Galilean superalgebra and supersymmetric field theory in 2+1 space

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1. Introduction

The Galilei group is the spacetime symmetry of non-relativistic systems as the Poincare group is the symmetry of relativistic systems. We can consider Galilei algebra in $2+1$ Minkowski space as subalgebra of the Poincare algebra in $3+1$ Minkowski space or as Inonu-Wigner contraction of Poincare algebra in $2+1$ Minkowski space [1].

The non-relativistic field theory is of vital importance in modern theoretical physics. First, most of non-relativistic theories are the non-relativistic limit of corresponding relativistic theories. Using non-relativistic theories we can study the generic field-theories concepts such as, for example, renormalization group in simpler way [3]. Second, Galilean field theories are second-quantized descriptions of quantum mechanics and they are rather suitable for description of many body problems in non-relativistic system, for example, Aharonov-Bohm scattering problem. Finally, Galilean field theories are also used in the relativistic superstring theory [2].

Supersymmetric generalization of the Galilean algebra were first proposed by Puzalowski in $3+1$ Minkowski space [4]. Puzalowski constructed representations of Galilean superalgebra with a single supercharge Q which can be understood as non-relativistic limit of $N = 1$ Poincare superalgebra.

In the present paper we consider representations of Galilean superalgebra and Galilean parasuperalgebra in 2+1 Minkowski space in terms of grassmannian variables and physical models invariant under the Galilean superalgebra.

2. Galilei superalgebra $N = 2$ in $2+1$ space

The Galilei superalgebra in $2+1$ space has seven operators of the Galilei algebra AG(1,2) P_0, P_i, K_i, J and M which satisfy the following commutation relations

$$[P_i, K_j] = i\delta_{ij}M, \quad [P_i, J] = -i\varepsilon_{ij}P_j, \quad [P_0, K_i] = iP_i, \quad [K_i, J] = -\varepsilon_{ij}K_j,$$

$$\varepsilon_{12} = -\varepsilon_{21} = 1, \qquad \varepsilon_{11} = \varepsilon_{22} = 0 \tag{1}$$

and two supercharges Q and R

$$\begin{aligned} &\{Q,Q^\dagger\}=M,\ \ \{R,R^\dagger\}=2P_0,\\ &[Q,J]=\frac{Q}{2},\ \ [R,J]=-\frac{R}{2},\\ &[R,K^-]=iQ,\ \ \{R,Q^\dagger\}=-P^+, \end{aligned} \tag{2}$$

where $K^-=K_1-iK_2$ and $P^+=P_1+iP_2$. The rest commutation relations between the generators of the Galilean superalgebra are equal to zero.

The Galilei superalgebra $N=2$ in $2+1$ space is subalgebra of the Poincare superalgebra N=1 in 3+1 Minkowski space. It can be verified using (2.2) and the following relations

$$\begin{aligned} &M=P_0+P_3, && P_0=\tfrac{1}{2}\,(P_0-P_3),\\ &K_i=J_{0i}+J_{3i}, && J=J_{12},\\ &Q=\frac{1}{\sqrt{2}}\,Q_2, && R=\tfrac{1}{\sqrt{2}}\,Q_1\,. \end{aligned} \tag{3}$$

The Galilei superalgebra in $2+1$ can be also found from $N=2$ Poincare superalgebra with nontrivial central charge in $2+1$ by the contraction. One can choose new basis

$$\begin{aligned} &\tilde{P}_0=-\frac{1}{\epsilon^2}\,M+\frac{1}{\epsilon}\,P_0,\\ &\tilde{P}_1=P_1,\qquad \tilde{P}_2=P_2,\\ &\tilde{J}=J_{12},\qquad \tilde{G}_1=\epsilon J_{01},\qquad \tilde{G}_2=\epsilon J_{02},\\ &\tilde{Q}_1^1=\frac{\sqrt{\epsilon}}{\sqrt{2}}\,(Q_1^1+Q_2^{\dagger 2}),\qquad \tilde{Q}_1^2=\frac{1}{\sqrt{2\epsilon}}\,(Q_1^1-Q_2^{\dagger 2}),\\ &\tilde{Q}_2^1=\frac{1}{\sqrt{2\epsilon}}\,(Q_1^2+Q_2^{\dagger 1}),\qquad \tilde{Q}_2^2=\frac{\sqrt{\epsilon}}{\sqrt{2}}\,(Q_1^2-Q_2^{\dagger 1}),\\ &Z=-\frac{2}{\epsilon^2}\,M \end{aligned} \tag{4}$$

(we suppose that M commutes with any generators of Poincare superalgebra). In the commutation and anticommutation relations of Poincare superalgebra in $2+1$ we put $\epsilon\longrightarrow 0$ and then denoting

$$R=2(\tilde{Q}_2^2+\tilde{Q}_1^{\dagger 2}),\quad Q=\tilde{Q}_2^1+\tilde{Q}_1^{\dagger 1}$$

we come to (1) and (2).

3. Representations of the Galilei superalgebra and parasuperalgebra $N = 2$ in $2+1$ and invariant physical models

Representations of the Galilei superalgebra in 2+1 in terms of grassmannian variables can be found using the representation of the Poincare superalgebra in $3+1$ in terms of grassmannian variables. The generators (1) and (2) have the form

$$P_0 = i\frac{\partial}{\partial t}, \quad P_i = i\frac{\partial}{\partial x_i}, \quad i = 1, 2, \tag{5}$$

$$K_1 = tP_1 - x_1 M + i\theta_1 \frac{\partial}{\partial \theta_2} + i\theta_1^\dagger \frac{\partial}{\partial \theta_2^\dagger}, \quad K_2 = tP_2 - x_2 M + \theta_1 \frac{\partial}{\partial \theta_2} - \theta_2^\dagger \frac{\partial}{\partial \theta_1^\dagger},$$
$$J = x_1 P_2 - x_2 P_1 + \frac{1}{2}(\theta_1 \frac{\partial}{\partial \theta_1} - \theta_2 \frac{\partial}{\partial \theta_2} - \theta_1^\dagger \frac{\partial}{\partial \theta_1^\dagger} + \theta_2^\dagger \frac{\partial}{\partial \theta_2^\dagger}) \tag{6}$$

and supercharges Q, R have the following form

$$\begin{aligned}
Q &= \frac{\partial}{\partial \theta_1} - 2i\theta_1^\dagger P_0 + i\theta_2^\dagger (P_1 + iP_2), \\
Q^\dagger &= -\frac{\partial}{\partial \theta_1^\dagger} - 2i\theta_1 P_0 - i\theta_2 (P_1 - iP_2), \\
R &= \frac{\partial}{\partial \theta_2} - i\theta_1^\dagger (P_1 - iP_2) - i\theta_2^\dagger M, \\
R^\dagger &= -\frac{\partial}{\partial \theta_2^\dagger} + i\theta_1 (P_1 + iP_2) + i\theta_2 M\,.
\end{aligned} \tag{7}$$

We can also define the covariant derivatives, which are used to build the physical models invariant under the Galilean superalgebra as

$$\begin{aligned}
D_1 &= \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \theta_1} - 2\theta_1^\dagger P_0 + \theta_2^\dagger (P_1 + iP_2)), \\
D_1^\dagger &= \frac{1}{\sqrt{2}}(-\frac{\partial}{\partial \theta_1^\dagger} + 2\theta_1 P_0 - \theta_2 (P_1 - iP_2)), \\
D_2 &= \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \theta_2} + \theta_1^\dagger (P_1 - iP_2) - \theta_2^\dagger M), \\
D_2^\dagger &= \frac{1}{\sqrt{2}}(\frac{\partial}{\partial \theta_2^\dagger} - \theta_1 (P_1 + iP_2) + \theta_2 M)\,.
\end{aligned} \tag{8}$$

Covariant derivatives (8) satisfy the following anticommutation relations

$$\begin{aligned}
&\{D_1, R\} = \{D_1, Q\} = \{D_2, R\} = \{D_2, Q\} = 0, \\
&\{D_1^\dagger, R\} = \{D_1^\dagger, Q\} = \{D_2^\dagger, R\} = \{D_2^\dagger, Q\} = 0, \\
&\{D_1, R^\dagger\} = \{D_1, Q^\dagger\} = \{D_2, R^\dagger\} = \{D_2, Q^\dagger\} = 0, \\
&\{D_1^\dagger, R^\dagger\} = \{D_1^\dagger, Q^\dagger\} = \{D_2^\dagger, R^\dagger\} = \{D_2^\dagger, Q^\dagger\} = 0\,.
\end{aligned} \tag{9}$$

4. Galilei superalgebra in 3+1 dimensions.

The minimal nontrivial generalization of $N=2$ Galilei superalgebra in 2+1 is $N=4$ superalgebra in $3+1$ dimensions with commutation relations

$$
\begin{aligned}
&\{\bar{Q}_1, Q_1\} = 2P_0, \ \ \{\bar{Q}_1, Q_2\} = 0, \ \ \{\bar{Q}_1, Q_3\} = P_3, \ \ \{\bar{Q}_1, Q_4\} = P_1 - iP_2, \\
&\{\bar{Q}_2, Q_1\} = 0, \ \ \{\bar{Q}_2, Q_2\} = 2P_0, \ \ \{\bar{Q}_2, Q_3\} = P_1 + iP_2, \ \ \{\bar{Q}_2, Q_4\} = -P_3, \\
&\{\bar{Q}_3, Q_1\} = P_3, \ \ \{\bar{Q}_3, Q_2\} = P_1 - iP_2, \ \ \{\bar{Q}_3, Q_3\} = 2M, \ \ \{\bar{Q}_3, Q_4\} = 0, \\
&\{\bar{Q}_4, Q_1\} = P_1 + iP_2, \ \ \{\bar{Q}_4, Q_2\} = -P_3, \ \ \{\bar{Q}_4, Q_3\} = 0, \ \ \{\bar{Q}_4, Q_4\} = 2M, \\
&[Q_1, J_1] = -\frac{1}{2} Q_2, \qquad [Q_1, J_2] = -\frac{i}{2} Q_2, \qquad [Q_1, J_3] = -\frac{1}{2} Q_1, \\
&[Q_2, J_1] = -\frac{1}{2} Q_1, \qquad [Q_2, J_2] = \frac{i}{2} Q_2, \qquad [Q_2, J_3] = \frac{1}{2} Q_2, \\
&[Q_3, J_1] = -\frac{1}{2} Q_4, \qquad [Q_3, J_2] = -\frac{i}{2} Q_4, \qquad [Q_3, J_3] = -\frac{1}{2} Q_3, \\
&[Q_4, J_1] = -\frac{1}{2} Q_3, \qquad [Q_4, J_2] = \frac{i}{2} Q_3, \qquad [Q_4, J_3] = \frac{1}{2} Q_4, \\
&[Q_1, G_1] = iQ_4, \qquad [Q_1, G_2] = -Q_4, \qquad [Q_1, G_3] = iQ_3, \\
&[Q_2, G_1] = iQ_3, \qquad [Q_2, G_2] = -Q_3, \qquad [Q_2, G_3] = -iQ_4.
\end{aligned}
\tag{10}
$$

In terms of grassmannian variables the supercharges Q_i $(i = 1, 2, 3, 4)$ can be written as

$$
\begin{aligned}
Q_1 &= -i\frac{\partial}{\partial\theta_1} + \bar{\theta}_1\frac{\partial}{\partial t} + \bar{\theta}_2\frac{\partial}{\partial x} + i\bar{\theta}_2\frac{\partial}{\partial y} + \bar{\theta}_1\frac{\partial}{\partial z}\,, \\
Q_2 &= -i\frac{\partial}{\partial\theta_2} + \bar{\theta}_2\frac{\partial}{\partial t} + \bar{\theta}_1\frac{\partial}{\partial x} + i\bar{\theta}_1\frac{\partial}{\partial y} + \bar{\theta}_2\frac{\partial}{\partial z}\,, \\
Q_3 &= -i\frac{\partial}{\partial\theta_3} - i\bar{\theta}_3 M + \bar{\theta}_2\frac{\partial}{\partial x} - i\bar{\theta}_2\frac{\partial}{\partial y} + \bar{\theta}_1\frac{\partial}{\partial z}\,, \\
Q_4 &= -i\frac{\partial}{\partial\theta_4} - i\bar{\theta}_4 M + \bar{\theta}_1\frac{\partial}{\partial x} + i\bar{\theta}_1\frac{\partial}{\partial y} - \bar{\theta}_2\frac{\partial}{\partial z}\,, \\
\bar{Q}_1 &= i\frac{\partial}{\partial\bar{\theta}_1} - \theta_1\frac{\partial}{\partial t} + \theta_2\frac{\partial}{\partial x} + i\theta_2\frac{\partial}{\partial y} + \theta_1\frac{\partial}{\partial z}\,, \\
\bar{Q}_2 &= i\frac{\partial}{\partial\bar{\theta}_2} - \theta_2\frac{\partial}{\partial t} + \theta_1\frac{\partial}{\partial x} + i\theta_1\frac{\partial}{\partial y} + \theta_2\frac{\partial}{\partial z}\,, \\
\bar{Q}_3 &= i\frac{\partial}{\partial\bar{\theta}_3} + \theta_3 M - \theta_2\frac{\partial}{\partial x} - i\theta_2\frac{\partial}{\partial y} - \theta_1\frac{\partial}{\partial z}\,, \\
\bar{Q}_4 &= i\frac{\partial}{\partial\bar{\theta}_4} + i\theta_4 M - \theta_1\frac{\partial}{\partial x} + i\theta_2\frac{\partial}{\partial y} + \theta_2\frac{\partial}{\partial z}\,.
\end{aligned}
\tag{11}
$$

The covariant derivatives take the following from

$$\begin{aligned}
D_1 &= -i\tfrac{\partial}{\partial\theta_1} + i\bar{\theta}_1\tfrac{\partial}{\partial t} + i\bar{\theta}_2\tfrac{\partial}{\partial x} - \bar{\theta}_2\tfrac{\partial}{\partial y} + i\bar{\theta}_1\tfrac{\partial}{\partial z}\,,\\
D_2 &= -i\tfrac{\partial}{\partial\theta_2} + i\bar{\theta}_2\tfrac{\partial}{\partial t} + i\bar{\theta}_1\tfrac{\partial}{\partial x} - \bar{\theta}_1\tfrac{\partial}{\partial y} + i\bar{\theta}_2\tfrac{\partial}{\partial z}\,,\\
D_3 &= -i\tfrac{\partial}{\partial\theta_3} + \bar{\theta}_3 M + i\bar{\theta}_2\tfrac{\partial}{\partial x} + \bar{\theta}_2\tfrac{\partial}{\partial y} + i\bar{\theta}_1\tfrac{\partial}{\partial z}\,,\\
D_4 &= -i\tfrac{\partial}{\partial\theta_4} + \bar{\theta}_4 M + i\bar{\theta}_1\tfrac{\partial}{\partial x} - \bar{\theta}_1\tfrac{\partial}{\partial y} - i\bar{\theta}_2\tfrac{\partial}{\partial z}\,,\\
\bar{D}_1 &= i\tfrac{\partial}{\partial\bar{\theta}_1} - i\theta_1\tfrac{\partial}{\partial t} + i\theta_2\tfrac{\partial}{\partial x} - \theta_2\tfrac{\partial}{\partial y} + i\theta_1\tfrac{\partial}{\partial z}\,,\\
\bar{D}_2 &= i\tfrac{\partial}{\partial\bar{\theta}_2} - i\theta_2\tfrac{\partial}{\partial t} + i\theta_1\tfrac{\partial}{\partial x} - \theta_1\tfrac{\partial}{\partial y} + i\theta_2\tfrac{\partial}{\partial z}\,,\\
\bar{D}_3 &= i\tfrac{\partial}{\partial\bar{\theta}_3} + i\theta_3 M - i\theta_2\tfrac{\partial}{\partial x} + \theta_2\tfrac{\partial}{\partial y} - i\theta_1\tfrac{\partial}{\partial z}\,,\\
\bar{D}_4 &= i\tfrac{\partial}{\partial\bar{\theta}_4} - \theta_4 M - i\theta_1\tfrac{\partial}{\partial x} - \theta_2\tfrac{\partial}{\partial y} + i\theta_2\tfrac{\partial}{\partial z}\,.
\end{aligned} \tag{12}$$

Below we consider the physical models invariant under the Galilei superalgebra.

5. Physical models invariant under the Galilean superalgebra.

To build the physical models in $2+1$ we introduce scalar superfield Φ, which has 16 component, in the form

$$\begin{aligned}
\Phi \;=\; & A(x) + \theta_\alpha\psi_\alpha + \theta^\dagger_\alpha\phi_\alpha + \theta_1\theta_2 B(x) + \theta_\alpha\theta^\dagger_\beta\Lambda_{\alpha\beta} + \theta^\dagger_1\theta^\dagger_2 C(x)\\
& + \theta_1\theta_2\theta^\dagger_\alpha\beta_\alpha + \theta_\alpha\theta^\dagger_1\theta^\dagger_2\varepsilon_\alpha + \theta_1\theta_2\theta^\dagger_1\theta^\dagger_2 D(x)\,,
\end{aligned} \tag{13}$$

where $x = (x_0, x_1, x_2)$.

Using covariant derivatives (8) we can build operator L which commutes with any generators of the Galilei superalgebra and act in the space of scalar superfields

$$L = D_1D_2 - aD^\dagger_1D^\dagger_2 + bD_2D^\dagger_2 + cM.$$

So we have following equation invariant under the Galilean superalgebra in $2+1$

$$L\Phi = g_1\Phi\Phi\Phi^\dagger, \tag{14}$$

where a, b, c and g_1 are constants.

Using covariant derivatives (12), we can build the operator L for the case of the Galilei superalgebra in $3+1$ dimensions

$$L = (D_1D_4 - D_2D_4) + a(\bar{D}_1\bar{D}_4 - \bar{D}_2\bar{D}_4) + bD_3D_4 + c\bar{D}_3\bar{D}_4 + fM\,, \tag{15}$$

where a, b, c and f are constant. The invariant equation (14) does not change the form in this case. The scalar superfield in $3+1$ dimensions $\Phi = \Phi(t, x, y, z, \theta_1, \theta_2, \theta_3, \theta_4, \bar{\theta_1}, \bar{\theta_2}, \bar{\theta_3}, \bar{\theta_4})$ has 64 components.

References

[1] Inonu E., Wigner T.P.,Proc. Nat. Acad Sci. U.S.**39**(1953) p.510.

[2] Thorn C.B., Nucl.Phys **B29**(1986) p.493.

[3] Bergman O.,Phys.Rev.**D46**(1992) p.5474

[4] Puzalowski R., Acta Physica Austriaca **50** (1978) p.45