathfrak{b}_m^{(\pm)}\ \mathrm{e}^{m\theta}\bigg)\ . \tag{6.114}$$

The reality condition for the connection coefficients $\beta_\pm^{(\mathrm{vac})}(\theta)$ suggests the Hermiticity

$$\big[\beta_\pm(\theta)\big]^\dagger = \beta_\pm(\theta^*)\ . \tag{6.115}$$

Using the definition (6.112), one can express the Wronskian $W\big[\,\tilde{\Psi}_-,\,\tilde{\Psi}_+\,\big]$ in terms of the connection coefficients $\beta_\pm^{(\mathrm{vac})}$. On the other hand, as follows from (6.110), it is equal to $\sin(2\pi \mathrm{e}^\theta)$. This yields the quantum Wronskian relation

$$\beta_+\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)\,\beta_-\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)-\beta_-\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)\,\beta_+\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big) = -4\mathrm{i}\,\sinh\big(2\pi\mathrm{e}^\theta\big)\ \sin\big(\tfrac{2\pi p_2}{n+2}\big)\ , \tag{6.116}$$

where the superscript "(vac)" is omitted, since we expect that it holds true for all eigenvalues of the operators $\beta_\pm$.

It turns out that the operators (6.114), acting in the Fock space $\mathcal{F}_{p_1,p_2}$, and satisfying the commutativity conditions

$$\big[\beta_\sigma(\theta),\,\beta_{\sigma'}(\theta')\,\big] = \big[\beta_\sigma(\theta),\,\tau(\theta')\,\big] = 0 \qquad \big(\sigma,\,\sigma'=\pm\big)\ , \tag{6.117}$$

can be defined explicitly. Their construction lies beyond the scope of this work. Here we just mention that it is similar to the construction of the $U_q(\hat{\mathfrak{sl}}_2)$ Q-operators from refs.[51, 52].

The last set of connection coefficients relates the basis in the space of solutions canonically defined at the singular point $\mathrm{e}^x=\infty$ with that defined at $\mathrm{e}^x=-1$, or, equivalently $\mathrm{e}^y=-1$ with $\mathrm{e}^y=0$. Let $\tilde{\Xi}(y)\equiv(1+\mathrm{e}^y)^{\frac{1}{2}}\ \Xi(x)|_{x(y)}$. Then it can be written in the form similar to (6.90) and (6.112):

$$\tilde{\Xi}(y\,|\,\theta) = \frac{\exp(\xi\,\mathrm{e}^\theta)}{\sin(\frac{2\pi p_2}{n+2})}\ \Big(a_+(\theta)\ \tilde{\Theta}_-\big(y\,|\,\mathrm{e}^{2\theta}\big)-a_-(\theta)\ \tilde{\Theta}_+\big(y\,|\,\mathrm{e}^{2\theta}\big)\Big)\ . \tag{6.118}$$

Since for real y, the solution $\tilde{\Xi}(y)$ is complex, the reality condition looks simpler if we introduce $\alpha_\pm^{(\mathrm{vac})}(\theta)$, such that

$$a_\pm(\theta) = \mathrm{i}\ \mathrm{e}^{\mp\mathrm{i}\pi p_2}\ \alpha_\pm^{(\mathrm{vac})}\big(\theta-\tfrac{\mathrm{i}\pi n}{2}\big)\ . \tag{6.119}$$

The latter turn out to be real functions for real θ and (p_1, p_2). Again, we interpret them as the vacuum eigenvalues of the Hermitian operators $\alpha_\pm(\theta)$,

$$\big[\alpha_\pm(\theta)\big]^\dagger = \alpha_\pm(\theta^*) \ , \tag{6.120}$$

which are also expected to be entire functions of θ. Similar to the operators $\zeta_\pm(\theta)$, one can obtain the quantum Wronskian relation

$$\alpha_+\big(\theta - \tfrac{\mathrm{i}\pi}{2}\big)\,\alpha_-\big(\theta + \tfrac{\mathrm{i}\pi}{2}\big) - \alpha_-\big(\theta - \tfrac{\mathrm{i}\pi}{2}\big)\,\alpha_+\big(\theta + \tfrac{\mathrm{i}\pi}{2}\big) = 2\mathrm{i}\,\sin\big(\tfrac{2\pi p_2}{n+2}\big) \tag{6.121}$$

and the $T-Q$ relations, which now have the canonical form

$$\tau(\mathrm{i}\lambda)\ \alpha_\pm(\theta) = \alpha_\pm(\theta + \mathrm{i}\pi) + \alpha_\pm(\theta - \mathrm{i}\pi) \tag{6.122}$$

$\big($recall that $\tau(\mathrm{i}\lambda) = \tau(-\mathrm{i}\lambda)$ and $\lambda = \mathrm{e}^{\frac{\theta}{n}}\,\big)$.

Using the WKB approximation, one can explore the asymptotic behaviour of the connection coefficients for $\theta \to +\infty$. This leads to the asymptotic expansions similar to formulae (6.62) for $\zeta_\pm(\theta)$. In particular, it is expected that for $\Re e(\theta) \to +\infty$ and $|\Im m(\theta)| < \frac{\pi}{2}\,(n+2)$,

$$\alpha_\pm(\theta) \asymp \mathrm{e}^{\mp \mathrm{i}\pi p_2}\ \tilde{\alpha}_\pm\Big(\mathrm{e}^{\frac{\mathrm{i}\pi}{2}}\,\mathrm{e}^{-\frac{\theta}{n+2}}\Big)\ \ \exp\Big(-\tfrac{\pi}{\sin(\frac{\pi n}{2})}\,\mathrm{e}^{\theta}\Big)\ , \tag{6.123}$$

where $\tilde{\alpha}_\pm(\tilde{\lambda})$ stand for the formal power series of the form

$$\tilde{\alpha}_\pm(\tilde{\lambda}) = \big(\tilde{\lambda}\big)^{\pm 2p_2}\ \exp\Big(-\sum_{m=1}^{\infty}\tilde{\mathfrak{s}}_m^{(\pm)}\ \tilde{\lambda}^{2m}\Big)\ . \tag{6.124}$$

The coefficients $\{\tilde{\mathfrak{s}}_m^{(\pm)}\}_{m=1}^{\infty}$ are dual nonlocal IM, which are algebraically expressed through the set $\{\tilde{\mathfrak{t}}_m^{(\pm)}\}_{m=1}^{\infty}$ by means of the formal $T-Q$ relation (to be compared with eq. (6.122)):

$$\tilde{\tau}(\tilde{\lambda})\,\tilde{\alpha}_\pm(\tilde{\lambda}) = \tilde{\alpha}_\pm\big(\tilde{q}\,\tilde{\lambda}\big) + \tilde{\alpha}_\pm\big(\tilde{q}^{-1}\tilde{\lambda}\big)\ , \tag{6.125}$$

where $\tilde{q} = \mathrm{e}^{-\frac{\mathrm{i}\pi}{n+2}}$. Substituting the formal power series (5.30) and (6.124) in (6.125), one can easily derive the relations between these two sets of dual nonlocal IM. For example

$$\tilde{\mathfrak{s}}_1^{(\pm)} = \frac{\tilde{\mathfrak{t}}_1}{[1][1 \pm 2p_2]}\ , \quad \tilde{\mathfrak{s}}_2^{(\pm)} = \frac{\tilde{\mathfrak{t}}_2}{[2][2 \pm 2p_2]} - \frac{\tilde{\mathfrak{t}}_1^2}{[1][2][1 \pm 2p_2][2 \pm 2p_2]} + \frac{\tilde{\mathfrak{t}}_1^2}{2[1]^2[1 \pm 2p_2]^2}\ , \tag{6.126}$$

where we use the shortcut notation $[x] = 2\,\sin(\frac{\pi x}{n+2})$.

Unlike for $\alpha_\pm$, the large-θ asymptotics of the operators $\beta_\pm$ include a contribution from the local IM; for $\Re e(\theta) \to +\infty$ and $|\Im m(\theta)| < \pi$, they read as

$$\beta_\pm(\theta) \asymp \tilde{\alpha}_\pm\Big(\mathrm{e}^{-\frac{\theta}{n+2}}\Big)\ \exp\Big(-2\theta\,\mathrm{e}^{\theta} - \sum_{j=1}^{\infty} \mathfrak{i}_{2j-1}\,\mathrm{e}^{-\theta(2j-1)}\Big)\ . \tag{6.127}$$

Finally, using the formulae (6.90), (6.111), (6.112), (6.118), (6.119), it is straightforward to show that the three sets of connection coefficients are not functionally independent, they satisfy the relations

$$\begin{aligned}
\zeta_+(\theta) &= \frac{\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\pi\mathrm{e}^{\theta}}}{2\sin(\frac{2\pi p_2}{n+2})}\ \Big[\mathrm{e}^{-\mathrm{i}\pi p_2}\ \beta_-(\theta)\,\alpha_+\big(\theta - \tfrac{\mathrm{i}\pi n}{2}\big) - \mathrm{e}^{+\mathrm{i}\pi p_2}\,\beta_+(\theta)\ \alpha_-\big(\theta - \tfrac{\mathrm{i}\pi n}{2}\big)\Big] \qquad (6.128)\\
\zeta_-(\theta) &= \frac{\mathrm{i}\,\mathrm{e}^{+\mathrm{i}\pi\mathrm{e}^{\theta}}}{2\sin(\frac{2\pi p_2}{n+2})}\ \Big[\mathrm{e}^{-\mathrm{i}\pi p_2}\ \beta_-(\theta+\mathrm{i}\pi)\,\alpha_+\big(\theta - \tfrac{\mathrm{i}\pi n}{2}\big) - \mathrm{e}^{+\mathrm{i}\pi p_2}\ \beta_+(\theta+\mathrm{i}\pi)\,\alpha_-\big(\theta - \tfrac{\mathrm{i}\pi n}{2}\big)\Big]\,.
\end{aligned}$$

It turns out these formulae and the quantum Wronskian relations, supplemented by the Hermiticity (6.65), (6.115), (6.120) and the analyticity of the operators $\alpha_\pm(\theta)$, $\beta_\pm(\theta)$, $\zeta_\pm(\theta)$ constitute a very restrictive set of conditions. In particular, it leads to the important relation (see Appendix F)

$$\alpha_\pm\big(\theta-\tfrac{\mathrm{i}\pi(n+1)}{2}\big)\alpha_\pm\big(\theta+\tfrac{\mathrm{i}\pi(n+1)}{2}\big)-\alpha_\pm\big(\theta-\tfrac{\mathrm{i}\pi(n-1)}{2}\big)\alpha_\pm\big(\theta+\tfrac{\mathrm{i}\pi(n-1)}{2}\big)=\beta_\pm\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)\beta_\pm\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)\,. \tag{6.129}$$

6.8 NLIE for the vacuum eigenvalues

The ODE/IQFT correspondence allows one, in principle, to find the spectrum of the commuting family of operators by numerically solving differential equations. Such a numerical procedure is especially convenient for the calculation of the eigenvalues of the operators $\beta_\pm(\theta)$. However, it is a highly non-trivial task to, say, extract the eigenvalues of the chiral transfer-matrix $\tau(\lambda)$ directly from the differential equations. Here we demonstrate how the functional relations and analytic conditions for the connection coefficients can be used to derive a system of Non-Linear Integral Equations (NLIE) which prove to be highly efficient in numerical work [131, 132, 133]. We will mostly focus on the vacuum eigenvalues.

For our purposes it is useful to rewrite eq. (6.129) using the notation

$$e^{i\varepsilon(\theta)} = \exp\big(2\pi i\,(e^\theta - p_2)\big)\ \frac{\alpha_+\big(\theta - \frac{i\pi n}{2}\big)}{\alpha_+\big(\theta + \frac{i\pi n}{2}\big)}\ . \tag{6.130}$$

Focusing on the case where the subscript in (6.129) is "+", one has

$$1 - e^{4\pi\, e^\theta + i\,\varepsilon\left(\theta + \frac{i\pi}{2}\right) - i\,\varepsilon\left(\theta - \frac{i\pi}{2}\right)} = \frac{\beta_+(\theta - \frac{i\pi}{2})\,\beta_+(\theta + \frac{i\pi}{2})}{\alpha_+\big(\theta - \frac{i\pi(n+1)}{2}\big)\,\alpha_+\big(\theta + \frac{i\pi(n+1)}{2}\big)}\ . \tag{6.131}$$

In order to define the operator $\varepsilon(\theta)$ itself, one should specify the branch of the logarithm $\frac{1}{i}\log(e^{i\varepsilon(\theta)})$. This can be done by supplementing (6.130) with the leading asymptotic behaviour

$$\varepsilon(\theta) \to 4\pi e^\theta - \tfrac{4\pi p_2}{n+2} + o(1) \qquad\quad \text{as} \quad \theta \to +\infty\ , \tag{6.132}$$

which is chosen to be consistent with the asymptotic formula (6.123). As follows from eq. (6.120), the operator $\varepsilon(\theta)$, thus defined and acting in the Fork space $\mathcal{F}_{p_1,p_2}$, satisfies the Hermiticity condition

$$\big[\varepsilon(\theta)\big]^\dagger = \varepsilon(\theta^*)\ . \tag{6.133}$$

Let us emphasize that (6.131)-(6.133) are operator relations for the commuting family, and therefore the same relations hold true for the eigenvalues corresponding to any common eigenvector $|\,\psi\,\rangle$ in the Fock space. Another important general property (which justifies the "strange" first factor in the definition (6.130)) concerns the zeroes of the eigenvalue $\beta_+^{(\psi)}(\theta)$. Namely it is easy to show that (see Appendix F)

$$\text{if}\ \ \theta_j:\ \ \beta_+^{(\psi)}(\theta_j) = 0\,,\ \ \text{then}\quad\ \exp\big(i\varepsilon^{(\psi)}(\theta_j - i\pi)\big) = -1\ . \tag{6.134}$$

Notice that since $\beta_+^{(\psi)}(\theta)$ is a real analytic and periodic function in θ with period $2\pi\mathrm{i}$, it is sufficient to consider its zeroes in the strip $0 \leq \Im m(\theta) \leq \pi$, only.

In the case of the vacuum eigenvalues, our numerical work suggests that for $\frac{2p_2}{n+2} > -\frac{1}{2}$ all the zeroes of $\beta_+^{(\mathrm{vac})}(\theta)$ in the strip $\big|\Im m(\theta)\big| \leq \pi$ are simple, located on the boundary $\Im m(\theta) = \pi$, accumulate toward $\Re e(\theta) \to +\infty$ and satisfy the condition

$$\varepsilon^{(\mathrm{vac})}(\theta_j - \mathrm{i}\pi) = \pi\,(2j-1) \qquad (j = 1,\, 2,\, \ldots)\,. \tag{6.135}$$

This "quantization condition" supplemented by the asymptotic formula (6.123), leads to an equation determining the vacuum roots of $\beta_+^{(\mathrm{vac})}(\theta)$ which is asymptotically exact as $j \to +\infty$:

$$\exp\big(\theta_j^{(\mathrm{vac})}\big) \equiv -\rho_j \;\;:\quad \rho_j \asymp \tfrac{1}{2}\,\big(j - \tfrac{1}{2} + \tfrac{2p_2}{n+2}\big) - \tfrac{1}{2\pi}\sum_{m=1}^{\infty} \tilde{s}_m(p_1,p_2)\;\sin\big(\tfrac{2\pi m}{n+2}\big)\;(\rho_j)^{-\frac{2m}{n+2}} \tag{6.136}$$

where $\tilde{s}_m(p_1,p_2)$ stands for the vacuum eigenvalues of the dual nonlocal IM $\tilde{\mathfrak{s}}_m^+$ in (6.124), (6.126). In particular

$$\tilde{s}_1(p_1,p_2) = -\Big(\frac{n+2}{2}\Big)^{\frac{2}{n+2}}\;\frac{\Gamma(\frac{1}{n+2})\Gamma(\frac{1}{2}-\frac{1}{n+2})}{4\sqrt{\pi}}\;\left[\frac{n}{n+4} - \frac{4p_1^2}{1-4p_2^2}\right]\frac{\Gamma(1+\frac{2p_2+1}{n+2})}{\Gamma(\frac{2p_2-1}{n+2})}\,. \tag{6.137}$$

Additional analytical input required for the derivation of the NLIE, is the behaviour at $\Re e(\theta) \to -\infty$. It can be studied along the following line: using the formulae (6.128) and the quantum Wronskian relations one can express $\alpha_+^{(\mathrm{vac})}$ in terms of $\zeta_\pm^{(\mathrm{vac})}$ and $\beta_+^{(\mathrm{vac})}$. The asymptotics of $\zeta_\pm^{(\mathrm{vac})}(\theta)$ at $\theta \to -\infty$ are given by eqs. (6.83). The general structure of the $\theta \to -\infty$ behaviour of $\beta_+^{(\mathrm{vac})}$ is dictated by the operator valued series expansion (6.114). Using the differential equation (6.106), it is not difficult to find the following explicit expressions for the vacuum eigenvalues of the first expansion coefficients:

$$b_\pm^{(\mathrm{vac})}(p_1,p_2) = (n+2)^{-\frac{1}{2}\mp\frac{2p_2}{n+2}}\quad \frac{2\pi\;\Gamma(1\pm 2p_2)}{\Gamma(\frac{1}{2}-\mathrm{i}p_1 \pm p_2)\;\Gamma(\frac{1}{2}+\mathrm{i}p_1 \pm p_2)\,\Gamma(1\pm\frac{2p_2}{n+2})} \tag{6.138}$$

and (to be compared with the coefficient $f_{1,0}$ in eq. (6.86))

$$b_1^{(\pm,\mathrm{vac})}(p_1,p_2) = -\psi\big(\tfrac{1}{2}-\mathrm{i}p_1\pm p_2\big)-\psi\big(\tfrac{1}{2}+\mathrm{i}p_1\pm p_2\big)+\psi\big(1+\tfrac{n}{2}\big)+\gamma_E+2\,\log 2-2\,. \quad (6.139)$$

Now it is straightforward to derive the leading asymptotic behaviour of the vacuum eigenvalue of the operator $\varepsilon(\theta)$ (6.130), (6.132). Below we list explicit formulae for $\varepsilon^{(\mathrm{vac})}(\theta)$ and for the vacuum eigenvalues of the operator

$$\omega(\theta) \equiv \log\Big(\mathrm{e}^{\mathrm{i}\varepsilon(\theta-\frac{\mathrm{i}\pi}{2})-\mathrm{i}\varepsilon(\theta+\frac{\mathrm{i}\pi}{2})-4\pi\mathrm{e}^{\theta}}-1\Big)\,, \qquad \omega(\theta)\to 4\pi\,\mathrm{e}^{\theta}+o(1) \quad \text{as} \quad \theta\to+\infty\,. \quad (6.140)$$

(a) For real $p_1\neq 0$ and $p_2>-\frac{1}{2}$:

$$\varepsilon^{(\mathrm{vac})}(\theta) = -2\pi p_2+\frac{1}{\mathrm{i}}\,\log\left[\frac{\sin\big(\frac{2p_1}{n}(\theta_0-\theta+\frac{\mathrm{i}\pi n}{2})\big)}{\sin\big(\frac{2p_1}{n}(\theta_0-\theta-\frac{\mathrm{i}\pi n}{2})\big)}\right]+o\big(\theta^{-\infty}\big) \quad (6.141)$$

$$\exp\big(-\omega^{(\mathrm{vac})}(\theta)\big) = \frac{\sinh^2\big(\frac{\pi(n-1)p_1}{n}\big)}{\sinh(2\pi p_1)\sinh\big(\frac{2\pi p_1}{n}\big)}+\frac{\sin^2\big(\frac{2p_1}{n}(\theta_0-\theta)\big)}{\sinh(2\pi p_1)\sinh\big(\frac{2\pi p_1}{n}\big)}+o\big(\theta^{-\infty}\big)$$

where

$$\theta_0=\log(n)+\frac{n}{4\mathrm{i}p_1}\,\log\left[\frac{\Gamma(1+2\mathrm{i}p_1)\Gamma(1+\frac{2\mathrm{i}p_1}{n})}{\Gamma(1-2\mathrm{i}p_1)\Gamma(1-\frac{2\mathrm{i}p_1}{n})}\,\frac{\Gamma^2(\frac{1}{2}+p_2-\mathrm{i}p_1)}{\Gamma^2(\frac{1}{2}+p_2+\mathrm{i}p_1)}\right]\,.$$

(b) For $p_1=0$ and $p_2>-\frac{1}{2}$:

$$\varepsilon^{(\mathrm{vac})}(\theta) = -2\pi p_2+\frac{1}{\mathrm{i}}\,\log\left[\frac{\theta_0-\theta+\frac{\mathrm{i}\pi n}{2}}{\theta_0-\theta-\frac{\mathrm{i}\pi n}{2}}\right]+o\big(\theta^{-\infty}\big)$$

$$\exp\big(-\omega^{(\mathrm{vac})}(\theta)\big) = \frac{(\theta_0-\theta)^2}{n\pi^2}+\frac{1}{4n}\,(n-1)^2+o\big(\theta^{-\infty}\big)\,, \quad (6.142)$$

where

$$\theta_0=\log(n)-(1+n)\,\gamma_E-n\,\psi\big(\tfrac{1}{2}+p_2\big)\,.$$

(c) For pure imaginary $p_1\equiv\frac{\mathrm{i}}{2}\,\mathfrak{m}$ with $0<\mathfrak{m}<\frac{n}{2}$ and real $p_2\equiv\mathfrak{j}+\frac{1}{2}$ such that $\mathfrak{j}-\frac{1}{2}\,\mathfrak{m}>-1$:

$$\varepsilon^{(\mathrm{vac})}(\theta) = -\pi\,(2\mathfrak{j}+1-\mathfrak{m})+\frac{2\pi\mathfrak{m}\,n^{-\frac{2\mathfrak{m}}{n}}}{\Gamma^2(1+\mathfrak{m})}\,\frac{\Gamma(1-\frac{\mathfrak{m}}{n})}{\Gamma(1+\frac{\mathfrak{m}}{n})}\,\frac{\Gamma^2(1+\mathfrak{j}+\frac{\mathfrak{m}}{2})}{\Gamma^2(1+\mathfrak{j}-\frac{\mathfrak{m}}{2})}\,\mathrm{e}^{\frac{2\mathfrak{m}\theta}{n}}+o\big(\mathrm{e}^{\frac{2\mathfrak{m}\theta}{n}}\big)$$

$$\omega^{(\mathrm{vac})}(\theta) = \frac{2\mathfrak{m}}{n}\,\theta-2\,\log\left[n^{\frac{1}{2}+\frac{\mathfrak{m}}{n}}\,\frac{\Gamma(1+\mathfrak{j}-\frac{\mathfrak{m}}{2})\Gamma(\mathfrak{m})\Gamma(1+\frac{\mathfrak{m}}{n})}{2\pi\Gamma(1+\mathfrak{j}+\frac{\mathfrak{m}}{2})}\right]+O\big(\mathrm{e}^{\frac{2\theta}{n}},\mathrm{e}^{\theta}\big)\,. \quad (6.143)$$

Notice that eqs. (6.142) with $p_2 = \mathfrak{j} + \frac{1}{2}$ and

$$\mathfrak{j} = \tfrac{1}{2},\, 1, \ldots, \tfrac{1}{2}\left[\tfrac{n}{2}\right] \qquad (n = 2,\, 3,\, 4, \ldots) \tag{6.144}$$

can be applied to the parafermion case with $\mathfrak{m} = 0$. For positive integer $\mathfrak{m}$, restricted to

$$1 \leq \mathfrak{m} \leq \left[\tfrac{n-1}{2}\right], \qquad \mathfrak{j} - \tfrac{1}{2}\,\mathfrak{m} \geq 0\ , \tag{6.145}$$

eqs. (6.143) should be used.

It is expected that for the cases (b) and (c), the function $\alpha^{(\mathrm{vac})}(\theta)$ does not have any zeroes in the strip $|\Im m(\theta)| < \frac{\pi}{2}\,(n+2)$. However, as it follows from the asymptotic behaviour (6.141) and formula (6.130), for $p_1 \neq 0$ and $p_2 > -\frac{1}{2}$ it has a sequence of zeroes, $\{\theta_m^{(\alpha)}\}_{m=1}^{\infty}$, extending towards $-\infty$ along the real axis such that

$$\theta_m^{(\alpha)} = \theta_0 - \frac{\pi n}{2p_1}\, m + o\big((m/p_1)^{-\infty}\big) \qquad (m = 1, 2, \ldots)\,. \tag{6.146}$$

Again, we expect that there are no other zeroes apart from $\{\theta_m^{(\alpha)}\}_{m=1}^{\infty}$ in the strip $|\Im m(\theta)| < \frac{\pi}{2}\,(n+2)$. In principle, these properties should be rigorously derived from the differential equation (6.106). Unfortunately, we don't have a proof at this moment and we formulate the statements as a conjecture. Notice that the domain of applicability of the large-θ asymptotic expansion (6.123) is also restricted to the same strip.

The outlined analytic properties fully determine all of the vacuum eigenvalues of the commuting operators acting in the Fock space $\mathcal{F}_{p_1,p_2}$. Practically, they can be used to derive a closed system of integral equations which involve the vacuum eigenvalues of $\varepsilon^{(\mathrm{vac})}(\theta)$ and $\omega^{(\mathrm{vac})}(\theta)$. A few useful formulae appearing in the intermediate steps of the derivation are given in appendices F and G. With the parameters p_1 and p_2 as in (b) and (c) above, the final result reads as follows:

$$
\begin{aligned}
\varepsilon(\theta-\mathrm{i}\gamma) &= 4\pi\,\mathrm{e}^{\theta-\mathrm{i}\gamma}-2\pi k+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[G(\theta-\theta'-2\mathrm{i}\gamma)\,\big(L(\theta'-\mathrm{i}\gamma)\big)^{*}\\
&- G(\theta-\theta')\,L(\theta'-\mathrm{i}\gamma)\Big]+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi}\,G_1(\theta-\theta'-\mathrm{i}\gamma)\,\log\big(1+\mathrm{e}^{-\omega(\theta')}\big)\\
\omega(\theta) &= 4\pi\,\mathrm{e}^{\theta}+\Im m\Big[\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,G_1(\theta-\theta'+\mathrm{i}\gamma)\,L(\theta'-\mathrm{i}\gamma)\Big] \qquad (6.147)\\
&- \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,G_2(\theta-\theta')\,\log\big(1+\mathrm{e}^{-\omega(\theta')}\big)\\
L(\theta) &= \log\big(1+\mathrm{e}^{-\mathrm{i}\varepsilon(\theta)}\big)\ .
\end{aligned}
$$

Here we drop the superscript "(vac)" in the notation for the vacuum eigenvalues, and the star "$*$" as usual, stands for complex conjugation. The constant γ is an arbitrary number belonging to the open segment $0<\gamma<\frac{\pi}{2}$. We also swap p_2 for

$$
k=\frac{2p_2}{n+2}\ , \qquad (6.148)
$$

and the kernels read explicitly as

$$
\begin{aligned}
G(\theta) &= \frac{\sin(\frac{2\pi}{n+2})}{(n+2)\sinh(\frac{\theta+\mathrm{i}\pi}{n+2})\sinh(\frac{\theta-\mathrm{i}\pi}{n+2})}\\
G_1(\theta) &= \frac{\sin(\frac{2\pi}{n+2})\sin(\frac{\pi}{n+2})\ \sinh(\frac{2\theta}{n+2})}{(n+2)\sinh(\frac{\theta+\frac{\mathrm{i}\pi}{2}}{n+2})\sinh(\frac{\theta-\frac{\mathrm{i}\pi}{2}}{n+2})\sinh(\frac{\theta+\frac{3\mathrm{i}\pi}{2}}{n+2})\sinh(\frac{\theta-\frac{3\mathrm{i}\pi}{2}}{n+2})} \qquad (6.149)\\
G_2(\theta) &= G(\theta)-\frac{\sin(\frac{4\pi}{n+2})}{2(n+2)\sinh(\frac{\theta+2\mathrm{i}\pi}{n+2})\sinh(\frac{\theta-2\mathrm{i}\pi}{n+2})}\ .
\end{aligned}
$$

It is important to keep in mind that the integral equations should be supplemented by the asymptotics for the vacuum eigenvalues given by eqs. (6.142)-(6.143). For real $p_1\neq 0$ and $p_2>-\frac{1}{2}$ (as in case (a) above) the integral equations must be modified by adding extra source terms to the r.h.s. The corresponding NLIE is given by formulae (G.1), (G.2) in appendix G along with some explanations.

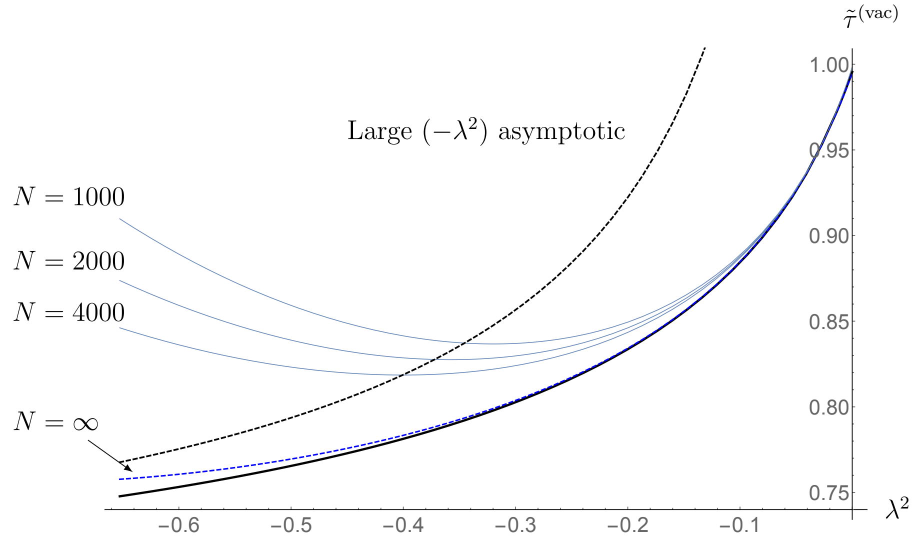


Figure 6.2: A plot of the scaling function $\tilde{\tau}^{(\mathrm{vac})} = \tau^{(\mathrm{vac})} \times \exp\big(2\pi\,(-\lambda^2)^{\frac{3}{2}}\big)$ for the parafermion vacuum with $n = 3$, and $2\mathfrak{j} = \mathfrak{m} = 1$. The thick black line comes from the numerical solution of the NLIE system (6.147). The blue curves are the same as in the plot appearing in the right panel of fig. D.1 from Appendix D. They were obtained from the numerical solution of the Bethe ansatz equations for finite N and subsequently interpolating the data to $N = \infty$. The large $(-\lambda^2)$ asymptotic is given by eqs. (5.29), (5.30).

Once the system of NLIE is solved, the numerical data can be used to reconstruct the vacuum eigenvalues of $\alpha^{(\mathrm{vac})}(\theta)$, $\beta^{(\mathrm{vac})}(\theta)$ and $\tau^{(\mathrm{vac})}(\mathrm{i}\lambda)$. The corresponding formulae are given by (F.6)-(F.7) from Appendix F. Expressions for the vacuum eigenvalues of the local and dual nonlocal integrals of motion are present there as well. In fig. 6.2 numerical results for $\tau^{(\mathrm{vac})}$ in the parafermionic vacuum with $n = 3$ and $2\mathfrak{j} = \mathfrak{m} = 1$ ($p_1 = \frac{\mathrm{i}}{2}$, $p_2 = 1$) are shown alongside the results from the Bethe ansatz. Also, for numerous cases, we compared the numerical results for the connection coefficients computed from the NLIE with those obtained by direct integration of the ordinary differential equations (6.106). The agreement we found in all cases justifies the assumptions made within the derivation of the NLIE system.

Chapter 7
Integrable structures in the sausage

We now turn to the main subject of interest – the sausage model. First of all, we recall some basic facts concerning this quantum field theory. For more details, see, e.g., [116].

7.1 Basic facts about the quantum sausage

In chapter 2, we briefly touched on the one loop renormalization in the sausage model. Let us introduce the renormalized coupling κ_r which substitutes the bare coupling κ (5.3):

$$\frac{1-\kappa_r}{1+\kappa_r} = (E_*/E)^{\frac{2}{n}} \ . \tag{7.1}$$

Here E stands for a typical energy scale, which, in the case under consideration, can be identified with the inverse of the circumference of the space-time cylinder, R^{-1}. Recall also that $n \equiv \frac{2\pi}{\hbar}$ and E_* is a RG invariant energy scale appearing in the theory through the mechanism of dimensional transmutation. Within the one-loop approximation the bare coupling is replaced by κ_r, so that the renormalized sausage metric is given by

$$G^{(\mathrm{ren})}_{ab}\, \mathrm{d}X^a \mathrm{d}X^b = \frac{n}{2\pi}\ \frac{(\mathrm{d}\phi)^2 + (\mathrm{d}\alpha)^2}{\frac{1}{2}\left(\kappa_r^{-1}+\kappa_r\right) + \frac{1}{2}\left(\kappa_r^{-1}-\kappa_r\right)\,\cosh(2\phi)}\ , \tag{7.2}$$

where we use the pair of real coordinates $X^a = (\phi, \alpha)$ (**??**). Consider the ultraviolet regime where $E \gg E^*$, i.e., $1-\kappa_r \ll 1$. In this case, the central region of the sausage depicted in fig. ?? looks like a long cylinder equipped with the flat metric. If one formally sets $\kappa_r = 1$ in (7.2), ignoring the presence of the two infinitely separated tips, then

$$G^{(\mathrm{ren})}_{ab}\,\mathrm{d}X^a\mathrm{d}X^b \approx \frac{n}{2\pi}\,\big((\mathrm{d}\phi)^2+(\mathrm{d}\alpha)^2\big)\ . \tag{7.3}$$

The NLSM action corresponding to this metric has the form of the massless Gaussian model[1]

$$\mathcal{A}_0 = \frac{n}{4\pi}\int \mathrm{d}^2x\,\big((\partial_\mu\phi)^2+(\partial_\mu\alpha)^2\big)\ . \tag{7.4}$$

We now apply the T-duality transformation to the field α, i.e., we replace α by the T-dual field ϑ such that $\partial_\mu\alpha = \frac{1}{\sqrt{n+2}}\,\epsilon_{\mu\nu}\partial_\nu\vartheta$ (the difference between n and $n+2$ is ignored here, since n is assumed to be large). The substitution of (ϕ,α) by the pair (φ,ϑ), with $\phi=\frac{1}{\sqrt{n}}\,\varphi$, brings the action to the form

$$\tilde{\mathcal{A}}_0 = \frac{1}{4\pi}\int \mathrm{d}^2x\,\big((\partial_\mu\varphi)^2+(\partial_\mu\vartheta)^2\big)\ . \tag{7.5}$$

The advantage of $\tilde{\mathcal{A}}_0$ compared to the action (7.4), is that it allows one to easily incorporate the effects of the sausage tips. The central region with the left tip of the sausage form the cigar, and the corresponding NLSM, as it was mentioned in sec. 5.2.2, admits the dual description in terms of the sine-Liouville action

$$\tilde{\mathcal{A}}^{(\mathrm{left})} = \tilde{\mathcal{A}}_0 + 2\mathcal{M}\int \mathrm{d}^2x\ \mathrm{e}^{-\sqrt{n}\varphi}\ \cos\big(\sqrt{n+2}\,\vartheta\big)\ . \tag{7.6}$$

Clearly, the cigar NLSM whose target space is glued from the right tip and the central region, is governed by the action which is related to (7.6) by the flip $\varphi\mapsto-\varphi$, i.e.,

$$\tilde{\mathcal{A}}^{(\mathrm{right})} = \tilde{\mathcal{A}}_0 + 2\mathcal{M}\int \mathrm{d}^2x\ \mathrm{e}^{+\sqrt{n}\varphi}\ \cos\big(\sqrt{n+2}\,\vartheta\big)\ . \tag{7.7}$$

At this point one can guess that the sausage NLSM admits the dual description by means of the renormalized action

$$\tilde{\mathcal{A}}^{(\mathrm{saus})} = \int \mathrm{d}^2x\,\Big(\tfrac{1}{4\pi}\,\big((\partial_\mu\varphi)^2+(\partial_\mu\vartheta)^2\big) + 4\mathcal{M}\,\cosh\big(\sqrt{n}\varphi\big)\cos\big(\sqrt{n+2}\,\vartheta\big)\Big)\ . \tag{7.8}$$

[1] In what follows the Planck constant $\hbar=\frac{2\pi}{n}$ will be always included in the action.

Remarkably, the naive guess turns out to be correct! The dual form of the sausage model was originally proposed by Aleosha Zamolodchikov and there are many arguments to support its validity, a few of which will be mentioned later in the text. Contrary to the sine-Liouville model where the dimensionless $\mathcal{M}$ is a somewhat fake parameter, the coupling $\mathcal{M}$ in (7.8) is an important, dimensionful characteristic of the theory. Notice that the field $\cosh\big(\sqrt{n}\varphi\big)\cos\big(\sqrt{n+2}\,\vartheta\big)$ has the scale dimensions equal to one w.r.t. the conventional energy momentum tensor of the "unperturbed" free theory (7.5). Therefore $\mathcal{M}$ has dimensions of energy, i.e., $\mathcal{M} \propto E_*$ up to some dimensionless constant.

Consider now the general structure of the Hilbert space of the sausage NLSM in finite volume equipped with the boundary conditions of the form (5.55). The quantum number m in that formula must take integer values only, it is a conserved charge associated with the $U(1)$-isometry of the sausage metric. In what follows we will focus on the neutral sector of the theory with $m = 0$, and therefore

$$\varphi(t, x+R) = \varphi(t, x)\ , \qquad \vartheta(t, x+R) = \vartheta(t, x)\ . \tag{7.9}$$

Since the action (7.8) and the boundary conditions are both invariant under the transformation $\vartheta \mapsto \vartheta + \frac{2\pi}{\sqrt{n+2}}$, the space of the neutral states of the sausage model is somewhat similar to the Hilbert space of a quantum particle in a periodic potential: it is split on the orthogonal subspaces $\mathcal{H}_k^{(K)}$ characterized by the quasimomentum restricted to the first Brillouin zone, $-\frac{1}{2} < k < \frac{1}{2}$, and a positive integer K – the band number. Our considerations below will be mostly restricted to the k-vacuum, $|\mathrm{vac}\rangle_k \in \mathcal{H}_k^{(1)}$ – the lowest energy neutral state in the first band.

In sec. 5.2.2 it was mentioned that the Hilbert space of the cigar/sine-Liouville theory is classified w.r.t. the action of a certain $\mathfrak{W} \otimes \overline{\mathfrak{W}}$-algebra. The W-algebras related by the reflection $\varphi \to -\varphi$ are algebraically isomorphic (for details see, e.g., [114, 116]). This property allows one to identify the spaces of states for the "left" and "right" sine-Liouville models (7.6) and (7.7) which can be then interpreted as an extended Hilbert space of the sausage NLSM. However, contrary to the case of the cigar NLSM, instead of the continuous summation as in (5.46), the zero-mode momentum p_1 in the sausage takes a certain discrete set of admissible values which depends on R. As $\mathcal{M}R \to 0$ the mechanism of the quantization of p_1 is similar to that in the sinh-Gordon model considered in ref. [134]. The discussions from this work can be easily adopted to our problem (see, e.g., [114]).

When $\mathcal{M}R \ll 1$, the quantization condition which determines the value of $p_1 = p_1(R)$ for the k-vacuum, $|\mathrm{vac}\rangle_k \in \mathcal{H}_k^{(1)}$, reads as follows

$$-\frac{8p_1}{n}\ \log\Big(\frac{\mathcal{M}R}{2}\Big) + 2\,\delta(p_1, p_2) = 2\pi\ . \tag{7.10}$$

Here $p_2 \equiv \frac{1}{2}(n+2)\,k$, $\delta(p_1, p_2) = -\mathrm{i}\,\log S_0(\mathbf{p})$ and $S_0(\mathbf{p})$ is the overall scalar factor for the S-matrix (5.52). This factor was first derived by A. and Al. Zamolodchikov [119] and, using the notation of B_{p_1,p_2} from (6.84), it is given by $S_0(p_1, p_2) = -B_{-p_1,p_2}/B_{p_1,p_2}$. Thus,

$$\begin{aligned}\delta(p_1, p_2) \ &= \ \frac{4p_1}{n}\ \log(n) \qquad\qquad (7.11)\\ &- \ \mathrm{i}\ \log\left[\frac{\Gamma(\frac{1}{2}-p_2-\mathrm{i}p_1)\,\Gamma(\frac{1}{2}+p_2-\mathrm{i}p_1)}{\Gamma(\frac{1}{2}-p_2+\mathrm{i}p_1)\Gamma(\frac{1}{2}+p_2+\mathrm{i}p_1)}\,\frac{\Gamma(1+2\mathrm{i}p_1)}{\Gamma(1-2\mathrm{i}p_1)}\,\frac{\Gamma(1+\frac{2\mathrm{i}p_1}{n})}{\Gamma(1-\frac{2\mathrm{i}p_1}{n})}\right]\end{aligned}$$

where the branch of the logarithm is chosen so that $\delta(0, p_2) = 0$. The vacuum energy in the sector $\mathcal{H}_k^{(1)}$ is approximately the corresponding vacuum energy of the free theory (7.5), with the zero mode momentum for the field φ determined by the quantization condition (7.10):

$$E_k^{(\mathrm{vac})} \approx \frac{\pi}{R}\,\Big(-\frac{1}{3} + \frac{4}{n}\ \big(p_1(R)\big)^2 + (n+2)\,k^2\Big)\ . \tag{7.12}$$

This result can be obtained both from the theory described by the action (7.8) and directly from the sausage NLSM within the so-called minisuperspace approximation, which, in fact, is a strong argument for the validity of the dual description of the quantum sausage.

A few comments to the formula (7.12) are in order. First of all a brief inspection of the scattering phase $\delta(p_1, p_2)$ (7.11), shows that the quantization condition can be applied literally only for $|p_2| < \frac{1}{2}$, or, equivalently, for $|k| < \frac{1}{n+2}$. For $p_2 = \frac{1}{2}$, notice that eq. (7.10) has an R-independent solution $p_1 = 0$, whereas for $\frac{1}{n+2} < |k| < \frac{1}{2}$, the precise form of the quantization condition, to the best of our knowledge, is currently not known. The next comment deals with corrections to the approximate formula (7.12). A superficial analysis shows that for $n > 2$ the main correction is of order $R^{\frac{4}{n}-1}$, while for $0 < n < 2$, a term $\propto R\log(R)$ dominates. The latter can be understood as the one-instanton contribution and has the following explicit form

$$\delta E^{(1-\text{inst})} = -\pi \mathcal{M}^2 R \, \Big(4\, \log\big(R\Lambda^{(\text{inst})}\big) + e(k) + O\big(p_1^2(R)\big)\Big) \, . \tag{7.13}$$

Here $\Lambda^{(\text{inst})}$ is a cut-off energy scale which regularizes the contribution of the small-size instantons [135, 136, 137, 66] and

$$e(k) = -2 - 2\,\psi\left(\tfrac{1}{2} - \tfrac{(n+2)k}{2}\right) - 2\psi\left(\tfrac{1}{2} + \tfrac{(n+2)k}{2}\right) \, . \tag{7.14}$$

Now, let us turn to the large-R limit. In this limit, $E_k^{(\text{vac})}$ contains an extensive part which is proportional to the spatial size of the system and does not depend on k. As follows from the results of ref.[66], the specific bulk energy $\mathcal{E} \equiv \lim\limits_{R\to\infty} E_k^{(\text{vac})}/R$ is given by

$$\mathcal{E} = \pi \mathcal{M}^2 \, \Big(\, 4\, \log\big(\mathcal{M}/\Lambda^{(\text{inst})}\big) + \pi \cot\left(\tfrac{\pi n}{2}\right) + 2\,\gamma_E + 2\,\psi\big(1 + \tfrac{n}{2}\big)\Big) \, . \tag{7.15}$$

The large-R behaviour of the difference $E_k^{(\text{vac})} - R\,\mathcal{E}$ is dictated by the factorized scattering theory associated with the model. In the original work [66], it was proposed that the spectrum of the sausage model in infinite volume consists of a triplet of particles of the same mass m. The mass scale is simply related to the dimensionful coupling in the renormalized action (7.8):

$$m = 4\pi\, \mathcal{M} \, . \tag{7.16}$$

Two of the particles A_+ and A_- carry the $U(1)$ charge $+1$ and -1, respectively, whereas the third particle A_0 is neutral. The factorized scattering is completely determined by the two-particle S-matrix which can be interpreted as a structure constant in the formal Zamolodchikov-Faddeev associative algebra:

$$A_a(\theta_1)\, A_b(\theta_2) = S_{ab}^{cd}(\theta_1 - \theta_2)\ A_d(\theta_2)\, A_c(\theta_1)\ . \tag{7.17}$$

For convenience, we collect in Appendix H explicit expressions for the scattering amplitude S_{ab}^{cd} proposed in [66]. Taking a closer look at these amplitudes, one can observe that they are trivialized for $n = 0$. This is consistent with the fact that the dual theory (7.8) can be understood, by use of the Coleman-Mandelstam bosonization, as an interacting theory of a massive scalar and Dirac fermion. At $n = 0$, the interaction disappears and we end up with a theory of three non-interacting particles. In fact, this was the starting point of Al. Zamolodchikov's proposal for the dual description of the sausage. He performed perturbative calculations for small n and found that the perturbative amplitudes match the small-n expansion of the exact two-particle S-matrix.

As usual for a massive quantum field theory, the large-R expansion of $E_k^{(\mathrm{vac})} - R\,\mathcal{E}$ can be represented in the form

$$E_k^{(\mathrm{vac})} - R\,\mathcal{E} = \Delta E^{1-\mathrm{particle}} + \Delta E^{2-\mathrm{particle}} + \ldots\ , \tag{7.18}$$

where the individual terms correspond to the virtual contributions of N-particle states. The one-particle contribution does not depend on the details of the interaction – it is the same as for the non-interacting theory and hence is given by

$$\Delta E^{1-\mathrm{particle}} = -\mathrm{Tr}_1\big(\boldsymbol{K}^{(1)}\big) \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta}{2\pi}\ \ m\cosh(\theta)\, \mathrm{e}^{-mR\cosh(\theta)}\ . \tag{7.19}$$

Here the trace is taken over the isotopic component of the one-particle sector of the theory in the infinite volume and $\boldsymbol{K}^{(1)}$ is a 3×3 diagonal matrix of the (complex) fugacities:

$$\boldsymbol{K}^{(1)} = \begin{pmatrix} \mathrm{e}^{+2\pi \mathrm{i} k} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{e}^{-2\pi \mathrm{i} k} \end{pmatrix} . \tag{7.20}$$

Following Lüscher [138, 139] (see also ref. [140]), one can derive the explicit formula for the two-particle contribution in eq. (7.18):

$$\begin{aligned} \Delta E^{2-\text{particle}} &= m \int_{-\infty}^{\infty} \frac{\mathrm{d}\theta \mathrm{d}\theta'}{(2\pi)^2} \cosh(\theta)\, \mathrm{e}^{-mR\cosh(\theta) - mR\cosh(\theta')} \\ &\times \ \mathrm{Tr}_2 \Big[\boldsymbol{K}^{(2)} \Big(\pi \boldsymbol{I}^{(2)}\, \delta(\theta-\theta') + \mathrm{i}\, \partial_\theta \log \boldsymbol{S}^{(2\mapsto 2)}(\theta - \theta') \Big) \Big] , \end{aligned} \tag{7.21}$$

where $\boldsymbol{S}^{(2\mapsto 2)}$ is the 9×9 matrix acting in the isotopic component of the two-particle sector, $\boldsymbol{K}^{(2)} = \boldsymbol{K}^{(1)} \otimes \boldsymbol{K}^{(1)}$ and $\boldsymbol{I}^{(2)}$ is the identity matrix.

For future reference let us make a short summary of the properties of the k-vacuum energy discussed above. For this purpose it is convenient to introduce the scaling variable

$$r = mR \tag{7.22}$$

and dimensionless scaling function

$$\mathfrak{F}(r,k) = \frac{R}{\pi} \left(E_k^{(\mathrm{vac})} - R\,\mathcal{E} \right) . \tag{7.23}$$

Notice that, $c_{\text{eff}} \equiv -6\,\mathfrak{F}(r,k)$ is sometimes interpreted as the effective central charge for the off-critical theory. As $r \to 0$ our discussion suggests that for $|k| < \frac{1}{n+2}$ and fixed n

$$\mathfrak{F}(r,k) = \begin{cases} -\frac{1}{3} + \frac{4}{n}\, p^2(r) + (n+2)\, k^2 + O(r^{\frac{4}{n}}) & \text{for} \quad n > 2 \\[2ex] -\frac{1}{3} + \frac{4}{n}\, p^2(r) + (n+2)\, k^2 + \delta\mathfrak{F}^{(1-\text{inst})} + O\big(r^2\, p^2(r)\big) & \text{for} \quad 0 < n < 2 \end{cases} \tag{7.24}$$

Here $p(r)$ is defined as the solution of the quantization condition

$$-\tfrac{8p}{n}\ \log\big(\tfrac{r}{8\pi}\big)+2\,\delta(p,p_2)=2\pi \qquad\qquad \big(p_2<\tfrac{1}{2}\big) \tag{7.25}$$

where $\delta(p_1,p_2)$ is given by (7.11), and

$$\delta\mathfrak{F}^{(1-\mathrm{inst})}=-\tfrac{r^2}{16\pi^2}\ \Big(4\ \log\big(\tfrac{r}{4\pi}\big)+\pi\ \cot\big(\tfrac{\pi n}{2}\big)+2\,\gamma_E+2\,\psi\big(1+\tfrac{n}{2}\big)+e_1(k)\Big) \tag{7.26}$$

with $e_1(k)$ defined in (7.14).

The large r-behaviour of the scaling function $\mathfrak{F}(r,k)$ is determined by eqs.(7.18)-(7.21). It can be equivalently described by the following formula which is convenient for numerical calculations:

$$\begin{aligned}\mathfrak{F}(r,k)\ &=\ -\frac{r}{\pi^2}\,\big(\,2\,c(2k)+1\big)\ K_1(r)+\frac{r}{2\pi^2}\ \big(\,2\,c(2k)+1\,\big)^2\ K_1(2r)\\ &-\ \ \frac{2r}{\pi^3}\,\big(\,1+c(2k)\,\big)\ \int_{-\infty}^{\infty}\mathrm{d}\nu\ \left(\frac{K_{1-\mathrm{i}\nu}(r)K_{\mathrm{i}\nu}(r)}{\sinh\big(\frac{\pi(n+2)\nu}{2}\big)}\right.\\ &\times\ \left.\Big[2c(2k)\sinh\big(\tfrac{\pi n\nu}{2}\big)-\sinh\big(\tfrac{\pi(n-2)\nu}{2}\big)\Big]\right)+O\big(\mathrm{e}^{-3r}\big)\end{aligned} \tag{7.27}$$

where $c(x)\equiv\cos(\pi x)$, $K_s(z)$ denotes the modified Bessel function of the second order,

$$K_s(z)=\frac{1}{2}\ \int_{-\infty}^{\infty}\mathrm{d}\theta\ \mathrm{e}^{s\theta-z\cosh(\theta)}\ , \tag{7.28}$$

and the symbol $O(\mathrm{e}^{-3r})$ stands for terms which decay faster than $\mathrm{e}^{-(3-\epsilon)r}$ as $r\to+\infty$, for any small $\epsilon>0$.

Finally, for $n=0$, the scaling function $\mathfrak{F}(r,k)$ is given explicitly by:

$$\mathfrak{F}(r,k)=-\frac{r}{2\pi^2}\ \int_{-\infty}^{\infty}\mathrm{d}\theta\ \mathrm{e}^{\pm\theta}\ \log\left(\frac{\big(1+\mathrm{e}^{2\pi\mathrm{i}k-r\cosh(\theta)}\big)\big(1+\mathrm{e}^{-2\pi\mathrm{i}k-r\cosh(\theta)}\big)}{1-\mathrm{e}^{-r\cosh(\theta)}}\right)\ . \tag{7.29}$$

Notice that the small r-asymptotic (7.24) can not be applied to this exact formula because of the noncommutativity of the limits $r\to0$ and $n\to0$.

7.2 NLIE for the k-vacuum eigenvalues in the sausage model

With some experience in working with nonlinear integral equations in integrable QFT, one expects that the generalization of the massless equations to the massive ones requires little effort. For this reason, before exploring the general integrable structures, we make a simple-minded shortcut and guess the NLIE describing the k-vacuum eigenvalues in the sausage model. Of course, this route requires careful consistency checks which will be the main subject of our discussion here.

As usual, the major modification required to get the massive NLIE is related to the source terms. In the case under consideration it is not difficult to guess that the system (6.147) should be modified to the following

$$\begin{aligned}
\varepsilon(\theta-\mathrm{i}\gamma) &= r\,\sinh(\theta-\mathrm{i}\gamma)-2\pi k+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[G(\theta-\theta'-2\mathrm{i}\gamma)\,\big(L(\theta'-\mathrm{i}\gamma)\big)^{*}\\
&- \;G(\theta-\theta')\,L(\theta'-\mathrm{i}\gamma)\Big]+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi}\,G_1(\theta-\theta'-\mathrm{i}\gamma)\,\log\big(1+\mathrm{e}^{-\omega(\theta')}\big)\\
\omega(\theta) &= r\,\cosh(\theta)+\Im m\Big[\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,G_1(\theta-\theta'+\mathrm{i}\gamma)\,L(\theta'-\mathrm{i}\gamma)\Big] \qquad (7.30)\\
&- \;\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,G_2(\theta-\theta')\,\log\big(1+\mathrm{e}^{-\omega(\theta')}\big)\\
L(\theta) &= \log\big(1+\mathrm{e}^{-\mathrm{i}\varepsilon(\theta)}\big)\,.
\end{aligned}$$

Unlike the massless case, there is no need to supplement these equations by the asymptotic conditions at $\theta\to-\infty$ – the source terms in (7.30) control the solution's behaviour both at $\theta\to\pm\infty$. In this subsection we will discuss the k-vacuum energy only. Having at hand the formula (F.9) for the vacuum eigenvalue of the conformal local IM $\mathfrak{i}_1$, one expects that for the massive case,

$$\mathfrak{F}(r,k)=\frac{r}{2\pi^2}\int_{-\infty}^{\infty}\mathrm{d}\theta\,\Big(\pm 2\,\Im m\Big[\mathrm{e}^{\pm(\theta-\mathrm{i}\gamma)}\,L(\theta-\mathrm{i}\gamma)\Big]-\mathrm{e}^{\pm\theta}\,\log\big(1+\mathrm{e}^{-\omega(\theta)}\big)\Big) \qquad (7.31)$$

and this should be valid for both choices of the sign $\pm$.

Some superficial observations can been made at this point. First we note that the kernels in (7.30) which are given by eqs. (6.149), can be expressed through the two-particle scattering amplitudes for the sausage model. Indeed, using the explicit formulae from Appendix H, it is straightforward to check that

$$G(\theta) = -\mathrm{i}\, \partial_\theta \log S(\theta)\,, \qquad\qquad G_1(\theta) = \partial_\theta \log t\big(\theta + \tfrac{\mathrm{i}\pi}{2}\big)$$
$$G_2(\theta) = -\tfrac{\mathrm{i}}{8}\, \partial_\theta \log\det\Big(\boldsymbol{S}^{(2\mapsto 2)}(\theta)\Big)\,. \tag{7.32}$$

The next observation is that the system (7.30) admits a simple solution for $n = 0$. In this case the kernels $G(\theta)$ and $G_1(\theta)$ vanish, whereas $G_2(\theta)$ turns to be $\pi\,\delta(\theta)$. This brings the NLIE to the form

$$\varepsilon(\theta) = r\,\sinh(\theta) - 2\pi k\,, \qquad \omega(\theta) = r\,\cosh(\theta) - \log\big(1 + \mathrm{e}^{-\omega(\theta)}\big)\,, \tag{7.33}$$

and using eq. (7.31), one arrives at (7.29). Furthermore, one can perturbatively solve the NLIE for small n, and compare the results to those from the weak coupling expansion based on the dual action (7.8) for the sausage model. We found complete agreement to the first non-trivial order in the expansion.

Much more effort is needed to derive directly from eqs.(7.30), (7.31) the asymptotic formulae (7.24)-(7.27) describing the behaviour of the k-vacuum energy at $r \to 0$ and $r \to +\infty$. It is, in fact, possible to do this analytically, but here we only present some evidence obtained through the numerical solution of the NLIE system (7.30) (see fig. 7.1 and tab. 7.1).

The remarkable feature of the formulae (7.30), (7.31) is that they do not depend explicitly on n. Hence, they can be applied to the case with n formally set to infinity, i.e., to the $O(3)$ sigma model. Nothing particularly special happens to the kernels

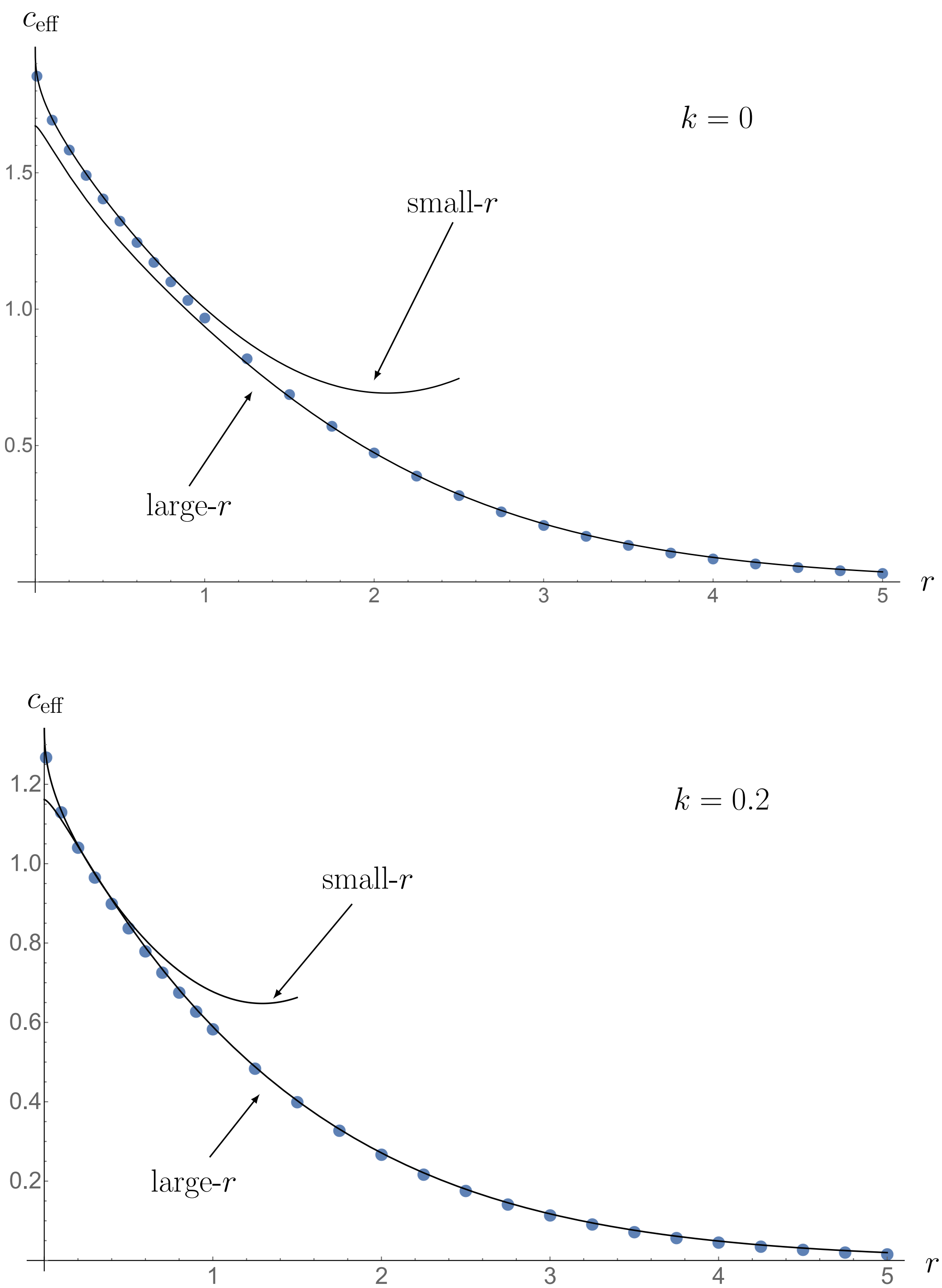


Figure 7.1: $c_{\text{eff}} \equiv -6\,\mathfrak{F}(r,k)$ (7.23) plotted as a function of $r = mR$ for $n = 0.5$ with $k = 0$ and $k = 0.2$. The dots were obtained from the numerical solution of the NLIE (7.30), (7.31). The small-r asymptotic comes from (7.24) while "large-r" represents (7.27). For the corresponding numerical data see tab. 7.1.

$k = 0$			
$r = mR$	c_{eff}	small-r	large-r
10^{-5}	1.963104810570	1.963104810585	$***$
10^{-4}	1.947406984923	1.947406984949	$***$
10^{-3}	1.919080208689	1.919080211060	$***$
0.01	1.859823363074	1.859823782193	$***$
0.10	1.698224051017	1.698316219415	$***$
0.20	1.589016773023	1.589515588521	$***$
0.40	1.409038525289	1.411851755076	$***$
0.60	1.250428169768	1.258361320834	1.1804081
0.80	1.105667568636	1.122485892386	1.0531828
1.00	0.973032006280	1.003537933054	0.9358090
1.50	0.691697724451	0.785383466436	0.6782200
2.00	0.477804027757	$***$	0.4735652
2.50	0.322319977959	$***$	0.3210983
3.00	0.213430553126	$***$	0.2130984
3.50	0.139339032762	$***$	0.1392523
4.00	0.089999431197	$***$	0.0899774
4.50	0.057660716365	$***$	0.0576552
5.00	0.036712137455	$***$	0.0367108
$k = 0.2$			
10^{-5}	1.364830335405	1.364830335417	$***$
10^{-4}	1.350312382232	1.350312382295	$***$
10^{-3}	1.324531940186	1.324531948818	$***$
0.01	1.271882443291	1.271883920213	$***$
0.10	1.134046123560	1.134350188655	1.11141992
0.20	1.044536780739	1.046137755580	1.04399668
0.40	0.902842376206	0.911610323133	0.90993729
0.60	0.783720297274	0.807969967819	0.78900239
0.80	0.679456178786	0.729950921532	0.68245438
1.00	0.587464397842	$***$	0.58894471
1.50	0.402767285740	$***$	0.40287319
2.00	0.271053581853	$***$	0.27100091
2.50	0.179589431194	$***$	0.17955720
3.00	0.117506056343	$***$	0.11749384
3.50	0.076124137247	$***$	0.07612023
4.00	0.048928716276	$***$	0.04892758
4.50	0.031251806913	$***$	0.03125149
5.00	0.019860097547	$***$	0.01986001

Table 7.1: The numerical data for $c_{\text{eff}} \equiv -6\,\mathfrak{F}(r,k)$ with $n = 0.5$, $k = 0$ and 0.2. The small-r asymptotic was obtained by (7.24) whereas the large-r asymptotic comes from (7.27).

(6.149); as $n \to \infty$ they just become the rational functions

$$\begin{aligned} G(\theta) &= \frac{2\pi}{(\theta+\mathrm{i}\pi)(\theta-\mathrm{i}\pi)} \\ G_1(\theta) &= \frac{4\pi^2\,\theta}{(\theta+\frac{\mathrm{i}\pi}{2})(\theta-\frac{\mathrm{i}\pi}{2})(\theta+\frac{3\mathrm{i}\pi}{2})(\theta-\frac{3\mathrm{i}\pi}{2})} \\ G_2(\theta) &= G(\theta) - \frac{2\pi}{(\theta+2\mathrm{i}\pi)(\theta-2\mathrm{i}\pi)}\ . \end{aligned} \tag{7.34}$$

Also the asymptotic formula (7.27) describing the large-r behaviour is, in the $O(3)$ limit,

$$\mathfrak{F}(r,k) = -\frac{r}{\pi^2}\,\big(2\,c(2k)+1\big)\ K_1(r) + \frac{r}{2\pi^2}\,\big(2\,c(2k)+1\big)^2\ K_1(2r) - \tag{7.35}$$
$$\frac{2r}{\pi^3}\,\big(1+c(2k)\big)\int_{-\infty}^{\infty}\mathrm{d}\nu\ K_{1-\mathrm{i}\nu}(r)K_{\mathrm{i}\nu}(r)\,\Big(2\,c(2k)\,\mathrm{e}^{-\pi|\nu|}-\mathrm{e}^{-2\pi|\nu|}\Big)+O\big(\mathrm{e}^{-3r}\big)\ .$$

The situation is much more subtle for the small-r asymptotic. Let us recall that for finite n the asymptotic formula (7.24) can be applied only for $|k| < \frac{1}{n+2}$. This implies that in the limit $n \to \infty$ the applicability of this formula is restricted to the case $k = 0$, and the only information it provides is that $\lim_{r\to 0}\mathfrak{F}(r,0) = -\frac{1}{3}$. As it follows from general perturbative arguments, $\mathfrak{F}(r,0)$ should admit the power series expansion in terms of the running coupling constant for the $O(3)$ NLSM. It is convenient to choose the RG scheme in which the running coupling $g = g(r)$ satisfies the RG flow equation [**?**, 141]

$$r\,\frac{\mathrm{d}g}{\mathrm{d}r} = \frac{g^2}{1-g} = g^2 + g^3 + \dots\ . \tag{7.36}$$

The solution to this equation which we will use is

$$g^{-1}\,\mathrm{e}^{-\frac{1}{g}} = \tfrac{1}{32\pi}\,\mathrm{e}^{\gamma_E+1}\,r\ . \tag{7.37}$$

The funny constant $\frac{1}{32\pi}\,\mathrm{e}^{\gamma_E+1} = 0.048\ldots$ is chosen following the convention from the works [142, 143, 144]. With this choice the gap between the vacuum and the first excited state energies in the $k = 0$ sector, ΔE_0, admits the perturbative expansion where the term $\propto g^2$ is absent: $R\Delta E_0/(2\pi) = g + g^3 + 1.19\,g^4 + O(g^5)$. The small-$r$ behaviour of $\mathfrak{F}(r,0)$ should admit the asymptotic expansion of the form

$$\mathfrak{F}(r,0) \asymp -\tfrac{1}{3} + a_1\, g(r) + a_2\, g^2(r) + a_3\, g^3(r) + a_4\;\, g^4(r) + \ldots\ . \tag{7.38}$$

The first coefficient in this series is known $a_1 = \frac{1}{2}$ [66]. All others can, in principle, be calculated within the renormalized perturbation theory for the $O(3)$ NLSM. Instead of doing so, we estimated their value by fitting the data obtained from the numerical solution of the NLIE. The fitting suggests that, in all likelihood, $a_2 = \frac{1}{4}$ and $a_3 \approx 1$. Also, our numerical results for $k = 0$ are in a full agreement with the numerical data quoted in ref. [145]. To the best of our knowledge, the vacuum energies with $0 < |k| \leq \frac{1}{2}$ have not been discussed in the literature.[2] One can expect that for non-zero k

$$\mathfrak{F}(r,k) \asymp a_0(k) + a_1(k)\, g(r) + a_2(k)\, g^2(r) + a_3(k)\, g^3(r) + \ldots \tag{7.39}$$

with

$$a_0(k) = -\tfrac{1}{3} + 2\,|k|\,\big(1-|k|\,\big) \qquad\quad \text{for} \quad |k| \leq \tfrac{1}{2}\ . \tag{7.40}$$

The last formula can be understood as follows. In the ultraviolet limit the effect of the target space curvature becomes negligible and $(-\frac{1}{3})$ here represents the contribution of two massless Goldstones. However, with non-zero k, the quasiperiodic boundary condition (4.67) implies the presence of conical singularities at the north and south poles of the 2-sphere. The result of the work [147] for a string propagating on a cone yields eq.(7.40). Our numerical data seems to be in agreement with this prediction. Some of the obtained results are depicted in fig. 7.2. Note that as k approaches to $\frac{1}{2}$, the calculations for small r require a considerable amount of computational resources.

[2] The case $k = \frac{1}{2}$ is of special interest for the application of resurgence theory to the problem of instanton summation in the $\mathbb{CP}^{N-1}$ NLSM [146].

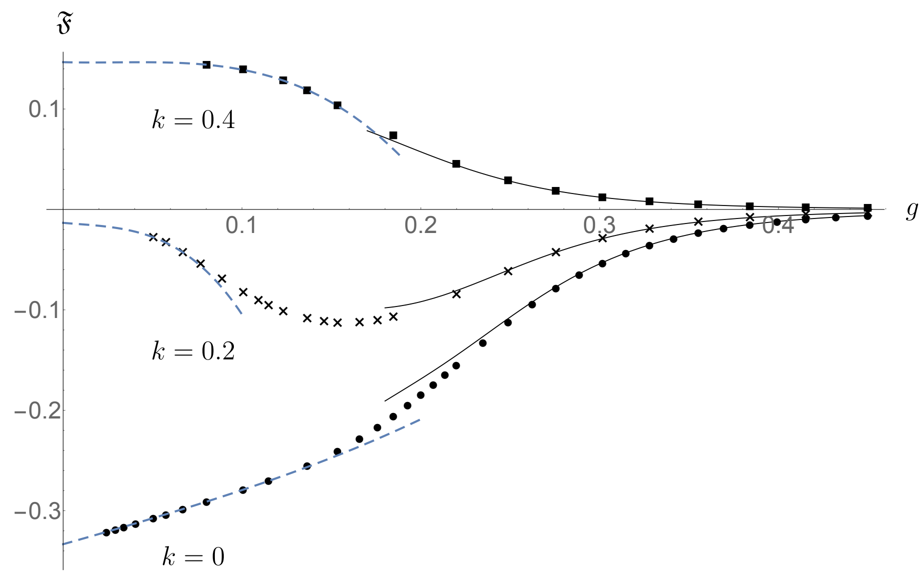


Figure 7.2: A plot of $\mathfrak{F}(r, k)$ with $k = 0$, 0.2 and 0.4 as a function of the running coupling constant $g(r)$ (7.37) for the $O(3)$ sigma model. The large-r asymptotics, depicted by the black curves, follow from eq. (7.35). For small r and $k = 0.2$, 0.4, the numerical data was fitted by a cubic polynomial of the form (7.39) with a_0 given by (7.40). The result of the fit is represented by the dashed line. For $k = 0$ a quartic fit was used (7.38) and the coefficients were found to be $(a_1, a_2, a_3) = (0.5, 0.25, 1.0)$. Note that the smallest value of the running coupling that we reached is $g = 0.0242\ \ldots$ (for $k = 0$), whereas the largest value is $g = 0.449\ldots$. These correspond to $r = 10^{-15}$ and $r = 5$, respectively.

To complete this subsection, let us return to the case of finite n. As has been already mentioned, the quantization condition (7.10) admits the R-independent solution $p_1 = 0$ for $p_2 = \frac{1}{2}$. The latter corresponds to $k = \frac{1}{n+2}$. For this case, as follows from (7.12), the value of the effective central charge at $r = 0$ is given by $\frac{2(n-1)}{n+2}$. For integer $n \geq 2$ this coincides with the central charge c_n (6.3) of the $\mathbb{Z}_n$ parafermions CFT. Based on the results of the work [148], one can expect that the k-vacuum energy with $k = \frac{1}{n+2}$ and $n = 2, 3, \ldots$ coincides with the ground state energy of the non-critical model referred to as $H_n^{(0)}$ in [148]. The model can be described by means of the Euclidean action

$$\mathcal{A}_{H_n^{(0)}} = \mathcal{A}_{\mathbb{Z}_n} - \lambda \int \mathrm{d}^2 x \, \left(\psi^+ \bar{\psi}^+ + \psi^- \bar{\psi}^-\right) , \tag{7.41}$$

which is the critical action of the $\mathbb{Z}_n$ parafermions CFT perturbed by the relevant operator of the scale dimension $d = 2 - \frac{2}{n}$. According to the work [148], the small-r expansion for the scaling function $\mathfrak{F}$ in this case reads as follows

$$\mathfrak{F}(r, \tfrac{1}{n+2}) = -\tfrac{1}{6}\, c_n + 2 \sum_{j=2}^{\infty} F_j \, r^{\frac{2j}{n}} + 2\, F_{(\log)} \, \left(\tfrac{r}{2\pi}\right)^2 \, \log(r) \, , \tag{7.42}$$

with

$$F_{(\log)} = \begin{cases} -\frac{n-1}{2n} & \text{for} \quad n \quad \text{odd} \\ -\frac{1}{2} & \text{for} \quad n \quad \text{even} \end{cases} . \tag{7.43}$$

For $n = 2$ the model $H_n^{(0)}$ coincides with the free theory of a massive Majorana fermion, and therefore

$$\mathfrak{F}(r, \tfrac{1}{4}) = -\frac{r}{2\pi^2} \int_{-\infty}^{\infty} \mathrm{d}\theta \, \cosh(\theta) \, \log\left(1 + \mathrm{e}^{-r\cosh(\theta)}\right) . \tag{7.44}$$

This was checked from the numerical solution of the NLIE (7.30).

7.3 $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{T}$

We are now ready to discuss the general integrable structures in the quantum sausage model. In fact, they are almost identical to those from the cigar/sine-Liouville CFT.

We will place a special emphasis on the aspects of the integrable structures which are related to the presence of the finite correlation length in the theory.

Recall that the sine-Liouville model possesses an infinite set of involutive local IM. At the end of sec. **??** we mentioned only the integrals of the left chirality. Of course, there are also the "right" local IM, so that the full commuting set is $\big\{\mathfrak{i}_{2j-1}, \bar{\mathfrak{i}}_{2j-1}\big\}_{j=1}^{\infty}$. Remarkably (see, e.g., [116] for details), all the local IM are invariant under the reflection $\varphi \mapsto -\varphi$, and therefore they can be interpreted as the local IM for both theories (7.6) and (7.7). This observation suggests that the quantum sausage NLSM possesses the infinite set of local IM $\big\{\mathbb{I}_{2j-1}, \bar{\mathbb{I}}_{2j-1}\big\}_{j=1}^{\infty}$ which can be thought of, in a certain sense, as a deformation of the conformal one [68]. In particular

$$\mathbb{I}_{2j-1} = \int_0^R \frac{\mathrm{d}x}{2\pi}\, \Big(\sum_{l+m=j} C_{lm}^{(j)}\ (\partial_+\varphi)^{2l}(\partial_+\vartheta)^{2m} + \ldots \Big) \tag{7.45}$$

and similar for $\bar{\mathbb{I}}_{2j-1}$ with ∂_+ replaced by ∂_-. Here we use the light cone variables $x^{\pm} = x^0 \pm x^1$, the constants $C_{lm}^{(j)}$ are the same as in eqs. (5.60)-(5.62), and the dots stand for monomials which include higher derivatives of φ and ϑ, as well as terms proportional to powers of $\mathcal{M}$. It should be emphasized that the φ and ϑ in (7.45) are *local* fields whose dynamics are governed by the dual action (7.8) and, if considering the neutral sector of the model, the periodic boundary conditions (7.9). A special rôle belongs to the integrals

$$\begin{aligned} \mathbb{I}_1 &= \int_0^R \frac{\mathrm{d}x}{2\pi}\, \Big((\partial_+\varphi)^2 + (\partial_+\vartheta)^2 - 4\mathcal{M}\, \cosh\big(\sqrt{n}\varphi\big) \cos\big(\sqrt{n+2}\,\vartheta\big) \Big) \\ \bar{\mathbb{I}}_1 &= \big(\, \partial_+ \mapsto \partial_-\, \big)\ , \end{aligned} \tag{7.46}$$

whose sum, $\mathbb{H}_R = \mathbb{I}_1 + \bar{\mathbb{I}}_1$, and difference, $\mathbb{P}_R = \mathbb{I}_1 - \bar{\mathbb{I}}_1$, coincide with the Hamiltonian and the total momentum, respectively. It is expected that the common eigenvectors of $\big\{\mathbb{I}_{2j-1}, \bar{\mathbb{I}}_{2j-1}\big\}_{j=1}^{\infty}$ form a basis in each invariant subspace $\mathcal{H}_k^{(K)}$ of the Hilbert space of the sausage NLSM. Let's denote the corresponding k-vacuum eigenvalues by

$\big\{\mathfrak{I}_{2j-1}, \bar{\mathfrak{I}}_{2j-1}\big\}_{j=1}^{\infty}$. For the k-vacuum the total momentum is zero, so

$$\mathfrak{I}_1 = \bar{\mathfrak{I}}_1 = \tfrac{1}{2}\, \mathcal{E} R + \tfrac{\pi}{2R}\, \mathfrak{F}(r,k)\ . \tag{7.47}$$

This relation together with (7.31), allows one to express the vacuum eigenvalues of $\mathbb{I}_1$ and $\bar{\mathbb{I}}_1$ in terms of the solution to the NLIE (7.30). It is not difficult to find similar expressions for the other local IM (to be compared with formulae (F.9) from Appendix F):

$$\mathfrak{I}_{2j-1} = \Big(\frac{m}{4}\Big)^{2j-1}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\bigg((-1)^j\, \mathrm{e}^{+(2j-1)\theta}\, \log\big(1+\mathrm{e}^{-\omega(\theta)}\big) + 2\,\Im m\Big[\mathrm{e}^{+(2j-1)(\theta-\mathrm{i}\gamma)}\, L(\theta-\mathrm{i}\gamma)\Big]\bigg)$$

$$\bar{\mathfrak{I}}_{2j-1} = \Big(\frac{m}{4}\Big)^{2j-1}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\bigg((-1)^j\, \mathrm{e}^{-(2j-1)\theta}\, \log\big(1+\mathrm{e}^{-\omega(\theta)}\big) - 2\,\Im m\Big[\mathrm{e}^{-(2j-1)(\theta-\mathrm{i}\gamma)}\, L(\theta-\mathrm{i}\gamma)\Big]\bigg) \tag{7.48}$$

For $r \ll 1$, similar to formula (7.24) for $\mathfrak{F}(r,k)$, the vacuum eigenvalues of the higher spin local IM can be approximated by

$$\mathfrak{I}_{2j-1} = \bar{\mathfrak{I}}_{2j-1} \approx \big(\tfrac{2\pi}{R}\big)^{2j-1}\ i_{2j-1}\big(p(r), \tfrac{1}{2}\,(n+2)\,k\,\big)\ . \tag{7.49}$$

Here, $i_{2j-1}(p_1,p_2)$ are the vacuum eigenvalues of the chiral local IM $\mathfrak{i}_{2j-1}$ and $p = p(r)$ is the solution of eq. (7.25). Tab. 7.2 demonstrates the quality of this approximation for the first few local IM.

Note that $\mathfrak{I}_{2j-1} = \bar{\mathfrak{I}}_{2j-1}$ for any $j = 1, 2, \ldots$. These relations can be easily understood since the model (7.8) is $\mathcal{P}$-invariant and that under the parity transformation

$$\mathcal{P}\,\mathbb{I}_{2j-1}\,\mathcal{P} = \bar{\mathbb{I}}_{2j-1}\ . \tag{7.50}$$

Another important global symmetry is $\mathcal{C}$-invariance. Acting on the local fields it flips the sign of ϑ while keeping φ unchanged. All the local IM are $\mathcal{C}$-invariant operators, i.e.,

$$\mathcal{C}\,\mathbb{I}_{2j-1}\,\mathcal{C} = \mathbb{I}_{2j-1}\ , \qquad \mathcal{C}\,\bar{\mathbb{I}}_{2j-1}\,\mathcal{C} = \bar{\mathbb{I}}_{2j-1}\ . \tag{7.51}$$

$r = mR$	$\left(\frac{R}{2\pi}\right)^3 \mathfrak{I}_3$	$i_3(p(r), p_2)$
10^{-1}	$3.45716396595 \times 10^{-4}$	$3.45832599760 \times 10^{-4}$
10^{-3}	$3.81476584833 \times 10^{-4}$	$3.81476594313 \times 10^{-4}$
10^{-5}	$3.92737566343 \times 10^{-4}$	$3.92737566334 \times 10^{-4}$
$r = mR$	$\left(\frac{R}{2\pi}\right)^5 \mathfrak{I}_5$	$i_5(p(r), p_2)$
10^{-1}	$-2.6148731 \times 10^{-5}$	$-2.6151868 \times 10^{-5}$
10^{-3}	$-2.8189874 \times 10^{-5}$	$-2.8189869 \times 10^{-5}$
10^{-5}	$-2.8833109 \times 10^{-5}$	$-2.8833124 \times 10^{-5}$

Table 7.2: The vacuum eigenvalues of the first two higher spin local IM in the sausage model for $n = 1$ and $p_2 = \frac{3}{2}\,k = \frac{5}{13}$. The numerical values were calculated from the solution of the NLIE and formula (7.48). The last column gives the vacuum eigenvalues of the chiral local IM, $i_3(p(r), p_2)$ and $i_5(p(r), p_2)$, where $p(r)$ is the solution to the quantization condition (7.25). The limiting values at $r = 0$ are given by $i_3(0, p_2) = \frac{39031}{95964960} = 4.067\ldots \times 10^{-4}$, $i_5(0, p_2) = -\frac{137442779}{4638370376640} = -2.963\ldots \times 10^{-5}$. Explicit expressions for i_3 and i_5 can be found in ref.[114].

Since $\mathcal{C}\,|\,\mathrm{vac}\,\rangle_k = |\,\mathrm{vac}\,\rangle_{-k}$, this explains the fact that the vacuum eigenvalues $\mathfrak{I}_{2j-1}$ are even functions of k. The last discrete symmetry that we shall consider is the invariance of the dual action (7.8) w.r.t. the transformation

$$\vartheta(r,t) \;\mapsto\; \mathbb{U}\,\vartheta(t,x)\,\mathbb{U}^{-1} \;=\; \vartheta(t,x) + \frac{2\pi}{\sqrt{n+2}}\;, \tag{7.52}$$

where the unitary operator $\mathbb{U}$ is the Flouquet-Bloch operator which is just a constant phase factor when it acts on the subspace $\mathcal{H}_k^{(K)}$:

$$\mathbb{U}\;\mathcal{H}_k^{(K)} = \mathrm{e}^{2\pi \mathrm{i} k}\;\mathcal{H}_k^{(K)}\;. \tag{7.53}$$

Of course, $[\mathbb{U}, \mathbb{I}_{2j-1}] = [\mathbb{U}, \bar{\mathbb{I}}_{2j-1}] = 0$.

Together with the local IM, the sausage model possesses the set of dual nonlocal IM $\left\{\mathbb{S}_j, \bar{\mathbb{S}}_j\right\}_{j=1}^{\infty}$, which again can be understood as a deformation of the corresponding conformal set. In contrast to the local IM, they are not $\mathcal{C}$-invariant operators. Instead

$r = mR$	$\left(\frac{R}{2\pi}\right)^{\frac{2}{n+2}}\,\mathfrak{S}_1$	$\tilde{s}_1(p(r), p_2)$	$\left(\frac{R}{2\pi}\right)^{\frac{2}{n+2}}\,\bar{\mathfrak{S}}_1$	$\tilde{s}_1(p(r), -p_2)$
10^{-1}	0.01936579	0.01877110	0.13129934	0.12726735
10^{-3}	0.03374979	0.03374058	0.22882227	0.22875988
10^{-5}	0.04003149	0.04003134	0.27141198	0.27141097

$r = mR$	$\left(\frac{r}{8\pi}\right)^{2k}\,\mathfrak{S}$	$S\big(p_2\,\vert\, \mathrm{i}p(r)\big)$
10^{-1}	0.35644580	0.35731884
10^{-3}	0.24462688	0.24463961
10^{-5}	0.19673087	0.19673105

Table 7.3: The vacuum eigenvalues of the dual nonlocal IM $\mathbb{S}_1$, $\bar{\mathbb{S}}_1$ and $\mathbb{S}$ for $n = \frac{9}{2}$ and $p_2 = \frac{13}{4}\,k = \frac{5}{13}$. Eq. (7.56) was used to find the numerical values of $\mathfrak{S}_1$, $\bar{\mathfrak{S}}_1$ from the solution to the NLIE, whereas the corresponding formula for $\mathfrak{S}$ is (7.59). The vacuum eigenvalues $\tilde{s}_1(p_1, p_2)$ of the chiral dual nonlocal IM are given by (6.137) and the expression for $S\big(p_2\big|\, q\big)$ is found in (7.61). Finally, $p(r)$ is the solution of the quantization condition (7.25).

they satisfy the relations

$$\mathcal{CP}\ \mathbb{S}_j\ \mathcal{CP} = \bar{\mathbb{S}}_j\ . \tag{7.54}$$

This implies that the analog of eq.(7.49) for the set $\big\{\mathfrak{S}_j, \bar{\mathfrak{S}}_j\big\}_{j=1}^{\infty}$ of k-vacuum eigenvalues of the dual nonlocal IM reads as follows

$$\mathfrak{S}_j \approx \left(\tfrac{2\pi}{R}\right)^{\frac{2j}{n+2}}\ \tilde{s}_j\big(p(r), +\tfrac{1}{2}\,(n+2)\,k\,\big)\ , \qquad \bar{\mathfrak{S}}_j \approx \left(\tfrac{2\pi}{R}\right)^{\frac{2j}{n+2}}\ \tilde{s}_j\big(p(r), -\tfrac{1}{2}\,(n+2)\,k\,\big) \tag{7.55}$$

Similarly as for the local IM, the vacuum eigenvalues of the dual nonlocal IM can be expressed through the solution of the NLIE:

$$\mathfrak{S}_j = \frac{2}{n+2}\Big(\frac{m}{4}\Big)^{\frac{2j}{n+2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\,\bigg(\sin\big(\tfrac{\pi j}{n+2}\big)\ \mathrm{e}^{+\frac{2j\theta}{n+2}}\ \log\big(1+\mathrm{e}^{-\omega(\theta)}\big) - \Im m\Big[\mathrm{e}^{+\frac{2j(\theta-\mathrm{i}\gamma)}{n+2}}\ L(\theta-\mathrm{i}\gamma)\Big]\bigg)$$

$$\bar{\mathfrak{S}}_j = \frac{2}{n+2}\Big(\frac{m}{4}\Big)^{\frac{2j}{n+2}}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\,\bigg(\sin\big(\tfrac{\pi j}{n+2}\big)\ \mathrm{e}^{-\frac{2j\theta}{n+2}}\ \log\big(1+\mathrm{e}^{-\omega(\theta)}\big) + \Im m\Big[\mathrm{e}^{-\frac{2j(\theta-\mathrm{i}\gamma)}{n+2}}\ L(\theta-\mathrm{i}\gamma)\Big]\bigg) \tag{7.56}$$

In tab. 7.3 we present numerical data illustrating formulae (7.55) for $\mathfrak{S}_1$ and $\bar{\mathfrak{S}}_1$.

We are now able to synthesize our study of the quantum sausage model in the form of the following conjecture. It is expected that the theory possesses the operators $\mathbb{A}(\theta)$, $\mathbb{B}(\theta)$ and $\mathbb{T}(\theta)$ satisfying the set of conditions:

(i) **Commutativity:** $\big[\mathbb{A}(\theta),\mathbb{A}(\theta')\big]=\big[\mathbb{B}(\theta),\mathbb{B}(\theta')\big]=\big[\mathbb{A}(\theta),\mathbb{B}(\theta')\big]$

(ii) **Analyticity:** The operators $\mathbb{A}(\theta)$, $\mathbb{B}(\theta)$ and $\mathbb{T}(\theta)$ are entire functions of the variable θ

(iii) **Global symmetries:**

$$\mathcal{CP}\,\mathbb{A}(\theta)\,\mathcal{CP}=\mathbb{A}(-\theta)\ ,\qquad \mathcal{CP}\,\mathbb{B}(\theta)\,\mathcal{CP}=\mathbb{B}(-\theta)$$

$$\mathcal{P}\,\mathbb{T}(\theta)\,\mathcal{P}=\mathbb{T}(-\theta)\ ,\qquad \mathcal{C}\,\mathbb{T}(\theta)\,\mathcal{C}=\mathbb{T}(\theta)$$

$$[\mathbb{U},\,\mathbb{A}(\theta)]=[\mathbb{U},\,\mathbb{B}(\theta)]=[\mathbb{U},\,\mathbb{T}(\theta)]=0$$

(iv) **(Quasi)periodicity:** $\mathbb{B}(\theta+\mathrm{i}\pi)=\mathbb{U}\,\mathbb{B}(\theta-\mathrm{i}\pi),\ \ \mathbb{T}(\theta+\mathrm{i}\pi n)=\mathbb{T}(\theta)$

(v) **Hermiticity:** $\mathbb{A}^{\dagger}(\theta)=\mathbb{A}(\theta^*)\ ,\ \ \mathbb{B}^{\dagger}(\theta)=\mathbb{B}(\theta^*)\ ,\ \ \mathbb{T}^{\dagger}(\theta)=\mathbb{T}(\theta^*)$

(vi) **Functional relation:**

$$\mathbb{A}\big(\theta-\tfrac{\mathrm{i}\pi(n+1)}{2}\big)\,\mathbb{A}\big(\theta+\tfrac{\mathrm{i}\pi(n+1)}{2}\big)-\mathbb{A}\big(\theta-\tfrac{\mathrm{i}\pi(n-1)}{2}\big)\,\mathbb{A}\big(\theta+\tfrac{\mathrm{i}\pi(n-1)}{2}\big)=\mathbb{B}\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)\,\mathbb{B}\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)$$

(vii) $T-Q$ **relation:** $\mathbb{T}\big(\theta+\tfrac{\mathrm{i}\pi n}{2}\big)\,\mathbb{A}(\theta)=\mathbb{U}^{-\frac{1}{2}}\ \mathbb{A}(\theta+\mathrm{i}\pi)+\mathbb{U}^{+\frac{1}{2}}\ \mathbb{A}(\theta-\mathrm{i}\pi)$

(viii) **Asymptotic behaviour of** $\mathbb{A}(\theta)$**:**

$$\mathbb{A}(\theta)\asymp\mathbb{S}^{\pm\frac{1}{2}}\ \exp\bigg(-\frac{r\,\cosh(\theta)}{4\sin(\frac{\pi n}{2})}\bigg)\ \exp\big(-a^{(\pm)}(\theta)\big)\qquad\text{as}\qquad\Re e(\theta)\to\pm\infty$$

with $|\Im m(\theta)|<\frac{\pi}{2}\,(n+2)$, and

$$a^{(+)}(\theta)=\sum_{j=1}^{\infty}\mathbb{S}_j\ \big(\tfrac{m}{4}\,\mathrm{e}^{+\theta}\big)^{-\frac{2j}{n+2}}\ ,\qquad a^{(-)}(\theta)=\sum_{j=1}^{\infty}\bar{\mathbb{S}}_j\ \big(\tfrac{m}{4}\,\mathrm{e}^{-\theta}\big)^{-\frac{2j}{n+2}}$$

(ix) **Asymptotic behaviour of $\mathbb{B}(\theta)$:**

$$\mathbb{B}(\theta) \asymp \mathbb{S}^{\pm\frac{1}{2}}\ \exp\left(-\frac{r\theta\sinh(\theta)}{2\pi}\right)\ \exp\big(-b^{(\pm)}(\theta)\big) \qquad \text{as} \qquad \Re e(\theta)\to\pm\infty$$

with $|\Im m(\theta)|<\pi$, and

$$\begin{aligned} b^{(+)}(\theta) &= \big(\mathbb{I}_1-\tfrac{1}{2}\,\mathcal{E}R\big)\ \tfrac{4}{m}\ \mathrm{e}^{-\theta}+\sum_{j=1}^{\infty}\Big(\mathbb{I}_{2j+1}\left(\tfrac{m}{4}\,\mathrm{e}^{+\theta}\right)^{-1-2j}+\mathbb{S}_j\ \left(\tfrac{m}{4}\,\mathrm{e}^{+\theta}\right)^{-\frac{2j}{n+2}}\Big)\\ b^{(-)}(\theta) &= \big(\bar{\mathbb{I}}_1-\tfrac{1}{2}\,\mathcal{E}R\big)\ \tfrac{4}{m}\ \mathrm{e}^{+\theta}+\sum_{j=1}^{\infty}\Big(\bar{\mathbb{I}}_{2j+1}\left(\tfrac{m}{4}\,\mathrm{e}^{-\theta}\right)^{-1-2j}+\bar{\mathbb{S}}_j\ \left(\tfrac{m}{4}\,\mathrm{e}^{-\theta}\right)^{-\frac{2j}{n+2}}\Big)\end{aligned}$$

(x) **Zeroes**: Let $\mathfrak{A}^{(\psi)}(\theta)$, $\mathfrak{B}^{(\psi)}(\theta)$, $\mathrm{e}^{2\pi\mathrm{i}k}$ be the eigenvalues of the operators $\mathbb{A}(\theta)$, $\mathbb{B}(\theta)$, $\mathbb{U}$, respectively, corresponding to a common eigenvector $|\,\psi\,\rangle$. If θ_j is a zero of $\mathfrak{B}^{(\psi)}(\theta)$, then

$$\exp\big(-\tfrac{\mathrm{i}}{2}\,r\sinh(\theta_j)-2\pi\mathrm{i}k\,\big)\ \frac{\mathfrak{A}^{(\psi)}\big(\theta_j-\mathrm{i}\pi-\frac{\mathrm{i}\pi n}{2}\big)}{\mathfrak{A}^{(\psi)}\big(\theta_j-\mathrm{i}\pi+\frac{\mathrm{i}\pi n}{2}\big)}=-1\ .$$

All zeroes of $\mathfrak{B}^{(\psi)}(\theta)$ are simple and accumulate towards infinity along the lines $\Im m(\theta)=\pi \pmod{2\pi}$.

Clearly, the conjectured properties of $\mathbb{A}(\theta)$, $\mathbb{B}(\theta)$ and $\mathbb{T}(\theta)$ are inspired by those of their chiral counterparts $\alpha_+(\theta)$, $\beta_+(\theta)$ and $\tau(\lambda)$ and the global symmetries of the model. Unlike the chiral case, the subscript was not included in the notation of operators $\mathbb{A}$ and $\mathbb{B}$. It can be restored by setting $\mathbb{A}_+\equiv\mathbb{A}$ and $\mathbb{B}_+\equiv\mathbb{B}$. The properties of the $\mathcal{C}$-conjugated operators $\mathbb{A}_-\equiv\mathcal{C}\,\mathbb{A}\,\mathcal{C}$, $\mathbb{B}_-\equiv\mathcal{C}\,\mathbb{B}\,\mathcal{C}$ can be easily deduced from (i)-(x). Perhaps only the θ-independent operator $\mathbb{S}$, which appears in the large-θ asymptotic expansions (viii) and (ix), requires some elucidations. Before presenting them, let us first discuss the vacuum eigenvalues of $\mathbb{A}(\theta)$ and $\mathbb{B}(\theta)$. The obvious counterparts to the formulae (F.6), (F.7) from Appendix F read as

$$\begin{aligned}\log\mathfrak{A}(\theta) &= -\frac{r\,\cosh(\theta)}{4\sin(\frac{\pi n}{2})}+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[F_1(\theta-\theta'+\mathrm{i}\gamma)\ L(\theta'-\mathrm{i}\gamma)\\ &- \ F_1(\theta-\theta'-\mathrm{i}\gamma)\,\Big(L(\theta'-\mathrm{i}\gamma)\Big)^*\,\Big]+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,F_2(\theta-\theta')\ \log\Big(1+\mathrm{e}^{-\omega(\theta')}\Big)\end{aligned} \tag{7.57}$$

where $|\Im m(\theta)| < \frac{\pi}{2}\,(n+2) - \gamma$, and

$$\log \mathfrak{B}(\theta) \;=\; -\frac{r\theta\sinh(\theta)}{2\pi} + \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[\,F_3(\theta-\theta'+\mathrm{i}\gamma)\;L(\theta'-\mathrm{i}\gamma) \qquad (7.58)$$

$$-\;\;F_3(\theta-\theta'-\mathrm{i}\gamma)\,\Big(L(\theta'-\mathrm{i}\gamma)\Big)^*\,\Big] - \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\;F_4(\theta-\theta')\;\log\Big(1+\mathrm{e}^{-\omega(\theta')}\Big)\,,$$

with $|\Im m(\theta)| < \pi - \gamma$. Now $\varepsilon(\theta)$ and $\omega(\theta)$ solve the massive NLIE (7.30), (6.149) and the explicit form of the functions $F_i(\theta)$ are given in (F.8). It is easy to see that these formulae combined with the asymptotics (viii) and (ix), yield the expressions (7.48) and (7.56) for the vacuum eigenvalues of the local and dual nonlocal IM. Notice that the term $(-k\theta)$ is absent in (7.57), (7.58) compared with the analogous formulae (F.6), (F.7). This is consistent with the absence of the factor $\mathrm{e}^{-k\theta}$ in the asymptotics (viii) and (ix) compared with the corresponding eqs. (6.123), (6.124) and (6.127) for the chiral case. In connection with this, note that the operator $\hat{k}$ is ill defined and only its exponent $\mathbb{U} = \exp(2\pi\mathrm{i}\hat{k})$ makes sense in the massive theory.

In the next subsection we will point out that the eigenvalues of the operator $\mathbb{S}$ play a special rôle in the ODE/IQFT correspondence. Eqs. (7.57), (7.58) predict that in the case of the k-vacuum states, its eigenvalue is given by

$$\mathfrak{S} = \exp\bigg(\frac{2}{n+2}\,\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\;\Im m\big(L(\theta-\mathrm{i}\gamma)\,\big)\bigg)\,. \qquad (7.59)$$

The small-r behaviour of $\mathfrak{S}$ is given by a formula similar to (7.49), (7.55) (see tab. 7.3):

$$\mathfrak{S} \approx \Big(\frac{8\pi}{r}\Big)^{2k}\;S\big(\tfrac{1}{2}\,(n+2)\,k\,|\,\mathrm{i}p(r)\big)\,, \qquad (7.60)$$

where

$$S(p_2|\,q) = \big(n+2\big)^{\frac{4p_2}{n+2}}\;\frac{\Gamma(\frac{1}{2}+p_2+q)\Gamma(\frac{1}{2}+p_2-q)}{\Gamma(\frac{1}{2}-p_2+q)\Gamma(\frac{1}{2}-p_2-q)}\,\frac{\Gamma(1-2p_2)}{\Gamma(1+2p_2)}\,\frac{\Gamma(1+\frac{2p_2}{n+2})}{\Gamma(1-\frac{2p_2}{n+2})}\,. \qquad (7.61)$$

Notice that $S(p_2|\,\mathrm{i}p_1)$ can be expressed in terms of the vacuum eigenvalues (6.138) of the operators $b_\pm$ defined in (6.114): $S(p_2|\,\mathrm{i}p_1) = b_-^{(\mathrm{vac})}(p_1,p_2)/b_+^{(\mathrm{vac})}(p_1,p_2)$.

The operator $\mathbb{T}(\theta)$ is the transfer-matrix in the sausage model – the quantum counterpart of the Wilson loop (3.2). By means of the $T-Q$ equation (vii) it is expressed in terms of the operator $\mathbb{A}(\theta)$ and, of course, commutes with both $\mathbb{A}$ and $\mathbb{B}$ for any values of the spectral parameter θ. We did not include the formula which described its large-θ asymptotic in the list (i)-(x) since it is an immediate consequence of the $T-Q$ equation and the asymptotic (viii) for $\mathbb{A}(\theta)$. Notice that unlike for the Toda-type theory, the transfer-matrix in the sausage model does not generate the local IM through its asymptotic expansion.

Finally we can turn to the case of the $O(3)$ NLSM. There is no reason to expect that the $n \to \infty$ limit is problematic for the operator $\mathbb{B}(\theta)$. Introduce the notation

$$\mathbb{B}_\infty(\theta) = \lim_{n\to\infty} \mathbb{B}(\theta)\ . \tag{7.62}$$

Using eqs.(7.58), (F.8), one finds the relation which expresses its vacuum eigenvalue in terms of the solution to the NLIE (7.30) with the kernels (7.34):

$$\begin{aligned} \log \mathfrak{B}_\infty(\theta) &= -\frac{r\theta\sinh(\theta)}{2\pi} + \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[\, f_3(\theta-\theta'+\mathrm{i}\gamma)\ L(\theta'-\mathrm{i}\gamma) \\ &- f_3(\theta-\theta'-\mathrm{i}\gamma)\ \Big(L(\theta'-\mathrm{i}\gamma)\Big)^*\,\Big] - \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\ f_4(\theta-\theta')\ \log\Big(1+\mathrm{e}^{-\omega(\theta')}\Big) \end{aligned} \tag{7.63}$$

where $|\Im m(\theta)| < \pi-\gamma$, and

$$f_3(\theta) = \frac{1}{\theta} - \frac{1}{\sinh(\theta)}\ , \qquad f_4(\theta) = \frac{\pi}{2(\theta+\frac{\mathrm{i}\pi}{2})(\theta-\frac{\mathrm{i}\pi}{2})} - \frac{1}{2\cosh(\theta)}\ . \tag{7.64}$$

The situation with the operators $\mathbb{A}(\theta)$ and $\mathbb{T}(\theta)$ is slightly more delicate. In this case, one can expect that the following limits exist

$$\begin{aligned} \mathbb{A}_\infty(\theta) &= \lim_{n\to\infty} \mathbb{A}\big(\theta-\tfrac{\mathrm{i}\pi n}{2}\big)\exp\Big(\tfrac{1}{4}\ r\cot\big(\tfrac{\pi n}{2}\big)\ \cosh(\theta)\Big) \\ \mathbb{T}_\infty(\theta) &= \lim_{n\to\infty} \mathbb{T}(\theta)\ \exp\Big(-\tfrac{1}{2}\ r\cot\big(\tfrac{\pi n}{2}\big)\ \cosh(\theta)\Big) \end{aligned} \tag{7.65}$$

and the limiting operators satisfy the $T-Q$ equation in the form

$$\mathbb{T}_\infty(\theta)\,\mathbb{A}_\infty(\theta) = \mathbb{U}^{-\frac{1}{2}}\ \ \mathbb{A}_\infty(\theta+\mathrm{i}\pi) + \mathbb{U}^{+\frac{1}{2}}\ \mathbb{A}_\infty(\theta-\mathrm{i}\pi) \tag{7.66}$$

(recall that in the sector $\mathcal{H}_k^{(K)}$ the operator $\mathbb{U}^{+\frac{1}{2}}$ becomes just a phase factor $(-1)^{K-1}\,\mathrm{e}^{\mathrm{i}\pi k}$). With eqs. (7.57), (F.8), it is easy to see that

$$\begin{aligned}\log\mathfrak{A}_\infty(\theta) &= \tfrac{\mathrm{i}}{4}\, r\sinh(\theta) + \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[\, f_1(\theta-\theta'+\mathrm{i}\gamma)\ L(\theta'-\mathrm{i}\gamma) \\ &- f_1(\theta-\theta'-\mathrm{i}\gamma)\,\Big(L(\theta'-\mathrm{i}\gamma)\Big)^*\,\Big] + \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\, f_2(\theta-\theta')\ \log\Big(1+\mathrm{e}^{-\omega(\theta')}\Big)\end{aligned} \tag{7.67}$$

with $\Im m(\theta)<\pi-\gamma$,

$$f_1(\theta)=\frac{1}{\theta+\mathrm{i}\pi}\ ,\qquad f_2(\theta)=-\frac{\pi}{2(\theta+\frac{3\mathrm{i}\pi}{2})(\theta+\frac{\mathrm{i}\pi}{2})}\ , \tag{7.68}$$

and $\varepsilon(\theta)$, $\omega(\theta)$ are defined through the solution of the NLIE (7.30), (7.34).

7.4 ODE/IQFT for the sausage model

In sec. 6.6 we briefly discussed the ODE/IM correspondence for the cigar NLSM. Recall that the correspondence relates the eigenvalues of the chiral transfer-matrices to the connection coefficients for the family of second order differential equations $\mathcal{D}(\theta)\,\psi=0$ with the operators $\mathcal{D}(\theta)$ of the form (6.104). The generalization of the construction to the sausage model is based on the ideas from the work [56] and goes along the following line.

As far as our attention was confined to the CFT, there was no need to separately consider the antiholomorphic operators, $\overline{\mathcal{D}}(\bar\theta)=-\partial_{\bar z}^2+\overline{T}_{\bar L}(\bar z)+\mathrm{e}^{2\bar\theta}\,\overline{\mathcal{P}}(\bar z)$, since there was only a nomenclature difference between the holomorphic and antiholomorphic cases. In the massive QFT, following [56], one should substitute the pair $\big(\mathcal{D}(\theta_0+\theta),\overline{\mathcal{D}}(\theta_0-\theta)\big)$ by a pair of (2×2)-matrix valued differential operators

$$\boldsymbol{D}(\theta)=\partial_z-\boldsymbol{A}_z\ ,\qquad \overline{\boldsymbol{D}}(\theta)=\partial_{\bar z}-\boldsymbol{A}_{\bar z} \tag{7.69}$$

with

$$\begin{aligned}\boldsymbol{A}_z &= -\tfrac{1}{2}\,\partial_z\eta\,\sigma_3+\sigma_+\,\mathrm{e}^{+\eta}+\sigma_-\,\mathrm{e}^{2\theta_0+2\theta}\,\mathcal{P}(z)\,\mathrm{e}^{-\eta} \\ \boldsymbol{A}_{\bar z} &= +\tfrac{1}{2}\,\partial_{\bar z}\eta\,\sigma_3+\sigma_-\,\mathrm{e}^{+\eta}+\sigma_+\,\mathrm{e}^{2\theta_0-2\theta}\,\overline{\mathcal{P}}(\bar z)\,\mathrm{e}^{-\eta}\ ,\end{aligned} \tag{7.70}$$

where $\sigma_3, \sigma_\pm = (\sigma_1 \pm \mathrm{i}\sigma_2)/2$ are the standard Pauli matrices and $\mathcal{P}(z)$ is given by (6.99). In fact, $(\boldsymbol{A}_z,\ \boldsymbol{A}_{\bar{z}})$ form a $\mathfrak{sl}(2)$ connection whose flatness is a necessary condition for the existence of a solution to the linear problem

$$\boldsymbol{D}(\theta)\,\boldsymbol{\Psi} = 0\ , \qquad \overline{\boldsymbol{D}}(\theta)\,\boldsymbol{\Psi} = 0\ . \tag{7.71}$$

The zero-curvature relation leads to the Modified Sinh-Gordon (MShG) equation:

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + \rho^4\ |\mathcal{P}(z)|^2\ \mathrm{e}^{-2\eta} = 0\ , \qquad \rho = \mathrm{e}^{\theta_0}\ . \tag{7.72}$$

In refs. [128, 90], a class of singular solutions to this partial differential equation distinguished by special monodromy properties of the associated linear problem (7.71) was introduced. Together with the singularities at $z = z_1,\ z_2,\ z_3$, the solutions are allowed to have the so-called apparent singularities, which do not affect the monodromy properties of the auxiliary linear problem (7.71). In the limit $\theta_0 \to -\infty$ with $\theta_+ = \theta_0 + \theta$ kept fixed, the system (7.71) can be reduced to $\mathcal{D}(\theta_+)\,\psi = 0,\ \partial_{\bar{z}}\psi = 0$ and the apparent singularities manifest themselves as the monodromy free singularities for the operator $\mathcal{D}(\theta_+)$ of the form (6.104). Parallel to this, the limit $\theta_0 \to -\infty$ with $\theta_- = \theta_0 - \theta$ kept fixed can be considered, which leads to the corresponding antiholomorphic equations $\overline{\mathcal{D}}(\theta_-)\,\psi = 0$ and $\partial_z\psi = 0$.

In the same works [128, 90], evidence was presented that the linear problem (7.71) built from the special singular solutions of the MShG equation makes up the ODE part for the ODE/IQFT correspondence where the IQFT counterpart is the so-called Fateev model [68]. The latter is governed by the Lagrangian

$$\mathcal{L} = \tfrac{1}{16\pi} \sum_{i=1}^{3} (\partial_\mu \varphi_i)^2 + 2\mathcal{M}\,\big(\,\mathrm{e}^{\mathrm{i}\,\alpha_3\varphi_3}\ \cos(\alpha_1\varphi_1 + \alpha_2\varphi_2) + \mathrm{e}^{-\mathrm{i}\alpha_3\varphi_3}\ \cos(\alpha_1\varphi_1 - \alpha_2\varphi_2)\,\big) \tag{7.73}$$

for the three scalar fields $\varphi_i = \varphi_i(t, x)$ which satisfy the periodic boundary conditions

$$\varphi_i(t, x + R) = \varphi_i(t, x)\ . \tag{7.74}$$

It is important that the dimensionless coupling constants α_i satisfy the linear constraint

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \tfrac{1}{2} \ , \tag{7.75}$$

so that the coupling $\mathcal{M}$ in the renormalized Lagrangian (8.4) has the dimensions of mass, $\mathcal{M} \sim [\,\mathrm{mass}\,]$. Within the ODE/IQFT correspondence the parameters are identified as follows

$$a_i = 4\,\alpha_i^2\,, \qquad\qquad (i = 1, 2, 3)\ , \tag{7.76}$$

whereas the relation between the dimensionless parameter $\mathcal{M}R$ and ρ from (8.20) is given by

$$\rho = \tfrac{1}{2}\ \mathcal{M}R\ . \tag{7.77}$$

Although the original considerations of refs. [128, 90] were focused on the ODE/IQFT correspondence with all three parameters a_1, a_2, a_3 positive, in the subsequent work [60] evidence was presented that the correspondence remains valid with minimum modifications to the case a_1, $a_2 > 0$ and $a_3 < 0$. In the recent works [149, 150], the same conclusion was reached for a_1, $a_2 > 0$ and $a_3 = 0$. Among the tasks of the current paper is to argue that the ODE/IQFT correspondence remains valid for

$$a_1 = -n, \qquad a_2 = n + 2\ , \qquad a_3 = 0 \qquad \text{with} \quad n > 0\,. \tag{7.78}$$

In this case, the coupling α_3 in the Lagrangian (8.4) vanishes and the field φ_3 is decoupled. The interaction part turns out to be the Lagrangian for the dual action of the sausage model (7.8) provided the identifications $\varphi_1 = 2\varphi$, $\varphi_2 = 2\vartheta$ are made. Notice that with the m - $\mathcal{M}$ relation for the sausage model (7.16), formula (7.77) can be re-written as

$$\rho = \frac{r}{8\pi}\ . \tag{7.79}$$

The ODE/IQFT correspondence suggests that for any common eigenvector $|\,\psi\,\rangle \in \mathcal{H}_k^{(1)}$ of the commuting family of operators $\mathbb{A}(\theta)$ and $\mathbb{B}(\theta)$, there exists a singular solution of the MShG equation (8.20) with $\mathcal{P}(z)$ given by (6.99) and the parameters a_i as in (7.78). The solution should be such that $\mathrm{e}^{-\eta}$ is a *smooth, single valued complex function without zeroes* on the punctured Riemann sphere. In the vicinity of $z = z_1,\ z_3$, the leading behaviour is described by

$$\mathrm{e}^{-\eta} \sim |\mathcal{P}(z)|^{-\frac{1}{2}} \qquad \text{as} \qquad |z - z_i| \to 0\ , \tag{7.80}$$

whereas in the neighbourhood of the second puncture

$$\mathrm{e}^{-\eta} \sim |z - z_2|^{1-(n+2)|k|} \qquad \text{as} \qquad |z - z_2| \to 0 \tag{7.81}$$

with $0 < |k| < \frac{1}{2}$.[3] The description of the apparent singularities involves some technical details that are completely analogous to those discussed in ref. [90]. In the case of the vacuum state, the apparent singularities are absent and the solution η is real. Notice that the point $z = \infty$ on the sphere is assumed to be regular, so that

$$\mathrm{e}^{-\eta} \sim |z|^2 \qquad \text{as} \qquad |z| \to \infty\ . \tag{7.82}$$

As it was mentioned in the previous subsection, the eigenvalue of the operator $\mathbb{S}$ which appears in the large θ-asymptotic formulae (viii) and (ix) is of special interest. Let us introduce the "regularized" value of the solution at the puncture $z = z_2$ as

$$\eta = \big(\,(n+2)\,|k| - 1\,\big)\ \log|z - z_2| + \eta^{(\mathrm{reg})} + o(1)\,. \tag{7.83}$$

Then for the solution corresponding to the eigenvector $|\,\psi\,\rangle \in \mathcal{H}_k^{(1)}$ with $0 < k < \frac{1}{2}$, the following formula holds:

$$\mathfrak{S}^{(\psi)} = \left(\frac{\rho}{n+2}\right)^{-2k} \frac{\Gamma(k)}{\Gamma(1-k)}\ \frac{\exp(\eta^{(\mathrm{reg})})}{(n+2)}\ \left|\frac{z_{13}}{z_{12}z_{23}}\right|^{-(n+2)k}\ , \tag{7.84}$$

where we use the shortcut notation $z_{ij} = z_i - z_j$.

[3] At $|k| = 0,\ \frac{1}{2}$ the leading asymptotic (7.81) involves logarithms. Here we ignore such subtleties.

We can now describe, in precise terms, the ODE/IQFT correspondence for the sausage model. Consider the auxiliary linear problem (7.71) associated with the singular solution of the MShG equation. The puncture $z = z_2$ is a regular singular point for this system of ODE. In the vicinity of this point, assuming that $0 < k < \frac{1}{2}$, one can introduce the basis solutions by means of the following asymptotic formulae as $z \to z_2$:

$$\begin{aligned}
\boldsymbol{\Theta}_-(z, \bar{z}\,|\,\theta\,) \quad &\to \quad \frac{\mathrm{e}^{+\mathrm{i}\beta_2}}{\sqrt{\sin(2\pi k)}}\ \mathrm{e}^{-k(\theta-\frac{\mathrm{i}\pi n}{2})} \left(\frac{z-z_2}{\bar{z}-\bar{z}_2}\right)^{+\frac{1}{4}\,(1-k(n+2))} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\boldsymbol{\Theta}_+(z, \bar{z}\,|\,\theta\,) \quad &\to \quad \frac{\mathrm{e}^{-\mathrm{i}\beta_2}}{\sqrt{\sin(2\pi k)}}\ \mathrm{e}^{+k(\theta-\frac{\mathrm{i}\pi n}{2})}\ \left(\frac{z-z_2}{\bar{z}-\bar{z}_2}\right)^{-\frac{1}{4}\,(1-k(n+2))} \begin{pmatrix} 0 \\ 1 \end{pmatrix}
\end{aligned} \tag{7.85}$$

where, for convenience, the constant phase factor is set to be

$$\mathrm{e}^{\mathrm{i}\beta_2} = \left(\frac{z_{12}z_{23}}{z_{13}}\ \frac{\bar{z}_{13}}{\bar{z}_{12}\bar{z}_{23}}\right)^{\frac{k}{4}(n+2)} . \tag{7.86}$$

Unlike $z = z_2$, the puncture at $z = z_1$ is an irregular singular point for the auxiliary linear problem. In its neighbourhood, and for $\frac{\pi}{2}\,(n-1) \leq \Im m(\theta) \leq \frac{\pi}{2}\,(n+1)$, another solution can be uniquely defined using the WKB asymptotic condition:

$$\begin{aligned}
\boldsymbol{\Xi}(z, \bar{z}|\,\theta\,) \quad \to \quad & |\mathcal{P}(z)|^{\frac{1}{4}}\ \exp\left(-\rho\mathrm{e}^{\theta}\int_{z_2}^{z}\mathrm{d}z\ \sqrt{\mathcal{P}(z)} - \rho\mathrm{e}^{-\theta}\int_{\bar{z}_2}^{\bar{z}}\mathrm{d}\bar{z}\ \sqrt{\overline{\mathcal{P}}(\bar{z})}\right) \\
\times & \begin{pmatrix} +\mathrm{e}^{-\frac{\theta}{2}}\,\big(\mathcal{P}(z)\big)^{-\frac{1}{4}} \\ -\mathrm{e}^{+\frac{\theta}{2}}\,\big(\overline{\mathcal{P}}(\bar{z})\big)^{-\frac{1}{4}} \end{pmatrix} \qquad \text{as} \quad z \to z_1\ .
\end{aligned} \tag{7.87}$$

There must be a linear relation between these three solutions and hence,

$$\boldsymbol{\Xi}\big(z, \bar{z}|\,\theta + \tfrac{\mathrm{i}\pi n}{2}\,\big) = \mathfrak{A}_+^{(\psi)}(\theta)\ \boldsymbol{\Theta}_-\big(z, \bar{z}|\,\theta + \tfrac{\mathrm{i}\pi n}{2}\,\big) + \mathfrak{A}_-^{(\psi)}(\theta)\ \boldsymbol{\Theta}_+\big(z, \bar{z}|\,\theta + \tfrac{\mathrm{i}\pi n}{2}\,\big)\ . \tag{7.88}$$

The ODE/IM correspondence states that the connection coefficients $\mathfrak{A}_+^{(\psi)}(\theta)$ and $\mathfrak{A}_-^{(\psi)}(\theta)$ coincide with the eigenvalues of the operators $\mathbb{A}(\theta)$ and $\mathcal{C}\,\mathbb{A}(\theta)\,\mathcal{C}$, for the common eigenvector $|\psi\rangle \in \mathcal{H}_k^{(1)}$ associated with the singular solution of the MShG equation. The eigenvalues of the transfer-matrices $\mathbb{T}_{\frac{1}{2}} \equiv \mathbb{T}$, and more generally $\mathbb{T}_j$ with $j = \frac{1}{2}, 1, \ldots$, can be obtained by the formulae similar to eqs. (6.92)-(6.94):

$$\begin{aligned} \Xi\big(z, \bar{z} \,|\, \theta + \mathrm{i}\pi(2j + \tfrac{1}{2})\big) \ &= \ \mathfrak{T}^{(\psi)}_j(\theta + \mathrm{i}\pi j)\ \Xi\big(z, \bar{z} \,|\, \theta + \tfrac{\mathrm{i}\pi}{2}\big) \\ &- \ \mathfrak{T}^{(\psi)}_{j-\frac{1}{2}}(\theta + \mathrm{i}\pi(j + \tfrac{1}{2}))\ \Xi\big(z, \bar{z} \,|\, \theta - \tfrac{\mathrm{i}\pi}{2}\big)\,. \end{aligned} \tag{7.89}$$

Finally, the eigenvalues of the operators $\mathbb{B}(\theta)$ and $\mathcal{C}\,\mathbb{B}(\theta)\,\mathcal{C}$ can also be expressed in terms of certain connection coefficients of the ODE system (7.71). For this purpose, one needs to introduce suitable basis solutions in the vicinity of the third puncture $z = z_3$. The corresponding formulae are simple generalizations of (6.112) and we do not present them here.

Chapter 8

Winding vacuum energies in a deformed $O(4)$ sigma model

8.1 Introduction

Recall that the 3D sausage model action is given by

$$\mathcal{A} = \int \mathrm{d}^2 x \ \frac{u\,\mathrm{Tr}(\partial_\mu \boldsymbol{U}\,\partial^\mu \boldsymbol{U}^{-1}) + 2l\,(L^3_\mu)^2 + 2r\,(R^3_\mu)^2}{4(u+r)(u+l) - rl\,\big(\mathrm{Tr}(\boldsymbol{U}\,\sigma_3\,\boldsymbol{U}^{-1}\,\sigma_3)\big)^2}\ . \tag{8.1}$$

The 3D sausage is a renormalizable NLSM within the three-dimensional space of couplings (u, r, l) at the one-loop level (here L^3_μ and R^3_μ stands for the left and right currents: $L^3_\mu := \frac{1}{2\mathrm{i}}\ \mathrm{Tr}(\partial_\mu \boldsymbol{U}\,\boldsymbol{U}^{-1}\sigma_3)$, $R^3_\mu := \frac{1}{2\mathrm{i}}\ \mathrm{Tr}(\boldsymbol{U}^{-1}\partial_\mu \boldsymbol{U}\sigma_3)$). The following combinations of parameters turned out to be renormalization group (RG) invariant:

$$a_1,\ a_2 > 0\ : \qquad a_1\,a_2 = \frac{\pi^2}{4\sqrt{(u+r)(u+l)rl}}\ , \qquad a_1^2 - a_2^2 = \frac{\pi^2}{4}\ \frac{u(r-l)}{(u+r)(u+l)rl} \tag{8.2}$$

Moreover, Fateev presented a set of convincing arguments in favor of the quantum integrability of the model (8.1). In particular, he argued that its spectrum is generated by two massive doublets of the same mass whose 2-particle S-matrix has the form of a direct product $(-S_{a_1} \otimes S_{a_2})$ of two $U(1)$-symmetric solutions of the S-matrix bootstrap equations. For this reason the above two-parameter deformation of the $O(4)$-sigma model is sometimes referred to as the SS-model. Also, it is worth noting, that S_a coincides with the soliton S-matrix [180] in the quantum sine-Gordon theory with the renormalized coupling constant a.

As usual we impose the twisted boundary condition for the matrix valued field $\boldsymbol{U}$,

$$\boldsymbol{U}(t, x+R) = \mathrm{e}^{\mathrm{i}\pi k_2 \sigma_3}\ \boldsymbol{U}(t,x)\ \mathrm{e}^{\mathrm{i}\pi k_1 \sigma_3}\ . \tag{8.3}$$

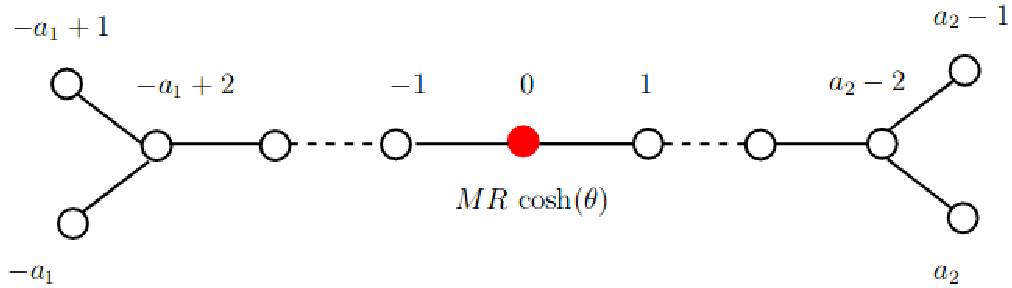


Figure 8.1: Incidence diagram for the TBA system describing the vacuum energy at the sector $k_1 = k_2 = 0$ in the case $a_1, a_2 = 2,\ 3,\ 4\dots$. The source term is indicated near the corresponding node.

The space of states of the theory then splits into sectors characterized by a pair of "winding" numbers, $\mathbf{k} = (k_1, k_2)$. The ground-state in each sector is referred to below as the k-vacuum and the corresponding energy is denoted by $E^{(\text{vac})}_{\mathbf{k}}$.

The lowest vacuum energy $E^{(\text{vac})}_{\mathbf{k}=0}$, can be calculated in the framework of the Thermodynamic Bethe Ansatz (TBA) approach. For the simplest case of integer parameters $a_1, a_2 = 2,\ 3, 4, \dots$, the required TBA equations were obtained in [68]. These equations are encoded by the incidence diagram shown in Fig.8.1, which has one massive node.[1] Subsequently, in Ref.[182], these equations were generalized to a system of Non-Linear Integral Equations (NLIE) [131, 183] which allows one to calculate $E^{(\text{vac})}_{\mathbf{k}=0}$ for any values of $a_1, a_2 \geq 2$. Moreover, the $\mathbf{k} = 0$ case of the undeformed $O(4)$-sigma model was separately considered in Refs.[184, 185, 186]. However, to the best of our knowledge, the problem of calculating the k-vacuum energies for general values of a_i and k_i is beyond the scope of traditional approaches of integrable quantum field theory. In this chapter we will discuss a conjectured exact formula for the k-vacuum energy in the 3D sausage for the general case.

[1] As noted in [181], if a model has an S-matrix in the form of a direct product $(-S_G \otimes S_H)$ and the TBA equations for the models described by S-matrices S_G and S_H are encoded by Dynkin-like diagrams of type G and H, each having one massive node, then the TBA equations for the model with the direct product S-matrix are obtained by "gluing" together the individual TBA equations at their massive nodes. This prescription, when applied to the SS-model with integer $a_1, a_2 \geq 2$, leads to a TBA system whose incidence diagram is shown in Fig.8.1.

8.2 UV/IR behavior of k-vacuum energy

Although $E_{\mathbf{k}}^{(\mathrm{vac})}$ is a rather complicated function of the parameters, its leading small-R (i.e., UV) and large-R (IR) behavior can be obtained via a simple and intuitive analysis which is based on the dual form of the 3D sausage proposed in Ref.[68].

The dual description is formulated in terms of three Bose fields governed by the Toda-like Lagrangian

$$\widetilde{\mathcal{L}}_{SS} = \frac{1}{16\pi}\sum_{i=1}^{3}(\partial_\mu\varphi_i\partial^\mu\varphi_i)^2 + 2\mu\,\left(\mathrm{e}^{b\varphi_3}\,\cos(\alpha_1\varphi_1+\alpha_2\varphi_2)+\mathrm{e}^{-b\varphi_3}\,\cos(\alpha_1\varphi_1-\alpha_2\varphi_2)\right) \tag{8.4}$$

where

$$\alpha_i = \frac{1}{2}\,\sqrt{a_i}\ , \qquad\qquad b = \frac{1}{2}\,\sqrt{a_1+a_2-2} \tag{8.5}$$

and the dimensionfull coupling μ is related to the soliton mass as

$$M = 2\mu\,\frac{\Gamma(2\alpha_1^2)\Gamma(2\alpha_2^2)}{\Gamma(2\alpha_1^2+2\alpha_2^2)}\ . \tag{8.6}$$

The soliton charges $q_i = 0, \pm1, \pm2\ldots$, corresponding to the factors S_{a_i} $(i = 1, 2)$ in the direct product $(-S_{a_1}\otimes S_{a_2})$, appear through the quasiperiodic boundary conditions imposed on the dual fields:

$$\varphi_1(x_1+R) = \varphi_1(x_1) + \frac{\pi}{\alpha_1}\,(q_2+q_1)\ , \qquad \varphi_2(x_1+R) = \varphi_2(x_1) + \frac{\pi}{\alpha_2}\,(q_2-q_1)\ . \tag{8.7}$$

In their turn, the winding numbers are interpreted as quasimomenta. Due to the periodicity of the potential terms in φ_j $(j = 1, 2)$, the stationary states can be chosen to be the Floquet states characterized by the pair $\mathbf{k} = (k_1, k_2)$:

$$\varphi_i \mapsto \varphi_i + 2\pi/\alpha_i\ :\quad \mid\Psi_{\mathbf{k}}\,\rangle \mapsto \mathrm{e}^{2\pi\mathrm{i}k_i}\mid\Psi_{\mathbf{k}}\,\rangle\ . \tag{8.8}$$

The form of the dual Lagrangian suggests that for small R

$$E_{\mathbf{k}}^{(\mathrm{vac})} \approx \frac{\pi}{R}\,\left(-\frac{1}{2}+\frac{p_0^2}{4b^2}+a_1k_1^2+a_2k_2^2\right)\ . \tag{8.9}$$

Since values of the field φ_3 is effectively restricted within the segment of length $\big(-2b\,\log(\mu R)\big)$, the corresponding "zero-mode momentum" p_0 is not arbitrary. It is determined through a certain quantization condition, similar to that discussed in Ref.[134] in the context of the quantum sinh-Gordon model. Assuming that

$$|a_1 k_1 \pm a_2 k_2| < 1\ , \tag{8.10}$$

the original consideration from [134] can be applied to the 3D sausage yielding

$$-\frac{p_0}{b^2}\ \log\Big(\frac{\mu R}{8b^2}\Big) + \delta^{(\mathbf{q}=0)}(p_0) \approx 2\pi\ , \tag{8.11}$$

with

$$\delta^{(\mathbf{q})}(p) = -\mathrm{i}\ \log\Big(S^{(q_1)}(p|a_1k_1 - a_2k_2)\, S^{(q_2)}(p|a_1k_1 + a_2k_2)\Big) \qquad \big(\delta^{(\mathbf{q})}(0) = 0\big) \tag{8.12}$$

Here $S^{(\mathbf{q})}(p|\lambda)$ stands for the so-called "reflection amplitude" for the sine-Liouville model [119]

$$S^{(q)}(p|\lambda) = \frac{\Gamma(\frac{1+|q|}{2} + \frac{\lambda}{2} - \frac{\mathrm{i}p}{2})\Gamma(\frac{1+|q|}{2} - \frac{\lambda}{2} - \frac{\mathrm{i}p}{2})}{\Gamma(\frac{1+|q|}{2} + \frac{\lambda}{2} + \frac{\mathrm{i}p}{2})\Gamma(\frac{1+|q|}{2} - \frac{\lambda}{2} + \frac{\mathrm{i}p}{2})}\ \frac{\Gamma(1+\mathrm{i}p)\Gamma(1+\frac{\mathrm{i}p}{4b^2})}{\Gamma(1-\mathrm{i}p)\Gamma(1-\frac{\mathrm{i}p}{4b^2})}\ . \tag{8.13}$$

In the IR limit the k-vacuum energy is composed of an extensive part proportional to the length of the system

$$E^{(\mathrm{vac})}_{\mathbf{k}} = R\,\mathcal{E}_0 + o(1) \qquad \text{as} \qquad R \to \infty\ . \tag{8.14}$$

The exact form of the specific bulk energy was found in [68]. It is expressed through the soliton mass M as

$$\mathcal{E}_0 = -\frac{M^2}{4}\ \frac{\sin(\frac{\pi}{2}a_1)\sin(\frac{\pi}{2}a_2)}{\sin(\frac{\pi}{2}(a_1+a_2))}\ . \tag{8.15}$$

In the case $a_1,\, a_2 > 1$, when the fundamental particles do not form bound states, the leading correction to (8.14) comes from virtual soliton and antisoliton trajectories winding once around the space circle. These trajectories should be counted with

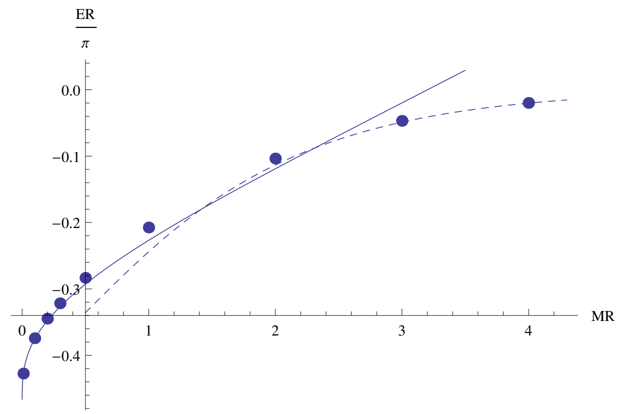


Figure 8.2: Numerical values of the dimensionless k-vacuum energy $\frac{R}{\pi}\,E^{(\mathrm{vac})}_{\mathbf{k}}$ versus MR for $a_1 = 2$, $a_2 = 3$ and $k_1 = k_2 = 0$. The solid and dashed lines follow from UV (8.9), and IR (8.16) asymptotic formulas, respectively. The heavy dots were obtained by means of a numerical solution of the TBA system encoded by the incidence diagram of Fig.8.1.

the phase factor $\mathrm{e}^{\mathrm{i}\pi(\sigma_1 k_1+\sigma_2 k_2)}$, where $\sigma_{1,2} = \pm 1$. Therefore, summing over the four possible sign combinations one obtains

$$E^{(\mathrm{vac})}_{\mathbf{k}} = R\,\mathcal{E}_0 - \tfrac{4}{\pi}\,\cos(\pi k_1)\,\cos(\pi k_2)\,M\,K_1(MR) + \big(\text{multiparticle}\big) \tag{8.16}$$

(here $a_{1,2} > 1$ and $K_1(r)$ stands for the conventional Bessel function). Note that similar arguments were originally applied to the quantum sine-Gordon model by Al. Zamolodchikov in Ref.[187].

In Fig. 8.2 the UV/IR asymptotic formulae are compared with the results of a numerical solution of the TBA system described by the incidence diagram from Fig.8.1.

8.3 Exact k-vacuum energy

8.3.1 3D sausage model

The model governed by the Lagrangian

$$\mathcal{L}_F = \frac{1}{16\pi}\,\sum_{i=1}^{3}(\partial_\mu\varphi_i\partial^\mu\varphi_i)^2 + 2\mu\,\left(\mathrm{e}^{\mathrm{i}\alpha_3\varphi_3}\,\cos(\alpha_1\varphi_1+\alpha_2\varphi_2) + \mathrm{e}^{-\mathrm{i}\alpha_3\varphi_3}\,\cos(\alpha_1\varphi_1-\alpha_2\varphi_2)\right), \tag{8.17}$$

where the coupling constants α_i are subjected to a single constraint

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \frac{1}{2} \ , \tag{8.18}$$

will be referred to below as the *Fateev model*. In the case when $\alpha_1,\ \alpha_2$ are real while α_3 is pure imaginary (*unitary regime*), the Lagrangian (8.17) is real and coincides with the dual Lagrangian $\widetilde{\mathcal{L}}$ provided $\alpha_3 = -\mathrm{i}b$. In the *symmetric regime* all the coupling constant α_i are real, the Lagrangian (8.17) is completely symmetric under simultaneous permutations of the real fields φ_i and couplings α_i. Despite that the theory is apparently non-unitary in this case, one can still address the problem of calculation of the k-vacuum energies. Since the Lagrangian $\mathcal{L}_F$ in the symmetric regime is invariant under the transformations $\varphi_i \mapsto \varphi_i + 2\pi\alpha_i$ with $i = 1, 2, 3$, the k-vacuum energies are labeled by the triple of quasimomenta $\mathbf{k} = (k_1, k_2, k_3)$ (contrary to the unitary regime where $\mathbf{k} = (k_1, k_2)$). The short distance expansion of $E_{\mathbf{k}}^{(\mathrm{vac})}$ in the symmetric regime is considerably simpler than in the unitary one. Its general structure follows from the fact that the potential term of $\mathcal{L}_F$ with $\alpha_i > 0$ is a uniformly bounded perturbation for any values of the dimensionless parameter μR. Therefore the conformal perturbation theory can be applied literally yielding an expansion

$$\text{Symmetric regime}\ :\qquad \frac{R}{\pi}\ E_{\mathbf{k}}^{(\mathrm{vac})} = -\frac{1}{2} + \sum_{i=1}^{3}(2\alpha_i k_i)^2 - \sum_{n=1}^{\infty} e_n\ (\mu R)^{4n}\ . \tag{8.19}$$

An exact formula for the k-vacuum energies in the symmetric regime was proposed in Ref.[128]. Below we argue that essentially the same formula actually holds in both regimes of the 3D sausage.

8.3.2 Regular solutions of the shG equation

Consider the classical partial differential equation

$$\partial_z\partial_{\bar{z}}\hat{\eta} - \rho^2\ |\mathcal{P}(z)|\left(\mathrm{e}^{2\hat{\eta}} - \mathrm{e}^{-2\hat{\eta}}\right) = 0 \tag{8.20}$$

where

$$\mathcal{P}(z) = \frac{(z_3 - z_2)^{a_1} (z_1 - z_3)^{a_2} (z_2 - z_1)^{a_3}}{(z - z_1)^{2-a_1} (z - z_2)^{2-a_2} (z - z_3)^{2-a_3}} \ . \tag{8.21}$$

and $\bar{z}$ denotes the complex conjugate of z. Here ρ is a real parameter and a_i $(i = 1, 2, 3)$ are also real and satisfy the condition

$$a_1 + a_2 + a_3 = 2 \ . \tag{8.22}$$

The variable z is regarded as a complex coordinate on $\mathbb{CP}^1 \backslash \{z_1, z_2, z_3\}$, the Riemann sphere with three punctures. Due to the relation (8.22), $\mathcal{P}(z)(\mathrm{d}z)^2$ is a quadratic differential under $\mathbb{PSL}(2, \mathbb{C})$ transformations, so that the punctures can be sent to any prescribed positions, say $(z_1, z_2, z_3) = (0, 1, \infty)$. Then the change of variables

$$w = \rho \int \mathrm{d}z \ z^{\frac{a_1}{2}-1} (1-z)^{\frac{a_2}{2}-1} \tag{8.23}$$

brings (8.20) to the standard form of the sinh-Gordon (shG) equation,

$$\partial_w \partial_{\bar{w}} \hat{\eta} - \mathrm{e}^{2\hat{\eta}} + \mathrm{e}^{-2\hat{\eta}} = 0 \ . \tag{8.24}$$

In the case when a_1, a_2, a_3 are all positive Eq.(8.23) defines the Schwarz-Christoffel mapping, transforming the upper and lower half-planes correspondingly to the triangles (w_1, w_2, w_3) and $(w_1, w_2, \bar{w}_3)$, depicted in Fig. 8.3a. Note, that the adjacent sides of the resulting polygon $(w_1,\ w_2,\ w_3,\ \bar{w}_3)$ should be identified to form a topological 2-sphere. In the case when $a_3 < 0$, but $a_1, a_2 > 0$, the image of the punctured sphere is shown in Fig. 8.3b. Again, the adjacent rays should be properly identified. In this way Eq.(8.20) on the thrice-punctured sphere can be equivalently formulated as the shG equation in the domains shown in Fig. 8.3a and Fig. 8.3b, corresponding to the two cases

$$\begin{aligned} &\text{Regime I} &&: \quad a_1 > 0,\ a_2 > 0\ , \quad a_3 = 2 - a_1 - a_2 > 0 \\ &\text{Regime II} &&: \quad a_1 > 0,\ a_2 > 0\ , \quad a_3 = 2 - a_1 - a_2 < 0\ . \end{aligned} \tag{8.25}$$

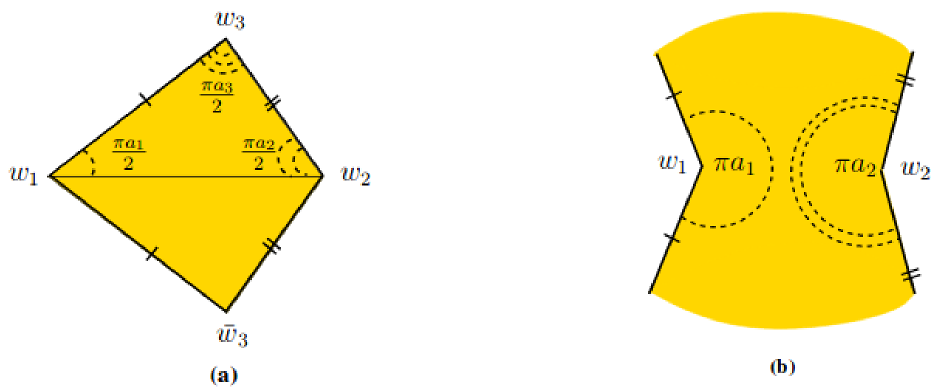


Figure 8.3: The image of the thrice-punctured sphere in the complex w-plain: (a) for the case $a_{1,2,3} > 0$ (regime I); (b) for the case $a_{1,2} > 0$ and $a_3 < 0$ (regime II).

We will consider *regular* solutions to (8.24), defined by the following two requirements. First, the regular solution should be a *smooth*, *single valued*, *real* function on the punctured sphere $\mathbb{CP}^1 \backslash \{z_1, z_2, z_3\}$ or, equivalently (when the complex coordinate w is employed) in the domains shown in Fig. 8.3 with properly identified edges. Second, the regular solution must develop the proper asymptotic behavior in the vicinity of the punctures. For regime I there is the freedom to control the asymptotic behavior of $\hat{\eta}$ at each of the three punctures, or, equivalently, at each vertex w_i in Fig.8.3a. Namely,

$$\hat{\eta} \to 2\, l_i \;\log |w - w_i| + O(1)\,, \quad \text{when} \;\; w \to w_i\,, \tag{8.26}$$

where

$$-\frac{1}{2} < l_i \leq 0 \tag{8.27}$$

denote free parameters[2]. For regime II, when $a_3 < 0$, the third puncture is mapped to the infinity of the domain, shown in Fig.8.3b, and we require that

$$\text{Regime II} \quad : \qquad \hat{\eta} \to 0 \qquad \text{as} \qquad |w| \to \infty\ , \tag{8.28}$$

[2]For $l_i = -\frac{1}{2}$ the leading asymptotics (8.26) should be replaced by

$$\hat{\eta} \to -\log\Big(|w - w_i| \log\big(\tfrac{4}{|w-w_i|}\big)\Big) + O(1)\ .$$

whereas the asymptotic behavior in the vicinity of $w = w_1$, w_2, is still described by (8.26) with two free parameters (8.27). It turns out that the solution of the shG equation, satisfying the above regularity conditions, exists and is unique for both regimes I and II.

8.3.3 Main conjecture

Define the functional

$$\mathfrak{F}(\rho) = -\frac{8}{\pi} \int \mathrm{d}^2 w \ \sinh^2(\hat{\eta}) + \sum_i a_i l_i^2 \ , \tag{8.29}$$

where $\hat{\eta}$ is a regular solution and the summation index i takes the values $i = 1, 2, 3$ and $i = 1, 2$ for the regimes I and II, respectively. The additive constant in (8.29) is chosen to provide the normalization condition

$$\lim_{\rho \to \infty} \mathfrak{F}(\rho) = 0 \ . \tag{8.30}$$

Now we can extend the conjecture of Ref.[128] and propose the expression for the k-vacuum energies, which is valid for both considered regimes,

$$\frac{R}{\pi} \ E_{\mathbf{k}}^{(\mathrm{vac})} = \mathfrak{F}(\rho) - 4\rho^2 \prod_{i=1}^{3} \gamma\big(\tfrac{a_i}{2}\big) \ , \tag{8.31}$$

where $\gamma(x) := \frac{\Gamma(x)}{\Gamma(1-x)}$. This formula should be supplemented with the relations between the parameters of quantum and classical problems:

$$\mu R = 2\,\rho \ , \qquad \alpha_i^2 = \frac{a_i}{4} \ , \qquad |k_i| = l_i + \frac{1}{2} \ . \tag{8.32}$$

In the case of the symmetric regime, formula (8.31) can be checked, in principle, perturbatively. Namely, let us return to the original variable z and replace $\hat{\eta}$ by $\eta = \hat{\eta} + \frac{1}{2} \ \log|\rho^2 \mathcal{P}|$. This brings (8.20) to the form of the modified shG equation:

$$\partial_z \partial_{\bar{z}} \eta - \mathrm{e}^{2\eta} + \rho^4 \ |\mathcal{P}|^2 \ \mathrm{e}^{-2\eta} = 0 \ . \tag{8.33}$$

For the regular solution the third term in (8.33) can be treated perturbatively even in the nearest neighbor of each puncture and the RHS of (8.31) admits a Taylor expansion (see [128] for details):

$$\text{Regime I} \quad : \qquad \mathfrak{F} - 4\rho^2 \prod_{i=1}^{3} \gamma\big(\tfrac{a_i}{2}\big) = \sum_{n=0}^{\infty} \mathfrak{f}_n \ \rho^{4n} \ . \tag{8.34}$$

On the other hand, the LHS of (8.31) possesses a series expansion (8.19) which, in principle, can be obtained using the conformal perturbation theory. Thus, in the symmetric regime (regime I), both sides of (8.31) can be understood perturbatively and the conjectured relation implies that the corresponding expansion coefficients are simply related: $\mathfrak{f}_n = -2^{4n}\ e_n$.

The situation is somewhat different in the unitary regime (regime II). Of course, the RHS of (8.31) in this regime is still well defined. However, the conformal perturbation theory cannot be applied literally in this case. More generally, at the moment, it is not entirely clear how one can calculate the LHS of (8.31) for arbitrary values of a_i and k_i in the SS-model. In particular, as was mentioned earlier, the knowledge of the exact S-matrix is not of much help in solving this problem. Therefore, as a first step in proving the correspondence (8.31), it would be desirable to *derive* the UV and IR asymptotics of $E_{\mathbf{k}}^{(\mathrm{vac})}$, discussed above, from the differential equation side. Fortunately, this could be done analytically by using an auxiliary linear problem associated with the shG equation (8.24). The derivation is rather technical and will be published elsewhere. Here we only present the results of our numerical work in support of the conjecture (8.31). The shG equation has been solved numerically for various sets of the parameters a_i and k_i. We found that the resulting values of the RHS of (8.31) are in good agreement with the UV and IR asymptotic formulae (8.9) and (8.16). Some (small) part of the available numerical data is presented in Fig. 8.4 and Table D.1.

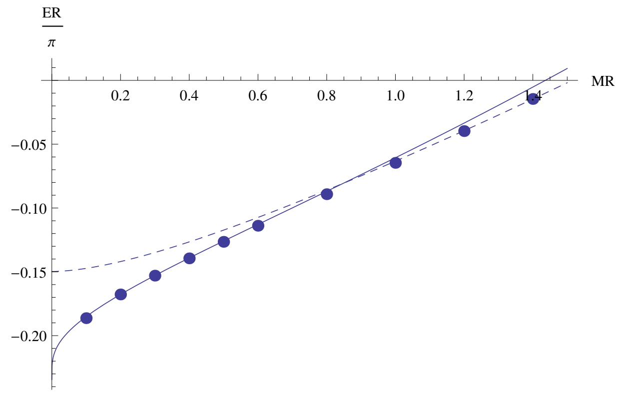


Figure 8.4: Numerical values of the dimensionless k-vacuum energy $\frac{R}{\pi}\,E^{(\mathrm{vac})}_{\mathbf{k}}$ versus the variable $r = MR$ for $a_1 = 1.7$, $a_2 = 1.5$, $k_1 = \frac{4}{17} = 0.235\ldots$, $k_2 = \frac{1}{3}$. The solid and dashed lines represent the small-R, (8.9), and large-R asymptotics (8.16), respectively. The heavy dots represent the LHS of (8.31) calculated from numerical solutions of the shG equation. The corresponding numerical values are presented in Table D.1.

MR	r.h.s of Eq.(8.31)	Eq.(8.9)	Eq.(8.16)
0.1	-0.18631	-0.18510	-0.14729
0.2	-0.16773	-0.16770	-0.14197
0.3	-0.15230	-0.15288	-0.13487
0.4	-0.13945	-0.13913	-0.12654
0.5	-0.12651	-0.12589	-0.11731
0.6	-0.11390	-0.11288	-0.10739
0.8	-0.08919	-0.08695	-0.08605
1.0	-0.06463	-0.06061	-0.06327
1.2	-0.03973	-0.03346	-0.03941
1.4	-0.01456	-0.00524	-0.01461

Table 8.1: The dimensionless k-vacuum energy $\frac{R}{\pi}\,E^{(\mathrm{vac})}_{\mathbf{k}}$ as a function of the variable MR.

Chapter 9

Discussion/Outlook

In this dissertation we considered the problem of the quantization of NLSM. Our principal examples were the 2D and 3D sausage models. To conclude, let's summarize and discuss the key points of the paper.

We started by introducing NLSM in $1+1$ space-time dimensions. A key feature of this class of models is renormalizability. The one-loop RG equations take the form of the Ricci flow, which arose independently in mathematics. A number of examples of one-loop renormalizable NLSM were provided, whose metric satisfies the Ricci flow equations. It was mentioned that the 2D and 3D sausage NLSM were originally found by explicitly solving the Ricci flow equations. It turned out that all these models were classically integrable field theories.

In the work [75], a classically integrable NLSM was constructed starting from an explicit ansatz for the form of the metric and the connection components entering into the Zero-Curvature Representation. This NLSM is a four parameter deformation of the SU(2) PCF with torsion that contains the 3D sausage model as a two parameter sub-family. Remarkably the model satisfies the generalized one-loop RG flow equations (2.45). This hints to a deep connection between classically integrability and one-loop renormalizability. It would be interesting to explore it further.

We emphasized the rôle of the Sklyanin exchange relations in an integrable field theory. They guarantee the Poisson commutativity of the infinite family of integrals of motion, which is an important component of classical integrability. Moreover, as the classical version of the quantum Yang-Baxter algebra, the Sklyanin exchange relations are crucial in the "first principles" quantization of the theory. We discussed the derivation of the Sklyanin exchange relations, which relies on the ultra-local form of the Poisson brackets for the flat connection (3.23). However, in the case of integrable NLSM, the connection components typically do not satisfy such relations, but contain in addition a term proportional to $\delta'(x-y)$. This, in turn, causes problems in the quantization of classically integrable NLSM.

The Poisson brackets of the flat connection depend on the gauge. We demonstrated that the classical sausage model admits the ultralocal gauge. Thus, "Hamiltonian Methods in the Theory of Solitons" [77] can be applied without modifications. In connection with this, we believe that the problem with ultralocality for other integrable NLSM should be revisited.

Another strategy was formulated for tracing the Sklyanin exchange relations in a non-ultralocal field theory. It is inspired by the age-old observation that the quantum monodromy operator is somehow better behaved than its classical counterpart. In the central example we recovered the Yang-Baxter Poisson algebra in a non-ultralocal system based on the $SU(2)$ current algebra by starting with an explicit quantum field theory realization of the Yang-Baxter relation and then taking the classical limit. As a result of the entangled interplay between the classical limit and the scaling one, which required ultraviolet regularization of the model, we found that the classical monodromy matrix is somewhat more cumbersome than its quantum counterpart. It turned out that the net result of the non-ultralocal structure for the Sklyanin exchange relations is the non-universal renormalization of the spectral parameter which occurs even at the classical level. This is somewhat in the spirit of Faddeev and Reshetikhin [48] who proposed to ignore the problem of non-ultralocality, arguing that it is a consequence of choosing the "false vacuum", and to restore the ultralocality of the current algebra by hand.

The example we elaborated is relevant to the Fateev model, an integrable two parameter deformation of the $SU(2)$ Principal Chiral Field. It provides evidence for the existence of the Sklyanin exchange relations for this remarkable non-linear sigma model, which was shown for several particular cases in the parameter space. We believe that unraveling the Sklyanin exchange relations for non-ultralocal systems is important in many respects. Of special interest is the Klimčík model and its reductions [92], which have recently attracted a great deal of attention in the context of the AdS/CFT correspondence [93, 94].

We considered the quantization of the 2D sausage model in detail. In our study of the quantum model we closely followed the ideas of the works [50, 51, 52]. We paid special attention to the integrable structures of the cigar NLSM — the CFT governing the ultraviolet behaviour of the quantum sausage. In particular we constructed the BLZ type representation for the chiral transfer-matrices in the quantum cigar.

The chiral transfer-matrices depend on a number of parameters and can be considered in the parameter domain where they are not directly related to the cigar NLSM. In this case, they are still of physical interest since they can be interpreted as the transfer-matrices for the minimal $\mathbb{Z}_n$ parafermionic models from ref. [121]. The situation here resembles the interplay between the quantum Liouville theory and the BPZ minimal models. We constructed lattice transfer-matrices and presented numerical evidence that in the scaling limit they become the chiral transfer-matrices in the parafermionic regime. We believe that it may hint as to how to proceed with the lattice formulation of the cigar and sausage models. To go further in this direction the most promising approach is, perhaps, the method of separation of variables [127] which was successfully applied to a similar problem appearing in the quantization of the sinh-Gordon model [151, 152, 153, 100, 154, 155, 156]. Another interesting possibility is related to the work [157], where some spectral properties of the cigar

NLSM were observed to appear in the scaling limit of a certain inhomogeneous version of the 6-vertex model. We have already carried out a preliminary study of the inhomogeneous 6-vertex model. Our main results are collected in the recent pre-print [61].

One of the most effective methods for the calculation of the spectrum of commuting families of operators including the transfer-matrices in integrable quantum field theory is based on the ODE/IQFT correspondence. From our study of the parafermionic transfer-matrix, we proposed the ODE counterpart in the correspondence for the cigar NLSM. It turns out to be identical to that which was introduced earlier in the context of the so-called paperclip model in the works [114, 115]. Based on the results of these papers, we derived non-linear integral equations for determining the vacuum eigenvalues of the chiral transfer-matrix which work both for the cigar and the parafermionic regimes. We believe that this might be a good starting point for applying the powerful fermionic methods [158, 159, 160, 161, 162] to the sausage/$O(3)$ NLSM.

In refs. [56, 90], a conceptual explanation was given of how the ODE/IQFT correspondence for integrable conformal field theory can be generalized to the massive IQFT. Following this route, we extended the ODE/IQFT correspondence from the cigar to the sausage NLSM. With the correspondence one can uncover the basic integrable structures by studying the properties of the connection coefficients of the ordinary differential equations. The properties of the commuting families of operators in the sausage model, which includes the quantum transfer-matrix, are given in the list (i)-(x) in sec. 7.3 . The technical result that deserves to be mentioned is the system of NLIE which describes the vacuum eigenvalues of the commuting families of operators. Among other things, it allows one to calculate the k-vacuum energies of the sausage/$O(3)$ NLSM.

There are many results in the literature concerning the energy spectrum of the

$O(3)$ sigma model in the sector with $k = 0$ [163, 145, 185, 184]. In ref. [66] a system of TBA equations was proposed which allows one to calculate the ground state energy for $k = 0$ and integer values of the dimensionless coupling $n \geq 3$ of the sausage model. Recently Ahn, Balog and Ravanini [140] transformed this system of TBA to a system of three non linear integral equations which, it is affirmed, works for any real positive n. Their main assumption is a periodicity condition for the Q-function given by eq. (3.16) from that paper. In our investigations, we did not find any trace of a Q-function satisfying such a strong periodic condition. Nevertheless, the numerical results presented in fig. 2 from that paper seem to be in agreement with the data obtained from the solution of our NLIE (7.30), (7.31) with $k = 0$ and $n = 1$. This situation needs to be clarified.

The ODE/IQFT correspondence was an invaluable tool in our study of the 2D sausage model. In [128], the ODE/IQFT correspondence was proposed for the 3D sausage. Based on the correspodence, a remarkable formula (8.31) was put forward that expresses the k-vacuum energy in terms of certain solutions to the MShG equation. In this thesis, we numerically verified it for the unitary regime. The results are given in fig. 8.4 and tab. 8.1. They show excellent agreement between the numerical data and the UV/IR asymptotics.

Let us briefly touch on some problems which have not been discussed in the thesis but are directly related to the subject of this work. We did not make any mention of the sausage model with the topological term equal to π which is also expected to be an integrable QFT [166, 66]. Another closely related model is the four-parameter integrable family of NLSM with torsion introduced in the work [75], which includes the 3D sausage as a two parameter subfamily. We believe that extending the ODE/IQFT approach to these models will be useful, both as a step in the development of the method, and in terms of applications. There are the remarkable works [167, 168, 169] on toroidal algebras, which are deeply connected to this field.

All the models mentioned above are based on the $\mathfrak{sl}(2)$-algebra and its associated integrable structures. Since the work of Klimč ík [69] there has been increasing interest in "deformed" integrable NLSM associated with higher rank Lie algebras [170, 171, 74, 172]. The first principles quantization of such theories seems to be a very interesting problem. In the recent work [173], an important step in this direction was taken where a one parameter deformation was found of the set of "circular brane" local integrals of motion introduced in ref. [174]. This offers the possibility for the quantization of the deformed $O(N)$ NLSM along the lines of this work.

Perhaps the main motivation for studying NLSM is based on the fact that certain types of SUSY sigma models are at the heart of the celebrated AdS/CFT correspondence, and integrability is an important possibility. In particular, the NLSM associated with the AdS side of the correspondence for $\mathcal{N} = 4$ SUSY Yang Mills theory was argued to be integrable [175, **?**]. As was already mentioned, the study of the first principles quantization of the NLSM by traditional techniques has proven to be difficult. A similar situation exists with sigma models on supergroups and superspaces, which are expected to provide theoretical descriptions of condensed matter systems with disorder [176]. That is where one is most tempted to try the power of the ODE/IQFT approach.

Appendix A

Poisson structure of the Klimčík model

Using the Lagrangian (2.41) one can show that the currents $\boldsymbol{I}_\pm = \sum_a I^a_\pm \mathtt{t}_a$ (3.13) obey the Poisson bracket relations

$$\mathrm{g}^{-2}\left\{I^a_\sigma(x),\, I^b_{\sigma'}(y)\right\} \;=\; \sigma\, q^{ab}\delta_{\sigma\sigma'}\,\delta'(x-y) + \sum_{\sigma''} F^{abc}(\sigma,\sigma'|\sigma'')\, q_{cd}\, I^d_{\sigma''}\,\delta(x-y)\,. \quad \text{(A.1)}$$

The structure constants are given by

$$\begin{aligned}
2F^{abc}(\pm\pm|\pm) &= +(1+\mathsf{b})\; f^{abc} \pm \mathrm{i}\varepsilon_2\left(\mathcal{R}^c{}_d\, f^{dba} + \mathcal{R}^b{}_d\, f^{dac} + \mathcal{R}^a{}_d\, f^{dcb}\right)\\
2F^{abc}(\pm\pm|\mp) &= -(1-\mathsf{b})\, f^{abc} \pm \mathrm{i}\varepsilon_2\,\mathcal{R}^c{}_d\, f^{dba} \qquad \text{(A.2)}\\
2F^{abc}(\pm\mp|\pm) &= +(1-\mathsf{b})\, f^{abc} \mp \mathrm{i}\varepsilon_2\,\mathcal{R}^b{}_d\, f^{dac}\\
2F^{abc}(\mp\pm|\pm) &= +(1-\mathsf{b})\, f^{abc} \mp \mathrm{i}\varepsilon_2\,\mathcal{R}^a{}_d\, f^{dcb}
\end{aligned}$$

with

$$\mathsf{b} = \tfrac{1}{2}\,\left(1+\varepsilon_1^2-\varepsilon_2^2\right)\,.$$

Also, $\mathcal{R}^b{}_a$ in the above formulae stands for the matrix elements of the Yang-Baxter operator

$$\hat{\boldsymbol{R}}(\mathtt{t}_a) = \mathtt{t}_b\,\mathcal{R}^b{}_a\,.$$

As was mentioned in the main body of the text, the currents $\boldsymbol{I}_\pm$ are related via the linear transformation (4.84), (4.85) to $\boldsymbol{J}_\pm = \sum_a J^a_\pm\,\mathtt{t}_a$ which form two independent copies of the current algebra (4.82). To write the explicit formulae for the matrix elements occurring in (4.85),

$$\boldsymbol{X}^A \equiv \begin{pmatrix} X^A_{++} & X^A_{+-} \\ X^A_{-+} & X^A_{--} \end{pmatrix} \qquad (A = \pm, 0)\,,$$

it is convenient to swap the deformation parameters ε_1, ε_2 for m_1, m_2 defined through the relations

$$\varepsilon_1 = \frac{(1-m_1^2)(1-m_2^2)}{(1+m_1^2)(1+m_2^2)}\,, \qquad \varepsilon_2 = \frac{4\,m_1 m_2}{(1+m_1^2)(1+m_2^2)}\ . \tag{A.3}$$

Then,

$$\begin{aligned}
\boldsymbol{X}^+ &= \frac{\mathrm{g}^2}{(1+m_1^2)(1+m_2^2)} \begin{pmatrix} (1-m_1m_2)^2 & (m_1-m_2)^2 \\ (m_1+m_2)^2 & (1+m_1m_2)^2 \end{pmatrix} \\
\boldsymbol{X}^- &= \frac{\mathrm{g}^2}{(1+m_1^2)(1+m_2^2)} \begin{pmatrix} (1+m_1m_2)^2 & (m_1+m_2)^2 \\ (m_1-m_2)^2 & (1-m_1m_2)^2 \end{pmatrix} \\
\boldsymbol{X}^0 &= \frac{\mathrm{g}^2}{(1+m_1^2)(1+m_2^2)} \begin{pmatrix} 1+m_1^2m_2^2 & m_1^2+m_2^2 \\ m_1^2+m_2^2 & 1+m_1^2m_2^2 \end{pmatrix}\ .
\end{aligned}$$

Finally we note that the Hamiltonian of the Klimčík model (4.80) is expressed in terms of the currents $\boldsymbol{K}_\pm$ as

$$H = \frac{\mathrm{g}^2}{4}\int \mathrm{d}x \sum_{\sigma,\sigma'=\pm} \Big(A^{\parallel}_{\sigma\sigma'}\,\langle\,\boldsymbol{K}^0_\sigma,\,\boldsymbol{K}^0_{\sigma'}\,\rangle + 2A^{\perp}_{\sigma\sigma'}\,\langle\,\boldsymbol{K}^+_\sigma,\,\boldsymbol{K}^-_{\sigma'}\,\rangle \Big)\ ,$$

where

$$A^{\parallel}_{\pm\pm} = 1+\varepsilon_1^2\ , \qquad A^{\parallel}_{\pm\mp} = 1-\varepsilon_1^2\,,$$

$$A^{\perp}_{\pm\pm} = 1+\varepsilon_1^2-\varepsilon_2^2\,, \qquad A^{\perp}_{\pm\mp} = (1+\varepsilon_1\mp\varepsilon_2)(1-\varepsilon_1\pm\varepsilon_2)\ .$$

Appendix B

Here we discuss some geometrical aspects of the Klimčík non-linear sigma model. The target space is topologically the same as $\mathfrak{G}$ (which below is assumed to be a compact simple Lie group) but equipped with a certain anisotropic metric $G_{\mu\nu}$. The latter can be thought of as a two-parameter deformation of the left/right invariant metric on the group manifold. In fact, the form of the Lagrangian (2.41) suggests that the target manifold is equipped with the affine connection Γ such that the metric is covariantly constant w.r.t. Γ, while its torsion is defined by the antisymmetric tensor $B_{\mu\nu}$. To be precise, the covariant torsion tensor

$$H_{\lambda\mu\nu} \;=\; G_{\lambda\rho}\,\big(\Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}\big) \tag{B.1}$$

(here $\Gamma^{\rho}{}_{\mu\nu}$ stands for the Christoffel symbol), is a closed 3-form with $B_{\mu\nu}$ playing the rôle of the torsion potential:

$$H_{\lambda\mu\nu} = \partial_\lambda B_{\mu\nu} + \partial_\nu B_{\lambda\mu} + \partial_\mu B_{\nu\lambda}\ .$$

A remarkable feature of the Klimč´ik target space background is that it admits a set of 1-forms which can be thought of as deformations of the Maurer-Cartan forms. Introduce two sets $\{\boldsymbol{e}^a_\mu(\sigma)\}_{a=1}^D$ ($D = \dim\mathfrak{G}$):

$$\mathtt{t}_a\,\boldsymbol{e}^a_\mu(\sigma)\,\mathrm{d}X^\mu \;=\; -2\,\mathrm{i}\,\hat{\boldsymbol{\Omega}}_\sigma^{-1}\big(\boldsymbol{U}^{-1}\,\mathrm{d}\boldsymbol{U}\big)\ . \tag{B.2}$$

Here $\hat{\boldsymbol{\Omega}}_\sigma$ stands for the linear operator acting in $\mathfrak{g}$,

$$\hat{\boldsymbol{\Omega}}_\sigma \;=\; \hat{\mathbf{1}} + \mathrm{i}\,\sigma\,\varepsilon_1\,\hat{\boldsymbol{R}}_{\boldsymbol{U}} + \mathrm{i}\,\sigma\,\varepsilon_2\,\hat{\boldsymbol{R}} \tag{B.3}$$

and σ takes two values $\pm$. It is not difficult to show that the metric can be written as

$$G_{\mu\nu} \;=\; \tfrac{1}{2\mathrm{g}^2}\; q_{ab}\; \boldsymbol{e}^a_\mu(+)\, \boldsymbol{e}^b_\nu(+) \;=\; \tfrac{1}{2\mathrm{g}^2}\; q_{ab}\; \boldsymbol{e}^a_\mu(-)\, \boldsymbol{e}^b_\nu(-)\;, \tag{B.4}$$

i.e., $\{\boldsymbol{e}^a_\mu(+)\}^D_{a=1}$ and $\{\boldsymbol{e}^a_\mu(-)\}^D_{a=1}$ are two vielbein sets in the cotangent space of the target manifold. Notice the following simple relations

$$G^{\mu\nu}\; \boldsymbol{e}^a_\mu(+)\, \boldsymbol{e}^b_\nu(+) \;=\; G^{\mu\nu}\; \boldsymbol{e}^a_\mu(-)\, \boldsymbol{e}^b_\nu(-) \;=\; 2\mathrm{g}^2\; q^{ab}$$

and

$$\sqrt{\det G_{\mu\nu}} \;=\; \big(\det \hat{\boldsymbol{\Omega}}_\sigma\big)^{-1} \;\times\; \sqrt{\det G^{(0)}_{\mu\nu}}\;, \tag{B.5}$$

where $G^{(0)}_{\mu\nu} = G_{\mu\nu}|_{\varepsilon_1=\varepsilon_2=0}$.

It turns out that the torsion also admits simple expressions involving $\boldsymbol{e}^a_\mu(\sigma)$ and the structure constants $F^{abc}(\sigma,\sigma'|\sigma'')$ (A.2) appearing in the Poisson algebra (A.1):

$$H_{\lambda\mu\nu} = +\frac{1}{4\mathrm{g}^2}\,\Big(F_{abc}(-\,+|+)\;\boldsymbol{e}^c_{[\lambda}(+)\boldsymbol{e}^a_\mu(-)\boldsymbol{e}^b_{\nu]}(+) - 2\,F_{abc}(+\,+|+)\;\boldsymbol{e}^a_\lambda(+)\boldsymbol{e}^b_\mu(+)\boldsymbol{e}^c_\nu(+)\Big) \tag{B.6a}$$

and

$$H_{\lambda\mu\nu} = -\frac{1}{4\mathrm{g}^2}\,\Big(F_{abc}(+\,-|-)\;\boldsymbol{e}^c_{[\lambda}(-)\boldsymbol{e}^a_\mu(+)\boldsymbol{e}^b_{\nu]}(-) - 2\,F_{abc}(-\,-|-)\;\boldsymbol{e}^a_\lambda(-)\boldsymbol{e}^b_\mu(-)\boldsymbol{e}^c_\nu(-)\Big)\,. \tag{B.6b}$$

Here the symbol $[\lambda\mu\nu]$ denotes the alternating summation over all possible permutations of the indices λ,μ and ν.

Before discussing the origin of the above formulae for the metric and torsion, let us first inspect the reality condition for the target space background. Consider the metric and the torsion as a function of ε_1 with the ratio $\varepsilon_2/\varepsilon_1$ a fixed real number. First of all it is easy to see that the determinant $\det\hat{\boldsymbol{\Omega}}_\sigma$ which appears in the formula

(B.5) does not depend on the choice of the sign factor σ – it is a polynomial in the variable ε_1^2 of degree coinciding with the integer part of half of $D \equiv \dim(\mathfrak{G})$:

$$\det \hat{\mathbf{\Omega}}_\sigma = 1 + \sum_{n=1}^{[\frac{D}{2}]} \omega^{(n)}\ \varepsilon_1^{2n}\ ,$$

where the coefficients $\omega^{(n)}$ are real as $\Im m(\varepsilon_2/\varepsilon_1) = 0$. In their turn, the components of the metric tensor and the torsion are rational functions of ε_1 of the form

$$\begin{aligned} G_{\mu\nu} &= \frac{1}{\det \hat{\mathbf{\Omega}}_\sigma} \sum_{n=0}^{[\frac{D-1}{2}]} g^{(n)}_{\mu\nu}\ \varepsilon_1^{2n} \\ H_{\lambda\mu\nu} &= \frac{\mathrm{i}\varepsilon_1}{(\det \hat{\mathbf{\Omega}}_\sigma)^2} \sum_{n=0}^{D-1} h^{(n)}_{\lambda\mu\nu}\ \varepsilon_1^{2n}\ . \end{aligned} \tag{B.7}$$

For pure imaginary ε_1, the 1-forms $\boldsymbol{e}^a_\mu(\sigma)$ are real and, as it follows from (B.4), the metric is positive definite. Formula (B.7) implies that it remains positive definite for sufficiently small real ε_1.[1] At the same time, as it follows from (B.6), (A.2) the torsion is real for pure imaginary ε_1. Therefore the expansion coefficients $h^{(n)}_{\lambda\mu\nu}$ turn out to be real as $\Im m(\varepsilon_2/\varepsilon_1) = 0$. However, $H_{\lambda\mu\nu}$ takes pure imaginary values for real ε_1 and ε_2, in particular for $0 < \varepsilon_1 < 1$, $0 < \varepsilon_2 < 1 - \varepsilon_1$. Notice that the case $\mathfrak{G} = SU(2)$ turns out to be somewhat special in that the torsion becomes zero identically [70]. The corresponding non-linear sigma model is equivalent to the model introduced by Fateev in ref.[68]. In the presence of non-vanishing torsion, the Lagrangian (2.41) is not invariant under the substitution $(t \pm x) \mapsto (t \mp x)$, i.e., the field theory is not P-invariant. However it is still invariant w.r.t. the special Lorentz transformation $(t \pm x) \mapsto \mathrm{e}^{\pm\theta}\,(t \pm x)$ with real θ.

Vielbeins

To clarify the special rôle of the 1-forms (B.2) for the Klimč´ik target space background let us make the following observations.

[1] Presumably the metric remains positive definite in the parameter domain $0 < \varepsilon_1 < 1\ ,\ 0 < \varepsilon_2 < 1 - \varepsilon_1$.

First we point out that the 1-forms $\boldsymbol{e}^a_\mu(+)$ are covariantly constant w.r.t. the spin-connection

$$\omega_{\nu,a}{}^b(+) \;=\; F_{ac}{}^b(+\,-\,|+)\,\boldsymbol{e}^c_\nu(-)\,,$$

i.e.,

$$\partial_\nu\,\boldsymbol{e}^a_\mu(+)\;-\;\Gamma^\lambda{}_{\mu\nu}\,\boldsymbol{e}^a_\lambda(+)+\omega_{\nu,b}{}^a(+)\,\boldsymbol{e}^b_\mu(+)=0\;. \tag{B.8}$$

A simple consequence of this fact is that the covariant derivative of the metric (B.4) is zero, as it should be. In a similar manner, the 1-forms $\boldsymbol{e}^a_\mu(-)$ satisfy the covariant constant condition

$$\partial_\nu\,\boldsymbol{e}^a_\mu(-)\;-\;\Gamma^\lambda{}_{\nu\mu}\,\boldsymbol{e}^a_\lambda(-)+\omega_{\nu,b}{}^a(-)\,\boldsymbol{e}^b_\mu(-)=0 \tag{B.9}$$

which involves another spin-connection

$$\omega_{\nu,a}{}^b(-) \;=\; F_{ac}{}^b(-\,+\,|-)\,\boldsymbol{e}^+_\nu(+)\,.$$

Finally, the covariantly constant 1-forms obey the Maurer-Cartan type equations:

$$\begin{aligned}
&\partial_{[\nu}\boldsymbol{e}^a_{\mu]}(+)-\tfrac{1}{2}\,\Big(q^{aa'}\,F_{a'bc}(+\,+\,|+)-\Theta^{aa'}\,F_{a'bc}(-\,+\,|+)\Big)\,\boldsymbol{e}^b_{[\nu}(+)\,\boldsymbol{e}^c_{\mu]}(+)=0\\
&\partial_{[\nu}\boldsymbol{e}^a_{\mu]}(-)-\tfrac{1}{2}\,\Big(q^{aa'}\,F_{a'bc}(-\,-\,|-)-\Theta^{a'a}\,F_{a'bc}(+\,-\,|-)\Big)\,\boldsymbol{e}^b_{[\nu}(-)\,\boldsymbol{e}^c_{\mu]}(-)=0
\end{aligned} \tag{B.10}$$

with

$$\Theta^{aa'}\;:\quad \boldsymbol{e}^a_\mu(+)=\Theta^a{}_b\,\boldsymbol{e}^b_\mu(-)\,,\qquad \Theta^{aa'}\;=\;\tfrac{1}{2\mathrm{g}^2}\,G^{\mu\nu}\,\boldsymbol{e}^a_\mu(+)\,\boldsymbol{e}^{a'}_\nu(-)\,,\qquad \Theta^a{}_c\,q^{cd}\,\Theta^b{}_d=q^{ab}\,.$$

Relations (B.8), (B.9) allow one to express the torsion in terms of $\boldsymbol{e}^a_\mu(\sigma)$. Namely, a simple calculation yields

$$\begin{aligned}
\Gamma_{\lambda\mu\nu}&=\tfrac{1}{2\mathrm{g}^2}\,q_{ab}\,\Big(\omega_{\nu,c}{}^a(+)\,\boldsymbol{e}^b_\lambda(+)\,\boldsymbol{e}^c_\mu(+)\;+\;\boldsymbol{e}^a_\lambda(+)\,\partial_\nu\,\boldsymbol{e}^b_\mu(+)\Big)\\
\Gamma_{\lambda\mu\nu}&=\tfrac{1}{2\mathrm{g}^2}\,q_{ab}\,\Big(\omega_{\mu,c}{}^a(-)\,\boldsymbol{e}^b_\lambda(-)\,\boldsymbol{e}^c_\nu(-)\;+\;\boldsymbol{e}^a_\lambda(-)\,\partial_\mu\,\boldsymbol{e}^b_\nu(-)\Big)\,.
\end{aligned} \tag{B.11}$$

These formulae, combined with (B.1) imply

$$H_{\lambda\mu\nu} \;=\; \tfrac{1}{2\mathrm{g}^2}\,\sigma\, q_{ab}\,\Big(\, \boldsymbol{e}^a_\lambda(\sigma)\,\big(\omega_{\nu,c}{}^b(\sigma)\,\boldsymbol{e}^c_\mu(\sigma) - \omega_{\mu,c}{}^b(\sigma)\,\boldsymbol{e}^c_\nu(\sigma)\big) \;+\; \boldsymbol{e}^a_\lambda(\sigma)\,\big(\,\partial_\nu\,\boldsymbol{e}^b_\mu(\sigma) - \partial_\mu\,\boldsymbol{e}^b_\nu(\sigma)\big)\Big)\,.$$

In the case under consideration, the torsion is a 3-form and the more elegant expressions (B.6) can be achieved by anti-symmetrizing w.r.t. the Greek indices and using the formula

$$q_{ab}\,\boldsymbol{e}^a_{[\lambda}(\sigma)\,\partial_\mu\,\boldsymbol{e}^b_{\nu]}(\sigma) - \tfrac{1}{2}\,\textstyle\sum_{\sigma'=\pm} F_{abc}(\sigma\sigma|\sigma')\;\boldsymbol{e}^a_{[\lambda}(\sigma)\,\boldsymbol{e}^b_\mu(\sigma)\,\boldsymbol{e}^c_{\nu]}(\sigma') = 0$$

valid for both choices of $\sigma = \pm$. The later is an immediate consequence of the Maurer-Cartan structure equations (B.10).

Formulae (B.4) and (B.6) can be made more transparent using the notation $\tilde F_{abc}(\sigma\,\sigma'\,\sigma'')$:

$$F_{abc}(\sigma\,\sigma'|\sigma'') \;=\; \mathrm{e}^{\frac{\mathrm{i}\pi}{4}(\sigma+\sigma'-\sigma'')}\,\tilde F_{abc}(\sigma\,\sigma'\,\sigma'')\ .$$

The advantage of $\tilde F_{abc}(\sigma\,\sigma'\,\sigma'')$ compared to $F_{abc}(\sigma\,\sigma'|\sigma'')$ is that it is a completely antisymmetric symbol w.r.t. the pair permutations $(a,\sigma)\leftrightarrow(b,\sigma')$ and $(b,\sigma')\leftrightarrow(c,\sigma'')$:

$$\tilde F_{abc}(\sigma\,\sigma'\,\sigma'') = -\tilde F_{bac}(\sigma'\,\sigma\,\sigma'') = -\tilde F_{acb}(\sigma\,\sigma''\,\sigma')\ .$$

Then (B.4), (B.6) can be re-written as

$$\begin{aligned} G_{\mu\nu} &= \frac{\mathrm{i}}{4\mathrm{g}^2}\sum_{\sigma=\pm}\sigma\, q_{ab}\;\mathcal{E}^a_\lambda(\sigma)\,\mathcal{E}^b_\mu(\sigma) \\ H_{\lambda\mu\nu} &= \frac{1}{4\mathrm{g}^2}\sum_{\sigma,\sigma',\sigma''=\pm}\mathrm{sgn}(\sigma+\sigma'+\sigma'')\,\tilde F_{abc}(\sigma\,\sigma'\,\sigma'')\;\mathcal{E}^a_\lambda(\sigma)\,\mathcal{E}^b_\mu(\sigma')\,\mathcal{E}^c_\nu(\sigma'')\,, \end{aligned}$$

where we also use

$$\mathcal{E}^a_\mu(\sigma)\;\equiv\;\mathrm{e}^{-\frac{\mathrm{i}\pi}{4}\sigma}\;\boldsymbol{e}^a_\mu(\sigma)\ .$$

Ricci tensor

Let $\mathsf{R}_{\mu\nu}$ be the Ricci tensor built from the affine connection Γ (B.11). For practical purposes, it is useful to express it in terms of the symmetric Ricci tensor $R_{\mu\nu}$ associated with the Levi-Civita connection.[2] Using the results from the work [73] one can show that

$$\begin{aligned}
\tfrac{1}{2}\,\mathsf{R}_{(\mu\nu)} &= R_{\mu\nu} - \tfrac{1}{4}\,H_\mu{}^{\sigma\rho}H_{\sigma\rho\nu} = \tfrac{1}{8}\big(1-(\varepsilon_1-\varepsilon_2)^2\big)\big(1-(\varepsilon_1+\varepsilon_2)^2\big)\sum_{\sigma=\pm} q_{ab}\,\boldsymbol{e}^a_\mu(\sigma)\,\boldsymbol{e}^b_\nu(-\sigma)\\
&- \nabla_\mu W_\nu - \nabla_\nu W_\mu \qquad\qquad (\text{B.12})\\
\tfrac{1}{2}\,\mathsf{R}_{[\mu\nu]} &= \tfrac{1}{2}\,\nabla_\lambda H^\lambda{}_{\mu\nu} = \tfrac{1}{8}\big(1-(\varepsilon_1-\varepsilon_2)^2\big)\big(1-(\varepsilon_1+\varepsilon_2)^2\big)\sum_{\sigma=\pm} q_{ab}\,\sigma\,\boldsymbol{e}^a_\mu(\sigma)\,\boldsymbol{e}^b_\nu(-\sigma)\\
&+ W_\lambda\,H^\lambda{}_{\mu\nu} + \partial_\mu W_\nu - \partial_\nu W_\mu\ .
\end{aligned}$$

Here

$$W_\mu = -\tfrac{1}{2}\,\partial_\mu \log\big(\det\hat{\boldsymbol{\Omega}}_\sigma\big) + w_\mu \qquad (\text{B.13})$$

with $\boldsymbol{\Omega}_\sigma$ given by (B.3) and

$$w_\mu = \pm\tfrac{\mathrm{i}}{4}\ \boldsymbol{e}^a_\mu(\pm)\ f_{ab}{}^c\ (\varepsilon_1\,\bar{\mathcal{R}} - \varepsilon_2\,\mathcal{R})^b{}_c\ .$$

The last formula holds true for any choice of the sign $\pm$ and we use the notation

$$\bar{\mathcal{R}}^b{}_c = (\mathcal{U}^{-1}\mathcal{R}\,\mathcal{U})^b{}_c = (\mathcal{U}^{-1})^b{}_{b'}\,\mathcal{R}^{b'}{}_{c'}\,\mathcal{U}^{c'}{}_c\ ,$$

where $\mathcal{U}^b{}_a$ stands for the $D\times D$ matrix of the group element $\boldsymbol{U}$ in the adjoint representation:

$$\boldsymbol{U}\,\mathtt{t}_a\,\boldsymbol{U}^{-1} = \mathtt{t}_b\,\mathcal{U}^b{}_a\ .$$

[2]Below, the Ricci tensor is defined as $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}$ where $R^\rho{}_{\lambda\mu\nu}$ is the Riemann tensor

$$R^\lambda{}_{\mu\rho\nu} = \partial_\rho\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\rho} + \Gamma^\lambda{}_{\sigma\rho}\Gamma^\sigma{}_{\mu\nu} - \Gamma^\lambda{}_{\sigma\nu}\Gamma^\sigma{}_{\mu\rho}$$

and $\Gamma^\sigma_{\mu\nu} = \Gamma^\sigma_{\nu\mu}$ stands for the Christoffel symbols for the Levi-Civita connection.

1-loop renormalization of the Klimč´ik NLSM

In the path-integral quantization, the general NLSM (2.41) should be equipped with a UV cutoff. A consistent removal of the UV divergences requires that the "bare" target space metric and torsion potential be given a certain dependence on the cutoff momentum Λ. To the first perturbative order in the Planck constant $\hbar$ the RG flow equations are given by [65, 71, 72]

$$\begin{aligned}
\partial_\tau G_{\mu\nu} &= -\hbar\Big(R_{\mu\nu} - \frac{1}{4}\, H_\mu{}^{\sigma\rho} H_{\sigma\rho\nu} + \nabla_\mu V_\nu + \nabla_\nu V_\mu\Big) + O(\hbar^2) \\
\partial_\tau B_{\mu\nu} &= -\hbar\Big(-\frac{1}{2}\, \nabla_\lambda H^\lambda{}_{\mu\nu} + V_\lambda\, H^\lambda{}_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu\Big) + O(\hbar^2)\ ,
\end{aligned} \tag{B.14}$$

where $\partial_\tau \equiv 2\pi\,\Lambda\,\frac{\partial}{\partial\Lambda}$. The infinitesimal variation of the Klimčík metric and torsion potential, assuming that the combinations of the couplings $\frac{\varepsilon_2}{\varepsilon_1}$, $\mathrm{g}^2\varepsilon_1$ are kept fixed, can be expressed as

$$\begin{aligned}
\delta G_{\mu\nu} &= +\frac{\delta\varepsilon_1}{4\mathrm{g}^2\varepsilon_1} \sum_{\sigma=\pm} q_{ab}\, \boldsymbol{e}^a_\mu(\sigma)\boldsymbol{e}^b_\nu(-\sigma) \\
\delta B_{\mu\nu} &= -\frac{\delta\varepsilon_1}{4\mathrm{g}^2\varepsilon_1} \sum_{\sigma=\pm} q_{ab}\, \sigma\, \boldsymbol{e}^a_\mu(\sigma)\boldsymbol{e}^b_\nu(-\sigma)\ .
\end{aligned}$$

With the explicit formulae for the Ricci tensor (B.12), it is easy to see that the general RG flow equations (B.14) are satisfied if $V_\mu = \Lambda_\mu = W_\mu$ with W_μ given by (B.13). Also it follows that the evolution of the bare couplings under a change in Λ is described by the system of ordinary differential equations (2.46).

Appendix C

In this Appendix we provide the explicit relation between the flat connection (3.15) for the case of the Fateev model ($\mathfrak{G} = SU(2)$) and that given in the work [75].

In that work a more general four parameter deformation of the $SU(2)$ principal chiral field is considered which contains the 3D sausage as a two-parameter subfamily. The deformation parameters were denoted by $(\eta, \nu^{(L)}, \sigma, q)$ and, for the case of the Fateev model, $\nu^{(L)}$ together with σ should be set to zero:

$$\nu^{(L)} = \sigma = 0\,.$$

Here the superscript L has been used to distinguish the parameter ν in ref.[75] with the one from this work. The remaining two parameters η and q are related to κ and ν in (2.49) as

$$\kappa = \frac{\vartheta_2^2(0,q^2)}{\vartheta_3^2(0,q^2)}\,, \qquad\qquad \nu = -\mathrm{i}\,\frac{\vartheta_1(\mathrm{i}\eta,q^2)}{\vartheta_4(\mathrm{i}\eta,q^2)}\,,$$

where ϑ_a stand for the conventional theta functions. In ref.[75] the co-ordinates v and w that appear in the Euler decomposition (3.33)-(3.34) are used, while ϕ from (3.34) is replaced by u, such that

$$\tanh(\phi) = \frac{\vartheta_2(u,q^2)\vartheta_3(0,q^2)}{\vartheta_3(u,q^2)\vartheta_2(0,q^2)} \qquad\qquad (0<u<\pi)\ .$$

The flat connection $\boldsymbol{A}_{\pm}^{(L)}$ found in [75] is defined by eqs. (1.6), (2.7) and (2.10)-(2.14) from that work, where λ is the spectral parameter and, for the 3D sausage, $\eta_+ = \eta_- = \eta$ and $\phi_\pm = 0$. Formulae (2.7), (2.10) involve the vielbein e^a_μ ($\mu = u, v, w$), which in turn are given by eqs. (2.28)-(2.32). Here, for the convenience of the reader, we reproduce the main equations needed for the computation of $\boldsymbol{A}_{\pm}^{(L)}$ specialized to the 3D sausage.

The non-vanishing components of the vielbein are given by

$$\begin{aligned} e^3_u &= \frac{\mathrm{i}}{\mathrm{g}} \frac{\vartheta_2(\mathrm{i}\eta, q)\,\vartheta_1'(0,q)}{\vartheta_1(\mathrm{i}\eta,q)\,\vartheta_2(0,q)} \\ e^{\pm}_v &= \mp\frac{\mathrm{i}}{\mathrm{g}} \frac{\vartheta_4(0,q^2)\,\vartheta_4(\mathrm{i}\eta \pm u, q^2)}{\vartheta_4(u,q^2)\,\vartheta_4(\mathrm{i}\eta,q^2)} \\ e^{\pm}_w &= \pm\frac{\mathrm{i}}{\mathrm{g}} \frac{\vartheta_4(0,q^2)\,\vartheta_1(\mathrm{i}\eta \pm u, q^2)}{\vartheta_4(u,q^2)\,\vartheta_1(\mathrm{i}\eta,q^2)} \;. \end{aligned}$$

Note that, with these expressions at hand, it is simple to re-write the Lagrangian of the 3D sausage in terms of the parameters (η, q) and the co-ordinates $X^\mu = (u, v, w)$ since

$$\mathcal{L}_F = 2\,G_{\mu\nu}\,\partial_+ X^\mu \partial_- X^\nu$$

and the non-zero components of the metric tensor $G_{\mu\nu}$ are

$$G_{uu} = (e^3_u)^2\,, \qquad G_{vv} = e^+_v e^-_v\,, \qquad G_{ww} = e^+_w e^-_w\,, \qquad G_{vw} = \tfrac{1}{2}\,(e^+_v\, e^-_w + e^-_v\, e^+_w)\,.$$

The connection is constructed from the matrix valued 1-form $\boldsymbol{\zeta}_\mu(\lambda)$ defined by

$$\boldsymbol{\zeta}_\mu(\lambda) = f_3(\lambda)\, e^3_\mu\, \sigma^3 + f_+(\lambda)\, e^+_\mu\, \sigma^- + f_-(\lambda)\, e^-_\mu\, \sigma^+\,,$$

where σ^3 and $\sigma^\pm = \frac{1}{2}(\sigma^1 \pm \mathrm{i}\sigma^2)$ are the standard Pauli matrices, while

$$\begin{aligned} f_+(\lambda) &= -f_-(-\lambda) = -\frac{\mathrm{g}}{2} \frac{\vartheta_1(u-\frac{\lambda}{2},q)\,\vartheta_1(\mathrm{i}\eta,q)\,\vartheta_2(0,q)}{\vartheta_1(u,q)\,\vartheta_2(\mathrm{i}\eta,q)\,\vartheta_1(\frac{\lambda}{2},q)} \\ f_3(\lambda) &= -\frac{\mathrm{g}}{2} \frac{\vartheta_1(\mathrm{i}\eta,q)\,\vartheta_2(0,q)\,\vartheta_1'(\frac{\lambda}{2},q)}{\vartheta_2(\mathrm{i}\eta,q)\,\vartheta_1'(0,q)\,\vartheta_1(\frac{\lambda}{2},q)} \;. \end{aligned}$$

In terms of this 1-form, the connection components $\boldsymbol{A}_{\pm}^{(L)}$ are expressed as

$$\begin{aligned} \boldsymbol{A}_+^{(L)} &= \frac{1}{2\mathrm{i}} \sum_\mu \big(\boldsymbol{\zeta}_\mu(\mathrm{i}\eta+\lambda) + \sigma^2\, \boldsymbol{\zeta}_\mu(\mathrm{i}\eta-\lambda)\,\sigma^2\big)\,\partial_+ X^\mu \\ \boldsymbol{A}_-^{(L)} &= \frac{1}{2\mathrm{i}} \sum_\mu \big(\boldsymbol{\zeta}_\mu(\mathrm{i}\eta+\lambda-\pi) + \sigma^2\, \boldsymbol{\zeta}_\mu(\mathrm{i}\eta-\lambda+\pi)\,\sigma^2\big)\,\partial_- X^\mu\,, \end{aligned}$$

where $X^\mu = (u, v, w)$. One should keep in mind that the zero curvature representation in [75] is

$$\left[\partial_+ + \boldsymbol{A}_+^{(L)},\, \partial_- + \boldsymbol{A}_-^{(L)}\right] = 0\,,$$

which differs from the convention used in this work (3.1) by the overall sign of $\boldsymbol{A}_\pm$.

The gauge transformation that maps the flat connection $\boldsymbol{A}_\pm^{(L)}$ to the one in (3.15), (3.13) with $\boldsymbol{U}$ understood as a matrix in the fundamental representation of $SU(2)$ (i.e., $\mathtt{h} = \sigma^3$, $\mathtt{e}_\pm = \sigma^\pm$), is described as follows:

$$\partial_\pm - \boldsymbol{A}_\pm \;=\; \boldsymbol{S}\left(\partial_\pm + \boldsymbol{A}_\pm^{(L)}\right)\boldsymbol{S}^{-1}\,,$$

where

$$\boldsymbol{S} \;=\; \sqrt{\frac{\vartheta_4(\lambda, q^2)\vartheta_4(0, q^2)}{2\vartheta_1(\lambda, q^2)\vartheta_4(u, q^2)}}\begin{pmatrix} \mathrm{e}^{\frac{\mathrm{i}w}{2}}\,\frac{\vartheta_2(\frac{1}{2}(\lambda-u),q)}{\vartheta_3(\frac{\lambda}{2},q)} & \mathrm{i}\,\mathrm{e}^{\frac{\mathrm{i}w}{2}}\,\frac{\vartheta_2(\frac{1}{2}(\lambda+u),q)}{\vartheta_3(\frac{\lambda}{2},q)} \\ \mathrm{i}\,\mathrm{e}^{-\frac{\mathrm{i}w}{2}}\,\frac{\vartheta_1(\frac{1}{2}(\lambda-u),q)}{\vartheta_4(\frac{\lambda}{2},q)} & \mathrm{e}^{-\frac{\mathrm{i}w}{2}}\,\frac{\vartheta_1(\frac{1}{2}(\lambda+u),q)}{\vartheta_4(\frac{\lambda}{2},q)} \end{pmatrix}$$

and $\boldsymbol{S}^{-1} = \sigma_2\,\boldsymbol{S}^T\sigma_2$ ($\det\boldsymbol{S} = 1$). The parameters $\rho_\pm$ are expressed in terms of the spectral parameter λ as

$$\frac{\rho_+}{\rho_-} = \frac{\vartheta_3^2(\frac{\lambda}{2}, q)}{\vartheta_4^2(\frac{\lambda}{2}, q)}\,, \qquad\qquad \rho_+\rho_- = \frac{\vartheta_4^2(\frac{\mathrm{i}\eta}{2}, q)}{\vartheta_3^2(\frac{\mathrm{i}\eta}{2}, q)}\,.$$

Finally note that m_1, m_2 which appear in eq. (A.3) can be elegantly written using q and η

$$m_1 = -\mathrm{i}\,\frac{\vartheta_1(\frac{\mathrm{i}\eta}{2}, q^2)\vartheta_2(\frac{\mathrm{i}\eta}{2}, q^2)}{\vartheta_3(\frac{\mathrm{i}\eta}{2}, q^2)\vartheta_4(\frac{\mathrm{i}\eta}{2}, q^2)}\,, \qquad m_2 = -\mathrm{i}\,\frac{\vartheta_1(\frac{\mathrm{i}\eta}{2}, q^2)\vartheta_3(\frac{\mathrm{i}\eta}{2}, q^2)}{\vartheta_2(\frac{\mathrm{i}\eta}{2}, q^2)\vartheta_4(\frac{\mathrm{i}\eta}{2}, q^2)}\,,$$

while

$$\varepsilon_1 = \frac{\vartheta_4^2(\mathrm{i}\eta, q^2)\,\vartheta_3(0, q^2)\,\vartheta_2(0, q^2)}{\vartheta_4^2(0, q^2)\,\vartheta_3(\mathrm{i}\eta, q^2)\,\vartheta_2(\mathrm{i}\eta, q^2)}\,, \qquad \varepsilon_2 = -\frac{\vartheta_1^2(\mathrm{i}\eta, q^2)\,\vartheta_3(0, q^2)\,\vartheta_2(0, q^2)}{\vartheta_4^2(0, q^2)\,\vartheta_3(\mathrm{i}\eta, q^2)\,\vartheta_2(\mathrm{i}\eta, q^2)}\,.$$

Appendix D

To investigate the scaling behaviour of $\mathcal{T}^{(N)}(\mu)$ (6.40)-(6.45), we conducted numerical work for integer n when the discretized operator is a finite dimensional matrix that can be diagonalized by means of the Bethe ansatz (see Appendix E for details). We focused only on the vacuum eigenvalue in the sector $\mathcal{H}^{(N)}_{\mathfrak{j}-\frac{\mathfrak{m}}{2}}$ and considered the cases with $n = 2, 3, \ldots, 6$ and all admissible values of $\mathfrak{j}, \mathfrak{m}$ (6.14). Let $\tau^{(\mathrm{vac})}(\lambda)$ be the vacuum eigenvalue of the chiral transfer-matrix in the parafermionic subspace $\mathcal{V}^{(\mathfrak{m})}_{\mathfrak{j}}$. We expect that it can be obtained from the vacuum eigenvalue of $\mathcal{T}^{(N)}(\mu)$ by using the formula (6.51) which explicitly describes the scaling limit of the discretized operator. To estimate numerical values of $\tau^{(\mathrm{vac})}(\lambda)$ we used data obtained for a set of finite N and then performed a certain interpolation procedure to $N = \infty$. The results were compared with predictions coming from the properties of $\tau^{(\mathrm{vac})}(\lambda)$ discussed in sec. 5.2, specialized to the values $p_1 = \frac{\mathrm{i}}{2}\,\mathfrak{m}$ and $p_2 = \mathfrak{j} + \frac{1}{2}$. Agreement was found in all cases considered. In this appendix, some of our numerical work is presented.

Let $\{u_l\}_{l=1}^{\infty}$ be the set of zeroes of $\tau^{(\mathrm{vac})}(\lambda)$ considered as a function of λ^2. From the numerical data it was found that all the zeroes are simple, real, positive, and

n	$\tilde{t}_2$	n	$\tilde{t}_2$
1	0	6	0.0658731
2	$\frac{\sqrt{2}}{48}$	7	0.0613178
3	0.0546105	8	0.0561029
4	0.0661040	9	0.0509101
5	0.0683646	10	0.0460445

Table D.1: Numerical values of $\tilde{t}_2$ for $\mathfrak{j} = \mathfrak{m} = 0$ (from ref. [115]).

root #	$N = 501$	$N = 1001$	$N = 1500$	$N = 2600$	$N = \infty$	$2u_l^{\frac{n}{2}}$
1	0.4818860	0.4818829	0.4818820	0.4818814	0.4818809	0.47349
2	1.4891566	1.4891424	1.4891392	1.4891372	1.4891359	1.48725
3	2.4919329	2.4918863	2.4918769	2.4918715	2.491868	2.49093
4	3.4935044	3.4933890	3.4933666	3.4933541	3.493348	3.49276
5	4.4946127	4.4943769	4.4943321	4.4943074	4.494295	4.49387
6	5.4955294	5.4951073	5.4950277	5.4949844	5.494962	5.49464
7	6.4963870	6.4956974	6.4955682	6.4954981	6.495463	6.49521
8	7.4972634	7.4962107	7.4960140	7.4959077	7.495854	7.49564
9	8.4982121	8.4966857	8.4964010	8.4962476	8.496171	8.49599
10	9.4992734	9.4971480	9.4967523	9.4965392	9.496432	9.49628
11	10.500481	10.497616	10.497084	10.496797	10.49665	10.49652
12	11.501864	11.498105	11.497406	11.497031	11.49684	11.49672
13	12.503448	12.498625	12.497729	12.497248	12.49701	12.49690
14	13.505258	13.499187	13.498060	13.497455	13.49715	13.49705
15	14.507318	14.499799	14.498404	14.497655	14.49728	14.49719

Table D.2: Numerical values of $\frac{2N}{\pi}\big[\mu_l^{(N)}\big]^n$, where $\mu_l^{(N)}>0$ are the roots of the vacuum eigenvalue of the discretized operator $\mathcal{T}^{(N)}(\mu)$ for $n=4$, $\mathfrak{j}=\mathfrak{m}=0$. The column "$N=\infty$" was obtained by interpolating the results for finite N. The entries in the last column were calculated by using the asymptotic formula (D.2) truncated at the first non-zero term in the series.

accumulate towards $\lambda^2 = \infty$ with the leading asymptotic behaviour

$$u_l \sim \left(\tfrac{1}{2}\right)^{\frac{2}{n}} \times \begin{cases} \left(l-\frac{1}{2}\right)^{\frac{2}{n}} & \text{for} \quad 0\leq 2\mathfrak{j} < \frac{n}{2} \\ \left(l-\frac{1}{2}+\frac{n}{n+2}\right)^{\frac{2}{n}} & \text{for} \quad 2\mathfrak{j} = \frac{n}{2} \qquad (n-\text{even}) \end{cases} \tag{D.1}$$

For $0\leq 2\mathfrak{j}<\frac{n}{2}$, this is consistent with the asymptotically exact formula,

$$u_l^{\frac{n}{2}} + \frac{1}{2\pi}\sum_{m=1}^{\infty} \tilde{g}_m\big(\tfrac{\mathrm{i}}{2}\,\mathfrak{m}, \mathfrak{j}+\tfrac{1}{2}\big)\ \sin\left(\tfrac{2\pi m}{n+2}\right)\ u_l^{-\frac{nm}{n+2}} \asymp \tfrac{1}{2}\left(l-\tfrac{1}{2}\right)\,, \tag{D.2}$$

which can be easily derived from eqs. (5.39)-(5.40). Knowledge of the coefficients $\tilde{g}_m$ allows us to compute systematic corrections to the leading asymptotic behaviour (D.1). As it follows from eq. (5.38), the first coefficient is

$$\tilde{g}_1(p_1,p_2) = \frac{\tilde{t}_1(p_1,p_2)}{2\,\cos\left(\frac{2\pi p_2}{n+2}\right)}\,, \tag{D.3}$$

with $\tilde{t}_1(p_1,p_2)$ – vacuum eigenvalue of $\tilde{\mathfrak{t}}_1$ – given by eq. (5.35). Notice that for $p_2=\mathfrak{j}+\frac{1}{2}=\frac{n+2}{4}$ (n-even), the denominator in (D.3) is zero so that (D.2) is no longer

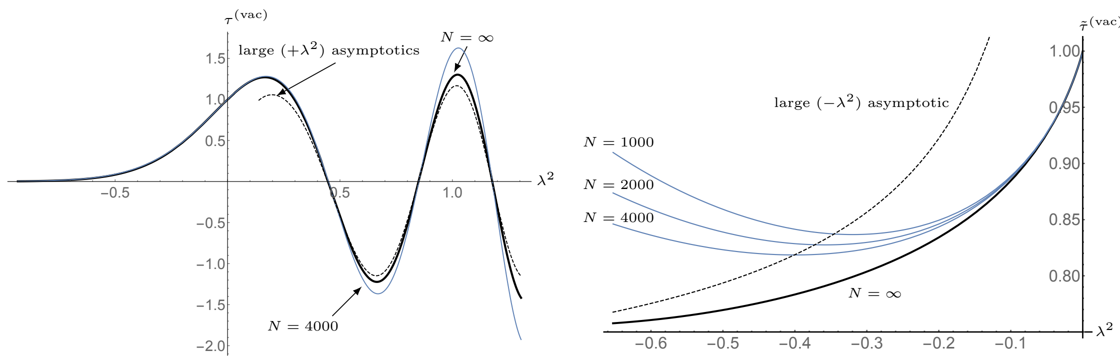


Figure D.1: On the left panel, a plot of $\tau^{(\mathrm{vac})}$ for $n = 3$, $2\mathfrak{j} = \mathfrak{m} = 1$ compared to its large $(+\lambda^2)$ asymptotic following from eq. (5.39). On the right panel, $\tilde{\tau}^{(\mathrm{vac})} = \tau^{(\mathrm{vac})} \exp\left(2\pi\,(-\lambda^2)^{\frac{3}{2}}\right)$ is plotted and compared with the large $(-\lambda^2)$ asymptotic derived from eqs. (5.29), (5.30). The scaling function was numerically estimated by interpolating to $N = \infty$ the data for $N = 500, 1000, 2000, 4000$.

valid. Also when $\mathfrak{j} = \mathfrak{m} = 0$, $\tilde{g}_1$ vanishes, but for this case the second term in the sum in (D.2) is known, since

$$\tilde{g}_2(0, \tfrac{1}{2}) = \frac{\tilde{t}_2(0, \frac{1}{2})}{2\,\cos\left(\frac{\pi}{n+2}\right)}$$

and numerical values of $\tilde{t}_2(0, \frac{1}{2})$ were calculated in ref.[115] and are reproduced in tab. D.1. Truncating the series in (D.2) at the first non vanishing term, we calculated the corrections to the leading asymptotic (D.1). This was compared to the zeroes of the vacuum eigenvalue of $\mathcal{T}^{(N)}(\mu)$ for increasing N. In all cases good agreement was observed. As an example, in tab. D.2 the results for $n = 4$, $\mathfrak{j} = \mathfrak{m} = 0$ are shown.

As $\lambda^2 \to -\infty$, the asymptotic behaviour of $\tau^{(\mathrm{vac})}$ is dictated by eqs. (5.29), (5.30). Truncating the sum in (5.30) at the first non-zero term and substituting $\tilde{\mathfrak{t}}_j$ by its vacuum eigenvalue, we compared this to the results of the $N = \infty$ interpolation. The agreement was good considering that the interpolation procedure becomes rapidly less efficient for increasing values of $(-\lambda^2)$. Fig. D.1 shows a plot of the estimated scaling function versus the asymptotics for $n = 3$ and $2\mathfrak{j} = \mathfrak{m} = 1$.

Another check that can be made is to consider the Taylor expansion of $\tau^{(\mathrm{vac})}(\lambda)$ at zero following from formulae (5.28) and (5.34). The coefficient $t_1(p_1, p_2)$ $(p_1 =$

$\frac{\mathrm{i}}{2}\,\mathfrak{m}, p_2 = \mathfrak{j}+\frac{1}{2})$ can be compared to the corresponding term in the vacuum eigenvalue of the discretized operator:

$$\mathcal{T}^{(N,\text{vac})}(\mu) = 2\cos\left(\tfrac{\mathfrak{m}\pi}{n}\right) + t_1^{(N)}\,\mu^2 + O(\mu^4)\,.$$

Note that $t_1^{(N)}$ is a divergent quantity for large N and must be regularized. According to eq. (6.51), for $n > 2$, the following limit exists and converges to t_1:

$$t_1\big(\tfrac{\mathrm{i}}{2}\,\mathfrak{m}, \mathfrak{j}+\tfrac{1}{2}\big) = \lim_{N\to\infty} t_1^{(N,\text{reg})}\,, \qquad t_1^{(N,\text{reg})} = \left(\tfrac{\pi}{N}\right)^{\frac{2}{n}} \left(t_1^{(N)} + 2N\,\frac{\cos(\frac{\mathfrak{m}\pi}{n})}{\cos(\frac{\pi}{n})}\right)\,. \tag{D.4}$$

We compared the value of $t_1\big(\frac{\mathrm{i}}{2}\,\mathfrak{m}, \mathfrak{j}+\frac{1}{2}\big)$ given by eq. (5.34) to the numerical values of $t_1^{(N,\text{reg})}$ and found good agreement for $n = 3, 4 \ldots, 6$ and all the allowed values of $\mathfrak{j}, \mathfrak{m}$. A few cases are presented in tab. D.3.

Finally, let us mention that for $n = 2$, analytic expressions exist for both $\tau^{(\text{vac})}$ and the vacuum eigenvalue of $\mathcal{T}^{(N)}$. In the case $\mathfrak{j} = \mathfrak{m} = 0$,

$$\mathcal{T}^{(N,\text{vac})}(\mu) = 2\prod_{m=1}^{N}\Big(1 - \mu^2\cot\left(\tfrac{\pi}{2N}\,(m-\tfrac{1}{2})\right)\Big)\,,$$

and using the formula (6.51), the scaling limit can be taken explicitly to yield

$$\tau^{(\text{vac})}(\lambda) = \left(\frac{\mathrm{e}}{2}\right)^{2\lambda^2} \frac{2\sqrt{\pi}}{\Gamma\big(\frac{1}{2} - 2\lambda^2\big)}\,.$$

It is easy to verify that this is consistent with the properties of the chiral transfer-matrix discussed in chapter. 5.2. For $n = 2$ and $2\mathfrak{j} = \mathfrak{m} = 1$, the discretized operator turns out to be zero for any N and hence, $\tau(\lambda) = 0$.

$n=6$	$N=100$	$N=200$	$N=400$	$N=800$	$N=\infty$	eq. (5.34)
$2\mathfrak{j}=\mathfrak{m}=0$	0.54474	0.54519	0.54542	0.54553	0.5456440	0.5456445
$2\mathfrak{j}=\mathfrak{m}=2$	0.43807	0.44710	0.45357	0.45818	0.469649	0.469446
$n=5$	$N=100$	$N=200$	$N=400$	$N=750$	$N=\infty$	eq. (5.34)
$2\mathfrak{j}=\mathfrak{m}=0$	0.86236	0.86271	0.86287	0.86294	0.8630048	0.8630049
$2\mathfrak{j}=\mathfrak{m}=2$	0.40173	0.40751	0.41144	0.41390	0.419808	0.419632

Table D.3: The regularized value $t_1^{(N,\text{reg})}$ (D.4) for a variety of cases and increasing N compared to the expression for $t_1\big(\frac{\mathrm{i}}{2}\,\mathfrak{m}, \mathfrak{j}+\frac{1}{2}\big)$ given by eq. (5.34). The column "$N=\infty$" was obtained by interpolation.

Appendix E

In this appendix we will consider the vacuum eigenvalue of the matrices $\mathcal{Z}_\pm(\mu)$ in the space $\mathcal{H}^{(N)}_{\mathsf{j}-\frac{\mathfrak{m}}{2}}$. Recall that $\mathcal{H}^{(N)}_{\mathsf{j}-\frac{\mathfrak{m}}{2}}$ denotes the eigenspace of the matrix $\pi_{\mathcal{H}^{(N)}}(\mathsf{Z})$ (6.42), (6.46) having eigenvalue $\omega^{\mathsf{j}-\frac{\mathfrak{m}}{2}}$, where j and $\mathfrak{m}$ are restricted as in (6.14). Our considerations are entirely based on the properties of $\mathcal{Z}_\pm(\mu)$ (i)-(v) listed in sec. 6.4.

Let $\mathcal{Z}^{(\psi)}_\pm(\mu)$ be the eigenvalue corresponding to a common eigenvector $|\psi\rangle$ of the commuting family $\mathcal{Z}_\pm(\mu)$. Using the analytical conditions (iv) and $\mu\to-\mu$ symmetry (v), it can be written in the form,

$$\mathcal{Z}^{(\psi)}_\pm(\mu) \;=\; B^{(N,\psi)}\;\mu^{\mathfrak{m}}\prod_{i=1}^{(n-1)N-2\mathsf{j}-\mathfrak{m}}\Big(1\mp\frac{\mu}{\mu_i}\Big)\qquad (n-\text{odd}) \tag{E.1}$$

and

$$\begin{aligned}\mathcal{Z}^{(\psi)}_+(\mu) \;&=\; B^{(N,\psi)}\;\mu^{\mathfrak{m}}\prod_{i=1}^{\frac{nN}{2}-\mathsf{j}-\frac{\mathfrak{m}}{2}}\Big(1-\frac{\mu^2}{v_i}\Big)\\ \mathcal{Z}^{(\psi)}_-(\mu) \;&=\; B^{(N,\psi)}\;\mu^{\mathfrak{m}}\prod_{i=1}^{\frac{(n-2)N}{2}-\mathsf{j}-\frac{\mathfrak{m}}{2}}\Big(1-\frac{\mu^2}{w_i}\Big)\end{aligned}\qquad (n-\text{ even}) \tag{E.2}$$

From the $T-Q$ type relations (iii), it follows that the overall coefficient $B^{(N,\psi)}$ (depending on the state $|\psi\rangle$) is the same for both $\mathcal{Z}^{(\psi)}_+$ and $\mathcal{Z}^{(\psi)}_-$. Another consequence of this relation is that the roots satisfy the following Bethe ansatz equations:

$$\prod_{i=1}^{(n-1)N-2\mathsf{j}-\mathfrak{m}}\frac{\mu_i+q^{-1}\,\mu_l}{\mu_i+q^{+1}\,\mu_l} \;=\; -q^{2\mathfrak{m}}\left(\frac{1-q^{+\frac{1}{2}}\,\mu_l}{1-q^{-\frac{1}{2}}\,\mu_l}\right)^{2N}\qquad (n-\text{odd}) \tag{E.3}$$

and

$$\prod_{i=1}^{\frac{nN}{2}-\mathfrak{j}-\frac{\mathfrak{m}}{2}} \frac{v_i - q^{-2}\, w_l}{v_i - q^{+2}\, w_l} \;=\; -q^{2\mathfrak{m}}$$

$$\prod_{i=1}^{\frac{(n-2)N}{2}-\mathfrak{j}-\frac{\mathfrak{m}}{2}} \frac{w_i - q^{-2}\, v_l}{w_i - q^{+2}\, v_l} \;=\; -q^{2\mathfrak{m}} \left(\frac{1-q^{+1}\, v_l}{1-q^{-1}\, v_l}\right)^{2N} \qquad (n - \text{even}) \qquad \text{(E.4)}$$

Similar equations for the Fateev-Zamolodchikov spin chain (6.48) with periodic boundary conditions were previously derived in the works [177] and [178] for odd and even n, respectively. Notice that the constant $B^{(N,\psi)}$ in (E.1), (E.2) is determined (up to an overall sign) by the quantum Wronskian type relations (ii).

The Bethe ansatz equations are valid for all integer $n \geq 2$ and $\mathfrak{j}, \mathfrak{m}$ restricted to (6.14), except for $2\mathfrak{j} = \mathfrak{m} = \frac{n}{2}$ (n even) which requires special attention. In this case, for certain sectors of $\mathcal{H}_0^{(N)}$ a significant simplification occurs; $\mathcal{Z}_-^{(\psi)}$ vanishes so that the $T-Q$ type relations (iii) become trivial and the quantum Wronskian type relations (ii) can be used to obtain much simpler equations for the roots. For instance, for the vacuum eigenvalue, $\mathcal{Z}_-^{(\mathrm{vac})}(\mu) = 0$ and $\mathcal{Z}_+^{(\mathrm{vac})}$ is given explicitly by

$$\mathcal{Z}_+^{(\mathrm{vac})}(\mu) = 2\sqrt{N}\, \mu^{\frac{n}{2}} \prod_{l=1}^{N-1} \Big(1 + \mu^n \, \cot\big(\tfrac{\pi l}{2N}\big)\Big)\,, \qquad \big(2\mathfrak{j} = \mathfrak{m} = \tfrac{n}{2}\,, \quad n - \text{ even}\big)\,. \quad \text{(E.5)}$$

Recall that the vacuum is defined as the lowest energy state of the Fateev-Zamolodchikov spin chain Hamiltonian (6.48), (6.49), which commutes with both $\mathcal{Z}_+(\mu)$ and $\mathcal{Z}_-(\mu)$ for any μ.

We studied the solutions to the Bethe ansatz equations corresponding to the low energy states $|\,\psi\,\rangle$ of the Fateev-Zamolodchikov spin chain. It was found that the roots accumulate along the rays given by (see fig. E.1)

$$\begin{aligned}
&\arg(\mu) \;= \pm\tfrac{\pi}{n}\, p\,, && p = 1, 3, \ldots, n-2 && (\mu_i - \text{roots})\\
&\arg(\mu^2) = \tfrac{2\pi}{n} p\,, && p = 1, 3, \ldots,\, n-1 && (v_i - \text{roots})\\
&\arg(\mu^2) = \tfrac{2\pi}{n}\, p\,, && p = 2, 4, \ldots, n-2 && (w_i - \text{roots})
\end{aligned}$$

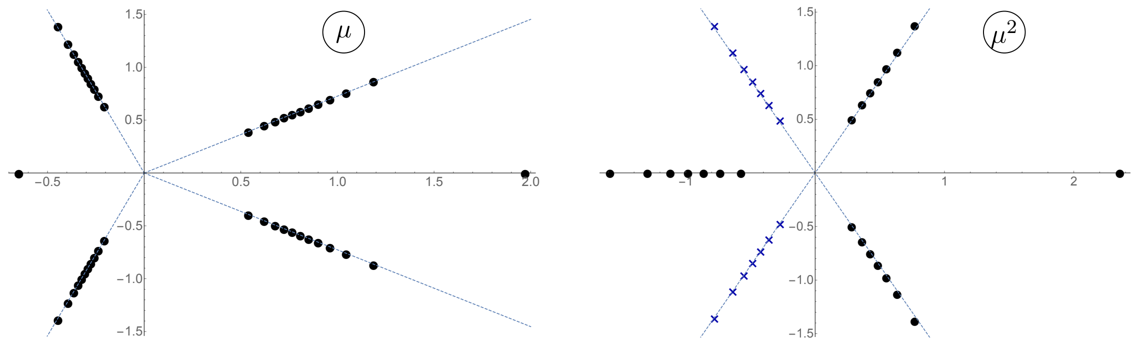


Figure E.1: On the left panel, the roots of $\mathcal{Z}_+^{(\mathrm{vac})}(\mu)$ are depicted in the complex plane for $n=5$, $\mathfrak{j}=1$, $\mathfrak{m}=0$ and $N=12$. On the right panel, the roots of $\mathcal{Z}_+^{(\mathrm{vac})}$ (circles) and $\mathcal{Z}_-^{(\mathrm{vac})}$ (crosses) as functions of μ^2 are shown for $n=6$, $2\mathfrak{j}=3$, $\mathfrak{m}=1$ and $N=8$.

In the scaling limit most of the roots become densely packed along the rays. However we observed that at the edges of the distribution, the roots exhibit a certain scaling behaviour. In particular, at the edge next to zero of the locus labeled by the integer p, with index i enumerating the roots ordered by increasing absolute value, the following limits exist

$$\lim_{\substack{N\to\infty \\ i-\text{fixed}}} N^{\frac{1}{n}}\,\mu_{i,p}^{(N,\psi)}\,, \qquad \lim_{\substack{N\to\infty \\ i-\text{fixed}}} N^{\frac{2}{n}}\,v_{i,p}^{(N,\psi)}\,, \qquad \lim_{\substack{N\to\infty \\ i-\text{fixed}}} N^{\frac{2}{n}}\,w_{i,p}^{(N,\psi)}\,.$$

Here we temporarily exhibit the dependence of the roots on N and the state $|\,\psi\,\rangle$. Also, the scaling limit can be defined for the coefficient $B^{(N,\psi)}$ in formulae (E.1), (E.2):

$$B^{(\psi)} = \operatorname*{slim}_{N\to\infty} (\pi/N)^{\frac{\mathfrak{m}}{n}}\,B^{(N,\psi)}\,. \tag{E.6}$$

Keeping N finite, consider the logarithm of the r.h.s of eqs. (6.54) and (6.55) for a given eigenvalue. With $\mathcal{Z}_\pm^{(\psi)}$ of the form (E.1), (E.2) it is straightforward to find their Taylor series at $\lambda=0$. In the case of odd n, the expansion coefficients are given by

$$
\begin{aligned}
M_m^{(N)} &= \frac{1}{m}\left(\frac{\pi}{N}\right)^{\frac{m}{n}}\left(\sum_i \mu_i^{-m} + \frac{(-1)^m\ N}{\cos\left(\frac{\pi m}{2n}\right)}\right) && (m<n)\\
M_n^{(N)} &= \frac{\pi}{nN}\sum_i \mu_i^{-n} + \frac{2}{n}\,(n-1)\ \log\left(\frac{N\mathrm{e}}{\pi}\right) && \\
M_m^{(N)} &= \frac{1}{m}\left(\frac{\pi}{N}\right)^{\frac{m}{n}}\sum_i \mu_i^{-m} && (m>n)
\end{aligned}
\tag{E.7}
$$

For even n,

$$
\begin{aligned}
V_m^{(N)} &= \frac{1}{m}\left(\frac{\pi}{N}\right)^{\frac{2m}{n}}\sum_i v_i^{-m} && \big(2m<n\big)\\
W_m^{(N)} &= \frac{1}{m}\left(\frac{\pi}{N}\right)^{\frac{2m}{n}}\left(\sum_i w_i^{-m} + \frac{N}{\cos\left(\frac{\pi m}{n}\right)}\right) && \big(2m<n\big)\\
V_{\frac{n}{2}}^{(N)} &= \frac{2\pi}{nN}\sum_i v_i^{-\frac{n}{2}} + 2\log\left(\frac{N\mathrm{e}}{\pi}\right) && \\
W_{\frac{n}{2}}^{(N)} &= \frac{2\pi}{nN}\sum_i w_i^{-\frac{n}{2}} - \frac{2}{n}\,(n-2)\ \log\left(\frac{N\mathrm{e}}{\pi}\right) && \\
V_m^{(N)} &= \frac{1}{m}\left(\frac{\pi}{N}\right)^{\frac{2m}{n}}\sum_i v_i^{-m}, \qquad W_m^{(N)} = \frac{1}{m}\left(\frac{\pi}{N}\right)^{\frac{2m}{n}}\sum_i w_i^{-m} && \big(2m>n\big)
\end{aligned}
\tag{E.8}
$$

It is expected that the following limits exist,

$$
\begin{aligned}
&M_m^{(\psi)} = \operatorname*{slim}_{N\to\infty} M_m^{(N)} && (n-\text{odd})\\
&V_m^{(\psi)} = \operatorname*{slim}_{N\to\infty} V_m^{(N)}, \qquad W_m^{(\psi)} = \operatorname*{slim}_{N\to\infty} W_m^{(N)} && (n-\text{even})
\end{aligned}
\tag{E.9}
$$

and coincide with the expansion coefficients in $\lambda \equiv \mathrm{e}^{\frac{\theta}{n}}$ of the CFT eigenvalues of $\log\zeta_\pm$:

$$
\log\zeta_\pm^{(\psi)}(\theta) = \log B^{(\psi)} + \tfrac{\mathfrak{m}}{n}\,\theta \mp \tfrac{2}{n}\,\theta\,\mathrm{e}^{\theta} - \sum_{m=1}^{\infty}(\pm 1)^m\ M_m^{(\psi)}\ \mathrm{e}^{\frac{m\theta}{n}} \qquad (n-\text{odd})
$$

$n=4,\ \mathfrak{j}=1,\ \mathfrak{m}=0$	$N=101$	$N=201$	$N=400$	$N=1001$	$N=\infty$	exact
$V_1^{(\mathrm{vac})}$	2.4301852	2.4299253	2.4298202	2.4297709	2.4297498	2.4297502
$V_2^{(\mathrm{vac})}-W_2^{(\mathrm{vac})}$	3.1094496	3.1094648	3.1094686	3.1094697	3.1094699	3.1094699
$n=4,\ 2\mathfrak{j}=\mathfrak{m}=1$	$N=201$	$N=401$	$N=1001$	$N=1500$	$N=\infty$	exact
$V_1^{(\mathrm{vac})}$	-0.970065	-0.962059	-0.955825	-0.954074	-0.948453	-0.948425
$W_1^{(\mathrm{vac})}$	2.020549	2.016718	2.013751	2.012921	2.010289	2.010250
$n=5,\ 2\mathfrak{j}=\mathfrak{m}=2$	$N=100$	$N=200$	$N=400$	$N=750$	$N=\infty$	exact
$M_1^{(\mathrm{vac})}$	-1.09540	-1.09018	-1.08669	-1.08453	-1.07962	-1.07956
$M_2^{(\mathrm{vac})}$	0.77960	0.77649	0.77445	0.77320	0.77054	0.77039

Table E.1: Numerical values of the coefficients (E.7), (E.8) for the vacuum of the Fateev-Zamolodchikov spin chain (6.48), (6.49). The column $N=\infty$ was obtained by interpolating the finite-N data. The last column lists the exact predictions given in (E.10), (E.11).

$$\log\zeta_+^{(\psi)}(\theta)=\log B^{(\psi)}\ +\tfrac{\mathfrak{m}}{n}\,\theta-\sum_{l=1}^{\infty}V_m^{(\psi)}\ \mathrm{e}^{\frac{2m\theta}{n}}\qquad\qquad (n-\text{even})$$

$$\log\zeta_-^{(\psi)}(\theta)=\log B^{(\psi)}\ +\tfrac{\mathfrak{m}}{n}\,\theta+\tfrac{4}{n}\,\theta\,\mathrm{e}^{\theta}-\sum_{m=1}^{\infty}W_m^{(\psi)}\ \mathrm{e}^{\frac{2m\theta}{n}}\ .$$

Recall that the symbol "slim" stands for the scaling limit which is applied for low energy eigenstates only. For numerical checks, we focused only on the vacuum of the Fateev-Zamolodchikov spin chain (6.48), (6.49). Our numerical work confirmed the existence of the limits (E.9) for $n=3,\,4,\,\ldots,6$ and all admissible values of $\mathfrak{j}$ and $\mathfrak{m}$ (6.14). Since a few of the expansion coefficients in (6.88) are available in explicit form, we have the following analytical predictions for some of the limits in (E.9).

Let $f_{0,1}=f_{0,1}(p_1,p_2)$ be defined by eq. (6.86) and $\gamma(x)\equiv\Gamma(x)/\Gamma(1-x)$. Then for $n>2$ one has (here the superscript "(vac)" in the notation for the coefficients (E.9) is omitted):

$$M_m(\mathfrak{m},\mathfrak{j}) = 0 \qquad (m = 1,3,\ldots,n-2-2\mathfrak{m})$$

$$M_1\big(\tfrac{n-1}{2},\tfrac{n-1}{4}\big) = -n^{-\frac{1}{n}}\ \gamma\big(\tfrac{1}{2}-\tfrac{1}{2n}\big)$$

$$M_2(\mathfrak{m},\mathfrak{j}) = -f_{0,1}\big(-\tfrac{\mathrm{i}\,\mathfrak{m}}{2},\mathfrak{j}+\tfrac{1}{2}\big) \qquad \big(\mathfrak{m}=0,1,2,\ldots,\tfrac{n-3}{2}\big)$$

$$M_2\big(\tfrac{n-1}{2},\tfrac{n-1}{4}\big) = -f_{0,1}\big(-\tfrac{\mathrm{i}(n-1)}{4},\mathfrak{j}+\tfrac{1}{2}\big)+\tfrac{1}{2}\,n^{-\frac{2}{n}}\ \gamma^2\big(\tfrac{1}{2}-\tfrac{1}{2n}\big) \tag{E.10}$$

$$M_n(0,\mathfrak{j}) = 2\log\big(\tfrac{\mathrm{e}}{2}\big)+\tfrac{2}{n}\big(\gamma_E-\log(n)\big)+4\psi(1+\mathfrak{j})-\psi(1+\tfrac{n}{2})+\gamma_E$$

$$V_m(\mathfrak{m},\mathfrak{j}) = W_m(\mathfrak{m},\mathfrak{j}) \qquad \big(m=1,2,\ldots,\tfrac{n-2}{2}-\mathfrak{m}\big)$$

$$V_1(\mathfrak{m},\mathfrak{j}) = W_1(\mathfrak{m},\mathfrak{j}) = -f_{0,1}\big(-\tfrac{\mathrm{i}\,\mathfrak{m}}{2},\mathfrak{j}+\tfrac{1}{2}\big) \qquad \big(\mathfrak{m}=0,1,2,\ldots,\tfrac{n}{2}-2\big)$$

$$V_1\big(\tfrac{n-2}{2},\mathfrak{j}\big) = -f_{0,1}\big(-\tfrac{\mathrm{i}(n-2)}{4},\mathfrak{j}+\tfrac{1}{2}\big)-n^{-\frac{2}{n}}\ \gamma\big(\tfrac{1}{2}-\tfrac{1}{n}\big) \tag{E.11}$$

$$W_1\big(\tfrac{n-2}{2},\mathfrak{j}\big) = -f_{0,1}\big(-\tfrac{\mathrm{i}(n-2)}{4},\mathfrak{j}+\tfrac{1}{2}\big)+n^{-\frac{2}{n}}\ \gamma\big(\tfrac{1}{2}-\tfrac{1}{n}\big)$$

$$V_{\frac{n}{2}}(0,\mathfrak{j})-W_{\frac{n}{2}}(0,\mathfrak{j}) = 4\log\big(\tfrac{\mathrm{e}}{2}\big)+\tfrac{4}{n}\big(\gamma_E-\log(n)\big)+8\,\psi(1+\mathfrak{j})-2\,\psi(1+\tfrac{n}{2})+2\gamma_E$$

The numerical data agreed with these explicit formulae. This is shown, for a few cases, in tab. E.1.

As was already mentioned, the constant $\big(B^{(N,\psi)}\big)^2$ can be found using the quantum Wronskian type relations (ii) from sec. 6.4. The r.h.s. of these relations is proportional to the lattice shift operator $\mathbb{P}^{(N)}$ (6.50) whose eigenvalues are pure phases (6.53). By explicit diagonalization of $\mathcal{Z}_\pm$ for small N we found that

$$\frac{1}{\pi}\ \arg\big(B^{(N,\psi)}\big) = \frac{1}{nN}\ \big((2\mathfrak{j}-s)\,s+n\,(L-\bar{L})\big)+s \qquad (\mathrm{mod}\,2)\ , \tag{E.12}$$

where $s=\mathfrak{j}-\frac{1}{2}\,\mathfrak{m}$ and L, $\bar{L}$ are non-negative integers depending on the state $|\psi\rangle$. For the vacuum state $L=\bar{L}=0$, and the overall sign of the limit $B^{(\mathrm{vac})}$ (E.6) is $(-1)^s$.

						$(-1)^s\, B_s(\mathfrak{m})$
$n=3,\ 2\mathfrak{j}=\mathfrak{m}=1$	$N=101$	$N=201$	$N=401$	$N=1001$	$N=\infty$	
	1.621564	1.623528	1.624666	1.625467	1.626213	1.626210
$n=6,\ 2\mathfrak{j}=\mathfrak{m}=2$	$N=100$	$N=200$	$N=400$	$N=800$	$N=\infty$	
	1.79398	1.80320	1.80971	1.81430	1.82531	1.82536
$n=6,\ 2\mathfrak{j}=3,\ \mathfrak{m}=1$	$N=2$	$N=4$	$N=6$	$N=8$	$N=\infty$	
	4.178	4.148	4.135	4.127	4.08	4.10

Table E.2: The absolute value of $(\pi/N)^{\frac{\mathfrak{m}}{n}}\, B^{(N,\mathrm{vac})}$ corresponding to the vacuum state of the Fateev-Zamolodchikov spin chain (6.48), (6.49). The column "$N=\infty$" contains the results of numerical interpolation from the finite N data. The analytical expression for $B_s(\mathfrak{m})$ is given by (6.87).

This coincides with the sign factor in $B_s(\mathfrak{m})$ (6.87). For large values of N, when direct diagonalization becomes impossible, we verified by means of the Bethe ansatz that the absolute value of $(\pi/N)^{\frac{\mathfrak{m}}{n}}\, B^{(N,\mathrm{vac})}$ converges to $(-1)^s\, B_s(\mathfrak{m})$ (see tab. E.2).

Recall that $2\mathfrak{j}=\mathfrak{m}=\frac{n}{2}$ with even n is a special case. Using eq. (E.5) the scaling functions can be found explicitly,

$$\zeta_+^{(\mathrm{vac})}(\theta)=\frac{2\sqrt{\pi}\ \mathrm{e}^{\frac{\theta}{2}}}{\Gamma\big(1+2\mathrm{e}^{\theta}\big)}\,,\qquad\qquad \zeta_-^{(\mathrm{vac})}(\theta)=0\,.$$

This formula can be applied for $n=2$. For the remaining $n=2$ case, $\mathfrak{j}=\mathfrak{m}=0$, it is easy to show that for finite N

$$\mathcal{Z}_+^{(\mathrm{vac})}(\mu)=\sqrt{2}\,\prod_{m=1}^{N}\Big(1-\mu^2\,\cot\big(\tfrac{\pi}{4N}\,(2m-1)\big)\Big)\,,\qquad\qquad \mathcal{Z}_-^{(\mathrm{vac})}(\mu)=\sqrt{2}\,,$$

so that the scaling functions are given by

$$\zeta_+^{(\mathrm{vac})}(\theta)=\frac{\sqrt{2\pi}}{\Gamma\big(\frac{1}{2}+2\mathrm{e}^{\theta}\big)}\,\left(\frac{2}{\mathrm{e}}\right)^{2\mathrm{e}^{\theta}}\,,\qquad\qquad \zeta_-^{(\mathrm{vac})}(\theta)=\sqrt{2}\ \exp\big(2\theta\,\mathrm{e}^{\theta}\big)\,.$$

Appendix F

In this appendix we sketch some technical details in the derivation of the system of NLIE (6.147).

Suppose θ, p_1^2 and p_2 are real, then eqs. (6.128), Hermiticity conditions (6.65), (6.115), (6.120) and the periodicity (6.113) imply that

$$\begin{aligned}
&\mathrm{e}^{-\mathrm{i}\pi \mathrm{e}^{\theta}}\Big[\mathrm{e}^{-\mathrm{i}\pi p_2}\,\beta_-(\theta)\,\alpha_+\big(\theta-\tfrac{\mathrm{i}\pi n}{2}\big)-\mathrm{e}^{+\mathrm{i}\pi p_2}\,\beta_+(\theta)\,\alpha_-\big(\theta-\tfrac{\mathrm{i}\pi n}{2}\big)\Big]=\\
&\mathrm{e}^{+\mathrm{i}\pi \mathrm{e}^{\theta}}\Big[\mathrm{e}^{-\mathrm{i}\pi p_2}\,\beta_+(\theta)\,\alpha_-\big(\theta+\tfrac{\mathrm{i}\pi n}{2}\big)-\mathrm{e}^{+\mathrm{i}\pi p_2}\,\beta_-(\theta)\,\alpha_+\big(\theta+\tfrac{\mathrm{i}\pi n}{2}\big)\Big]\,, \qquad \text{(F.1)}\\
&\mathrm{e}^{+\mathrm{i}\pi \mathrm{e}^{\theta}}\Big[\mathrm{e}^{-\mathrm{i}\pi p_2}\,\beta_-(\theta+\mathrm{i}\pi)\,\alpha_+\big(\theta-\tfrac{\mathrm{i}\pi n}{2}\big)-\mathrm{e}^{+\mathrm{i}\pi p_2}\,\beta_+(\theta+\mathrm{i}\pi)\,\alpha_-\big(\theta-\tfrac{\mathrm{i}\pi n}{2}\big)\Big]=\\
&\mathrm{e}^{-\mathrm{i}\pi \mathrm{e}^{\theta}}\Big[\mathrm{e}^{-\mathrm{i}\pi p_2}\,\beta_+(\theta+\mathrm{i}\pi)\,\alpha_-\big(\theta+\tfrac{\mathrm{i}\pi n}{2}\big)-\mathrm{e}^{+\mathrm{i}\pi p_2}\,\beta_-(\theta+\mathrm{i}\pi)\,\alpha_+\big(\theta+\tfrac{\mathrm{i}\pi n}{2}\big)\Big]\,.
\end{aligned}$$

Due to the analyticity of the operators $\alpha_\pm(\theta)$ and $\beta_\pm(\theta)$, these relations should be satisfied for any complex θ. Let us introduce the shortcut notations

$$B_0=\frac{\beta_+(\theta)}{\beta_-(\theta)}\ ,\quad B_1=\frac{\beta_+(\theta+\mathrm{i}\pi)}{\beta_-(\theta+\mathrm{i}\pi)}\ ,\quad U=\mathrm{e}^{2\pi\mathrm{i}p_2}\ \frac{\alpha_+(\theta+\frac{\mathrm{i}\pi n}{2})}{\alpha_-(\theta+\frac{\mathrm{i}\pi n}{2})}\ ,\quad A_\pm=\mathrm{e}^{\mp 2\pi\mathrm{i}p_2}\ \frac{\alpha_\pm(\theta-\frac{\mathrm{i}\pi n}{2})}{\alpha_\pm(\theta+\frac{\mathrm{i}\pi n}{2})}$$

and $\Lambda=\exp\big(2\pi\mathrm{i}\,\mathrm{e}^{\theta}\big)$. Then (F.1) can be rewritten as

$$B_0=U\ \frac{1+\Lambda^{-1}A_+}{1+\Lambda^{-1}A_-}\ ,\qquad\qquad B_1=U\ \frac{1+\Lambda\,A_+}{1+\Lambda\,A_-}\ .$$

Solving these equations w.r.t. A_+ and A_-, one finds

$$A_+=-\tfrac{1}{2}\big(\Lambda+\Lambda^{-1}\big)+\frac{\Lambda-\Lambda^{-1}}{B_1-B_0}\ \big(\,B_0B_1\,U^{-1}-\tfrac{1}{2}\,B_0-\tfrac{1}{2}\,B_1\,\big)$$

and similar for A_-. This formula, combined with the quantum Wronskian relation (6.116) written in the form

$$\frac{\Lambda-\Lambda^{-1}}{B_1-B_0}=-\frac{\beta_-(\theta)\,\beta_-(\theta+\mathrm{i}\pi)}{2\mathrm{i}\,\sin\big(\frac{2\pi p_2}{n+2}\big)}\ ,$$

leads to

$$A_+(\theta) = -\cos\big(2\pi e^{\theta}\big) - \frac{\beta_+(\theta)\,\beta_+(\theta+\mathrm{i}\pi)}{2\mathrm{i}\,U(\theta)\sin\big(\frac{2\pi p_2}{n+2}\big)} + \frac{\beta_+(\theta)\,\beta_-(\theta+\mathrm{i}\pi)+\beta_-(\theta)\,\beta_+(\theta+\mathrm{i}\pi)}{4\mathrm{i}\ \sin\big(\frac{2\pi p_2}{n+2}\big)}\ .$$

Together with the periodicity condition $\beta_\pm(\theta+\mathrm{i}\pi)=\beta_\pm(\theta-\mathrm{i}\pi)$ the last equation implies

$$A_+\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)-A_+\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)=\frac{\beta_+\big(\theta+\frac{\mathrm{i}\pi}{2}\big)\beta_-\big(\theta-\frac{\mathrm{i}\pi}{2}\big)}{2\mathrm{i}\ \sin\big(\frac{2\pi p_2}{n+2}\big)}\ \Big(U^{-1}\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)-U^{-1}\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)\Big)\ .$$

As it follows from the quantum Wronskian relation (6.121):

$$U^{-1}\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)-U^{-1}\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)=\frac{2\mathrm{i}\ e^{-2\pi\mathrm{i}p_2}\ \sin\big(\frac{2\pi p_2}{n+2}\big)}{\alpha_+\big(\theta+\frac{\mathrm{i}\pi(n-1)}{2}\big)\alpha_+\big(\theta+\frac{\mathrm{i}\pi(n+1)}{2}\big)}\ .$$

This can be substituted into the previous formula, yielding eq. (6.129) with the subscript "+". Of course the formula is valid for the "−" case also.

Let us now take a closer look at the second equation in (F.1) specialized to the eigenvalues corresponding to a common eigenvector $|\,\psi\,\rangle$. Suppose θ_j is a zero of $\beta_+^{(\psi)}(\theta)$. As follows from the quantum Wronskian relation (6.116), $\beta_-^{(\psi)}(\theta_j)\neq 0$, and therefore we conclude that

$$e^{-\mathrm{i}\pi(e^{\theta_j}+p_2)}\ \alpha_+^{(\psi)}\big(\theta_j-\mathrm{i}\pi-\tfrac{\mathrm{i}\pi n}{2}\big)=-e^{\mathrm{i}\pi(e^{\theta_j}+p_2)}\ \alpha_+^{(\psi)}\big(\theta_j-\mathrm{i}\pi+\tfrac{\mathrm{i}\pi n}{2}\big)\ ,$$

which can be equivalently written in the form (6.134).

As was mentioned in the main body of the text, the zeroes of the entire periodic function $\beta_+^{(\mathrm{vac})}(\theta)=\beta_+^{(\mathrm{vac})}(\theta+\mathrm{i}\pi)$ are simple, located on the lines $\Im m(\theta)=\pi\,(2m+1)$, $m\in\mathbb{Z}$, and accumulate toward $\Re e(\theta)\to+\infty$. Also, assuming that the parameters p_1 and p_2 are restricted as in cases (b), (c) from sec. 6.8, it is expected that the entire function $\alpha_+^{(\mathrm{vac})}(\theta)$ does not have any zeroes within the strip $|\Im m(\theta)|<\frac{\pi}{2}\,(n+2)$. Therefore, as follows from the definition (6.130), $\varepsilon^{(\mathrm{vac})}(\theta)$ is an analytic function for $|\Im m(\theta)|<\pi$ where it has the leading asymptotic behaviour (6.132) at $\Re e(\theta)\to+\infty$.

Combining this analytic information with the "quantization condition" (6.135) for the zeroes of $\beta_+^{(\mathrm{vac})}(\theta)$ and the asymptotic behaviour $\big($see eq. (6.127)$\big)$

$$\log\beta_+^{(\mathrm{vac})}(\theta) = -2\theta\,\mathrm{e}^{\theta} - k\,\theta + o(1) \qquad \text{as} \qquad \Re e(\theta)\to+\infty \quad \& \quad \big|\Im m(\theta)\big| < \pi\ ,$$

with $k=\frac{2p_2}{n+2}$, it is a simple exercise (see however appendix G) to derive a dispersion-type relation

$$\log\Big(\beta_+^{(\mathrm{vac})}\big(\theta-\tfrac{\mathrm{i}\pi}{2}\big)\,\beta_+^{(\mathrm{vac})}\big(\theta+\tfrac{\mathrm{i}\pi}{2}\big)\Big) = 2\pi\,\mathrm{e}^{\theta} - 2k\,\theta - \tag{F.2}$$
$$\Im m\bigg[\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,\frac{1}{1+\mathrm{e}^{2\theta-2\theta'+2\mathrm{i}\gamma}}\Big(2L^{(\mathrm{vac})}(\theta'-\mathrm{i}\gamma)+\mathrm{i}\big(\varepsilon^{(\mathrm{vac})}(\theta'-\mathrm{i}\gamma)-4\pi\,\mathrm{e}^{\theta'-\mathrm{i}\gamma}+2\pi k\big)\Big)\bigg].$$

Here $\gamma\in(0,\frac{\pi}{2})$ is an arbitrary constant and the notation

$$L^{(\mathrm{vac})}(\theta) = \log\Big(1+\exp\big(-\mathrm{i}\varepsilon^{(\mathrm{vac})}(\theta)\big)\Big) \tag{F.3}$$

is used.

The next important property employed in the derivation of the system of integral equations (6.147)-(6.149) is that $\varepsilon^{(\mathrm{vac})}(\theta)$ can be written in terms of the Fourier integral

$$\varepsilon^{(\mathrm{vac})}(\theta) = 4\pi\,\mathrm{e}^{\theta} - 2\pi k + \int_{\mathbb{R}+\mathrm{i}0}\frac{\mathrm{d}\nu}{2\pi}\,\mathrm{e}^{\mathrm{i}\nu\theta}\,\tilde{\varepsilon}(\nu)\ . \tag{F.4}$$

Notice that the existence of the Fourier transform is ensured by the asymptotic behaviour (6.132) at $\theta\to+\infty$, and formulae (6.142), (6.143) for $\theta\to-\infty$. One can expect that the function $\tilde{\varepsilon}(\nu)$ decays sufficiently fast as $\nu\to\pm\infty$, so that the integral in (F.4) converges for any θ in the strip of analyticity $|\Im m(\theta)|<\pi$. It is not difficult to see now that

$$\log\Big[\alpha_+^{(\mathrm{vac})}\big(\theta-\tfrac{\mathrm{i}\pi(n+1)}{2}\big)\alpha_+^{(\mathrm{vac})}\big(\theta+\tfrac{\mathrm{i}\pi(n+1)}{2}\big)\Big] = 2\pi\mathrm{e}^{\theta}-2k\theta+\mathrm{i}\int_{\mathbb{R}+\mathrm{i}0}\frac{\mathrm{d}\nu}{2\pi}\,\mathrm{e}^{\mathrm{i}\nu\theta}\,\frac{\cosh(\frac{\pi(n+1)\nu}{2})}{\sinh(\frac{\pi n\nu}{2})}\,\tilde{\varepsilon}(\nu) \tag{F.5}$$

and also that the imaginary part of the function (F.3) with θ having infinitesimally small negative imaginary part, can be represented by the convergent integral

$$\Im m\big(L^{(\mathrm{vac})}(\theta-\mathrm{i}0)\big)=\int\limits_{\mathbb{R}+\mathrm{i}0}\frac{\mathrm{d}\nu}{2\pi}\,\mathrm{e}^{\mathrm{i}\nu\theta}\,\tilde{L}(\nu)\ .$$

Similarly for the function $\omega(\theta)$ (6.140) with θ real, one has

$$\omega^{(\mathrm{vac})}(\theta)=4\pi\,\mathrm{e}^{\theta}+\int\limits_{\mathbb{R}+\mathrm{i}0}\frac{\mathrm{d}\nu}{2\pi}\,\mathrm{e}^{\mathrm{i}\nu\theta}\,\tilde{\omega}(\nu)\,,\qquad\qquad \log\big(1+\mathrm{e}^{-\omega^{(\mathrm{vac})}(\theta)}\big)=\int\limits_{\mathbb{R}+\mathrm{i}0}\frac{\mathrm{d}\nu}{2\pi}\,\mathrm{e}^{\mathrm{i}\nu\theta}\,\tilde{M}(\nu)\ .$$

The remaining part of the derivation of the NLIE consists of straightforward manipulations with the Fourier images $\tilde{\varepsilon}$, $\tilde{L}$, $\tilde{\omega}$, $\tilde{M}$. Finally, going back to functions of the variable θ, one derives the system of integral equations (6.147)-(6.149).

Knowing the functions $\varepsilon^{(\mathrm{vac})}(\theta)$, $\omega^{(\mathrm{vac})}(\theta)$ from the solution of the NLIE, and the asymptotic formulae (6.123), (6.127), one can recover the vacuum eigenvalues of the operators $\alpha_+(\theta)$ and $\beta_+(\theta)$ from (F.2), (F.5). The corresponding explicit relations are given below, where we drop the superscript "(vac)" like in the NLIE (6.147)-(6.149):

$$\log\alpha_+(\theta)=-\frac{\pi}{\sin(\frac{\pi n}{2})}\,\mathrm{e}^{\theta}-k\theta+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[\,F_1^{(\mathrm{CFT})}(\theta-\theta'+\mathrm{i}\gamma)\,L(\theta'-\mathrm{i}\gamma) \tag{F.6}$$
$$-F_1^{(\mathrm{CFT})}(\theta-\theta'-\mathrm{i}\gamma)\,\Big(L(\theta'-\mathrm{i}\gamma)\Big)^*\,\Big]+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,F_2(\theta-\theta')\,\log\Big(1+\mathrm{e}^{-\omega(\theta')}\Big)$$

valid for $|\Im m(\theta)|<\frac{\pi}{2}\,(n+2)-\gamma$, and

$$\log\beta_+(\theta)=-2\theta\,\mathrm{e}^{\theta}-k\theta+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[\,F_3^{(\mathrm{CFT})}(\theta-\theta'+\mathrm{i}\gamma)\,L(\theta'-\mathrm{i}\gamma) \tag{F.7}$$
$$-F_3^{(\mathrm{CFT})}(\theta-\theta'-\mathrm{i}\gamma)\,\Big(L(\theta'-\mathrm{i}\gamma)\Big)^*\,\Big]-\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,F_4(\theta-\theta')\,\log\Big(1+\mathrm{e}^{-\omega(\theta')}\Big)$$

for $|\Im m(\theta)|<\pi-\gamma$. Here the kernels are given by $F_1^{(\mathrm{CFT})}(\theta)=F_1(\theta)-\frac{1}{n+2}$,

$F_3^{(\mathrm{CFT})}(\theta) = F_3(\theta) - \frac{1}{n+2}$ with

$$F_1(\theta) = \frac{1}{n+2}\ \tanh\Big(\frac{\theta}{n+2}\Big)\ , \qquad F_2(\theta) = \frac{\sin(\frac{\pi}{n+2})}{2(n+2)\cosh\big(\frac{\theta+\frac{\mathrm{i}\pi}{2}}{n+2}\big)\cosh\big(\frac{\theta-\frac{\mathrm{i}\pi}{2}}{n+2}\big)}$$

$$F_3(\theta) = \frac{1}{n+2}\ \coth\Big(\frac{\theta}{n+2}\Big) - \frac{1}{\sinh(\theta)} \tag{F.8}$$

$$F_4(\theta) = \frac{\sin(\frac{\pi}{n+2})}{2(n+2)\sinh(\frac{\theta+\frac{\mathrm{i}\pi}{2}}{n+2})\sinh(\frac{\theta-\frac{\mathrm{i}\pi}{2}}{n+2})} - \frac{1}{2\cosh(\theta)}\ .$$

The vacuum eigenvalues of the chiral transfer-matrix can be obtained using the $T-Q$ relation

$$\tau^{(\mathrm{vac})}(\mathrm{i}\lambda) = \frac{\alpha_+(\theta+\mathrm{i}\pi)}{\alpha_+(\theta)} + \frac{\alpha_+(\theta-\mathrm{i}\pi)}{\alpha_+(\theta)} \qquad \text{with} \qquad \lambda = \mathrm{e}^{\frac{\theta}{n}}\ .$$

Combining (F.6), (F.7) with the general asymptotic expansions at $\Re e(\theta) \to +\infty$ found in (6.123), (6.127), the expressions for the local and dual nonlocal integrals of motion follow

$$i_{2m-1}(p_1,p_2) = \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\Big(2\,\Im m\Big[\mathrm{e}^{(2m-1)(\theta-\mathrm{i}\gamma)}L(\theta-\mathrm{i}\gamma)\Big] + (-1)^m\mathrm{e}^{(2m-1)\theta}\log\big(1+\mathrm{e}^{-\omega(\theta)}\big)\Big) \tag{F.9}$$

$$\tilde{s}_m(p_1,p_2) = -\frac{2}{n+2}\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta}{\pi}\Big(\Im m\Big[\mathrm{e}^{\frac{2m(\theta-\mathrm{i}\gamma)}{n+2}}L(\theta-\mathrm{i}\gamma)\Big] - \sin\big(\tfrac{\pi m}{n+2}\big)\,\mathrm{e}^{\frac{2m\theta}{n+2}}\ \log\big(1+\mathrm{e}^{-\omega(\theta)}\big)\Big).$$

Appendix G

Here we discuss the modifications to the integral equations (6.147)-(6.149) for the case of real $p_1 \neq 0$ and $p_2 > -\frac{1}{2}$, when the asymptotics of the functions $\varepsilon^{(\mathrm{vac})}(\theta)$ and $\omega^{(\mathrm{vac})}(\theta)$ at $\theta \to -\infty$ oscillate (6.141).

The first important difference in this case is that $\alpha_+^{(\mathrm{vac})}(\theta)$ has a set of zeroes $\{\theta_m^{(\alpha)}\}_{m=1}^{\infty}$ in the strip $|\Im m(\theta)| < \frac{\pi}{2}\,(n+2)$ whose asymptotic behaviour is given by relation (6.146). Secondly, in the derivation of (6.147) presented in the previous appendix, we implicitly assumed that all values θ_* on the real axis, such that $\varepsilon^{(\mathrm{vac})}(\theta_*) = \pi \pmod{2\pi}$ arise from the quantization condition (6.135), i.e., $\theta_* = \theta_j^{(\mathrm{vac})} - \mathrm{i}\pi$ for some $j = 1, 2, \ldots$ (recall that $\Im m(\theta_j) = \pi$). In other words all such θ_* are related to the zeroes of $\beta_+^{(\mathrm{vac})}(\theta)$ and, therefore, form an increasing semi-infinite sequence extending towards $+\infty$ on the real axis (see (6.136)). For the oscillating asymptotics (6.141) this is no longer true. Indeed, it is easy to check from (6.141) that the condition

$$\varepsilon^{(\mathrm{vac})}(\tilde{\theta}_m) = -\pi\,(2m-1) \qquad \text{with} \quad m = 1,\, 2, \ldots$$

is satisfied for an infinite set of values $\big\{\tilde{\theta}_m\big\}_{m=1}^{\infty}$ which extend towards $-\infty$ such that

$$\tilde{\theta}_m = -\frac{n}{2p_1}\Big(\pi m - \frac{1}{2}\,\delta(p_1,p_2)\Big) + o\big((m/p_1)^{-\infty}\big)\,,$$

valid up to an exponentially small correction. Here

$$\delta(p_1,p_2) = 4p_1\theta_0/n + \mathrm{i}\,\log\big[\cos(\pi(p_2+\mathrm{i}p_1))/\cos(\pi(p_2-\mathrm{i}p_1)\big]$$

coincides with the scattering phase defined by eq. (7.11). In the terminology of the Bethe ansatz we have an infinite number of "holes" where the phase passes a resonant value without a corresponding zero θ_j. Therefore the integrals in the r.h.s. of (6.147) contain spurious contributions from non-existent roots. To exclude these unwanted contributions one needs to add extra source terms to the r.h.s. of eqs. (6.147).

Introduce the notation

$$\begin{aligned} J^{(\varepsilon)}(\theta) &= -\mathrm{i}\sum_{m=1}^{\infty}\log\left[\frac{S(\theta-\tilde{\theta}_m)}{S\big(\theta+\frac{\mathrm{i}\pi}{2}(n+2)-\theta_m^{(\alpha)}\big)}\right] \\ J^{(\omega)}(\theta) &= -\sum_{m=1}^{\infty}\log\left[\frac{t\big(\theta+\frac{\mathrm{i}\pi}{2}-\tilde{\theta}_m\big)}{t\big(\theta+\frac{\mathrm{i}\pi}{2}+\frac{\mathrm{i}\pi}{2}(n+2)-\theta_m^{(\alpha)}\big)}\right] \end{aligned} \tag{G.1}$$

where $S(\theta)$ and $t(\theta)$ are defined in (H.1) below. Then the modified equations (6.147) can be written as

$$\begin{aligned} \varepsilon(\theta-\mathrm{i}\gamma) &= 4\pi\,\mathrm{e}^{\theta-\mathrm{i}\gamma}-2\pi k+J^{(\varepsilon)}(\theta-\mathrm{i}\gamma)+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi\mathrm{i}}\,\Big[G(\theta-\theta'-2\mathrm{i}\gamma)\,\big(L(\theta'-\mathrm{i}\gamma)\big)^{*} \\ &- G(\theta-\theta')\,L(\theta'-\mathrm{i}\gamma)\Big]+\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{2\pi}\,G_1(\theta-\theta'-\mathrm{i}\gamma)\,\log\big(1+\mathrm{e}^{-\omega(\theta')}\big) \\ \omega(\theta) &= 4\pi\,\mathrm{e}^{\theta}+J^{(\omega)}(\theta)+\Im m\Big[\int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,G_1(\theta-\theta'+\mathrm{i}\gamma)\,L(\theta'-\mathrm{i}\gamma)\Big] \\ &- \int_{-\infty}^{\infty}\frac{\mathrm{d}\theta'}{\pi}\,G_2(\theta-\theta')\,\log\big(1+\mathrm{e}^{-\omega(\theta')}\big) \\ L(\theta) &= \log\big(1+\mathrm{e}^{-\mathrm{i}\varepsilon(\theta)}\big)\,. \end{aligned} \tag{G.2}$$

One can check that the leading terms in the asymptotics (6.141) solves these equations at $\theta\to-\infty$, i.e., when the exponential terms proportional to e^{θ} in the r.h.s. are omitted.

Appendix H

Here, we present the explicit form of the two particle scattering amplitudes for the sausage model [66]. The S-matrix satisfies the Yang-Baxter equation and was originally introduced as the Boltzmann weights of the so-called 19-vertex model [179].

$$
\begin{aligned}
S(\theta) &= S_{++}^{++}(\theta) = S_{--}^{--}(\theta) = -\frac{\sinh(\frac{\mathrm{i}\pi-\theta}{n+2})}{\sinh(\frac{\mathrm{i}\pi+\theta}{n+2})} \\
T(\theta) &= S_{+-}^{+-}(\theta) = S_{-+}^{-+}(\theta) = S(\mathrm{i}\pi-\theta) \\
t(\theta) &= S_{+0}^{+0}(\theta) = S_{0+}^{0+}(\theta) = S_{-0}^{-0}(\theta) = S_{0-}^{0-}(\theta) = \frac{\sinh(\frac{\theta}{n+2})\,\sinh(\frac{\mathrm{i}\pi-\theta}{n+2})}{\sinh(\frac{2\mathrm{i}\pi-\theta}{n+2})\,\sinh(\frac{\mathrm{i}\pi+\theta}{n+2})} \\
r(\theta) &= a(\mathrm{i}\pi-\theta) = S_{+0}^{0+}(\theta) = S_{0+}^{+0}(\theta) = S_{-0}^{0-}(\theta) = S_{0-}^{-0}(\theta) = S_{00}^{+-}(\mathrm{i}\pi-\theta) \\
&= S_{00}^{-+}(\mathrm{i}\pi-\theta) = S_{+-}^{00}(\mathrm{i}\pi-\theta) = S_{-+}^{00}(\mathrm{i}\pi-\theta) = -\mathrm{i}\,\frac{\sin(\frac{2\pi}{n+2})\,\sinh(\frac{\mathrm{i}\pi-\theta}{n+2})}{\sinh(\frac{2\mathrm{i}\pi-\theta}{n+2})\,\sinh(\frac{\mathrm{i}\pi+\theta}{n+2})} \\
R(\theta) &= S_{-+}^{+-}(\theta) = S_{+-}^{-+}(\theta) = \frac{\sin(\frac{\pi}{n+2})\sin(\frac{2\pi}{n+2})}{\sinh(\frac{2\mathrm{i}\pi-\theta}{n+2})\sinh(\frac{\mathrm{i}\pi+\theta}{n+2})} \\
\sigma(\theta) &= S_{00}^{00}(\theta) = S_{+0}^{+0}(\theta) + S_{-+}^{+-}(\theta)
\end{aligned}
\tag{H.1}
$$

As a 9×9 matrix $\boldsymbol{S}^{(2\mapsto 2)}$ satisfies the conditions

$$
\begin{aligned}
&\left(\boldsymbol{S}^{(2\mapsto 2)}\right)^{\dagger}\boldsymbol{S}^{(2\mapsto 2)} = \boldsymbol{I}^{(2)} \qquad \text{for } \Im m(\theta) = 0 \\
&\det \boldsymbol{S}^{(2\mapsto 2)}(\theta) = \left[\frac{\sinh^2(\frac{\mathrm{i}\pi-\theta}{n+2})}{\sinh^2(\frac{\mathrm{i}\pi+\theta}{n+2})}\;\frac{\sinh(\frac{2\mathrm{i}\pi+\theta}{n+2})}{\sinh(\frac{2\mathrm{i}\pi-\theta}{n+2})}\right]^4 .
\end{aligned}
\tag{H.2}
$$

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