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Symmetry reduction on non-expanding horizons

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Abstract. Local symmetries of a non-expanding horizon have been investigated in the 1st order formulation of gravity. When applied to a spherically symmetric horizons, only a U(1) subgroup of the Lorentz group survives as residual local symmetry that one can make use of in constructing an effective theory on the horizon.

In this paper, we have explored the local symmetries of a non-expanding horizon (NEH), a 3-dimensional null hypersurface, which imitates properties of a black hole horizon. Precise definition of NEH can be found in [1, 2], sufficient to characterize an NEH to be a lightlike hypersurface Δ imbedded in spacetime, such that the unique (up to scaling by a function) lightlike, real vector field l tangential to Δ is expansion, shear and twist-free. Since l is also normal to Δ, it is geodesic as well. These properties of Δ are independent of the scaling of l [1, 3]. Let us further assume that Δ is topologically equivalent to $\mathbb{S} \times \mathbb{R}$, where $\mathbb{S}$ is a 2-sphere. To understand local symmetries, it is imperative that the gravitational dynamics be described by first order formulation of gravity, so that we are also able to decipher those Lorentz transformations, which survive as symmetries on this null surface. Incidentally, since the definition of NEH is very general with minimal number of local conditions, our symmetry analysis, as done in the following paragraphs will survive even for Killing horizons and event horizons.

Gravity is invariant, apart from diffeomorphisms, under the local Lorentz group $SL(2,\mathbb{C})$, which is explicit in the first order formulation only. Here, our specific interest is to investigate whether the definition of NEH (more precisely the boundary conditions on it) leads to breaking of the bulk local Lorentz symmetries on Δ. This suspicion is motivated by the very well known examples of breaking of other local symmetries such as diffeomorphisms through boundary conditions in general relativity.

We have systematically analyzed to find out if there is indeed breaking of local Lorentz symmetry on NEH. Based on the residual gauge group that we have found through a kinematical analysis, we have proposed an effective theory on the horizon. This is worked out using the pre-symplectic structure on the covariant phase-space of the first order theory. It is supposed that subsequent quantization of that theory, with loop quantum gravity in the bulk would yield the quantum states of a black hole. There is a recent upsurge of interest in such effective theories, where an $SU(2)$ Chern-Simons theory has been proposed [4, 5, 6] in correlation with previous works [7, 8, 9, 10] as the effective quantum theory on the horizon in contrast to a $U(1)$ theory [11, 12, 13]. However, all of these analyses are based on isolated horizon definitions, which need some more geometric structures over our prime interest, the NEH conditions. In this medley, we have put the relevance of our paper through the symmetry reduction mechanism on NEH as a conclusive answer from $SL(2,\mathbb{C}) \rightarrow ISO(2) \ltimes \mathbb{R}$. A farther, rather dramatic reduction to $U(1)$ follows due to some special properties of the lie algebra $\mathfrak{iso}(2)$.

First, let us see how a NEH Δ reduces the local Lorentz symmetry. In order to facilitate our study, we have classified the Lorentz transformation matrices into the conjugacy classes of $SL(2,\mathbb{C})$ as:

$$\Lambda_{IJ} = -\,\xi l_I n_J - \xi^{-1} n_I l_J + 2m_{(I}\bar{m}_{J)}, \tag{1}$$

$$\Lambda_{IJ} = -\,2l_{(I}n_{J)} + (e^{i\theta} m_I \bar{m}_J + c.c.), \tag{2}$$

$$\begin{aligned}\Lambda_{IJ} = &-\,l_I n_J - (n_I - c m_I - \bar{c} m_I + |c|^2 l_I) l_J \\ &+ (m_I - \bar{c} l_I)\bar{m}_J + (\bar{m}_I - c l_I) m_J, \quad \textit{and}\end{aligned} \tag{3}$$

$$\begin{aligned}\Lambda_{IJ} = &-\,l_I n_J - (l_I - b m_I - \bar{b} m_I + |b|^2 n_I) n_J \\ &+ (m_I - \bar{b} n_I)\bar{m}_J + (\bar{m}_I - b n_I) m_J.\end{aligned} \tag{4}$$

Let us consider the Palatini connection $\mathbb{A}_{IJ}$, and in the interior of the spacetime, let us expand $\mathbb{A}_{IJ}$ in the internal Lorentz basis as:

$$\mathbb{A}_{IJ} = -2\mathbb{W} l_{[I}n_{J]} + 2\mathbb{V} m_{[I}\bar{m}_{J]} + 2(\bar{\mathbb{N}} n_{[I}m_{J]} + c.c.) + 2(\bar{\mathbb{U}} l_{[I}m_{J]} + c.c.), \tag{5}$$

where $\mathbb{W}$, $\mathbb{V}$, $\mathbb{N}$, $\mathbb{U}$ are connection 1-forms; as defined, $\mathbb{W}$ is real, $\mathbb{V}$ is imaginary and $\mathbb{N}, \mathbb{U}$ are complex (in all, there are six of them associated with the six generators). For the rest of our analysis, we have fixed an internal Lorentz frame for which $l_I, n_I, m_I, \bar{m}_I$ are constants. However, our results will be unaffected by such a choice.

The pull-back of the Palatini connection to the NEH Δ is of the form:

$$A_{IJ} \triangleq -2W l_{[I}n_{J]} + 2V m_{[I}\bar{m}_{J]} + 2(\bar{U} l_{[I}m_{J]} + c.c.), \tag{6}$$

where W, V, U are respectively the pull-backs of $\mathbb{W}, \mathbb{V}, \mathbb{U}$. Clearly, the 1-form N, which is the pull-back of $\mathbb{N}$, vanishes on Δ by the NEH boundary conditions. Proof: The simplest way to show this is to relate the connection 1-forms to the Newman-Penrose coefficients (the constant $l_I, n_I, m_I, \bar{m}_I$ basis simplifies these relations):

$$\mathbb{W} = -(\gamma + \bar{\gamma})l - (\epsilon + \bar{\epsilon})n + (\alpha + \bar{\beta})m + (\bar{\alpha} + \beta)\bar{m}\ , \tag{7}$$

$$\mathbb{V} = -(\gamma - \bar{\gamma})l - (\epsilon - \bar{\epsilon})n + (\alpha - \bar{\beta})m + (\beta - \bar{\alpha})\bar{m}\ , \tag{8}$$

$$\mathbb{U} = -\bar{\nu} l - \bar{\pi} n + \bar{\mu} m + \bar{\lambda}\bar{m}\ , \quad \textit{and} \tag{9}$$

$$\mathbb{N} = \tau l + \kappa_{\rm NP} n - \rho m - \sigma \bar{m}\ . \tag{10}$$

l_a pulled back to Δ vanishes. κ, ρ and σ also vanishe on Δ, as a consequence of the non-expanding nature of Δ; hence, does N. So, only three independent connection 1-forms W, V, U survive on Δ. This indicates that the pulled back connection given in Eq. (6) does not take the value in the full $\mathfrak{sl}(2,\mathbb{C})$, but rather in a sub-algebra of it. The following analysis puts it into firm ground. Under the local Lorentz transformations given in Eqs. (1), (2), (3) and (4), the Palatini connection in Eq. (5) transforms as:

$$\mathbb{A}_{IJ} \mapsto \Lambda_I{}^K \mathbb{A}_{KL} \Lambda_J{}^L + \Lambda_{IK} d\Lambda_J{}^K. \tag{11}$$

A lengthy but straightforward calculation shows that under the Lorentz transformations given in Eqs. (1), (2) and (3), the connection 1-forms transform as:

$$\mathbb{W} \mapsto \mathbb{W} - d\ln\xi, \quad \mathbb{V} \mapsto \mathbb{V}, \quad \mathbb{U} \mapsto \xi\mathbb{U}, \quad \mathbb{N} \mapsto \xi^{-1}\mathbb{N}. \tag{12}$$

$$\mathbb{W} \mapsto \mathbb{W}, \quad \mathbb{V} \mapsto \mathbb{V} - id\theta, \quad \mathbb{U} \mapsto e^{-i\theta}\mathbb{U}, \quad \mathbb{N} \mapsto e^{-i\theta}\mathbb{N}. \tag{13}$$

$$\begin{aligned} &\mathbb{W} \mapsto \mathbb{W} - c\mathbb{N} - \bar{c}\bar{\mathbb{N}}, \quad \mathbb{V} \mapsto \mathbb{V} - c\mathbb{N} + \bar{c}\bar{\mathbb{N}}, \\ &\qquad \mathbb{U} \mapsto \mathbb{U} - d\bar{c} + \bar{c}(\mathbb{W} - \mathbb{V}) - \bar{c}^2\bar{\mathbb{N}}, \quad \mathbb{N} \mapsto \mathbb{N}. \end{aligned} \tag{14}$$

Since $\mathbb{N}$ transforms homogeneously, its pull-back $N \triangleq 0$ in one frame implies that it vanishes in all Lorentz frames related by Eqs. (1), (2) and (3). However, under Eq. (4), the connection 1-forms transform as:

$$\begin{aligned} &\mathbb{W} \mapsto \mathbb{W} + b\mathbb{U} + \bar{b}\bar{\mathbb{U}}, \quad \mathbb{V} \mapsto \mathbb{V} - b\mathbb{U} + \bar{b}\bar{\mathbb{U}}\ , \\ &\mathbb{U} \mapsto \mathbb{U}, \quad \mathbb{N} \mapsto \mathbb{N} + d\bar{b} - \bar{b}(\mathbb{W} + \mathbb{V}) - \bar{b}^2\bar{\mathbb{U}}. \end{aligned} \tag{15}$$

Clearly, in this case, $N \triangleq 0$ if and only if b satisfies the equation $db \triangleq b(W - V + b\bar{U}) =: bY$, where Y is a 1-form. This equation has a non-trivial solution if and only if Y is a closed 1-form. However, we have shown that the equation admits only the trivial solution, $b = 0$. (For a detailed proof, see [14].) This proves conclusively that out of the four transformations given in Eqs. (1), (2), (3) and (4), the fourth one is not allowed on Δ, due to non-trivial boundary conditions on it. Apart from the fourth transformation, the rest three are generated by a Borel sub-algebra [15] of $\mathfrak{sl}(2,\mathbb{C})$ and the corresponding group is $ISO(2) \ltimes \mathbb{R}$ connection [14]. It is interesting to note that the Cartan Killing metric of this Lie algebra has two zero modes and two non-zero modes, which makes it non semi-simple.

For later convenience, we have fixed a basis for the Lie algebra $\mathfrak{iso}(2) \ltimes \mathbb{R}$ as:

$$B_{IJ} = (\partial\Lambda_{IJ}/\partial\xi)_{\xi=1} = -2l_{[I}n_{J]}, \tag{16}$$

$$R_{IJ} = (\partial\Lambda_{IJ}/\partial\theta)_{\theta=0} = 2im_{[I}\bar{m}_{J]}, \tag{17}$$

$$P_{IJ} = (\partial\Lambda_{IJ}/\partial\mathrm{Re}\, c)_{c=0} = 2m_{[I}l_{J]} + 2\bar{m}_{[I}l_{J]}, \quad \textit{and} \tag{18}$$

$$Q_{IJ} = (\partial\Lambda_{IJ}/\partial\mathrm{Im}\, c)_{c=0} = 2im_{[I}l_{J]} - 2i\bar{m}_{[I}l_{J]}. \tag{19}$$

Hence, we can expand A in this basis as:

$$A_{IJ} = 2A_B B_{IJ} + 2A_R R_{IJ} + 2A_P P_{IJ} + 2A_Q Q_{IJ}, \tag{20}$$

where $2A_B = W$, $2A_R = -iV$, $2A_P = -\mathrm{Re}\, U$, and $2A_Q = \mathrm{Im}\, U$. The connection 1-forms A_B, A_R, A_P, A_Q will turn out to be more useful in the context of an effective theory on the horizon.

Let us now turn our attention to the symplectic structures. The Hölst action [16] gives rise to the symplectic current 3-form (in units of $4\pi G\gamma_B = 1$ and $\mathbb{E}^I$ is the spacetime tetrad 1-form):

$$\mathbb{J}(\delta_1, \delta_2) = -\frac{1}{4}\mathrm{Tr}\left(\delta_1(\mathbb{E} \wedge \mathbb{E}) \wedge \delta_2\mathbb{H} - (1 \leftrightarrow 2)\right). \tag{21}$$

Here, the trace involves the $\mathfrak{sl}(2,\mathbb{C})$ Cartan-Killing metric. The expansion of the tetrad in the null tetrad basis is $\mathbb{E}^I = -nl^I - ln^I + m\bar{m}^I + \bar{m}m^I$. So the two-form $\mathbb{E}^I \wedge \mathbb{E}^J$ pulled back onto Δ, and expanded in the $\mathfrak{iso}(2) \ltimes \mathbb{R}$ basis is given by:

$$E^I \wedge E^J \triangleq {}^2\epsilon R^{IJ} + \mathrm{Re}(n \wedge m)P^{IJ} - \mathrm{Im}(n \wedge m)Q^{IJ}, \tag{22}$$

where ${}^2\epsilon = im \wedge \bar{m}$. Now, the symplectic current given in Eq. (21) is a closed spacetime 3-form $d\,\mathbb{J} = 0$. Now, consider a region $\mathcal{M}$ of the spacetime, with the partially Cauchy surfaces $M_\pm$ and the horizon Δ enclosing it. Moreover, let $M_\pm$ intersect Δ at circles $S_\pm$. Integrating $d\,\mathbb{J}$ over $\mathcal{M}$, we have found that the sum-total contribution of the symplectic current from the boundaries of $\mathcal{M}$ must vanish as:

$$\int_{M_+ \cup M_- \cup \Delta \cup i^0} \mathbb{J}(\delta_1, \delta_2) = 0. \tag{23}$$

We have assumed that the boundary conditions at infinity are such that the contribution of i^0 to the integral in Eq. (23) vanishes. We must also ensure that the symplectic structure is independent of the choice of our foliation by the partial Cauchy slices. Using Eqs. (20) and (22), and the fact that the trace in Eq. (21) is taken over a degenerate Killing metric, the pull-back of the symplectic current in Eq. (21) is:

$$J(\delta_1, \delta_2) \triangleq \frac{1}{2}\delta_1 {}^2\epsilon \wedge \delta_2(iV + \gamma_B W) - (1 \leftrightarrow 2). \tag{24}$$

It is easy to see why only the combination $iV + \gamma_B W$ survives the pull-back: The pull-back of the connection A, hence also of H, have all the $\mathfrak{iso}(2) \ltimes R$ components. However, the pull-back $E^I \wedge E^J$ is only $\mathfrak{iso}(2)$-valued, as is obvious from Eq. (22). Furthermore, only the RR and BB components survive the tracing, because of the degeneracy of the metric. Since $E^I \wedge E^J$ has no B-component, only the RR components survive in the symplectic current, which gives rise to the combination $iV + \gamma_B W$ in Eq. (24), where ${}^2\epsilon$ is the area 2-form of some spherical cross-section of Δ. In the derivation of the symplectic current, it is sufficient to assume that the spherical cross-section foliates Δ and is not necessarily a geometric 2-sphere. However, for the rest of our analysis, we have restricted ourselves to the unique foliation of Δ, in which each leaf is a geometric 2-sphere; this is possible if and only if the horizon Δ is spherically symmetric. For such a horizon with a fixed area $\mathcal{A} = \int {}^2\epsilon$, the 1-form W is closed and dV is proportional to ${}^2\epsilon$ [17, 18], and

$$dW \triangleq 0, \quad \textit{and} \quad dV \triangleq \frac{4\pi i}{\mathcal{A}}\,{}^2\epsilon, \tag{25}$$

where d is the exterior derivative intrinsic to Δ. Using Eq. (25), we have found that the symplectic current 3-form is exact on Δ, given by:

$$\begin{aligned} &J(\delta_1, \delta_2) \triangleq dj(\delta_1, \delta_2), \text{ where} \\ &j(\delta_1, \delta_2) = -\frac{\mathcal{A}}{8\pi}\delta_1(iV + \gamma_B W) \wedge \delta_2(iV + \gamma_B W). \end{aligned} \tag{26}$$

It is to be noted that in the $\mathfrak{iso}(2) \ltimes \mathbb{R}$ basis, the 1-form $iV + \gamma_B W = -2(A_R - \gamma_B A_B) =: -2A_{\rm CS}$.

We have now chosen a particular orientation of the relevant spacetime boundaries M_+, M_- and Δ, such that the current conservation given in Eq. (23) reduces to:

$$\left(\int_{M_+} - \int_{M_-}\right)\mathbb{J}(\delta_1, \delta_2) = \frac{\mathcal{A}}{2\pi}\left(\int_{S_-} - \int_{S_+}\right)(\delta_1 A_{\rm CS} \wedge \delta_2 A_{\rm CS})\,.$$

This gives a foliation independent symplectic structure, whose boundary part is given by (putting back $4\pi G\gamma_B = 1$):

$$\Omega(\delta_1, \delta_2) = -\frac{\mathcal{A}}{8\pi^2 G\gamma_B}\int_{\mathbb{S}} \delta_1 A_{\rm CS} \wedge \delta_2 A_{\rm CS}, \tag{27}$$

where $\mathbb{S}$ is the unique spherical cross-section of Δ and $A_{\rm CS} = A_R - \gamma_B A_B$.

The form in Eq. (27) suggests that on a spherically symmetric NEH, one can take the effective boundary theory as a $U(1)$ Chern-Simons theory. Two distinct cases of $U(1)$ arise: (i) If either the pull-back of A_B vanishes on $\mathbb{S}$ [19] or one restricts the gauge freedom to a constant class ($\xi =$ constant, as has been the original choice [11]) then one gets a compact $U(1)$, and (ii) In general, if no restrictions are imposed, then one gets a non-compact $U(1)$.

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